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Dominant dimensions of algebras

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Eigenständigkeitserklärung

Hiermit bestätige ich, dass ich die vorliegende Arbeit selbständig verfasst und keine anderen als die angegebenen Hilfsmittel benutzt habe. Die Stellen der Arbeit, die dem Wortlaut oder dem Sinn nach anderen Werken entnommen sind, wurden unter Angabe der Quelle kenntlich gemacht.

Contents

Abstract

We always assume that our algebras are finite dimensional connected algebras over a field K. We furthermore assume that all algebras are non-semisimple, if nothing is stated otherwise.
In this thesis we prove several new results about finite dimensional algebras and their dominant dimensions and related concepts. In the second chapter we start with the preliminaries. In [Abr], Abrar asked whether the dominant dimension of a non-selfinjective Nakayama algebra with $n \geq 3$ simple modules is bounded by $2n-3$. In chapter three we show that $2n-2$ instead of $2n-3$ is the correct bound and generalize the statement to a much more general class of algebras, giving a proof of a conjecture of Yamagata for this class of algebras. We furthermore give formulas to compute the dominant and Gorenstein dimensions of Nakayama algebras that are Morita algebras in the sense of [KerYam]. In the fourth chapter we give a new characterisation of gendo-symmetric algebras, which are algebras isomorphic to endomorphism rings of generators over symmetric algebras as first introduced in [FanKoe]. We show that an algebra is gendo-symmetric iff $(A, D(A))$ is a bocs. Further results include a description of the bocs module category and some new results about gendo-symmetric algebras using the theory of bocses.
Chapter five gives a new method to construct gendo-symmetric (nonselfinjective) Gorenstein algebras from symmetric algebras. We give the general construction, including explicit values for the dominant and Gorenstein dimensions, and then specialize to symmetric Nakayama algebras for examples.
Chapter 6 gives a new construction of an infinite series of algebras with dominant dimension at least two from any given finite dimensional algebra. We look at this construction in detail for gendo-symmetric algebras and give partial results about homological dimensions, which leads to a conjecture. We furthermore generalise the classical formulas $\tau \cong \Omega^2$ and $\tau^{-1} \cong \Omega^{-2}$ from the class of symmetric algebras to the more general class of gendo-symmetric algebras.
The last chapter is about a conjecture of Hongxing Chen and Changchang Xi in [CX] stated there as conjecture 2. We give a counterexample to this conjecture.

Zusammenfassung

In dieser Arbeit seien alle Algebren endlich dimensional und zusammenhängend über einem Körper K. Wir nehmen desweiteren an, dass alle Algebren nicht halbeinfach sind, sofern nichts anderes gesagt wird. In dieser Arbeit werden einige neue Resultate über endlich dimensionale Algebren, deren dominante Dimension und verwandte Konzepte bewiesen. Das zweite Kapitel beinhaltet die Grundlagen. In der Arbeit [Abr] hat Abrar die Vermutung geäußert, dass die dominante Dimension von nicht-selbstinjektiven Nakayamaalgebren mit $n \geq 3$ einfachen Moduln durch $2n-3$ beschränkt ist. Im dritten Kapitel zeigen wir, dass $2n-2$ anstatt $2n-3$ die korrekte Schranke ist und verallgemeinern die Aussage für eine viel größere Klasse von Algebren. Dies zeigt auch eine Vermutung von Yamagata für diese Klasse von Algebren. Wir zeigen außerdem Formeln für die dominante Dimension und Gorensteindimension von Nakayamaalgebren, die gleichzeitig Moritaalgebren im Sinne von [KerYam] sind. Im vierten Kapitel geben wir eine neue Charakterisierung von gendo-symmetrischen Algebren an. Solche Algebren sind definiert als Endomorphismenringe von Generatoren über symmetrischen Algebren und wurden in [FanKoe] eingeführt. Wir zeigen, dass eine Algebra genau dann gendo-symmetrisch ist wenn $(A, D(A))$ ein Koring im Sinne von [BreWis] ist. Weitere Resultate sind die Beschreibung der Koring Modulkategorie und einige neue Resultate über gendo-symmetrische Algebren mit Hilfe der Theorie von Koringen. Kapitel fünf gibt neue Methoden zur Konstruktion von gendo-symmetrischen (nichtselbstinjektiven) Gorensteinalgebren aus symmetrischen Algebren an. Wir beschreiben die allgemeine Konstruktion und geben explizite Werte für die dominante Dimension und die Gorensteindimension an. Wir spezialisieren dann auf symmetrische Nakayamaalgebren und geben eine große Klasse von Beispielen an. Kapitel 6 beschreibt eine neue Konstruktion einer unendliche Sequenz von Algebren mit dominanter Dimension größergleich zwei aus einer gegebenen endlich dimensionalen Algebra. Wir betrachten diese Konstruktion für gendo-symmetrische Algebren und zeigen Teilresultate über homologische Dimensionen, die zu einer Vermutung führen. Außerdem verallgemeinern wir die klassischen Formeln $\tau \cong \Omega^2$ und $\tau^{-1} \cong \Omega^{-2}$ für symmetrische Algebren auf gendo-symmetrische Algebren. Das letzte Kapitel gibt ein Gegenbeispiel zu Vermutung 2 von Hongxing Chen und Changchang Xi in [CX].

1 Introduction

This thesis mainly studies finite dimensional algebras and their dominant dimensions. The dominant dimension of a finite dimensional algebra A is defined as follows:
Let $0 \to A \to I_0 \to I_1 \to I_2 \to \cdots$ be a minimal injective resolution of the right regular module A, then the dominant dimension of A is defined to be zero in case I_0 is not projective and defined to be $sup\{n|I_n$ is projective for $i = 0, 1, ..., n\} + 1$ in case I_0 is projective. There are several cases where the dominant dimension plays a significant role in the field of representation theory:
Double centraliser properties:
Let M be a right A-module. Then M is a left B-module, when $B = End_A(M)$. M is said to have the double centralizer property in case the canonical homomorphism of algebras $f : A \to End_{B^{op}}(M)^{op}$ is an isomorphism, where $f(a)(x) = xa$. The most important situation appears when $M = eA$ for some idempotent e, such that the module eA is a minimal faithful projective-injective right module. It can be shown, see for example [KerYam2], that such an M has the double centraliser property if and only if A has dominant dimension at least two. Famous algebras having dominant dimension at least two, and thus having such a double centraliser property, are the Schur algebras $S(n, r)$ for $n \geq r$ or the higher Auslander algebras, defined by Iyama in [Iya2]. We recall the definition of the Schur algebra: Let V be an n-dimensional vector space and S_r the symmetric group on r letters. Then the group algebra KS_r for some field K operates on $V^{\otimes r}$ from the right in a natural way. Define $S(n, r) := End_{KS_r}(V^{\otimes r})$. The restriction $n \geq r$ is needed to make sure that $V^{\otimes r}$ is really a generator. Double centraliser properties are useful to relate certain subcategories of the module category of an algebra A with subcategories of the module category of an algebra B and compare homological properties of those subcategories.
Definition of special algebras:
Some important classes of algebras are defined via the use of dominant dimension. Most prominent examples are the higher Auslander algebras A, defined in [Iya2] as algebras having the property that $gldim(A) \leq n \leq domdim(A)$ for some $n \geq 2$, where $gldim(A)$ denotes the global dimension and $domdim(A)$ the dominant dimension. Those higher Auslander algebras generalise the classical Auslander algebras, which are defined as the endomorphism rings of the direct sum of all indecomposable modules over a representation-finite algebra. Other examples of algebras defined or characterised via dominant dimension are the gendo-symmetric algebras first defined in [FanKoe], which are defined as algebras A of dominant dimension at least two with a minimal faithful

projective-injective module eA for an idempotent e, such that eAe is symmetric. Such algebras generalise symmetric algebras and famous examples of gendo-symmetric algebras are the Schur algebras $S(n, r)$ for $n \geq r$.

Also the recently introduced Auslander-Gorenstein algebras in [IyaSol] are characterised via the use of domiant dimensions.
Homological conjectures:
One of the most famous conjectures in the homological part of representation theory is the Nakayama conjecture, introduced in [Nak]. The Nakayama conjecture states that every non-selfinjective finite dimensional algebra has finite dominant dimension. The conjecture is wide open and only known for small classes of algebras such as representation-finite algebras. A stronger conjecture was given by Yamagata in [Yam]:

Conjecture
(Yamagata in [Yam] on page 876) The dominant dimension of the class of non-selfinjective algebras with a given number of simples is bounded by a function depending on this number of simples.

This motivates the search for upper bounds for the dominant dimension for certain classes of algebras.
Recall that Nakayama algebras are defined as algebras having the property that every indecomposable module is uniserial. Nakayama algebras often occur in the representation theory of finite dimensional algebras. For example a famous result in the modular representation theory of finite groups is that every representation-finite block of a group algebra is stable and derived equivalent to a symmetric Nakayama algebra, see for example [Zi] chapter 5 and 6. Thus the homological theory of such representation-finite blocks is determined by the structure of symmetric Nakayama algebras. In [Abr], Abrar calculated for a large class of Nakayama algebras the dominant dimension and made the following conjecture, which can be viewed as a very special case of Yamagata's conjecture:

Conjecture
(Abrar in [Abr], 4.3.21.) The dominant dimension of a non-selfinjective Nakayama algebra with $n \geq 3$ simple modules is bounded by $2n - 3$.

For every $n \geq 2$, we will find a Nakayama algebra with n simple modules having dominant dimension $2n-2$, showing that Abrar's conjecture is not quite true. The following theorem, which is the main result of chapter 3, corrects and proves the conjecture of Abrar noting that every Nakayama algebras satisfies that eAe is again a Nakayama algebra for any idempotent e:

Theorem
(see 3.1.15) Let A be a finite dimensional nonselfinjective algebra with dominant dimension at least 1 and minimal faithful injective-projective module eA. Let s be the number of nonisomorphic indecomposable injective-projective modules in mod-A and assume that eAe is a Nakayama algebra. Then the dominant dimension of A is bounded by $2s$.

As a corollary of this theorem, every non-selfinjective Nakayama algebra with n simple modules has its dominant dimension bounded by $2n-2$ and this bound is optimal.
The Gorenstein dimension of an algebra is defined as the injective dimension of the right regular module. Our next main theorem describes how to compute dominant and Gorenstein dimensions of Nakayama algebras that are Morita algebras in the sense of [KerYam]. For simplicity, we formulate it here just in case the algebra is even gendo-symmetric and refer to the main text for a more general version and the relevant definitions:

Theorem
(see 3.2.4 and 3.2.11) Let A be a symmetric Nakayama algebra with Loewy length $w \equiv_n 1$ and n simple modules. Let $M = \bigoplus_{i=0}^{n-1} e_i A \oplus \bigoplus_{i=1}^{r} e_{x_i}A/e_{x_i}J^{w-1}$ with the x_i pairwise different for all $i \in \{1,...,r\}$. The x_i in the quiver of A are called special points. Then $B := End_A(M)$ is a Nakayama algebra and the following holds:

$$\operatorname{domdim}(B) = 2\inf\{s \geq 1 \mid \exists i,j : x_i + s \equiv_n x_j\}$$

So the dominant dimension is just twice the (directed) graph theoretical minimal distance of two special points which appear in M. Furthermore B has Gorenstein dimension

$$2\sup\{u_i \mid u_i = \inf\{b \geq 1 \mid \exists j:\ x_i + b \equiv_n x_j\}\},$$

which is twice the maximal distance between two neighboring special points.

A bocs is a generalization of the notion of coalgebra over a field. Bocses are also known under the name coring (see the book [BreWis]). A famous application of bocses has been the proof of the tame and wild dichotomy theorem by Drozd for finite dimensional algebras over an algebraically closed field (see [Dro] and the book [BSZ]). The main result of chapter 4 is the next theorem:

Theorem
(see 4.2.2) An algebra A is gendo-symmetric if and only if the bimodule $D(A)$ is a bocs.

This motivates to study gendo-symmetric algebras with tools from bocs theory. We indeed prove some new results about gendo-symmetric algebras using the theory of bocses in chapter 4.
An algebra A is called Gorenstein, in case the right and left injective dimension of the regular module coincide and are finite. It is desirable to find classes of nonselfinjective Gorenstein algebras of infinite global dimension and study concepts such as Gorenstein projective modules or the singularity category. In the literature there are not many examples of Gorenstein algebras with infinite global dimension, where the Gorenstein dimension is easy to describe.

Chapter 5 gives a construction of such Gorenstein algebras from symmetric algebras. Those algebras are gendo-symmetric and their Gorenstein dimension has an easy graph-theoretic interpretation. We call those newly constructed algebras circle gendo-symmetric algebras, since one can associate a directed cyclic graph with black and white points to those algebras, which determines their properties. We refer to chapter 5 for the technical details and just state one of the main properties of those algebras:

Theorem
(see 5.1.8) Let A be a circle gendo-symmetric algebra. Then A is Gorenstein and the Gorenstein dimension can be calculated as the maximal distance between two neighboring black points in the directed graph. The dominant dimension can be calculated as the minimal distance between two black points in the directed graph.

Chapter 6 introduces SGC-extensions, which associates to any finite dimensional algebra an infinite series of algebras of dominant dimension at least two. Namely, let $A = A_0$ be an arbitrary algebra and define for $i \geq 0$ $A_{i+1} = End_{A_i}(Ba(A_i \oplus D(A_i))$, where $Ba(M)$ of a module M denotes the basic version of this module. Call A_i then the i-th SGC-extension of A. We look at this construction for gendo-symmetric algebras and show how the dominant and Gorenstein dimension behave under those extensions for $i = 1$.

Theorem
(see 6.1.8) Let $A_0 = A$ be a gendo-symmetric algebra with finite dominant dimension and A_1 the first SGC-extension of A_0.

1. Then $domdim(A_1) = domdim(A_0)$.
2. $Gordim(A_1) = Gordim(A_0)$.

We conjecture in fact that the equality of the dominant and Gorenstein dimension continues to hold for all i-th SGC-extensions of gendo-symmetric algebras with finite dominant dimension. We also generalise the classical formulas $\tau \cong \Omega^2$ and $\tau^{-1} \cong \Omega^{-2}$ from symmetric algebra to gendo-symmetric algebras and give applications to the construction of ortho-symmetric modules and the construction of algebras having reflexive Auslander-Reiten sequences.

Chapter 7 gives a counterexample to the following conjecture 2 in [CX]:

Conjecture
Let A be an algebra of dominant dimension $n \geq 1$ with I_0 being the minimal injective hull of the regular module. Then the algebra $B := End_A(I_0 \oplus \Omega^{-n}(A))$ has dominant dimension equal to n.

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2 Preliminaries

2.1 Notation and basic definitions

Throughout an algebra A is a finite dimensional, connected algebra over a field K. Furthermore, we assume that A is not semisimple. We always work with finite dimensional right modules, if not stated otherwise. By $mod - A$, we denote the category of finite dimensional right A-modules. J will always denote the Jacobson radical of an algebra. We assume the reader to be familiar with the basic notions of representation theory and homological algebra of finite dimensional algebras and refer to [ASS] or [SkoYam] for more details. $D := Hom_K(-, K)$ denotes the K-duality of an algebra A over the field K. For a module M, $add(M)$ denotes the full subcategory of $mod - A$ consisting of direct summands of M^n for some $n \geq 1$. A module M is called *uniserial* if it has a unique composition series. A module M is called *basic* in case $M \cong M_1 \oplus M_2 \oplus ... \oplus M_n$, where every M_i is indecomposable and M_i is not isomorphic to M_j for $i \neq j$. The *basic version* of a module N is the unique (up to isomorphim) module M such that $add(M) = add(N)$ and such that M is basic. We denote by $S_i = e_iA/e_iJ$, $P_i = e_iA$ and $I_i = D(Ae_i)$ the simple, indecomposable projective and indecomposable injective module, respectively, corresponding to the primitive idempotent e_i.

The *dominant dimension* domdim(M) of a module M with a minimal injective resolution

$(I_i) : 0 \to M \to I_0 \to I_1 \to ...$ is defined as:

domdim(M):=sup$\{n|I_i$ is projective for $i = 0, 1, ..., n\}$+1, if I_0 is projective, and domdim(M):=0, if I_0 is not projective.

The *codominant dimension* of a module M is defined as the dominant dimension of the A^{op}-module $D(M)$. The dominant dimension of a finite dimensional algebra is defined as the dominant dimension of the regular module. It can be shown that the dominant dimension of an algebra always equals the dominant dimension of the opposite algebra, see for example [Ta]. So domdim$(A) \geq 1$ means that the injective hull of the regular module A is projective or equivalently, that there exists an idempotent e such that eA is a minimal faithful projective-injective module. Unless otherwise stated, e without an index will always denote the idempotent such that eA is the minimal faithful injective-projective A-module in case A has dominant dimension at least one. Algebras with dominant dimension larger than or equal to 1 are called *QF-3 algebras*. *Nakayama algebras* are defined as algebras such that every indecomposable module is uniserial or equivalently that the indecomposable projective left or right modules are uniserial. All Nakayama algebras are QF-3 algebras (see [Abr], Proposition 4.2.2 and Proposition 4.3.3). For more information on dominant dimensions and QF-3 algebras, we refer to [Ta]. The Morita-Tachikawa correspondence says that an algebra A has dominant dimension at least two iff $A \cong End_B(M)$ for some algebra B and generator-cogenerator M. Recall that an algebra is *selfinjective* in case every projective module is injective and *symmetric* in case the regular module A is isomorphic to $D(A)$ as bimodules.

Definition 2.1.1
A is called a *Morita algebra* iff it has dominant dimension larger than or equal to 2 and $D(Ae) \cong eA$ as A-right modules. This is equivalent to A being isomorphic to $End_B(M)$, where B is a selfinjective algebra and M a generator of mod-B and in this case $B = eAe$ and $M = D(eA)$ (see [KerYam]). A is called a *gendo-symmetric algebra* iff it has dominant dimension larger than or equal to 2 and $D(Ae) \cong eA$ as $(eAe, A)-$bimodules iff it has dominant dimension larger than or equal to 2 and $D(eA) \cong Ae$ as (A, eAe)-bimodules. This is equivalent to A being isomorphic to $End_B(M)$, where B is a symmetric algebra and M a generator of mod-B and in this case $B = eAe$ and $M = Ae$ (see [FanKoe]).

An algebra is called *Gorenstein* in case $injdim(A) = projdim(D(A)) < \infty$. In this case $Gdim(A)$ is called the *Gorenstein dimension* of A and we say that A has infinite Gorenstein dimension if $injdim(A) = \infty$. Note that $Gdim(A) = max\{injdim(e_iA) | e_i \text{ a primitive idempotent}\}$ and $domdim(A) = min\{domdim(e_iA) | e_i \text{ a primitive idempotent }\}$. By an *acyclic algebra* we denote quiver algebras whose quiver is acyclic.

2.2 Nakayama algebras

We now recall some results on Nakayama algebras, see chapter 32 in [AnFul] or chapter 5 in [ASS] for more on this topic. When talking about Nakayama algebras, we assume that they are given by quivers and relations (meaning that they are basic and split algebras). This is not really a restriction since the dominant dimension is invariant under Morita equivalence and field extensions, see for example [Mue] Lemma 5. A Nakayama algebra with an acyclic quiver is called *LNakayama algebra* (L for line) and with a cyclic quiver *CNakayama algebra* (C for circle). The quiver of a CNakayama algebra:

$$Q = \begin{tikzcd}
 & \circ^0 \arrow[r] & \circ^1 \arrow[dr] & \\
\circ^{n-1} \arrow[ur] & & & \circ^2 \arrow[d] \\
\circ^{n-2} \arrow[u] & & & \circ^3 \arrow[dl] \\
\ddots & \circ^5 & \circ^4 \arrow[l] &
\end{tikzcd}$$

The quiver of an LNakayama algebra:

$$Q = \circ^0 \longrightarrow \circ^1 \longrightarrow \circ^2 \quad \cdots \quad \circ^{n-2} \longrightarrow \circ^{n-1}$$

For connected CNakayama algebras with n simple modules the simple modules are numbered from 0 to n-1 clockwise (corresponding to e_iA, the projective indecomposable modules at the point i). $\mathbb{Z}/n$ denotes the cyclic group of order n and $l_r(i)$ the length of the projective indecomposable right module at the point i (so l_r is a function from $\mathbb{Z}/n$ to the natural numbers). $l_l(i)$ gives the length of the projective indecomposable left module at i. Recall that the lengths of the projective indecomposable modules determine the Nakayama algebra uniquely. We often denote $l_r(i)$ by c_i and $l_l(i)$ by d_i. In the case of a non-selfinjective CNakayama algebra, one can order the c_i such that $c_{n-1} = c_0 + 1$ and $c_i - 1 \leq c_{i+1}$ for $0 \leq i \leq n-2$ and then $(c_0, c_1, ..., c_{n-1})$ is called the *Kupisch series* of the Nakayama algebra. A Nakayama algebra A is selfinjective iff the $c_i = l_r(i)$ are all equal and the quiver of A is a circle. Every indecomposable module of a Nakayama algebra is uniserial, which means that the chain of submodules of an indecomposable module coincides with its radical series. Thus one can write every indecomposable module of a Nakayama algebra as a quotient of an indecomposable projective module P by a radical power of P. Two Nakayama algebras A (with Kupisch series $(c_0, c_1, ..., c_{n-1})$) and B (with Kupisch series $(C_0, C_1, ..., C_{m-1})$) are said to be in the same *difference class*, if $n = m$ and $c_i \equiv_n C_i$ for all $i = 0, 1, ..., n-1$. Given a Nakayama algebra with n simple modules, the largest number of the c_i minus the smallest number is less than n. Therefore there are only finitely many difference classes of Nakayama algebras with a fixed number of simple modules.

Lemma 2.2.1
The dimension of the indecomposable projective left module Ae_i at a vertex i (and, therefore, the length of the indecomposable injective right module at i) satisfies:

$$d_i = \inf\{k \geq 1 | k \geq c_{i-k}\}.$$

Furthermore, the values c_i are a permutation of the values of the d_j.

Proof. See [Ful] Theorem 2.2. □

Lemma 2.2.2
Let $M := e_iA/e_iJ^m$ be an indecomposable module of the Nakayama algebra A with $m = dim(M) \leq c_i$. Then M is injective iff $c_{i-1} \leq m$. Especially: e_iA is injective iff $c_{i-1} \leq c_i$.

Proof. See [AnFul] Theorem 32.6. □

3 Dominant dimensions of Nakayama and related algebras

3.1 Nakayama algebras and related algebras

In this chapter we prove, besides other things, that the dominant dimension of a nonselfinjective Nakayama algebra A is bounded by $2s$, where s is the number of nonisomorphic projective-injective indecomposable modules of A. Later we will provide examples which show that the number $2s$ is attained by some Nakayama algebras with s nonisomorphic projective-injective indecomposable modules. Then we can correct and prove a sharpened version of a conjecture of Abrar, who conjectured that the dominant dimension of a nonselfinjective Nakayama algebra with n simple modules is bounded by $2n-3$ (see [Abr]). We will in fact show that the correct bound is $2n-2$ and this value is attained in an example. Recall our convention that Nakayama algebras are assumed to be given by quiver and relations.

3.1.1 Resolutions for Nakayama algebras

In this subsection homological results about Nakayama algebras will be collected. A will always be a Nakayama algebra with n simple modules and indices of primitive idempotents are integers modulo n.

Let $M := e_iA/e_iJ^k$ be an indecomposable module of A. The projective cover of M is obviously e_iA and $\Omega(M) = e_iJ^k$. Then $\Omega(e_iJ^k) = e_{i+k}J^{l_r(i)-k}$, since

$$\begin{aligned} top(e_iJ^k) &= e_iJ^k/e_iJ^{k+1} \cong S_{i+k}, \\ \dim(e_iJ^k) &= l_r(i) - k \text{ and} \\ \dim(e_{i+k}J^{l_r(i)-k}) &= l_r(i+k) - (l_r(i) - k), \end{aligned}$$

which determines $\Omega(e_iJ^k)$ uniquely. To see this, recall that the submodules of $e_{i+k}A$ form a chain and $\dim(e_iJ^k)+\dim\ \Omega(e_iJ^k)=\dim(e_{i+k}A)=l_r(i+k)$. We set $j := i+k$ and define a function $f : \mathbb{Z}/n \to \mathbb{Z}/n, f(x) := x + l_r(x)$, which seems to be due to [Gus]. Then the minimal projective resolution of M has the following form by repeating the above process (f^e denotes the function f taken e times for a natural number $e \geq 0$):

$$\cdots \longrightarrow e_{f^r(j)}A \longrightarrow e_{f^r(i)}A \longrightarrow \cdots \longrightarrow e_{f^2(j)}A \longrightarrow e_{f^2(i)}A \longrightarrow e_{f^1(j)}A \longrightarrow e_{f^1(i)}A \longrightarrow e_jA \longrightarrow e_iA \longrightarrow M \longrightarrow 0$$

If we denote $e_x J^y$ for short by $(x, y) \in \mathbb{Z}/n \times \mathbb{N}$, then $\Omega(e_x J^y) = (x+y, c_x - y)$. Like this we can calculate the syzygies successively with a simple formula depending only on the Kupisch series of the Nakayama algebra. Dually, we get a minimal injective resolution (with $k = c_i$, if M is projective): We have $\mathrm{soc}(M){=}S_{i+k-1}$ (the simple module corresponding to the point $i{+}k{-}1$). Therefore, the injective hull of M is $D(Ae_{i+k-1})$ and we get $\Omega^{-1}(M) = D(J^k e_{i+k-1})$ and $\Omega^{-1}(D(J^k e_{i+k-1})) = D(J^{l_l(i+k-1)-k} e_{i-1})$, again by comparing dimensions and using that submodules form a chain. Defining $g : \mathbb{Z}/n \to \mathbb{Z}/n$ as $g(x) := x - l_l(x)$, the minimal injective resolution of M looks like this by repeating the above process:

$$0 \longrightarrow M \longrightarrow D(Ae_{j-1}) \longrightarrow D(Ae_{i-1}) \longrightarrow D(Ae_{g(j-1)}) \longrightarrow D(Ae_{g(i-1)}) \longrightarrow$$
$$\longrightarrow D(Ae_{g^2(j-1)}) \longrightarrow D(Ae_{g^2(i-1)}) \longrightarrow \cdots \longrightarrow D(Ae_{g^e(j-1)}) \longrightarrow D(Ae_{g^e(i-1)}) \longrightarrow \cdots$$

If we denote $D(J^y e_x)$ for short by $[x, y] \in \mathbb{Z}/n{\times}\mathbb{N}$ then we get that $\Omega^{-1}(D(J^y e_x)) = [x - y, d_x - y]$. Like this we can calculate the cosyzygies successively.
Now we specialize to selfinjective Nakayama algebras with Loewy length k. We give the minimal projective resolution of a general indecomposable nonprojective module M and a formula for $Ext^i(M, M)$ for arbitrary $i \geq 1$. Without loss of generality, we can assume that $M = e_0 J^s$, for $1 \leq s \leq k-1$. The minimal projective resolution of M then looks like this:

$$\cdots \longrightarrow e_{(i+1)k+s}A \xrightarrow{L_{(i+1)k,s}} e_{(i+1)k}A \xrightarrow{L_{ik+s,k-s}} e_{ik+s}A \xrightarrow{L_{ik,s}} e_{ik}A \longrightarrow$$
$$\longrightarrow \cdots \longrightarrow e_k A \xrightarrow{L_{s,k-s}} e_s A \xrightarrow{L_{0,s}} M \longrightarrow 0.$$

Here, we denote by $L_{x,y}$ the left multiplication by $w_{x,y}$, where $w_{x,y}$ is the unique path starting at x and having length y. If we apply the functor $Hom(-, M)$ to this minimal projective resolution (with M deleted), then we get the complex:

$$0 \longrightarrow e_0 J^s e_s \xrightarrow{R_{s,k-s}} e_0 J^s e_k \xrightarrow{R_{k,s}} e_0 J^s e_{k+s} \longrightarrow \cdots \longrightarrow$$
$$\longrightarrow e_0 J^s e_{ik} \xrightarrow{R_{ik,s}} e_0 J^s e_{ik+s} \xrightarrow{R_{ik+s,k-s}} e_0 J^s e_{(i+1)k} \xrightarrow{R_{(i+1)k,s}} e_0 J^s e_{(i+1)k+s} \longrightarrow \cdots$$

Here, $R_{x,y}$ is right multiplication by $w_{x,y}$. We see that $R_{ik+s,k-s} = 0$ for all $i \geq 1$, since we map paths of length at least s to paths of length at least k (and $J^k = 0$). Therefore, we have for all $i \geq 1$:
$Ext^{2i-1}(M, M) = ker(R_{ik,s}) \neq 0$, iff there is a path of length larger than or equal to $k - s$ in $e_0 J^s e_{ik}$ and $Ext^{2i}(M, M) = e_0 J^s e_{ik+s}/Im(R_{ik,s})$.

Calculating dominant dimensions of Nakayama algebras
The following theorem shows that the dominant dimension of a given Nakayama algebra depends only on its difference class:

Theorem 3.1.1
Let A be a Nakayama algebra with n simple modules and $M = e_i A / e_i J^k$ be an

A-module. The dominant dimension of M depends only on the difference class of A and on the $i \bmod n$ and $k \bmod n$. Especially, the dominant dimension of A depends only on the difference class of A.

Proof. We may assume that M is not injective. First we see that in a given difference class of Nakayama algebras, e_iA is injective iff $c_{i-1} \leq c_i$, so the position of the injective-projective modules doesn't depend on the choice of A inside a given difference class. In order to determine the dominant dimension of M, we calculate a minimal injective resolution (I_i) and the cosyzygies of M. Note that $\Omega^{-1}(M) = D(J^k e_{i+k-1})$ and that calculating syzygies of modules of the form $[x,y] = D(J^y e_x)$ is done by $\Omega^{-1}[x,y] = [x-y, d_x - y]$. If $\Omega^{-j}(M) = D(J^p e_q)$, then $I_{j-1} \cong D(Ae_q)$. We see that all those calculations only depend on i, $k \bmod n$ and the difference class (which determines the $d_i \bmod n$) of the algebra. Now there are two cases to consider:
<u>Case 1</u>: $\Omega^{-j}(M) \neq 0$ for every $j \geq 1$. Then by the above the indices of the socles of I_i do not depend on the difference class and $i \bmod n$ and $k \bmod n$. Thus the calculation of the dominant dimension of M is also independent of the difference class and $i \bmod n$ and $k \bmod n$.
<u>Case 2:</u> Assume now that $\Omega^{-j}(M) = 0$ in one algebra of a given difference class, but $\Omega^{-j}(M) \neq 0$ in another algebra in the given difference class for a module M of the form e_iA/e_iJ^k, for some $j \geq 1$. When $\Omega^{-j}(M) = 0$ happens for some $j \geq 1$ for the first time, there must have been an I_l with $l \leq j-1$, which is not projective. Otherwise we would have a minimal injective resolution (I_i), with the properties that all terms are also projective and that its ending has the following form:

$$\cdots \to I_{j-2} \xrightarrow{f} I_{j-1} \to 0.$$

Therefore, the surjective map f between projective modules would be split, contradicting the minimality of the resolution. So calculating the dominant dimension of M involves only those terms I_l for $1 \leq l \leq j-1$ in the minimal injective resolution of M until $\Omega^{-j}(M) = 0$ happens for the first time. But those terms in the injective resolution depend only on the difference class of A and $i \bmod n$ and $k \bmod n$ and so does the dominant dimension of M.

$\square$

Example 3.1.2
We calculate the dominant dimension of a Nakayama algebra A in the difference class of Nakayama algebras with Kupisch series $(c_0, c_1, c_2) = (3k+2, 3k+2, 3k+3)$, for $k \geq 0$. First we calculate the dimension of the injective indecomposable modules:
$l_l(0) = \inf\{s \geq 3k+2 \mid s \geq l_r(-s)\} = 3k+2$ and likewise $l_l(1) = 3k+3$ and $l_l(2) = 3k+2$. Thus $(d_0, d_1, d_2) = (3k+2, 3k+3, 3k+2)$. With $\mathrm{soc}(e_1A)$=$e_1J^{3k+1} \cong S_2$, it follows that e_1A embeds into $D(Ae_2)$. But, since e_1A and $D(Ae_2)$ have the same dimension, they are isomorphic.
With $\mathrm{soc}(e_2A) = e_2J^{3k+2} \cong S_1$, it follows that e_2A embeds into $D(Ae_1)$ and as above both are isomorphic, because they have the same dimension. Thus the projective-injective indecomposable modules are $e_1A \cong D(Ae_2)$ and $e_2A \cong D(Ae_1)$. Now its enough to look at an injective resolution of e_0A. Since

$soc(e_0A) = e_0J^{3k+1} \cong S_1$, e_0A embeds into $D(Ae_1)$ with cokernel equal to $D(J^{3k+2}e_1) = [1, 3k+2]$. Then $\Omega^{-1}([1, 3k+2]) = [1-(3k+2), d_1-(3k+2)] = [2, 1]$ and $\Omega^{-1}([2, 1]) = [2-1, d_2-1] = [1, 3k+1]$ and $\Omega^{-1}([1, 3k+1]) = (1-(3k+1), d_1-(3k+1)) = [0, 2]$. The minimal injective resolution of e_0A starts as follows:

$$0 \to e_0A \to D(Ae_1) \to D(Ae_2) \to D(Ae_1) \to D(Ae_0) \to \cdots .$$

Since $D(Ae_0)$ is not projective, the dominant dimension of e_0A is equal to 3, as is the dominant dimension of A, since e_0A is the only indecomposable projective and not injective module. Note that if A has Kupisch series $(2, 2, 3)$, then $D(J^2e_0) = [0, 2]$=0, while for $k \geq 1$, that module is nonzero. Also note that the Gorenstein dimension is not independent of the difference class of the Nakayama algebra: If A has Kupisch series $(2, 2, 3)$, then, by the above, the Gorenstein dimension is equal to the dominant dimension and finite. But, if A has Kupisch series $(3k+2, 3k+2, 3k+3)$ for a $k \geq 1$, then continuing as above, one gets: $\Omega^{-1}([0, 2]) = [1, 3k]$, $\Omega^{-1}([1, 3k]) = [1, 3]$, $\Omega^{-1}([1, 3]) = [1, 3k+2] = \Omega^{-1}(e_0A)$, and the resolution gets periodic and is, therefore, infinite.

3.1.2 Gorenstein-projective modules

In this subsection, A denotes a finite dimensional algebra. See [Che] Section 2, for an elementary introduction to Gorenstein homological algebra. We take our definitions and lemmas from this source.

Definition 3.1.3
A complex $P^\bullet : ... \to P^{n-1} \xrightarrow{d^{n-1}} P^n \xrightarrow{d^n} P^{n+1} \to ...$ of projective A-modules is called *totally acyclic*, if it is exact and the complex $Hom(P^\bullet, A)$ is also exact. An A-module M is called *Gorenstein-projective*, if there is a totally acyclic complex of projective modules such that $M = ker(d^0)$. We denote by A-gproj the full subcategory of mod-A of Gorenstein-projective modules and we denote by ${}^\perp A$ the full subcategory of mod-A of all modules N with $Ext^i(N, A) = 0$, for all $i \geq 1$. $D(A)^\perp$ denotes the subcategory of mod-A of all modules N with $Ext^i(D(A), N) = 0$ for all $i \geq 1$.

Lemma 3.1.4
(see [Che] Corollary 2.1.9. and 2.2.17.)
Let A be a finite dimensional algebra and M an A-module.

1. A-gproj $\subseteq {}^\perp A$.

2. An A-module N is in A-gproj, in case there is an n, such that $Ext^i(N, A) = 0$, for all $i = 1, ..., n$, and $\Omega^n(N) = N$.

3. If $Ext^i(N, A) = 0$ for all $i = 1, ..., d$ and $\Omega^d(N)$ is Gorenstein-projective, then also N is Gorenstein-projective.

Lemma 3.1.5
If A is a Nakayama algebra, then A-gproj $= {}^\perp A$.

Proof. We know that A-gproj $\subseteq {}^{\perp}A$. Now let $M \in {}^{\perp}A$ with M indecomposable. Since all syzygies over a Nakayama algebra of an indecomposable module are also indecomposable and since there is only a finite number of indecomposable modules, there exist numbers k, n with : $\Omega^n(\Omega^k(M)) = \Omega^k(M)$. Since we also have $\Omega^k(M) \in {}^{\perp}A$ by the formula $Ext^i(\Omega^k(M), A) = Ext^{i+k}(M, A) = 0$, we know that $\Omega^k(M)$ is Gorenstein-projective by 2. of the above lemma. Now by 3. of the above lemma also M is Gorenstein-projective. $\square$

Lemma 3.1.6
Let M be an indecomposable Gorenstein-projective A-module which is not projective. Then there is an exact sequence $0 \to M \to P \to N \to 0$ such that P is projective.

Proof. By the definition of Gorenstein-projective, M can be embedded in a projective module P. $\square$

Corollary 3.1.7
An indecomposable injective and Gorenstein projective-module is projective.

3.1.3 CoGen-dimension and dominant dimension

Definition 3.1.8
For a finite dimensional algebra A and a module M we define ϕ_M as
$\phi_M := \inf\{r \geq 1 | Ext_A^r(M, M) \neq 0\}$ with the convention $\inf(\emptyset) = \infty$. We call a module M which is a generator and a cogenerator for short a *CoGen*. We also define $\Delta_A := \sup\{\phi_M | M$ is a nonprojective CoGen $\}$.

We remark that for a nonselfinjective algebra A
$\Delta_A = \inf\{r \geq 1 | Ext_A^r(D(A), A) \neq 0\}$, and for a selfinjective algebra A
$\Delta_A = \sup\{\phi_M | M$ is a non-projective indecomposable A-module$\}$.

Theorem 3.1.9
(see [Mue]) If M is a CoGen of A, then the dominant dimension of $B := End_A(M)$ is equal to $\phi_M + 1$.

We often refer to the previous theorem as Mueller's theorem.
The next conjecture is known as the Nakayama conjecture.

Conjecture
The Nakayama conjecture states that every nonselfinjective finite dimensional algebra has finite dominant dimension.

As a corollary of Mueller's theorem, the Nakayama conjecture is equivalent to the finiteness of Δ_A, for every finite dimensional algebra A.

Conjecture
Yamagata (in [Yam]) states the even stronger conjecture that the dominant dimensions of nonselfinjective algebras with a fixed number of simple modules are bounded by a function of the number of simple modules of A.

In this section, we will prove Yamagata's conjecture in case eAe is a Nakayama algebra or a quiver algebra with an acyclic quiver, when eA is the minimal faithful injective-projective A-module.

Lemma 3.1.10
Let A be a nonselfinjective connected algebra of finite injective dimension $g = injdim(A)$. Then $\Delta_A \leq g$.

Proof. We have

$$g = injdim(A) = projdim(D(A)) =$$
$$\sup\{r \geq 1 | Ext_A^r(D(A), A) \neq 0\} \geq \inf\{r \geq 1 | Ext_A^r(D(A), A) \neq 0\}$$
$$= \Delta_A,$$

where we used $projdim(M) = sup\{r \geq 1 | Ext_A^r(M, A) \neq 0\}$, in case M has finite projective dimension. □

The following generalizes and gives an easier proof of Theorem 1.2.3 of [Abr], which states 3. of the following Corollary.

Corollary 3.1.11
1. Let A be an connected acyclic algebra with $d \geq 2$ simple modules. Then $gldim(A) \leq d-1$ and, therefore, $\Delta_A \leq d-1$.

2. Let A be a QF-3 algebra with s projective-injective indecomposable modules such that eAe is acyclic, where eA is the minimal faithful injective-projective module. Then $\text{domdim}(A) \leq s$.

3. Let A be an acyclic algebra with s indecomposable injective-projective modules, then $\text{domdim}(A) \leq s-1$.

4. For an LNakayama algebra A with n simple modules, the following holds: $\Delta_A \leq n-1$.

Proof.
1. For an elementary proof of $gldim(A) \leq d-1$, see e.g. [Farn]. Then $\Delta_A \leq d-1$ follows from the previous lemma, since the equality $gldim(A) = injdim(A)$ holds, in case A has finite global dimension.

2. In case A has dominant dimension equal to one, the statement is clear. Now assume A has dominant dimension at least two. Then $A \cong End_B(M)$ for some algebra B and generator-cogenerator M by the Morita-Tachikawa correspondence. By Mueller's theorem and the previous part (1) the statement follows.

3. This holds, since with A also eAe is acyclic for every idempotent $e \in A$.

4. This is clear by 3., since LNakayama algebras are acyclic.

□

We give the following example of a nonacyclic algebra Λ such that $C = e\Lambda e$ is acyclic to show that the above is really a generalisation of Theorem 1.2.3 of [Abr].

Example 3.1.12
Take any acyclic endowild algebra C (this means that for every finite dimensional algebra R, there is a finite dimensional C-module N with $R \cong End_C(N)$. Examples of such algebras C are wild hereditary algebras over an algebraically closed field, see [SimSko3] page 329) and a basic C-module M such that $End_C(M)$ is not acyclic. We claim that then the algebra $\Lambda = End_C(B(C \oplus D(C) \oplus M))$ is not acyclic, where $B(X)$ denotes the basic version of a module X. Denoting here by f the projection from $B(C \oplus D(C) \oplus M)$ onto $B(M)$, the algebra $f\Lambda f \cong End_C(B(M))$ is not acyclic and thus Λ is not acyclic. On the other hand, denoting by $e \in \Lambda$ the idempotent such that $e\Lambda$ is the minimal faithful projective-injective right module, the algebra $e\Lambda e \cong C$ is acyclic.

For the main lemma, recall the following result:

Lemma 3.1.13
Let A be a finite dimensional algebra, N be an indecomposable A-module and S a simple A-module. Let (P_i) be a minimal projective resolution of N and (I_i) a minimal injective resolution of N.

1. For $l \geq 0$, $Ext^l(N,S) \neq 0$ iff S is a quotient of P_l.
2. For $l \geq 0$, $Ext^l(S,N) \neq 0$ iff S is a submodule of I_l.

Proof. See [Ben] Corollary 2.5.4. □

Lemma 3.1.14
(Mainlemma) Let A be a finite dimensional nonselfinjective Nakayama algebra with $n \geq 2$ simple modules. Let N be an A-module and S a simple A-module.

1. Assume that $Ext^l(N,S) \neq 0$ for some $l \geq 1$. Then $\inf\{s \geq 1 | Ext^s(N,S) \neq 0\} \leq 2n-2$.
2. Assume that $Ext^l(S,N) \neq 0$ for some $l \geq 1$. Then $\inf\{s \geq 1 | Ext^s(S,N) \neq 0\} \leq 2n-2$.

Proof. We only prove 1. since 2. follows dually.
We can assume that $Ext^1(N,S) = 0$, since the result is obvious in case $Ext^1(N,S) \neq 0$.
So in the following we look at the problem of determining the smallest possible finite value $s \geq 2$ with respect to the following properties:
$Ext^s(N,S) \neq 0$, but $Ext^i(N,S) = 0$, for $i = 1, ..., s-1$, for an indecomposable module N. $Ext^s(N,S) \neq 0$ simply means that in the minimal projective resolution (P_i) of N there is a direct summand of P_s isomorphic to the projective cover of $S = S_r := e_r A / e_r J$ by the previous lemma. The minimal projective resolution has the form:

$$\cdots \longrightarrow e_{f^r(j)}A \longrightarrow e_{f^r(i)}A \longrightarrow \cdots \longrightarrow e_{f^2(j)}A \longrightarrow$$
$$\longrightarrow e_{f^2(i)}A \longrightarrow e_{f^1(j)}A \longrightarrow e_{f^1(i)}A \longrightarrow e_jA \longrightarrow e_iA \longrightarrow N \longrightarrow 0$$

Claim1. f is not surjective.
Proof: If f were surjective, it would be bijective and, because of $soc(e_iA) = S_{f(i)-1}$, for every i, A would be selfinjective (see [SkoYam] Chapter IV. Theorem 6.1.), contradicting our assumption that A is not selfinjective. So Claim 1 is proved.

Now, $Ext^u(N, S) \neq 0$, for some $u \geq 1$, tells us $e_rA \cong P_u$.

Claim2. The smallest index i with $e_rA \cong P_i$ must be smaller than or equal to $2n-2$.
Proof: Since f is a mapping from a finite set to a finite set, there is a minimal number w with $Im(f^w) = Im(f^{w+1})$. Define $X := Im(f^w)$. Note that the cardinality of X is smaller than or equal to $n-w$, since $f : \mathbb{Z}/n \to \mathbb{Z}/n$ is not surjective and therefore the number of elements in $Im(f^i)$ decreases by at least 1 as long as $i < w$ in the sequence $Im(f) \supset Im(f^2) \supset \cdots \supset Im(f^w)$.
f is a bijection from X to X, since $f\big|_X : X \to X$ is surjective (and X a finite set). When we have reached

$$\cdots \to e_{f^{w+1}(j)}A \to e_{f^{w+1}(i)}A \to e_{f^w(j)}A \to e_{f^w(i)}A \to \cdots,$$

f acts as a cyclic permutation on the $\{f^l(i)\}$ and $\{f^l(j)\}$ for $l \geq w$. Note that after the term $e_{f^{w+n-w-1}(i)}A = e_{f^{n-1}(i)}A = P_{2n-2}$ (recall that X has cardinality at most $n-w$) some index $q \in X$ must exist such that $P_{2n-1} = e_qA$ and this indecomposable projective module e_qA is also isomorphic to P_u for an $u \leq 2n-2$. Therefore, the index r must occur in the first $2n-2$ terms, since later there are only indices which already occurred before. □

Theorem 3.1.15
Let A be an algebra with dominant dimension larger than 1 and the property that eA is a minimal faithful projective injective module and eAe is a Nakayama-algebra with s simple modules. Then:

$$domdim(A) \leq \Delta_{eAe} + 1 \leq 2s.$$

Especially, Yamagata's conjecture is true in this special case.

Proof. We first prove the following lemma:

Lemma 3.1.16
For a Nakayama algebra with n simple modules, the following holds: $\Delta_A \leq 2n-1$.

We split the proof of this lemma in two cases: one case of a nonselfinjective Nakyama algebra and in the other case the Nakayama algebra is selfinjective.

<u>Case 1:</u> For a nonselfinjective CNakayama algebra A with n simple modules, $\Delta_A \leq 2n-1$.

Proof. There is an injective indecomposable module I of the form $I = eA/soc(eA)$: 2.2.2 tells us that the module $I = eA/soc(eA)$ is injective, when i is chosen such that $c_{i-1} \leq c_i - 1$ and $e := e_i$. This module is not Gorenstein-projective by Corollary 3.1.7, since it is injective, but not projective. By Lemma 3.1.5 I is not in ${}^{\perp}A$. Therefore, there is a smallest index $k \geq 1$ with $Ext^k(I, A) \neq 0$. Thus also $Ext^k(D(A), A) \neq 0$ which implies the theorem, if we show that $k \leq 2n-1$. If $Ext^1(I, A) \neq 0$, there is nothing to prove. So we assume that $Ext^1(I, A) = 0$. Denote soc(eA) by $S = S_r$ (which means that the projective cover of S is $e_r A$). In general we have $Ext^i(S, A) = Ext^i(\Omega(I), A) = Ext^{i+1}(I, A)$. Therefore, we will look in the following for the smallest index s with $Ext^s(S, A) \neq 0$. But by the main lemma $\inf\{s \geq 1 | Ext^s(S, A) \neq 0\} \leq 2n - 2$.
Because of $Ext^i(S, A) = Ext^i(\Omega(I), A) = Ext^{i+1}(I, A)$ and $2n-2+1 = 2n-1$ we proved case 1.

□

For the next case, recall the results of 3.1.1 about calculating minimal projective resolutions and $Ext^i(M, M)$ for an indecomposable module M in a selfinjective Nakayama algebra.
<u>Case 2:</u> A selfinjective Nakayama algebra A with n simple modules, satisfies:

$$\Delta_A \leq 2n - 1.$$

Proof. To prove this, we have to show that $\phi_M \leq 2n - 1$ for all nonprojective indecomposable modules M.
We can assume that A has Loewy length k and $M = e_0 J^s$, with $1 \leq s \leq k-1$. We consider two cases:
<u>First case:</u> k is a zero divisor in $\mathbb{Z}/n$. Then there is a q with $kq \equiv_n n \equiv_n 0$ and $1 \leq q \leq n-1$. We know that $\Omega^{2i}(M) = e_{ik} J^s$ and, therefore, $\Omega^{2q}(M) = e_{qk} J^s = e_0 J^s = M$. Consequently, $Ext^{2q}(M, M) = \underline{Hom}(\Omega^{2q}(M), M) = \underline{Hom}(M, M) \neq 0$.
<u>Second case:</u> k is not a zero divisor in $\mathbb{Z}/n$ and, therefore, a unit. We have $Ext^{2i-1}(M, M)$
$= ker(R_{ik,s}) \neq 0$, iff there is a path of length larger than or equal to $k - s$ in $e_0 J^s e_{ik}$. But, for $i = 1, ..., n$, the integers ik are all different from one another mod n. This is why there surely is a path of length larger than or equal to $k - s$ in $e_0 J^s e_{ik}$ for some $i \leq n$. □

Now we return to the proof of 3.1.15:
Combining Case 1 and Case 2, we have proved the lemma 3.1.16. To get a proof of theorem 3.1.15, we use Mueller's theorem and the fact that the number of nonisomorphic indecomposable projective-injective modules equals the number of simple modules of eAe to get that

$$domdim(A) \leq \Delta_{eAe} + 1 \leq 2s.$$

This finishes the proof of 3.1.15.

□

3.1.4 Dominant dimension of Nakayama algebras

We will prove the bound $2s$ for the dominant dimension of a non-selfinjective Nakayama algebra with s projective-injective indecomposable modules in this section:

Theorem 3.1.17
Let B be a nonselfinjective Nakayama algebra with s projective-injective indecomposable modules. Then the dominant dimension of B is bounded above by $2s$.

Note that s in the previous theorem is always bounded by $n-1$, when n denotes the number of simple B-modules. We will also show in the next section that there is a Nakayama algebra such that the maximal value $2(n-1)$ (if B has n simple modules) is attained, see Corollary 2.1.4. Therefore the maximal possible value of the dominant dimension of a nonselfinjective Nakayama algebra with n simple modules is $2(n-1)$. This corrects and proves a conjecture of Abrar, who conjectured that the maximal value is $2n-3$ (see [Abr] Conjecture 4.3.21).

Lemma 3.1.18
If B is a Nakayama algebra, then $A := eBe$ is a Nakayama algebra for every idempotent e of B.

Proof. If J is the radical of B, the radical of eBe is eJe (see [Lam] Theorem 21.10). If $e = e_1 + \ldots + e_n$, with primitive orthogonal idempotents e_i of B, then those e_i are a complete system of primitive orthogonal idempotents in eBe.
We have $e_i(rad(eBe)/(rad^2(eBe)))e_j = e_iJe_j/(e_iJeJe_j)$. Now e_iJe_j are those paths starting at i and ending at j with length larger than or equal to 1. e_iJeJe_j are those paths of the form $\alpha\beta$, where α is a path (of length larger than or equal to 1) from i to a point in e and β is a path (of length larger than or equal to 1) from a point in e to j. We see that there is at most one arrow starting at i and at most one arrow ending at j in the quiver of eBe, since $e_i(rad(eBe)/(rad^2(eBe)))e_j = e_iJe_j/(e_iJeJe_j) \neq 0$, iff there is no e_k, which is a summand of e, between e_i and e_j. But the property that there is at most one arrow starting at e_i and at most one arrow ending at e_j in the quiver of eBe for all points e_i, e_j characterises Nakayama algebras.

Therefore, Theorem 3.1.17 follows from 3.1.15 and the theorem of Mueller, since $A := eBe$ is a Nakayama algebra with s simple modules and $\Delta_A \leq 2s-1$. So we have by Mueller's theorem: $\mathrm{domdim}(B) = \Delta_A + 1 \leq 2s$. □

3.1.5 Finitistic dominant dimension of Nakayama algebras

Using the main lemma, we show in this section that we can give a bound of the finitistic dominant dimension for Nakayama algebras.

Definition 3.1.19
The *finitistic dominant dimension* of a finite dimensional algebra A is

$$\text{fdomdim}(A) := \sup\{\text{domdim}(M) | \text{domdim}(M) < \infty\}$$

Example 3.1.20
If A has global dimension g, then fdomdim$(A) \leq g$, since for every noninjective module M domdim$(M) \leq$injdim$(M) \leq g$ holds.

The following theorem gives again the bound $2n-2$ for the dominant dimension of Nakayama algebras.

Theorem 3.1.21
Let A be a nonselfinjective Nakayama algebra with $n \geq 2$ simple modules. Then fdomdim$(A) \leq 2n-2$.

Proof. Clearly we can assume that A is a CNakayama algebra, since an LNakayama algebra has global dimension at most $n-1$. So assume now that A is a CNakayama algebra and M an indecomposable A-module with finite dominant dimension. Note that domdim$(M) = \inf\{i | Ext^i(S,M) \neq 0$ for a simple module S with nonprojective injective hull$\}$. We can assume that M has dominant dimension larger than or equal to 1. Let S be a simple module with nonprojective injective envelope such that $Ext^i(S,M) \neq 0$ for an $i \geq 1$. Then by the main lemma $\inf\{s \geq 1 | Ext^s(S,N) \neq 0\} \leq 2n-2$ and thus domdim$(M) \leq 2n-2$. □

Example 3.1.22
Take the CNakayama algebra A with Kupisch series $(3s+1, 3s+2, 3s+2)$, $s \geq 1$. We first calculate the Gorenstein dimension and the dominant dimension of A and then the finitistic dominant dimension of A. First note that $e_1A \cong D(Ae_2)$ is injective. Also $e_2A \cong D(Ae_0)$ is injective. The only noninjective indecomposable projective module is then e_0A and the only nonprojective injective indecomposable module is $D(Ae_1)$. We have the following injective resolution:

$$0 \to e_0A \to D(Ae_0) \to D(Ae_2) \to D(Ae_1) \to 0.$$

Thus the dominant dimension and the Gorenstein dimension of A are both 2. Now take an indecomposable module $M = e_aA/e_aJ^k$ and calculate the minimal injective presentation of M: $0 \to M \to D(Ae_{a+k-1}) \to D(Ae_{a-1})$. Thus M has dominant dimension larger than or equal to 2 iff $a+k-1 \in \{0,2\}$ mod 3 and $a-1 \in \{0,2\}$ mod 3 iff ($a = 0$ mod 3 and $k \in \{0,1\}$ mod 3) or ($a = 1$ mod 3 and $k \in \{0,2\}$ mod 3). The following table gives the relevant values of the dominant dimensions:

a	0	1
$k \equiv_3 0$	4	2
$k \equiv_3 1$	2	-
$k \equiv_3 2$	-	3

Thus the finitistic dominant dimension equals 4, while the finitistic dimension equals the Gorenstein dimension which is 2.

3.2 Nakayama algebras which are Morita algebras and their dominant and Gorenstein dimension

In this section we calculate the dominant dimension of all Nakayama algebras that are Morita algebras and give the promised example of a nonselfinjective Nakayama algebra having n simple modules and dominant dimension $2n-2$. We also show how to calculate the Gorenstein dimension of such algebras and give a surprising interpretation of the dominant and Gorenstein dimension for gendo-symmetric Nakayama algebras. This is the main results of this chapter and we will need several subsections of preliminary results before we can come to the proof of the main result.

3.2.1 Calculating the dominant dimensions of Nakayama algebras that are Morita algebras

Definition 3.2.1
A finite dimensional algebra B is called a Morita algebra, if it is isomorphic to the endomorphism ring of a module M, which is a generator of a selfinjective algebra A (see [KerYam]). If A is even symmetric, then B is called a gendo-symmetric algebra (see [FanKoe]).

The following is a special case of a result of Yamagata in [Yam2].

Theorem 3.2.2
Let A be a nonsemisimple selfinjective Nakayama algebra with Loewy length w and $M = \bigoplus_{i=0}^{n-1} e_i A \oplus \bigoplus_{i=1}^{r} e_{x_i} A/e_{x_i} J^{k_i}$. Then $B := End_A(M)$ is a basic nonselfinjective Nakayama algebra, iff all the x_i are pairwise different and $k_i = w-1$ for all $i \in \{1, ..., r\}$ and $r \geq 1$.

Keep this notation for B and call points of the form x_i special points. We say that two special points are *neighboring* in case there is no other special point between them.

Proposition 3.2.3
Let $r \geq 1$. Let A be a selfinjective Nakayama algebra with Loewy length w and n simple modules. Let $M = \bigoplus_{i=0}^{n-1} e_i A \oplus \bigoplus_{i=1}^{r} e_{x_i} A/e_{x_i} J^{w-1}$ with the x_i different for all $i \in \{1, ..., r\}$. Then:

$$\begin{aligned} domdim(B) &= \phi_M + 1 \\ &= \inf\{k \geq 1 | \exists x_i, x_j : Ext^k(e_{x_i}A/e_{x_i}J^{w-1}, e_{x_j}A/e_{x_j}J^{w-1}) \neq 0\} + 1 \\ &= \inf\{k \geq 1 | \exists x_i, x_j : x_j + w - 1 \equiv_n x_i + [\frac{k+1}{2}]w - g_k\} + 1. \end{aligned}$$

Here, we set $g_k = 1$, if k is even, and $g_k = 0$, if k is odd. $[l]$ is equal to l, if l is an integer, and otherwise equal to the smallest integer larger than l (for example $[1.5] = 2$).

Proof. Note that the first equality is by Mueller's theorem and we just have to show the last equality. Lemma 1.3.7 says that for a module M and a simple module S $Ext^i(M,S) \neq 0$ iff S is a direct summand of the top of the module P_i, where P_i is the i-th term in a minimal projective resolution of M. Note that

$$Ext^k(e_{x_i}A/e_{x_i}J^{w-1}, e_{x_j}A/e_{x_j}J^{w-1}) = Ext^k(e_{x_i}J^{w-1}, e_{x_j}J^{w-1}),$$

which is what we want to calculate.
Observe that $e_{x_i}J^{w-1} \cong S_{x_i+w-1}$ is a simple module. The minimal projective resolution of $e_{x_i}J^{w-1}$ then looks like this:

$$\cdots \longrightarrow e_{x_i+(l+1)w+w-1}A \longrightarrow e_{x_i+(l+1)w}A \longrightarrow e_{x_i+lw+w-1}A \longrightarrow e_{x_i+lw}A \longrightarrow$$
$$\longrightarrow \cdots \longrightarrow e_{x_i+w}A \longrightarrow e_{x_i+w-1}A \longrightarrow e_{x_i}J^{w-1} \longrightarrow 0$$

Thus the k-th term in the minimal projective resolution of $e_{x_i}J^{w-1}$ is equal to

$$P_k = e_{x_i+[\frac{k+1}{2}]w-g_k}A.$$

Then $Ext^k(e_{x_i}A/e_{x_i}J^{w-1}, e_{x_j}A/e_{x_j}J^{w-1}) \neq 0$, iff $x_j+w-1 \equiv_n x_i+[\frac{k+1}{2}]w-g_k$, for a $k \geq 1$. □

Corollary 3.2.4
Let A be a symmetric Nakayama algebra with Loewy length $w \equiv_n 1$ and n simple modules. Let $M = \bigoplus_{i=0}^{n-1} e_iA \oplus \bigoplus_{i=1}^{r} e_{x_i}A/e_{x_i}J^{w-1}$ with the x_i different for all $i \in \{1,...,r\}$. Then for $B = End_A(M)$:

$$\text{domdim}(B) = 2\inf\{s \geq 1 \mid \exists i,j : x_i + s \equiv_n x_j\}$$

So the dominant dimension is just twice the (directed) graph theoretical minimal distance of two special points which appear in M.

Proof. The formula takes for $w \equiv_n 1$ an especially nice form:

$$\text{domdim}(B) = \inf\{k \geq 1 \mid \exists i,j : x_i + [\frac{k+1}{2}] - g_k \equiv_n x_j\} + 1.$$

For $k = 2s+1$ and $k = 2s+2$ the value of $[\frac{k+1}{2}] - g_k$ is the same. This means that the infimum is attained at an odd number of the form $k = 2s-1$ and the formula simplifies to

$$\begin{aligned}\text{domdim}(B) &= \inf\{2s-1 \geq 1 \mid \exists i,j : x_i + s \equiv_n x_j\} + 1 \\ &= 2\inf\{s \geq 1 \mid \exists i,j : x_i + s \equiv_n x_j\}.\end{aligned}$$

□

We can now state the corrected conjecture of Abrar as the next proposition by showing that the bound is optimal:

Proposition 3.2.5
A non-selfinjective Nakayama algebra with $n \geq 2$ simple modules has its dominant dimension bounded above by $2n-2$ and this bound is optimal for every $n \geq 2$, that is, there exists a non-selfinjective Nakayama algebra with n simple modules and dominant dimension $2n-2$. This Nakayama algebra has $n-1$ injective-projective indecomposable modules.

Proof. The bound was given in Theorem 3.1.17. Using the previous corollary, we take the algebra $C = End_A(A \oplus e_0 A / e_0 J^{w-1})$, where A is a symmetric Nakayama algebra with $n-1$ simple modules and Loewy length w. Then C is a Nakayama algebra with n simple modules and dominant dimension $2n-2$. □

We give another application, showing how to construct algebras of arbitrary dominant dimension larger than or equal to two:

Corollary 3.2.6
Let $w \equiv_n 2$ and B as above. Then B has dominant dimension
$domdim(B) = \inf\{k \geq 1 | \exists x_i, x_j : x_j \equiv_n x_i + k\} + 1$.

Proof. This follows, since in case $w \equiv_n 2$:
$[\frac{k+1}{2}]w - g_k = k+1$ and therefore:
$domdim(B) = \inf\{k \geq 1 | \exists x_i, x_j : x_j + w - 1 \equiv_n x_i + [\frac{k+1}{2}]w - g_k\} + 1 =$
$\inf\{k \geq 1 | \exists x_i, x_j : x_j + 1 \equiv_n x_i + k + 1\} + 1 =$
$\inf\{k \geq 1 | \exists x_i, x_j : x_j \equiv_n x_i + k\} + 1.$ □

So in this case the dominant dimension is simply equal to one plus the minimal distance of two special points. Like this, one can construct a family of Nakayama algebras with dominant dimension an arbitrary number larger than or equal to two.

3.2.2 Gorenstein dimensions of Nakayama algebras which are Morita algebras

We first recall definitions and standard facts about approximations. Note that by maps we always mean A-homomorphisms, when we speak about modules.

Definition 3.2.7
Let M and N be A-modules. Recall that a map $g : M \to N$ is called right minimal in case $gh = g$ implies that h is an isomorphism for any map $h : M \to M$. A map $f : M_0 \to X$, with $M_0 \in \text{add}(M)$, is called a right add(M)-approximation of X iff the induced map $Hom(N, M_0) \to Hom(N, X)$ is surjective for every $N \in \text{add}(M)$. Note that in case M is a generator, such an f must be surjective. When f is a right minimal homomorphism, we call it a minimal right add(M)-approximation. Note that minimal right add(M)-approximations always exist for finite dimensional algebras. The kernel of such a minimal right add(M)-approximationl f is denoted by $\Omega_M(X)$. Inductively we define $\Omega_M^0(X) := X$ and $\Omega_M^n(X) := \Omega_M(\Omega_M^{n-1}(X))$. The add$(M)$-resolution

dimension of a module X is a defined as:

$$M\text{-resdim}(X) := \inf\{n \geq 0 | \Omega_M^n(X) \in \text{add}(M)\}$$

and we sometimes also use the notation $add(M)$-resdim(X).

We use the following Proposition 3.11. from [CheKoe] in order to calculate the Gorenstein dimensions:

Proposition 3.2.8
Let A be a finite dimensional algebra and M a CoGen of mod-A and define $B := End_A(M)$. Let B have dominant dimension $z+2$, with $z \geq 0$. Then, for the right injective dimension of B the following holds:

$$\text{injdim}(B) = z + 2 \ + \ M\text{-resdim}(\tau_{z+1}(M) \oplus D(A)).$$

Here we use the common notation $\tau_{z+1} = \tau\Omega^z$, introduced by Iyama (see [Iya]).

We note that the Gorenstein symmetry conjecture (which says that the injective dimensions of A and A^{op} are the same) is known to hold for algebras with finite finitistic dimension (see [ARS] page 410, conjecture 13), and thus for Nakayama algebras which are our main examples. Therefore, we will only look at the right injective dimension at such examples.
We now fix our notation as in the previous section: A is a selfinjective Nakyama algebra with n simple modules, Loewy length w and $M = \bigoplus_{i=0}^{n-1} e_i A \oplus \bigoplus_{i=1}^{r} e_{x_i} A / e_{x_i} J^{w-1}$. Using the same notation as in the above theorem, we note that B is derived equivalent to $C = End_A(A \oplus N)$ (see [HuXi] Corollary 1.3. (2)), with the semisimple module $N = \Omega^1(M) = \bigoplus_{i=1}^{r} e_{x_i} J^{w-1}$. We also set $W := A \oplus N$ and we fix that notation for the rest of this section.

Lemma 3.2.9
The above mentioned derived equivalence between B and C preserves dominant dimension and Gorenstein dimension.

Proof. In [HuXi] Corollary 1.2., it is proved that such a kind of derived equivalence preserves dominant dimension and finitistic dimension. If the Gorenstein dimension is finite, it is equal to the finitistic dimension. Since a derived equivalence also preserves the finiteness of Gorenstein dimension, the result follows. □

We see that we need to know how to calculate minimal right add(W)-approximations of an arbitrary module in a selfinjective Nakayama algebra. For this we have the following lemma:

Lemma 3.2.10
Let $e_a J^y$ be an arbitrary non-projective indecomposable module in the selfinjective Nakayama algebra A and assume that this module is not contained in add(N).
1.If $a \neq x_i$ for all $i = 1, ..., r$, then the projective cover $e_{a+y}A \to e_a J^y \to 0$ is a

minimal right add(W)-approximation of $e_a J^y$.
2.If there is an x_i with $a = x_i$, then we have the following short exact sequence:

$$0 \to e_{x_i+y} J^{w-(y+1)} \to e_{x_i+y} A \oplus e_{x_i} J^{w-1} \to e_{x_i} J^y \to 0.$$

Here, the map $e_{x_i+y} A \oplus e_{x_i} J^{w-1} \to e_{x_i} J^y$ is the sum of the projective cover of $e_{x_i} J^y$ and the socle inclusion of $e_{x_i} J^{w-1}$ in $e_{x_i} J^y$. Then the surjective map in the above short exact sequence is a minimal right add(W)-approximation.

Proof. 1.The projective cover is clearly minimal. The kernel of the projective cover is $e_{a+y} J^{w-y}$ and we have to show $Ext^1(Z, e_{a+y} J^{w-y}) = 0$ for every $Z \in$ add(W). Since W is a direct sum of simple and projective modules, this simply means that I_1 (the first term in a minimal injective resolution of $e_{a+y} J^{w-y}$) has a socle, which does not lie in add(W). But this is true because of $I_1 = e_a A$ and our assumption in i).
2.Again, the minimality is obvious. At first we show that the short exact sequence exists. What is left to show is that the kernel is really $e_{x_i+y} J^{w-(y+1)}$. With

$$e_{x_i} J^y \cong e_{x_i+y} A / e_{x_i+y} J^{w-y}$$

and

$$e_{x_i} J^{w-1} \cong e_{x_i+y} J^{w-y-1} / e_{x_i+y} J^{w-y}$$

we see that the map of interest has up to isomorphism the following form:

$$f : e_{x_i+y} A \oplus e_{x_i+y} J^{w-y-1} / e_{x_i+y} J^{w-y} \to e_{x_i+y} A / e_{x_i+y} J^{w-y}.$$

We have $f(w_1, \overline{w_2}) = \overline{w_1} + \overline{w_2}$, when $\overline{w}$ denotes the residue class of an element w. A basis of the kernel is thus given by the elements

$$\{(\phi_{x_i+y,l}, 0) \mid w-1 \geq l \geq w-y\} \cup \{(\phi_{x_i+y,w-y-1}, -\overline{\phi_{x_i+y,w-y-1})}\},$$

when we denote by $\phi_{c,d}$ the unique path starting at c and having length d.

A basis of the socle of the kernel is given by $(\phi_{x_i+y,w-1}, 0)$ and thus the kernel is isomorphic to $e_{x_i+y} J^{w-(y+1)}$ (by comparing dimension and socle). We now have to show that the induced map $Hom(G, e_{x_i} J^{w-1} \oplus e_{x_i+y} A) \to Hom(G, e_{x_i} J^y)$ is surjective for every $G \in add(W)$. Note that we can assume that G has no simple summands S which are not isomorphic to $e_{x_i} J^{w-1}$, since we would have $Hom(S, e_{x_i} J^y) = 0$ then. With this assumption we get

$$Ext^1(G, e_{x_i+y} J^{w-(y+1)}) = 0, \text{ iff } Ext^1(e_{x_i} J^{w-1}, e_{x_i+y} J^{w-(y+1)}) = 0,$$

and this is true, since the minimal injective presentation of $e_{x_i+y} J^{w-(y+1)}$ is the following:

$$0 \to e_{x_i+y} J^{w-(y+1)} \to e_{x_i+y} A \to e_{x_i-1} A.$$

Then $Ext^1(G, e_{x_i+y} J^{w-(y+1)}) = 0$ and thus, the induced map
$Hom(G, e_{x_i} J^{w-1} \oplus e_{x_i+y} A) \to Hom(G, e_{x_i} J^y)$ is surjective. $\square$

Now we will use this result to calculate the Gorenstein dimensions of gendo-symmetric Nakayama algebras. We note that for a simple module S, $\tau_{z+1}(S)$ is always a simple module, if the dominant dimension of B is even. It is a radical of a projective indecomposable module, if the dominant dimension of B is odd. So, in order to calculate the Gorenstein dimension, it is enough to calculate the minimal right add(W)-resolutions for modules of the form $(a, w-1)$ and $(a, 1)$ for a point a. A diagram of the form

$$\begin{tikzcd} A' \arrow[d, dash, "1"'] \arrow[dr, dash, "2"] & \\ B' & C' \end{tikzcd}$$

means that the kernel of a W-approximation of the indecomposable nonprojective module $A' = e_a J^k$ is B', in case $e_a J^{w-1}$ is not a summand of W (always corresponding to the arrow with a 1), and the kernel is C' otherwise (always corresponding to an arrow with a 2).
So for a general module $(a, k) = e_a J^k$, not in add(W), the diagram looks as follows in the first step:

$$\begin{tikzcd} (a,k) \arrow[d, dash, "1"'] \arrow[dr, dash, "2"] & \\ (a+k, w-k) & (a+k, w-(k+1)) \end{tikzcd}$$

We also set $B' = stop$, if B' is a summand of W. Dots like $\cdots$ indicate that it is clear how the resolution continues from this point on.

Theorem 3.2.11
Let $w \equiv_n 1$ (which is equivalent to A being a symmetric Nakayama algebra). Then B has Gorenstein dimension

$$2 \sup\{u_i \mid u_i = \inf\{b \geq 1 \mid \exists j : \ x_i + b \equiv_n x_j\}\},$$

which is two times the maximal distance between two neighboring special points.

Proof. By 3.2.4, B has dominant dimension equal to $2\inf\{s \geq 1 \mid \exists i, j : x_i + s \equiv_n x_j\}$, which is equal to two times the smallest distance of two special points. Denote by d the smallest distance between two special points. Using 3.2.8 and $\tau \cong \Omega^2$ (since A is symmetric), the Gorenstein dimension is equal to $2d + W$-resdim$(\Omega^{2d}(W))$, with $W = \bigoplus_{i=0}^{n-1} e_i A \oplus \bigoplus_{i=1}^{r} e_{x_i} J^{w-1}$. Note that $\Omega^{2d}(W) = \bigoplus_{i=1}^{r} e_{x_i+d} J^{w-1}$ and so we have to calculate W-resdim$(\Omega^{2d}(W))$. Since the resolution dimension of a direct sum of modules equals the supremum of the resolution dimensions of the indecomposable summands, it is enough to look at a resolution of a single simple module of the form $(x_j + d, w-1)$:

$$\begin{tikzcd}
(x_j+d, w-1) \arrow[d, dash, "1"'] \arrow[dr, dash, "2"] & \\
(x_j+d, 1) \arrow[d, dash, "1"'] & \mathit{stop} \\
(x_j+d+1, w-1) \arrow[d, dash, "1"'] \arrow[dr, dash, "2"] & \\
(x_j+d+1, 1) \arrow[d, dash, "1"'] & \mathit{stop} \\
(x_j+d+2, w-1) \arrow[d, dash, "1"'] \arrow[dr, dash, "2"] & \\
\cdots & \mathit{stop}
\end{tikzcd}$$

Considering this diagram, we see now that the resolution finishes exactly when the kernel is of the form $(x_j + d + i, w - 1)$ with the smallest $i \geq 0$ such that $e_{x_j+d+i}J^{w-1}$ is a summand of W. This takes $2i$ steps. Now the result is clear. $\square$

It follows that the dominant dimension (Gorenstein dimension) of a nonsymmetric gendo-symmetric Nakayama algebra A can be calculated purely graph theoretically:
It is two times the minimal (two times the maximal) distance of special points in the quiver of the symmetric Nakayama algebra eAe, when e is a primitive idempotent, such that eA is a minimal faithful projective-injective module of A.
Combining these results, we get the following geometric characterisation when the dominant dimension equals the Gorenstein dimension for a nonselfinjective gendo-symmetric Nakayama algebra:

Corollary 3.2.12
In the situation of the above theorem, injdim(B)=domdim(B) iff all the special points in M have the same distance from one another.

The next result shows a nice interplay between the Gorenstein dimension, dominant dimension and some number theory:

Proposition 3.2.13
Let $w > 2$ and $n \geq 3$. Let A be a selfinjective Nakayama algebra with Loewy length w and n simple modules. Let $B := End_A(A \oplus S)$ for some simple module $S = e_i J^{w-1}$. Then the following are equivalent:

1. B is Gorenstein.

2. The dominant dimension of B is even.

3. w is a unit in $\mathbb{Z}/n\mathbb{Z}$.

In this case, the Gorenstein dimension equals the dominant dimension $2\inf\{s \geq 0 | sw+1 \equiv 0 \bmod n\}+2$.

Proof. Because of the symmetry of the problem we can assume that $S = e_0 J^{w-1}$, which corresponds to $(0, w-1)$ in our notation. If not stated otherwise, $\equiv$ will always denote equality mod n in the following. In the case of just one indecomposable non-projective summand, the formula for the dominant dimension d in 3.2.3 simplifies to give the following: $d = \inf\{k \geq 1 | w-1 \equiv_n [\frac{k+1}{2}]w - g_k\} + 1$. Splitting into even and odd cases, this can be simpliesfied to give $d = min\{d_e, d_o\}$, where $d_o = 2\inf\{s \geq 1 | sw \equiv 0\} + 1$ and $d_e = 2\inf\{s \geq 0 | ws+1 \equiv 0\} + 2$. Now assume $d = d_e$ is even. Then one has $ws \equiv -1$ for some s and therefore w is a unit in $\mathbb{Z}/n\mathbb{Z}$. In case the dominant dimension is odd, one has $sw \equiv 0$ for some s and so w is a zero-divisor and can not be a unit. This shows the equivalence of 2. and 3. Now we show that 2. implies 1. Assume that the dominant dimension d is even. This means $d = d_e = 2\inf\{s \geq 0 | s \equiv -w^{-1}\} + 2$. Then $\tau(\Omega^{d-2}((0, w-1))) = (-ww^{-1}+1, w-1) = (0, w-1)$ and thus the configuration is Gorenstein, with Gorenstein dimension coinciding with the dominant dimension $2\inf\{s \geq 0 | s \equiv -w^{-1}\} + 2$. What is left to show is that the configuration is never Gorenstein in case the dominant dimension is odd. So assume now that $d = d_o = 2\inf\{s \geq 1 | sw \equiv 0\} + 1$. Then $\tau(\Omega^{d-2}((0, w-1))) = (0+w-1+(s-1)w+1, 1) = (0, 1)$. The next beginning of a resolution now shows that this module $(0, 1)$ has infinite resolution dimension in case w is not -1 mod n. $(0,1) \xrightarrow{2} (1, w-2) \xrightarrow{1} (w-1, 2) \xrightarrow{1} (w+1, w-2) \xrightarrow{1} (2w-1, 2) \xrightarrow{1} (2w+1, w-2) \xrightarrow{1} (3w-1, 2) \xrightarrow{1} (3w+1, w-2) \xrightarrow{1} ....$ It is clear that the first coordinate will always be of the form $pw \pm 1$ from there on and therefore the resolution dimension is infinite, since w is a zero divisor and therefore $pw \pm 1$ is never 0 in $\mathbb{Z}/n\mathbb{Z}$.

□

3.3 Yamagata's conjecture for monomial algebras and finitistic dominant dimension

In this short section we show how to extent the bounds on the dominant dimension of Nakayama algebras to the much larger class of monomial algebras and we also show that the finitistic dominant dimension is finite for a large class of algebras, including monomial algebras. Recall that a *monomial algebra* is by definition a quiver algebra with only zero relations.

Proposition 3.3.1
Let A be a non-selfinjective monomial algebra with minimal faithful projective-injective module eA, then eAe is a Nakayama algebra.

Proof. Let $e = e_1 + e_2 + ... + e_r$ be a decomposition of the idempotent e into primitive orthogonal idempotents e_i. To show that eAe is a Nakayama algebra, it is enough to show that its quiver is a directed line or a directed circle, which equivalently can be formulated as $dim(rad(e_iAe)/rad^2(e_iAe)) = 0$

or $= 1$ for every i and dually $dim(rad(eAe_i)/rad^2(eAe_i)) = 0$ or $= 1$. We show $dim(rad(e_iAe)/rad^2(e_iAe)) = 0$ or $= 1$ in the following, while the dual property can be proven dually or by going over to the opposite algebra (which, of course, is still monomial). Since eA is injective, all modules e_iA are injective and thus have a simple socle. Note that $rad(eAe) = eJe$ and thus $rad(e_iAe) = e_iJe$ and $rad^2(e_iAe) = e_iJeJe$. Assume that the dimension of e_iJe/e_iJeJe is at least two for some i. Since A is monomial and e_iA has a simple socle, there can be only one arrow α starting at e_i. The target of α can not be an idempotent of the form e_s for $1 \leq s \leq r$, or else e_iJe/e_iJeJe would be at most one-dimensional. Thus there exists two paths $r_1 = \alpha p_1$ and $r_2 = \alpha p_2$ of smallest length starting at e_i and going to e_x and e_y respectively, where e_xA and e_yA are summands of eA. But this contradicts the fact that the socle of e_iA is simple and A being monomial, since there is no commutativity relation so that there are two different socle elements having the paths r_1 and r_2 as factors. Thus $dim(rad(e_iAe)/rad^2(e_iAe)) = 0$ or $= 1$ and eAe has to be a Nakayama algebra. □

Theorem 3.3.2
Let A be a non-selfinjective monomial algebra with s indecomposable projective-injective modules and $n \geq 2$ simple modules. Then the dominant dimension of A is bounded by $2s \leq 2n-2$.

Proof. In case the the dominant dimension is less than or equal to 1 there is nothing to show. Thus assume A has dominant dimension at least two. By 3.3.1, $A \cong End_B(M)$, where B is some Nakayama algebra with generator-cogenerator M. But by 3.1.15, the dominant dimension of such algebras is bounded by $2s$. $2s \leq 2n-2$ follows from the assumption that the algebra is non-selfinjective. □

$\Omega^i(mod-A)$ denotes the full subcategory of projective modules or modules being i-th syzygies of some other modules. We call a subcategory here representation-finite in case it has only finitely many indecomposable objects. Next we show that the finitistic dominant dimension of a large class of algebras, including monomial algebras, is finite. We will need the next result:

Theorem 3.3.3
Let A be a monomial algebra, then $\Omega^2(mod-A)$ is representation-finite.

Proof. See for example [But]. □

Proposition 3.3.4
Let A be an algebra such that the subcategory $\Omega^n(mod-A)$ is representation-finite for some $n \geq 1$, then A has finite finitistic dominant dimension.

Proof. Assume there is a module M of finite dominant dimension, which is larger than or equal to n. Then the start of minimal injective resolution of M looks as follows:

$$0 \to M \to I_0 \to I_1 \to ... \to I_{n-1} \to \Omega^{-n}(M) \to 0.$$

Since I_i are projective-injective for i with $0 \leq i \leq n-1$, one has that $M \cong \Omega^n(\Omega^{-n}(M))$ and thus $M \in \Omega^n(mod - A)$. But since $\Omega^n(mod - A)$ is representation-finite, there are only finitely many candidates for modules having finite dominant dimension larger than or equal to n. Thus the finitistic dominant dimension is finite. □

Corollary 3.3.5
Let A be a monomial algebra. Then A has finite finitistic dominant dimension.

Proof. This follows by applying 3.3.4 to monomial algebras, using 3.3.3. □

Proposition 3.3.6
The finitistic dominant dimension is always larger than or equal to the dominant dimension for a non-selfinjective algebra.

Proof. This is trivial, in case the dominant dimension is finite. Now assume the dominant dimension is infinite. Since the dominant dimension equals the codominant dimension, there is an indecomposable nonprojective injective module N with infinite codominant dimension. Now the module $\Omega^i(N)$ has finite dominant dimension equal to i for $i \geq 1$ and thus also the finitistic dominant dimension is infinite. □

Dually, one can define the *finitistic codominant dimension* as the supremum of all codominant dimensions of modules having finite codominant dimension. However, both dimensions coincide:

Proposition 3.3.7
The finitistic dominant dimension coincides with the finitistic codominant dimension.

Proof. By symmetry, we just have to show that the finitistic codominant dimension is larger than or equal to the finitstic dominant dimension. In case the finitistic dominant dimension is zero, there is nothing to show. Let M be a module with dominant dimension $n \geq 1$ and minimal injective resolution (I_i). Then the following exact sequence exists, with I_i being projective for $i = 0, 1, ..., n-1$:

$$0 \to M \to I_0 \to I_1 \to ... \to I_{n-1} \to \Omega^{-n}(M) \to 0.$$

This shows that $\Omega^{-n}(M)$ has codominant dimension at least n.

□

We give an interesting consequence for algebras to have finite finitistic dominant dimension:

Proposition 3.3.8
Let A be an algebra of finite finitistic dominant dimension. Then a module has infinite dominant dimension iff it has infinite codominant dimension.

Proof. Assume M has infinite codominant dimension but finite dominant dimension. Then there is a combined minimal projective resolution and injective resolution as follows:

$$... \to P_n \to P_{n-1} \to ... \to P_1 \to P_0 \to M \to I_0 \to ... \to I_{n-1} \to I_n \to ....$$

Here the terms P_i are projective-injective and the terms I_i are projective-injective for $i = 0, 1, ..., n-1$, but I_n is not projective. Thus the modules $\Omega^j(M)$ have arbitrary large finite dominant dimension for increasing j and thus the finitistic dominant dimension can not be finite. This contradicts our assumptions. The proof of the other direction is dual. □

We give an example of a non-projective module having infinite dominant dimension:

Example 3.3.9
Let A be the Nakayama algebra with Kupisch series $[3, 4]$ and $M = e_0 J^1 \cong e_1 J^2$. Then the unique indecomposable projective injective module is $e_1 A$. Note that $\Omega^1(M) \cong M$. Now it is clear that M has infinite codominant dimension, since M has projective cover equal to $e_1 A$. Thus by the previous proposition M also has infinite dominant dimension.

We remark that we are not aware of an algebra with infinite finitistic dominant dimension. By 3.3.6, the finiteness of the finitistic dominant dimension for any algebra would imply the Nakayama conjecture.

4 A bocs-theoretic characterisation of gendo-symmetric algebras

4.1 Preliminaries

We collect here all needed definitions and lemmas to prove the main theorems in this chapter. $mod-A$ denotes the category of finite dimensional right A-modules and $proj$ (inj) denotes the subcategory of finitely generated projective (injective) A-modules. We note that we often omit the index in a tensor product, when we calculate with elements. We often identify $A \otimes_A X \cong X$ for an A-module X without explicitly mentioning the natural isomorphism. The Nakayama functor $\nu : mod-A \to mod-A$ is defined as $DHom_A(-, A)$ and is isomorphic to the functor $(-) \otimes_A D(A)$. The inverse Nakayama functor $\nu^{-1} : mod-A \to mod-A$ is defined as $Hom_{A^{op}}(-, A)D$ and is isomorphic to the functor $Hom_A(D(A), -)$ (see [SkoYam] Chapter III section 5 for details). The Nakayama functors play a prominent role in the representation theory of finite dimensional algebras, since $\nu : proj \to inj$ is an equivalence with quasi-inverse ν^{-1}. For example they appear in the definition of the Auslander-Reiten translates τ and τ^{-1} (see [SkoYam] Chapter III. for the definitions):

Proposition 4.1.1
Let M be an A-module with a minimal injective presentation $0 \to M \to I_0 \to I_1$. Then the following sequence is exact:
$0 \to \nu^{-1}(M) \to \nu^{-1}(I_0) \to \nu^{-1}(I_1) \to \tau^{-1}(M) \to 0$.

Proof. See [SkoYam], Chapter III. Proposition 5.3. (ii). □

The full subcategory of modules of dominant dimension at least $i \geq 1$ is denoted by Dom_i.

Proposition 4.1.2
Let A be a gendo-symmetric algebra and M an A-module. Then M has dominant dimension larger than or equal to two iff $\nu^{-1}(M) \cong M$.

Proof. See [FanKoe2], proposition 3.3. □

The following result gives a formula for the dominant dimension of Morita algebras:

Proposition 4.1.3
Let A be a Morita algebra with a minimal faithful projective-injective module eA and M an A-module. Then $domdim(M) = \inf\{i \geq 0 | Ext^i(A/AeA, M) \neq 0\}$. Especially, $Hom_A(A/AeA, A) = 0$ for every Morita algebra, since they always have dominant dimension at least 2.

Proof. This is a special case of [APT], Proposition 2.6. □

The following lemma gives another characterization of gendo-symmetric algebras, which is used in the proof of the main theorem.

Lemma 4.1.4
Let A be a finite dimensional algebra. Then A is a gendo-symmetric algebra iff $D(A) \otimes_A D(A) \cong D(A)$ as A-bimodules. Assume eA is the minimal faithful projective-injective module. In case A is gendo-symmetric, $D(A) \cong Ae \otimes_{eAe} eA$ as A-bimodules.

Proof. See [FanKoe2] Theorem 3.2. and [FanKoe] in the construction of the comultiplication following Definition 2.3. □

Lemma 4.1.5
An A-module P is projective iff there are elements $p_1, p_2, ..., p_n \in P$ and elements $\pi_1, \pi_2, ..., \pi_n \in Hom_A(P, A)$ such that the following condition holds:
$x = \sum\limits_{i=1}^{n} p_i \pi_i(x)$ for every $x \in P$.
We then call the $p_1, ..., p_n$ a *projective basis* and $\pi_1, ..., \pi_n$ a *dual projective basis* of P.

Proof. See [Rot] Propostion 3.10. □

Example 4.1.6
Let $P = eA$, for an idempotent e. Then a projective basis is given by $p_1 = e$ and the dual projective basis is given by $\pi_1 = l_e \in Hom_A(eA, A)$, which is left multiplication by e. l_e can be identified with e under the (A, eAe)-bimodule isomorphism $Ae \cong Hom_A(eA, A)$.

Proposition 4.1.7
1. $Hom_A(D(A), A)$ is a faithful right A-module iff there is an idempotent e, such that eA and Ae are faithful and injective.
2. Let A be an algebra with $Hom_A(D(A), A) \cong A$ as right A-modules, then A is a Morita algebra.

Proof.
1. See [KerYam], Theorem 1.
2. See [KerYam], Theorem 3.

□

Lemma 4.1.8
Let Y and Z be A-bimodules. Then the following is an isomorphism of A-bimodules:

$$Hom_A(Y, D(Z)) \cong D(Y \otimes_A Z).$$

Proof. See [ASS] Appendix 4, Proposition 4.11. □

Definition 4.1.9
Let A be a finite dimensional algebra and W an A-bimodule and let $c_r : W \to A \otimes_A W$ and $c_l : W \to W \otimes_A A$ be the canonical isomorphisms. Then the pair $\mathcal{B} := (A, W)$ is called a *bocs* (see [Kue]) or the module W is called an A-coring (see [BreWis]) if there are A-bimodule maps $\mu : W \to W \otimes_A W$ (the comultiplication) and $\epsilon : W \to A$ (the counit) with the following properties: $(1_W \otimes_A \epsilon)\mu = c_l, (\epsilon \otimes_A 1_W)\mu = c_r$ and $(\mu \otimes_A 1_W)\mu = (1_W \otimes_A \mu)\mu$. We often say for short that W is a bocs, if A (and μ and ϵ) are clear from the context. The category of the finite dimensional bocs modules is defined as follows:
Objects are the finite dimensional right A-modules.
Homomorphism spaces are $Hom_{\mathcal{B}}(M, N) := Hom_A(M, Hom_A(W, N))$ with the following composition $*$ and units:
Let $g : M \to Hom_A(W, N) \in Hom_{\mathcal{B}}(M, N)$ and $f : L \to Hom_A(W, M) \in Hom_{\mathcal{B}}(L, M)$. Then $g * f := Hom_A(\mu, N)\psi Hom_A(W, g)f$, where ψ is the adjunction isomorphism $Hom_A(W, Hom_A(W, N)) \to Hom_A(W \otimes_A W, N)$. The units $1_M \in Hom_{\mathcal{B}}(M, M)$ are defined as follows: $1_M := Hom_A(\epsilon, M)\xi$, where $\xi : M \to Hom_A(A, M)$ is the canonical isomorphism. Note that the module category of a bocs is K-linear. We refer to [Kue] for other equivalent descriptions of the bocs module category and more information.

Examples 4.1.10
1. (A, A) is always a bocs with the obvious multiplication and comultiplication. The next natural bimodule to look for a bocs-structure is $D(A)$. We will see that $(A, D(A))$ is not a bocs for arbitrary finite dimensional algebras.
2. The next example can be found in 17.6. in [BreWis], to which we refer for more details. Let P be a (B, A)-bimodule for two finite dimensional algebras B and A such that P is projective as a right A-module and let $P^* := Hom(P, A)$, which is then a (A, B)-bimodule. Let $p_1, p_2, ..., p_n$ be a projective basis for P and $\pi_1, \pi_2, ..., \pi_n$ a dual projective basis of the projective A-module P. Denote the A-bimodule $P^* \otimes_B P$ by W and define the comultiplication $\mu : W \to W \otimes_A W$ as follows: Let $f \in P^*$ and $p \in P$, then $\mu(f \otimes p) = \sum_{i=1}^{n} (f \otimes p_i) \otimes (\pi_i \otimes p)$. Define the counit $\epsilon : W \to A$ as follows: $\epsilon(f \otimes p) = f(p)$. Now specialise to $P = eA$, for an idempotent e and identify $Hom_A(eA, A) = Ae$. Then $\mu(ae \otimes eb) = (ae \otimes e) \otimes (e \otimes eb)$ and $\epsilon(ae \otimes eb) = aeb$. We will use this special case in the next section to show that $(A, D(A))$ is always a bocs for a gendo-symmetric algebra.
3. Let (A_1, W_1) and (A_2, W_2) be bocses, then $(A_1 \otimes_K A_2, W_1 \otimes_K W_2)$ is again a bocs. See [BreWis] 24.1. for a proof.

4.2 Characterization of gendo-symmetric algebras

The following lemma, will be important for proving the main theorem of this chapter.

Lemma 4.2.1
Assume that $Hom_A(D(A), A) \cong A \oplus X$ as right A-modules for some right A-module X, then $domdim(A) \geq 2$ and $X = 0$.

Proof. By assumption $Hom_A(D(A), A)$ is faithful and so there is an idempotent e with eA and Ae faithful and injective by 4.1.7 1., which implies that A has dominant dimension at least 1. Choose e minimal such that those properties hold. Now look at the minimal injective presentation $0 \to A \to I_0 \to I_1$ of A and note that $I_0 \in add(eA)$. Using 4.1.1, there is the following exact sequence: $0 \to \nu^{-1}(A) \to \nu^{-1}(I_0) \to \nu^{-1}(I_1) \to \tau^{-1}(A) \to 0$. But $\nu^{-1}(A) \cong Hom_A(D(A), A) \cong A \oplus X$ and so there is the embedding: $0 \to A \oplus X \to \nu^{-1}(I_0)$. Note that $\nu^{-1}(I_0) \in add(eA)$ is the injective hull of $A \oplus X$, since $\nu^{-1} : inj \to proj$ is an equivalence and eA is the minimal faithful projective injective module. Thus $\nu^{-1}(I_0)$ has the same number of indecomposable direct summands as I_0. Therefore $soc(X) = 0$ and so $X = 0$, since every indecomposable summand of the socle of the module provides an indecomposable direct summand of the injective hull of that module. Thus $Hom_A(D(A), A) \cong A$ and A is a Morita algebra by 4.1.7 2. and so A has dominant dimension at least 2. $\square$

We now give a bocs-theoretic characterization of gendo-symmetric algebras.

Theorem 4.2.2
Let A be a finite dimensional algebra. Then the following are equivalent:

1. A is gendo-symmetric.
2. There is a comultiplication and counit such that $\mathcal{B} = (A, D(A))$ is a bocs.

Proof. We first show that 1. implies 2.:
Assume that A is gendo-symmetric with a minimal faithful projective-injective module eA. Set $P := eA$ and apply the second example in 4.1.10, with $B := eAe$, to see that $\mathcal{B} := (A, Ae \otimes_{eAe} eA)$ has the structure of a bocs. Now note that by 4.1.4 $D(A) \cong Ae \otimes_{eAe} eA$ as A-bimodules and one can use this to get a bocs structure for $(A, D(A))$.
Now we show that 2. implies 1.:
Assume that $(A, D(A))$ is a bocs with comultiplication μ and counit ϵ. Note first that the comultiplication μ always has to be injective because in the identity $(\epsilon \otimes_A 1_W)\mu = c_r$ appearing the definition of a bocs, c_r is an isomorphism. So there is a injection $\mu : D(A) \to D(A) \otimes_A D(A)$ which gives a surjection $D(\mu) : D(D(A) \otimes_A D(A)) \to A$. Now using 4.1.8 we see that $D(D(A) \otimes_A D(A)) \cong Hom_A(D(A), A)$ as A-bimodules.
Since A is projective, $D(\mu)$ is split and $Hom_A(D(A), A) \cong A \oplus X$ for some A-right module X. By 4.2.1, this implies $Hom_A(D(A), A) \cong A$ and comparing dimensions, $D(\mu)$ and thus also μ have to be isomorphisms. By 4.1.4, A is gendo-symmetric.

$\square$

The following proposition gives an application:

Proposition 4.2.3
Let A and B be gendo-symmetric K-algebras. Then $A \otimes_K B$ is again a gendo-symmetric K-algebra. In particular, let F be a field extension of K and A a gendo-symmetric K-algebra. Then $A \otimes_K F$ is again gendo-symmetric.

Proof. Let A and B two gendo-symmetric algebras. Then $\mathcal{B}_1 = (A, D(A))$ and $\mathcal{B}_2 = (B, D(A))$ are bocses. By example 3 of 4.1.10 also the tensor product of $\mathcal{B}_1$ and $\mathcal{B}_2$ are bocses, it is the bocs $\mathcal{C} = (A \otimes_K B, D(A) \otimes_K D(B))$. Recall the well known formula $(D(A) \otimes_K D(B)) \cong D(A \otimes_K B)$, which can be found as exercise 12. of chapter II. in [SkoYam]. Using this isomorphism one can find a bocs structure on $(A \otimes_K B, D(A \otimes_K B))$ using the bocs structure on $\mathcal{C}$. Thus by our bocs-theoretic characterization of gendo-symmetric algebras, also $A \otimes_K B$ is gendo-symmetric. The second part follows since every field is a symmetric and thus gendo-symmetric algebra. $\square$

Let $A^e := A^{op} \otimes_K A$ denote the enveloping algebra of a given algebra A. The following proposition can be found in [BreWis], 17.8.

Proposition 4.2.4
Let (A, W) be a bocs and $c \in W$ with $\mu(c) = \sum_{i=1}^{n} c_{1,i} \otimes c_{2,i}$.

1. $Hom_A(W, A)$ has a ring structure with unit ϵ and product $*^r$, given as follows for $f, g \in Hom_A(W, A)$:
   $f *^r g = g(f \otimes_A id_W)\mu$.
   There is a ring anti-morphism $\zeta : A \to Hom_A(W, A)$, given by $\zeta(a) = \epsilon(a(-))$.

2. $Hom_{A^e}(W, A)$ has a ring structure with unit ϵ and multiplication $*$ given as follows for $f, g \in Hom_{A^e}(W, A)$:
   $f * g(c) = \sum_{i=1}^{n} f(c_{1,i})g(c_{2,i})$.

We now describe the ring structures on $Hom_{A^e}(D(A), A)$ and $Hom_A(D(A), A)$.

Proposition 4.2.5
Let A be gendo-symmetric algebra.

1. ζ, as defined in the previous proposition, is a ring anti-isomorphism $\zeta : A \to Hom_A(D(A), A)$.

2. With the ring structure on $Hom_{A^e}(D(A), A)$ as defined in the previous proposition, $Hom_{A^e}(D(A), A)$ is isomorphic to the center $Z(A)$ of A.

Proof. We use the isomorphism of A-bimodules $D(A) \cong Ae \otimes_{eAe} eA$.

1. Since A and $Hom_A(D(A), A)$ have the same K-dimension, the only thing left to show is that ζ is injective. So assume that $\zeta(a) = \epsilon(a(-)) = 0$, for some $a \in A$. This is equivalent to $\epsilon(ax) = 0$ for every $x = ce \otimes ed \in$

$Ae \otimes eA$. Now $\epsilon(a(ce \otimes ed)) = \epsilon(ace \otimes ed) = aced$. Thus, since c, d were arbitrary, $aAeA$=0. This means that a is in the left annihilator $L(AeA)$ of the two-sided ideal AeA. But $L(AeA) = 0$, since $Hom_A(A/AeA, A) = 0$, by 4.1.3 and thus $a = 0$. Therefore ζ is injective.

2. Define $\psi : Hom_{A^e}(D(A), A) \to Z(eAe)$ by $\psi(f) = f(e \otimes e)$, for $f \in Hom_{A^e}(D(A), A)$. First, we show that this is well-defined, that is $f(e \otimes e)$ is really in the center of $Z(eAe)$. Let $x \in eAe$. Then $xf(e \otimes e) = f(xe \otimes e) = f(e \otimes ex) = f(e \otimes e)x$ and therefore $f(e \otimes e) \in Z(eAe)$. Clearly, ψ is K-linear. Now we show that the map is injective: Assume $\psi(f) = 0$, which is equivalent to $f(e \otimes e) = 0$. Then for any $a, b \in A$: $f(ae \otimes eb) = 0$, and thus $f = 0$.
Now we show that ψ is surjective. Let $z \in Z(eAe)$ be given. Then define a map $f_z \in Hom_{A^e}(D(A), A)$ by $f_z(ae \otimes eb) = zaeb$. Then, since z is in the center of eAe, f is A-bilinear and obviously $\psi(f_z) = f_z(e \otimes e) = ze = z$. ψ also preserves the unit and multiplication:
$\psi(\epsilon) = \epsilon(e \otimes e) = e^2 = e$ and for two given $f, g \in Hom_{A^e}(D(A), A)$: $\phi(f * g) =$
$(f * g)(e \otimes e) = (f * g)(e \otimes e) = f(e \otimes e)g(e \otimes e)$, by the definition of $*$.
To finish the proof, we use the result from [FanKoe], Lemma 2.2., that the map $\phi : Z(A) \to Z(eAe)$, $\phi(z) = eze$ is a ring isomorphism in case A is gendo-symmetric.

□

4.3 Description of the module category of the bocs $(A, D(A))$ for a gendo-symmetric algebra

Let A be a gendo-symmetric algebra. In this section we describe the module category of the bocs $\mathcal{B} = (A, D(A))$ as a K-linear category. We will use the A-bimodule isomorphism $Ae \otimes_{eAe} eA \cong D(A)$ often without mentioning. Let M be an arbitrary A-module. Define for a given M the map $I_M : M \to Hom_A(D(A), M)$ by $I_M(m) = u_m$ for any $m \in M$, where $u_m : D(A) \to M$ is the map $u_m(ae \otimes eb) = maeb$ for any $a, b \in A$. Before we get into explicit calculation, let us recall how $*$ is defined in this special case. Let $f \in Hom_{\mathcal{B}}(L, M)$ and $g \in Hom_{\mathcal{B}}(M, N)$, then for $l \in L$ and $a, b \in A$: $(g * f)(l)(ae \otimes eb) = g(f(l)(ae \otimes e))(e \otimes eb)$.

Proposition 4.3.1 1. I_M is well defined.

2. I_M is injective, iff M has dominant dimension larger than or equal to 1.

3. I_M is bijective, iff M has dominant dimension larger than or equal to 2.

Proof. 1. We have to show two things: First, u_m is A-linear for any $m \in M$: $u_m((ae \otimes eb)c) = u_m(ae \otimes ebc) = maebc = (maeb)c = u_m(ae \otimes eb)c$. Second, I_M is also A-linear: $I_M(mc)(ae \otimes eb) = u_{mc}(ae \otimes eb) = mcaeb = u_m(cae \otimes eb) = (u_m c)(ae \otimes eb) = (I_M(m)c)(ae \otimes eb)$.

2. I_M is injective iff $(m = 0 \Leftrightarrow u_m = 0)$. Now $u_m = 0$ is equivalent to $maeb = 0$ for any $a, b \in A$. This is equivalent to the condition that the two-sided ideal AeA annihilates m. Thus there is a nonzero m with $u_m = 0$ iff $Hom_A(A/AeA, M) \neq 0$ iff M has dominant dimension zero by 4.1.3.

3. By 4.1.2 M has dominant dimension larger than or equal to two iff $M \cong \nu^{-1}(M)$.
Thus 3. follows by 2. since an injective map between modules of the same dimension is a bijective map.

□

Lemma 4.3.2
For any module M, there is an isomorphism
$Hom_A(\mu, M)\psi : Hom_A(D(A), Hom_A(D(A), M)) \rightarrow Hom(D(A), M)$ and thus $\nu^{-1}(M) \cong \nu^{-2}(M)$. It follows that every module of the form $\nu^{-1}(M)$ has dominant dimension at least two.

Proof. The result follows, since ψ is the canonical isomorphism
$\psi : Hom_A(D(A), Hom_A(D(A), M)) \rightarrow Hom_A(D(A) \otimes_A D(A), M)$ and since μ is an isomorphism also $Hom_A(\mu, M)$ is an isomorphism. That $\nu^{-1}(M)$ has dominant dimension at least two, follows now from 4.1.2. □

We define a functor $\phi : mod - A \rightarrow mod - \mathcal{B}$ by $\phi(M) = M$ and $\phi(f) = I_N f$ for an A-homomorphism $f : M \rightarrow N$. ϕ is obviously K-linear. The next result shows that it really is a functor and calculates its kernel on objects.

Theorem 4.3.3 1. ϕ is a K-linear functor.

2. $\phi(M) = 0$ iff the two-sided ideal AeA annihilates M, that is M is a an A/AeA-module. All modules M that are annihilated by AeA have dominant dimension zero.

3. By restricting ϕ to Dom_2, one gets an equivalence of K-linear categories $Dom_2 \rightarrow Dom_2^{\mathcal{B}}$, where $Dom_2^{\mathcal{B}}$ denotes the full subcategory of $mod - \mathcal{B}$ having objects all modules of dominant dimension at least 2.

4. Any module A-module M is isomorphic to $\nu^{-1}(M)$ in $\mathcal{B}$-mod and thus $\mathcal{B}$-mod is equivalent to Dom_2 as K-linear categories, which is equivalent to the module category mod-eAe.

Proof. 1. It was noted above that ϕ is K-linear. We have to show $\phi(id_M) = Hom(\epsilon, M)\zeta$, where $\zeta : M \rightarrow Hom_A(A, M)$ is the canconical isomorphism, and $\phi(g \circ f) = I_N(g) * I_M(f)$, where $f : L \rightarrow M$ and $g : M \rightarrow N$ are A-module homomorphisms. To show the first equality $\phi(id_M) = Hom(\epsilon, M)\zeta$, just note that $Hom(\epsilon, M)\zeta(m)(ae \otimes eb) = l_m(\epsilon(ae \otimes eb)) = maeb = I_M(m)(ae \otimes eb)$, where $l_m : A \rightarrow M$ is left multiplication by m. Next we show the above equality $\phi(g \circ f) = I_N(g) * I_M(f)$:
Let $l \in L$ and $a, b \in A$. First, we calculate $\phi(g \circ f)(l)(ae \otimes eb) =$

$g(f(l))aeb$.
Second, $I_N(g) * I_M(f)(l)(ae \otimes eb) = I_N(g)(I_M(f)(l)(ae \otimes e))(e \otimes eb) = I_N(g)(u_{f(l)}(ae \otimes e))(e \otimes eb) = I_N(g)(f(l)(ae))(e \otimes eb) = g(f(l))aeb$.
Thus $\phi(g \circ f) = I_N(g) * I_M(f)$ is shown.

2. A module M is zero in the K-category mod-$\mathcal{B}$ iff its endomorphism ring $End_{\mathcal{B}}(M)$ is zero iff the identity of $End_{\mathcal{B}}(M)$ is zero. Thus M is zero in mod-$\mathcal{B}$ iff $I_M(m) = 0$ for every $m \in M$. But $I_M(m) = 0$ iff $mAeA = 0$ and so $\phi(M) = 0$ iff $MAeA = 0$. To see that such an M must have dominant dimension zero, note that AeA annihilates no element of M iff M has dominant dimension larger than or equal to 1 by 4.1.3.

3. Restricting ϕ to Dom_2, ϕ is obviously still dense by the definition of $Dom_2^{\mathcal{B}}$. Now recall that by the previous proposition a module M has dominant dimension at least two iff I_M is an isomorphism. Let now $h \in Hom_{\mathcal{B}}(M, N)$ be given with $M, N \in Dom_2^{\mathcal{B}}$. Then $\phi(I_N^{-1}h) = I_N(I_N^{-1}h)) = h$ and ϕ is full. Assume $\phi(h) = I_N h = 0$, then $h = 0$, since I_N is an isomorphism, and so ϕ is faithful.

4. Define $f \in Hom_{\mathcal{B}}(M, \nu^{-1}(M))$ as $f = (Hom_A(\mu, M)\psi)^{-1} I_M$ and $g \in Hom_{\mathcal{B}}(\nu^{-1}(M), M)$ as $g = id_{\nu^{-1}(M)}$. We show that $f * g = I_{\nu^{-1}}(M)$ and $g * f = I_M$, which by 1. are the identities of $Hom_{\mathcal{B}}(\nu^{-1}(M), \nu^{-1}(M))$ and $Hom_{\mathcal{B}}(M, M)$. This shows that any module M is isomorphic to $\nu^{-1}(M)$ in $\mathcal{B}$-mod.
Let $m \in M$ and $a, b \in A$.
Then $(g * f)(m)(ae \otimes eb) = g(f(m)(ae \otimes e))(e \otimes eb) = ((Hom_A(\mu, M)\psi)^{-1} I_M(m))((ae \otimes e))(e \otimes eb)) = maeb = I_M(m)(ae \otimes eb)$, where we used that g is the identity on $\nu^{-1}(M)$. Next we show that $f * g = I_{\nu^{-1}}(M)$: Let $l \in \nu^{-1}(M) = Hom_A(D(A), M)$.
First, note that by definition $I_{\nu^{-1}}(M)(l)(ae \otimes eb)(a'e \otimes eb') = (laeb)(a'e \otimes eb') = l(aeba'e \otimes eb')$. Next $(f * g)(l)(ae \otimes eb)(a'e \otimes eb') = f(g(l)(ae \otimes eb)(a'e \otimes eb') = f(l(ae \otimes eb))(a'e \otimes eb') = (Hom_A(\mu, M)\psi)^{-1} I_M(l(ae \otimes eb)(a'e \otimes eb') = l(ae \otimes eba'eb') = l(aeba'e \otimes eb')$, where we used in the last step that we tensor over eAe.
Now we use 4.3.2, to show that every module of the form $\nu^{-1}(M)$ has dominant dimension at least two. Since every module M is isomorphic to $\nu^{-1}(M)$, $\mathcal{B} - mod$ is isomorphic to $Dom_2^{\mathcal{B}}$, which is isomorphic to Dom_2 by 3. Now recall that there is an equivalence of categories mod-$eAe \to Dom_2$ (this is a special case of [APT] Lemma 3.1.). Combining all those equivalences, we get that $\mathcal{B} - mod$ is equivalent to the module category mod-eAe.

□

Corollary 4.3.4
In case an A-module M has dominant dimension larger than or equal to 2, the map
$Hom_A(M, I_M) : End_A(M) \to End_{\mathcal{B}}(M)$ is a K-algebra isomorphism. In par-

ticular $A \cong End_A(A) \cong End_{\mathcal{B}}(A)$, since A has dominant dimension at least 2.

Proof. This follows since I_M is an isomorphism, in case M has dominant dimension at least two by 4.3.1 3.

□

Example 4.3.5
Let $n \geq 2$ and $A := K[x]/(x^n)$ and J the Jacobson radical of A. Let
$M := A \oplus \bigoplus_{k=1}^{n-1} J^k$ and $B := End_A(M)$. Then B is the Auslander algebra of A and B has n simple modules. The idempotent e is in this case primitive and corresponds to the unique indecomposable projective-injective module $Hom_A(M, A)$. By the previous theorem, the kernel of ϕ is isomorphic to the module category $mod - (A/AeA)$. Here A/AeA is isomorphic to the preprojective algebra of type A_{n-1} by [DR] chapter 7.
We describe the bocs module category $\mathcal{B}$-mod of $(B, D(B))$ for $n = 2$ explicitly. In this case B is isomorphic to the Nakayama algebra with Kupisch series $[2, 3]$. Then B has five indecomposable modules. Let e_0 be the primitive idempotent corresponding to the indecomposable projective module with dimension two and e_1 the primitive idempotent corresponding to the indecomosable projective module with dimension three. Then $e_1 A$ is the unique minimal faithful indecomposable projective-injective module. Let S_i denote the simple B-modules. The only indecomposable module annihilated by Be_1B is S_0, which is therefore isomorphic to zero in the bocs module category. The two indecomposable projective modules $P_0 = e_0 B$ and $P_1 = e_1 B$ have dominant dimension at least two and thus are not isomorphic. The only indecomposable module of dominant dimension 1 is S_1 and the only indecomposable module of dominant dimension zero, which is not isomorphic to zero in $\mathcal{B}$-mod, is $D(Be_0)$. Now let $X = S_1$ or $X = D(Be_0)$, then $\nu^{-1}(X) = Hom_B(D(B), X) \cong e_0 B$. Thus in $\mathcal{B}$-mod $S_1 \cong e_0 B \cong D(Be_0)$ and $e_1 B$ are up to isomorphism the unique indecomposable objects.

5 A new construction of gendo-symmetric Gorenstein algebras from symmetric algebras

5.1 Construction for general symmetric algebras

This section introduces circle gendo-symmetric algebras, which provide a new construction of nonselfinjective Gorenstein algebras from symmetric algebras. Furthermore, for this class of algebras one can explicitly calculate the dominant and Gorenstein dimensions in a very nice way as a graph theoretic distance. In this chapter, m-periodic modules are those modules M with $\Omega^m(M) \cong M$ and no $1 \leq n < m$ with $\Omega^n(M) \cong M$.

Definition 5.1.1
Let A be a symmetric algebra and M a nonprojective indecomposable A-module. Then M is called *m-special* for some $m \geq 3$, if M is m-periodic for some $m \geq 3$ and $Ext^j(M,M) \neq 0$ for $j \geq 1$ is equivalent to $j \equiv -1 \mod m$ or $j \equiv 0 \mod m$. We often say just special instead of m-special when m is clear or not relevant. Then for $N = A \oplus \bigoplus_{k=0}^{r} \Omega^{x_k}(M)$, with $0 \leq x_0 < x_1 < x_2 < x_3 < ... < x_r < m$ and $x_i \neq x_j + 1 \mod m$ for all i,j, the algebra $B := End_A(N)$ is called a *circle gendo-symmetric algebra* and the module N is called *m-multispecial* or just multispecial for short.

Note that M is special iff $\Omega^i(M)$ is special for some $i \in \mathbb{Z}$. For the convenience of the reader, we recall the following properties for the module category of a symmetric algebra.

Lemma 5.1.2
Let A be a symmetric algebra and M, N A-modules. Then the following holds:

1. Ω^i is a stable equivalence of $\underline{mod} - A$ with inverse Ω^{-i} for every $i \geq 1$.

2. $\tau \cong \Omega^2, \tau^{-1} \cong \Omega^{-2}$.

3. $Ext^i(M,N) \cong \underline{Hom}(\Omega^i(M),N) \cong \underline{Hom}(M,\Omega^{-i}(N))$ as vector spaces for every $i \geq 0$.

4. $Ext^1(M,N) \cong \underline{Hom}(\Omega^{-2}(N),M)$ as vector spaces.

5. $Ext^1(M,N) \cong Ext^1(N, \Omega^3(M))$ as vector spaces.

Proof. For a proof of 1., 2., 3. and 4. see [SkoYam] chapter IV, where 4. is part of the Auslander-Reiten formulas in the special case of a symmetric algebra. We give a proof of 5. using the previous results: $Ext^1(M,N) \cong \underline{Hom}(\Omega^{-2}(N), M) \cong \underline{Hom}(\Omega^1(N), \Omega^3(M)) \cong Ext^1(N, \Omega^3(M))$. □

Proposition 5.1.3
Let A be a symmetric algebra and M an indecomposable module with period $n \geq 2$. Then $Ext^{n-1}(M,M) \cong Ext^n(M,M) \neq 0$.

Proof. First note that $Ext^n(M,M) \cong \underline{Hom}(\Omega^n(M), M) \cong \underline{Hom}(M,M) \neq 0$, since the identity does not factor over a projective. Using the previous lemma 5.1.2 and the periodicity of M, we get the following isomorphisms of k-vector spaces:
$Ext^{n-1}(M,M) \cong Ext^1(\Omega^{n-2}(M), M) \cong Ext^1(M, \Omega^{n+1}(M))$
$\cong Ext^1(M, \Omega^1(M)) \cong Ext^1(M, \Omega^{-(n-1)}(M)) \cong Ext^n(M,M) \neq 0$. □

Using the previous proposition, one obtains following easy characterisation of a special module.

Corollary 5.1.4
Let A be a symmetric algebra and M an indecomposable module with period $n \geq 3$. Then M is n-special iff $Ext^i(M,M) = 0$ for all $i = 1, 2, ..., n-2$.

Proof. In case M is n-special, $Ext^i(M,M) = 0$ for all $i = 1, 2, ..., n-2$ is clear. Now assume $Ext^i(M,M) = 0$ for all $i = 1, 2, ..., n-2$. Recall that $Ext^i(M,M) \cong \underline{Hom}(\Omega^i(M), M)$. This shows that
$Ext^i(M,M) = 0$ for $i \neq -1$ or 0 mod n, using the periodicity of M. By the previous proposition and the periodicity, $Ext^i(M,M) \neq 0$ is clear in case $i \equiv -1$ or 0 mod n. □

We call a module M *p-rigid* in case $Ext^i(M,M) = 0$ for $i = 1, 2, ..., p$. We often call 1-rigid modules just rigid for short. Note that all special modules are 1-rigid by definition. For a fixed algebra A and a special A-module M with period $m \geq 3$, we picture the algebra $B := End_A(N)$ (or the module N) with $N = A \oplus \bigoplus_{k=0}^{r} \Omega^{x_k}(M)$ in a directed circle with black or white points numbered from 0 to $m-1$ and directed clockwise, where a point i is black iff $i = x_k$ for some k as above and white else. We call this circle the *associated circle to N*. Viewing this circle as a directed graph, we can talk about distances of points as the minimal positive number of edges between those two points in the directed graph. We say that two black points are *neighboring points* in case there is no other black point between them.

Let A, M, B and N always be as in the previous definition 5.1.1 for the rest of this section. We want to calculate the dominant dimension of B and to do this, we have to calculate the dominant dimensions of the projective and noninjective indecomposable B-modules, which are those isomorphic to $Hom_A(N, \Omega^{x_p}(M))$ (this is for example explained in [Yam], section 3.2.). We recall the following theorem which is essentially due to Mueller, see [Mue], and can also be found in proposition 3.7. in [APT].

Theorem 5.1.5
Let C be a symmetric algebra and $N = \bigoplus_{l=1}^{r} N_l$ a generator-cogenerator of $mod-C$, where N_i are indecomposable. Let $B := End_C(N)$. Then the dominant dimension of the B-module $Hom_C(N, N_l)$ equals $inf\{i \geq 1 | Ext^i(N, N_i) \neq 0\} + 1$.

We recall the following proposition, which is a special case of corollary 5.4. in [CheKoe]. We refer to 3.2.7 for the relevant defintions.

Proposition 5.1.6
Let A be a symmetric algebra and N a nonprojective generator-cogenerator, which is p-rigid but not $p+1$-rigid. Then $End_A(N)$ has dominant dimension equal to $d = p + 2$ (this is due the previous theorem of Mueller) and right Gorenstein dimension equal to $d + add(N) - resdim(\Omega^d(N))$ and the left Gorenstein dimension is equal to $d + add(N) - coresdim(\Omega^{-d}(N))$.

Since the calculations are completely analogous, we restrict to calculate right Gorenstein dimensions in the following. The left Gorenstein dimension will always coincide with the right Gorenstein dimension here. We note that modules N, which are generator-cogenerators such that $End_A(N)$ has Gorenstein dimension equal to the dominant dimension equal to d, are called $(d-2, 0)-$*orthosymmetric* in [CheKoe] and precluster-tilting by Iyama and Solberg in [IyaSol].

Lemma 5.1.7
Let $R \cong \Omega^i(M)$ for some $0 \leq i \leq m-1$ and $x_k \neq i-1$ mod m for all k. Then the minimal right $add(N)$-approximation of R equals the identity if $R \in add(N)$ and equals the projective cover of R if $R \notin add(N)$.

Proof. The statement is clear when $R \in add(N)$. Thus assume that $R \notin add(N)$, which means $i \neq x_k$ mod m for all k. Let (P_i) be a minimal projective resolution of M. Then the projective cover of R is determined by the exact sequence $0 \to \Omega^{i+1}(M) \to P_i \to \Omega^i(M) \to 0$. Applying the functor $Hom(N, -)$ to this exact sequence gives:

$$0 \to Hom(N, \Omega^{i+1}(M)) \to Hom(N, P_i) \to Hom(N, \Omega^i(M)) \to Ext^1(N, \Omega^{i+1}(M)) \to 0.$$

Thus $Hom(N, P_i) \to Hom(N, \Omega^i(M))$ is surjective iff $Ext^1(N, \Omega^{i+1}(M)) = 0$.
We now show that $Ext^1(N, \Omega^{i+1}(M)) = 0$ or equivalently:
$Ext^1(\Omega^{x_k}(M), \Omega^{i+1}(M)) = 0$ for all $k = 0, .., r$. We consider two cases:
<u>Case 1 $x_k > i+1$</u>:
Then $Ext^1(\Omega^{x_k}(M), \Omega^{i+1}(M)) = Ext^{1+x_k-(i+1)}(M, M) = 0$, since $1 + x_k - (i+1) = x_k - i$ is not equal to -1 or 0 mod m.
<u>Case 2 $x_k < i+1$</u>:
Then $Ext^1(\Omega^{x_k}(M), \Omega^{i+1}(M)) = Ext^1(\Omega^{m+x_k-(i+1)}(M), M) =$
$Ext^{m+x_k-i}(M, M) = 0$, since $m + x_k - i (\equiv x_k - i$ mod $m)$ is not equal to -1 or 0 mod m.
In case $x_k = i+1$, $Ext^1(\Omega^{i+1}(M), \Omega^{i+1}(M)) = 0$, since Ω is an equivalence and M is 1-rigid. $\square$

Theorem 5.1.8
Let $B = End_A(N)$ be as in 5.1.1

1. The dominant dimension of B equals the minimal distance between black points in the associated circle to N.

2. The Gorenstein dimension of B equals the maximal distance between neighboring black points in the associated circle to N.

3. N is $(n,0)$-orthosymmetric iff all black points have the same distance $n+2$ between each other.

Proof. Recall that m denotes the period of M.

1. We use 5.1.5 and calculate first
$a_{q,p} := \inf\{i \geq 1 | Ext^i(\Omega^{x_q}(M), \Omega^{x_p}(M)) \neq 0\}$. We consider 3 cases:
Case $x_p = x_q$: In this case we have $a_{p,p} = m-1$, since Ω is an equivalence of the stable category and M is special.
Case $x_q > x_p$: In this case $Ext^i(\Omega^{x_q}(M), \Omega^{x_p}(M)) \cong Ext^i(\Omega^{x_q - x_p}(M), M) \cong Ext^{i+x_q-x_p}(M,M) \neq 0$ for the first time for $i + x_q - x_p = m-1$ and thus $a_{q,p} = m - 1 - x_q + x_p$ here.
Case $x_q < x_p$: In this case $Ext^i(\Omega^{x_q}(M), \Omega^{x_p}(M)) \cong Ext^i(\Omega^{m+x_q-x_p}(M), M)$. Now note that with $\hat{x_q} := m + x_q - x_p$ and $\hat{x_p} = 0$, we are in the previous case and therefore, $Ext^i(\Omega^{x_q}(M), \Omega^{x_p}(M)) \neq 0$ for the first time for $m - 1 - \hat{x_q} - \hat{x_p} = x_p - x_q - 1$. Using 5.1.5, 1. is clear now, since the dominant dimension of the algebra equals the minimum of the dominant dimensions of the projective modules.

2. This follows from the previous lemma 5.1.7 in combination with 5.1.6: The Gorenstein dimension is the minimal distance between black points plus $add(N) - resdim(\Omega^d(N))$. We calculate $add(N) - resdim(\Omega^d(N)) = \sup\{add(N) - resdim(\Omega^d(\Omega^{x_k}(M)))\}$: By 5.1.7, $add(N) - resdim(\Omega^d(\Omega^{x_k}(M)))$ just equals the smallest $i \geq 0$ such that $\Omega^{d+i}(\Omega^{x_k}(M)) \in add(N)$, which shows 2.

3. This is clear by combination of 1. and 2.

□

5.2 1-rigid modules over symmetric Nakayama algebras

Before we classify the special modules of symmetric Nakayama algebras in the next section, we classify the indecomposable modules M here in such algebras with $Ext^1(M,M) = 0$. In the following we write $A_{n,z}$ for short for the symmetric Nakayama algebra with n simple modules and Loewy length $w := zn + 1$ with a $z \geq 1$. We assume $n \geq 2$, since the local symmetric Nakayama algebras are not interesting for our theory because there $Ext^1(M,M) \neq 0$ for every

nonprojective module M. We introduce some notation. Let x, y be points of the quiver of a symmetric Nakayama algebra A in the following definitions and u a natural number.
We denote by $\phi_{x,y}$ the unique path of smallest length larger than or equal to 1 connecting x and y and by $\mathcal{L}_{x,y}$ its length. For example we have $\mathcal{L}_{x,x} = n$ for every x.
We denote by $L_{x,u}$ the unique map (a module endomorphism), which is given by left multiplication by the path starting at x and having length u.
We denote by $R_{x,u}$ the unique map, which is given by right multiplication by the path starting at x and having length u.

Lemma 5.2.1 1. For $A = A_{n,z}$ and $q \geq p$ we have:
$Ext^1(e_c J^p, e_d J^q) = 0$ iff $max(q, w-p) > \mathcal{L}_{d,c+1} + (z-1)n$.

2. For $A = A_{n,z}$ and $p \geq q$ we have:
$Ext^1(e_c J^p, e_d J^q) = 0$ iff $max(p, w-q) > \mathcal{L}_{d+q,c+p+1} + (z-1)n$

Proof. 1. We look at the start of a minimal projective resolution of $e_c J^p$:
$e_{c+p+1}A \overset{L_{c+1,p}}{\to} e_{c+1}A \overset{L_{c+p,w-p}}{\to} e_{c+p}A \to e_c J^p \to 0$. Now we apply the functor $Hom(-, e_d J^q)$ to this projective resolution (with $e_c J^p$ deleted) and we get the complex:
$0 \to e_d J^q e_{c+p} \overset{R_{c+p,w-p}}{\to} e_d J^q e_{c+1} \overset{R_{c+1,p}}{\to} e_d J^q e_{c+p+1}$. Note that since $q \geq p$ we have $R_{c+p,w-p} = 0$, since $R_{c+p,w-p}$ maps paths in $e_d J^q e_{c+p}$(having length at least q) to paths of length $w-p+q \geq w$, and in $A_{n,z}$ we have $J^w = 0$. Therefore, we have $Ext^1(e_c J^p, e_d J^q) = ker(R_{c+1,p})$.
Now $e_d J^q e_{c+1} = \langle\{\phi_{d,c+1}\phi^i_{c+1,c+1} | i = 0,..,z-1 \text{ and } \mathcal{L}_{d,c+1} + ni \geq q\}\rangle$ (the vector space span of these elements) and $ker(R_{c+1,p}) = \langle\{\phi_{d,c+1}\phi^i_{c+1,c+1} | i = 0,..,z-1 \text{ and } \mathcal{L}_{d,c+1} + ni \geq q \text{ and } \mathcal{L}_{d,c+1} + ni + p \geq w\}\rangle$. Therefore, we get $Ext^1(e_c J^p, e_d J^q) = ker(R_{c+1,p}) = 0$ iff $max(q, w-p) > \mathcal{L}_{d,c+1} + (z-1)n$, since then there is no path in $e_d J^q e_{c+1}$ fullfiling the extra condition $\mathcal{L}_{d,c+1} + ni + p \geq w$.

2. Assume now that $p \geq q$. Then
$Ext^1(e_c J^p, e_d J^q) = Ext^1(\Omega^1(e_c J^p), \Omega^1(e_d J^q)) = Ext^1(e_{c+p} J^{w-p}, e_{d+q} J^{w-q})$ (here we used that Ω is an equivalence in the stable category). Note that here $w-q \geq w-p$ because of $p \geq q$. Therefore, we are in the situation of part a) of the lemma and we get using 1.:
$Ext^1(e_c J^p, e_d J^q) = 0$ iff $max(p, w-q) > \mathcal{L}_{d+q,c+p+1} + (z-1)n$.

□

By substituting and combining the above conditions we get the

Corollary 5.2.2 1. For $y \geq x$ we have $Ext^1(e_a J^x, e_b J^y) = 0 = Ext^1(e_b J^y, e_a J^x)$ iff $max(w-x, y) > max(\mathcal{L}_{b,a+1}, \mathcal{L}_{a+x,b+y+1}) + (z-1)n$.

2. For $x \geq y$ we have $Ext^1(e_a J^x, e_b J^y) = 0 = Ext^1(e_b J^y, e_a J^x)$ iff $max(w-y, x) > max(\mathcal{L}_{a,b+1}, \mathcal{L}_{b+y,a+x+1}) + (z-1)n$.

3. We also note the following special case:
   $a = b, x = y$: The condition for $Ext^1(e_a J^x, e_a J^x) = 0$, then is:
   $max(w-x, x) > max(\mathcal{L}_{a,a+1}, \mathcal{L}_{a+x,a+x+1}) + (z-1)n = 1 + (z-1)n$, since $\mathcal{L}_{a,a+1} = \mathcal{L}_{a+x,a+x+1} = 1$.
   So x has to satisfy one of the conditions: $2+(z-1)n \leq x \leq zn$ or $1 \leq x \leq n-1$. This gives all indecomposable nonprojective rigid modules in $A_{n,z}$.

5.3 Special modules in symmetric Nakayama algebras

In this section we specialize the theory of the first section of this chapter to the case when the symmetric algebra is a symmetric Nakayama algebra with at least two simple modules. Our goal is to classify all special modules over a symmetric Nakayama algebra.

Lemma 5.3.1
Let A be a symmetric Nakayama algebra with $n \geq 2$ simple modules and M an indecomposable non-projective module, which is not in the Ω-orbit of a simple A-module. Then $Ext^2(M, M) \neq 0$.

Proof. Assume A has Loewy length k and without loss of generality $M = e_0 J^s$ for some s with $2 \leq s \leq k-2$. In section 3.1.1, we saw that $Ext^2(M, M) = 0$ iff the map $R_{k,s} : e_0 J^s e_k \rightarrow e_0 J^s e_{k+s}$ is surjective, where $R_{k,s}$ denotes right multiplication by the path starting at k with length s. Thus we have to show that $R_{k,s}$ is not surjective in order to prove the lemma. To see this just note that $k \equiv 1 \bmod n$ and thus $R_{k,s} : e_0 J^s e_1 \rightarrow e_0 J^s e_{1+s}$ can not be surjective, since the path from 0 to $1+s$ of smallest positive length $1+s$ (which is nonzero since $1+s \leq k-1$) is not in the image of $R_{k,s}$, since $s \geq 2$ and thus all paths in the image of $R_{k,s}$ have length at least $s+s = 2s > s+1$, noting that the smallest path in $e_0 J^s e_1$ has length at least s.

□

Lemma 5.3.2
Let A be a symmetric Nakayama algebra with $n \geq 2$ simple modules and Loewy length w and let $M = e_i J^s$ be an indecomposabe nonprojective A-module.

1. Every indecomposable nonprojective A-module M satisfies $\Omega^{2n}(M) \cong M$.

2. A module in the Ω-orbit of a simple A-module has period $2n$.

3. M is 1-rigid iff $1 \leq s \leq n-1$ or $w-n+1 \leq s \leq w-1$.

4. A 1-rigid module M has period at least four except in the case $n = 3$ and $w = 4$ and M being in the Ω-orbit of the module $e_0 J^2$.

Proof. 1. This follows by the fact that $\Omega^2(e_i J^s) = e_{i+1} J^s$.

2. This easily follows from the fact that $w - 1 \neq 1$ and the formulas for Ω in section 3.1.1 and thus $\Omega^l(e_i J^s) \neq e_i J^s$ for some l with $1 \leq l \leq 2n - 1$.

3. This was shown in 5.2.2.

4. We can assume without loss of generality that $M = e_0 J^s$ with $1 \leq s \leq n - 1$ by applying Ω^1 one time if necessary. Then $\Omega^1(M) = e_s J^{w-s}, \Omega^2(M) = e_1 J^s$ and $\Omega^3(M) = e_{1+s} J^{w-s}$. Thus $M \cong \Omega^i(M)$ for some i with $1 \leq i \leq 3$ is only possible in case $M \cong \Omega^1(M)$ or $M \cong \Omega^3(M)$. $M \cong \Omega^1(M) = e_s J^{w-s}$ is only possible in case $s \equiv 0 \bmod n$ and $w - s = s$, which forces $w \equiv 0 \bmod n$, which is not possible since A is assumed to be symmetric and thus $w \equiv 1 \bmod n$. The only remaining possiblity is $M \cong \Omega^3(M) = e_{1+s} J^{w-s}$, which forces $1 + s \equiv 0 \bmod n$ and $w - s = s$. This is only possible for $s = n - 1$ and $w = 2n - 2$, which forces $n = 3$, since otherwise $w \neq qn + 1$ for some $q \geq 1$.

□

Proposition 5.3.3
An indecomposable module over a symmetric Nakayama algebra A with at least two simple modules is special iff it is in the Ω-orbit of a simple module (which is $2n$-periodic) or n=3 and the algebra has Loewy length 4, and then every module in the Ω-orbit of $e_0 J^2$ is special and 3-periodic.

Proof. Let w be the Loewy length of A. Since a simple module in a symmetric Nakayama algebra has period $2n$, a module N in the Ω-orbit of a simple module is $2n$-periodic and has $Ext^i(N, N) = 0$ for all $i = 1, 2, ..., 2n-2$ (this is a special case of 3.2.3) and thus is special. By the previous lemma, $Ext^2(e_i J^k, e_i J^k) \neq 0$ for any indecomposable module $e_i J^k$ not in the Ω-orbit of a simple module. The period of a 1-rigid module $e_i J^k$ is at least 4 except in the case $n = 3$ and $w = 4$, where the modules in the Ω-orbit of the module $e_0 J^2$ have period 3. But special modules L with period at least 4 have $Ext^2(L, L) = 0$ and thus the module of the form $e_i J^k$, not in a Ω-orbit of a simple module, can not be special unless $n = 3, w = 4$ and $k = 2$. □

6 SGC-extensions of algebras

6.1 SGC-extensions for gendo-symmetric algebras

Starting with a given finite dimensional algebra A, we propose a new construction associating infinitely many new algebras to A. For a given module M, we denote by $Ba(M)$ the basic version of the module.

Definition 6.1.1
Let $A = A_0$ be an algebra. For $i \geq 0$, define
$A_{i+1} := End_{A_i}(Ba(A_i \oplus D(A_i)))$. We call A_i the *i-th SGC-extension* of A_0.

Note that SGC stands for smallest generator-cogenerator and that $Ba(A_i \oplus D(A_i))$ is the smallest generator-cogenerator in mod-A_i, in the sense that every generator-cogenerator has $Ba(A_i \oplus D(A_i))$ as a direct summand. SGC-extensions seem to be used for the first time by Yamagata in [Yam], where he produces algebras with arbitrary large dominant dimension using this construction. We will need the following results on gendo-symmetric algebras, where we refer to [FanKoe2] for proofs:

Theorem 6.1.2
Let A be a gendo-symmetric algebra and M an A-module. Then $domdim(M) \geq 2$ iff $\nu^{-1}(M) \cong M$ and in this case $domdim(M) = \inf\{i \geq 1 | Ext^i(D(A), M) \neq 0\}+1$. Dually, $codomdim(M) \geq 2$ iff $\nu(M) \cong M$ and in this case $codomdim(M) = \inf\{i \geq 1 | Ext^i(M, A) \neq 0\} + 1$.

A generator-cogenerator $M \in mod - A$ is called *m-cluster tilting* object (or module instead of object), in case $add(M) = \{X | Ext^i(X, M) = 0$ for all $0 < i < m\}$. An algebra A is called *higher Auslander algebra* in case $\infty > domdim(A) = gldim(A) \geq 2$ and in this case A is isomorphic to the endomorphism ring of some cluster tilting object, see [Iya] for more information. In [Iya], Iyama showed that M is an m-cluster tilting object for some m iff $End_A(M)$ is a higher Auslander algebra and he also introduced the functors $\tau_{r+1} := \tau\Omega^r$ and $\tau_{-(r+1)} := \tau^{-1}\Omega^{-r}$. Recall that an algebra A is (isomorphic to) a quiver algebra, iff it is basic and elementary, see for example [ARS].

Lemma 6.1.3
Let A be a quiver algebra, then also $B = End_A(Ba(A \oplus D(A)))$ is a quiver algebra.

Proof. Since $Ba(A \oplus D(A))$ is basic, also its endomorphism ring is basic. Recall that a basic algebra is elementary if all the simple modules have dimension one. Now the simple modules of B are those of the form $End_A(N)/rad(End_A(N))$, when N is an indecomposable direct summand of $Ba(A \oplus D(A))$. Assume that N is indecomposable projective and thus $N = e_i A$ for some primitive idempotent e_i. Then $End_A(N)/rad(End_A(N)) \cong e_i A e_i / e_i J e_i$ is one dimensional. Dually $End_A(N)/rad(End_A(N))$ is one-dimensional, in case $N = D(Ae_i)$ is an indecomposable injective modules for some primitive idempotent e_i. □

We need the following lemma for the next theorem, where one direction is inspired by [FanKoe3], Lemma 3.4.:

Lemma 6.1.4
Let A be a gendo-symmetric algebra and M an A-module.

1. Let M be non-projective. M has codominant dimension larger than or equal to 2 iff $\tau(M) \cong \Omega^2(M)$.

2. Let M be non-injective. M has dominant dimension larger than or equal to 2 iff $\tau^{-1}(M) \cong \Omega^{-2}(M)$.

Proof. Our main tool for the proof is 6.1.2. We prove only (1), since the proof of (2) is dual. Assume that M has codominant dimension larger than or equal to 2. Assume that $P_1 \to P_0 \to M \to 0$ is the minimal projective presentation of M. There is the following exact sequence, which exists by the definition of τ:
$0 \to \tau(M) \to \nu(P_1) \to \nu(P_0) \to \nu(M) \to 0$. But since P_1, P_0 are also injective and thus have codominant dimension larger than or equal to 2: $\nu(P_1) \cong P_1$ and $\nu(P_0) \cong P_0$. As M also has codominant dimension larger than or equal to 2: $\nu(M) \cong M$ and the exact sequence looks like the beginning of a minimal projective resolution of M:
$0 \to \tau(M) \to P_1 \to P_0 \to M \to 0$. Thus $\Omega^2(M) \cong \tau(M)$.
Assume now that $\Omega^2(M) \cong \tau(M)$. Looking at the minimal projective presentation $P_1 \to P_0 \to M \to 0$ of M, we get a minimal injective coresolution of $\tau(M)$ as follows: $0 \to \tau(M) \to \nu(P_1) \to \nu(P_0) \to \nu(M) \to 0$. But now since $\Omega^2(mod{-}A) = Dom_2$ (see for example [MarVil] proposition 4), $\tau(M) \cong \Omega^2(M)$ has dominant dimension at least two and thus in the above minimal injective coresolution $\nu(P_1)$ and $\nu(P_0)$ are projective-injective, which implies that P_0 and P_1 are also projective-injective, since in a gendo-symmetric algebra a module P is projective-injective iff $P \in add(eA)$, when eA is the minimal faithful projective-injective A-module. Thus M has codominant dimension at least two. □

Note that the previous lemma generalizes the well-known fact that $\Omega^2(M) \cong \tau(M)$ for any module M in a symmetric algebra.

Theorem 6.1.5
Let A be a gendo-symmetric algebra, M an A-module and $r \geq 0$ a natural number.

1. Let M be non-projective. Then M has codominant dimension at least $r+2$ iff $\Omega^{l+2}(M) \cong \tau_{l+1}(M)$ for all $l = 0, 1, ..., r$.

2. Let M be non-injective. Then M has dominant dimension at least $r+2$ iff $\Omega^{-l-2}(M) \cong \tau_{-(l+1)}(M)$ for all $l = 0, 1, ..., r$.

Proof. We prove only (1), since the proof of (2) is dual.
$codomdim(M) \geq r+2 \iff codomdim(\Omega^l(M)) \geq 2$ for all $l = 0, 1, ..., r$
$\iff \tau(\Omega^l(M)) \cong \Omega^2(\Omega^l(M))$ for all $l = 0, 1, ..., r$ (here we used 6.1.4)
$\iff \tau_{l+1}(M) \cong \Omega^{l+2}(M)$ for all $l = 0, 1, ..., r$. □

We note a special case of the previous theorem as a corollary.:

Corollary 6.1.6
Let A be a nonsymmetric, gendo-symmetric algebra.
Then $domdim(A) = 2 + \sup\{r \geq 0 | \Omega^{l+2}(D(A)) \cong \tau_{l+1}(D(A))$ for all $l = 0, 1, ..., r\}$
$= 2 + \sup\{r \geq 0 | \Omega^{-l-2}(A) \cong \tau_{-(l+1)}(A)$ for all $l = 0, 1, ..., r\}$.

Proof. This follows from 6.1.5 in combination with the fact that the dominant dimension of an algebra coincides with the codominant dimension of that algebra. □

We need a lemma for the next theorem:

Lemma 6.1.7
Let A be a gendo-symmetric algebra. Then the inverse Nakayama functor ν^{-1} is isomorphic to the identity functor on the subcategory Dom_2.

Proof. Let A be gendo-symmetric with minimal faithful projective-injective module eA. Then by 4.1.4, $D(A) \cong Ae \otimes_{eAe} eA$ and we use this bimodule isomorphism in the following. For a module $M \in Dom_2$, define $I_M : M \to \nu^{-1}(M)$ as $I_M(m)(ae \otimes eb) = maeb$. We saw in 4.3.1 that I_M is an isomorphism when M has dominant dimension at least two. Then I defines a natural transformation, which is an isomorphism, $I : id_{|Dom_2} \to \nu^{-1}_{|Dom_2}$, since for a map $f : M \to N$ between two modules of dominant dimension at least two: $I_N f(m) = \nu^{-1}(f) I_M(m)$ for every $m \in M$, because this equality is equivalent to $f(m)aeb = f(maeb)$ for any $a, b \in A$ which is true since f is A-linear.

□

Theorem 6.1.8
Let $A_0 = A$ be a gendo-symmetric algebra with finite dominant dimension.

1. Then $domdim(A_1) = domdim(A_0)$.

2. $Gordim(A_1) = Gordim(A_0)$.

Proof.
1. $domdim(A_1) = inf\{l \geq 1 | Ext^l(D(A), A) \neq 0\} + 1 = domdim(A_0)$, by 6.1.2 in combination with Mueller's theorem.

2. Now we calculate the Gorenstein dimension using formula 3.2.8. We just do it for the right Gorenstein dimension, since the left Gorenstein dimension can be treated in the same way. Let n be such that $n+2 = domdim(A_0)$. Then $\tau_{n+1} = \tau\Omega^n(D(A)) = \Omega^{n+2}(D(A))$, using

6.1.4. For $M = A \oplus D(A)$, we have to calculate a minimal add(M)-resolution of $\Omega^l(D(A))$ for $l \geq n+2$. Let $(P_k)_k$ be a minimal projective resolution of $D(A)$. Now note that for $k \geq 2$, the exact sequence $0 \to \Omega^{k+1}(D(A)) \to P_k \to \Omega^k(D(A)) \to 0$ involves only modules with dominant dimension at least two and thus $\nu^{-1}(N) \cong N$ for all those modules, by 6.1.2. Now the exact sequence stays exact after applying the functor $Hom(D(A) \oplus A, -) \cong Hom(D(A), -) \oplus Hom(A, -)$ since $Hom(D(A), -) \cong \nu^{-1}$ is isomorphic to the identity functor on Dom_2 and $Hom(A, -)$ is exact anyway. Thus such minimal approximations are given by the usual projective covers. Thus the Gorenstein dimension of A_1 equals $\inf\{k \geq n+2 | \Omega^k(D(A)) \in add(D(A) \oplus A)\}$. Now note that for $k \geq 2$, $\Omega^k(D(A))$ can not have a summand isomorphic to an injective nonprojective module I, since I has dominant dimension 0, but $\Omega^k(D(A))$ has dominant dimension at least two. Thus $\inf\{k \geq n+2 | \Omega^k(D(A)) \in add(D(A) \oplus A)\}$ equals the right projective dimension of $D(A)$, which is the right Gorenstein dimension of A_0.

□

We have the following conjecture, which is based on many computer experiments with the GAP package QPA, generalising the previous theorem:

Conjecture
Let $A = A_0$ be a nonsymmetric, gendo-symmetric algebra. Then for all $i \geq 1$: $domdim(A_0) = domdim(A_i)$ and $Gordim(A_0) = Gordim(A_i)$.

In general, the previous theorem is not true for algebras, which are not gendo-symmetric. It is not even true for Morita algebras, as the next example shows.

Example 6.1.9
Let C be the selfinjective Nakayama algebra with Kupisch series $[4,4]$ and $L = rad(P)$ for an indecomposable projective C-module. Then $A_0 := End_C(C \oplus L)$ is a Nakayama algebra with Kupisch series $[5,5,6]$ and has dominant dimension 3 and infinite Gorenstein dimension, but A_1 has dominant dimension and Gorenstein dimension equal to two, while A_2 has again dominant dimension 3 and infinite Gorenstein dimension. Note that A_1 and A_2 are Nakayama algebras with Kupisch series $[6,7,7,7]$ and $[8,8,8,8,9]$.

We give a concrete example, where we calculate all A_i for $i \geq 1$ for a given class of gendo-symmetric algebras A_0.

Example 6.1.10
Let B be the symmetric Nakayama algebra with Kupisch series $[3,3]$. Define $A_0 = A = End_B(B \oplus rad(P))$, where P is an arbitrary projective indecomposable B-module. Then $A_0 = A$ is again a Nakayama algebra with Kupisch series $[3,4,4]$. A_0 has dominant dimension equal to the global dimension equal to 4. Recall that the Gorenstein dimension coincides with the global dimension, in case the global dimension is finite and not equal to zero. For all $i \geq 1$, A_i is again a Nakayama algebra with $3+i$ simple modules and with Kupisch

series $[3+i, 3+i, ..., 3+i, 4+i, 4+i]$, where $3+i$ appears $i+1$ times. Thus A_i has $i+3$ simple modules. A_i never has finite global dimension for $i \geq 1$, but dominant and Gorenstein dimension equal to 4 for $i \geq 1$ as we will show now. By 2.2.2, there is only one indecomposable injective non-projective module. This module is $D(A_i e_i)$. In fact the top of the module $D(A_i e_i)$ is equal to $D(soc(A_i e_i)) \cong S_{i+1}$. Thus the projective cover of $D(A_i e_i)$ is $e_{i+1} A_i$ with kernel S_{i+1}. Using the notation and methods in 3.1.1., we have to calculate a minimal projective resolution of $S_{i+1} = (i+1, i+3)$. Now the calculation of syzygies shows that the codominant dimension and the projective dimension of $D(A)$ equals 4: $(i+1, i+3) \rightarrow (2i+5, 1) \rightarrow (0, i+3)$. To show that the global dimension of A_i is infinite for $i \geq 1$, we show that the module $e_0 J^1 = (0,1)$ has infinite projective dimension: $(0,1) \rightarrow (1, i+2) \rightarrow (0,1)$. Thus $e_0 J^1$ is 2-periodic and therefore has infinite projective dimension.

In fact we are not aware of an example of a gendo-symmetric algebra A_0 with finite non-zero global dimension, such that A_1 also has finite global dimension.

6.2 Applications

In the following we give two applications of SGC-extensions of gendo-symmetric algebras.

Recall the following definition, see for example [ARS], chapter IV.3.: A module M is *reflexive* in case $Ext^i(D(A), \tau(M)) = 0$ for $i = 1, 2$.

We start with the following definition given by Tachikawa in [Ta2]:

Definition 6.2.1
An Auslander-Reiten sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ is called *reflexive*, if X, Y and Z are reflexive modules. An algebra A is said to have *reflexive Auslander-Reiten sequences* iff all Auslander-Reiten sequences $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$, with projective and noninjective X are reflexive.

In the same article [Ta2], Tachikawa proved the following result in theorem 2.4.:

Theorem 6.2.2
The following are equivalent for a finite dimensional algebra A:

1. A has reflexive Auslander-Reiten sequences.

2. domdim$(A) \geq 2$ and domdim$(R) \geq 4$, where $R := End_A(A \oplus D(A))$.

We note the following proposition, which gives alot of examples of algebras having reflexive Auslander-Reiten sequences.

Proposition 6.2.3
A gendosymmetric algebra A has reflexive Auslander-Reiten sequences iff it has dominant dimension larger than or equal to 4.

Proof. This is immediate by the above theorem and the characterisation of the dominant dimension in gendosymmetric algebras (6.1.2) and Mueller's theorem: domdim(A)=$inf\{i \geq 1 | Ext^i(D(A), A) \neq 0\} + 1 =$
domdim($End_A(A \oplus D(A))$).

□

The next corollary applies SGC-extensions to the construction of new orthosymmetric modules from known ones.

Corollary 6.2.4
Let $A_0 = A = End_B(M)$ be a gendo-symmetric algebra and assume the generator-cogenerator M over the symmetric algebra B is n-orthosymmetric. Then $Ba(D(A_0) \oplus A_0)$ is n-orthosymmetric over the algebra A_0.

Proof. This follows by 6.1.8, since the dominant and Gorenstein dimensions of A_0 and A_1 coincide. □

Of course, if the conjecture 6.1 is true then the corollary could be strengthened to the statement that even $Ba(D(A_i) \oplus A_i)$ is n-orthosymmetric for all $i \geq 0$.

Example 6.2.5
The algebras A_i in example 6.1.10 all have $Ba(D(A_i) \oplus A_i)$ as a 2-orthosymmetric module. Furthermore, they always have reflexive Auslander-Reiten sequences, since they have dominant dimension equal to 4 for all $i \geq 0$.

We end this chapter with a remark on SGC-extensions for other classes of algebras:

Remark 6.2.6
We calculated the SGC-extensions for several hereditary quiver algebras by hand and with a computer and found no counterexample to the following possible result:
A hereditary algebra A_0 is representation-finite iff infinitely many SGC-extensions A_i of A_0 are higher Auslander algebras.

7 A counterexample to a conjecture about dominant dimension

In this short chapter we provide a counterexample to the following conjecture stated in [CX] as conjecture 2:

Conjecture
Let A be a finite dimensional algebra with finite dominant dimension $n \geq 1$. Let $0 \to A \to I_0 \to I_1 \to ...$ be a minimal injective resolution of A, then $B := End_A(I_0 \oplus \Omega^{-n}(A))$ has dominant dimension n.

Take for A the Nakayama algebra with Kupisch series $[3, 4]$. The quiver of A looks as follows:

$$\begin{tikzcd} \bullet^0 \arrow[r, bend right, "a_0"'] & \bullet^1 \arrow[l, bend right, "a_1"'] \end{tikzcd}$$

Let e_0A and e_1A be the indecomposable projective A-modules. Then $soc(e_1A) = e_1J^3 = S_0$ and by comparing dimensions $e_1A \cong D(Ae_0)$ is projective-injective. Note that $soc(e_0A) = e_0J^2 \cong S_0$ and thus the injective hull of e_0A is $e_1A \cong D(Ae_0)$. Comparing dimensions, one gets $e_0A \cong e_1J$. The short exact sequence for the injective hull of e_0A looks as follows:
$0 \to e_0A \to e_1A \to S_1 \to 0$. Thus $\Omega^{-1}(A) = \Omega^{-1}(e_0A) = S_1$ and the injective hull I_0 of A is e_1A^2. A has dominant dimension 1, since the injective hull of S_1 is not projective. Note that in case $add(M) = add(N), End_A(M)$ and $End_A(N)$ are Morita equivalent by [SkoYam], lemma 6.12. Since dominant dimension is preserved under Morita equivalence, we calculate the dominant dimension of $B := End_A(e_1A \oplus S_1)$. First we calculate the quiver and relations of B: $Hom_A(e_1A, e_1A) \cong e_1Ae_1$ has dimension two and radical e_1Je_1. $Hom_A(S_1, S_1)$ has dimension one and radical zero. $Hom_A(S_1, e_1A)$ is zero and $Hom_A(e_1A, S_1)$ is one dimensional. Thus B has dimension 4 and B is a radical square zero algebra with the following quiver:

$$\begin{tikzcd} \bullet^2 \arrow[r, "\beta"'] & \bullet^1 \arrow[loop right, "\alpha"] \end{tikzcd}$$

Now the indecomposable injective B-modules have dimension one and three, while the indecomposable projective B-modules both have dimension two. Thus there exists no indecomposable projective-injective B-module and therefore B has dominant dimension zero. This gives a counterexample to the conjecture with $n = 1$.

Bibliography

[Abr] Abrar, Muhammad : *Dominant dimensions of finite dimensional algebras. http://elib.uni-stuttgart.de/opus/volltexte/2012/7548/pdf/Abrar.pdf*, PhD thesis, Stuttgart 2012.

[AnFul] Anderson, Frank W.; Fuller, Kent R.: *Rings and Categories of Modules.*Graduate Texts in Mathematics, Volume 13, Springer-Verlag, (1992).

[ASS] Assem, Ibrahim; Simson, Daniel; Skowronski, Andrzej: *Elements of the Representation Theory of Associative Algebras, Volume 1.* London Mathematical Society Student Texts, Volume 72, (2007).

[APT] Auslander, Maurice; Platzeck, Maria Ines; Todorov, Gordana: *Homological theory of idempotent Ideals.* Transactions of the American Mathematical Society, Volume 332, Number 2 , August 1992.

[ARS] Auslander, Maurice; Reiten, Idun; Smalo, Sverre: *Representation Theory of Artin Algebras.* Cambridge Studies in Advanced Mathematics, Volume 36, Cambridge University Press, 1997.

[BSZ] Bautista, Raymundo; Salmeron, Leonardo; Zuazua Vega, Rita: *Differential Tensor Algebras and their Module Categories.* London Mathematical Society Lecture Note Series 362.

[Ben] Benson, David J.: *Representations and cohomology I: Basic representation theory of finite groups and associative algebras.* Cambridge Studies in Advanced Mathematics, Volume 30, Cambridge University Press, 1991.

[BreWis] Brzezinski, Tomasz; Wisbauer, Robert: *Corings and Comodules.* London Mathematical Society Lecture Note Series 309.

[But] Butler, Michael: *The syzygy theorem for monomial algebras.* Trends in the representation theory of finite-dimensional algebras (Seattle, WA, 1997), 111-116, Contemp. Math., 229, Amer. Math. Soc., Providence, RI, 1998.

[ChMar] Chan, Aaron; Marczinzik, René: *On representation-finite gendo-symmetric biserial algebras.* http://arxiv.org/abs/1607.05965.

[CheKoe] Chen, Hongxing; Koenig, Steffen: *Ortho-symmetric modules, Gorenstein algebras and derived equivalences.* International

Mathematics Research Notices (2016), electronically published. doi:10.1093/imrn/rnv368.

[CX] Chen, Hongxing; Xi,Changchang: *Dominant dimensions, derived equivalences and tilting modules.* Israel Journal of Mathematics (2016), 1-44 DOI: 10.1007/s11856-016-1327-4.

[Che] Chen, Xiao-Wu: *Gorenstein Homological Algebra of Artin Algebras.* *http://home.ustc.edu.cn/~xwchen/Personal%20Papers/postdoc-Xiao-Wu%20Chen%202010.pdf*, retrieved 18.06.2016.

[DR] Dlab, Vlastimil; Ringel, Claus Michael: *The Module Theoretical Approach to Quasi-hereditary Algebras.* in Representations of Algebras and Related Topics, London Mathematical Society Lecture Note Series 168, 200-224.

[Dro] Drozd, Yuriy: *Tame and wild matrix problems.* Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979), pp. 242-258, Lecture Notes in Math., 832, Springer, Berlin-New York, 1980.

[FanKoe] Fang, Ming; Koenig, Steffen: *Gendo-symmetric algebras, canonical comultiplication, bar cocomplex and dominant dimension.* Trans. Amer. Math. Soc. 368 (2016), no. 7, 5037-5055.

[FanKoe2] Fang, Ming, Koenig, Steffen: *Endomorphism algebras of generators over symmetric algebras.* J. Algebra 332 (2011), 428-433.

[FanKoe3] Fang, Ming, Koenig, Steffen: *Schur functors and dominant dimension.* Trans. Amer. Math. Soc. 363 (2011), no. 3, 1555-1576.

[Farn] Farnsteiner, Rolf: *Algebras of Finite Global Dimension: Acyclic Quivers.* *https://www.math.uni-bielefeld.de/~sek/select/Acyclic.pdf*, retrieved 2015-06-18.

[Ful] Fuller, Kent R.: *Generalized Uniserial Rings and their Kupisch Series.* Math. Zeitschr. Volume 106, pages 248-260, (1968).

[Gus] Gustafson, William H.: *Global dimension in serial rings.* Journal of Algebra, Volume 97, pages 14-16, 1985.

[HuXi] Hu, Wei; Xi, Changchang: *Derived equivalences and stable equivalences of Morita type, I.* Nagoya Math. J. 200 (2010), 107-152.

[Iya] Iyama, Osamu: *Higher-dimensional Auslander-Reiten theory on maximal orthogonal subcategories.* Advances in Mathematics, Volume 210, pages 22-50, 2007.

[Iya2] Iyama, Osamu: *Auslander correspondence.* Adv. Math. 210 (2007), no. 1, 51-82.

[IyaSol] Iyama, Osamu; Solberg, Øyvind: *Auslander-Gorenstein algebras and precluster tilting.* http://arxiv.org/abs/1608.04179.

[KerYam] Kerner, Otto; Yamagata, Kunio: *Morita algebras.* Journal of Algebra, Volume 382, pages 185-202, (2013).

[KerYam2] Kerner, Otto; Yamagata, Kunio: *Morita theory, revisited.* Expository lectures on representation theory, 85-96, Contemp. Math., 607, Amer. Math. Soc., Providence, RI, 2014.

[KSX] Koenig, Steffen; Slungård, Inger Heidi;Xi, Changchang: *Double centralizer properties, dominant dimension, and tilting modules.* J. Algebra 240 (2001), no. 1, 393-412.

[Kue] Külshammer, Julian: *In the bocs seat: Quasi-hereditary algebras and representation type.* To appear in SPP 1388 Conference Proceedings, 2016.

[Lam] Lam, Tsi-Yuen: *A First Course in Noncommutative Rings.* Graduate Texts in Mathematics, Volume 131, Springer-Verlag, 1991.

[MarVil] Martinez-Villa, Roberto: *Modules of dominant and codominant dimension.* Comm. Algebra 20 (1992), no. 12, 3515-3540.

[Mue] Mueller, Bruno: *The classification of algebras by dominant dimension.* Canadian Journal of Mathematics, Volume 20, pages 398-409, 1968.

[Nak] Nakayama, Tadasi: *On algebras with complete homology.* Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg (1958), 22: 300–307.

[Rot] Rotman, Joseph : *An Introduction to Homological Algebra (Universitext).* Springer; 2nd edition, 2009.

[SimSko3] Simson, Daniel; Skowronski, Andrzej: *Elements of the Representation Theory of Associative Algebras, Volume 3: Representation-Infinite Tilted Algebras.* London Mathematical Society Student Texts, Volume 72, 2007.

[SkoYam] Skowronski, Andrzej; Yamagata, Kunio: *Frobenius Algebras I: Basic Representation Theory.* EMS Textbooks in Mathematics, (2011).

[Ta] Tachikawa, Hiroyuki: *Quasi-Frobenius Rings and Generalizations: QF-3 and QF-1 Rings (Lecture Notes in Mathematics 351).* Springer; (1973).

[Ta2] Tachikawa, Hiroyuki: *Reflexive Auslander-Reiten sequences.* Perspectives in ring theory (Antwerp, 1987), 311-320, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 233, Kluwer Acad. Publ., Dordrecht, 1988.

[Yam] Yamagata, Kunio: *Frobenius Algebras.* Hazewinkel, M. (editor): Handbook of Algebra, North-Holland, Amsterdam, Volume I, pages 841-887, 1996.

[Yam2] Yamagata, Kunio: *Modules with serial Noetherian endomorphism rings.* Journal of Algebra, Volume 127, pages 462-469, 1989.

[Zi] Zimmermann, Alexander: *Representation theory. A homological algebra point of view.* Algebra and Applications, Springer 2014.