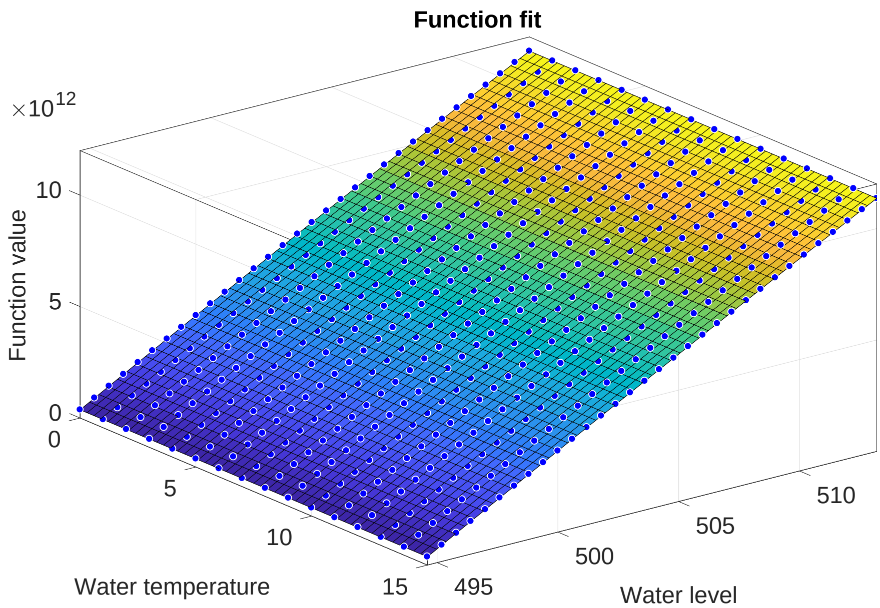
Universität Stuttgart

Geodätisches Institut

Observing the gravitational constant G underneath two pump storage reservoirs near Vianden (Luxembourg)



Bachelorarbeit im Studiengang

Geodäsie und Geoinformatik

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Stuttgart, 30. September 2017 Unterschrift

Abstract

The aim of this thesis is to extract the gravitational constant G from relative gravity measurements underneath two artificial reservoirs in Vianden (Luxembourg). If one wants to determine the Newtonian constant G from measurements many variables play a role, like water level, density and temperature. All of them are measured, while the gravimeter is measuring the gravitational attraction. To extract the gravitational constant G it is important to know the gravitational attraction of each point inside the reservoir. For this calculation the 3D models of the lakes are filled with tiny cubes, from which the exact integral to calculate their gravitational attraction to the gravimeter is known. Calculated values are compared to the measured ones which makes it possible to extract the searched gravitational constant G.

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Chapter 1

Introduction

Gravity is the first spotted force of the currently known four fundamental forces. Also it is the first force that was mathematically described in an exact way of these four - by Newtons I. Theory of Gravity (1687). The gravitational force F between two masses m_1 and m_2 with the distance r is calculated by,

$$F = G\frac{m_1 m_2}{r^2} \tag{1.1}$$

where G is the gravitational constant.
In today´s physics, 330 years later, the gravitation is getting a more and more special status in the context of nature forces. When we analyse the measured values for the gravitational constant G in the last 25 years, it can be seen that all the values are not very precise. The value used today (CODATA 2002) is just a average over the measured values, seen in figure 1.1.

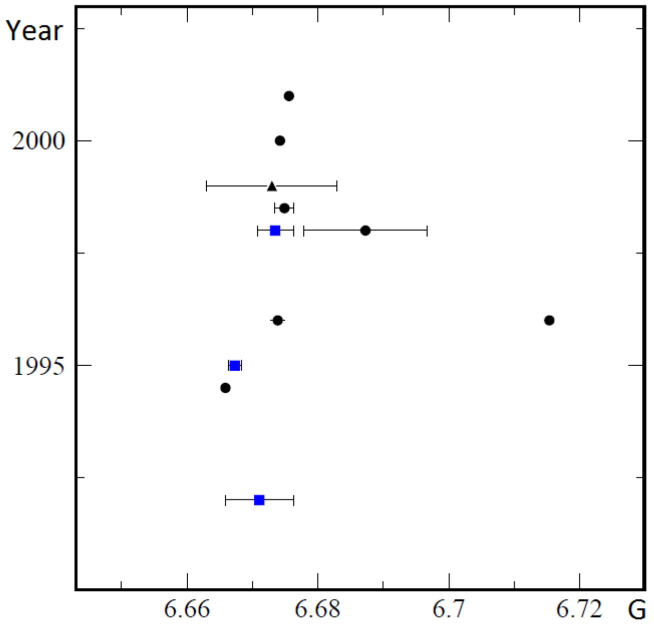

Figure 1.1: *Determined Newtonian gravitational constant G* $\left[10^{-11}\frac{\mathrm{m}^3}{\mathrm{kg}\cdot\mathrm{s}^2}\right]$ *values between 1990 and 2002 (Kleinevoß, 2002)*

In conclusion the question is how to get G determined with an higher accuracy?
In the past various experiments have attempted to determine a more exact gravitational constant G. The best known attempt was performed in 1797 by the British scientist Henry Cavendish.

1.1 Cavendish experiment

As Figure 1.2 indicates, Cavendish used a torsion balance apparatus, which was invented by the cleric John Mitchell. On this he got two lead spheres with a mass of 1.46 kg together. These two masses are connected by a rod, which is hanging free on a rope. Next to the lead sphere were two bigger masses of 316 kg together, which attract the small masses by the bigger masses leads to a torsion of the rope of $1°$. The outside masses are arranged so that the attraction to each other is vertical with the gravitational attraction. To calculate the gravitational constant, the torsion momentum of the rope has to be noticeably. In this case the Torsion momentum can be defined over a oscillation period measurement with known inertia momentum. The value for the mass of the earth calculated by Cavendish is $5.48 \cdot m_{\text{Water}}$,that gives a value for the gravitational constant G of $G_{\text{Cavendish}} = 6.754 \cdot 10^{-11} \frac{\text{m}^3}{\text{kg} \cdot \text{s}^2}$. This value fails the value we use today by only 1.2 %.

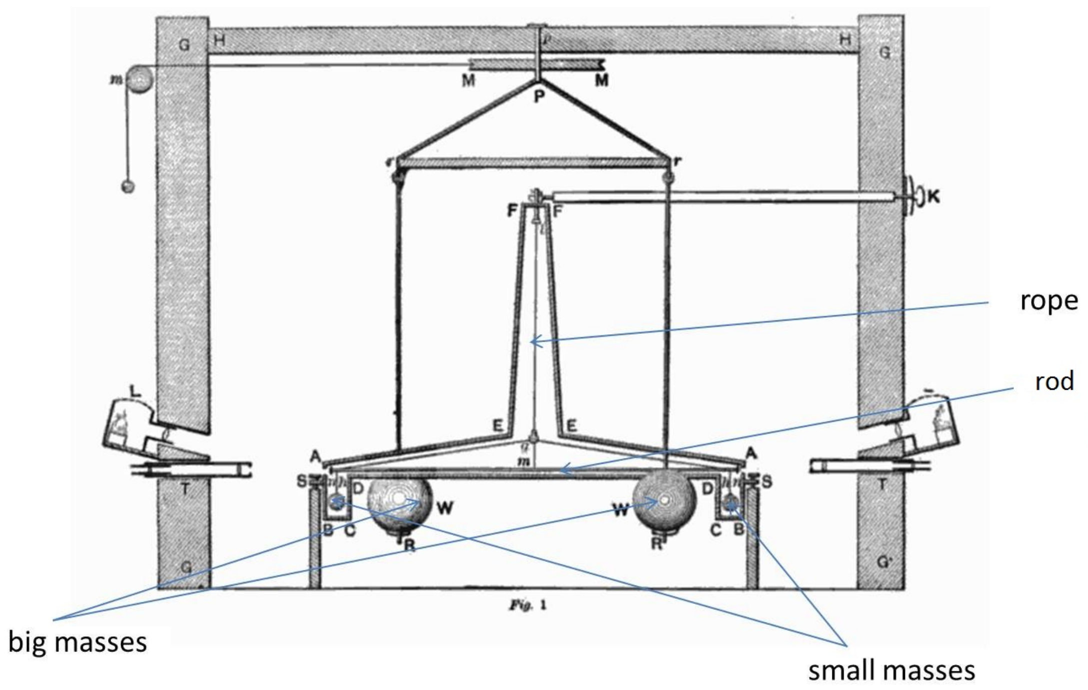

__Figure 1.2:__ Cavendish apparatus to define the gravitational constant G, with torsion balance (Kleinevoß, 2002)

1.2 Large scale experiments

The idea is monitoring the modification of the gravitational attraction next to a big mass. For example in 1774 the British scientist Nevil Maskelyne did an experiment in the Scottish highlands based on this idea. In figure 1.3 his breadboard is sketched.

Figure 1.3: *Nevil Maskelyne experiment in the Scottish highlands*

The deflection of the verticals beside Schiehallion mountain (with assumed known mass) is measured, to calculate the mass of the earth. With this experiment G was defined as $7.8 \cdot 10^{-11} \frac{m^3}{kg \cdot s^2}$. The weakness of this kind of experiments is the unknown mass distribution of the mountain. Soon the systematic errors were discovered and as a result other ways to calculate the gravitational constant G were investigated. Another concept was to measure the gravitational attraction in the depths of the Pacific or in a mine. To calculate the constant the mass of a spherical shell and the mass of the whole earth is set into relation. In a mine in Australia 1984 the best value was measured, with this kind of experiment, as $6.712 \pm 0.0037 \cdot 10^{-11} \frac{m^3}{kg \cdot s^2}$, (Kleinevoß, 2002).

1.3 Vianden experiment

The idea of the experiment in Vianden (Luxembourg) is to put a relative gravimeter under two man-made lakes. The gravimeter measures the gravitational attraction of the lakes every hour. Also the water temperature and level are observed. As the lakes are used as a pump storage station, they change from full to empty more than once a day, in exact numbers 14–16 m. The water in the reservoir also is well mixed, i.e. not stratified. Thus the variation of the gravitational attraction changes about 400 μGal (Gal equals $1 \cdot \frac{cm}{s^2}$) more than once a day. With the measurements of water level and temperature, it is also achievable to calculate the attraction that the gravimeter measures. Both compared to each other give the chance to calculate G. The following pages describe the project more precisely.

Chapter 2

Composition of the experiment in Vianden

Figure 2.1 shows the general composition of the experiment.

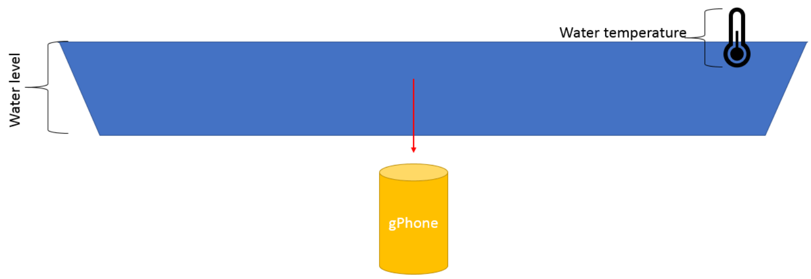

Figure 2.1: *Sketch of the Vianden project*

In general it is a relative gravimeter under a lake complex in Vianden. The lakes are used for generating energy by a pump storage system. The advantage is that the shape of the lakes is well known. Also the walls on the edges and the bottom are made of concrete, so there is no water in the soil. Thus the volume and volume changes of the lakes can be calculated very precisely. As a relative gravimeter is used, only the gravitational attraction changes are measured. The measured values depend on water level and density $g_{\text{measured}}(\text{Temperature}, \text{Level})$. Each time the relative gravimeter measures the attraction, the temperature and water level are measured simultaneously.

As a result the following measurement files are created, as shown in table 2.1.

***Table 2.1:** Measurements from the 1th December 2016*

Time	Water Temperature [°C]	gravitational attraction [μGal]	Water Level [m]
00 00 00	8.39	1.70	498.850
01 00 00	8.38	120.13	500.068
02 00 00	8.38	54.44	501.711
03 00 00	8.37	-8.83	503.306
04 00 00	8.36	-71.56	504.896
05 00 00	8.35	-124.03	506.294
06 00 00	8.34	-126.40	506.227
07 00 00	8.34	-51.80	504.003
08 00 00	8.33	32.41	501.965
09 00 00	8.32	76.70	500.999
10 00 00	8.31	84.09	500.809
11 00 00	8.30	51.82	501.665
12 00 00	8.29	19.31	502.514
13 00 00	8.28	-8.01	503.194
14 00 00	8.27	-25.39	503.657
15 00 00	8.27	-30.18	503.799
16 00 00	8.26	-99.15	503.062
17 00 00	8.25	71.06	501.158
18 00 00	8.24	121.87	499.872
19 00 00	8.23	152.45	499.122
20 00 00	8.22	169.11	498.741
21 00 00	8.21	206.05	497.754
22 00 00	8.20	225.60	497.240
23 00 00	8.19	183.49	498.377

With the software T-Soft (*T-Soft*, 1.1.0.1) it is possible to look at these measurements in a smoother way, see figure 2.2.

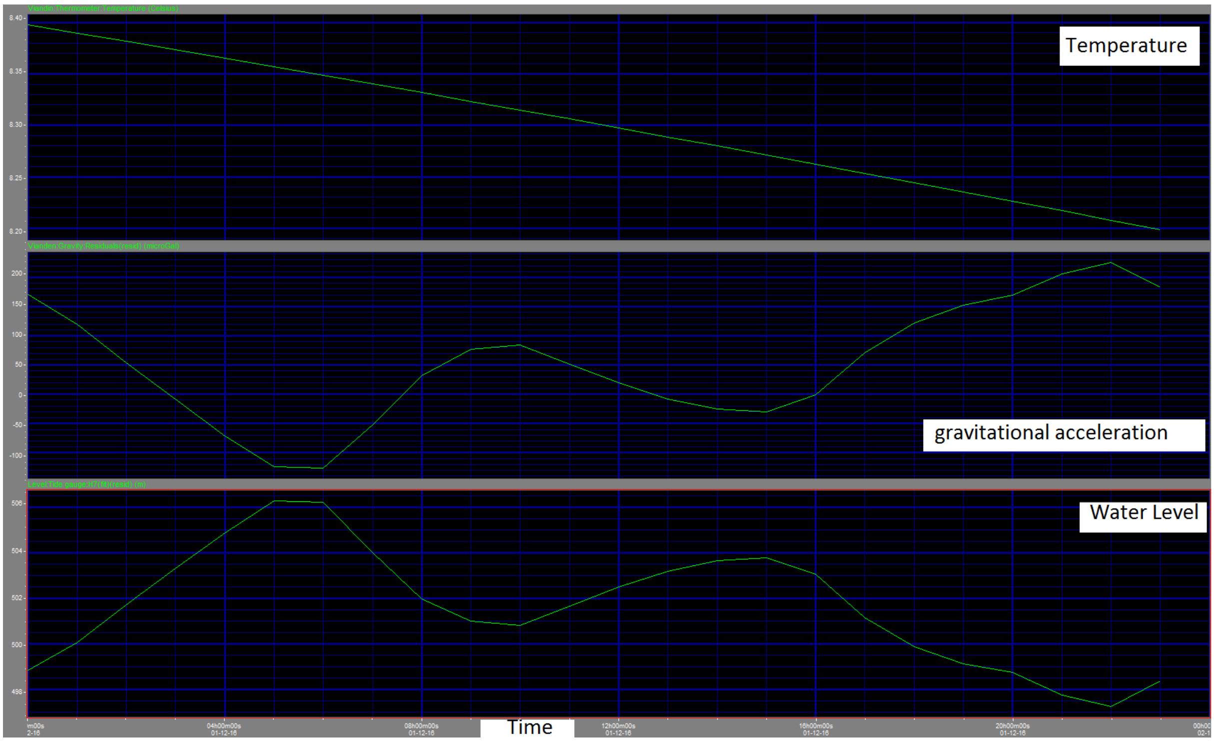

Figure 2.2: *Signals of the 1th December 2016 (table 2.1), plotted in the software T-Soft (Screenshot). The first plot shows the Temperature in* [°C] *between 8.20 and 8.40, the second shows the gravitational attraction in* [μGal] *between -100 and 200, the third shows the water level in* [m] *between 498 and 506, over the time.*

In the figure above it is discernible, that the gravity and water level are anti-correlated with each other. To shield the project from external effects, the calculation of G is realized using a differential, as shown in the following equation.

$$G = \frac{g_{\text{measured}}(t_0) - g_{\text{measured}}(t)}{f(t_0) - f(t)} \tag{2.1}$$

The next step is to characterize the function f. The gravimeter is just measuring in Z-direction. In figure 2.3 it is shown what the gravimeter measures and which attraction originate in the cubes. Every cube has an attraction component in x, y and z, but the gravimeter's measurement of g is only in Z-direction. In other words, the Z-direction is the gravity direction, i.e. along plumb line, at least locally. It does not necessarily coincide with the Z-direction of all the individual cubes. Hypothesizing that all of the plumb lines are parallel is the first approximation made.

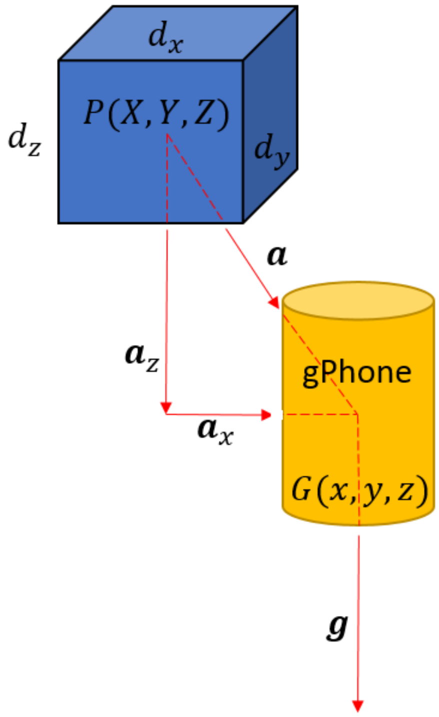

Figure 2.3: *Sketch of the cubes and the gPhone gravimeter*

Hence it is relevant to calculate the attraction of, for an example, a point mass only in Z-direction.
That implies the standard equation for the attraction of a point mass in x_2, y_2, z_2 to a point in x_1, y_1, z_1 with:

$$\mathbf{a} = -G \cdot \frac{m}{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{3/2}} \cdot \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix} \tag{2.2}$$

This can be simplified with $r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ to:

$$a_z = -G \frac{m \cdot (z_2 - z_1)}{r^3} \tag{2.3}$$

As the aim is to calculate G. The used function should not depend on the Newtonian constant G. In conclusion the gravitational attraction equation is used without G. The used part is described by the following equation.

$$f = \frac{m \cdot (z_2 - z_1)}{r^3} \tag{2.4}$$

2.1 Contours of the lakes

Figure 2.4 shows the contours of the reservoirs that geodetic engineers observed in Vianden. The figure was generated with the use of MatLab (*MatLab*, 2017b).

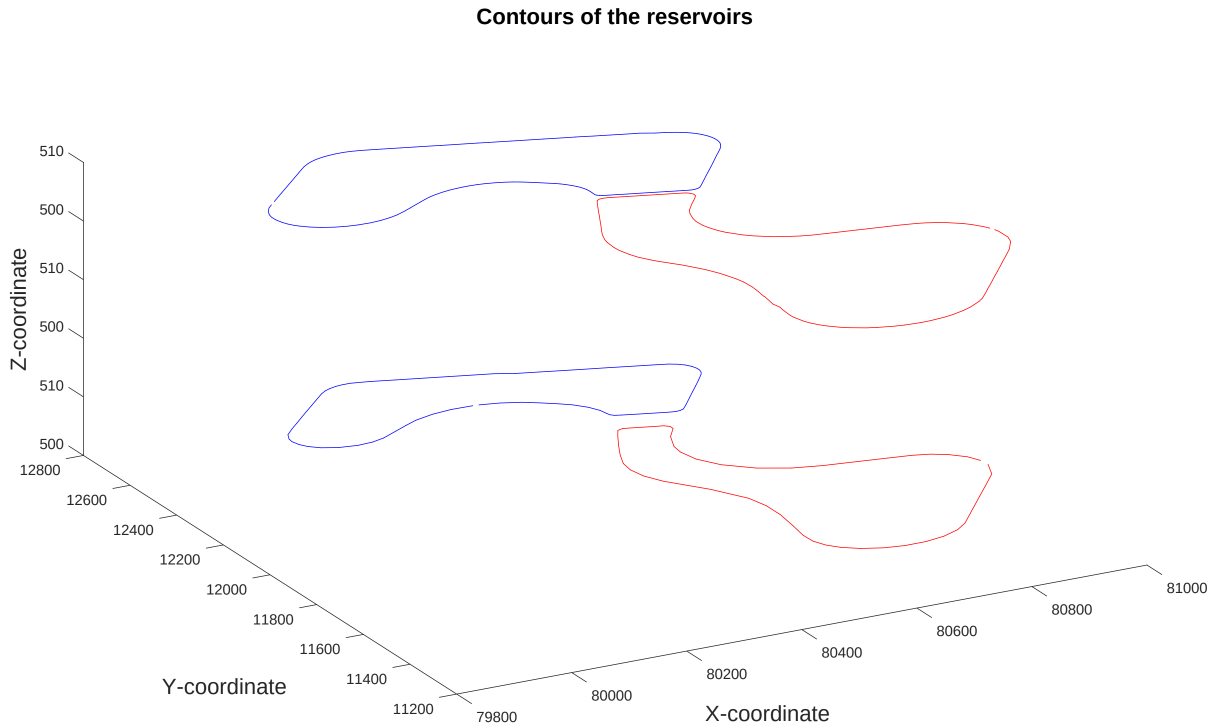

Figure 2.4: *Contours of the Lakes in Vianden, generated by MatLab* (MatLab*, 2017b)*

The coordinate system used is a local system. It is obvious on this plot both lakes combined are more than 800 m in X-direction and 1 500 m in Y-direction. The water level difference is up to 19 m.

2.2 3D Model

The next step is to create a 3D model of the Lakes, because it is necessary to voxelize them. The Lake will be modelled by the use of small cubes. The gravitational attraction of cubes can be calculated perfectly, which helps to find the function $F(\text{Level}, \text{Temperature})$. The creation of the 3D Model is made by IGMAS (*IGMAS*, 1.2.102.1), a software from the University of Kiel. The figure 2.5 shows how the 3D Model of the south-eastern Lake looks like.

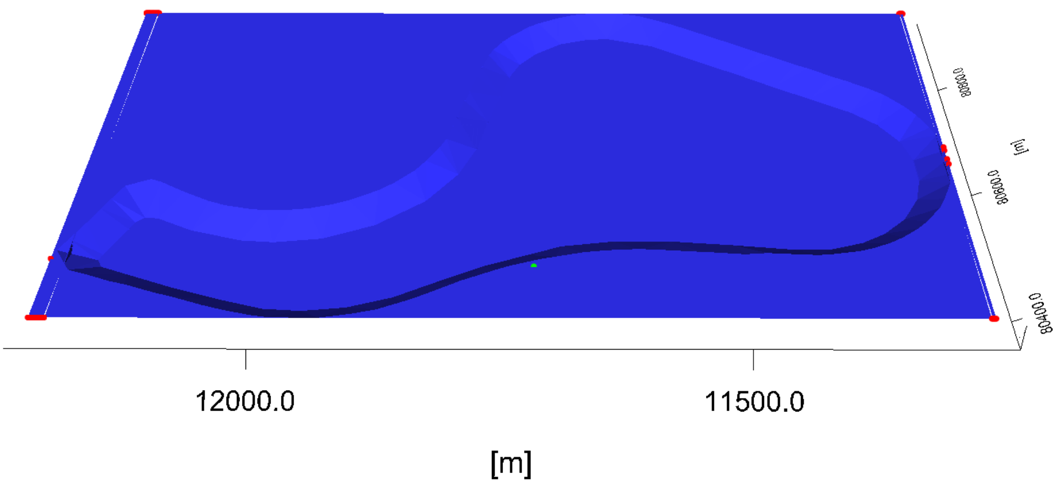

Figure 2.5: *IGMAS produces a 3D Model of the south-eastern lake (Screenshot from IGMAS)*

The next task is to fill the 3D Model with cubes. The resolution is limited by IGMAS. The maximum number of cubes is 40 000 000, this gives the maximum resolution for the reservoirs of 0.6 m. Even this task is possible with IGMAS (*IGMAS*, 1.2.102.1), figure 2.6 shows how the borders of the lake look like.

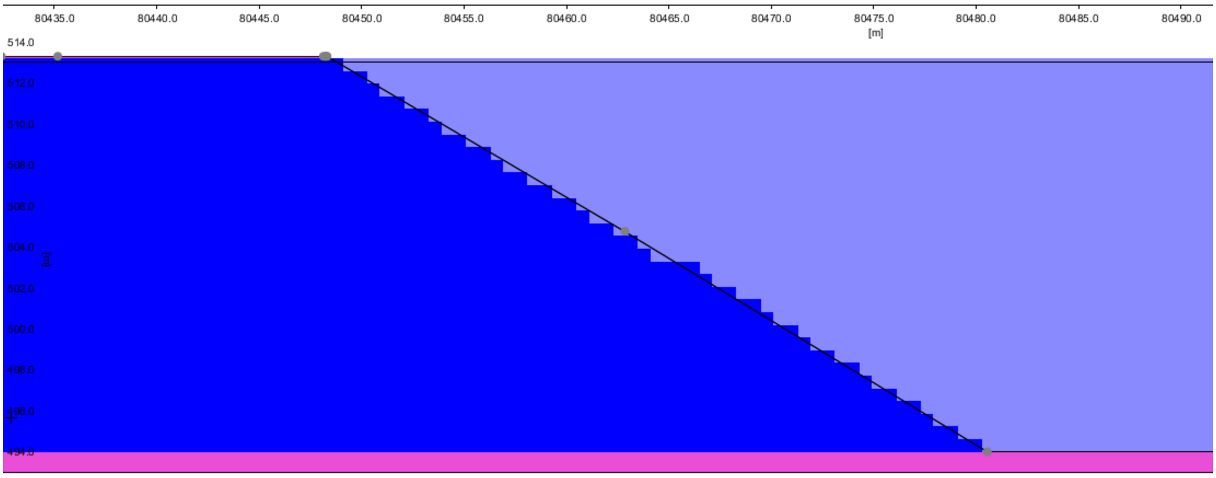

Figure 2.6: *Voxelized Border of the Lake (Screenshot from IGMAS)*

The exported voxel cube textfile from IGMAS (*IGMAS*, 1.2.102.1) contains the coordinates of the origin and the side lengths of the cube. With all these cubes it is possible to calculate the function for any water level. Table 2.2 shows the design of the voxel file dataset.

Table 2.2: *Part of the voxel file*

X-coordinate [m]	Y-coordinate [m]	Z-coordinate [m]	dx [m]	dy [m]	dz [m]
80626	11664	500.9	0.6	0.6	0.6
80627	11664	500.9	0.6	0.6	0.6
80628	11664	500.9	0.6	0.6	0.6
80628	11664	500.9	0.6	0.6	0.6
80629	11664	500.9	0.6	0.6	0.6
80629	11664	500.9	0.6	0.6	0.6
80630	11664	500.9	0.6	0.6	0.6
80631	11664	500.9	0.6	0.6	0.6
80631	11664	500.9	0.6	0.6	0.6
80632	11664	500.9	0.6	0.6	0.6
80632	11664	500.9	0.6	0.6	0.6
80633	11664	500.9	0.6	0.6	0.6
80634	11664	500.9	0.6	0.6	0.6
80634	11664	500.9	0.6	0.6	0.6
80635	11664	500.9	0.6	0.6	0.6
80635	11664	500.9	0.6	0.6	0.6
80636	11664	500.9	0.6	0.6	0.6
80637	11664	500.9	0.6	0.6	0.6

The voxels are saved in a text file from the type .vox, the files are about 2 GB big and more than 36 Million lines long. The expansions in x, y, z are constant 0.6 m.

Chapter 3

Approach to calculate the function

In this chapter the function depending on water level and temperature will be described. The first choice is which formula should be used to calculate the attraction in Z-direction.

3.1 Comparison of equations

There are different ways to calculate the attraction of a mass to a point in a 3D coordinate system.

1. Calculation with the point mass formula:

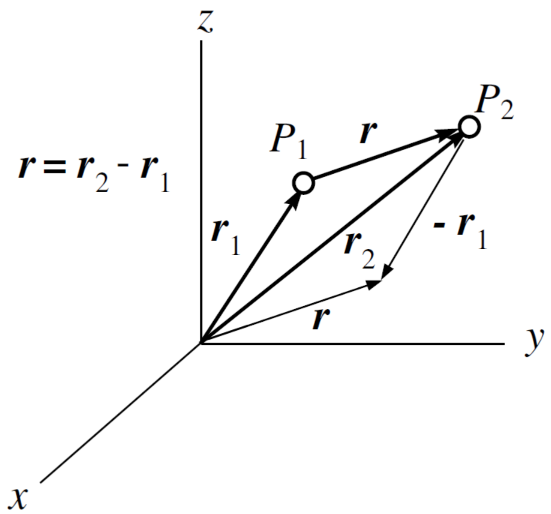

Figure 3.1: *Attraction of a point mass in $P2(x_2, y_2, z_2)$ to the point $P1(x_1, y_1, z_1)$, with the space between them described by $r = \sqrt{x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ (Sneeuw, 2006)*

The equation which describes what is shown in 3.1 is:

$$\mathbf{a} = -G \cdot \frac{m}{r^3} \cdot \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix} \tag{3.1}$$

As described earlier only the attraction in Z-direction is important for the gPhone gravimeter. The parts of the attraction vector in x and y can be ignored. Calculating the attraction in z-direction is possible with the following equation.

$$a_z = -G \cdot \frac{m \cdot (z_2 - z_1)}{r^3} \tag{3.2}$$

2. Calculation with the discrete right rectangular prism integral, (Tsoulis, 1999).

$$a_z = G \cdot \rho \, ||| \, x \cdot \ln(y + r) + y \cdot \ln(x + r) - z \cdot \arctan\left(\frac{xy}{zr}\right) \Big|_{X+\frac{dx}{2}}^{X-\frac{dx}{2}} \Big|_{Y+\frac{dy}{2}}^{Y-\frac{dy}{2}} \Big|_{Z+\frac{dz}{2}}^{Z-\frac{dz}{2}} \tag{3.3}$$

This integral is more complex, but also it is a better approximation.

To find out if the point mass approximation is sufficient to calculate the attraction of the lakes to the gravimeter, a relative precision is made.
For this the equations are used to calculate the attraction of both voxelized reservoirs. The same water level and temperature is used for both calculations.
For an example:

Table 3.1: *Attraction calculated by the different equations*

Equation	Attraction [μGal]	Water level [m]	Water temperature [°C]
(3.2)	637.26243245	19	4
(3.3)	637.31878509	19	4

Based on this example, the relative precision can be calculated.

$$T_D = \left(\frac{3.3 - 3.2}{3.3} - 1\right) \cdot 100 \tag{3.4}$$

The total relative precision is 99.991157 %, while the point mass formula is a rough approximation, but the equation of Tsoulis is a better one. Given this result it is important which of the equations is picked. To determine the Newtonian constant G with 4 or 5 digits, i.e. a relative accuracy of 10^{-5} is needed. The differences here are at the 4th digit as seen in the table 3.1.

3.2 Function calculation

The reservoirs are filled with 36 991 286 voxel cubes. Even the data file they are saved in, is around 2 GB big. To complete the function for any water level and water temperature much memory is needed. The aim now is to generate a polynomial fit for every water level and temperature the reservoirs ever present. First of all it is important to calculate the attraction for every water level and temperature.
Means that every voxel in dz defines a water level, as you can see in the graphic 3.2 below.

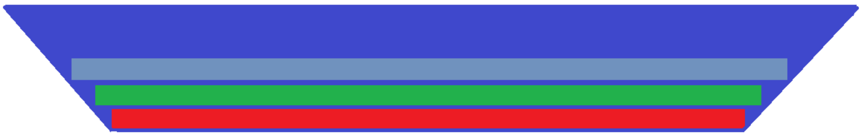

Figure 3.2: *Water level described by voxel layers*

Every layer is 0.6 m in height. The maximum water level is 19 m. As a conclusion there are 31 of these voxel layers.

The water temperature in the lakes varies between 1°C and 16°C. In table 3.2 you can see the relationship between temperature difference and modeled attraction.

Table 3.2: *Attraction for different temperatures*

Water level [m]	Water temperature [°C]	Attraction [μGal]
0.6	1.0	24.624
0.6	2.0	24.626
0.6	3.0	24.627
0.6	4.0	24.627
⋮	⋮	⋮
0.6	13.0	24.616
0.6	14.0	24.613
0.6	15.0	24.609
0.6	16.0	24.606

This has to be done for every voxel layer. As a result a function can be fitted to the measurements.
With the software MatLab (*MatLab*, 2017b), it is possible to perform a function fit. The aim of this fit is that, in fast ways it calculates for a water level and temperature the function value. The function that is fitted is a polynomial with 5 degrees in x and 5 degrees in y. In MatLab (*MatLab*, 2017b) there is a toolbox for polynomial fitting, named „Curve Fitting Tool ". The figure 3.3 shows the fitted function with water level (h) in x, water temperature (t) in y and the function values in z.

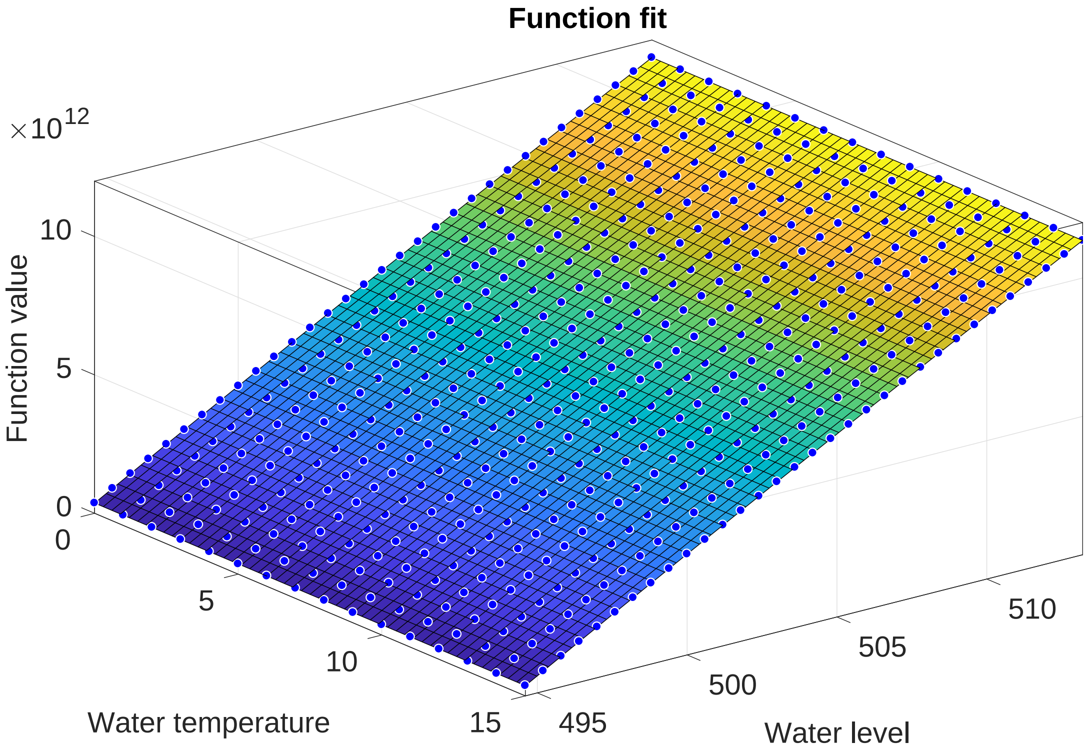

Figure 3.3: *Plot of the function used to calculate the gravitational constant*

The equation that generates the figure 4.2 is:

$$\begin{aligned} f(h,t) = {} & p_{00} + p_{10} \cdot h + p_{01} \cdot t + p_{20} \cdot h^2 + p_{11} \cdot h \cdot t \\ & + p_{02} \cdot t^2 + p_{30} \cdot h^3 + p_{21} \cdot h^2 \cdot t + p_{12} \cdot h \cdot t^2 \\ & + p_{03} \cdot t^3 + p_{40} \cdot h^4 + p_{31} \cdot h^3 \cdot t + p_{22} \cdot h^2 \cdot t^2 \\ & + p_{13} \cdot h \cdot t^3 + p_{04} \cdot t^4 + p_{50} \cdot h^5 + p_{41} \cdot h^4 \cdot t \\ & + p_{32} \cdot h^3 \cdot t^2 + p_{23} \cdot h^2 \cdot t^3 + p_{14} \cdot h \cdot t^4 \\ & + p_{05} \cdot t^5 \end{aligned} \tag{3.5}$$

A 5th order polynomial is used to calculate the function, because the variances with this fit are better than with any other fitting option. In the following table 3.3 one can take a look at the parameters.

Table 3.3: *Parameters for the function (3.4)*

Parameter	Value
p_{00}	$5.968 \cdot 10^{12}$
p_{10}	$3.275 \cdot 10^{12}$
p_{01}	$-1.467 \cdot 10^{9}$
p_{20}	$-3.558 \cdot 10^{10}$
p_{11}	$-8.062 \cdot 10^{8}$
p_{02}	$-9.134 \cdot 10^{8}$
p_{30}	$1.173 \cdot 10^{9}$
p_{21}	$8.748 \cdot 10^{6}$
p_{12}	$-5.013 \cdot 10^{8}$
p_{03}	$3.910 \cdot 10^{7}$
p_{40}	$-4.158 \cdot 10^{7}$
p_{31}	$-2.738 \cdot 10^{5}$
p_{22}	$5.497 \cdot 10^{6}$
p_{13}	$2.279 \cdot 10^{7}$
p_{04}	$-1.903 \cdot 10^{6}$
p_{50}	$4.384 \cdot 10^{5}$
p_{41}	$1.005 \cdot 10^{4}$
p_{32}	$-1.788 \cdot 10^{5}$
p_{23}	$-2.416 \cdot 10^{5}$
p_{14}	$-1.055 \cdot 10^{6}$
p_{05}	$7.324 \cdot 10^{5}$

To check how well the function fits the measured values, it makes sense to calculate the attraction for every measurement file. Figure 3.4 shows the fit between measured and function calculated values for the gravitational attraction.

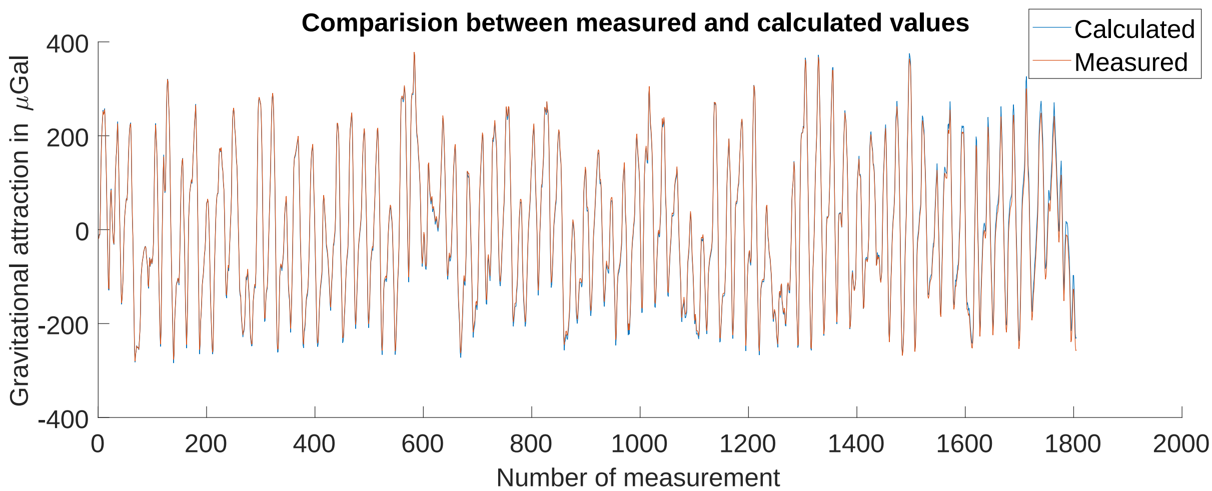

Figure 3.4: *Plot of the measured (red) and calculated (blue) gravitational attractions in* [μGal] *over time*

The Figure shows that the values at the beginning fit quite perfectly, but in the end the difference is getting bigger and bigger. The result gets obvious by a look at the residuals.

$$residuals = g_{\text{measured}} - g_{\text{calculated}} \tag{3.6}$$

The plot is even more revealing, as one can see in figure 3.5.

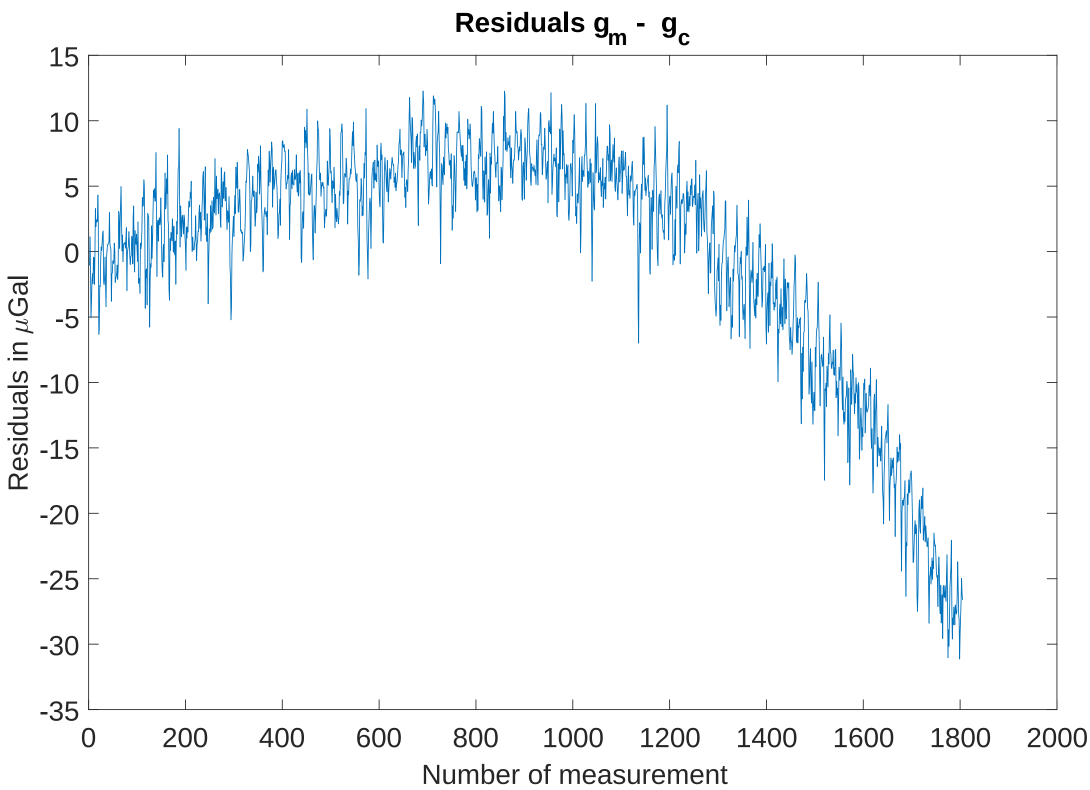

Figure 3.5: *Plot of the residuals between measured and calculated attraction values*

The drift of the gravimeter is visible in the residuals. Relative gravimeters drift. To get rid of the drift the gravimeters get calibrated by a measured value from an absolute gravimeter. The relative gravimeter is localized under two lakes, which makes the calibration complicated if not impossible. Ordinarily the drift gets erased by connecting the relative measurements with absolute gravimeter measurements. That practice of calibration is not possible under the Vianden reservoirs.

Chapter 4

Calculation of the gravitational constant

4.1 Theory

To calculate the gravitational constant the following equation is beeing used.

$$G = \frac{g(h,t)}{f(h,t)} \tag{4.1}$$

In order to calculate the gravitational constant, it is important that the calculated attractions should not have G involved. So as a conclusion it is necessary to calculate the function without today's known gravitational constant $G = 6.67408 \cdot 10^{-11} \frac{\mathrm{m}^3}{\mathrm{kg} \cdot \mathrm{s}^2}$.
As seen both values depend on water temperature and level, but only the measured values g contain the gravitational constant. To get more accurate results, it is important to calculate with a difference of gravitational measurements. This shields the calculation against outside effects.

$$G = \frac{g(h_0,t_0) - g(h,t)}{f(h_0,t_0) - f(h,t)} \tag{4.2}$$

With this equation it is easily possible to calculate the first values for the gravitational constants.

4.2 Calculations

The MatLab (*MatLab*, 2017b) code reads the measurement files, which were composed by T-Soft. Every variable like temperature, level and attraction gets saved in a separate array. For every level and temperature in the measurement file, the function value is calculated. As a result an array with the values for the gravitational constant G is generated. In the figure 4.1 all of the G values are plotted.

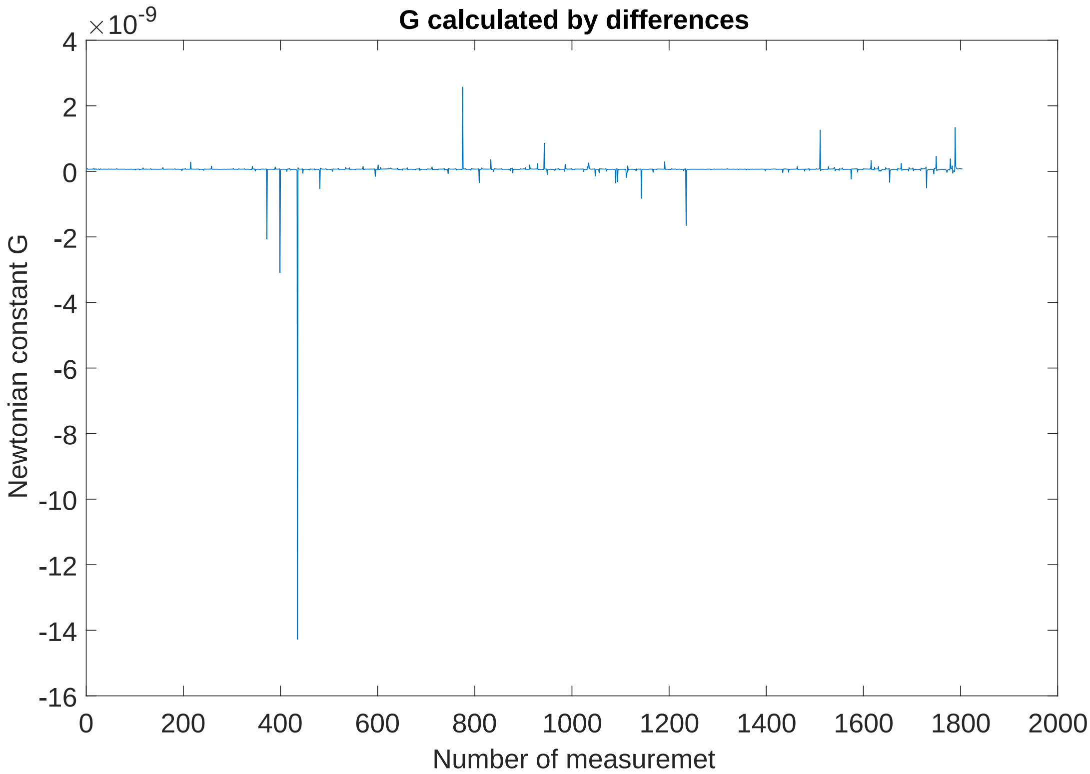

Figure 4.1: *Plot of the G values in* $\left[\frac{\mathrm{m}^3}{\mathrm{kg}\cdot\mathrm{s}^2}\right]$ *over time*

The figure also shows several outliers. To get rid of them in an easy way, two limits are added. So that the remaining values are all between $6.4 \cdot 10^{-11}$ and $7.0 \cdot 10^{-11}$. To have a look on the signal in a neat visual way, a smoothing spline is applied. This smoothing spline filters the values of G. MatLab (*MatLab*, 2017b) offers a function for this, the smoothing parameter is $1 \cdot 10^{-6}$. The result is the following figure 4.2.

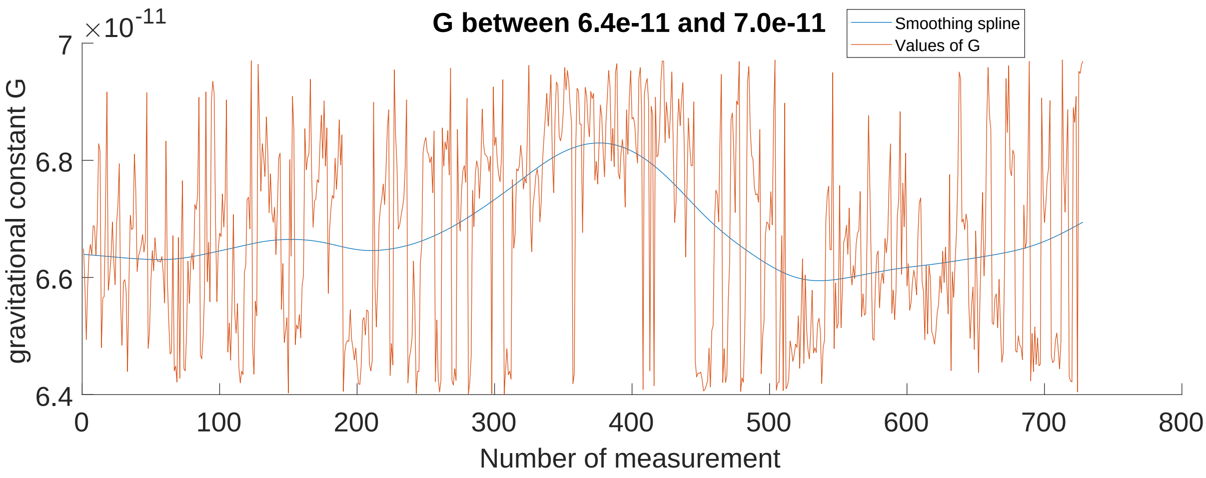

Figure 4.2: *Plot of the G values in between* $6.4 \cdot 10^{-11}$ *and* $7.0 \cdot 10^{-11}$ *in red and the smoothing spline in blue*

To extract the constant the arithmetic mean of the original measurements (with removed outliers) is used.

$$G_{\text{mean}} = \frac{\sum_1^n G}{n} = 6.67\,466 \cdot 10^{-11} \frac{\text{m}^3}{\text{kg} \cdot \text{s}^2} \tag{4.3}$$

To find out in what accuracy area the calculated value for G is, it is recommended to take a look at the standard deviation. The standard deviation is calculated by the following procedure.

$$\sigma = \sqrt{\frac{\sum_1^n (G_\text{n} - G_{\text{mean}})^2}{n-1}} = 6.57 \cdot 10^{-13} \frac{\text{m}^3}{\text{kg} \cdot \text{s}^2} \tag{4.4}$$

To get the relative uncertainty it is necessary to divide the standard deviation σ through the mean value G_{mean}.

$$\text{Uncertainty} = \frac{\sigma}{G_{\text{mean}}} = 0.0098 \tag{4.5}$$

This means that the gravitational constant calculated is wrong by 0.98%. As notation this gives the following value for the Newtonian gravitational constant.

$$G = 6.6747 \pm 0.0657 \cdot 10^{-11} \frac{\text{m}^3}{\text{kg} \cdot \text{s}^2} \tag{4.6}$$

Chapter 5

Error influences

Many effects influence the quality of the fitted function as well of the calculated gravitational constant G. In the following pages the influences are explained and calculated.

5.1 Voxel resolution

The voxels are created with the support of the software IGMAS. The software just allows 40 000 000 voxels, which limits the resolution on a maximum of $dx, dy, dz = 0.6$ m.
To find out what the influence is between a lower and the highest resolution possible. Both voxel files are used to calculate the function. Then the calculated functions are compared in a plot. The figure 5.1 shows this plot.

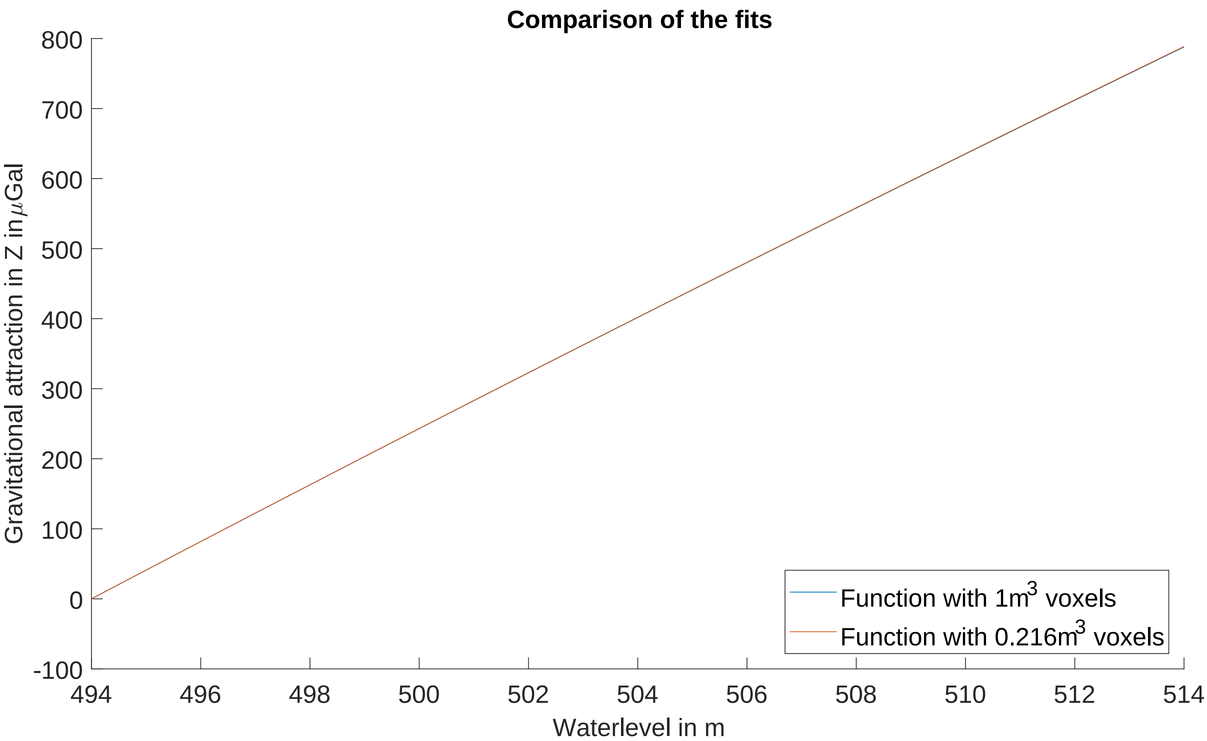

Figure 5.1: *Plot of the gravitational attractions* $[\mu\text{Gal}]$ *depending on the water level* $[\text{m}]$ *calculated by different voxel resolutions concrete with* $dx \cdot dy \cdot dz = 1\text{m}^3$ *(blue) and* $dx \cdot dy \cdot dz = 0.216\text{m}^3$ *(red)*

This plot is 2D because there are no temperature changes, this was applied for code runtime improvement.
Figure 5.1 does not contain that much information, because it is not really possible to see any differences between the resolutions. Thus it is necessary to have a look at the residuals, which you can see in figure 5.2.

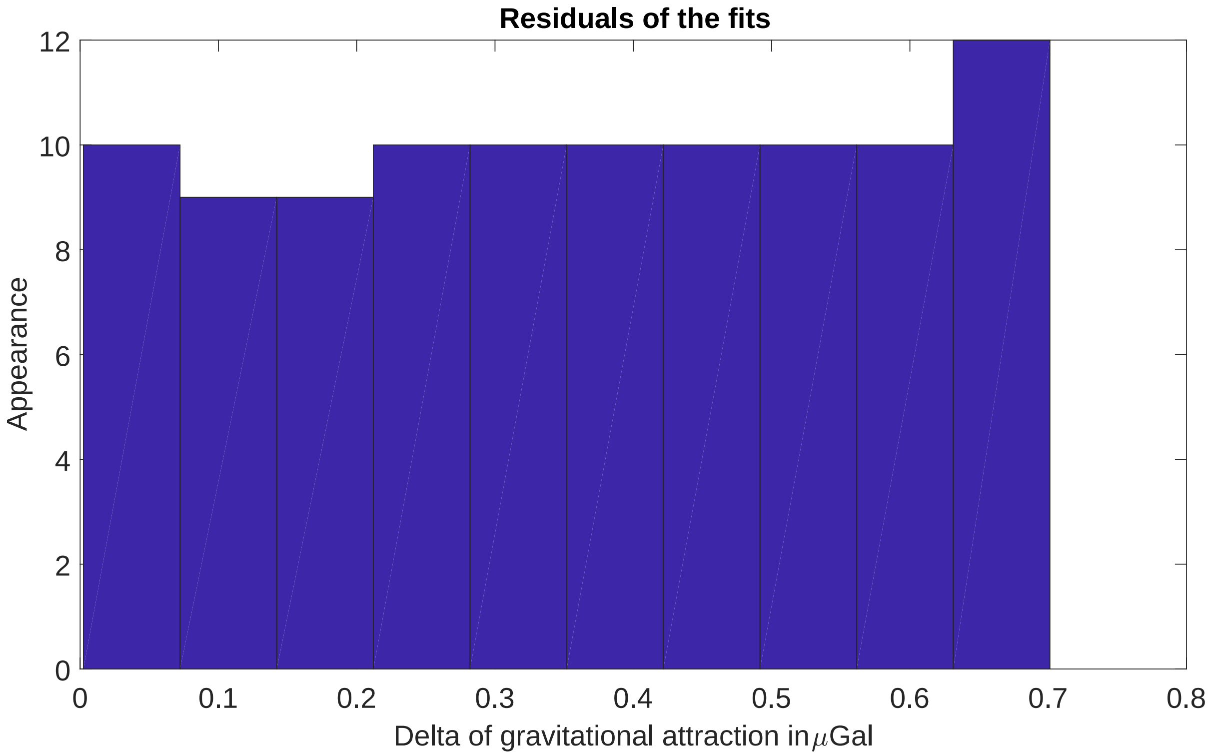

Figure 5.2: *Residuals of the two functions calculated by different voxel resolutions, in Y is the Appearance and in X is the Error of the gravitational attraction in* [μGal]

The figure 5.2 shows that the maximum difference between the two voxel resolution functions is 0.7 μGal. This is fine, because without a big amount of extra work, it is not possible to raise the resolution above 0.6 m.

5.2 Gravimeter position

The next influence that is investigated is the accuracy of the measured gravimeter position. Thus an error is added to the position in X- and Y-direction. The reason that there is no error that is added in Z-direction, is because of the relativity between Z-coordinate and water level. In the relative gravimeter measurements in Vianden the effect of an increase in water level or the elevation of the gravimeter is the same. The water level effect will be evaluated in the next section. Beginning with the gravimeter position, the errors will be increased in every iteration. You can see the procedure in the following table 5.1.

Table 5.1: *Procedure of adding errors to the right gravimeter position*

Iteration	X-coordinate [m]	Y-coordinate [m]	error in X [m]	error in Y [m]
0	80691.1290	11555.1020	0	0
1	80691.1290	11555.1020	0.1	0.1
⋮	⋮	⋮	⋮	⋮
50	80691.1290	11555.1020	5	5

With every wrong position the function $F(h,t)$ will be calculated. The residuals, between the calculated attraction value and the measured value, will give an indication of the influence of a wrongly measured gravimeter position.
To get to know how big the average influence of a wrong gravimeter position by 1 m is, a differential calculation is made.

$$D = \frac{0.1172\ \mu\text{Gal} - 0.1048\ \mu\text{Gal}}{5\ \text{m}} = 2.48 \cdot 10^{-3}\ \frac{\mu\text{Gal}}{\text{m}} \tag{5.1}$$

The error of an inexactitude of 1 m in the measured gravimeter position (in x and y) is $2.48 \cdot 10^{-3} \mu$Gal.

5.3 Temperature measurements

To find out how big the influence of a wrongly measured temperature is, the same procedure as the one used for the gravimeter position is applied. The table 5.2 shows precisely which measurement errors were used.

Table 5.2: *Procedure of adding errors to the right temperature measurement*

Iteration	Temperature T [°C]	error in T [°C]
0	0	0
1	0	0.01
⋮	⋮	⋮
50	0	10.0

As already done in the gravimeter position calculations, for every temperature the Function $F(h, t)$ will be calculated. To extract the influence of the temperature a constant water level of 500 m is used. Thus the calculated function is $F(500\text{ m}, t)$. Figure 5.3 shows the influence of the water temperature on the attraction while the water level stays constant.

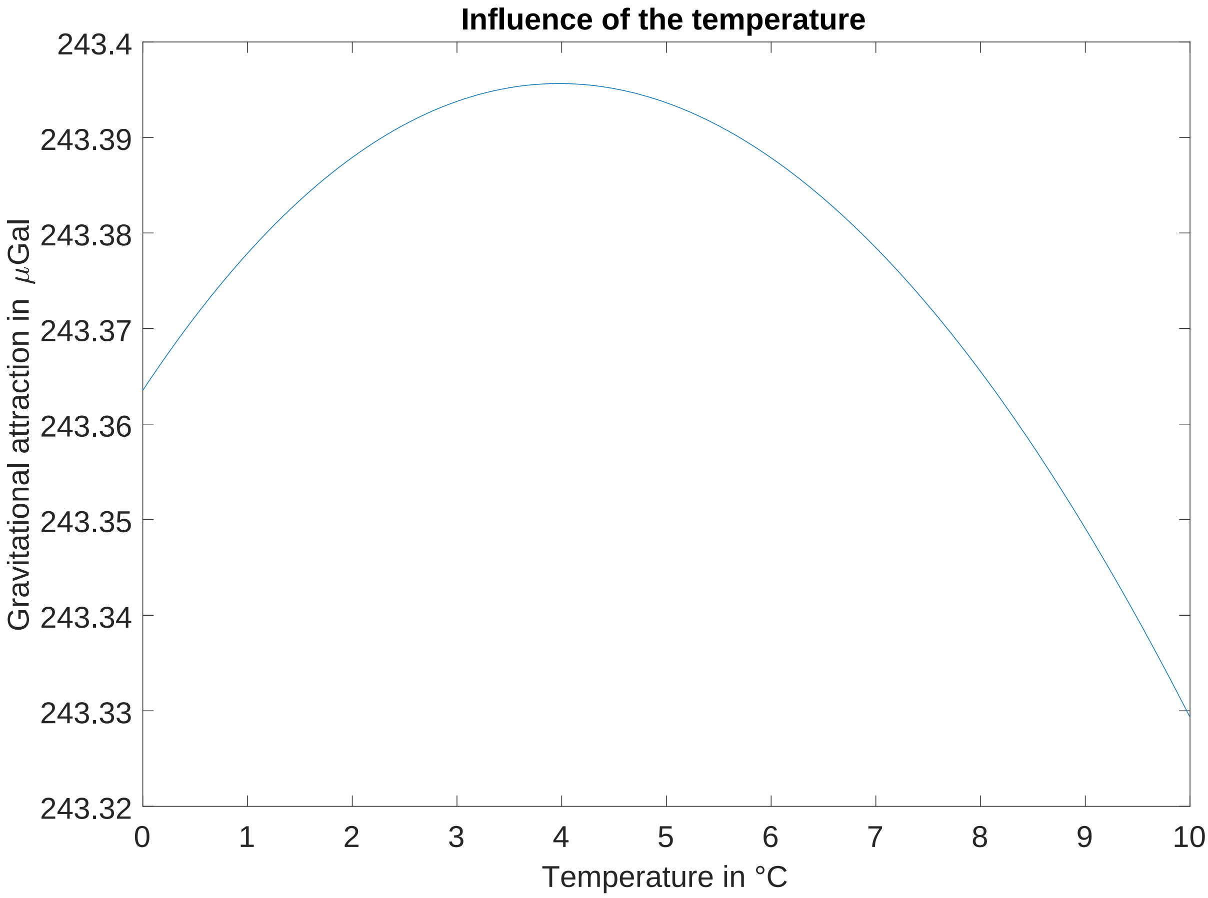

Figure 5.3: *Influence of the temperature on the attraction, by a constant water level*

To find out how big the average influence of a wrong measured temperature by 1 °C is, a difference calculation is made.

$$D = \frac{243.4\ \mu\text{Gal} - 243.33\ \mu\text{Gal}}{10\ ^\circ\text{C}} = 7 \cdot 10^{-3}\ \frac{\mu\text{Gal}}{^\circ\text{C}} \tag{5.2}$$

The figure 5.3 also shows the anomaly of the water. The density of water at 4 °C is the highest, so the biggest influence in wrong temperature measurements is $7 \cdot 10^{-3}\ \frac{\mu\text{Gal}}{^\circ\text{C}}$.

5.4 Level measurements

The same procedure as last time is used, which means that an error is added on the correct water level. In table 5.3 you can see the added error and the initial water level.

Table 5.3: *Procedure of adding errors to the right water level measurement*

Iteration	Water level h [m]	error in h [m]
0	495	0
1	495	0.001
$\vdots$	$\vdots$	$\vdots$
1000	495	1

With the wrong measurements the function $F(L, 10\,°C)$ is calculated and plotted in a figure 5.4 which you can see below.

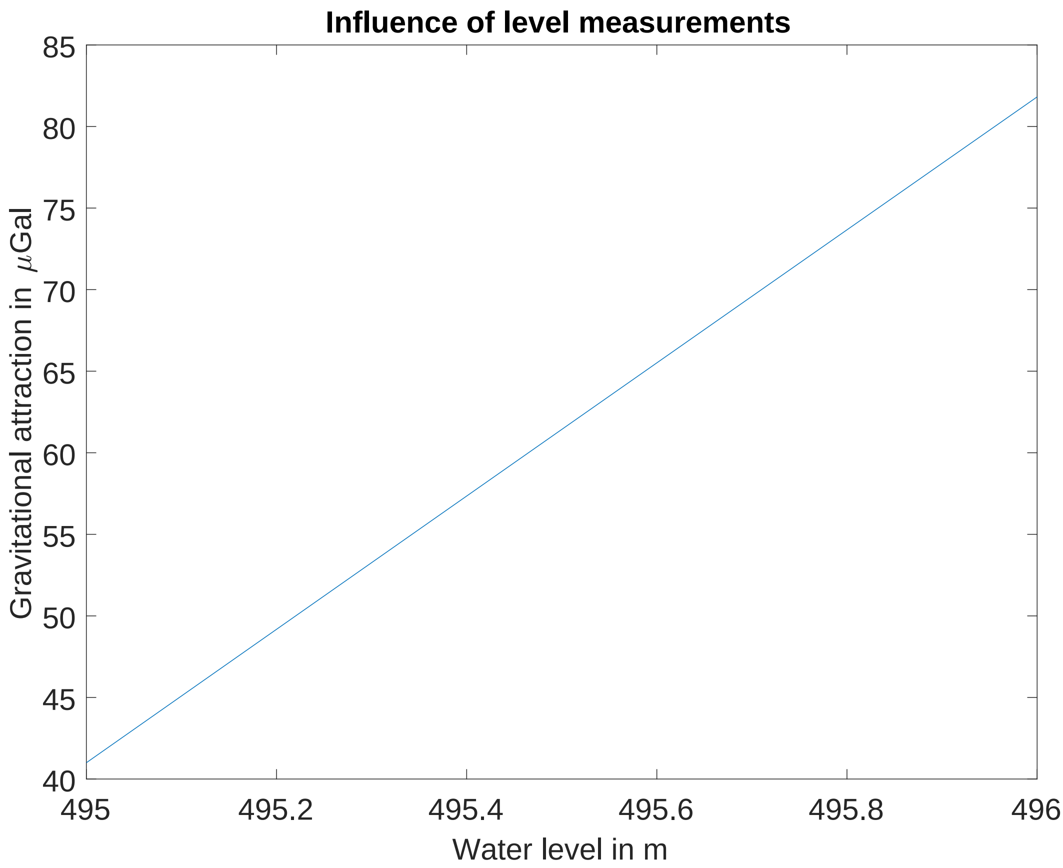

Figure 5.4: *Influence of a wrong observed water level [m] on the attraction [μGal], while the water temperature is constant*

For clarification about how big the influence of a wrong measured water level is, the average difference is calculated.

$$D = \frac{82\ \mu\text{Gal} - 41\ \mu\text{Gal}}{1\ \text{m}} = 41\ \frac{\mu\text{Gal}}{\text{m}} \tag{5.3}$$

That means the influence of a wrong measured water level is extremely high, in numbers $41\,\frac{\mu\text{Gal}}{\text{m}}$.

5.5 Error influences overview

You can see the error influences altogether in the table 5.4.

Table 5.4: *Rated overview of the single error influences*

Error	Influence	Rating
Resolution	maximal 0.7 μGal	$\surd$
Gravimeter position	$2.48 \cdot 10^{-3}\,\frac{\mu\text{Gal}}{\text{m}}$	$\surd$
Temperature	$7 \cdot 10^{-3}\,\frac{\mu\text{Gal}}{^\circ\text{C}}$	$\surd$
Level	$41\,\frac{\mu\text{Gal}}{\text{m}}$	$\times$

For a closer look every of the influence will be shortly discussed.

- Resolution
  The resolution influence is with 0.7 μGal, indeed not that good. But with the software of the University of Kiel it is the best resolution possible.
- Gravimeter position
  The influence of the gravimeter position is good. Even with the idea in mind that it is easy to check the position in X-and Y-direction, the influences stays good.
- Temperature
  The influence of the temperature is very low, which is good, because until now the temperature anomalies in the lake are not observed.
- Water level
  The biggest influence is the influence in Z-direction, either in the water level measurements or the Z-coordinate of the gravimeter. A high accurate leveling has been done, but the measurements are not included in the calculations until now.

Chapter 6

Proposal

In the sixth chapter the proposals for further work on this experiment will be discussed. To give an overview in every section one of the influences will be analysed.

6.1 Contours of the reservoirs

The contour files of the two artificial reservoirs in Vianden were measured in 1995. That brings a lot of challenges with it, because the dams were raised, turrets were added and the contour it self may have changed. As already known the influence of the resolution is with 0.7 μGal quite big. But without getting new measurements and data from the lakes this is the best solution possible. It is also limited by the software that provides the voxels, but still the contours need to be re-examined.
The more accurate the better is the solution, but as a reference $\sim$ 1–2 cm shall be enough. As the used voxels have a resolution of 0.6 m the smallest detectable change in the calculations is 0.6 m. With other solutions it is surely possible to detect changes between $\sim$ 1–2 cm. Looking forward the accuracy is enough to create good fitting voxels.
One idea to create voxels from the more accurate measurements is to compute a water tight .obj file in MatLab (*MatLab*, 2017b) and voxelize it with one of the many codes from MatLab (*MatLab*, 2017b) „Fileexchange“. With this workflow it should be possible to create more accurate models.

6.2 Gravimeter position

The gravimeter position is not even known, which means that the X- and Y-coordinate never were measured. That should be done. Even though the influence of a wrong gravimeter position is quite small, with $2.48 \cdot 10^{-3}\ \frac{\mu\text{Gal}}{\text{m}}$ the influence is still recognizable.

6.3 Influence of the river Our

The river Our is located in the east of the reservoirs. The river changes the water level very quickly in the area of influence. That is caused by dams that stow the water. It is very complex to model the river in 3D and even more complex to voxelize it. In the thesis of Dr.-Ing. Sperling the influence of the river Our is calculated with 3 μGal, (Sperling, 1994). The influence for the current gravimeter position is smaller, but still significant 1.2 μGal. The old contours intersect the idea of modelling the river, that is why they do not find any reference in the calculations made in this thesis.

6.4 Position and height of the towers

The height, position and thickness of the turrets is not in the calculation, until now. The challenges with the towers is that the measurement files from 1995 are corrupted. They can not be correct. Every tower represses some water in the reservoirs, which causes trouble. The hitch in fact is, that the water level changes won't change the attraction of the tower. There out follows that the towers have to be modelled, so the missing attraction of the whole basin can be calculated with the model of the turrets. The accuracy should be set to $\sim$ 1–2 cm, because of the same reason like in the contours. The $\sim$ 1–2 cm should be fine to be detected by voxelized models.

6.5 Level measurements

That is the hardest part, because of the 41 $\frac{\mu\text{Gal}}{\text{m}}$ error influence. The fact that the effect of wrong measured water level and wrong measured Z-coordinate is the same, caused by relative gravimeter measurements exposes the background of this big influence. The level measurements have to be very accurate, in numbers $\sim$ 1–2 mm.
The second part of the big influence is the gravimeter Z-coordinate. In April 2017 a high accuracy levelling was done, which gives the opportunity to clear out one part of this error influence. The level measurements still hold some difficulty, which is the measurement method of the water level. To get the best results the Z-coordinate and the water level shall be in the same coordinate system, synchronized and very accurate.

6.6 Temperature measurements

The temperature is the smallest influence. But still in a reservoir with two big basins there are temperature anomalies. They are not measured until now. Currently at just one point inside the reservoir the temperature is measured.
That should be increased, with an accuracy of $\sim$ 0.01–0.1°C.

6.7 Load effect

The load effect is an error influence that is not modeled until now. As more load (water) is in the test area the whole area gets lowered. The result is that the project measures a wrong water level. It is therefor the biggest error influence. In figure 6.1 you can see an idea of measuring the load effect.

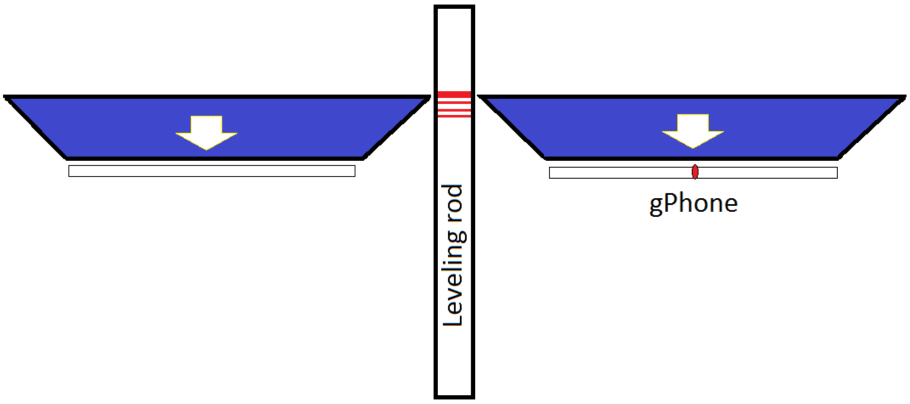

Figure 6.1: *Sketch of the load effect measurement idea*

As you see there is a long measurement rod, this rod is solid rooted in the soil under the basins. At time stamp t_0 a reference value for the water level will be generated. Every time a water level is measured the value from the rod will be added. It therefor gives the calculation a water level independence from the load effect. The measurement of the rod shall be very accurate, because the effect of the Z-direction is 41 $\frac{\mu\text{Gal}}{\text{m}}$. If the accuracy is about $\sim$ 1–2 mm the influence still is 0.041 μGal.

6.8 Drift of the gravimeter

The drift of relative gravimeters is a big challenge, ordinarily the drift gets erased by connecting the relative measurements with a absolute gravimeter measurements. That practice of calibration is not possible under the Vianden reservoirs. To get rid of the drift, a reference value can be generated.
In greater detail at a fix water level and temperature the attraction value will be listed. So for an example, take a look at table 6.1.

Table 6.1: *Example of a reference value*

Water Level [m]	Water Temperature [°C]	Measured Attraction by the gPhone [μGal]
500	10	243.3322

Every time the reservoirs reaches the same values as the listed reference value, the value that the gravimeter measures will be set on the value that it should measure, in this example 243.3322 μGal. That should help to remove the drift or at least minimize it.

Chapter 7

Conclusion

In this chapter a discussion about the lab and the work there will be made.

The lab in Vianden is impressive and with some minor changes in the measuring concept, it is possible to calculate a precise gravitational constant G. These mentioned changes were discussed in the Proposal Chapter. Vianden and the connected experiment is the first idea of calculating G by a relative gravimeter under artificial basins. The result is, in respect of the limited time, the drift and the load effect, amazing. Surely with more time, another software for the voxels and a few more measurements it will be possible to find a more accurate gravitational constant. The project is stunning and hopefully it is moving forward by this thesis.

Acknowledgements

Last but surely not least this thesis is the expression of many terrific people. Starting with Prof. Dr.-Ing. Nico Sneeuw, my supervisor for the thesis and the man who made Luxembourg possible for me. In Luxembourg Prof. Dr. Olivier Francis taught me everything about the project and a lot more about gravitation and geophysics. He gave me the opportunity to work with a great international team, not to say the best. Every one of them helped me a lot with my thesis. The last people I want to mention is my family, Jürgen, Andrea, Lukas and Svenja, without you guys all of this would not be possible. **Thank you**.

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