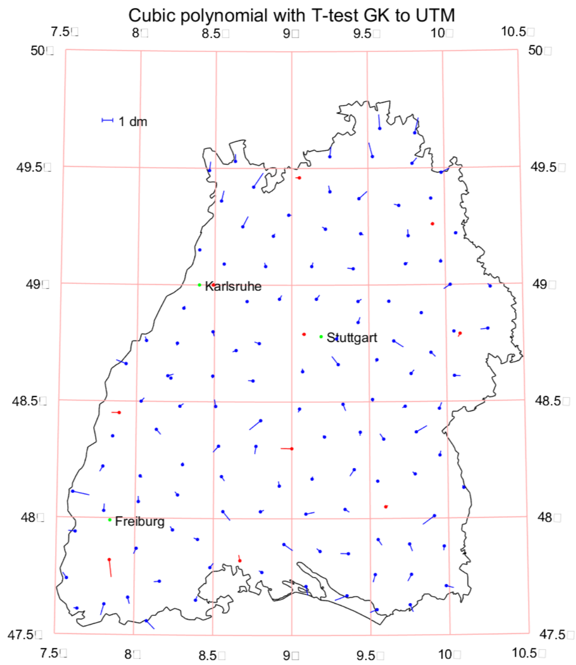
Universität Stuttgart

Geodätisches Institut

Analysis of coordinate transformation with different polynomial models



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Abstract

The main task of geodesy is providing geodetic networks with fixed points in order to create a uniform geographical spatial reference frame as a fundament for the data collection by the official geodesy survey institutes. A german geodesy survey institute called AdV (Arbeitsgemeinschaft der Vermessungsverwaltungen der Länder der Bundesrepublik Deutschland) declared in 1991 that the ETRS89 datum should be introduced in Germany as a reference system.

In order to transform the already exiting coordinate informations in the Gauß-Krüger coordinate system into the later introduced UTM coordinate system, different transformation models have been developed and discussed. Besides the most commonly used 7-parameter Helmert transformation and 6-parameter affine transformation models, polynomial transformation models can also be applied. A method for improving the transformation results of a polynomial model will be discussed, with which a significance test (T-test) for each parameter will be done and the polynomial terms with lower significance to the model will be eliminated in order to get the optimal polynomial model.

Here different transformation models are reviewed and the transformation results based on these models with the Least Squares estimation method are compared and analysed.

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Chapter 1

Introduction

The Gauß-Krüger coordinate system is a Cartesian coordinate cystem, which is named after Carl Friedrich Gauss and Johann Heinrich Louis Krüger. After publication the Gauß-Krüger coordinate system has became a commonly used reference system in Germany.

The UTM coordinate cystem was developed in 1947 by the Armed Forces of the United States(Markus Penzkofer 2017). It was originally used by the American military, but with the development and popularization of the UTM coordinate cystem, it was finally also introduced into Germany.

The Gauß-Krüger coordinate system is a transverse Mecator map projection. The cylinder is longitudinal along a certain meridian. Because of such a map projection the Gauß-Krüger coordinate system is known as a conformal projection that does not remain true directions. The reference ellipsoid of the Gauß-Krüger coordinate system is the Bessel ellipsoid (1841). In this system the earth is divided into 120 meridional zones. Each meridional zone is 3 degrees wide and reaches from the North Pole to the South Pole parallel to its central meridian. The origin of the Gauß-Krüger coordinate system is the intersection of the central meridian and the equator. The x-coordinate is called Hochwert H, which counts positive from the origin parallel to the north direction of the central meridian. And the y-coordinate is called Rechtswert R, which is the modified value from the origin to east along the equator. Germany uses the meridional zones with the central meridians $6°, 9°, 12°, 15°$.

The UTM coordinate system is an universal transversal Mercator projection based on the ETRS89 datum. It divides the earth into 60 zones, each spanning of lontitude and having its own central meridian. The origin for each zone is the intersection of its central meridian and the equator. To eliminate negative coordinates, the coordinate system alters the coordinate values at the origin. The value given to the central meridian is the false easting, and the value assigned to the equator is the false northing. A false easting of 500,000 meters is applied. A north zone has a false northing of zero, while a south zone has a false northing of 10,000,000 meters.

In oder to introduce ETRS89 into Baden-Würtemberg the transformation from the Gauß-Krüger coordinates in German geodetic reference system into UTM coordinates in the ETRS89 is necessary.

For the coordinate transformation different transformation models have been widely developed and discussed. Under the circumstance when the 6-parameter affine transformation and 7-parameter helmerttransformation can not achieve the required precision, the polynomial models are usually applied. In oder to find the best polynomial combination, a significance test of the transformation parameter is needed. For the estimation of the transformation parameters the Least Squares method and Total Least-Squares method are usually used. There are also other estimation methods, but here the Least Squares method will be applied as the estimation method of the transformation parameters.

131 collocated points in both coordinate system (Gauß-Krüger Coordinate System and UTM Coordinate System) participate in the transformation, among which 121 points are the collocated points and 10 points are the interpolated points.

The transformation results of the 6-parameter affine transformation, 7-parameter helmerttransformation, different polynomial models and the selected polynomial models with significance test (T-test) will be analysed and compared. With the analysis the practicability and the reliability of the polynomial models will be checked.

Chapter 2

Transformation Models

2.1 2-D Transformation Models

2.1.1 6-Parameter Affine Transformation

Two-dimensional transformation models are directly based on the map coordinates, which are resulted from the projection of the reference ellipsoid into plane. Here the coordinate transformation from Gauß-Krüger coordinate system in DHDN into UTM coordinate system in the ETRS89 datum in Baden-Würtemberg is discussed. The coordinates in both systems are georeferenced plane coordinates and quite different from the non-georeferenced plane coordinates such as Cartesian coordinates and polar coordinates, which can be easily transformed from one coordinate system to the other with direct mathematical relationships, which has the consequence that the Gauß-Krüger coordinates of the net points of the DHDN can be transformed only over collocated points into UTM coordinates related to ETRS89. The most commonly used two-dimensional transformation models are the 4-parameter similarity transformation and 6-parameter affine transformation models. Here the 6-parameter affine transformation model will be used for the plane coordinate transformation based on the 121 collocated points in Baden-Würtemberg.

With the 6-parameter affine transformation, both coordinate axis are not only rotated with two different angles α and β, but also scaled with two different scale corrections m_H and m_R. Besides, the origin also changes its position with two translation parameters in both coordinate directions. The 6-parameter affine transformation results in a change of the original angles or lengths. However, it preserves collinearity and ratios of distances.

The 6-parameter affine transformation model from Gauß-Krüger coordinates to UTM coordinates can be written as

$$\begin{bmatrix} N \\ E \end{bmatrix} = \begin{bmatrix} m_H cos\alpha & -m_R sin\beta \\ m_H sin\alpha & m_R cos\beta \end{bmatrix} \begin{bmatrix} H \\ R \end{bmatrix} + \begin{bmatrix} t_N \\ t_E \end{bmatrix} \tag{2.1}$$

with:
t_N and t_E: translation parameters

α and β: rotation angles

m_H and m_R: scale corrections

The 6-parameter affine transformation model can be simplified as

$$\begin{bmatrix} N \\ E \end{bmatrix} = \begin{bmatrix} a & b \\ g & f \end{bmatrix} \begin{bmatrix} H \\ R \end{bmatrix} + \begin{bmatrix} t_N \\ t_E \end{bmatrix} \tag{2.2}$$

with:
$a = m_H cos\alpha$

$b = -m_R sin\beta$

$g = m_H sin\alpha$

$f = m_R cos\beta$

2 translation parameters: t_N and t_E

2 rotation parameters: $\alpha = arctan(\frac{g}{a})$ and $\beta = arctan(\frac{b}{f})$

2 scale corrections: $m_H = \sqrt{a^2 + g^2}$ and $m_R = \sqrt{b^2 + f^2}$

When $a \cdot f - b \cdot g \neq 0$ and there are at least 3 collocated points, the 6 transformation parameters t_N, t_E, α, β, m_H, m_H will have unique solution.

The final linearized model can be written as

$$\begin{bmatrix} E \\ N \end{bmatrix} = \begin{bmatrix} 1 & 0 & R & H & 0 & 0 \\ 0 & 1 & 0 & 0 & R & H \end{bmatrix} \begin{bmatrix} t_E \\ t_N \\ f \\ g \\ b \\ a \end{bmatrix} \tag{2.3}$$

All the collocated points should be located in the same meridional zone of the Gauß-Krüger coordinate system and in the same zone of the UTM coordinate system, otherwise a transformation between different zones is also necessary.

2.1.2 2D Multiple Linear Regression

Compared with the 4-parameter similarity transformation and 6-parameter affine transformation models, whose mainly task is to determine the physical relationship between different datums, the coordinate transformation model with multiple linear regression in 2D coordinate transformation is to find a series of best-fit equations, which provides the local shifts in latitude and longitude as a function of position. While the equations might be physically meaningless, it may extremely valuable for the coordinate transformation.

The main advantages of the multiple regression equation method over 6-parameter affine transformation model is that a better fit over continental size land areas can be achieved.The main disadvantage of this method is that the results outside area of the collocated points can be extremely unreliable (Cai, 2009). As a consequence , the collocated points should cover the boundaries of the area in which the transformation is to process.

The two-dimensional multiple linear regression model can be defined as

$$\begin{aligned} \Delta\phi &= A_0 + A_1U + A_2V + A_3U^2 + A_4UV + A_5V^2 + ... + A_{99}U^9V^9 \\ \Delta\lambda &= B_0 + B_1U + B_2V + B_3U^2 + B_4UV + B_5V^2 + ... + B_{99}U^9V^9 \end{aligned} \tag{2.4}$$

where
A_0, B_0 = constant

A_0,B_0, A_1,B_1,..., A_{nn},B_{nn} = coefficients determined in the development

U = k(ϕ - ϕ_m) = normalized geodetic latitude of the computation point

V = k(λ - λ_m) = normalized geodetic longitude of the computation point

k = scale factor, and degree-to-radian conversion

ϕ, λ = local geodetic latitude and local geodetic longitude(in degrees), respectively, of the computation point

ϕ_m, λ_m = mid-latitude and mid-longitude values, respectively, of the local geodetic datum area(in degrees)

Conversion of the equation(2.4) in Gauß-Markov model l = Ax:

$$\begin{bmatrix} \Delta\phi_{G1} \\ \Delta\phi_{G2} \\ \vdots \\ \Delta\phi_{G(n-1)} \\ \Delta\phi_{Gn} \end{bmatrix} = \begin{bmatrix} 1 & U_{L1} & V_{L1} & U_{L1}^2 & U_{L1}V_{L1} & \cdots & U_{L1}^9V_{L1}^9 \\ 1 & U_{L2} & V_{L2} & U_{L2}^2 & U_{L2}V_{L2} & \cdots & U_{L2}^9V_{L2}^9 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & U_{L(n-1)} & V_{L(n-1)} & U_{L(n-1)}^2 & U_{L(n-1)}V_{L(n-1)} & \cdots & U_{L(n-1)}^9V_{L(n-1)}^9 \\ 1 & U_{Ln} & V_{Ln} & U_{Ln}^2 & U_{Ln}V_{L1} & \cdots & U_{Ln}^9V_{Ln}^9 \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ \vdots \\ A_{99} \end{bmatrix} \tag{2.5}$$

$$\begin{bmatrix} \Delta\lambda_{G1} \\ \Delta\lambda_{G2} \\ \vdots \\ \Delta\lambda_{G(n-1)} \\ \Delta\lambda_{Gn} \end{bmatrix} = \begin{bmatrix} 1 & U_{L1} & V_{L1} & U_{L1}^2 & U_{L1}V_{L1} & \cdots & U_{L1}^9V_{L1}^9 \\ 1 & U_{L2} & V_{L2} & U_{L2}^2 & U_{L2}V_{L2} & \cdots & U_{L2}^9V_{L2}^9 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & U_{L(n-1)} & V_{L(n-1)} & U_{L(n-1)}^2 & U_{L(n-1)}V_{L(n-1)} & \cdots & U_{L(n-1)}^9V_{L(n-1)}^9 \\ 1 & U_{Ln} & V_{Ln} & U_{Ln}^2 & U_{Ln}V_{L1} & \cdots & U_{Ln}^9V_{Ln}^9 \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ \vdots \\ B_{99} \end{bmatrix} \tag{2.6}$$

The polynomial parameters will be estimated with Least Squares Method.

2.2 3-D Transformation Models

2.2.1 7-Parameter- Helmerttransformation

When the collocated points are based in a 3-D Cartesian coordinate system, a three-dimensional transformation model is then required instead of a 2D 6-parameter affine transformation model. In most applications of three-dimensional transformation seven parameter similarity transformation model, which is also usually named as 7-parameter Helmert transformation model, is suggested to be applied. Compared with 8-parameter Vanicek-Well model, 9-parameter Hotine model and 10-parameter Krakiwsky-Thomson model, the 7-parameter Helmert transformation model has more advantages. It allows a direct physical interpretation of the origin shifts and performs a conformal transformation, where the ratios of distances and the angles preserve invariantly (Cai, 2009).

The 7-parameter Helmert transformation allows three-dimensional coordinates to be transformed from one geodetic datum to another on the basis of transformation parameters estimated from at least three common points. The common points should be selected in such a way that they homogeneously cover the entire region which is to be transformed. The number, distribution and coordinate quality of the common points determine the achievable accuracy of the transformation.

With the 7-parameter Helmert transformation model, the orthogonality condition of three coordinate axes remain unchanged. Threee axes rotate individually with three angles α, β, γ. The origin of the coordinate system has a translation T_X, T_Y, T_Z in three axes direction. Besides, three axes scale with the same scale correction λ.

In a conclusion, the 7-parameter Helmert transformation model between two Cartesian system can be written as

$$\begin{bmatrix} X_G \\ Y_G \\ Z_G \end{bmatrix} = \lambda \boldsymbol{R_3}(\gamma)\boldsymbol{R_2}(\beta)\boldsymbol{R_1}(\alpha) \begin{bmatrix} X_L \\ Y_L \\ Z_L \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix} \tag{2.7}$$

with

$$\boldsymbol{R_1}(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\alpha & sin\alpha \\ 0 & -sin\alpha & cos\alpha \end{bmatrix}$$

$$\boldsymbol{R_2}(\beta) = \begin{bmatrix} cos\beta & 0 & -sin\beta \\ 0 & 1 & 0 \\ sin\beta & 0 & cos\beta \end{bmatrix}$$

$$\boldsymbol{R_3}(\gamma) = \begin{bmatrix} cos\gamma & sin\gamma & 0 \\ -sin\gamma & cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

X_G, Y_G, Z_G: coordinates of the common points in global system

X_L, Y_L, Z_L: coordinates of the common points in global system

7 parameters in the transormation model:

T_X, T_Y, T_Z: translation parameters

α, β, γ: rotation parameters of three coordinate axes

λ : scale correction

In order to simplify the product of three rotation matrices, some characters of the matrices and rotation angles should be considered.
The three rotation matrices are orthogonal matrices, so there is $\boldsymbol{R^{-1}} = \boldsymbol{R^T}$
Generally, the rotation angles α, β, γ are small, so

$\alpha = \delta\alpha, \beta = \delta\beta, \gamma = \delta\gamma$

$sin\alpha = \alpha, sin\beta = \beta, sin\gamma = \gamma$

$cos\alpha = cos\beta = cos\gamma = 1$

$\delta\alpha\delta\beta = \delta\beta\delta\gamma = \delta\alpha\delta\gamma = 0$

With the above conditions the product of three rotation matrices can be approximated as

$$\boldsymbol{R_3}(\gamma)\boldsymbol{R_2}(\beta)\boldsymbol{R_1}(\alpha) \approx \begin{bmatrix} 1 & \gamma & -\beta \\ -\gamma & 1 & \alpha \\ \beta & -\alpha & 1 \end{bmatrix}$$

And the scale correction is written as $\lambda = 1 + \delta\lambda$ The transformation model (2.7) can be converted as

$$\begin{bmatrix} X_G \\ Y_G \\ Z_G \end{bmatrix} = (1+\delta\lambda) \begin{bmatrix} 1 & \delta\gamma & -\delta\beta \\ -\delta\gamma & 1 & \delta\alpha \\ \delta\beta & -\delta\alpha & 1 \end{bmatrix} \begin{bmatrix} X_L \\ Y_L \\ Z_L \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix} \tag{2.8}$$

In order to determine the 7 parameter, the transformation model (2.8) still needs to be converted in Gauß-Markov model l = Ax:

$$\begin{aligned} \begin{bmatrix} X_G \\ Y_G \\ Z_G \end{bmatrix} &= \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix} + \begin{bmatrix} 1+\delta\lambda & \delta\gamma & -\delta\beta \\ -\delta\gamma & 1+\delta\lambda & \delta\alpha \\ \delta\beta & -\delta\alpha & 1+\delta\lambda \end{bmatrix} \begin{bmatrix} X_L \\ Y_L \\ Z_L \end{bmatrix} \\ &= \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix} + \begin{bmatrix} X_L \\ Y_L \\ Z_L \end{bmatrix} + \begin{bmatrix} \delta\lambda & \delta\gamma & -\delta\beta \\ -\delta\gamma & \delta\lambda & \delta\alpha \\ \delta\beta & -\delta\alpha & \delta\lambda \end{bmatrix} \begin{bmatrix} X_L \\ Y_L \\ Z_L \end{bmatrix} \end{aligned}$$

The final form of the 7-parameter transformation model can be written as

$$\begin{bmatrix} X_G \\ Y_G \\ Z_G \end{bmatrix} - \begin{bmatrix} X_L \\ Y_L \\ Z_L \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & -Z_L & Y_L & X_L \\ 0 & 1 & 0 & Z_L & 0 & -X_L & Y_L \\ 0 & 0 & 1 & -Y_L & X_L & 0 & Z_L \end{bmatrix} \begin{bmatrix} T_X \\ T_Y \\ T_Z \\ \delta\alpha \\ \delta\beta \\ \delta\gamma \\ \delta\lambda \end{bmatrix} \tag{2.9}$$

2.2.2 Quadratic Polynomial Transformation Model

The concept of modelling the 3D network distortions with 7-parameter transformation has been explained. Mikhail developed simultaneous three dimensional transformation with polynomial of higher degree and the general polynomial in three-dimensions for the coordinate transformation.

Similar with 2D multiple linear regression, quadratic polynomial model is based on the directly expression of changes in the curvilinear coordinates between two datums. Here the transformation with the quadratic polynomial model from one Cartesian coordinate system to another is discussed.
The quadratic polynomial model can be written as

$$\begin{aligned} X_G =& \beta_{X0} + \beta_{X1}X_L + \beta_{X2}Y_L + \beta_{X3}Z_L + \beta_{X4}X_L^2 + \beta_{X5}Y_L^2 + \beta_{X6}Z_L^2 + \\ & \beta_{X7}(X_LY_L) + \beta_{X8}(X_LZ_L) + \beta_{X9}(Y_LZ_L) \end{aligned} \tag{2.10}$$

$$\begin{aligned} Y_G =& \beta_{Y0} + \beta_{Y1}X_L + \beta_{Y2}Y_L + \beta_{Y3}Z_L + \beta_{Y4}X_L^2 + \beta_{Y5}Y_L^2 + \beta_{Y6}Z_L^2 + \\ & \beta_{Y7}(X_LY_L) + \beta_{Y8}(X_LZ_L) + \beta_{Y9}(Y_LZ_L) \end{aligned} \tag{2.11}$$

$$\begin{aligned} Z_G =& \beta_{Z0} + \beta_{Z1}X_L + \beta_{Z2}Y_L + \beta_{Z3}Z_L + \beta_{Z4}X_L^2 + \beta_{Z5}Y_L^2 + \beta_{Z6}Z_L^2 + \\ & \beta_{Z7}(X_LY_L) + \beta_{Z8}(X_LZ_L) + \beta_{Z9}(Y_LZ_L) \end{aligned} \tag{2.12}$$

The transformation model(2.10), (2.11), (2.12) can also be written in Gauß-Markov model l = Ax

$$\begin{bmatrix} X_{G1} \\ X_{G2} \\ X_{G3} \\ \vdots \\ X_{Gn} \end{bmatrix} = \begin{bmatrix} 1 & X_{L1} & Y_{L1} & Z_{L1} & X_{L1}^2 & Y_{L1}^2 & Z_{L1}^2 & X_{L1}Y_{L1} & X_{L1}Z_{L1} & Y_{L1}Z_{L1} \\ 1 & X_{L2} & Y_{L2} & Z_{L2} & X_{L2}^2 & Y_{L2}^2 & Z_{L2}^2 & X_{L2}Y_{L2} & X_{L2}Z_{L2} & Y_{L2}Z_{L2} \\ 1 & X_{L3} & Y_{L3} & Z_{L3} & X_{L3}^2 & Y_{L3}^2 & Z_{L3}^2 & X_{L3}Y_{L3} & X_{L3}Z_{L3} & Y_{L3}Z_{L3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & X_{Ln} & Y_{Ln} & Z_{Ln} & X_{Ln}^2 & Y_{Ln}^2 & Z_{Ln}^2 & X_{Ln}Y_{Ln} & X_{Ln}Z_{Ln} & Y_{Ln}Z_{Ln} \end{bmatrix} \begin{bmatrix} \beta_{X0} \\ \beta_{X1} \\ \beta_{X2} \\ \beta_{X3} \\ \beta_{X4} \\ \beta_{X5} \\ \beta_{X6} \\ \beta_{X7} \\ \beta_{X8} \\ \beta_{X9} \end{bmatrix} \tag{2.13}$$

$$\begin{bmatrix} Y_{G1} \\ Y_{G2} \\ Y_{G3} \\ \vdots \\ Y_{Gn} \end{bmatrix} = \begin{bmatrix} 1 & X_{L1} & Y_{L1} & Z_{L1} & X_{L1}^2 & Y_{L1}^2 & Z_{L1}^2 & X_{L1}Y_{L1} & X_{L1}Z_{L1} & Y_{L1}Z_{L1} \\ 1 & X_{L2} & Y_{L2} & Z_{L2} & X_{L2}^2 & Y_{L2}^2 & Z_{L2}^2 & X_{L2}Y_{L2} & X_{L2}Z_{L2} & Y_{L2}Z_{L2} \\ 1 & X_{L3} & Y_{L3} & Z_{L3} & X_{L3}^2 & Y_{L3}^2 & Z_{L3}^2 & X_{L3}Y_{L3} & X_{L3}Z_{L3} & Y_{L3}Z_{L3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & X_{Ln} & Y_{Ln} & Z_{Ln} & X_{Ln}^2 & Y_{Ln}^2 & Z_{Ln}^2 & X_{Ln}Y_{Ln} & X_{Ln}Z_{Ln} & Y_{Ln}Z_{Ln} \end{bmatrix} \begin{bmatrix} \beta_{Y0} \\ \beta_{Y1} \\ \beta_{Y2} \\ \beta_{Y3} \\ \beta_{Y4} \\ \beta_{Y5} \\ \beta_{Y6} \\ \beta_{Y7} \\ \beta_{Y8} \\ \beta_{Y9} \end{bmatrix} \tag{2.14}$$

$$\begin{bmatrix} Z_{G1} \\ Z_{G2} \\ Z_{G3} \\ \vdots \\ Z_{Gn} \end{bmatrix} = \begin{bmatrix} 1 & X_{L1} & Y_{L1} & Z_{L1} & X_{L1}^2 & Y_{L1}^2 & Z_{L1}^2 & X_{L1}Y_{L1} & X_{L1}Z_{L1} & Y_{L1}Z_{L1} \\ 1 & X_{L2} & Y_{L2} & Z_{L2} & X_{L2}^2 & Y_{L2}^2 & Z_{L2}^2 & X_{L2}Y_{L2} & X_{L2}Z_{L2} & Y_{L2}Z_{L2} \\ 1 & X_{L3} & Y_{L3} & Z_{L3} & X_{L3}^2 & Y_{L3}^2 & Z_{L3}^2 & X_{L3}Y_{L3} & X_{L3}Z_{L3} & Y_{L3}Z_{L3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & X_{Ln} & Y_{Ln} & Z_{Ln} & X_{Ln}^2 & Y_{Ln}^2 & Z_{Ln}^2 & X_{Ln}Y_{Ln} & X_{Ln}Z_{Ln} & Y_{Ln}Z_{Ln} \end{bmatrix} \begin{bmatrix} \beta_{Z0} \\ \beta_{Z1} \\ \beta_{Z2} \\ \beta_{Z3} \\ \beta_{Z4} \\ \beta_{Z5} \\ \beta_{Z6} \\ \beta_{Z7} \\ \beta_{Z8} \\ \beta_{Z9} \end{bmatrix} \tag{2.15}$$

With the converted transformation models (2,12), (2,13), (2,14) and at least 10 collocated points($n \geqslant 10$) in both Cartesian systems, the 30 polynomial parameters can be individually estimated.

2.2.3 Cubic Polynomial Transformation Model

After the discussion of the quadratic polynomial model in the previous subsection, a question appears spontaneously, if polynomial models with higher degrees can achieve a better accuracy? In this subsection the cubic polynomial model will be discussed so as to solve this question.

The structure of a cubic polynomial model is similar to the structure of a quadratic polynomial model, except that the highest degree of a cubic polynomial model is 3 instead of 2. As a consequence, a cubic polynomial model consists of 60 terms instead of 30. 60 polynomial parameters need to be estimated in a cubic polynomial model instead of 30.

$$\begin{aligned} X_G = &\beta_{X0} + \beta_{X1}X_L + \beta_{X2}Y_L + \beta_{X3}Z_L + \beta_{X4}X_L^2 + \beta_{X5}Y_L^2 + \beta_{X6}Z_L^2 + \beta_{X7}(X_LY_L) + \beta_{X8}(X_LZ_L) + \\ &\beta_{X9}(Y_LZ_L) + \beta_{X10}X_L^3 + \beta_{X11}Y_L^3 + \beta_{X12}Z_L^3 + \beta_{X14}(X_LY_LZ_L) + \beta_{X14}(X_L^2Y_L) + \\ &\beta_{X15}(X_L^2Z_L) + \beta_{X16}(X_LY_L^2) + \beta_{X17}(Y_L^2Z_L) + \beta_{X18}(X_LZ_L^2) + \beta_{X19}(Y_LZ_L^2) \end{aligned} \tag{2.16}$$

$$
\begin{aligned}
Y_G =&\beta_{Y0} + \beta_{Y1}X_L + \beta_{Y2}Y_L + \beta_{Y3}Z_L + \beta_{Y4}X_L^2 + \beta_{Y5}Y_L^2 + \beta_{Y6}Z_L^2 + \beta_{Y7}(X_LY_L) + \beta_{Y8}(X_LZ_L) + \\
&\beta_{Y9}(Y_LZ_L) + \beta_{Y10}X_L^3 + \beta_{Y11}Y_L^3 + \beta_{Y12}Z_L^3 + \beta_{Y13}(X_LY_LZ_L) + \beta_{Y14}(X_L^2Y_L) + \\
&\beta_{Y15}(X_L^2Z_L) + \beta_{Y16}(X_LY_L^2) + \beta_{Y17}(Y_L^2Z_L) + \beta_{Y18}(X_LZ_L^2) + \beta_{Y19}(Y_LZ_L^2)
\end{aligned}
\tag{2.17}
$$

$$
\begin{aligned}
Z_G =&\beta_{Z0} + \beta_{Z1}X_L + \beta_{Z2}Y_L + \beta_{Z3}Z_L + \beta_{Z4}X_L^2 + \beta_{Z5}Y_L^2 + \beta_{Z6}Z_L^2 + \beta_{Z7}(X_LY_L) + \beta_{Z8}(X_LZ_L) + \\
&\beta_{Z9}(Y_LZ_L) + \beta_{Z10}X_L^3 + \beta_{Z11}Y_L^3 + \beta_{Z12}Z_L^3 + \beta_{Z13}(X_LY_LZ_L) + \beta_{Z14}(X_L^2Y_L) + \\
&\beta_{Z15}(X_L^2Z_L) + \beta_{Z16}(X_LY_L^2) + \beta_{Z17}(Y_L^2Z_L) + \beta_{Z18}(X_LZ_L^2) + \beta_{Z19}(Y_LZ_L^2)
\end{aligned}
\tag{2.18}
$$

The transformation model(2.16), (2.17), (2.18) can also be written in Gauß-Markov model l = Ax

$$
\begin{bmatrix} X_{G1} \\ X_{G2} \\ X_{G3} \\ \vdots \\ X_{Gn} \end{bmatrix} =
\begin{bmatrix}
1 & X_{L1} & Y_{L1} & Z_{L1} & X_{L1}^2 & Y_{L1}^2 & \cdots & Y_{L1}^2Z_{L1} & X_{L1}Z_{L1}^2 & Y_{L1}Z_{L1}^2 \\
1 & X_{L2} & Y_{L2} & Z_{L2} & X_{L2}^2 & Y_{L2}^2 & \cdots & Y_{L2}^2Z_{L2} & X_{L2}Z_{L2}^2 & Y_{L2}Z_{L2}^2 \\
1 & X_{L3} & Y_{L3} & Z_{L3} & X_{L3}^2 & Y_{L3}^2 & \cdots & Y_{L3}^2Z_{L3} & X_{L3}Z_{L3}^2 & Y_{L3}Z_{L3}^2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 & X_{Ln} & Y_{Ln} & Z_{Ln} & X_{Ln}^2 & Y_{Ln}^2 & \cdots & Y_{Ln}^2Z_{Ln} & X_{Ln}Z_{Ln}^2 & Y_{Ln}Z_{Ln}^2
\end{bmatrix}
\begin{bmatrix} \beta_{X0} \\ \beta_{X1} \\ \beta_{X2} \\ \beta_{X3} \\ \beta_{X4} \\ \vdots \\ \beta_{X16} \\ \beta_{X17} \\ \beta_{X18} \\ \beta_{X19} \end{bmatrix}
\tag{2.19}
$$

$$
\begin{bmatrix} Y_{G1} \\ Y_{G2} \\ Y_{G3} \\ \vdots \\ Y_{Gn} \end{bmatrix} =
\begin{bmatrix}
1 & X_{L1} & Y_{L1} & Z_{L1} & X_{L1}^2 & Y_{L1}^2 & \cdots & Y_{L1}^2Z_{L1} & X_{L1}Z_{L1}^2 & Y_{L1}Z_{L1}^2 \\
1 & X_{L2} & Y_{L2} & Z_{L2} & X_{L2}^2 & Y_{L2}^2 & \cdots & Y_{L2}^2Z_{L2} & X_{L2}Z_{L2}^2 & Y_{L2}Z_{L2}^2 \\
1 & X_{L3} & Y_{L3} & Z_{L3} & X_{L3}^2 & Y_{L3}^2 & \cdots & Y_{L3}^2Z_{L3} & X_{L3}Z_{L3}^2 & Y_{L3}Z_{L3}^2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 & X_{Ln} & Y_{Ln} & Z_{Ln} & X_{Ln}^2 & Y_{Ln}^2 & \cdots & Y_{Ln}^2Z_{Ln} & X_{Ln}Z_{Ln}^2 & Y_{Ln}Z_{Ln}^2
\end{bmatrix}
\begin{bmatrix} \beta_{Y0} \\ \beta_{Y1} \\ \beta_{Y2} \\ \beta_{Y3} \\ \beta_{Y4} \\ \vdots \\ \beta_{Y16} \\ \beta_{Y17} \\ \beta_{Y18} \\ \beta_{Y19} \end{bmatrix}
\tag{2.20}
$$

$$\begin{bmatrix} Z_{G1} \\ Z_{G2} \\ Z_{G3} \\ \vdots \\ Z_{Gn} \end{bmatrix} = \begin{bmatrix} 1 & X_{L1} & Y_{L1} & Z_{L1} & X_{L1}^2 & Y_{L1}^2 & \cdots & Y_{L1}^2 Z_{L1} & X_{L1} Z_{L1}^2 & Y_{L1} Z_{L1}^2 \\ 1 & X_{L2} & Y_{L2} & Z_{L2} & X_{L2}^2 & Y_{L2}^2 & \cdots & Y_{L2}^2 Z_{L2} & X_{L2} Z_{L2}^2 & Y_{L2} Z_{L2}^2 \\ 1 & X_{L3} & Y_{L3} & Z_{L3} & X_{L3}^2 & Y_{L3}^2 & \cdots & Y_{L3}^2 Z_{L3} & X_{L3} Z_{L3}^2 & Y_{L3} Z_{L3}^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & X_{Ln} & Y_{Ln} & Z_{Ln} & X_{Ln}^2 & Y_{Ln}^2 & \cdots & Y_{Ln}^2 Z_{Ln} & X_{Ln} Z_{Ln}^2 & Y_{Ln} Z_{Ln}^2 \end{bmatrix} \begin{bmatrix} \beta_{Z0} \\ \beta_{Z1} \\ \beta_{Z2} \\ \beta_{Z3} \\ \beta_{Z4} \\ \vdots \\ \beta_{Z16} \\ \beta_{Z17} \\ \beta_{Z18} \\ \beta_{Z19} \end{bmatrix} \tag{2.21}$$

With the converted transformation models (2,19), (2,20), (2,21) and at least 20 collocated points(n $\geqslant$ 20) in both Cartesian systems, the 60 polynomial parameters can be individually estimated.

2.2.4 Quadratic Polynomial Transformation Model with Orthogonal Polynomial

As discussed in the previous subsection, there is a huge possibility that the A matrix is an ill-posed matrix and the determined polynomial parameters can not be precise enough with the Least Squares method. Therefore the orthogonal polynomial will be introduced in the polynomial transformation model, in order to test if the introduction of orthogonal polynomials has a positive effect on the transformation result.

Orthogonal polynomials are classes of polynomials $P_n(x)$ defined over a range [a,b] that obey an orthogonality relation

$$< P_m, P_n >= \int_a^b P_m(x) P_n(x) \omega(x) dx \tag{2.22}$$

$$or$$

$$< P_m, P_n >= \sum_{i=1}^{n} P_m(x_i) P_n(x_i) \omega(x_i) \tag{2.23}$$

Where $\omega(x)$ is a positive function, which can be also called a weighting function. The form (2.23) is to be used when the function $P_n(x)$ and $P_n(x)$ is discontinuous. The two functions $P_n(x)$ and $P_n(x)$ are orthogonal if

$$< P_m, P_n >= \begin{cases} 0, m \neq n \\ A_k, m = n \end{cases} \tag{2.24}$$

For coordinate transformation the Legendre polynomials, which belong to the orthogonal polynomials, will be applied in the quadratic polynomial transformation model.

The Legendre polynomials are defined with

$$P_n(x) = \frac{1}{2^n \cdot n!} \cdot \frac{d^n}{dx^n} [(x^2 - 1)^n] \tag{2.25}$$

where x is continious in the interval [-1, 1]

Because the highest degree of the quadratic polynomial transformation is 2, so we only need the Legendre polynomials with n = 0, 1, 2

$$P_0 = 1 \tag{2.26}$$

$$P_1 = x \tag{2.27}$$

$$P_2 = \frac{1}{2}(3x^2 - 1) \tag{2.28}$$

The quadratic polynomial model with Legendre polynomials can be written as

$$\begin{aligned} X_G =&\beta_{X0} + \beta_{X1}X_L + \beta_{X2}Y_L + \beta_{X3}Z_L + \beta_{X4}(\frac{1}{2}(3X_L^2 - 1)) + \beta_{X5}(\frac{1}{2}(3Y_L^2 - 1)) + \\ &\beta_{X6}(\frac{1}{2}(3Z_L^2 - 1)) + \beta_{X7}(X_L Y_L) + \beta_{X8}(X_L Z_L) + \beta_{X9}(Y_L Z_L) \end{aligned} \tag{2.29}$$

$$\begin{aligned} Y_G =&\beta_{Y0} + \beta_{Y1}X_L + \beta_{Y2}Y_L + \beta_{Y3}Z_L + \beta_{Y4}(\frac{1}{2}(3X_L^2 - 1)) + \beta_{Y5}(\frac{1}{2}(3Y_L^2 - 1)) + \\ &\beta_{Y6}(\frac{1}{2}(3Z_L^2 - 1)) + \beta_{Y7}(X_L Y_L) + \beta_{Y8}(X_L Z_L) + \beta_{Y9}(Y_L Z_L) \end{aligned} \tag{2.30}$$

$$\begin{aligned} Z_G =&\beta_{Z0} + \beta_{Z1}X_L + \beta_{Z2}Y_L + \beta_{Z3}Z_L + \beta_{Z4}\frac{1}{2}(3X_L^2 - 1) + \beta_{Z5}\frac{1}{2}(3Y_L^2 - 1) + \\ &\beta_{Z6}\frac{1}{2}(3Z_L^2 - 1) + \beta_{Z7}(X_L Y_L) + \beta_{Z8}(X_L Z_L) + \beta_{Z9}(Y_L Z_L) \end{aligned} \tag{2.31}$$

The transformation model(2.29), (2.30), (2.31) can also be written in Gauß-Markov model l = Ax

$$\begin{bmatrix} X_{G1} \\ X_{G2} \\ X_{G3} \\ \vdots \\ X_{Gn} \end{bmatrix} = \begin{bmatrix} 1 & \cdots & \frac{1}{2}(3X_{L1}^2 - 1) & \frac{1}{2}(3Y_{L1}^2 - 1) & \frac{1}{2}(3Z_{L1}^2 - 1) & \cdots & Y_{L1}Z_{L1} \\ 1 & \cdots & \frac{1}{2}(3X_{L2}^2 - 1) & \frac{1}{2}(3Y_{L2}^2 - 1) & \frac{1}{2}(3Z_{L2}^2 - 1) & \cdots & Y_{L2}Z_{L2} \\ 1 & \cdots & \frac{1}{2}(3X_{L3}^2 - 1) & \frac{1}{2}(3Y_{L3}^2 - 1) & \frac{1}{2}(3Z_{L3}^2 - 1) & \cdots & Y_{L3}Z_{L3} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & \frac{1}{2}(3X_{Ln}^2 - 1) & \frac{1}{2}(3Y_{Ln}^2 - 1) & \frac{1}{2}(3Z_{Ln}^2 - 1) & \cdots & Y_{Ln}Z_{Ln} \end{bmatrix} \begin{bmatrix} \beta_{X0} \\ \beta_{X1} \\ \beta_{X2} \\ \beta_{X3} \\ \beta_{X4} \\ \beta_{X5} \\ \beta_{X6} \\ \beta_{X7} \\ \beta_{X8} \\ \beta_{X9} \end{bmatrix} \tag{2.32}$$

$$\begin{bmatrix} Y_{G1} \\ Y_{G2} \\ Y_{G3} \\ \vdots \\ Y_{Gn} \end{bmatrix} = \begin{bmatrix} 1 & \cdots & \frac{1}{2}(3X_{L1}^2 - 1) & \frac{1}{2}(3Y_{L1}^2 - 1) & \frac{1}{2}(3Z_{L1}^2 - 1) & \cdots & Y_{L1}Z_{L1} \\ 1 & \cdots & \frac{1}{2}(3X_{L2}^2 - 1) & \frac{1}{2}(3Y_{L2}^2 - 1) & \frac{1}{2}(3Z_{L2}^2 - 1) & \cdots & Y_{L2}Z_{L2} \\ 1 & \cdots & \frac{1}{2}(3X_{L3}^2 - 1) & \frac{1}{2}(3Y_{L3}^2 - 1) & \frac{1}{2}(3Z_{L3}^2 - 1) & \cdots & Y_{L3}Z_{L3} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & \frac{1}{2}(3X_{Ln}^2 - 1) & \frac{1}{2}(3Y_{Ln}^2 - 1) & \frac{1}{2}(3Z_{Ln}^2 - 1) & \cdots & Y_{Ln}Z_{Ln} \end{bmatrix} \begin{bmatrix} \beta_{Y0} \\ \beta_{Y1} \\ \beta_{Y2} \\ \beta_{Y3} \\ \beta_{Y4} \\ \beta_{Y5} \\ \beta_{Y6} \\ \beta_{Y7} \\ \beta_{Y8} \\ \beta_{Y9} \end{bmatrix} \tag{2.33}$$

$$\begin{bmatrix} Z_{G1} \\ Z_{G2} \\ Z_{G3} \\ \vdots \\ Z_{Gn} \end{bmatrix} = \begin{bmatrix} 1 & \cdots & \frac{1}{2}(3X_{L1}^2 - 1) & \frac{1}{2}(3Y_{L1}^2 - 1) & \frac{1}{2}(3Z_{L1}^2 - 1) & \cdots & Y_{L1}Z_{L1} \\ 1 & \cdots & \frac{1}{2}(3X_{L2}^2 - 1) & \frac{1}{2}(3Y_{L2}^2 - 1) & \frac{1}{2}(3Z_{L2}^2 - 1) & \cdots & Y_{L2}Z_{L2} \\ 1 & \cdots & \frac{1}{2}(3X_{L3}^2 - 1) & \frac{1}{2}(3Y_{L3}^2 - 1) & \frac{1}{2}(3Z_{L3}^2 - 1) & \cdots & Y_{L3}Z_{L3} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & \frac{1}{2}(3X_{Ln}^2 - 1) & \frac{1}{2}(3Y_{Ln}^2 - 1) & \frac{1}{2}(3Z_{Ln}^2 - 1) & \cdots & Y_{Ln}Z_{Ln} \end{bmatrix} \begin{bmatrix} \beta_{Z0} \\ \beta_{Z1} \\ \beta_{Z2} \\ \beta_{Z3} \\ \beta_{Z4} \\ \beta_{Z5} \\ \beta_{Z6} \\ \beta_{Z7} \\ \beta_{Z8} \\ \beta_{Z9} \end{bmatrix} \tag{2.34}$$

With the converted transformation models (2,32), (2,33), (2,34) and at least 10 collocated points($n \geqslant 10$) in both Cartesian systems, the 30 polynomial parameters can be individually estimated.

Chapter 3

Paramter Estimation with Least Squares Method

3.1 Data Pre-processing

For the coordinate transformation there are 131 available common points in Baden-Würtemberg, among which 121 points are selected as collocated points for the estimation of transformation parameters and 10 points as interpolated points. Their coordinates are based in UTM and Gau-Krüger coordinate systems. The UTM coordinate system is regarded as global coordinate System while the Gau-Krüger coordinate systems as local coordinate system.

Firstly, the UTM and Gau-Krüger coordinates should be transformed to elliptical coordinates latitude (B) and longitude (L), which will be used in the two dimensional Multiple Linear Regression.

Secondly, the transformed elliptical coordinates should be further transformed to three dimensional Cartesian coordinate X, Y, Z with the following equation

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} (N+H)cosBcosL \\ (N+H)cosBsinL \\ [N(1-e^2)+H]sinB \end{bmatrix} \tag{3.1}$$

where
B, L, H: ellipsoidal coordinates
N: normal curvature
e^2; eccentricity
H: ellipsoid height
$H = h_s + N_{NN}$: h_s is the normal-orthometric height and N_{NN} is Normal-Null surface undulation

All the data Pre-processing is prepared with the above operations for the following transformation models.

3.2 The Least Squares Method

The method of least squares is a standard approach in regression analysis to the approximate solution of overdetermined systems, which means in sets of equations there are more equations than unknowns. "Least squares" means that the overall solution minimizes the sum of the squares of the residuals made in the results of every single equation.(wiki) When an equation in the form of Gauß-Markov model $\boldsymbol{l}_{(n\times1)} = \boldsymbol{A}_{(n\times u)}\boldsymbol{x}_{(u\times1)}$ already exists, the parameters $\boldsymbol{x}$ will be estimated as follows:

n: number of the variables in the target system

u: number of transformation parameters in the transformation model

$\boldsymbol{l}$: vector of the observed values of the response variable in the target system

$\boldsymbol{A}$: matrix of the observed values in the start system

$\boldsymbol{P}$: weight matrix of the observed values in the start system

$\boldsymbol{x}$: vector of the transformation parameters

estimated parameters $\widehat{\boldsymbol{x}}$: $\widehat{\boldsymbol{x}} = (\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{l}$

the residual $\widehat{\boldsymbol{v}}$ after a transformation: $\widehat{\boldsymbol{v}} = \boldsymbol{l} - \boldsymbol{A}\widehat{\boldsymbol{x}}$

the standard deviation $\widehat{\sigma}$: $\widehat{\sigma} = \sqrt{\frac{1}{n-u}(\widehat{\boldsymbol{v}}^T \cdot \widehat{\boldsymbol{v}})}$

3.3 2-D Coordinate Transformation

3.3.1 6-Parameter Affine Transformation

As discussed in the previous chapter, the 6-parameter affine transformation model(2.1) is finally converted into a transformation model(2.3) in Gauß-Markov model $\boldsymbol{l}$=$\boldsymbol{A}\boldsymbol{x}$

$$\begin{bmatrix} E \\ N \end{bmatrix} = \begin{bmatrix} 1 & 0 & R & H & 0 & 0 \\ 0 & 1 & 0 & 0 & R & H \end{bmatrix} \begin{bmatrix} t_E \\ t_N \\ f \\ g \\ b \\ a \end{bmatrix}$$

$$\boldsymbol{l} = \boldsymbol{A}\boldsymbol{x}$$

where
E, N: the UTM coordinates of 121 collocated points
R, H: the Gauß-Krüger coordinates
$\boldsymbol{l}$: vector of the observed values of the response variable in the UTM coordinate system
$\boldsymbol{A}$: matrix of the observed values in the Gau Krüger coordinate system
$\boldsymbol{x}$: vector of the transformation parameters

Before the estimation of transformation parameters, E, N, R, H should be centralized:

$$\begin{aligned} N_c &= N - mean(N) \\ E_c &= E - mean(E) \\ H_c &= H - mean(H) \\ R_c &= R - mean(R) \end{aligned} \tag{3.2}$$

The original coordinates N, E, H, R in the equation(2.3) will be substituted with the centralized coordinates N_c, E_c, H_c, R_c

The transformation parameters will be estimated with Least Squares Method.
The 6 estimated transformation parameters and their standard derivation with 121 collocated points are listed as follows

6-parameter affine transformation GK to UTM							
t_N(m)	t_E(m)	$\alpha('')$	$\beta('')$	m_1	m_2	QMR(m)	$\widehat{\sigma}$(m)
437.3896	119.889076	0.1598	-0.2074	0.9996	0.9996	0.1192	0.1208

Table 3.1: *Transformation parameters of 6-parameter affine transformation and the deviation*

The horizontal residuals v_E and v_N and their directions of 121 collocated points and 10 interpolated points are showed in the figure bellow:

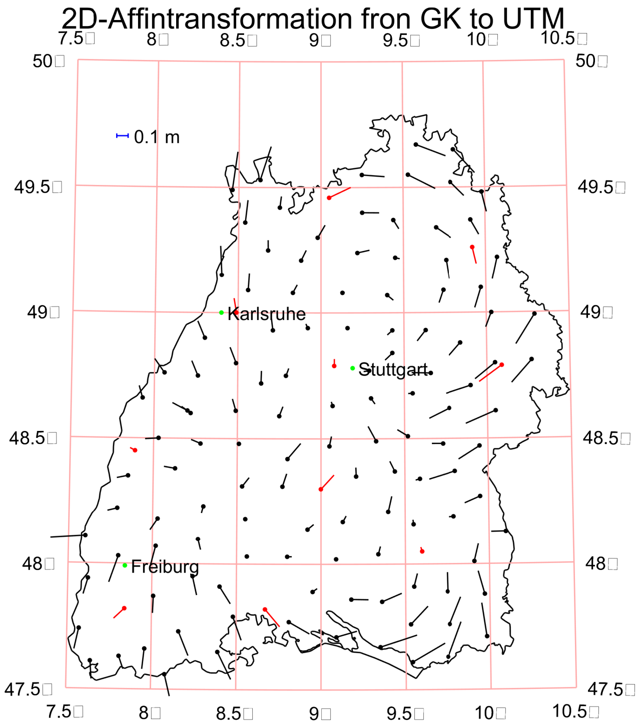

Figure 3.1: *Horizontal residuals of 121 collocated points and 10 interpolated points with 6-parameter affine transformation*

3.3.2 2D Multiple Linear Regression of 9th Order

The US Defense Mapping Agency (DMA) published a series of multiple regression equations (MREs) for transforming some local datums to the World Geodetic System 1984 in Appendix

D of its Technical Report 8350.2, the third edition of which is NIMA (2004). Among the MREs provided by NIMA, the following MREs are used for transforming the European Datum 1950 to WGS 84 in Western Europe.

The following MREs come from the full range 9th-order MREs (2.4) and the selection of variables U^iV^j used in the following MREs is based on the statistical significance of their contribution to the known datum shifts at control points. The method used is normally the 'stepwise' multiple regression procedure. This is described in Section 7.2.4.3.3 of DMA (1987a) and by Appelbaum (1982). The full range of multiple regression procedures can be found in Draper and Smith (1966). (A. C. Ruffhead)

$$\begin{aligned}\Delta\phi = &-2.65261 + 2.06392U + 0.77921V + 0.26743U^2 + 0.10706UV + 0.76407U^3 - 0.95430U^2V + \\ &0.17197U^4 + 1.04974U^4V - 0.22899U^5V^2 - 0.05401V^8 - 0.78909U^9 - 0.10572U^2V^7 + \\ &0.05283UV^9 + 0.02445U^3V^9\end{aligned} \tag{3.3}$$

$$\begin{aligned}\Delta\lambda = &-4.13447 - 1.50572U + 1.94075V - 1.37600U^2 + 1.98425UV + 0.30068V^2 - 2.31939U^3 - \\ &1.70401U^4 - 5.48711UV^3 + 7.41956U^5 - 1.61351U^2V^3 + 5.92923UV^4 - 1.97974V^5 + \\ &1.57701U^6 - 6.52522U^3V^3 + 16.85976U^2V^4 - 1.79701UV^5 - 3.08344U^7 - 14.32516U^6V + \\ &4.49096U^4V^4 + 9.98750U^8V + 7.80215U^7V^2 - 2.26917U^2V^7 + 0.16438V^9 - 17.45428U^4V^6 - \\ &8.25844U^9V^2 + 5.28734U^8V^3 + 8.87141U^5V^7 - 3.48015U^9V^4 + 0.71041U^4V^9\end{aligned} \tag{3.4}$$

Where:

K = 0.05235988

U = K[ϕ_L - 52°]

V = K[λ_L - 10°]

$\Delta\phi = \phi_G - \phi_L$

$\Delta\lambda = \lambda_G - \lambda_L$

ϕ_L, λ_L: the elliptical coordinates in European Datum 1950

ϕ_G, λ_G: the elliptical coordinates in WGS84

For the datum transformation from Bessel ellipsoid 1841 to ETRS 89 in Baden-Würtemberg, we can apply the above-mentioned MREs, which is suitable for Western Europe. The transformation parameter should be recalculated with 121 collocated points in Baden-Würtemberg. The Multiple Linear Regression model is written as follows

$$\begin{aligned}\Delta\phi = &A_{00} + A_{10}U + A_{01}V + A_{20}U^2 + A_{11}UV + A_{30}U^3 + A_{21}U^2V + A_{40}U^4 + A_{41}U^4V + \\ &A_{52}U^5V^2 + A_{08}V^8 + A_{90}U^9 + A_{27}U^2V^7 + A_{19}UV^9 + A_{39}U^3V^9\end{aligned} \tag{3.5}$$

$$
\begin{aligned}
\Delta\lambda = & B_{00} + B_{10}U + B_{01}V + B_{20}U^2 + B_{11}UV + B_{02}V^2 + B_{30}U^3 + B_{40}U^4 + B_{13}UV^3 + B_{50}U^5 + \\
& B_{23}U^2V^3 + B_{14}UV^4 + B_{05}V^5 + B_{60}U^6 + B_{33}U^3V^3 + B_{24}U^2V^4 + B_{15}UV^5 + B_{70}U^7 + \\
& B_{61}U^6V + B_{44}U^4V^4 + B_{81}U^8V + B_{72}U^7V^2 + B_{27}U^2V^7 + B_{09}V^9 + B_{46}U^4V^6 + \\
& B_{92}U^9V^2 + B_{83}U^8V^3 + B_{57}U^5V^7 + B_{94}U^9V^4 + B_{49}U^4V^9
\end{aligned}
\tag{3.6}
$$

Where U and V should be centralized:

K = 0.05235988

U = K[ϕ_L - (mean)ϕ_L]

V = K[λ_L - (mean)λ_L]

$\Delta\phi = \phi_G - \phi_L$ in [rad]

$\Delta\lambda = \lambda_G - \lambda_L$ in [rad]

ϕ_L, λ_L: the elliptical coordinates of 121 collocated points converted from the Gauß-Krüger coordinates

ϕ_G, λ_G: the elliptical coordinates of 121 collocated points converted from the UTM coordinates

The equations (3.3), (3.4) will be then converted in Gauß-Markov model l = Ax and the transformation parameters will be estimated with Least Squares Method.

The estimated transformation parameters in λ, ϕ components and their standard derivation with 121 collocated points are listed as follows:

2D 9th Order Multiple Linear Regression Transformation Parameters			
A_{00}	-1.8087×10^{-5}	B_{00}	-1.7968×10^{-5}
A_{10}	-3.7331×10^{-5}	B_{10}	-5.6167×10^{-6}
A_{01}	2.6368×10^{-6}	B_{01}	-4.6961×10^{-5}
A_{20}	2.2821×10^{-6}	B_{20}	1.4008×10^{-5}
A_{11}	-1.2915×10^{-5}	B_{11}	-7.8630×10^{-6}
A_{30}	-2.2690×10^{-5}	B_{02}	-9.8752×10^{-6}
A_{21}	1.1512×10^{-4}	B_{30}	-4.8613×10^{-4}
A_{40}	-0.0019	B_{40}	-0.0017
A_{41}	0.0024	B_{13}	0.0030
A_{52}	-6.7821	B_{50}	0.2987
A_{08}	9.5410	B_{23}	0.0469
A_{90}	-353.1089	B_{14}	-1.8087×10^{-5}
A_{27}	-2301.8230	B_{05}	-0.0367
A_{19}	-496.7397	B_{60}	3.2684
A_{39}	-5.744×10^{-6}	B_{33}	-1.6149
		B_{24}	-0.9816
		B_{15}	-0.7890
		B_{70}	-3.0316
		B_{61}	15.2678
		B_{44}	-2.1722
		B_{81}	-958.5143
		B_{72}	-6755.9401
		B_{27}	-6158.7628
		B_{09}	1979.0664
		B_{46}	140923.4686
		B_{92}	1257980.5270
		B_{83}	-1419456.7285
		B_{57}	13308805.9393
		B_{94}	318467433.6728
		B_{49}	363882024.4378
		QMR(m)	0.0402
		$\widehat{\sigma}(m)$	0.0445

Table 3.2: *Transformation parameters of 2D 9th order multiple linear regression and the deviation*

The coefficients A_0 and B_0 are actually two translation parameters and the unit of their value in the above table is radian. We can convert their value into meter in north and east directions in order to see the translations more clearly.
Translation in north direction:
$A_0 = -1.8087 \times 10^{-5} \times 6378137 = -115.3614m$
Translation in east direction:
$B_0 = -1.7968 \times 10^{-5} \times 6378137 = -114.6024m$

The horizontal residuals v_E and v_N and their directions of 121 collocated points and 10 interpolated points are showed in the figure bellow:

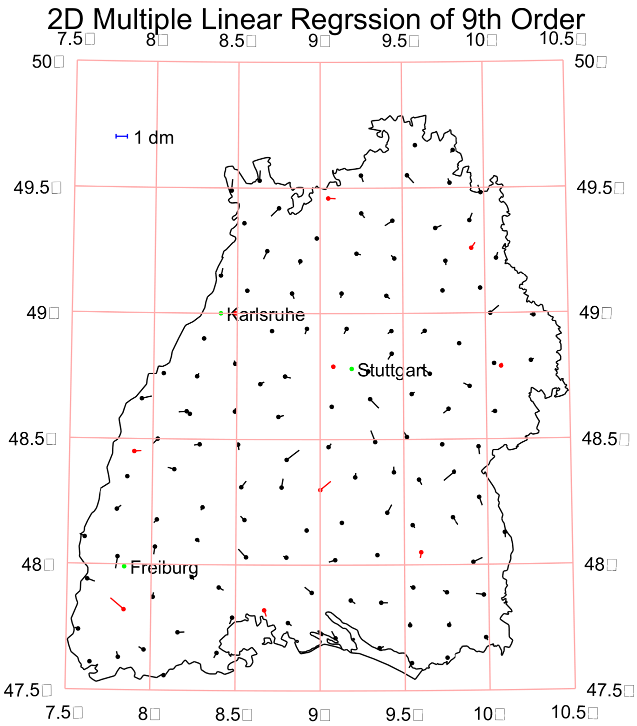

Figure 3.2: *Horizontal residuals of 121 collocated points and 10 interpolated points with 9th order multiple linear regression*

3.3.3 2D Multiple Linear Regression of 5th Order

In the previous section we used the 9th-order MREs (3.5) and (3.6) for the transformation. In the 9th-order MREs, there are 45 polynomial terms in total, in which 15 terms are in the ϕ-component and 30 terms are in the λ-component. The huge number of polynomial terms increases not only the computation complexity but also the number of needed collocated points comparing with MREs of lower order.

So in this section the 9th-order MREs will be cut to a 5th-order MREs, which means the polynomial terms in 9th-order MREs lower than or equal to 5th-order will be reserved, but the polynomial terms higher than 5th-order will be removed, in order to check if the transformation accuracy with 5th-order is good enough to replace the complicated 9th-order MREs.
The 5th-order MREs are written as follows:

$$\Delta\phi = A_{00} + A_{10}U + A_{01}V + A_{20}U^2 + A_{11}UV + A_{30}U^3 + A_{21}U^2V + A_{40}U^4 + A_{41}U^4V \quad (3.7)$$

$$\begin{aligned}\Delta\lambda = & B_{00} + B_{10}U + B_{01}V + B_{20}U^2 + B_{11}UV + B_{02}V^2 + B_{30}U^3 + B_{40}U^4 + B_{13}UV^3 + B_{50}U^5 + \\ & B_{23}U^2V^3 + B_{14}UV^4 + B_{05}V^5\end{aligned} \quad (3.8)$$

Where U and V should be centralized:

K = 0.05235988

U = K[ϕ_L - (mean)ϕ_L]

V = K[λ_L - (mean)λ_L]

$\Delta\phi = \phi_G - \phi_L$ in [rad]

$\Delta\lambda = \lambda_G - \lambda_L$ in [rad]

ϕ_L, λ_L: the elliptical coordinates of 121 collocated points converted from the Gauß-Krüger coordinates

ϕ_G, λ_G: the elliptical coordinates of 121 collocated points converted from the UTM coordinates

The equations (3.5), (3.6) will be then converted in Gauß-Markov model l = Ax and the transformation parameters will be estimated with Least Squares Method.

The estimated transformation parameters in λ, ϕ components and their standard derivation with 121 collocated points are listed as follows:

2D 5th Order Multiple Linear Regression Transformation Parameters			
A_{00}	-1.8086×10^{-5}	B_{00}	-1.7966×10^{-5}
A_{10}	-3.7429×10^{-5}	B_{10}	-5.8875×10^{-6}
A_{01}	2.6469×10^{-6}	B_{01}	-4.7107×10^{-5}
A_{20}	-1.0428×10^{-6}	B_{20}	1.0634×10^{-5}
A_{11}	-1.2773×10^{-5}	B_{11}	-2.1376×10^{-5}
A_{30}	9.3580×10^{-5}	B_{02}	-1.2413×10^{-5}
A_{21}	7.2003×10^{-5}	B_{30}	1.2534×10^{-4}
A_{40}	0.0016	B_{40}	0.0059
A_{41}	-0.0070	B_{13}	0.0058
		B_{50}	0.0495
		B_{23}	0.0618
		B_{14}	-0.0060
		B_{05}	0.0151
		QMR(m)	0.0554
		$\widehat{\sigma}(m)$	0.0581

Table 3.3: *Transformation parameters of 2D 5th order multiple linear regression and the deviation*

The coefficients A_0 and B_0 are actually two translation parameters and the unit of their value in the above table is radian. We can convert their value into meter in north and east directions in order to see the translations more clearly.
Translation in north direction:
$A_0 = -1.8087 \times 10^{-5} \times 6378137 = -115.3550m$
Translation in east direction:
$B_0 = -1.7968 \times 10^{-5} \times 6378137 = -114.5896m$

The horizontal residuals v_E and v_N and their directions of 121 collocated points and 10 interpolated points are showed in the figure bellow:

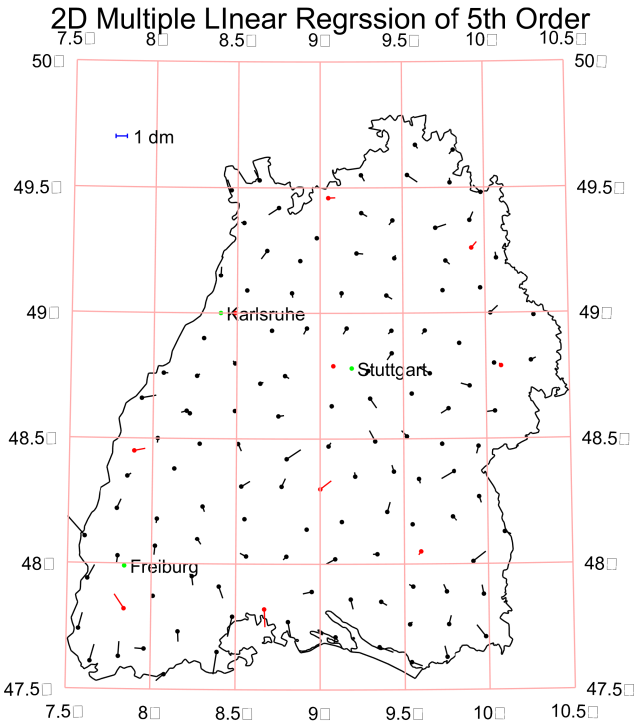

Figure 3.3: Horizontal residuals of 121 collocated points and 10 interpolated points with 5th order multiple linear regression

3.4 3-D Coordinate Transformation

3.4.1 7-Parameter- Helmerttransformation

As already derivated in the chapter 2, the equation (2.8) in the form of Gauß-Markov model can be used for the estimation of the transformation parameters.

$$\begin{bmatrix} X_G \\ Y_G \\ Z_G \end{bmatrix} - \begin{bmatrix} X_L \\ Y_L \\ Z_L \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & -Z_L & Y_L & X_L \\ 0 & 1 & 0 & Z_L & 0 & -X_L & Y_L \\ 0 & 0 & 1 & -Y_L & X_L & 0 & Z_L \end{bmatrix} \begin{bmatrix} T_X \\ T_Y \\ T_Z \\ \delta\alpha \\ \delta\beta \\ \delta\gamma \\ \delta\lambda \end{bmatrix}$$

$$\boldsymbol{l} = \boldsymbol{Ax}$$

where

X_L, Y_L, Z_L: Cartesian coordinates converted from the Gau Krüger coordinates

X_G, Y_G, Z_G: Cartesian coordinates converted from the UTM coordinates

Both coordinates should be scaled into interval [-1,1] in order to compare the transformation results of 7 parameter Helmert transformation and quadratic polynomial transformation with Legendre polynomials. (The domain of Legendre polynomials is [-1,1])

The transformation parameters will be estimated with Least Squares Method: $\widehat{\boldsymbol{x}} = (\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{l}$
$\boldsymbol{P}$ is an identity matrix in the formula above, because here every observation is equal weighted.

The transformation parameters and their deviation estimated with 121 collocated points are listed bellow:

7-parameter helmert transformation GK to UTM								
T_X(m)	T_Y(m)	T_Z(m)	$\delta\alpha('')$	$\delta\beta('')$	$\delta\gamma('')$	$\delta\lambda$	QMR(m)	$\widehat{\sigma}$(m)
582.9539	112.2600	405.7222	-2.2607	-0.336845	2.065813	9.0963×10^{-6}	0.1253	0.1036

Table 3.4: *Transformation parameters of 7-parameter Helmert transformation and the deviation*

The horizontal residuals v_E and v_N and their directions of 121 collocated points and 10 interpolated points are showed in the figure bellow:

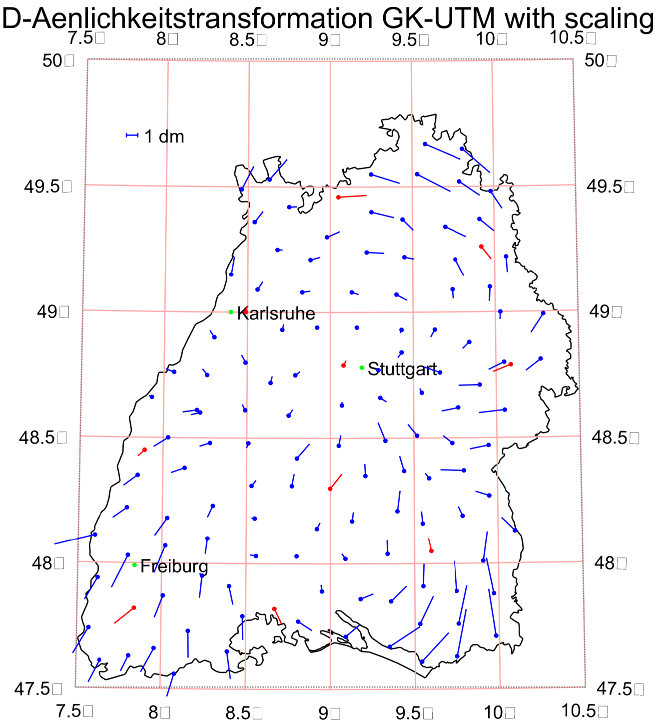

Figure 3.4: Horizontal residuals of 121 collocated points and 10 interpolated points with 7-parameter Helmert transformation

3.4.2 Quadratic Polynomial Transformation

The quadratic polynomial transformation model in Gauß-Markov model is the in chapter 2 derivated equations (2.12), (2.13) and (2.14)

$$\begin{bmatrix} X_{G1} \\ X_{G2} \\ X_{G3} \\ \vdots \\ X_{Gn} \end{bmatrix} = \begin{bmatrix} 1 & X_{L1} & Y_{L1} & Z_{L1} & X_{L1}^2 & Y_{L1}^2 & Z_{L1}^2 & X_{L1}Y_{L1} & X_{L1}Z_{L1} & Y_{L1}Z_{L1} \\ 1 & X_{L2} & Y_{L2} & Z_{L2} & X_{L2}^2 & Y_{L2}^2 & Z_{L2}^2 & X_{L2}Y_{L2} & X_{L2}Z_{L2} & Y_{L2}Z_{L2} \\ 1 & X_{L3} & Y_{L3} & Z_{L3} & X_{L3}^2 & Y_{L3}^2 & Z_{L3}^2 & X_{L3}Y_{L3} & X_{L3}Z_{L3} & Y_{L3}Z_{L3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & X_{Ln} & Y_{Ln} & Z_{Ln} & X_{Ln}^2 & Y_{Ln}^2 & Z_{Ln}^2 & X_{Ln}Y_{Ln} & X_{Ln}Z_{Ln} & Y_{Ln}Z_{Ln} \end{bmatrix} \begin{bmatrix} \beta_{X0} \\ \beta_{X1} \\ \beta_{X2} \\ \beta_{X3} \\ \beta_{X4} \\ \beta_{X5} \\ \beta_{X6} \\ \beta_{X7} \\ \beta_{X8} \\ \beta_{X9} \end{bmatrix}$$

$$\boldsymbol{l}_x = \boldsymbol{A}\boldsymbol{x}_x$$

$$\begin{bmatrix} Y_{G1} \\ Y_{G2} \\ Y_{G3} \\ \vdots \\ Y_{Gn} \end{bmatrix} = \begin{bmatrix} 1 & X_{L1} & Y_{L1} & Z_{L1} & X_{L1}^2 & Y_{L1}^2 & Z_{L1}^2 & X_{L1}Y_{L1} & X_{L1}Z_{L1} & Y_{L1}Z_{L1} \\ 1 & X_{L2} & Y_{L2} & Z_{L2} & X_{L2}^2 & Y_{L2}^2 & Z_{L2}^2 & X_{L2}Y_{L2} & X_{L2}Z_{L2} & Y_{L2}Z_{L2} \\ 1 & X_{L3} & Y_{L3} & Z_{L3} & X_{L3}^2 & Y_{L3}^2 & Z_{L3}^2 & X_{L3}Y_{L3} & X_{L3}Z_{L3} & Y_{L3}Z_{L3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & X_{Ln} & Y_{Ln} & Z_{Ln} & X_{Ln}^2 & Y_{Ln}^2 & Z_{Ln}^2 & X_{Ln}Y_{Ln} & X_{Ln}Z_{Ln} & Y_{Ln}Z_{Ln} \end{bmatrix} \begin{bmatrix} \beta_{Y0} \\ \beta_{Y1} \\ \beta_{Y2} \\ \beta_{Y3} \\ \beta_{Y4} \\ \beta_{Y5} \\ \beta_{Y6} \\ \beta_{Y7} \\ \beta_{Y8} \\ \beta_{Y9} \end{bmatrix}$$

$$\boldsymbol{l}_y = \boldsymbol{A}\boldsymbol{x}_y$$

$$\begin{bmatrix} Z_{G1} \\ Z_{G2} \\ Z_{G3} \\ \vdots \\ Z_{Gn} \end{bmatrix} = \begin{bmatrix} 1 & X_{L1} & Y_{L1} & Z_{L1} & X_{L1}^2 & Y_{L1}^2 & Z_{L1}^2 & X_{L1}Y_{L1} & X_{L1}Z_{L1} & Y_{L1}Z_{L1} \\ 1 & X_{L2} & Y_{L2} & Z_{L2} & X_{L2}^2 & Y_{L2}^2 & Z_{L2}^2 & X_{L2}Y_{L2} & X_{L2}Z_{L2} & Y_{L2}Z_{L2} \\ 1 & X_{L3} & Y_{L3} & Z_{L3} & X_{L3}^2 & Y_{L3}^2 & Z_{L3}^2 & X_{L3}Y_{L3} & X_{L3}Z_{L3} & Y_{L3}Z_{L3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & X_{Ln} & Y_{Ln} & Z_{Ln} & X_{Ln}^2 & Y_{Ln}^2 & Z_{Ln}^2 & X_{Ln}Y_{Ln} & X_{Ln}Z_{Ln} & Y_{Ln}Z_{Ln} \end{bmatrix} \begin{bmatrix} \beta_{Z0} \\ \beta_{Z1} \\ \beta_{Z2} \\ \beta_{Z3} \\ \beta_{Z4} \\ \beta_{Z5} \\ \beta_{Z6} \\ \beta_{Z7} \\ \beta_{Z8} \\ \beta_{Z9} \end{bmatrix}$$

$$\boldsymbol{l}_z = \boldsymbol{A}\boldsymbol{x}_z$$

In order to compare the transformation results of the quadratic polynomial and quadratic polynomial model with Legendre polynomials, the original Cartesian coordinates in both systems should be scaled into the interval [-1,1].

Besides, all the scaled global coordinate in the Gauß-Markov model will be substituted with the difference values of the scaled global coordinates and local coordinates:

$$\begin{aligned} X_{Gc} &= X_G - X_L \\ Y_{Gc} &= Y_G - Y_L \\ Z_{Gc} &= Z_G - Z_L \end{aligned}$$

After the above mentioned process with the coordinates in both systems, the 30 transformation parameters can be then separately estimated with Least Squares method.
The estimated transformation parameters in λ, ϕ components and their standard derivation with 121 collocated points are listed as follows:

3D quadratic polynomial transformation					
β_{X0}	635.2211	β_{Y0}	24.3160	β_{Z0}	449.4206
β_{X1}	7.0016	β_{Y1}	-7.9253	β_{Z1}	1.5622
β_{X2}	1.8300	β_{Y2}	-0.1883	β_{Z2}	1.2965
β_{X3}	5.8459.	β_{Y3}	-7.3017	β_{Z3}	2.3141
β_{X4}	-390.8437	β_{Y4}	91.5024	β_{Z4}	255.0302
β_{X5}	-9.0002	β_{Y5}	0.8947	β_{Z5}	5.9370
β_{X6}	-336.0679	β_{Y6}	77.3813	β_{Z6}	221.7853
β_{X7}	-119.3618	β_{Y7}	21.6543	β_{Z7}	77.8467
β_{X8}	-724.9722	β_{Y8}	128.2169	β_{Z8}	475.6438
β_{X9}	-110.3723	β_{Y9}	19.8538	β_{Z9}	72.3052
				QMR(m)	0.0478
				$\widehat{\sigma}$(m)	0.0412

Table 3.5: *Transformation parameters of the 3D quadratic polynomial and the deviation*

The horizontal residuals v_E and v_N and their directions of 121 collocated points and 10 interpolated points are showed in the figure bellow:

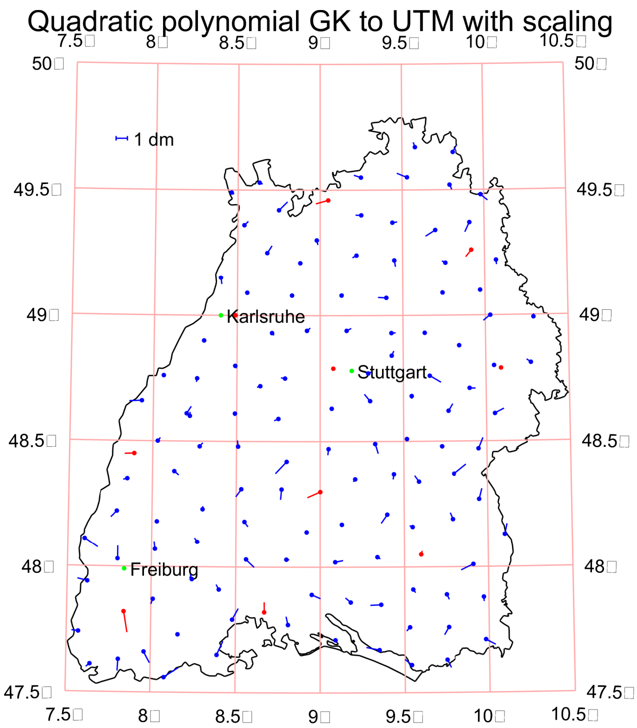

Figure 3.5: *Horizontal residuals of 121 collocated points and 10 interpolated points with quadratic polynomial model*

3.4.3 Cubic Polynomial Transformation

The parameter estimation in the cubic polynomial transformation is similar with that in the quadratic polynomial transformation.

$$\begin{bmatrix} X_{G1} \\ X_{G2} \\ X_{G3} \\ \vdots \\ X_{Gn} \end{bmatrix} = \begin{bmatrix} 1 & X_{L1} & Y_{L1} & Z_{L1} & X_{L1}^2 & Y_{L1}^2 & \cdots & Y_{L1}^2 Z_{L1} & X_{L1} Z_{L1}^2 & Y_{L1} Z_{L1}^2 \\ 1 & X_{L2} & Y_{L2} & Z_{L2} & X_{L2}^2 & Y_{L2}^2 & \cdots & Y_{L2}^2 Z_{L2} & X_{L2} Z_{L2}^2 & Y_{L2} Z_{L2}^2 \\ 1 & X_{L3} & Y_{L3} & Z_{L3} & X_{L3}^2 & Y_{L3}^2 & \cdots & Y_{L3}^2 Z_{L3} & X_{L3} Z_{L3}^2 & Y_{L3} Z_{L3}^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & X_{Ln} & Y_{Ln} & Z_{Ln} & X_{Ln}^2 & Y_{Ln}^2 & \cdots & Y_{Ln}^2 Z_{Ln} & X_{Ln} Z_{Ln}^2 & Y_{Ln} Z_{Ln}^2 \end{bmatrix} \begin{bmatrix} \beta_{X0} \\ \beta_{X1} \\ \beta_{X2} \\ \beta_{X3} \\ \beta_{X4} \\ \vdots \\ \beta_{X16} \\ \beta_{X17} \\ \beta_{X18} \\ \beta_{X19} \end{bmatrix}$$

$$\begin{bmatrix} Y_{G1} \\ Y_{G2} \\ Y_{G3} \\ \vdots \\ Y_{Gn} \end{bmatrix} = \begin{bmatrix} 1 & X_{L1} & Y_{L1} & Z_{L1} & X_{L1}^2 & Y_{L1}^2 & \cdots & Y_{L1}^2 Z_{L1} & X_{L1} Z_{L1}^2 & Y_{L1} Z_{L1}^2 \\ 1 & X_{L2} & Y_{L2} & Z_{L2} & X_{L2}^2 & Y_{L2}^2 & \cdots & Y_{L2}^2 Z_{L2} & X_{L2} Z_{L2}^2 & Y_{L2} Z_{L2}^2 \\ 1 & X_{L3} & Y_{L3} & Z_{L3} & X_{L3}^2 & Y_{L3}^2 & \cdots & Y_{L3}^2 Z_{L3} & X_{L3} Z_{L3}^2 & Y_{L3} Z_{L3}^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & X_{Ln} & Y_{Ln} & Z_{Ln} & X_{Ln}^2 & Y_{Ln}^2 & \cdots & Y_{Ln}^2 Z_{Ln} & X_{Ln} Z_{Ln}^2 & Y_{Ln} Z_{Ln}^2 \end{bmatrix} \begin{bmatrix} \beta_{Y0} \\ \beta_{Y1} \\ \beta_{Y2} \\ \beta_{Y3} \\ \beta_{Y4} \\ \vdots \\ \beta_{Y16} \\ \beta_{Y17} \\ \beta_{Y18} \\ \beta_{Y19} \end{bmatrix}$$

$$\begin{bmatrix} Z_{G1} \\ Z_{G2} \\ Z_{G3} \\ \vdots \\ Z_{Gn} \end{bmatrix} = \begin{bmatrix} 1 & X_{L1} & Y_{L1} & Z_{L1} & X_{L1}^2 & Y_{L1}^2 & \cdots & Y_{L1}^2 Z_{L1} & X_{L1} Z_{L1}^2 & Y_{L1} Z_{L1}^2 \\ 1 & X_{L2} & Y_{L2} & Z_{L2} & X_{L2}^2 & Y_{L2}^2 & \cdots & Y_{L2}^2 Z_{L2} & X_{L2} Z_{L2}^2 & Y_{L2} Z_{L2}^2 \\ 1 & X_{L3} & Y_{L3} & Z_{L3} & X_{L3}^2 & Y_{L3}^2 & \cdots & Y_{L3}^2 Z_{L3} & X_{L3} Z_{L3}^2 & Y_{L3} Z_{L3}^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & X_{Ln} & Y_{Ln} & Z_{Ln} & X_{Ln}^2 & Y_{Ln}^2 & \cdots & Y_{Ln}^2 Z_{Ln} & X_{Ln} Z_{Ln}^2 & Y_{Ln} Z_{Ln}^2 \end{bmatrix} \begin{bmatrix} \beta_{Z0} \\ \beta_{Z1} \\ \beta_{Z2} \\ \beta_{Z3} \\ \beta_{Z4} \\ \vdots \\ \beta_{Z16} \\ \beta_{Z17} \\ \beta_{Z18} \\ \beta_{Z19} \end{bmatrix}$$

The coordinates in both systems should also be processed as in the quadratic polynomial transformation before they are used for the parameter estimation: The original Cartesian coordinates in both systems should be scaled into the interval [-1,1]. Besides, all the scaled global coordinates in the Gauß-Markov model will be substituted with the difference values of the scaled global coordinates and local coordinates:

$$X_{Gc} = X_G - X_L$$
$$Y_{Gc} = Y_G - Y_L$$
$$Z_{Gc} = Z_G - Z_L$$

After the above mentioned process with the coordinates in both systems, the 60 transformation parameters can be then separately estimated with Least Squares Method.
The transformation parameters of the cubic polynomial and their deviation are listed bellow:

3D cubic polynomial transformation					
β_{X0}	635.2568	β_{Y0}	24.2847	β_{Z0}	449.4108
β_{X1}	-6.7250	β_{Y1}	-1.6466	β_{Z1}	10.2579
β_{X2}	-0.2529	β_{Y2}	0.7842	β_{Z2}	2.6461
β_{X3}	-6.9949	β_{Y3}	-1.4914	β_{Z3}	10.5275
β_{X4}	652.4432	β_{Y4}	-301.0340	β_{Z4}	-548.3575
β_{X5}	14.5154	β_{Y5}	-8.1300	β_{Z5}	-12.4114
β_{X6}	577.8018	β_{Y6}	-259.7650	β_{Z6}	-498.5000
β_{X7}	194.2975	β_{Y7}	-97.7464	β_{Z7}	-165.1389
β_{X8}	1227.9815	β_{Y8}	-559.4723	β_{Z8}	-1036.2760
β_{X9}	183.2154	β_{Y9}	-90.8292	β_{Z9}	-156.3493
β_{X10}	-25126.0230	β_{Y10}	-294.3041	β_{Z10}	21104.4602
β_{X11}	-77.7067	β_{Y11}	-7.9296	β_{Z11}	66.2855
β_{X12}	-19820.5606	β_{Y12}	-439.4980	β_{Z12}	16650.2692
β_{X13}	-20569.4211	β_{Y13}	-812.6765	β_{Z13}	17346.4087
β_{X14}	-11128.0518	β_{Y14}	-399.5948	β_{Z14}	9383.7360
β_{X15}	-69679.2754	β_{Y15}	-1052.0950	β_{Z15}	58530.5954
β_{X16}	-1622.5569	β_{Y16}	-105.6164	β_{Z16}	1375.4449
β_{X17}	-1499.1791	β_{Y17}	-103.3560	β_{Z17}	1270.8404
β_{X18}	-64382.8524	β_{Y18}	-1196.5861	β_{Z18}	54084.0128
β_{X19}	-9501.1087	β_{Y19}	-410.6011	β_{Z19}	8012.6912
				QMR(m)	0.0404
				$\widehat{\sigma}$(m)	0.0372

Table 3.6: *Transformation parameters of the 3D cubic polynomial and the deviation with scaled coordinates*

The horizontal residuals v_E and v_N and their directions of 121 collocated points and 10 interpolated points are showed in the figure bellow:

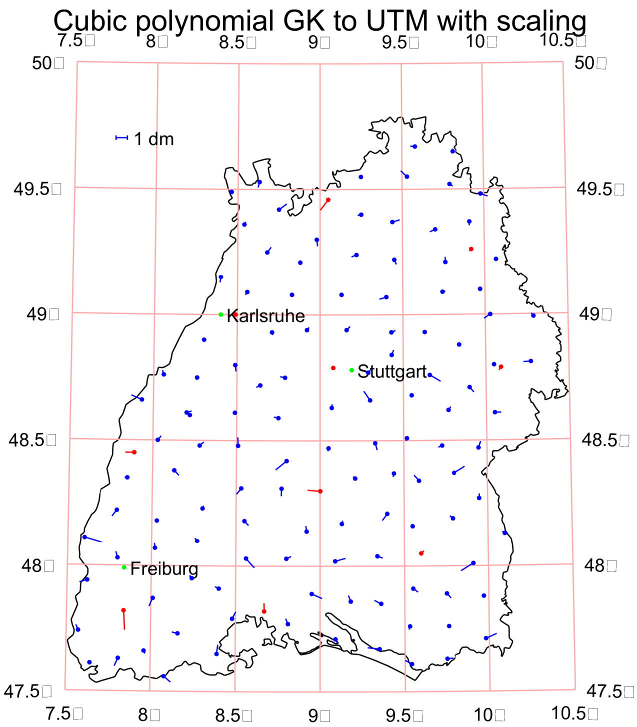

Figure 3.6: *Horizontal residuals of 121 collocated points and 10 interpolated points with cubic polynomial model*

Another interesting result appeared when non-scaled coordinates were used in the 3D cubic polynomial transformation. The following table shows the new transformation parameters estimated with the non-scaled coordinates:

3D cubic polynomial transformation					
β_{X0}	635.3826	β_{Y0}	24.3473	β_{Z0}	449.4603
β_{X1}	-33.4044	β_{Y1}	-68.3393	β_{Z1}	80.0037
β_{X2}	2.8350	β_{Y2}	-0.7801	β_{Z2}	24.4557
β_{X3}	-46.6997	β_{Y3}	-78.6524	β_{Z3}	103.1593
β_{X4}	10716.9027	β_{Y4}	-32938.7845	β_{Z4}	-8716.6676
β_{X5}	358.8223	β_{Y5}	-1034.8678	β_{Z5}	-303.2434
β_{X6}	16379.8228	β_{Y6}	-43860.1669	β_{Z6}	-13969.9047
β_{X7}	3818.6145	β_{Y7}	-11550.1617	β_{Z7}	-3230.7390
β_{X8}	26539.3318	β_{Y8}	-76037.4778	β_{Z8}	-22158.5807
β_{X9}	4738.0467	β_{Y9}	-13325.9157	β_{Z9}	-4092.0364
β_{X10}	-29059385.0660	β_{Y10}	-349034.6701	β_{Z10}	24406028.2524
β_{X11}	-99021.9057	β_{Y11}	-9895.8614	β_{Z11}	84425.5585
β_{X12}	-43405675.6628	β_{Y12}	-964189.7655	β_{Z12}	36459456.1321
β_{X13}	-30387801.9862	β_{Y13}	-1188508.3346	β_{Z13}	25621884.1114
β_{X14}	-13288487.5582	β_{Y14}	-473110.6336	β_{Z14}	11203577.7869
β_{X15}	-99696691.9877	β_{Y15}	-1526729.6104	β_{Z15}	83739052.5578
β_{X16}	-2001175.7613	β_{Y16}	-128121.6451	β_{Z16}	1695840.4059
β_{X17}	-2287481.8353	β_{Y17}	-155020.5551	β_{Z17}	1938479.1111
β_{X18}	-113965485.4097	β_{Y18}	-2132774.6134	β_{Z18}	95726697.2639
β_{X19}	-17364982.2879	β_{Y19}	-741888.4958	β_{Z19}	14642167.6910
				QMR(m)	0.0613
				$\widehat{\sigma}$(m)	0.1627

Table 3.7: *Transformation parameters of the 3D cubic polynomial and the deviation before scaling the input coordinates*

The horizontal residuals v_E and v_N and their directions of 121 collocated points and 10 interpolated points are showed in the figure bellow:

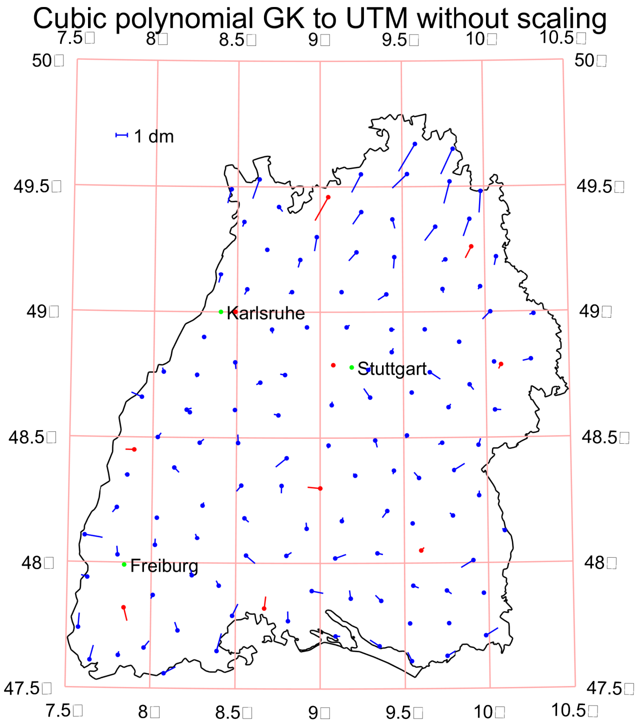

Figure 3.7: *Horizontal residuals of 121 collocated points and 10 interpolated points with cubic polynomial model*

Comparing the figures (3.6) and (3.7) we can see that the horizontal residuals of most points using the cubic polynomial transformation with non-scaled coordinates are bigger than using the cubic polynomial transforamtion with scaled coordinates. The horizontal residuals of the collocated points near the boundary of Baden-Wütemberg after coordinate transformation are especially huge.

Here a more clear numerical comparison of the 3D cubic polynomial transformation results with scaled and non-scaled collocated coordinates is presented in the table below:

Transformation model	Absolute mean of Residuals (m)		Max. absolute Residuals(m)		QMR (m)	Standard deviation $\hat{\sigma}$ (m)
	$[V_N]$	$[V_E]$	$[V_N]$	$[V_E]$		
with scaled coordinates	0.0318	0.0321	0.0820	0.1553	0.0404	0.0372
with non-scaled coordinates	0.0526	0.0369	0.2419	0.1621	0.0613	0.1627

Table 3.8: *Numerical comparison of cubic polynomial transformation results with 121 scaled and non-scaled coordinates*

This interesting result shows that the cubic polynomial coordinate transformation can get a better result with scaled collocated coordinates. In the next chapter a significance test for each polynomial term in the cubic polynomial transformation model with non-scaled collocated coordinates will be processed in order to check if the transformation results can be improved, when some polynomial terms with lower contributions to the transformation model are deleted.

3.4.4 Quadratic Polynomial Transformation with Legendre Polynomial

The transformation model with quadratic orthogonal polynomial is also similar with the quadratic polynomial model, except that the local coordinates X_L, Y_L, Z_L should be replaced with local coordinates in the form of Legendre polynomials.

$$\begin{bmatrix} X_{G1} \\ X_{G2} \\ X_{G3} \\ \vdots \\ X_{Gn} \end{bmatrix} = \begin{bmatrix} 1 & \cdots & \frac{1}{2}(3X_{L1}^2-1) & \frac{1}{2}(3Y_{L1}^2-1) & \frac{1}{2}(3Z_{L1}^2-1) & \cdots & Y_{L1}Z_{L1} \\ 1 & \cdots & \frac{1}{2}(3X_{L2}^2-1) & \frac{1}{2}(3Y_{L2}^2-1) & \frac{1}{2}(3Z_{L2}^2-1) & \cdots & Y_{L2}Z_{L2} \\ 1 & \cdots & \frac{1}{2}(3X_{L3}^2-1) & \frac{1}{2}(3Y_{L3}^2-1) & \frac{1}{2}(3Z_{L3}^2-1) & \cdots & Y_{L3}Z_{L3} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & \frac{1}{2}(3X_{Ln}^2-1) & \frac{1}{2}(3Y_{Ln}^2-1) & \frac{1}{2}(3Z_{Ln}^2-1) & \cdots & Y_{Ln}Z_{Ln} \end{bmatrix} \begin{bmatrix} \beta_{X0} \\ \beta_{X1} \\ \beta_{X2} \\ \beta_{X3} \\ \beta_{X4} \\ \beta_{X5} \\ \beta_{X6} \\ \beta_{X7} \\ \beta_{X8} \\ \beta_{X9} \end{bmatrix}$$

$$\begin{bmatrix} Y_{G1} \\ Y_{G2} \\ Y_{G3} \\ \vdots \\ Y_{Gn} \end{bmatrix} = \begin{bmatrix} 1 & \cdots & \frac{1}{2}(3X_{L1}^2-1) & \frac{1}{2}(3Y_{L1}^2-1) & \frac{1}{2}(3Z_{L1}^2-1) & \cdots & Y_{L1}Z_{L1} \\ 1 & \cdots & \frac{1}{2}(3X_{L2}^2-1) & \frac{1}{2}(3Y_{L2}^2-1) & \frac{1}{2}(3Z_{L2}^2-1) & \cdots & Y_{L2}Z_{L2} \\ 1 & \cdots & \frac{1}{2}(3X_{L3}^2-1) & \frac{1}{2}(3Y_{L3}^2-1) & \frac{1}{2}(3Z_{L3}^2-1) & \cdots & Y_{L3}Z_{L3} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & \frac{1}{2}(3X_{Ln}^2-1) & \frac{1}{2}(3Y_{Ln}^2-1) & \frac{1}{2}(3Z_{Ln}^2-1) & \cdots & Y_{Ln}Z_{Ln} \end{bmatrix} \begin{bmatrix} \beta_{Y0} \\ \beta_{Y1} \\ \beta_{Y2} \\ \beta_{Y3} \\ \beta_{Y4} \\ \beta_{Y5} \\ \beta_{Y6} \\ \beta_{Y7} \\ \beta_{Y8} \\ \beta_{Y9} \end{bmatrix}$$

$$\begin{bmatrix} Z_{G1} \\ Z_{G2} \\ Z_{G3} \\ \vdots \\ Z_{Gn} \end{bmatrix} = \begin{bmatrix} 1 & \cdots & \frac{1}{2}(3X_{L1}^2-1) & \frac{1}{2}(3Y_{L1}^2-1) & \frac{1}{2}(3Z_{L1}^2-1) & \cdots & Y_{L1}Z_{L1} \\ 1 & \cdots & \frac{1}{2}(3X_{L2}^2-1) & \frac{1}{2}(3Y_{L2}^2-1) & \frac{1}{2}(3Z_{L2}^2-1) & \cdots & Y_{L2}Z_{L2} \\ 1 & \cdots & \frac{1}{2}(3X_{L3}^2-1) & \frac{1}{2}(3Y_{L3}^2-1) & \frac{1}{2}(3Z_{L3}^2-1) & \cdots & Y_{L3}Z_{L3} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & \frac{1}{2}(3X_{Ln}^2-1) & \frac{1}{2}(3Y_{Ln}^2-1) & \frac{1}{2}(3Z_{Ln}^2-1) & \cdots & Y_{Ln}Z_{Ln} \end{bmatrix} \begin{bmatrix} \beta_{Z0} \\ \beta_{Z1} \\ \beta_{Z2} \\ \beta_{Z3} \\ \beta_{Z4} \\ \beta_{Z5} \\ \beta_{Z6} \\ \beta_{Z7} \\ \beta_{Z8} \\ \beta_{Z9} \end{bmatrix}$$

In order to use the orthogonality of the Legendre polynomials in the transformation models,

the local coordinates in $\boldsymbol{A}$ matrix should be scaled in the interval [-1,1].
The global coordinates should be substituted with the difference of global coordinates and local coordinates.
With the above mentioned two steps the transformation parameters can be then estimated with Least Squares Method.
The estimated transformation parameters in λ, ϕ components and their standard derivation with 121 collocated points are listed as follows:

transformation parameters of 3D quadratic polynomial with legendre polynomials					
β_{X0}	389.9172	β_{Y0}	80.9088	β_{Z0}	610.3381
β_{X1}	7.0016	β_{Y1}	-7.9253	β_{Z1}	1.5622
β_{X2}	1.8300	β_{Y2}	-0.1883	β_{Z2}	1.2965
β_{X3}	5.8459	β_{Y3}	-7.3017	β_{Z3}	2.3141
β_{X4}	-260.5624	β_{Y4}	61.0016	β_{Z4}	170.0201
β_{X5}	-6.0001	β_{Y5}	0.5965	β_{Z5}	3.9580
β_{X6}	-224.0452	β_{Y6}	51.5875	β_{Z6}	147.8569
β_{X7}	-119.3618	β_{Y7}	21.6543	β_{Z7}	77.8467
β_{X8}	-724.9742	β_{Y8}	168.2169	β_{Z8}	475.6438
β_{X9}	-110.3723	β_{Y9}	19.8538	β_{Z9}	72.3052
				QMR(m)	0.047751
				$\widehat{\sigma}$(m)	0.0412146

Table 3.9: *Transformation parameters of quadratic polynomial with legendre polynomials and the deviation*

The horizontal residuals v_E and v_N and their directions of 121 collocated points and 10 interpolated points are showed in the figure bellow:

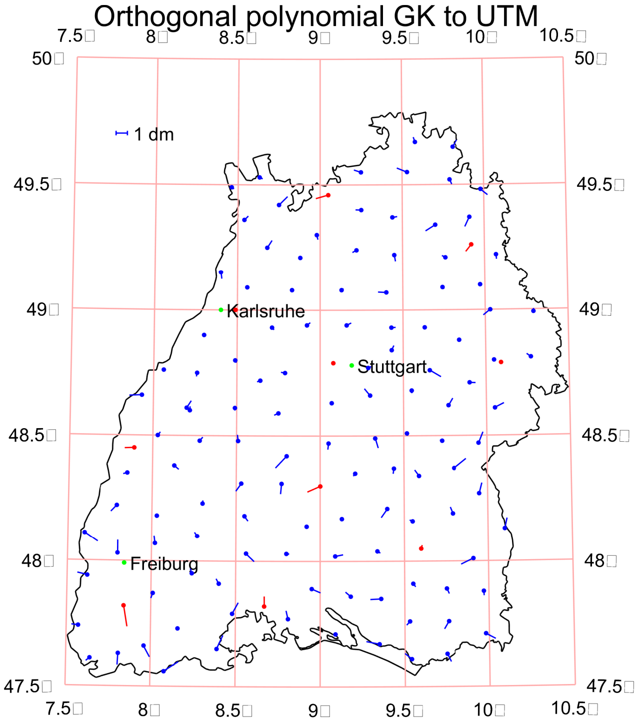

Figure 3.8: Horizontal residuals of 121 collocated points and 10 interpolated points with orthogonal polynomial model

3.5 Analysis of the Transformation Results

3.5.1 Analysis of the Results with 2D Transformation Models

The transformation results of the 6-parameter affine transformation and multiple linear regression with 121 collocated points are listed in the table bellow:

Transformation model	Absolute mean of Residuals (m) $[V_N]$	$[V_E]$	Max. absolute Residuals(m) $[V_N]$	$[V_E]$	QMR (m)	Standard deviation $\hat{\sigma}(m)$
6-Para Transformation	0.1053	0.0801	0.3348	0.3203	0.1192	0.1208
9th-order MREs	0.0342	0.0302	0.1122	0.1138	0.0402	0.0445
5th-order MREs	0.0477	0.0368	0.1942	0.1532	0.0554	0.0581

Table 3.10: Numerical comparison of the 6-parameter affine transformation and multiple linear regression with 121 control points

The transformation results of the 6-parameter affine transformation and multiple linear regression with 10 interpolated points are also showed in the table bellow:

Transformation model	Absolute mean of Residuals (m)		Max. absolute Residuals(m)		QMR (m)
	$[V_N]$	$[V_E]$	$[V_N]$	$[V_E]$	
6-Parameter transformation	0.0998	0.0842	0.1544	0.1983	0.1093
9th-order MREs	0.0384	0.0437	0.0944	0.1162	0.0534
5th-order MREs	0.0504	0.0431	0.1572	0.0984	0.0648

Table 3.11: *Numerical comparison of the 6-parameter affine transformation and multiple linear regression with 10 interpolated points*

From the table (3.10) and (3.11) we can see, the distortion and the deformation are well modelized with the multiple linear regression of 9th order and the quadratic mean of the residuals (QMR) of 121 collocated points are reduced from 11.92cm to 4.02cm while the QMR of 10 interpolated points are reduced from 10.93cm to 5.34cm, which are the significant improvements.

With the 5th order linear regression model the standard deviation ($\hat{\sigma}$) and the quadratic mean of the residuals (QMR) of 131 points are only 1cm-1.5cm worse than that with the 9th order linear regression model, and using the 5th order linear regression for a coordinate transformation is much more efficient than using the 9th order linear regression, because there are much less regression terms in the 5th order multiple regression model, which means we need less collocated points for the transformation.

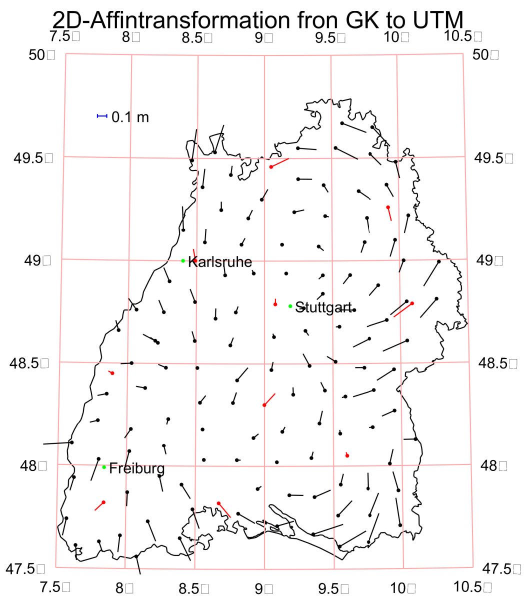

Figure 3.9: *Horizontal residuals of 121 collocated points and 10 interpolated points with 6-parameter transformation*

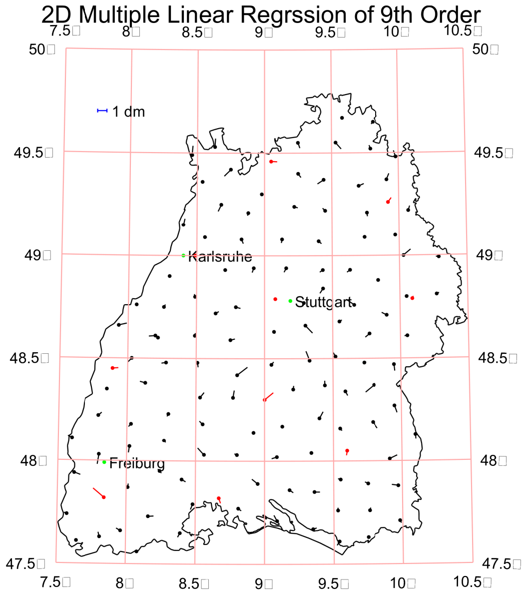

Figure 3.10: *Horizontal residuals of 121 collocated points and 10 interpolated points with 9th order multiple regression*

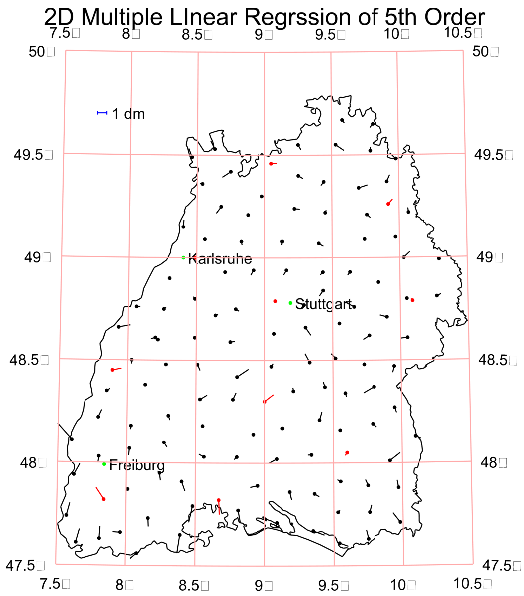

Figure 3.11: *Horizontal residuals of 121 collocated points and 10 interpolated points with 5th order multiple linear regression*

The mean and maximal value of the absolute residuals in northing and easting directions of 131 points are also listed in the table, in which 121 points are collocated points and 10 points are interpolated points. From the figure we can see that the horizontal residuals of 121 collocated points have been decreased for 8.67 cm and the horizontal residuals of 10 interpolated points have been reduced for 7.24 cm in average by using the 9th order multiple linear regression model instead of the 6-parameter transformation model. We can see the improvement in the figure (3.10) obviously.

3.5.2 Analysis of the Results with 3D Transformation Models

Comparison of 4 three dimensional transformations with 121 collocated points in the table bellow:

Transformation model	Absolute mean of Residuals (m)		Max. absolute Residuals(m)		QMR (m)	Standard deviation $\hat{\sigma}$ (m)
	$[V_N]$	$[V_E]$	$[V_N]$	$[V_E]$		
7-Parameter transformation	0.1063	0.0841	0.4164	0.3595	0.1253	0.1036
Quadratic polynomial	0.0387	0.0359	0.1207	0.1298	0.0478	0.0412
Cubic polynomial	0.0318	0.0321	0.0820	0.1553	0.0404	0.0372
Orthogonal polynomial	0.0387	0.0359	0.1207	0.1298	0.0477	0.0412

Table 3.12: *Numerical comparison of 4 three dimensional transformations with 121 control points*

The transformation results of 4 three dimensional transformations with 10 interpolated points are also showed in the table bellow:

Transformation model	Absolute mean of Residuals (m)		Max. absolute Residuals(m)		QMR (m)
	$[V_N]$	$[V_E]$	$[V_N]$	$[V_E]$	
7-Parameter transformation	0.0860	0.0949	0.1432	0.2473	0.1087
Quadratic polynomial	0.0477	0.0413	0.1865	0.1124	0.0657
Cubic polynomial	0.0425	0.0344	0.1733	0.1107	0.0594
orthogonal polynomial	0.0477	0.0413	0.1865	0.1124	0.0657

Table 3.13: *Numerical comparison of 4 three dimensional transformations with 10 interpolated points*

From the table(3.12) we can see that under the same input coordinates condition the best transformation result comes from the cubic polynomial transformation model. Theoretically the cubic polynomial transformation should indeed get the best transformation results, because the distortion and the deformation will be better fitted with higher polynomial degree and more terms.

When comparing the quadratic polynomial transformation results with and without orthogonal polynomials, we see that the standard deviations and quadratic means of the residuals of 121 collocated points with these two transformation models are almost the same. This shows that the Legendre polynomials make no big differences in the quadratic polynomial transformation model and in the determination of its inverse matrix of the design matrix. The reason for it can be that, the coordinates of 121 collocated points are not continuous in the interval [-1,1], therefore the condition for the orthogonality of the Legendre polynomials can not be satisfied.

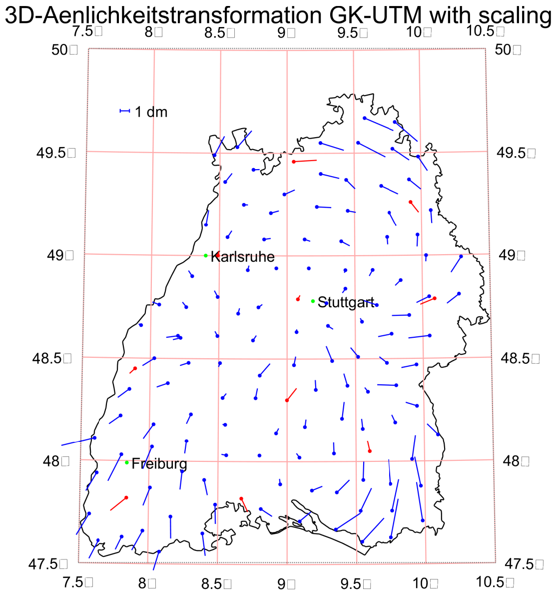

Figure 3.12: *Horizontal residuals of 121 collocated points and 10 interpolated points with 7-parameter transformation*

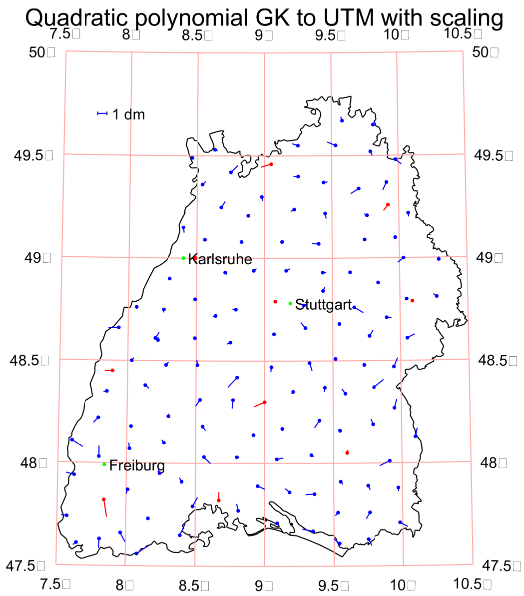

Figure 3.13: *Horizontal residuals of 121 collocated points and 10 interpolated points with quadratic polynomial model*

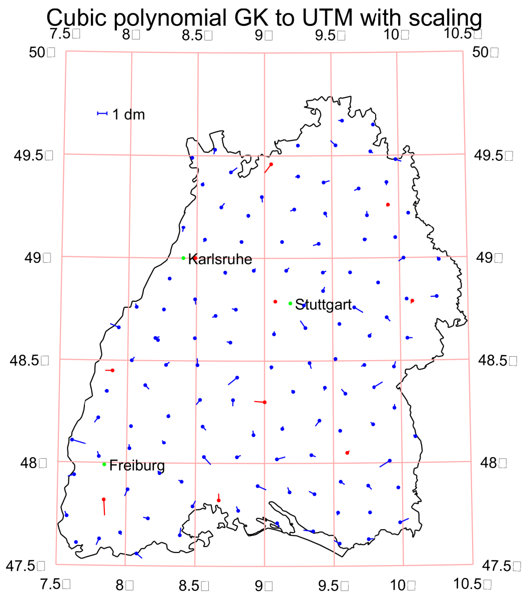

Figure 3.14: *Horizontal residuals of 121 collocated points and 10 interpolated points with cubic polynomial*

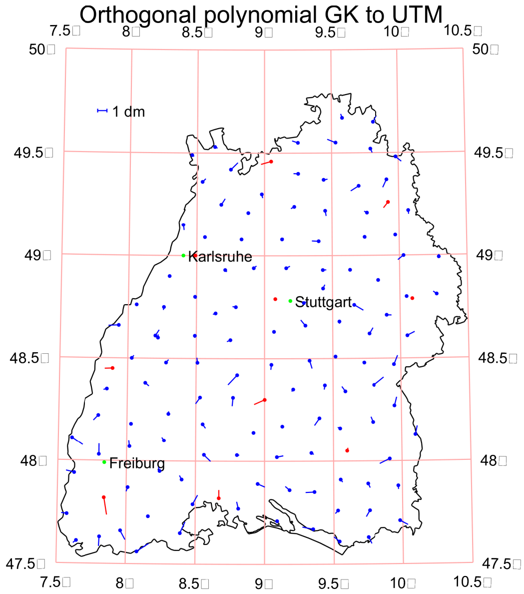

Figure 3.15: Horizontal residuals of 121 collocated points and 10 interpolated points with orthogonal polynomial model

And we can also directly see from the figure (3.12) and (3.14) that the horizontal residuals in northern and east directions at each collocated point are obviously decreased using 3D cubic polynomial transformation with scaled collocated coordinates compared with the 7-parameter transformation. Besides, the horizontal residuals of the points which lies in the boundary of Baden-Würtemberg are greater than those in the inner area.

Chapter 4

Selecting the Best Combination of the Polynomial Terms

4.1 Introduction

When fitting a multiple linear regression model, it is necessary to make a significance test for all the regression parameters.
For a basic regression model:

$$\boldsymbol{\omega} = \beta_0 + \beta_1 \boldsymbol{u}_1 + \beta_2 \boldsymbol{u}_2 + \cdots + \beta_m \boldsymbol{u}_m$$

can be also written as:

$$\boldsymbol{\omega}_{(n\times 1)} = \boldsymbol{U}_{(n\times m)} \boldsymbol{\beta}_{(m\times 1)}$$

Although the regression model consists of m+1 polynomial terms, but not every term is of the same importance in predicting the dependent variable $\boldsymbol{\omega}$. After eliminating the unimportant terms in the polynomial model with the significance test the variance of the noise can be reduced. In this chapter the polynomial models will be processed with the significance test(t test) to find the equation with a least number of polynomial terms and a minimum variance of the noise.

The process for selecting the best polynomial model can be realized as follows:

- determination of the estimated parameters $\widehat{\boldsymbol{\beta}}$: $\widehat{\boldsymbol{\beta}} = (\boldsymbol{U}^T \boldsymbol{P} \boldsymbol{U})^{-1} \boldsymbol{U}^T \boldsymbol{P} \boldsymbol{l}$ with $\boldsymbol{P}$ is the identity matrix
- determination of the residual $\widehat{\boldsymbol{e}}$: $\widehat{\boldsymbol{e}} = \boldsymbol{\omega} - \boldsymbol{U}\widehat{\boldsymbol{\beta}}$
- determination of the variance of the noise σ^2: $\sigma^2 = \frac{1}{n-m}(\widehat{\boldsymbol{e}}^T \cdot \widehat{\boldsymbol{e}})$
- estimation of the variance of $\widehat{\beta}_m$: $v\widehat{a}r(\widehat{\beta}_m) = C_m^2 \widehat{\sigma}^2$, where C_m^2 is the mth element on the diagonal of $(\boldsymbol{U}^T \boldsymbol{U})^{-1}$
- determination of the t test value: $t_m = \frac{\widehat{\beta}_m}{\sqrt{v\widehat{a}r(\widehat{\beta}_m)}}$
- find out the transformation parameter β_m with the minimum t_m

- deletion of the parameter β_m from the complete polynomial model and repetition of the process above until a polynomial model with the minimum variance of the noise σ^2 is found.

4.2 Processing with 2D Multiple Linear Regression of 9th Order

The in the section 3.3.2 performed 9th-order MREs are actually derived by NIMA for the transformation from European Datum 1950 to WGS 84, but here they are used for the transformation from Bessel ellipsoid 1841 to ETRS 89, so it is possible that the polynomial combination in the 9th-order MREs is not perfect for the transformation from Gaušs-krüger coordinates to UTM coordinates. A significance test is now necessary for all the polynomial terms to obtain the best combined 9th-order MREs.

There are 15 terms in ϕ component while 30 terms in λ component in the original 9th-order multiple linear regression. Because of the huge number the completely selecting processes with the t test value for each parameter in both components of the regression model will not be listed in a table here. Instead, a simple process of the elimination of parameters will be showed.

Eliminated parameters of the regression model in ϕ component in sequence:

$$A_{19} \rightarrow A_{30} \rightarrow A_{41} \rightarrow A_{20} \rightarrow A_{08}$$

Eliminated parameters of the regression model in λ component in sequence:

$$B_{14} \rightarrow B_{44} \rightarrow B_{70} \rightarrow B_{81} \rightarrow B_{40} \rightarrow B_{15} \rightarrow B_{57} \rightarrow B_{13} \rightarrow B_{23}$$

New 2D multiple linear regression transformation parameters			
A_{00}	-1.8086×10^{-5}	B_{00}	-1.7968×10^{-5}
A_{10}	-3.7371×10^{-5}	B_{10}	-5.6365×10^{-6}
A_{01}	2.6414×10^{-6}	B_{01}	-4.6941×10^{-5}
A_{20}	-	B_{20}	1.3210×10^{-5}
A_{11}	-1.2521×10^{-5}	B_{11}	-6.1845×10^{-6}
A_{30}	-	B_{02}	-9.7972×10^{-6}
A_{21}	1.3331×10^{-4}	B_{30}	-4.3601×10^{-4}
A_{40}	-9.7157×10^{-4}	B_{40}	-
A_{41}	-	B_{13}	-
A_{52}	-6.7494	B_{50}	0.3106
A_{08}	-	B_{23}	-
A_{90}	-179.2922	B_{14}	-
A_{27}	-2839.3710	B_{05}	-0.0381
A_{19}	-	B_{60}	3.4649
A_{39}	-6.8583×10^{-6}	B_{33}	-2.0298
		B_{24}	-1.0057
		B_{15}	-
		B_{70}	-
		B_{61}	12.8413
		B_{44}	-
		B_{81}	-
		B_{72}	-6114.4954
		B_{27}	-3225.4982
		B_{09}	2037.9727
		B_{46}	14818.4654
		B_{92}	1127920.6456
		B_{83}	-1802231.9764
		B_{57}	-
		B_{94}	320630360.4146
		B_{49}	191640450.9034
		$\widehat{\sigma}(m)$	0.0445

Table 4.1: *New transformation parameters of 2D multiple linear regression and the deviation*

From the table we can see that, 5 parameters in ϕ component are eliminated while 9 parameters in λ component are eliminated. The standard deviation decreases from 0.0445 m to 0.0433 m after polynomial selection.
The best 9th-order MREs is then:

$$\begin{aligned}\Delta\phi =& A_{00} + A_{10}U + A_{01}V + A_{11}UV + A_{21}U^2V + A_{40}U^4 + A_{52}U^5V^2 + \\ & A_{90}U^9 + A_{27}U^2V^7 + A_{39}U^3V^9\end{aligned} \tag{4.1}$$

$$\begin{aligned}\Delta\lambda = & B_{00} + B_{10}U + B_{01}V + B_{20}U^2 + B_{11}UV + B_{02}V^2 + B_{30}U^3 + B_{50}U^5 + B_{05}V^5 + \\ & B_{60}U^6 + B_{33}U^3V^3 + B_{24}U^2V^4 + B_{61}U^6V + B_{72}U^7V^2 + B_{27}U^2V^7 + B_{09}V^9 + \\ & B_{46}U^4V^6 + B_{92}U^9V^2 + B_{83}U^8V^3 + B_{94}U^9V^4 + B_{49}U^4V^9\end{aligned} \tag{4.2}$$

The transformation results of the 9th-order MREs before and after polynomial selection with 121 collocated points are listed in the table bellow:

Transformation model	Absolute mean of Residuals (m) $[V_N]$	$[V_E]$	Max. absolute Residuals(m) $[V_N]$	$[V_E]$	Standard deviation $\hat{\sigma}(m)$
before polynomial selection	0.0342	0.0302	0.1122	0.1138	0.0445
after polynomial selection	0.0344	0.0299	0.1120	0.1141	0.0433

Table 4.2: Numerical comparison of the 9th-order MREs before and after polynomial selection with 121 control points

The transformation results of the 9th-order MREs before and after polynomial selection with 10 interpolated points are also showed in the table bellow:

Transformation model	Absolute mean of Residuals (m) $[V_N]$	$[V_E]$	Max. absolute Residuals(m) $[V_N]$	$[V_E]$	QMR (m)
before polynomial selection	0.0384	0.0437	0.0944	0.1162	0.0534
after polynomial selection	0.0381	0.0418	0.0901	0.1051	0.0516

Table 4.3: Numerical comparison of the 9th-order MREs before and after polynomial selection with 10 interpolated points

From table (4.2) and (4.3) we can see that the standard deviation of 121 collocated points decreases from 0.0445 m to 0.0433 m after polynomial selection. The absolute mean of residuals of 10 interpolates points in north and east direction reduce by about a few millimeters, and the QMR of 10 interpolated points also decreases from 0.0534 m to 0.0516 m.

The horizontal residuals v_E and v_N and their directions of 121 collocated points and 10 interpolated points before and after selecting the best polynomial degree and terms are showed in the figure as follows:

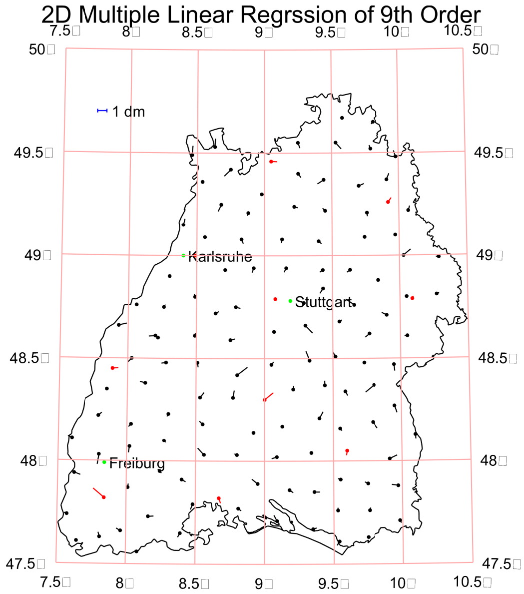

Figure 4.1: *Horizontal residuals of 121 collocated points and 10 interpolated points before selecting the best polynomial*

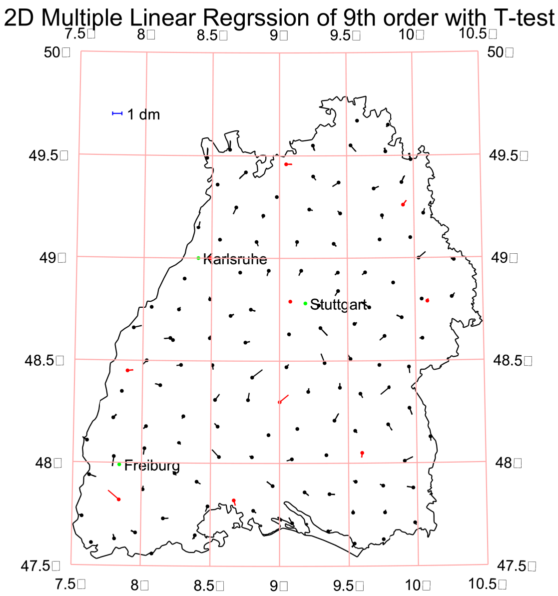

Figure 4.2: *Horizontal residuals of 121 collocated points and 10 interpolated points after selecting the best polynomial*

4.3 Processing with 3D Quadratic Polynomial transformation

Here the 3D quadratic polynomial model with scaled input coordinates is tested. The process of selecting the best 3D quadratic polynomial is implemented in 3 components. The processes in X-,Y-,Z-component are showed seperately in 3 table bellow:

t test value for each parameter in each turn		
	1. Turn	2. Turn
β_{X0}	40103.8644	84331.8675
β_{X1}	4.4344	76.7415
β_{X2}	7.6656	100.2980
β_{X3}	3.9577	-
β_{X4}	-5.9722	-4.5893
β_{X5}	-6.0808	-4.8228
β_{X6}	-5.9830	-4.6157
β_{X7}	-6.0573	-4.7211
β_{X8}	-5.9780	-4.6023
β_{X9}	-6.0468	-4.7180
[vv]$(m)^2$	0.1699	0.1939
$\widehat{\sigma}_X(m)$	0.03913	0.0461

Table 4.4: *selecting the best quadratic polynomial in X-component*

From the table we can see, when no parameters are eliminated, the quadratic polynomial in X-component gets the minimum $\widehat{\sigma}_X$.
The best quadratic polynomial in X-component is then:

$$\begin{aligned} X_G =& \beta_{X0} + \beta_{X1}X_L + \beta_{X2}Y_L + \beta_{X3}Z_L + \beta_{X4}X_L^2 + \beta_{X5}Y_L^2 + \beta_{X6}Z_L^2 + \\ & \beta_{X7}(X_LY_L) + \beta_{X8}(X_LZ_L) + \beta_{X9}(Y_LZ_L) \end{aligned} \tag{4.3}$$

Similar processes are implemented in Y- and Z-components.
The process of selecting the best polynomial in Y-component:

t test value for each parameter in each turn				
	1. Turn	2. Turn	3. Turn	4.Turn
β_{Y0}	1254.3460	1275.6417	2888.5443	2571.1504
β_{Y1}	-4.1013	-4.1582	-107.4954	-96.5810
β_{Y2}	-0.6446	-0.5247	-	-
β_{Y3}	-4.0391	-4.0929	-100.9582	-91.1689
β_{Y4}	1.1424	5.6855	6.4458	4.3840
β_{Y5}	0.4939	-	-	-
β_{Y6}	1.1256	5.7828	5.9015	-
β_{Y7}	0.8979	6.5413	7.8454	12.2027
β_{Y8}	1.1334	5.5238	6.1083	4.2764
β_{Y9}	0.8887	6.2030	7.0253	11.5938
$[vv](m)^2$	0.2546	0.2551	0.2557	0.3314
$\widehat{\sigma}_Y(m)$	0.0479	0.0477	0.0476	0.0539

Table 4.5: *selecting the best quadratic polynomial in Y-component*

From the table we can see, when only the parameter β_{Y2} and β_{Y5} is eliminated, which means that the quadratic polynomial in Y-component gets the minimum $\widehat{\sigma}_Y$ when the two terms Y_L and Y_L^2 are eliminated.
The best quadratic polynomial in Y-component is then:

$$\begin{aligned} Y_G = &\beta_{Y0} + \beta_{Y1}X_L + \beta_{Y3}Z_L + \beta_{Y4}X_L^2 + \beta_{Y6}Z_L^2 + \beta_{Y7}(X_LY_L) + \\ &\beta_{Y8}(X_LZ_L) + \beta_{Y9}(Y_LZ_L) \end{aligned} \tag{4.4}$$

The process of selecting the best polynomial in Z-component:

t test value for each parameter in each turn		
	1. Turn	2. Turn
β_{Z0}	31133.3334	69585.1610
β_{Z1}	1.0856	-
β_{Z2}	5.9589	146.2114
β_{Z3}	1.7190	108.3340
β_{Z4}	4.2760	4.9888
β_{Z5}	4.4014	4.9519
β_{Z6}	4.3225	5.0260
β_{Z7}	4.3347	4.9834
β_{Z8}	4.3036	5.0076
β_{Z9}	4.3466	4.9860
$[vv](m)^2$	0.1412	0.1426
$\widehat{\sigma}_Z(m)$	0.03565	0.037569

Table 4.6: *selecting the best quadratic polynomial in Z-component*

The quadratic polynomial in Z-component gets the minimum $\widehat{\sigma}_Z$ only when the original polynomial model in Z-component is completely kept.
The best quadratic polynomial in Z-component is then:

$$\begin{aligned} Z_G =&\beta_{Z0} + \beta_{Z1}X_L + \beta_{Z2}Y_L + \beta_{Z3}Z_L + \beta_{Z4}X_L^2 + \beta_{Z5}Y_L^2 + \beta_{Z6}Z_L^2 + \\ &\beta_{Z7}(X_LY_L) + \beta_{Z8}(X_LZ_L) + \beta_{Z9}(Y_LZ_L) \end{aligned} \tag{4.5}$$

The transformation parameters should be recalculated with the best selected polynomial model, which consists of the equations (4.3), (4.4) and (4.5).

The new parameters are listed bellow:

3D quadratic polynomial transformation with the best selected polynomial model					
β_{X0}	635.2211	β_{Y0}	24.3055	β_{Z0}	449.4206
β_{X1}	7.0016	β_{Y1}	-6.67755	β_{Z1}	1.5622
β_{X2}	1.8300	β_{Y2}	-	β_{Z2}	1.2965
β_{X3}	5.8459	β_{Y3}	-6.1324	β_{Z3}	2.3141
β_{X4}	-390.8437	β_{Y4}	49.5957	β_{Z4}	255.0302
β_{X5}	-9.0002	β_{Y5}	-	β_{Z5}	5.9370
β_{X6}	-336.0679	β_{Y6}	41.6983	β_{Z6}	221.7853
β_{X7}	-119.3618	β_{Y7}	9.2911	β_{Z7}	77.8467
β_{X8}	-724.9722	β_{Y8}	90.8435	β_{Z8}	475.6438
β_{X9}	-110.3723	β_{Y9}	8.4437	β_{Z9}	72.3052
				$\widehat{\sigma}$(m)	0.0411

Table 4.7: *New transformation parameters of the 3D quadratic polynomial and the deviation*

The transformation results of the quadratic polynomial before and after polynomial selection with 121 collocated points are listed in the table bellow:

Transformation model	Absolute mean of Residuals (m) $[V_N]$	$[V_E]$	Max. absolute Residuals(m) $[V_N]$	$[V_E]$	Standard deviation $\widehat{\sigma}(m)$
before polynomial selection	0.0387	0.0359	0.1207	0.1298	0.0412
after polynomial selection	0.0386	0.0359	0.1205	0.1251	0.0411

Table 4.8: *Numerical comparison of the quadratic polynomial before and after polynomial selection with 121 control points*

The transformation results of the quadratic polynomial before and after polynomial selection with 10 interpolated points are also showed in the table bellow:

Transformation model	Absolute mean of Residuals (m) $[V_N]$	$[V_E]$	Max. absolute Residuals(m) $[V_N]$	$[V_E]$	QMR (m)
before polynomial selection	0.0477	0.0413	0.1865	0.1124	0.0657
after polynomial selection	0.0478	0.0420	0.1880	0.1152	0.0659

Table 4.9: *Numerical comparison of the quadratic polynomial before and after polynomial selection with 10 interpolated points*

Compared with the old quadratic polynomial model, the standard deviation of 121 collocated points decreases from 4.12 cm to 4.11 cm. In contrast, the QMR of 10 interpolated points increases from 0.0657 m to 0.0659 m. Although a obvious reduce of the standard deviation does not appear after the selection of the best polynomial model, it provides an idea to improve the quality of the polynomial model.

The horizontal residuals v_E and v_N and their directions of 121 collocated points and 10 interpolated points before and after selecting the best polynomial degree and terms are showed in the figure as follows:

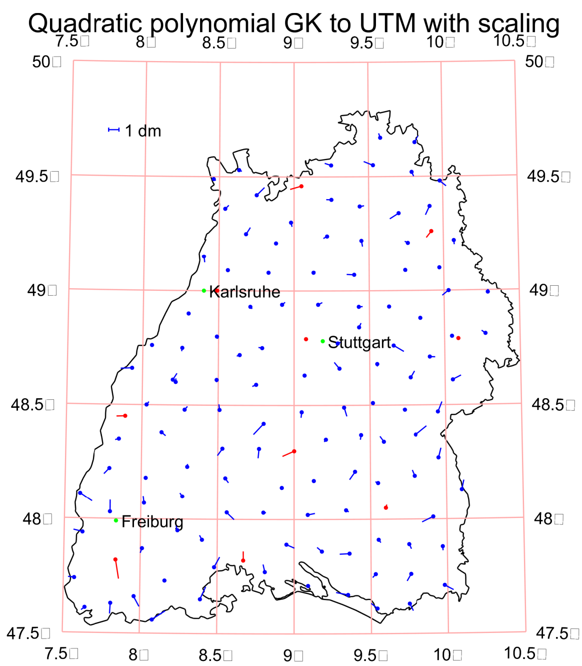

Figure 4.3: *Horizontal residuals of 121 collocated points and 10 interpolated points before selecting the best polynomial*

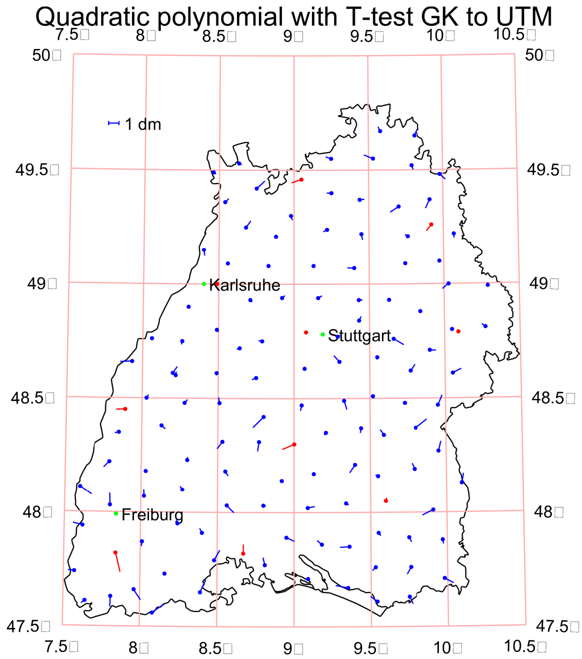

Figure 4.4: *Horizontal residuals of 121 collocated points and 10 interpolated points after selecting the best polynomial*

4.4 Processing with 3D Cubic Polynomial Transformation Model

Here the 3D cubic polynomial model with non-scaled input coordinates is tested. The process of selecting the best combination of the cubic polynomial terms is also implemented in X-, Y-, Z-components separately. As proceed in the quadratic polynomial, the selecting results are shown in the following tables stepwise.

t test value for each parameter in each turn			
	1. Turn	2. Turn	3. Turn
β_{X0}	9.6353×10^3	3.7598×10^3	3.9467×10^3
β_{X1}	-0.2472	-1.0181	-20.2141
β_{X2}	0.1319	0.4767	-
β_{X3}	-0.3003	-1.2277	-22.3191
β_{X4}	0.0572	-	-
β_{X5}	0.0755	0.8520	0.7026
β_{X6}	0.0666	2.0078	1.9363
β_{X7}	0.0640	0.6537	0.4791
β_{X8}	0.0618	1.9919	1.9217
β_{X9}	0.0693	1.1167	1.0032
β_{X10}	-0.4305	-2.0675	-2.0442
β_{X11}	-0.3921	-1.9718	-1.9644
β_{X12}	-0.4262	-2.0548	-2.0317
β_{X13}	-0.4216	-2.0507	-2.0329
β_{X14}	-0.4229	-2.0545	-2.0366
β_{X15}	-0.4292	-2.0636	-2.0404
β_{X16}	-0.4103	-2.0231	-2.0106
β_{X17}	-0.4089	-2.0195	-2.0070
β_{X18}	-0.4278	-2.0594	-2.0362
β_{X19}	-0.4201	-2.0466	-2.0289
$[vv](m)^2$	5.2058	0.3509	0.3621
$\widehat{\sigma}_X(m)$	0.2270	0.0587	0.0593

Table 4.10: *selecting the best cubic polynomial in X-component*

From the selecting result in X-component, we can see obviously that a minimum $\widehat{\sigma}_X$ can be achieved only when the transformation parameter β_{X4} (term X_L^2) is deleted. The best combination of the cubic polynomial in X-component is then:

$$\begin{aligned} X_G =& \beta_{X0} + \beta_{X1}X_L + \beta_{X2}Y_L + \beta_{X3}Z_L + \beta_{X5}Y_L^2 + \beta_{X6}Z_L^2 + \beta_{X7}(X_LY_L) + \beta_{X8}(X_LZ_L) + \\ & \beta_{X9}(Y_LZ_L) + \beta_{X10}X_L^3 + \beta_{X11}Y_L^3 + \beta_{X12}Z_L^3 + \beta_{X14}(X_LY_LZ_L) + \beta_{X14}(X_L^2Y_L) + \\ & \beta_{X15}(X_L^2Z_L) + \beta_{X16}(X_LY_L^2) + \beta_{X17}(Y_L^2Z_L) + \beta_{X18}(X_LZ_L^2) + \beta_{X19}(Y_LZ_L^2) \end{aligned} \tag{4.6}$$

The process of selecting the best combination of the cubic polynomial in Y-component is shown in the following table:

t test value for each parameter in each turn						
	1. Turn	2. Turn	3. Turn	$\cdots$	7. Turn	8. Turn
β_{Y0}	1.7665×10^3	1.8634×10^3	2.0321×10^3	$\cdots$	2.4672×10^3	2.4633×10^3
β_{Y1}	-2.4200	-2.4780	-32.1361	$\cdots$	-38.0724	-69.8960
β_{Y2}	-0.1737	-0.1756	-	$\cdots$	-	-
β_{Y3}	-2.4195	-2.4774	-31.0992	$\cdots$	-37.5293	-67.0638
β_{Y4}	-0.8418	-0.9832	-0.9721	$\cdots$	-	-
β_{Y5}	-1.0411	-1.1978	-1.1933	$\cdots$	-3.2716	-3.7302
β_{Y6}	-0.8527	-0.9921	-0.9811	$\cdots$	-	-
β_{Y7}	-0.9265	-1.0745	-1.0655	$\cdots$	-2.6406	-2.6172
β_{Y8}	-0.8475	-0.9880	-0.9769	$\cdots$	-3.3462	-3.4069
β_{Y9}	-0.9323	-1.0793	-1.0704	$\cdots$	-	-
β_{Y10}	-0.0247	-	-	$\cdots$	-2.6450	-2.6233
β_{Y11}	-0.1785	-1.2978	-1.3046	$\cdots$	-1.3565	-
β_{Y12}	-0.0453	-1.2396	-1.2372	$\cdots$	-1.8268	-1.7445
β_{Y13}	-0.0789	-1.3634	-1.3811	$\cdots$	-	-
β_{Y14}	-0.0720	-1.1542	-1.1574	$\cdots$	-1.9202	-1.8248
β_{Y15}	-0.0314	-1.2319	-1.2301	$\cdots$	-1.8478	-1.7663
β_{Y16}	-0.1257	-1.2192	-1.2238	$\cdots$	-	-
β_{Y17}	-0.1326	-1.3260	-1.3376	$\cdots$	-1.9793	-1.8745
β_{Y18}	-0.0383	-1.2362	-1.2341	$\cdots$	-1.8376	-1.7557
β_{Y19}	-0.0859	-1.5558	-1.5881	$\cdots$	-1.9095	-1.8118
$[vv](m)^2$	0.2274	0.2207	0.2208	$\cdots$	0.2266	0.2305
$\widehat{\sigma}_Y(m)$	0.0475	0.0465	0.0463	$\cdots$	0.0460	0.0462

Table 4.11: *selecting the best cubic polynomial in Y-component*

There are totally 8 turns for selecting the best polynomial combination in Y-component. The 4th, 5th and 6th turns in the table above are omitted. The final result shows that β_{Y2} (term Y_L), β_{Y4} (term X_L^2), β_{Y6} (term Z_L^2), β_{Y9} (term $Y_L Z_L$), β_{Y13} (term $X_L Y_L Z_L$) and β_{Y16} (term $X_L Y_L^2$) should be removed from the polynomial in Y-component, so that the new polynomial combination can get the most accurate transformation result.

The final polynomial model in Y-component is then:

$$\begin{aligned} Y_G = & \beta_{Y0} + \beta_{Y1} X_L + \beta_{Y3} Z_L + \beta_{Y5} Y_L^2 + \beta_{Y7}(X_L Y_L) + \beta_{Y8}(X_L Z_L) + \beta_{Y10} X_L^3 + \beta_{Y11} Y_L^3 + \\ & \beta_{Y12} Z_L^3 + \beta_{Y14}(X_L^2 Y_L) + \beta_{Y15}(X_L^2 Z_L) + \beta_{Y17}(Y_L^2 Z_L) + \beta_{Y18}(X_L Z_L^2) + \beta_{Y19}(Y_L Z_L^2) \end{aligned} \tag{4.7}$$

The process of selecting the best combination of the cubic polynomial in Z-component is shown in the following table:

t test value for each parameter in each turn						
	1. Turn	2. Turn	3. Turn	···	6. Turn	7. Turn
β_{Z0}	9.6682×10^3	1.9202×10^3	3.9027×10^4	···	9.3832×10^3	9.3383×10^3
β_{Z1}	0.8399	1.7333	3.7617	···	5.5146	5.9166
β_{Z2}	1.6142	3.2922	6.8875	···	10.5998	11.1021
β_{Z3}	0.9408	1.9369	4.1815	···	6.1770	6.6117
β_{Z4}	-0.0660	-	-	···	-	-
β_{Z5}	-0.0904	0.5643	-	···	-	-
β_{Z6}	-0.0805	0.2230	2.9836	···	3.2933	3.0373
β_{Z7}	-0.0868	0.6106	3.2574	···	3.9153	3.8245
β_{Z8}	-0.0732	0.3963	2.9836	···	3.2701	3.0066
β_{Z9}	-0.0849	0.2457	0.7890	···	-	-
β_{Z10}	0.5129	1.2429	2.2662	···	1.5173	-
β_{Z11}	0.4742	1.2047	2.1871	···	-	-
β_{Z12}	0.5078	1.2355	2.2532	···	-	-
β_{Z13}	0.5042	1.2378	2.2558	···	5.0577	6.1575
β_{Z14}	0.5058	1.2400	2.2596	···	3.5467	6.1504
β_{Z15}	0.5114	1.2407	2.2623	···	1.5573	4.2157
β_{Z16}	0.4932	1.2277	2.2342	···	5.2668	6.4353
β_{Z17}	0.4916	1.2255	2.2303	···	5.6873	6.1249
β_{Z18}	0.5097	1.2382	2.2580	···	1.5999	4.3181
β_{Z19}	0.5025	1.2354	2.2516	···	5.5242	5.9775
$[vv](m)^2$	2.5873	0.6812	0.2501	···	0.0905	0.0924
$\widehat{\sigma}_Z(m)$	0.1601	0.0817	0.0446	···	0.0292	0.0294

Table 4.12: *selecting the best cubic polynomial in Z-component*

There are totally 7 turns for selecting the best polynomial combination in Z-component. The 5th and 6th turns in the table above are omitted. The final result shows that β_{Z4} (term X_L^2), β_{Z5} (term Y_L^2), β_{Z9} (term $Y_L Z_L$), β_{Z11} (term Y_L^3) and β_{Z12} (term Z_L^3)should be removed from the polynomial in Y-component, so that the new polynomial combination can get the most accurate transformation result.

The final polynomial model in Z-component is then:

$$\begin{aligned} Z_G = &\beta_{Z0} + \beta_{Z1} X_L + \beta_{Z2} Y_L + \beta_{Z3} Z_L + \beta_{Z6} Z_L^2 + \beta_{Z7}(X_L Y_L) + \beta_{Z8}(X_L Z_L) + \beta_{Z10} X_L^3 + \beta_{Z13}(X_L Y_L Z_L) + \\ &\beta_{Z14}(X_L^2 Y_L) + \beta_{Z15}(X_L^2 Z_L) + \beta_{Z16}(X_L Y_L^2) + \beta_{Z17}(Y_L^2 Z_L) + \beta_{Z18}(X_L Z_L^2) + \beta_{Z19}(Y_L Z_L^2) \end{aligned} \tag{4.8}$$

The transformation parameters should be recalculated with the selected polynomial models in X-, Y- and Z, which mean the equations (4.6),(4.7) and (4.8).

The new parameters after selections are listed below:

new 3D cubic polynomial transformation parameters after selection					
β_{X0}	635.3832	β_{Y0}	24.3398	β_{Z0}	449.4476
β_{X1}	-34.8828	β_{Y1}	-64.5069	β_{Z1}	80.3611
β_{X2}	2.5973	β_{Y2}	-	β_{Z2}	24.6409
β_{X3}	-48.4141	β_{Y3}	-74.2448	β_{Z3}	103.6351
β_{X4}	-	β_{Y4}	-	β_{Z4}	-
β_{X5}	87.4352	β_{Y5}	-154.8895	β_{Z5}	-
β_{X6}	2292.6708	β_{Y6}	-	β_{Z6}	1228.8510
β_{X7}	406.2981	β_{Y7}	-795.2031	β_{Z7}	234.0361
β_{X8}	1965.7152	β_{Y8}	-22.8728	β_{Z8}	1408.9624
β_{X9}	825.6772	β_{Y9}	-	β_{Z9}	-
β_{X10}	-30995292.8737	β_{Y10}	-919.7570	β_{Z10}	1788594.5460
β_{X11}	-106928.1274	β_{Y11}	-94.7880	β_{Z11}	-
β_{X12}	-46351257.5207	β_{Y12}	-417702.7167	β_{Z12}	-
β_{X13}	-323512504.5345	β_{Y13}	-	β_{Z13}	230418.9193
β_{X14}	-14212313.6566	β_{Y14}	-102979.2492	β_{Z14}	127553.8801
β_{X15}	-106375962.0016	β_{Y15}	-321058.2675	β_{Z15}	421827.5601
β_{X16}	-2148821.1662	β_{Y16}	-	β_{Z16}	16335.1952
β_{X17}	-2457235.1536	β_{Y17}	-8140.2991	β_{Z17}	13572.2065
β_{X18}	-121646610.2750	β_{Y18}	-732703.3418	β_{Z18}	249349.1309
β_{X19}	-18586785.4765	β_{Y19}	-117457.3961	β_{Z19}	94400.2077
				$\widehat{\sigma}$(m)	0.0460

Table 4.13: *New transformation parameters of the 3D cubic polynomial and the deviation*

In chapter 3 we have discussed the transformation results of the quadratic polynomial model, the orthogonal quadratic polynomial model and the cubic polynomial model. The transformation accuracy of the cubic polynomial model is much worse than that of the quadratic polynomial model and the orthogonal quadratic polynomial model. The standard deviation of the two quadratic polynomial models is 4.12 cm, while the cubic polynomial model has a standard deviation of 16.27 cm.

From the table we can see that, the transformation accuracy of the cubic polynomial model has a great improvement after selecting the polynomial terms. Compared with the original cubic polynomial model, the standard deviation decreases from 16.27 cm to 4.60 cm after deleting 1 term in X-component, 6 terms in Y-component and 5 terms in Z-component. Compared with the quadratic polynomial and the orthogonal polynomial model after selecting the polynomial terms, which both have a standard deviation of 4.11 cm, we are able to say that the accuracy of the new cubic polynomial model is now at the same level.

The transformation results of the cubic polynomial before and after polynomial selection with 121 collocated points are listed in the table bellow:

Transformation model	Absolute mean of Residuals (m)		Max. absolute Residuals(m)		Standard deviation $\widehat{\sigma}(m)$
	$[V_N]$	$[V_E]$	$[V_N]$	$[V_E]$	
before polynomial selection	0.0526	0.0369	0.2419	0.1621	0.1627
after polynomial selection	0.0405	0.0331	0.1454	0.1585	0.0460

Table 4.14: *Numerical comparison of the cubic polynomial before and after polynomial selection with 121 control points*

The transformation results of the cubic polynomial before and after polynomial selection with 10 interpolated points are also showed in the table bellow:

Transformation model	Absolute mean of Residuals (m)		Max. absolute Residuals(m)		QMR (m)
	$[V_N]$	$[V_E]$	$[V_N]$	$[V_E]$	
before polynomial selection	0.0637	0.0471	0.2085	0.1176	0.0722
after polynomial selection	0.0296	0.0327	0.1664	0.1053	0.0511

Table 4.15: *Numerical comparison of the cubic polynomial before and after polynomial selection with 10 interpolated points*

Compared with the old quadratic polynomial model, the absolute mean of residuals of 10 interpolates points in north and east directions both decreases by several centimeter and the QMR of 10 interpolated points also decreases from 7.22 cm to 5.11 cm. As a result, the transformation accuracy of the cubic polynomial model has a great improvement after the polynomial selection.

The horizontal residuals v_E and v_N and their directions of 121 collocated points and 10 interpolated points before and after selecting the best polynomial degree and terms are showed in the figure as follows:

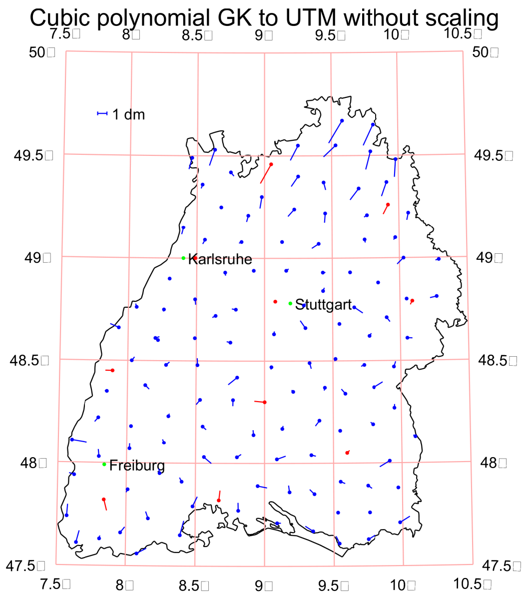

Figure 4.5: *Horizontal residuals of 121 collocated points and 10 interpolated points before selecting the best polynomial*

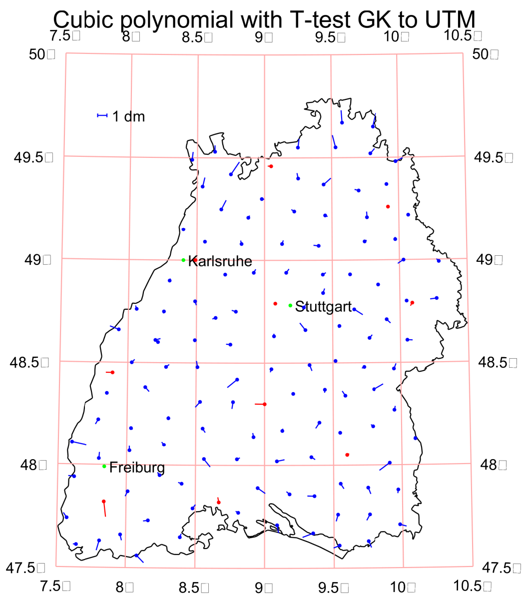

Figure 4.6: *Horizontal residuals of 121 collocated points and 10 interpolated points after selecting the best polynomial*

4.5 Processing with 3D Orthogonal Polynomial Transformation Model

With the reference of the previous section, the selecting process in orthogonal polynomial can be simply completed.
The final selecting result of the orthogonal polynomial terms s is showed as follows:

new transformation parameters of 3D quadratic polynomial with legendre polynomials					
β_{X0}	389.9172	β_{Y0}	54.7368	β_{Z0}	610.3381
β_{X1}	7.0016	β_{Y1}	-6.6755	β_{Z1}	1.5622
β_{X2}	1.8300	β_{Y2}	-	β_{Z2}	1.2965
β_{X3}	5.8459	β_{Y3}	-6.1324	β_{Z3}	2.3141
β_{X4}	-260.5624	β_{Y4}	33.0638	β_{Z4}	170.0201
β_{X5}	-6.0001	β_{Y5}	-	β_{Z5}	3.9580
β_{X6}	-224.0452	β_{Y6}	27.7988	β_{Z6}	147.8569
β_{X7}	-119.3618	β_{Y7}	9.2911	β_{Z7}	77.8467
β_{X8}	-724.9742	β_{Y8}	90.8435	β_{Z8}	475.6438
β_{X9}	-110.3723	β_{Y9}	8.4437	β_{Z9}	72.3052
				$\widehat{\sigma}$(m)	0.04113

Table 4.16: *New transformation parameters of quadratic polynomial with legendre polynomials and the deviation*

The table shows that there are no eliminations in the polynomial models in X- and Z- components. In Y-component two parameters β_{Y2} and β_{Y5} are eliminated.Compared with the old orthogonal polynomial model, the standard deviation decreases from 4.12 cm to 4.11 cm.
The new polynomial model in Y-component is:

$$\begin{aligned} Y_G =& \beta_{Y0} + \beta_{Y1}X_L + \beta_{Y3}Z_L + \beta_{Y4}(\frac{1}{2}(3X_L^2 - 1)) + \beta_{Y6}(\frac{1}{2}(3Z_L^2 - 1)) + \\ & \beta_{Y7}(X_L Y_L) + \beta_{Y8}(X_L Z_L) + \beta_{Y9}(Y_L Z_L) \end{aligned} \tag{4.9}$$

The transformation results of the orthogonal polynomial before and after polynomial selection with 121 collocated points are listed in the table bellow:

Transformation model	Absolute mean of Residuals (m)		Max. absolute Residuals(m)		Standard deviation $\widehat{\sigma}(m)$
	$[V_N]$	$[V_E]$	$[V_N]$	$[V_E]$	
before polynomial selection	0.0387	0.0359	0.1207	0.1298	0.0412
after polynomial selection	0.0386	0.0420	0.1205	0.1251	0.0411

Table 4.17: *Numerical comparison of the orthogonal polynomial before and after polynomial selection with 121 control points*

The transformation results of the orthogonal polynomial before and after polynomial selection with 10 interpolated points are also showed in the table bellow:

Transformation model	Absolute mean of Residuals (m) $[V_N]$	$[V_E]$	Max. absolute Residuals(m) $[V_N]$	$[V_E]$	QMR (m)
before polynomial selection	0.0477	0.0413	0.1865	0.1124	0.0657
after polynomial selection	0.0474	0.0421	0.1865	0.1138	0.0659

Table 4.18: *Numerical comparison of the orthogonal polynomial before and after polynomial selection with 10 interpolated points*

We can see from table (4.17) and (4.18), the standard deviation of 121 collocated points decreases from 4.12 cm to 4.11 cm. In contrast, the QMR of 10 interpolated points increases from 6.57 cm to 6.59 cm, and the absolute mean of residuals of 10 interpolates points in east direction also increases, which means the polynomial selection does not have an obvious effect on the orthogonal polynomial model.

The horizontal residuals v_E and v_N and their directions of 121 collocated points and 10 interpolated points before and after selecting the best polynomial degree and terms are showed in the figure as follows:

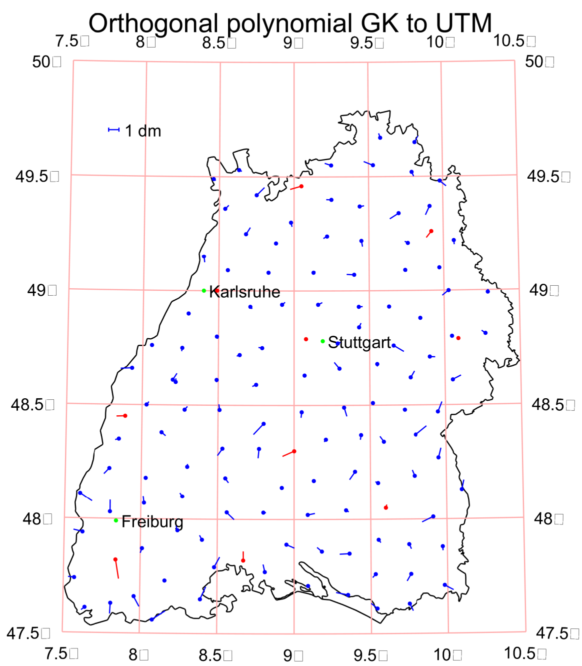

Figure 4.7: *Horizontal residuals of 121 collocated points and 10 interpolated points before selecting the best polynomial*

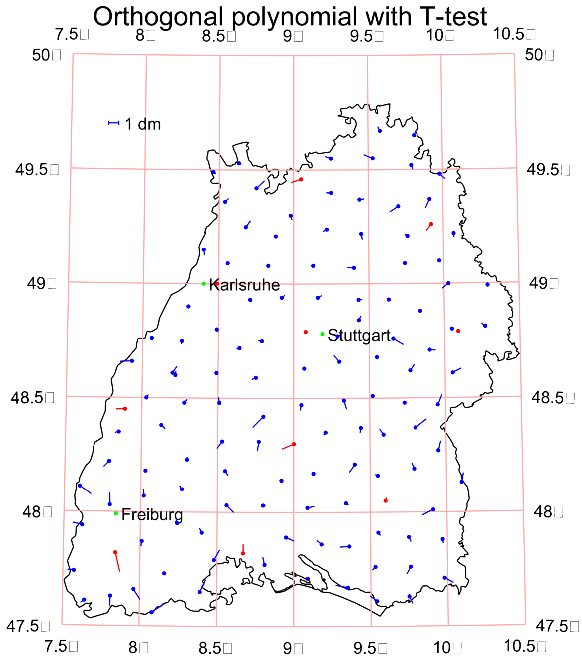

Figure 4.8: *Horizontal residuals of 121 collocated points and 10 interpolated points after selecting the best polynomial*

Chapter 5

Conclusion

In order to transform the coordinates from Gauß-Krüger coordinate system into UTM coordinate system 7 different transformation models are performed.

In the 2-D coordinate transformation the 6-parameter affine transformation model, the 9th-order multiple regression equations and the 5th-order multiple regressions have been performed and their transformation results have been discussed. From the comparison of their transformation results in Baden-Wütermberg we come to a conclusion that the 9th-order and the 5th-order multiple regression equations reached better accuracy than the 6-parameter affine transformation model. The transformation accuracy of the 6-parameter affine transformation model can reach the decimeter level, while the 9th-order and the 5th-order multiple regression equations can reach a centimeter level. When we compare the 9th-order with the 5th-order multiple regression equations, we can make a conclusion that although the transformation accuracy of the 9th-order multiple regression equations is about 1cm better than the 5th-order multiple regression equations, the computation complexity of 5th-order multiple regression equations is much lower than the 9th-order multiple regression equations.

As for the 3-D coordinate transformation using the Least Squares method, the 7-parameter Helmert transformation model, the quadratic polynomial model, the cubic polynomial model and the quadratic model with Legendre polynomial have been performed and discussed. From the comparison of their transformation results we can draw a conclusion that the 7-parameter Helmert transformation model get the worst transformation results while the cubic polynomial model using scaled collocated coordinates get the best transformation results. With the cubic polynomial model the transformation accuracy can reach the centimeter level, while the 7-parameter Helmert transformation model can only get a accuracy in decimeter level. By comparing the transformation results of the quadratic polynomial model and the quadratic model with Legendre polynomial we can conclude that the Legendre polynomial in the quadratic polynomial does not have a positive effect on the transformation accuracy, because the condition of independent variables for the orthogonal property of the Legendre polynomials is not met. The coordinates are not continuous in the interval [-1,1]. When comparing the transformation results of the quadratic polynomial model and the cubic polynomial model using scaled coordinates we can make a conclusion that the cubic polynomial model can achieve a better accuracy, because the distortion and the deformation will be better fitted with higher polynomial degree and more terms, but only under the condition that the cubic polynomial transformation model use scaled coordinates.

The method to select the best polynomial terms are also performed here. The main task of this method is to decrease the number of the polynomial terms, besides, the accuracy of the transformation result should be improved or at least remain unchanged. The process of selecting the best combination of the polynomial terms is meaningful because when the number of terms are decreased, the dimensions of the design matrix and the number of transformation parameters are also reduced. Although the application of this method in the 9th-order multiple regression equations, the quadratic polynomial model and the quadratic model with Legendre polynomial does not make great improvements to the transformation accuracy, but it has a great improvement to the transformation accuracy of the cubic polynomial model using non-scaled coordinates, which has been improved for about 2 cm. As a consequence, the the method to select the best polynomial terms is of great significance when a polynomial is full range or it has overmany polynomial terms.

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