Master Thesis 56

Indefinite Linear Quadratic Optimal Control: Periodic Dissipativity and Turnpike Properties

Master Thesis

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Abstract

In the present thesis, we study discrete-time indefinite linear quadratic (LQ) optimal control problems in the presence of constraints on states and inputs. Due to their high relevance in both theory and practical applications, these problems are under ongoing research. In the recent literature, a characterization of the optimal trajectories of LQ-problems was given in terms of strict dissipativity and turnpike properties at steady-states, provided that the stage cost is positive semidefinite. We extend these results to *periodic* dissipativity and turnpike properties in LQ-problems with *indefinite* cost functions. The contribution of this thesis is threefold:

First, we state sufficient conditions for periodic dissipativity and turnpike properties in compactly constrained indefinite LQ-problems. It is shown that the corresponding optimal periodic orbit can be computed explicitly using a non-strict dissipation inequality and is, in many cases, located on the boundary of the constraints.

A similar technique is applied to strict dissipativity and turnpike properties at steady-states, where some of the arguments simplify. Sufficient conditions for strict dissipativity at steady-states are given in terms of linear matrix inequalities, for which the compactness assumption made in the periodic case is not required to hold. Two kinds of conditions are proposed, one of which can handle more general cases while the other one establishes a direct link between negative cost eigenvalues, the shape of the constraints, and the exact location of the resulting optimal steady-state. Moreover, in the absence of state constraints and for ellipsoidal input constraints, necessary conditions for strict dissipativity with a quadratic storage function are stated.

The third approach does not explicitly take the constraints into account and relies on the notion of P-step systems, which have been used in the recent economic model predictive control literature for handling periodic optimality. It is shown that, if the stage cost accumulated over multiple consecutive time steps satisfies a certain convexity assumption, then the occurrence of strict dissipativity and turnpike properties can be characterized by spectral properties of the involved matrices.

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Notation

$\mathbb{R}$	set of real numbers
$\mathbb{R}_{\geq 0}$	set of nonnegative real numbers
$\mathbb{I}_{\geq 0}$	set of nonnegative integers
$\mathbb{I}_{[a,b]}$	set of integers in the interval $[a, b] \subseteq \mathbb{R}$
$\mathbb{C}$	set of complex numbers
$\mathbb{C}_{=1}$	set of complex numbers on the unit circle, i.e., $\mathbb{C}_{=1} := \{z \in \mathbb{C} \mid \lvert z\rvert = 1\}$
$C(\mathbb{R}^n)$	set of continuous functions $f : \mathbb{R}^n \to \mathbb{R}$
$\lVert x\rVert$	Euclidean norm of $x \in \mathbb{R}^n$
$\lvert x\rvert_V$	Euclidean distance of $x \in \mathbb{R}^n$ to a set $V \subseteq \mathbb{R}^n$, i.e., $\lvert x\rvert_V := \inf_{v\in V} \lVert x - v\rVert$
$A^\top$	transpose of a matrix $A \in \mathbb{R}^{n\times m}$
$A \succ 0\ (A \succeq 0)$	matrix $A \in \mathbb{R}^{n\times n}$ is positive definite (semidefinite)
$A \prec 0\ (A \preceq 0)$	matrix $A \in \mathbb{R}^{n\times n}$ is negative definite (semidefinite)
$\mathcal{K}$	A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class $\mathcal{K}$ function, i.e., $\alpha \in \mathcal{K}$, if α is continuous, strictly increasing, and $\alpha(0) = 0$.
$\mathcal{K}_\infty$	A function $\alpha \in \mathcal{K}$ is a class $\mathcal{K}_\infty$ function, i.e., $\alpha \in \mathcal{K}_\infty$, if, in addition, α is unbounded.
$\#L$	cardinality of a set $L \subseteq \mathbb{I}_{\geq 0}$
$\mathrm{int}(X)$	interior of a set $X \subseteq \mathbb{R}^n$
∂X	boundary of a set $X \subseteq \mathbb{R}^n$

1. Introduction

1.1. Motivation

Since the pioneering work of Kalman [13], the theory of linear quadratic (LQ) optimal control has become a mature research field. Contributions such as the classical paper by Willems [27] provided not only a characterization of the optimal behavior for zero or free endpoint constraints, but also relations to relevant system theoretic properties. A particularly relevant property is dissipativity, which was introduced by Willems [28, 29] and can be interpreted in the way that a system cannot create energy by itself. Although powerful and easily applicable, the traditional approaches in LQ optimal control have an important drawback: They cannot deal with constrained problems. A more promising tool for this is provided by the so-called *turnpike property*. The turnpike property has been introduced in the context of economics in the 1950's to describe a particular phenomenon that occurs in dynamic optimization problems [7]. Loosely speaking, it states that the optimal trajectory of an optimal control problem stays near an optimal point "most of the time". It took several decades until its significance in economic model predictive control (MPC) was discovered in [9]. MPC is a control method that solves a finite-horizon optimal control problem repeatedly and then applies only the first part of the computed optimal input [25]. In economic MPC, as opposed to stabilizing MPC, one is interested in minimizing a given cost function that might not be positive definite [8]. Under certain further assumptions, the turnpike property can be used to establish both performance bounds and convergence of the closed-loop [8,9]. The relation between strict dissipativity and the turnpike property has been worked out in detail in [11], where it was shown that, under an additional technical assumption, the two are equivalent. For discrete-time LQ-problems with convex cost, this connection was analyzed and related to geometric system properties in the very recent work [10]. The main result of this paper is that, when the cost satisfies a suitable convexity assumption, then strict dissipativity, the turnpike property and certain spectral properties involving the system matrices are equivalent.

Typically, when the control objective involves some kind of maximization, the corresponding stage cost is non-convex, i.e., (in the linear quadratic case) the cost matrices are not positive semidefinite. Such *indefinite* cost functions have been treated extensively in the classical theory of linear quadratic optimal control, cf. e.g. [26, 27] and [24] for contributions on continuous and discrete time problems, respectively. As in the convex case, however, these classical results do not apply in the presence of constraints on states and inputs. Recent research in economic MPC included the consideration of indefinite cost functions in constrained LQ-problems (cf. [22, 30]), thereby providing conditions for closed-loop asymptotic stability in the indefinite case. However, if one is interested in minimizing an economic performance objective rather than stabilizing the closed-loop, then the optimal trajectory might as well be periodic. The synthesis of control laws which achieve convergence to an optimal periodic orbit has been investigated in the recent literature [19, 20, 30], where periodic dissipativity and turnpike properties emerged as key ingredients. Nevertheless, for steady-states as well as for periodic orbits, conditions for strict dissipativity and turnpike properties in constrained indefinite optimal control problems remain to be characterized.

1.2. Contributions and outline of the thesis

The goal of the present thesis is to explore the optimal behavior of indefinite discrete-time LQ-problems subject to constraints on states and inputs. By means of sufficient and (partly) necessary conditions, we aim at closing the gap mentioned in the above introduction. We characterize strict dissipativity and turnpike properties w.r.t. steady-states and periodic orbits. There are two substantially different approaches which are employed in this thesis: First, we analyze strict dissipativity w.r.t. periodic orbits by considering a certain non-strict dissipation inequality. We state sufficient conditions as well as an explicit computational procedure for constructing the corresponding optimal periodic orbit. Herein, the central idea is that, in the presence of negative cost eigenvalues, the corresponding modes are in many cases on the boundary of the constraints and thus the particular shape of the constraints plays an important role.

Since steady-states are periodic orbits with period one, this approach can as well be applied to steady-states, where, however, the application of the results simplifies. The second approach is based on P-step systems. We provide sufficient and necessary conditions for strict dissipativity and turnpike properties w.r.t. steady-states, when the cost accumulated over several consecutive time steps satisfies a suitable convexity assumption. This allows us to characterize the above properties when the original stage cost is non-convex in the state.

The thesis is structured as follows: In Chapter 2, we set the stage by introducing the considered framework as well as important definitions and existing results. Chapter 3 provides conditions for strict dissipativity w.r.t. periodic orbits which imply the occurrence of periodic turnpikes as well. Thereafter, in Chapter 4, we explore strict dissipativity and turnpike properties w.r.t. steady-states by first employing similar techniques as in the periodic case. Moreover, using the framework of P-step systems, we obtain further conditions characterizing these properties. We conclude the thesis in Chapter 5, followed by a short appendix containing two auxiliary Lemmas.

2. Background

2.1. Setting

In the present thesis, we investigate discrete-time finite-horizon linear quadratic optimal control problems subject to state and input constraints. The stage cost is of the form

$$\ell(x,u) = x^\top Qx + 2x^\top S^\top u + u^\top Ru + s^\top x + v^\top u + c$$

with

$$Q \in \mathbb{R}^{n\times n},\ \ S \in \mathbb{R}^{m\times n},\ \ R \in \mathbb{R}^{m\times m}, s \in \mathbb{R}^n,\ \ v \in \mathbb{R}^m,\ \ c \in \mathbb{R}.$$

Note that we do not pose any assumptions on the definiteness of Q, R or

$$\begin{pmatrix} Q & S^\top \\ S & R \end{pmatrix}.$$

We consider linear system dynamics $x(k+1) = Ax(k) + Bu(k)$ with initial condition $x(0) = x_0$, as well as state and input constraints $\mathbb{X}$ and $\mathbb{U}$, respectively, where

$$A \in \mathbb{R}^{n\times n},\ \ B \in \mathbb{R}^{n\times m},\ \ x_0 \in \mathbb{R}^n,\ \ \mathbb{X} \subseteq \mathbb{R}^n,\ \ \mathbb{U} \subseteq \mathbb{R}^m.$$

Throughout this thesis, the constraint sets $\mathbb{X}$ and $\mathbb{U}$ are assumed to be closed and convex. For a fixed initial value x_0 and an input trajectory $u \in \mathbb{U}^N$, the solution of $x(k+1) = Ax(k) + Bu(k)$ is denoted by $x_u(\cdot, x_0)$ and the corresponding running cost is given as

$$J_N(x_0, u) = \sum_{k=0}^{N-1} \ell\left(x_u(k, x_0), u(k)\right)$$

for some $N \in \mathbb{N}$. Moreover, we define the set of feasible input trajectories

$$\mathbb{U}^N(x_0) = \left\{u \in \mathbb{U}^N \mid x_u(k, x_0) \in \mathbb{X},\ k \in \mathbb{I}_{[1,N]}\right\}$$

as well as the feasible state-input pairs

$$\mathbb{Z} = \{(x, u) \in \mathbb{X} \times \mathbb{U} \mid Ax + Bu \in \mathbb{X}\}.$$

We set up the following optimization problem which will be of central interest throughout this thesis:

$$V_N(x_0) = \underset{u \in \mathbb{U}^N(x_0)}{minimize} \; J_N(x_0, u). \tag{2.1.1}$$

The optimal state and input trajectories of (2.1.1) for the initial value x_0 are denoted by $x^*(\cdot, x_0)$ and $u^*(\cdot, x_0)$, respectively.

Definition 2.1.1. (i) We say that $(x^e, u^e) \in \mathbb{X} \times \mathbb{U}$ is a *steady-state* or an *equilibrium* for the dynamical system $x(k+1) = Ax(k) + Bu(k)$, if $x^e = Ax^e + Bu^e$.

(ii) We say that the set $\Pi = \left\{(x^1, u^1), \ldots, (x^P, u^P)\right\} \subset \mathbb{X} \times \mathbb{U}$ is a *periodic orbit* for the dynamical system $x(k+1) = Ax(k) + Bu(k)$, if $Ax^i + Bu^i = x^{i+1}$ for all $i \in \mathbb{I}_{[1,P-1]}$ and $Ax^P + Bu^P = x^1$. P is called the *period* or *length* of the periodic orbit Π. Π is called a *minimal periodic orbit*, if $x^i \neq x^j$ for all $i, j \in \mathbb{I}_{[1,P]}$ with $i \neq j$. The average cost on a periodic orbit Π is denoted by $\ell_\Pi := \frac{1}{P} \sum_{i=1}^{P} \ell(x^i, u^i)$.

For technical reasons, we make the standing assumption that there exists a steady-state in the interior of the constraints, i.e., there exist $(x, u) \in \text{int}\,(\mathbb{X} \times \mathbb{U})$ such that $x = Ax + Bu$.

2.2. Related work

As mentioned in the introduction, strict dissipativity and the turnpike property have emerged as useful ingredients in the analysis of constrained finite-horizon optimal control problems. In this section, we state rigorous definitions of the two properties and review their relationship which has been investigated in the recent economic MPC literature. The following definitions are adopted from [10].

Definition 2.2.1. (i) We call the LQ-problem *strictly cyclo-dissipative* [1] at an equilibrium $(x^e, u^e) \in \mathbb{X} \times \mathbb{U}$ w.r.t. the supply rate $\ell(x, u) - \ell(x^e, u^e)$, if there exist a storage function $\lambda : \mathbb{X} \to \mathbb{R}$ which is bounded on bounded subsets of $\mathbb{X}$, and a function $\alpha \in \mathcal{K}_\infty$ such that the dissipation inequality

$$\ell(x, u) - \ell(x^e, u^e) + \lambda(x) - \lambda(Ax + Bu) \geq \alpha(\|x - x^e\|) \tag{2.2.1}$$

[1] In [10], the authors call the property defined in Definition 2.2.1 (i) *strict pre-dissipativity*. The name *strict cyclo-dissipativity*, which is preferred in the present thesis, originates from the classical paper [12].

holds for all $(x,u) \in \mathbb{Z}$.

(ii) We call the LQ-problem *strictly dissipative* at an equilibrium $(x^e, u^e) \in \mathbb{X} \times \mathbb{U}$ w.r.t. the supply rate $\ell(x,u) - \ell(x^e, u^e)$, if it is strictly cyclo-dissipative in the sense of (i) and λ is bounded from below on $\mathbb{X}$.

Since, throughout this thesis, the only supply rate under consideration is $\ell(x,u) - \ell(x^e, u^e)$, we do not mention the supply rate appearing in the above dissipation inequality explicitly. A well-established result in economic MPC (cf. [1]) is that strict (cyclo-) dissipativity at an equilibrium (x^e, u^e) implies that this equilibrium is the *optimal steady-state* in the sense that

$$\ell(x^e, u^e) = \min_{\substack{x \in \mathbb{X}, u \in \mathbb{U} \\ x = Ax + Bu}} \ell(x,u).$$

Therefore, we will use the notions of optimal steady-states and steady-states at which strict dissipativity holds interchangeably.

Definition 2.2.2. (i) We say that the optimal control problem (2.1.1) has the *turnpike property* at an equilibrium $(x^e, u^e) \in \mathbb{X} \times \mathbb{U}$ on a set $\mathbb{X}_{tp} \subset \mathbb{X}$, if for each compact set $K \subset \mathbb{X}_{tp}$ and for each $\varepsilon > 0$ there exists a constant $C_{K,\varepsilon} > 0$ such that for all $x \in K$ and all $N \in \mathbb{N}$ the optimal trajectories $x^*(\cdot, x)$ of (2.1.1) with initial value x satisfy

$$\#\{k \in \{0, \ldots, N-1\} \mid \|x^*(k,x) - x^e\| > \varepsilon\} \leq C_{K,\varepsilon}.$$

(ii) We say that the optimal control problem (2.1.1) has the *near equilibrium turnpike property* at an equilibrium $(x^e, u^e) \in \mathbb{X} \times \mathbb{U}$, if for each $\rho > 0, \varepsilon > 0$ and $\delta > 0$ there exists a constant $C_{\rho,\varepsilon,\delta} > 0$ such that for all $x \in \mathbb{X}$ with $\|x - x^e\| \leq \rho$, all $N \in \mathbb{N}$, and all trajectories $x_u(\cdot, x)$ satisfying $J_N(x,u) \leq N\ell(x^e, u^e) + \delta$ for some $u \in \mathbb{U}^N(x)$, the inequality

$$\#\{k \in \{0, \ldots, N-1\} \mid \|x_u(k,x) - x^e\| > \varepsilon\} \leq C_{\rho,\varepsilon,\delta}$$

holds.

Loosely speaking, the turnpike property (Definition 2.2.2 (i)) at some equilibrium states that the optimal trajectories stay near this equilibrium for all but a finite number of time instances which is independent of the horizon N. For the near equilibrium turnpike property (Definition 2.2.2 (ii)) this needs to hold for trajectories with cost close to the steady-state cost under consideration.

It is known since the pioneering work of Willems that the behavior of optimal solutions of unconstrained LQ-problems and dissipativity are closely related [27].

This was extended to the constrained case by the recent developments in the economic MPC literature (cf. [9,11]), where it was shown that strict dissipativity is sufficient and, under a mild local controllability assumption, also necessary for the near equilibrium turnpike property. For LQ-problems with $Q \succeq 0$ and $R \succ 0$, this relationship was extended by the authors in [10] who established a connection between the above definitions and spectral properties of the matrix pair (A, Q). In particular, in the absence of state constraints, they showed that, for steady-states (x^e, u^e) with $u^e \in \operatorname{int}(\mathbb{U})$, strict dissipativity, the turnpike property and detectability [2] of the pair (A, Q) are equivalent (cf. [10, Theorem 8.1]). Again, in view of the classical literature on linear quadratic optimal control, these results came as no surprise. It is well-known that, under certain additional assumptions, detectability of the above matrix pair implies the existence of a unique optimal and stabilizing solution to the algebraic Riccati equation, which yields an optimal stabilizing feedback law. A more surprising results is that, for compact state constraints, strict dissipativity, the near equilibrium turnpike property and observability of the matrix pair (A, Q) for all eigenvalues on the complex unit circle $\mathbb{C}_{=1}$ are equivalent (cf. [10, Theorem 8.3]). Thus, unstable unobservable eigenvalues can be "compensated" by boundedness of the constraints.

In summary, the characterization of steady-state dissipativity and turnpikes in the interior of the constraints for $Q \succeq 0, R \succ 0$ can be considered as fairly complete. A generalization of the results from [10] is the main goal of the present thesis. In particular, we characterize strict dissipativity and turnpike properties in linear quadratic optimal control problems

- for non-convex stage costs,
- w.r.t. steady-states *and* periodic orbits,
- under explicit consideration of the constraints.

[2] As can be deduced from Lemma A.1.1, detectability of (A, Q) and (A, C) for some C with $C^\top C = Q$ are equivalent. Therefore, we will use the two notions interchangeably throughout this thesis

3. Strict dissipativity w.r.t. periodic orbits

As discussed in Section 2.2, the connection of strict dissipativity and the turnpike property to geometric system properties has been thoroughly investigated for linear quadratic systems under the assumptions that the cost matrices satisfy $Q \geq 0$, $R > 0, S = 0$ and when the corresponding optimal steady-state lies in the interior of the constraints. In this section and also in the remainder of the thesis, we generalize these results into several directions. First of all, we drop the convexity assumptions on the cost, i.e., the matrices Q, R and

$$\begin{pmatrix} Q & S^\top \\ S & R \end{pmatrix}$$

do, in general, not need to satisfy any definiteness properties. Moreover, we extend the notions of strict dissipativity and turnpike properties to sets more general than steady-states. In [19], it is shown that the sufficiency of strict dissipativity for the near equilibrium turnpike property can be directly adapted to periodic orbits. Therefore, we are interested in finding computationally tractable and, if possible, geometric systems theoretic conditions for strict dissipativity of LQ-problems w.r.t. periodic orbits. To do this, we assume compact state and input constraints. When considering optimal periodic orbits, this is not too restrictive. It has been known since the early days of LQ-control and dissipativity that, when there are no constraints and the optimal cost stays bounded, the optimal value function is quadratic in the initial condition and the optimal feedback is linear in the state. Thus, in this case, the optimal trajectories are solutions of a linear time-invariant difference equation which cannot exhibit periodic behavior. Nevertheless, the compactness assumption is mainly due to technical reasons and we conjecture that all subsequent results can be relaxed to the postulation that only modes which are not positively weighted in the cost function must be bounded. In Chapter 4, we show that this can, indeed, be done in the steady-state case

The remainder of this chapter is structured as follows. First, we define the notions of turnpike properties and strict dissipativity w.r.t. periodic orbits in a slightly different manner than in the recent economic MPC literature. After discussing the connection between the different dissipativity properties and their relation to the periodic turnpike property, we present necessary and sufficient conditions for strict dissipativity w.r.t.

periodic orbits in the form of an explicit computational procedure. Using the S-procedure, this in general computationally intractable procedure can be relaxed to a convex problem, which yields sufficient conditions for finding optimal periodic orbits, when the state and input constraints are assumed to be of a specific quadratic form. In this case, the particular shape of the constraints plays an important role in determining the location of the periodic orbit. We conclude the section by presenting a purely computational approach for showing strict dissipativity w.r.t. a given periodic orbit based on sum-of-squares (SOS) programming.

3.1. Preliminaries

The following definition is a straightforward generalization of Definition 2.2.1 (ii) to periodic orbits.

Definition 3.1.1. We call the LQ-problem *strictly dissipative w.r.t. a periodic orbit* $\Pi = \{(x^1, u^1), \dots, (x^P, u^P)\} \subset \mathbb{X} \times \mathbb{U}$ and the supply rate $l(x,u) - l_\Pi$, if there exist a storage function $\lambda : \mathbb{X} \to \mathbb{R}$ which is bounded on bounded subsets of $\mathbb{X}$ and bounded from below, and a function $\alpha \in \mathcal{K}_\infty$ such that the dissipation inequality

$$\ell(x,u) - \ell_\Pi + \lambda(x) - \lambda(Ax + Bu) \geq \alpha\left(\left|(x,u)\right|_\Pi\right) \tag{3.1.1}$$

holds for all $(x,u) \in \mathbb{Z}$.

As in the steady-state case, Definition 3.1.1 implies that Π is an *optimal periodic orbit*, meaning that there is no other periodic orbit with performance superior to Π [31]. [1] From the strictness of the dissipation inequality (3.1.1), it follows that strict dissipativity w.r.t. periodic orbits is unique, i.e., whenever the LQ-problem is strictly dissipative w.r.t. two periodic orbits, then the orbits must coincide.

Note that, although Definition 3.1.1 extends Definition 2.2.1 to more general sets, the former requires strictness of the inequality w.r.t. both input and state, whereas the latter only takes the state into account. The reason for choosing strictness in input and state is twofold: First, Definition 3.1.1 is inspired by the work in [19], where this stronger assumption is indeed needed for some of the results. Furthermore, strict dissipativity w.r.t. periodic orbits is certainly more involved than strict dissipativity w.r.t. steady-states. In particular, given a set of x-coordinates that constitute a periodic orbit, determining the corresponding inputs that connect these states is not trivial, even when they are uniquely determined. Therefore, it is desirable to incorporate the input into the notion of strict dissipativity. This has the advantage that the tools we

[1] As a matter of fact, it even implies that there is no other *trajectory* with performance superior to Π [31].

develop in the following allow us to compute both the states and the corresponding inputs of the periodic orbit w.r.t. which the system is strictly dissipative. Similarly, we extend the notion of the near equilibrium turnpike property (Definition 2.2.2 (ii)) to periodic orbits.

Definition 3.1.2. We say that the LQ-problem (2.1.1) has the *turnpike property w.r.t. the periodic orbit* $\Pi = \left\{(x^1, u^1), \ldots, (x^P, u^P)\right\} \subset \mathbb{X} \times \mathbb{U}$, if for each $\varepsilon > 0$ and $\delta > 0$ there exists a constant $C_{\varepsilon,\delta} > 0$ such that for all $x \in \mathbb{X}$, all $N \in \mathbb{N}$, and all trajectories $x_u(\cdot, x)$ satisfying $J_N(x, u) \leq N\ell_\Pi + \delta$ for some $u \in \mathbb{U}^N(x)$, the inequality

$$\#\left\{k \in \{0, \ldots, N-1\} \mid \left|(x_u(k, x), u(k))\right|_\Pi > \varepsilon\right\} \leq C_{\varepsilon,\delta}$$

holds.

In the recent economic MPC literature, the most commonly used turnpike property is the one from Definition 2.2.2 (ii) and thus it is natural to consider Definition 3.1.2 as an extension of the steady-state case in the sense that trajectories with average performance close to ℓ_Π stay near Π most of the time. Therefore, we do not generalize the turnpike property in the sense of Definition 2.2.2 (i) and mean in the context of periodic orbits, if not indicated otherwise, by *turnpike property* always properties in the above sense.

Definition 3.1.1 differs from the common definitions of periodic strict dissipativity in the recent economic MPC literature. In [31], the authors chose to define strict dissipativity w.r.t. a periodic orbit of length P by demanding that there exist P (possibly different) storage functions satisfying P dissipation inequalities along trajectories of length P. A seemingly weaker definition than the one from [31] is given in [19]. In this work, the authors postulate the existence of a storage function such that an appropriately defined P-step system [2] is strictly dissipative. Summing up the dissipation inequality (3.1.1) along any feasible trajectory of length P, it is readily derived that the definition from [19] is implied by Definition 3.1.1. A more surprising result is that, for compact constraints, the definition from [19] implies Definition 3.1.1, i.e., both are equivalent [14]. Moreover, according to [14], they are equivalent to the one from [31] as well. Since the results of this thesis only require the fact that Definition 3.1.1 implies the one from [19], we do not discuss the other results in more detail.

The application of [19, Theorem 12] reveals that strict dissipativity in the sense of Definition 3.1.1 implies the turnpike property (Definition 3.1.2). This fact serves as the main motivation for the remainder of this chapter. Since, also in the periodic case, strict dissipativity is sufficient for the turnpike property, we investigate sufficient

[2] Cf. Section 4.2 for a rigorous definition of P-step systems

conditions for strict dissipativity w.r.t. periodic orbits and provide tools that allow for an explicit construction of such orbits.

3.2. Strict dissipativity via non-strict dissipation inequalities

In this section, we state necessary and sufficient conditions for strict dissipativity w.r.t. periodic orbits as well as an explicit procedure for constructing such orbits without a priori knowledge of the corresponding period P. For our approach, we require compactness of state and input constraints, and thus we assume that $\mathbb{X}$ and $\mathbb{U}$ are compact for the remainder of this section. We begin with a motivating example that illustrates the idea of the upcoming results. Clearly, whenever a system is strictly dissipative w.r.t. some set Π, inequality (3.1.1) implies the non-strict dissipation inequality

$$\ell(x,u) - \ell_\Pi + \lambda(x) - \lambda(Ax+Bu) \geq 0 \tag{3.2.1}$$

for all $(x,u) \in \mathbb{Z}$. As it turns out, the points for which (3.2.1) holds with equality constitute the set w.r.t. which the system is strictly dissipative.

Example 3.2.1. Consider the indefinite LQ-problem with

$$x(k+1) = x(k) + u(k),\ \ell(x,u) = -u^2,\ \mathbb{X} = [-1,1],\ \mathbb{U} = [-2,2].$$

It is not difficult to see that the optimal periodic orbit of this system is $\Pi = \{(1,-2),(-1,2)\}$, i.e., $\ell_\Pi = -4$. The non-strict dissipation inequality (3.2.1) with storage function $\lambda(x) = -x^2$ reads

$$-u^2 + 4 - x^2 + (x+u)^2 = 4 - 2xu \geq 0.$$

Indeed, this inequality holds on $\mathbb{Z}$, with equality if and only if $(x,u) \in \Pi$. Hence, by applying Lemma A.1.2, we conclude that the strict dissipation inequality (3.1.1) holds. Consequently, the above LQ-problem is strictly dissipative w.r.t. Π.

The observation that we can deduce strict dissipativity from the non-strict dissipation inequality is exploited extensively throughout this section. In particular, the following algorithm provides a constructive procedure for finding periodic orbits w.r.t. which the system is strictly dissipative.

Algorithm 3.2.2. Consider the LQ-problem (2.1.1). Given a set of functions $\Lambda \subseteq C(\mathbb{R}^n)$, solve the following optimization problem:

$$\begin{aligned} \underset{\gamma\in\mathbb{R},\, \lambda\in\Lambda}{\text{maximize}} \quad & \gamma \\ \text{s.t.} \quad & \ell(x,u) - \gamma + \lambda(x) - \lambda(Ax+Bu) \geq 0 \\ & \text{for all } (x,u) \in \mathbb{Z}. \end{aligned} \tag{3.2.2}$$

Denote the optimal quantities by γ^*, λ^*. Define the set of points for which the constraint in (3.2.2) holds with equality as

$$M := \{(x,u) \in \mathbb{Z} \mid \ell(x,u) - \gamma^* + \lambda^*(x) - \lambda^*(Ax+Bu) = 0\}.$$

Return M.

Given a set of functions $\Lambda \subseteq C(\mathbb{R}^n)$, we denote the corresponding output of Algorithm 3.2.2 by $\Pi^\Lambda = M$. For simplicity, we restrict the search for a storage function in the above procedure to continuous ones. Since $\mathbb{Z}$ is compact, this implies that the optimization problem (3.2.2) is always feasible. We note that the same optimization problem appears in the recent publication [6], albeit with a different motivation and usage. Next, we state the main result of this section. It provides necessary and sufficient conditions for strict dissipativity w.r.t. periodic orbits.

Theorem 3.2.3. Consider the LQ-problem (2.1.1) with compact state and input constraints and denote the output of Algorithm 3.2.2, given some $\Lambda \subseteq C(\mathbb{R}^n)$, by Π^Λ. Then, the following statements hold:

(i) If, for some $\Lambda \subseteq C(\mathbb{R}^n)$, Π^Λ is a periodic orbit, then the LQ-problem is strictly dissipative and has the turnpike property w.r.t. Π^Λ.

(ii) Suppose that the system is strictly dissipative w.r.t. some periodic orbit Π with storage function λ. If $\lambda \in \Lambda \subseteq C(\mathbb{R}^n)$, then $\gamma^* = \ell_\Pi$ and $\Pi \subseteq \Pi^\Lambda$.

Proof. **(i):** Denote $\Pi^\Lambda = \left\{(x^1,u^1),\ldots,(x^P,u^P)\right\}$ and define

$$\tilde{\ell}^*(x,u) = \ell(x,u) - \gamma^* + \lambda^*(x) - \lambda^*(Ax+Bu),$$

where the asterisks denote the optimal quantities obtained from Algorithm 3.2.2. Due to the maximization in (3.2.2), Π^Λ is not empty. Summing up $\tilde{\ell}^*(x,u) = 0$ along

Π^Λ and using that Π^Λ is a periodic orbit, we obtain

$$\sum_{i=1}^{P} \tilde{\ell}^*(x^i, u^i) = P \cdot (\ell_{\Pi^\Lambda} - \gamma^*) = 0,$$

i.e., $\ell_{\Pi^\Lambda} = \gamma^*$. Moreover, from the definition of Π^Λ, we have

$$\begin{aligned} \tilde{\ell}^*(x,u) &> 0 \ \forall\, (x,u) \in \mathbb{Z} \setminus \Pi^\Lambda, \\ \tilde{\ell}^*(x,u) &= 0 \ \forall\, (x,u) \in \Pi^\Lambda. \end{aligned}$$

Thus, by Lemma A.1.2, there exists a class $\mathcal{K}_\infty$-function α such that

$$\ell(x,u) - \ell_{\Pi^\Lambda} + \lambda^*(x) - \lambda^*(Ax+Bu) \geq \alpha\left(|(x,u)|_{\Pi^\Lambda}\right) \ \ \forall\, (x,u) \in \mathbb{Z}.$$

Due to compactness of $\mathbb{X}$, the storage function λ is bounded. Hence, the LQ-problem is strictly dissipative w.r.t. Π^Λ. The statement about the turnpike property follows from [19, Theorem 12].

(ii): Let $\Pi = \left\{(x^1,u^1), \ldots, (x^P,u^P)\right\}$. Since $\lambda \in \Lambda$, we can use λ as well as the average cost ℓ_Π on Π as candidate solutions for the optimization problem (3.2.2) to conclude $\gamma^* \geq \ell_\Pi$. Define

$$\begin{aligned} \tilde{\ell}(x,u) &= \ell(x,u) - \ell_\Pi + \lambda(x) - \lambda(Ax+Bu), \\ \tilde{\ell}^*(x,u) &= \ell(x,u) - \gamma^* + \lambda^*(x) - \lambda^*(Ax+Bu), \end{aligned}$$

where, again, asterisks denote optimal solutions of (3.2.2). Summing up $\tilde{\ell}^*(x,u) \geq 0$ along Π and using that Π is a periodic orbit, we obtain

$$\begin{aligned} \sum_{i=1}^{P} \tilde{\ell}^*(x^i,u^i) = \sum_{i=1}^{P} & \tilde{\ell}(x^i,u^i) + \lambda^*(x^i) - \lambda(x^i) - \lambda^*(Ax^i + Bu^i) \\ & + \lambda(Ax^i + Bu^i) - \gamma^* + \ell_\Pi = P \cdot (\ell_\Pi - \gamma^*) \geq 0, \end{aligned} \tag{3.2.3}$$

i.e., $\ell_\Pi \geq \gamma^*$ and hence $\ell_\Pi = \gamma^*$. Consequently, the above inequality (3.2.3) holds with equality. Since $\tilde{\ell}^*(x,u) \geq 0$ for any $(x,u) \in \mathbb{Z}$, each term in the sum $\sum_{i=1}^{P} \tilde{\ell}^*(x^i,u^i)$ must vanish and hence $\tilde{\ell}^*(x,u)$ is zero on Π, i.e., $\Pi \subseteq \Pi^\Lambda$. □

Remark 3.2.4. The main idea of Algorithm 3.2.2 and thus of Theorem 3.2.3 lies in the optimization problem (3.2.2). Given a set of functions Λ, the optimal value γ^* is the maximum average cost a periodic orbit may have such that the problem can be strictly dissipative w.r.t. it, using a storage function from Λ. To be more precise, consider the case where Π^Λ does not contain a periodic orbit. In this case, by construction, the dissipation inequality w.r.t. Π^Λ still holds, i.e.,

$$\begin{aligned} \tilde{\ell}^*(x,u) &> 0 \ \forall\, (x,u) \in \mathbb{Z} \setminus \Pi^\Lambda, \\ \tilde{\ell}^*(x,u) &= 0 \ \forall\, (x,u) \in \Pi^\Lambda. \end{aligned}$$

Hence, summing $\tilde{\ell}^*(x,u)$ along an arbitrary periodic orbit

$$\Pi = \left\{(x^1,u^1),\ldots,(x^P,u^P)\right\} \not\subseteq \Pi^\Lambda$$

yields

$$\begin{aligned} \sum_{i=1}^{P} \tilde{\ell}^*(x^i,u^i) &= \sum_{i=1}^{P} \ell(x^i,u^i) - \gamma^* + \lambda^*(x^i) - \lambda^*(Ax^i + Bu^i) \\ &= P \cdot (\ell_\Pi - \gamma^*) > 0, \end{aligned}$$

i.e., $\ell_\Pi > \gamma^*$ and thus, since γ^* is optimal for (3.2.2), ℓ_Π is not feasible for the optimization problem. Since this holds for any periodic orbit Π, we conclude that, if Π^Λ does not contain a periodic orbit, then there is no periodic orbit with sufficiently low average cost to render the non-strict dissipation inequality feasible.

Theorem 3.2.3 is a powerful tool in that it provides sufficient as well as necessary conditions for the existence of periodic orbits w.r.t. which a system is strictly dissipative. The application is straightforward: First, one chooses a set of storage functions, usually parametrized by real parameters. Then, Algorithm 3.2.2 is executed in order to obtain the corresponding set Π^Λ. If Π^Λ is a periodic orbit, we can directly conclude that the LQ-problem is strictly dissipative and has the turnpike property w.r.t. it. On the contrary, if Π^Λ does not contain a periodic orbit with average cost γ^*, then the system cannot be strictly dissipative w.r.t. any periodic orbit using a storage function in Λ.

Of course, this view is idealized: Even when we assume that Λ is the set of quadratic functions and that $\mathbb{Z}$ is of some quadratic form, the optimization problem (3.2.2) is a *semi-infinite optimization problem* which can, in general, not be solved efficiently. It is the purpose of the next section to use a standard tool from convex optimization, the S-procedure, to arrive at computationally tractable sufficient conditions for strict dissipativity w.r.t. periodic orbits. Nevertheless, for simple problems, the idea of Theorem 3.2.3 can be applied to give insight into the optimal behavior of the LQ-problem. This is illustrated by the following two examples.

Example 3.2.5. Consider the indefinite LQ-problem with

$$x(k+1) = x(k) + u(k),\ \ell(x,u) = x^2 - u^2,\ \mathbb{X} = [0,1],\ \mathbb{U} = [-1,1].$$

It is readily seen that the optimal periodic orbit is $\Pi = \{(1,-1),(0,1)\}$, i.e., $\ell_\Pi = -\frac{1}{2}$. Choosing the storage function $\lambda(x) = -\frac{1}{2}x^2$, the dissipation inequality takes the form

$$x^2 + xu - \frac{1}{2}u^2 + \frac{1}{2} \geq 0,$$

which holds on $\mathbb{Z}$, with equality if and only if $(x,u) \in \Pi$. Hence, due to Lemma A.1.2, the strict dissipation inequality (3.1.1) holds and thus the LQ-problem is strictly dissipative w.r.t. Π. Note that we could apply the presented approach although, contrary to Example 3.2.1, the stage cost was not constant along Π.

In the following example, we apply the idea of Theorem 3.2.3 to a different but related problem, namely to showing strict dissipativity w.r.t. a linear subspace. Without going into technical details, i.e., defining an average cost and the notion of strict dissipativity on subspaces, we simply illustrate how such a scenario can be treated in the presented framework. A systematic extension of our approach to strict dissipativity w.r.t. subspaces seems promising, but is beyond the scope of this thesis.

Example 3.2.6. Consider the indefinite LQ-problem with

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(k),$$
$$\ell(x,u) = -x_2^2,\ \mathbb{X} = \mathbb{R} \times [0,1],\ \mathbb{U} = [0,1].$$

From a practical point of view, one can think of x_1 as the position, of x_2 as the velocity and of u as the acceleration of a rigid body. Then, the above LQ-problem aims at maximizing the velocity while keeping velocity and acceleration constrained. The position, however, may be arbitrarily large. It is not difficult to see that the optimal behavior consists of steering x_2 to the boundary $x_2 = 1$ and then applying $u = 0$. This implies that the optimal stage cost is -1. Using the storage function $\lambda(x) = -x_2^2$, the dissipation inequality reads

$$1 - x_2^2 + 2x_2u + u^2 \geq 0. \tag{3.2.4}$$

It is straightforward to check that this, indeed, holds for all $(x,u) \in \mathbb{Z}$. Moreover, equality holds if and only if $(x,u) \in V := \{(a,1,0),\ a \in \mathbb{R}\}$ and, thus, (3.2.4) can be interpreted as strict dissipativity w.r.t. V, which is indeed a linear subspace of $\mathbb{R}^3$.

Remark 3.2.7. We conclude the section by noting that Theorem 3.2.3 applies to sets more general than periodic orbits. First of all, we observe that none of our results require minimality of the periodic orbit (cf. Definition 2.1.1 (ii)). An example for a non-minimal periodic orbit w.r.t. which the LQ-problem can be shown to be strictly dissipative appears in the setting

$$x(k+1) = u(k),\ \ell(x,u) = -u^2,\ \mathbb{X} = \mathbb{U} = [-1,1].$$

The application of Theorem 3.2.3 reveals that this problem is strictly dissipative w.r.t. $\Pi = \{(1,1),(1,-1),(-1,-1),(-1,1)\}$. Although Π contains two steady-states, it fulfills the definition of a periodic orbit. Note, however, that Π is not a minimal periodic orbit since the state $x = 1$ appears twice in it. Furthermore, Theorem 3.2.3 applies to any set which is a disjoint union of periodic orbits, e.g., one can show strict dissipativity of the LQ-problem with

$$x(k+1) = x(k) + u(k),\ \ell(x,u) = -x^2,\ \mathbb{X} = [-1,1],\ \mathbb{U} = [-1,1]$$

w.r.t. $\Pi = \{(-1,0),(1,0)\}$, which consists of two steady-states.

This assumption on Π can be relaxed even further: In general, the developed tools apply to any set Π which is invariant, i.e., in which we have that for all $(x,u) \in \Pi$ there exists an input $\bar{u}$ such that $(Ax + Bu, \bar{u}) \in \Pi$. In this case, the average cost on Π would be defined as the average cost of all trajectories that stay in Π. Moreover, as Example 3.2.6 shows, we can even consider uncountably infinite sets such as linear subspaces in this framework. Nevertheless, since periodic orbits of length ≥ 2 are, after steady-states, the most common sets w.r.t. which dissipativity is considered in the existing literature, we postpone the discussion of strict dissipativity w.r.t. more general sets to future research.

3.3. Convex relaxation via the S-procedure

As described in the previous section, Theorem 3.2.3 provides powerful conditions for strict dissipativity, which are, however, in general not computationally tractable. It is the purpose of this section to adapt Algorithm 3.2.2 such that it can be executed using tools from convex optimization. First, we present and discuss one of the main results of this thesis which achieves a convex relaxation to Algorithm 3.2.2. Thereafter, we focus on the application of this relaxation to the special case of *symmetric* LQ-problems.

3.3.1. The main result

Throughout this section, we will make extensive use of the so-called *S-procedure*. The S-procedure allows for a reformulation of an inequality on a quadratic set into an inequality on the whole euclidean space. It can be summarized as follows (cf. [4]): Given some quadratic functions $F_i : \mathbb{R}^n \to \mathbb{R}$ with

$$F_i(x) = \begin{pmatrix} x \\ 1 \end{pmatrix}^\top \begin{pmatrix} P_i & q_i^\top \\ q_i & c_i \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix}, \; i \in \mathbb{I}_{[0,r]},$$

we have $F_0(x) \geq 0 \; \forall x \in \left\{ x \mid F_i(x) \leq 0, i \in \mathbb{I}_{[1,r]} \right\}$ if there exist constant multipliers $\lambda_i \geq 0, i \in \mathbb{I}_{[1,r]}$ such that

$$F_0(x) + \sum_{i=1}^{r} \lambda_i F_i(x) \geq 0 \tag{3.3.1}$$

for all $x \in \mathbb{R}^n$. Note that the latter inequality can be checked efficiently using linear matrix inequality (LMI) techniques. In case that $r = 1$, the S-procedure is also necessary in the sense that $F_0(x) \geq 0 \; \forall x \in \{x \mid F_1(x) \leq 0\}$ implies the existence of a multiplier $\lambda_1 \geq 0$ such that (3.3.1) holds [4]. Before we state the main result of this section, we make an additional assumption on the constraints which are involved in the LQ-problem: We assume compact constraints of a specific, quadratic form, namely

$$\begin{aligned} \mathbb{X} &= \left\{ x \in \mathbb{R}^n \mid g_i^x(x) = x^\top P_i^x x + x^\top q_i^x + c_i^x \leq 0, i \in \mathbb{I}_{[1,l^x]} \right\}, \\ \mathbb{U} &= \left\{ u \in \mathbb{R}^m \mid g_j^u(u) = u^\top P_j^u u + u^\top q_j^u + c_j^u \leq 0, j \in \mathbb{I}_{[1,l^u]} \right\} \end{aligned} \tag{3.3.2}$$

for some

$$P_i^x \in \mathbb{R}^{n \times n}, P_j^u \in \mathbb{R}^{m \times m}, q_i^x \in \mathbb{R}^n, q_j^u \in \mathbb{R}^m, c_i^x \in \mathbb{R}, c_j^u \in \mathbb{R}$$

with $P_i^x, P_j^u \geq 0$.

Remark 3.3.1. The particular form of the above constraints will be employed almost everywhere throughout this thesis. They incorporate the most common choices of constraint sets, such as ellipsoidal, hyperbox or polytopic sets (and combinations thereof), by which we mean sets of the form

$$\left\{ x \in \mathbb{R}^n \mid x^\top P x + x^\top q + c \leq 0 \right\}, \tag{3.3.3}$$

$$\begin{aligned}&[a_1, b_1] \times \cdots \times [a_n, b_n] \\ &= \left\{x \in \mathbb{R}^n \mid (x_1 - c_1)^2 - d_1 \leq 0, \ldots, (x_n - c_n)^2 - d_n \leq 0\right\},\end{aligned} \tag{3.3.4}$$

and

$$\{x \in \mathbb{R}^n \mid Hx \leq b\}, \tag{3.3.5}$$

respectively, for suitably defined $P \in \mathbb{R}^{n \times n}, q \in \mathbb{R}^n, a_i, b_i, c_i, d_i, c \in \mathbb{R}$, $H \in \mathbb{R}^{l \times n}$, $b \in \mathbb{R}^l$. However, as we will see in the following, the upcoming results mainly take advantage of the quadratic parts of the functions g_i^x, g_j^u in (3.3.2). Therefore, for their application, the above constraints should be parametrized such that the matrices P_i^x, P_j^u do not vanish. Clearly, as it can be seen from (3.3.3) and (3.3.4), this is possible for any ellipsoidal or hyperbox set and combinations thereof. Moreover, convex polytopes are defined as the intersection of finitely many half-spaces. Thus, given that they are compact, they can as well be considered as the intersection of finitely many rotated hyperboxes, i.e., as the intersection of sets of the form

$$\left\{x \in \mathbb{R}^n \mid (z_1 - \tilde{c}_1)^2 - \tilde{d}_1 \leq 0, \ldots, (z_n - \tilde{c}_n)^2 - \tilde{d}_n \leq 0, z = Tx\right\} \tag{3.3.6}$$

for suitably defined $\tilde{c}_i, \tilde{d}_i \in \mathbb{R}$ and an appropriate rotation matrix $T \in \mathbb{R}^{n \times n}$. Any set of the form (3.3.6) can be written in the form (3.3.2) with non-vanishing matrices P_i^x, P_j^u. Thus, since convex compact polytopes are intersections of such sets, they can be written in the form (3.3.2) with non-zero P_i^x, P_j^u as well.

The following algorithm provides an explicit computational procedure for finding periodic orbits w.r.t. which the LQ-problem is strictly dissipative, similar to Algorithm 3.2.2. The main difference is that we employ the S-procedure to render the underlying optimization problem computationally tractable.

Algorithm 3.3.2. Consider the LQ-problem (2.1.1) with constraints of the form (3.3.2). Given a set of functions $\Lambda \subseteq C(\mathbb{R}^n)$, solve the following optimization problem:

$$\begin{aligned} \underset{\substack{\lambda_i^x, \lambda_i^f, \lambda_j^u \geq 0 \\ \gamma \in \mathbb{R},\, \lambda \in \Lambda}}{\mathit{maximize}} \quad & \gamma \\ \text{s.t.} \quad & \ell(x,u) - \gamma + \lambda(x) - \lambda(Ax+Bu) \\ & + \sum_{i=1}^{l^x} \lambda_i^x g_i^x(x) + \lambda_i^f g_i^x(Ax+Bu) + \sum_{j=1}^{l^u} \lambda_j^u g_j^u(u) \geq 0 \\ & \text{for all } (x,u) \in \mathbb{R}^{n+m}. \end{aligned} \tag{3.3.7}$$

Denote the optimal quantities by $\lambda_i^{x*}, \lambda_i^{f^*}, \lambda_j^{u*}, \gamma^*, \lambda^*$ and define

$$\begin{aligned} \tilde{\ell}_\lambda^*(x,u) = {} & \ell(x,u) - \gamma^* + \lambda^*(x) - \lambda^*(Ax+Bu) \\ & + \sum_{i=1}^{l^x} \lambda_i^{x*} g_i^x(x) + \lambda_i^{f^*} g_i^x(Ax+Bu) + \sum_{j=1}^{l^u} \lambda_j^{u*} g_j^u(u). \end{aligned}$$

Further, define the set of points for which the constraint in (3.3.7) holds with equality as

$$M := \left\{ (x,u) \in \mathbb{R}^{n+m} \mid \tilde{\ell}_\lambda^*(x,u) = 0 \right\}.$$

Consider all points in M that satisfy the *complementary slackness* condition:

$$\begin{aligned} M_c := \Big\{ (x,u) \in M \mid \lambda_i^{x*} g_i^x(x) = 0, \lambda_i^{f^*} g_i^x(Ax+Bu) = 0, \lambda_j^{u*} g_j^u(u) = 0 \\ \text{for } i \in \mathbb{I}_{[1,l^x]}, j \in \mathbb{I}_{[1,l^u]} \Big\}. \end{aligned}$$

Return $\mathbb{Z} \cap M_c$.

Given a set of functions $\Lambda \subseteq C(\mathbb{R}^n)$, we denote the corresponding output of Algorithm 3.3.2 by $\Pi^\Lambda = \mathbb{Z} \cap M_c$. The following theorem is the main result of this section. It is an adaption of Theorem 3.2.3 to Algorithm 3.3.2 with two essential differences. First, the sufficient conditions for strict dissipativity w.r.t. a periodic orbit are, at least for polynomial storage functions, computationally tractable. The converse result, however, does not provide necessary conditions for the existence of a periodic orbit w.r.t. which the LQ-problem is strictly dissipative.

Theorem 3.3.3. Consider the LQ-problem (2.1.1) with compact state and input constraints of the form (3.3.2) and denote the output of Algorithm 3.3.2, given some $\Lambda \subseteq C(\mathbb{R}^n)$, by Π^Λ. Then, the following statements hold:

(i) If, for some $\Lambda \subseteq C(\mathbb{R}^n)$, $\Pi^\Lambda \neq \emptyset$ is a periodic orbit, then the LQ-problem is strictly dissipative and has the turnpike property w.r.t. Π^Λ.

(ii) Suppose that the system is strictly dissipative w.r.t. some periodic orbit Π with storage function λ. If $\Pi^{\{\lambda\}} \neq \emptyset$, [3] then $\Pi = \Pi^{\{\lambda\}}$.

Proof. **(i):** Since $\Pi^\Lambda \neq \emptyset$, there are optimal parameters for which the constraint in (3.3.7) holds. Therefore, by the S-procedure,

$$\tilde{\ell}^*(x,u) := \ell(x,u) - \gamma^* + \lambda^*(x) - \lambda^*(Ax+Bu) \geq 0 \;\forall\,(x,u) \in \mathbb{Z}, \tag{3.3.8}$$

where the asterisks denote the optimal quantities obtained from Algorithm 3.3.2. Further, using the complementary slackness conditions in M_c, the non-strict inequality (3.3.8) is fulfilled with equality exactly on $\mathbb{Z} \cap M_c$, i.e.,

$$\begin{aligned} \tilde{\ell}^*(x,u) &> 0 \;\forall\,(x,u) \in \mathbb{Z} \setminus M_c, \\ \tilde{\ell}^*(x,u) &= 0 \;\forall\,(x,u) \in \mathbb{Z} \cap M_c = \Pi^\Lambda. \end{aligned}$$

The remainder of the proof is completely analogous to the proof of Theorem 3.2.3 and therefore omitted.

(ii): First, we note that this statement does not trivially follow from the uniqueness of strict dissipativity w.r.t. periodic orbits since we have not yet proven that $\Pi^{\{\lambda\}}$ is a periodic orbit. Let $\Pi = \left\{(x^1,u^1),\dots,(x^P,u^P)\right\}$ and $\tilde{\ell}(x,u) := \ell(x,u) - \ell_\Pi + \lambda(x) - \lambda(Ax+Bu)$. Then,

$$\begin{aligned} \sum_{i=1}^{P} \tilde{\ell}^*(x^i,u^i) &= \sum_{i=1}^{P} \ell(x^i,u^i) - \gamma^* + \lambda(x^i) - \lambda(Ax^i + Bu^i) \\ &= P \cdot (\ell_\Pi - \gamma^*) \geq 0. \end{aligned}$$

Hence, $\tilde{\ell}^*(x,u) \geq \tilde{\ell}(x,u) \geq 0$ for any $(x,u) \in \mathbb{Z}$. By definition, $\tilde{\ell}^*(x,u)$ vanishes on $\Pi^{\{\lambda\}}$ and therefore the above inequalities hold with equality, i.e., $\tilde{\ell}^*(x,u) = 0$

[3] $\Pi^{\{\lambda\}} = \Pi^\Lambda$ where Λ consists only of the function λ, i.e., the only admissible storage function that we allow for in Algorithm 3.3.2 is λ.

as well as $\tilde{\ell}(x,u) = 0$ for any $(x,u) \in \Pi^{\{\lambda\}}$. Since $\tilde{\ell}^*(x,u)$ and $\tilde{\ell}(x,u)$ differ by a constant, this constant must be zero, i.e., $\ell_\Pi = \gamma^*$, and thus $\tilde{\ell}^*(x,u) = \tilde{\ell}(x,u)$ for any $(x,u) \in \mathbb{Z}$. Finally, considering the strictness of the dissipation inequalities, one readily verifies that $\Pi = \Pi^{\{\lambda\}}$. □

Remark 3.3.4. If we consider only quadratic storage functions of the form $\lambda(x) = x^\top P x + q^\top x$, then the optimization problem (3.3.7) can be solved efficiently by reformulating the constraint as the LMI

$$\begin{pmatrix} Q_\lambda & S_\lambda^\top & \frac{1}{2}s_\lambda \\ S_\lambda & R_\lambda & \frac{1}{2}v_\lambda \\ \frac{1}{2}s_\lambda^\top & \frac{1}{2}v_\lambda^\top & c_\lambda \end{pmatrix} \geq 0, \tag{3.3.9}$$

where

$$Q_\lambda = P - A^\top P A + Q + \sum_{i=1}^{l^x} \lambda_i^x P_i^x + \lambda_i^f A^\top P_i^x A,$$

$$S_\lambda = S - B^\top P A + \sum_{i=1}^{l^x} \lambda_i^f B^\top P_i^x A,$$

$$R_\lambda = R - B^\top P B + \sum_{i=1}^{l^x} \lambda_i^f B^\top P_i^x B + \sum_{j=1}^{l^u} \lambda_j^u P_j^u,$$

$$s_\lambda = q - A^\top q + s + \sum_{i=1}^{l^x} \lambda_i^x q_i^x + \lambda_i^f A^\top q_i^x,$$

$$v_\lambda = v - B^\top q + \sum_{i=1}^{l^x} \lambda_i^f B^\top q_i^x + \sum_{j=1}^{l^u} \lambda_j^u q_j^u,$$

$$c_\lambda = c - \gamma + \sum_{i=1}^{l^x} \lambda_i^x c_i^x + \lambda_i^f c_i^x + \sum_{j=1}^{l^u} \lambda_j^u c_j^u.$$

After maximizing γ subject to the LMI (3.3.9), Algorithm 3.3.2 amounts to determining M as well as M_c which involves the solution of algebraic equations. Assuming a polynomial storage function, one could also reformulate the optimization problem (3.3.7) as an SOS program, thereby keeping the algorithm tractable. In general, we can restrict the choice of Λ to any set of functions that simplifies the computations. Of course, one should keep in mind that this introduces conservatism since there might exist feasible storage functions which are not in the chosen set Λ. Nevertheless, we only need to find a set of parameters that renders the constraint of problem (3.3.7) feasible and for which $\mathbb{Z} \cap M_c$ is a non-empty periodic orbit. In this case, due to the uniqueness of periodic strict dissipativity, $\mathbb{Z} \cap M_c$ is *the* periodic orbit w.r.t. which the system is strictly dissipative. The above LMI (3.3.9) confirms the claim made in Remark 3.3.1: Since the matrices $P_i^x, P_j^u \geq 0$ appear mainly on the block-diagonal terms in (3.3.9), the feasibility of the LMI is enhanced when they are non-zero. Thus, the constraints (3.3.2) should be parametrized such that they do not vanish.

Remark 3.3.5. Necessary conditions for the LMI (3.3.9) are $Q_\lambda \geq 0$ and $R_\lambda \geq 0$. When the cost is assumed to be indefinite, in most cases some of the multipliers $\lambda_i^x, \lambda_i^f, \lambda_j^u$ have to be non-zero in order to render these two LMIs feasible. However, for showing strict dissipativity w.r.t. a periodic orbit with Theorem 3.3.3, the complementary slackness condition must hold on the orbit resulting from the algorithm. Hence, for all non-zero multipliers, the corresponding constraint functions must be zero, i.e., the respective modes lie on the boundary of the constraints. Thus, when we allow for negative eigenvalues in the cost and consider quadratic storage functions, then the optimal periodic orbit resulting from Theorem 3.3.3 is on the boundary of the constraints.

Remark 3.3.6. The assumptions $\Lambda = \{\lambda\}$ and $\Pi^\Lambda \neq \emptyset$ in part (ii) of Theorem 3.3.3 are considerably stronger than simply demanding $\lambda \in \Lambda$ as it was done in Theorem 3.2.3. In particular, the assumption that $\Pi^{\{\lambda\}}$ is non-empty indicates that Theorem 3.3.3 (ii) fails to provide truly necessary conditions for strict dissipativity w.r.t. a periodic orbit. When $\Pi^{\{\lambda\}} = \emptyset$, we cannot use Algorithm 3.3.2 to conclude anything about the existence of a periodic orbit w.r.t. which the LQ-problem is strictly dissipative. There are multiple situations in which this might occur: First of all, it might happen due to the failure of the S-procedure to provide necessary conditions. To see this, suppose that the system is strictly dissipative w.r.t. some periodic orbit Π. Clearly, $\Pi^{\{\lambda\}} = \emptyset$ implies $\tilde{\ell}^*(x,u) > 0$ for all $(x,u) \in \mathbb{Z}$. Summing this up along Π, we conclude that $\ell_\Pi > \gamma^*$. It is readily seen that $\ell_\Pi > \gamma^*$ is the case if and only if the converse direction of the S-procedure fails, i.e., when $\tilde{\ell}(x,u) \geq 0$ for all $(x,u) \in \mathbb{Z}$ does not imply the existence of multipliers such that $\tilde{\ell}(x,u) + \sum_{i=1}^{l^x} \lambda_i^x g_i^x(x) + \lambda_i^f g_i^x(Ax+Bu) + \sum_{j=1}^{l^u} \lambda_j^u g_j^u(u) \geq 0$ for all $(x,u) \in \mathbb{R}^{n+m}$ and thus, we cannot use ℓ_Π as candidate solution for (3.3.7).

Another reason for the failure of Algorithm 3.3.2 might be due to the complementary slackness conditions. As discussed in Remark 3.3.5, for indefinite cost functions, all elements of the optimal periodic orbit must lie on the boundary of the constraints. Thus, as the following example illustrates, when parts of the periodic orbit are in the interior of the constraints, then Algorithm 3.3.2 cannot be used to find this orbit.

Example 3.3.7. Consider the indefinite LQ-problem with

$$x(k+1) = x(k) + u(k),\ \ell(x,u) = -x^2 - 4u^2,\ \mathbb{X} = [0,2],\ \mathbb{U} = [-1,1].$$

It is straightforward to see that the optimal trajectory of this system is the periodic orbit $\Pi = \{(2,-1),(1,1)\}$. Considering only quadratic storage functions, an application of Algorithm 3.3.2 yields three non-zero multipliers. Hence, the states and inputs in M_c can only consist of boundary points of $\mathbb{X}$ and $\mathbb{U}$, respectively. Therefore, it is impossible to arrive at the periodic orbit Π from above, using Theorem 3.3.3. This is confirmed by the execution of Algorithm 3.3.2 which reveals $M_c = \emptyset$.

To conclude, we cannot expect results on necessity from Algorithm 3.3.2 since, even if we knew the storage function of a periodic orbit w.r.t. which the LQ-problem is strictly dissipative, the algorithm might still fail to find this orbit.

The complementary slackness condition in Algorithm 3.3.2, which is well-known from Lagrange duality theory, indicates a connection between the present approach and some constrained optimization problem. This connection becomes more transparent when considering strict dissipativity w.r.t. steady-states, since the latter is basically a procedure for solving an optimal steady-state problem. Therefore, we discuss the appearance of the complementary slackness condition in Section 4.1 (Remark 4.1.7) and take it, for now, simply as an algebraic condition that ensures strict dissipativity.

Example 3.3.8. Consider the indefinite LQ-problem with

$$\begin{aligned}&x(k+1) = -2x(k) + u(k),\ \ell(x,u) = -x^2,\ \mathbb{X} = [-10,10],\ \mathbb{U} = [-1,1],\\ &\text{i.e., } g_1^x(x) = x^2 - 100,\ g_1^u(u) = u^2 - 1.\end{aligned}$$

In order to apply Algorithm 3.3.2, we consider Λ to be the set of quadratic functions, i.e.,

$$\Lambda = \left\{\lambda \in C\left(\mathbb{R}^n\right) \mid \lambda(x) = x^\top P x + q^\top x,\ P \in \mathbb{R}^{n\times n},\ q \in \mathbb{R}^n\right\}.$$

Maximizing γ subject to the LMI (3.3.9) yields $\gamma^* = -1, \lambda_1^{x*} = \lambda_1^{f^*} = 0,\ \lambda_1^{u*} = 1, P = -1, q = 0$, and the constraint function of the optimization problem (3.3.7) for these optimal parameters becomes

$$\ell_\lambda^*(x,u) = \begin{pmatrix} x \\ u \end{pmatrix}^\top \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}.$$

Clearly, the corresponding kernel is

$$M = \operatorname{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}.$$

Furthermore, by including the complementary slackness condition, we arrive at

$$M_c = \left\{(x,u) \in M \mid g_1^u(u) = u^2 - 1 = 0\right\} = \left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\end{pmatrix}\right\}.$$

Consequently, $\Pi^{\Lambda} = \mathbb{Z} \cap M_c = M_c \neq \emptyset$ and hence the system is strictly dissipative w.r.t. the periodic orbit Π^{Λ} and has the turnpike property w.r.t. Π^{Λ}. This is confirmed by a simulation where the optimal control problem was solved numerically using the MATLAB-solver *fmincon*, cf. Figure 3.1. One can see that the trajectory first converges to a neighborhood of the periodic orbit and then leaves the periodic orbit towards the end of the simulation horizon.

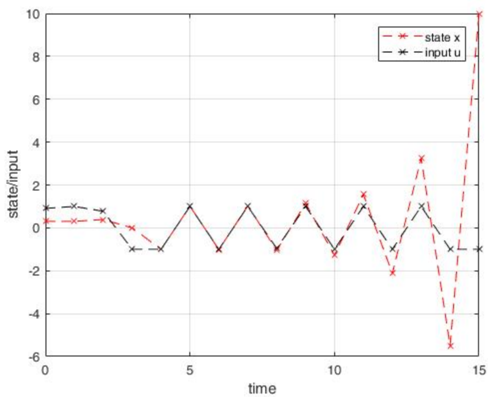

Figure 3.1.: Optimal trajectories for the optimal control problem stated in Example 3.3.8 with $N = 15, x_0 = 0.3$.

3.3.2. Symmetric LQ-problems

Example 3.3.8 exhibits an interesting connection between the optimal periodic orbit and the null-set of the non-strict dissipation inequality. It can be seen that the periodic orbit spans the set of points for which the modified cost $\ell_\lambda^*(x,u)$ is equal to zero, i.e., it spans the *kernel* of the matrix

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

The occurrence of such a case simplifies the application of Algorithm 3.3.2, since M can be computed by finding the kernel of a matrix, i.e., by solving a system of linear equations, and M_c is then the intersection of this linear subspace with the complementary slackness condition. Moreover, the above phenomenon indicates a relation between strict dissipativity of a constrained LQ-problem w.r.t. a periodic orbit and strict dissipativity of an unconstrained LQ-problem with modified cost $\ell_\lambda^*(x,u)$ w.r.t. the subspace spanned by the periodic orbit. Therefore, it is of interest to find situations in which this phenomenon appears. Example 3.3.8 shows a certain symmetry, i.e., $\ell(-x,-u) = \ell(x,u)$ and $\mathbb{X}$ and $\mathbb{U}$ are symmetric w.r.t. zero. In fact, one sees from the LMI (3.3.9) that, when assuming a quadratic storage function, this symmetry assumption implies that M is a linear subspace: As already mentioned in Remark 3.3.1, the matrices P_i^x and P_j^u should be chosen preferably non-zero and therefore we assume that, for symmetric constraints, $q_i^x = 0$, $q_j^u = 0$ $(i \in \mathbb{I}_{[1,l^x]}$, $j \in \mathbb{I}_{[1,l^u]})$. Moreover, due to the symmetry in the cost, we have $s = 0$ as well as $v = 0$. Hence, in view of the LMI (3.3.9), there is no use in choosing a storage function $\lambda(x) = x^\top Px + q^\top x$ with $q \neq 0$ since q, $A^\top q$ and $B^\top q$ are the only terms appearing in the $(1,3),(2,3),(3,1),(3,2)$-blocks of the LMI. [4] Consequently, all linear terms in the dissipation inequality vanish. Moreover, due to the maximization in γ, any optimal solution of (3.3.7) satisfies $c_\lambda = 0$. To conclude, in the symmetric case, we can replace the LMI (3.3.9) by

$$H_\lambda = \begin{pmatrix} Q_\lambda & S_\lambda^\top \\ S_\lambda & R_\lambda \end{pmatrix} \geq 0 \quad \text{and} \quad c_\lambda = 0 \tag{3.3.10}$$

with $Q_\lambda, S_\lambda, R_\lambda, c_\lambda$ as in (3.3.9). Then, given the optimal multipliers $\lambda_i^{x*}, \lambda_i^{f^*}, \lambda_j^{u*}$, the set M is readily computed as the kernel of H_λ, i.e., $M = \ker(H_\lambda)$. Thereafter,

[4] Block-diagonal entries of a positive semidefinite matrix are positive semidefinite themselves. Therefore, positive semidefiniteness of the matrix in (3.3.9) for some s_λ, v_λ implies positive semidefiniteness of the same matrix with $s_\lambda = 0, v_\lambda = 0$.

it only remains to check whether the complementary slackness conditions reduce $\ker(H_\lambda)$ to a periodic orbit.

Since $\ker(H_\lambda)$ is a linear subspace of $\mathbb{R}^{n+m}$ and the constraints are assumed to be symmetric, we have that, whenever $(x,u) \in \Pi$ for some periodic orbit Π resulting from Algorithm 3.3.2, then $-(x,u) \in \Pi$, i.e., $\Pi = -\Pi$. [5] This symmetry property of the periodic orbit holds even in a more general case when we allow for non-quadratic storage functions. Although the proof involves only simple algebraic arguments, we consider it important enough to state it as an explicit result.

Proposition 3.3.9. Consider the LQ-problem (2.1.1) with even stage cost $\ell(x,u)$ and symmetric constraints $\mathbb{X}$ and $\mathbb{U}$ of the form (3.3.2), i.e., $\ell(x,u) = \ell(-x,-u)$ for all $(x,u) \in \mathbb{Z}$, as well as $\mathbb{X} = -\mathbb{X}$ and $\mathbb{U} = -\mathbb{U}$. Suppose that the system is strictly dissipative w.r.t. the periodic orbit Π. Then, Π is symmetric, i.e., $\Pi = -\Pi$.

Proof. Using the symmetry assumptions on $\ell, \mathbb{X}, \mathbb{U}$, as well as the fact that $\tilde{\ell}(x,u) = \ell(x,u) - \ell_\Pi + \lambda(x) - \lambda(Ax+Bu)$ is zero on Π, summing up $\tilde{\ell}(x,u)$ along $-\Pi$ yields

$$\begin{aligned}
\sum_{i=1}^{P} \tilde{\ell}(-x^i,-u^i) &= \sum_{i=1}^{P} \ell(-x^i,-u^i) - \ell_\Pi + \lambda(-x^i) - \lambda(-Ax^i - Bu^i) \\
&= \sum_{i=1}^{P} \ell(x^i,u^i) - \ell_\Pi + \lambda(-x^i) - \lambda(-Ax^i - Bu^i) \\
&= \sum_{i=1}^{P} \tilde{\ell}(x^i,u^i) + \lambda(-x^i) - \lambda(-Ax^i - Bu^i) \\
&\quad - \lambda(x^i) + \lambda(Ax^i + Bu^i) \\
&= 0.
\end{aligned}$$

Since $\tilde{\ell}(-x^i,-u^i) \geq 0$, we conclude that $\tilde{\ell}(-x^i,-u^i) = 0$ for any $i \in \mathbb{I}_{[1,P]}$, i.e., $\tilde{\ell}(x,u)$ vanishes on $-\Pi$ and thus $-\Pi \subseteq \Pi$. Now, take any $(x,u) \in \Pi$. Then, due to the above work, $-(x,u) \in \Pi$ and hence $(x,u) \in -\Pi$. This implies $\Pi = -\Pi$, which concludes the proof. □

Remark 3.3.10. First, we note that the result from Proposition 3.3.9 can be extended to more general constraints than (3.3.2), possibly non-compact, with the only requirement that they are symmetric. We can even relax the linearity assumption on the system dynamics - the proof works for any dynamical system $x(k+1) = f(x(k),u(k))$

[5] Given a subset L of some euclidean space $\mathbb{R}^a$, we define $-L := \{l \in \mathbb{R}^a \mid -l \in L\}$.

with a vector field f which is odd in both arguments, i.e., $f(x,u) = -f(-x,-u)$. Although the proof of Proposition 3.3.9 requires only algebraic manipulations of the dissipation inequality, it gives interesting insights into the nature of optimal periodic orbits for symmetric LQ-problems. In particular, under the assumptions of Proposition 3.3.9, the following statements hold true:

- The period P of the optimal periodic orbit Π is even if and only if $\Pi \neq \{(0,0)\}$.
- If the LQ-problem is strictly dissipative w.r.t. a steady-state, then this steady-state is $(0,0)$.
- For $P = 2$, the stage cost is constant along Π.

3.4. A direct computational approach via sum-of-squares

This section presents an alternative approach for showing strict dissipativity w.r.t. a *given* periodic orbit. The main idea lies in the approximation of the point-to-set distance $|(x,u)|_\Pi$ by a polynomial function. This allows for the verification of the dissipation inequality using methods from SOS programming. Given a periodic orbit $\Pi = \{(x^1,u^1),\dots,(x^P,u^P)\}$, we define a distance-like function as

$$d_\Pi(x,u) = \left\| \begin{pmatrix} x \\ u \end{pmatrix} - \begin{pmatrix} x^1 \\ u^1 \end{pmatrix} \right\|^2 \cdot \ldots \cdot \left\| \begin{pmatrix} x \\ u \end{pmatrix} - \begin{pmatrix} x^P \\ u^P \end{pmatrix} \right\|^2 .$$

Obviously, $d_\Pi(x,u)$ is nonnegative, zero if and only if $(x,u) \in \Pi$, and radially unbounded in (x,u). Thus, it qualifies as distance w.r.t. the set Π. Strict dissipativity w.r.t. Π then amounts to finding a storage function λ which is bounded appropriately, and a scalar $\gamma > 0$ such that the following inequality holds for all $(x,u) \in \mathbb{Z}$:

$$\ell(x,u) - \ell_\Pi + \lambda(x) - \lambda(Ax + Bu) \geq \gamma \cdot d_\Pi(x,u). \tag{3.4.1}$$

It follows from Lemma A.1.2 that, for compact constraints, inequality (3.4.1) implies the existence of a class $\mathcal{K}_\infty$ function such that the periodic dissipation inequality (3.1.1) holds. Thus, the approach presented in the following can be seen as a sufficient condition for periodic dissipativity, when the corresponding periodic orbit is known in advance. In the following, we assume that the constraints are compact and of the form (3.3.2). Assuming a polynomial storage function, inequality (3.4.1) can then be verified computationally using SOS programming. A polynomial $p : \mathbb{R}^n \to \mathbb{R}$ is called SOS if there exist polynomials $q_i : \mathbb{R}^n \to \mathbb{R}, i \in \mathbb{I}_{[1,r]}$ such that

$$p(x) = \sum_{i=1}^{r} q_i(x)^2.$$

Note that, if p is SOS, then $p(x) \geq 0$ for all $x \in \mathbb{R}^n$. However, as already shown by Hilbert in the 19th century, the converse is in general not true. As detailed in [16], the question whether a given polynomial is SOS can be answered using LMI techniques. The main idea lies in writing p as a, in general non-unique, factorization

$$p(x) = h(x)^\top M h(x)$$

with $h(x)$ being the vector of monomials up to a certain degree. The problem then amounts to finding a matrix M which is positive semidefinite. This technique even allows p to contain decision variables that enter M in an affine fashion. Now, one can combine this setting with S-procedure-like conditions for checking nonnegativity of polynomials on a given set of the form $\mathbb{P} = \left\{(x \in \mathbb{R}^n \mid g_i(x) \leq 0,\ i \in \mathbb{I}_{[1,l]}\right\}$ for arbitrary polynomial functions g_i: A polynomial p is nonnegative on $\mathbb{P}$ if there exist SOS polynomials Λ_i such that $p + \sum_{i=1}^{l} \Lambda_i g_i$ is SOS. For practical purposes, we restrict the degree of the Λ_i's to be less than or equal to the degree of p. Then, a sufficient condition for inequality (3.4.1) to hold on constraints of the form (3.3.2) can be given in terms of LMIs. This is illustrated in the following example. All computations were performed using the sum-of-squares module of the MATLAB toolbox Yalmip [15, 16] in combination with the solver MOSEK [18].

Example 3.4.1. Consider the indefinite LQ-problem from Example 3.2.1, i.e.,

$$x(k+1) = x(k) + u(k),\ \ell(x,u) = -u^2,\ \mathbb{X} = [-1,1],\ \mathbb{U} = [-2,2].$$

As discussed in Example 3.2.1, the optimal periodic orbit is $\Pi = \{(-1,2),(1,-2)\}$ with $\ell_\Pi = -4$. We confirm this statement by verifying inequality (3.4.1) via SOS programming. To do so, we define the distance-like function

$$\begin{aligned} d_\Pi(x,u) &= \left\| \begin{pmatrix} x \\ u \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\|^2 \cdot \left\| \begin{pmatrix} x \\ u \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\|^2 \\ &= \left((x+1)^2 + (u-2)^2\right) \cdot \left((x-1)^2 + (u+2)^2\right). \end{aligned}$$

Moreover, the constraints are written as

$$\begin{aligned} \mathbb{Z} = \Big\{(x,u) \in \mathbb{R}^2 \mid\ & g_1(x,u) = x^2 - 1 \leq 0, g_2(x,u) = u^2 - 4 \leq 0, \\ & g_3(x,u) = (x+u)^2 - 1 \leq 0\Big\}. \end{aligned}$$

Finally, to conclude that inequality (3.4.1) holds, we need to find a polynomial storage function $\lambda : \mathbb{R} \to \mathbb{R}$, SOS polynomials $\Lambda_i : \mathbb{R}^2 \to \mathbb{R}, i \in \mathbb{I}_{[1,3]}$, as well as some scalar $\gamma > 0$ such that the function

$$\ell(x,u) - \ell_\Pi + \lambda(x) - \lambda(Ax + Bu) - \gamma \cdot d_\Pi(x,u) + \sum_{i=1}^{3} \Lambda_i(x,u) g_i(x,u)$$

is SOS. Using Yalmip with the solver MOSEK, we arrive at the following parameters which render inequality (3.4.1) feasible: $\gamma = 0.12, \lambda(x) = -0.5x^2$ and the SOS polynomials

$$\begin{aligned}
\Lambda_1(x,u) &= 0.261x^4 - 0.068x^3u + 0.162x^2u^2 + 0.118xu^3 + 0.083u^4 \\
&\quad + 0.12x^2 - 0.065xu + 0.034u^2 + 0.119, \\
\Lambda_2(x,u) &= 0.064x^4 + 0.067x^3u + 0.113x^2u^2 + 0.082xu^3 + 0.031u^4 \\
&\quad - 0.129x^2 - 0.067xu - 0.001u^2 + 0.064, \\
\Lambda_3(x,u) &= 0.188x^4 + 0.086x^3u + 0.144x^2u^2 + 0.174xu^3 + 0.098u^4 \\
&\quad + 0.066x^2 + 0.352xu + 0.14u^2 + 0.245.
\end{aligned}$$

It is straightforward to verify that these polynomials, indeed, render the above-defined function nonnegative. Hence, we conclude that inequality (3.4.1) holds on $\mathbb{Z}$, i.e., the system is strictly dissipative w.r.t. Π in the sense of inequality (3.4.1).

4. Strict dissipativity and turnpike properties w.r.t. steady-states

This chapter is devoted to the analysis of strict dissipativity and turnpike properties w.r.t. steady-states. As explained in Section 2.2, this problem has been solved for positive (semi-)definite cost functions and steady-states in the interior of the constraints in the recent work [10]. We extend these results into several directions using two different approaches. Both of them do, in general, not require that $Q \geq 0$ and $R > 0$. To some extent, the first method can be viewed as an application of the framework from Chapter 3 to steady-states. However, since we want to deal with possibly unbounded constraints, we develop a slightly different framework which is essentially a combination of the S-procedure and the results from [10]. Nevertheless, it is in good accordance with the results for periodic orbits, given that the constraints are compact. The second method which we present in this chapter applies the theory from [10] to P-step systems, thereby relaxing the condition that Q needs to be positive semidefinite.

4.1. An S-procedure approach

Any steady-state is a periodic orbit with period one. Therefore, the theory from Section 3 can as well be applied to steady-states. This theory, however, relied heavily on Lemma A.1.2 which requires compact constraints. As we will see in the remainder of this section, we can establish computationally tractable and, partly, geometric conditions for strict dissipativity w.r.t. steady-states without making this assumption. To do so, we take a slightly different approach to the problem which is mainly a combination of the S-procedure and the theory from [10]. Nevertheless, the particular shape of the quadratic constraints (3.3.2) will play an important role. Furthermore, in this section, we consider the definitions of strict dissipativity and near equilibrium turnpikes from Section 2.2 which require strictness only in the state. This is mainly due to the fact that most of the upcoming results are based on [10], which uses the latter dissipativity notion. Moreover, one motivation for using strictness w.r.t. the input and the state in the previous chapter was the fact that, given a set of states that constitute a periodic orbit, it might not be obvious which inputs are required to patch these together in the correct order. Clearly, this is not an issue in the steady-state case. Nevertheless, we note that all subsequent results can be easily adapted to dissipativity with strictness in state and input.

The remainder of the section is structured as follows: First, we provide sufficient conditions for strict dissipativity based on LMIs and give an explicit convex computational procedure to compute the corresponding optimal steady-state. Herein, the key observation is that negative eigenvalues can be compensated by the specific shape of the constraints when the turnpike equilibrium lies on their boundary. In this case, the relation between negative eigenvalues and the exact location of the turnpike is even more direct than in the periodic case in Chapter 3. Although the result stated in Section 4.1.1 allows for an intuitive geometric characterization, we state an improved version in Section 4.1.2 which allows for showing strict dissipativity (and thus the near equilibrium turnpike property) for a significantly wider variety of problems. The section closes with a result on necessary and sufficient conditions for strict dissipativity with quadratic storage functions in case that there is only one input constraint.

4.1.1. A geometric characterization

The following theorem is a combination of the S-procedure and the recent work from [10]. For its statement, we need to assume that the coupling cost matrix S vanishes. Problems with non-vanishing coupling costs are the subject of Section 4.1.2. The theorem provides sufficient conditions for the existence of steady-states w.r.t. which the LQ-problem is strictly dissipative and thus has the near equilibrium turnpike property.

Theorem 4.1.1. Consider the LQ-problem (2.1.1) with $S = 0$ and constraints of the form (3.3.2). Suppose there exist $\lambda_i^x, \lambda_j^u \geq 0$ $\left(i \in \mathbb{I}_{[1,l^x]}, j \in \mathbb{I}_{[1,l^u]}\right)$ and a symmetric matrix P such that the LMIs

$$\begin{aligned} &Q + \sum_{i=1}^{l^x} \lambda_i^x P_i^x \geq 0, \quad R + \sum_{j=1}^{l^u} \lambda_j^u P_j^u \succ 0, \\ &Q + \sum_{i=1}^{l^x} \lambda_i^x P_i^x + P - A^\top P A \succ 0 \end{aligned} \tag{4.1.1}$$

hold. Assume further that $\mathbb{X}$ is bounded or there exist $\lambda_i^c \geq 0$ $\left(i \in \mathbb{I}_{[1,l^x]}\right)$ such that the LMI

$$P + \sum_{i=1}^{l^x} \lambda_i^c P_i^x \succ 0 \tag{4.1.2}$$

holds. Then, for those λ_i^x, λ_j^u, the modified LQ-problem with cost $\ell_\lambda(x,u) = \ell(x,u) + \sum_{i=1}^{l^x} \lambda_i^x g_i^x(x) + \sum_{j=1}^{l^u} \lambda_j^u g_j^u(u)$ is strictly dissipative at the steady-state

$$(x^e, u^e) = \underset{\substack{x \in \mathbb{X}, u \in \mathbb{U} \\ x = Ax + Bu}}{\operatorname{argmin}} \ \ell_\lambda(x,u). \tag{4.1.3}$$

Moreover, if the complementary slackness conditions

$$\lambda_i^x g_i^x(x^e) = 0, \ \ \lambda_j^u g_j^u(u^e) = 0 \tag{4.1.4}$$

hold for all $i \in \mathbb{I}_{[1,l^x]}$, $j \in \mathbb{I}_{[1,l^u]}$, then the original system with stage cost $\ell(x,u)$ is strictly dissipative at (x^e, u^e) and has the near equilibrium turnpike property.

Proof. The fact that, given the assumptions, the modified optimal control problem with stage cost $\ell_\lambda(x,u)$ is strictly cyclo-dissipative at (x^e, u^e) from (4.1.3), can be proven in analogy to the proof of [10, Lemma 4.1], where we employ the definiteness properties of the modified Q and R matrices. The corresponding storage function is then quadratic and of the form $\lambda(x) = \gamma \cdot x^\top P x + q^\top x$ for some $q \in \mathbb{R}^n, \gamma > 0$. By strict cyclo-dissipativity, there exists $\alpha \in \mathcal{K}_\infty$ such that

$$\ell_\lambda(x,u) - \ell_\lambda(x^e,u^e) + \lambda(x) - \lambda(Ax+Bu) \geq \alpha\left(\|x - x^e\|\right)$$

for all $(x,u) \in \mathbb{Z}$. Thus, due to $g_i^x(x) \leq 0, g_j^u(u) \leq 0$ as well as $\lambda_i^x g_i^x(x^e) = 0$, $\lambda_j^u g_j^u(u^e) = 0$ for all $i \in \mathbb{I}_{[1,l^x]}$, $j \in \mathbb{I}_{[1,l^u]}$, we arrive at

$$\begin{aligned}
&\ell(x,u) - \ell(x^e,u^e) + \lambda(x) - \lambda(Ax+Bu) \\
&\geq \sum_{i=1}^{l^x} \lambda_i^x \left(g_i^x(x^e) - g_i^x(x)\right) + \sum_{j=1}^{l^u} \lambda_j^u \left(g_j^u(u^e) - g_j^u(u)\right) + \alpha\left(\|x - x^e\|\right) \\
&\geq \sum_{i=1}^{l^x} \lambda_i^x g_i^x(x^e) + \sum_{j=1}^{l^u} \lambda_j^u g_j^u(u^e) + \alpha\left(\|x - x^e\|\right) = \alpha\left(\|x - x^e\|\right).
\end{aligned}$$

Hence, the original system is strictly cyclo-dissipative at (x^e, u^e). It remains to show that the storage function λ is bounded from below on $\mathbb{X}$. In case that $\mathbb{X}$ is bounded, this is trivial. Otherwise, we need the additional LMI $P + \sum_{i=1}^{l^x} \lambda_i^c P_i^x \succ 0$ which implies that the function $\tilde{\lambda}(x) = \lambda(x) + \gamma \sum_{i=1}^{l^x} \lambda_i^c g_i^x(x)$ is bounded from below on $\mathbb{R}^n$, i.e., there exists some constant $C \in \mathbb{R}$ such that $\tilde{\lambda}(x) \geq C \; \forall x \in \mathbb{R}^n$. Applying the S-procedure, we conclude that $\lambda(x) \geq C$ for all $x \in \mathbb{X}$, i.e., λ is bounded from below on $\mathbb{X}$. Combining these results, the LQ-problem is strictly dissipative and therefore has the near equilibrium turnpike property at (x^e, u^e) due to [9, Theorem 5.3]. □

Remark 4.1.2. Note that, for unbounded state constraints $\mathbb{X}$, the LMI (4.1.2) is indeed important in the proof of Theorem 4.1.1 since we need boundedness of the storage function from below to establish strict dissipativity and thus the near equilibrium turnpike property. Obviously, feasibility of this LMI does not require boundedness of $\mathbb{X}$. An intuitive explanation for this fact is that we do not need $\mathbb{X}$ to be bounded in the directions where P is positive definite. This allows us to consider certain cases where some modes corresponding to negative eigenvalues of Q are bounded and other (detectable) modes corresponding to nonnegative eigenvalues of Q are not - in these cases, when Theorem 4.1.1 is applicable, the latter modes span a subspace on which P is positive definite. It is worth noting that boundedness of $\mathbb{X}$ is not sufficient for the existence of multipliers λ_i^c such that $P + \sum_{i=1}^{l^x} \lambda_i^c P_i^x \succ 0$, since we can construct any bounded polytope using only affine terms, i.e., with $P_i^x = 0$ for all i.

Remark 4.1.3. First we note that, concerning the non-unique representation of constraints of the form (3.3.2), we arrive at the same conclusion as in Section 3.3: In view of the LMIs (4.1.1) and (4.1.2), it becomes apparent that the constraint functions g_i^x, g_j^u should be chosen such that the matrices P_i^x, P_j^u do not vanish. As discussed in Remark 3.3.1, this is possible for any ellipsoidal, hyperbox or convex compact polytopic set, as well as intersections thereof. Furthermore, the first two LMIs in (4.1.1) have the following intuitive geometric interpretation:

- $Q + \sum_{i=1}^{l^x} \lambda_i^x P_i^x \succeq 0$:

  Suppose Q has an eigenvalue $\lambda < 0$ with corresponding eigenvector v, i.e., $v^* Q v = \lambda \|v\|^2 < 0$. Then, in order to render the above LMI feasible, there must be some constraint with index k such that $\lambda_k^x v^* P_k^x v > 0$. Thus, $\lambda_k^x > 0$ as well as $v^* P_k^x v > 0$. From $v^* P_k^x v > 0$ we conclude that, to fulfill the sufficient conditions of Theorem 4.1.1, the constraints $\mathbb{X}$ must be bounded in direction v. Moreover, $\lambda_k^x \neq 0$ implies that the optimal steady-state of (4.1.3) has to satisfy $g_k^x(x^e) = 0$, i.e., x^e is located on the boundary of the k-th state constraint.

- $R + \sum_{j=1}^{l^u} \lambda_j^u P_j^u \succ 0$:

  This LMI allows for the same interpretation as $Q + \sum_{i=1}^{l^x} \lambda_i^x P_i^x \succeq 0$: If R has nonpositive eigenvalues and Theorem 4.1.1 can be applied, then the constraints must be bounded in direction of the corresponding eigenvectors and the optimal equilibrium input lies on the boundary.

This illustrates the main idea of the presented approach: Negative eigenvalues of the cost can be compensated by the boundary of the constraints. These interpretations are in good accordance with the results discussed in Remark 3.3.5, where we observed similar phenomena when using non-strict dissipation inequalities and complementary slackness to show strict dissipativity w.r.t. periodic orbits.

Example 4.1.4. Consider the indefinite LQ-problem with

$$x(k+1) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} x(k) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(k),\ \ell(x,u) = x_1^2 - 2x_2^2 + u^2,$$

$$\mathbb{X} = \left\{ x \in \mathbb{R}^2 \mid g_1^x(x) = x_1^2 + (x_2 - 1)^2 - 1 \leq 0 \right\},$$

$$\mathbb{U} = \left\{ u \in \mathbb{R} \mid g_1^u(u) = u^2 - 1 \leq 0 \right\}.$$

It is readily seen that $\mathbb{X}$ is bounded and the LMIs (4.1.1) in Theorem 4.1.1 are fulfilled for $P = 0, \lambda_1^x = 3, \lambda_1^u = 0$. The optimal steady-state from (4.1.3) can then be calculated as $(x^e, u^e) = ((0,2)^\top, 0)$. Since $g_1^x(x^e) = 0$ and $\lambda_1^u = 0$, the complementary slackness conditions (4.1.4) are satisfied and hence the system is strictly dissipative and has the near equilibrium turnpike property at (x^e, u^e). This can also be seen directly: $\mathbb{X}$ is a circle with radius 1 around $(0,1)$, thus $0 \leq x_2 \leq 2$. Therefore, there is no feasible point with lower cost than $x_1 = 0, x_2 = 2, u = 0$. The location of this optimal steady-state confirms the statement of Remark 4.1.3: Strict dissipativity occurs on the boundary of the constraints, in the direction corresponding to negative eigenvalues of Q and R. In principle, we could have chosen any $\lambda_1^x > 2$ to fulfill the LMIs (4.1.1). However, if λ_1^x is chosen too big (i.e. $\lambda_1^x > 4$), the optimal steady-state of (4.1.3) differs from the optimal steady-state of the original problem. Moreover, the LMIs (4.1.1) are also feasible for any $\lambda_1^u > 0$. In both cases, we could not conclude strict dissipativity since the complementary slackness conditions (4.1.4) would not be satisfied.

From Example 4.1.4, we deduce that there is a need for a systematic procedure of choosing the multipliers λ_i^x and λ_j^u before solving the optimal steady-state problem (4.1.3). Intuitively, they should be as small as possible (subject to the constraint that the LMIs (4.1.1) are feasible) to avoid that the modified optimal steady-state differs from the original one. Since the optimal steady-state is not known beforehand, we cannot decide which multipliers have to be zero in order to fulfill the complementary slackness conditions (4.1.4). Therefore, we propose an iterative procedure that solves problem (4.1.3) and then sets λ_i^x and λ_j^u to zero for those indices i and j for which (4.1.4) does not hold. For some real, nonnegative multipliers $\lambda_i^x, \lambda_j^u, \lambda_i^c$, we introduce the abbreviations $\Lambda^x = \left\{\lambda_1^x, \dots, \lambda_{l^x}^x\right\}, \Lambda^u = \left\{\lambda_1^u, \dots, \lambda_{l^u}^u\right\}$. The algorithm can then be stated as follows:

Algorithm 4.1.5.

- Step 0: Set $\mathbb{I}^x = \mathbb{I}^u = \emptyset$.
- Step 1: Choose Λ^x, Λ^u via the optimization problem

$$\begin{aligned} \underset{\substack{\lambda_i^x, \lambda_j^u, \lambda_i^c \geq 0 \\ P = P^\top \in \mathbb{R}^{n \times n}}}{\text{minimize}} \quad & \sum_{i=1}^{l^x} \lambda_i^x + \sum_{j=1}^{l^u} \lambda_j^u \\ \text{s.t.} \quad & \text{(4.1.1) and (if } \mathbb{X} \text{ unbounded) (4.1.2)}, \\ & \lambda_i^x = \lambda_j^u = 0 \ \left(i \in \mathbb{I}^x, j \in \mathbb{I}^u\right). \end{aligned} \tag{4.1.5}$$

  - Case 1.1 - (4.1.5) is feasible: Go to Step 2.
  - Case 1.2 - (4.1.5) is infeasible: Stop the algorithm.
- Step 2: Solve problem (4.1.3) with $\Lambda^x, \Lambda^u \to \left(x^e, u^e\right)$.
- Step 3: Check conditions (4.1.4) for Λ^x, Λ^u and $\left(x^e, u^e\right)$:
  - Case 3.1 - They hold: Stop the algorithm.
  - Case 3.2 - They do not hold for indices $I^x \subseteq \mathbb{I}_{[1,l^x]}$, $I^u \subseteq \mathbb{I}_{[1,l^u]}$: Set $\mathbb{I}^x = \mathbb{I}^x \cup I^x, \mathbb{I}^u = \mathbb{I}^u \cup I^u$ and go back to Step 1.

Since there are only finitely many constraint functions g_i^x, g_j^u, Algorithm 4.1.5 will always terminate, either in Case 1.2 or in Case 3.1. Assuming the latter, we have found a steady-state at which the LQ-problem (2.1.1) is strictly dissipative and has the near equilibrium turnpike property. Contrary, in Case 1.2, the algorithm cannot be used to draw conclusions about the properties of interest.

Remark 4.1.6. Note that a similar problem as in Example 4.1.4 occurs in the application of Theorem 3.3.3, when considering strict dissipativity w.r.t. periodic orbits. Before solving the optimization problem (3.3.7) in Algorithm 3.3.2, it is not known which multipliers need to be zero in order to render the complementary slackness condition feasible. Thus, when applying Theorem 3.3.3, one should also use an iterative procedure in the sense of Algorithm 4.1.5, which incorporates the complementary slackness conditions into the optimization problem by setting appropriate multipliers to zero.

Remark 4.1.7. The term *complementary slackness* is well-known from the standard literature on convex optimization (cf. e.g. [4]). In Theorem 4.1.1, it implies that the duality gap between the primal and dual optimal value of the optimal steady-state problem vanishes. To be more precise, consider the problem of finding the optimal steady-state of the original LQ-problem (2.1.1), i.e.,

$$\min_{\substack{x\in\mathbb{X},u\in\mathbb{U}\\ x=Ax+Bu}} \ell(x,u).$$

It follows directly from weak duality that this value is always greater than or equal to the optimal value of (4.1.3), i.e.,

$$\min_{\substack{x\in\mathbb{X},u\in\mathbb{U}\\ x=Ax+Bu}} \ell(x,u) \geq \min_{\substack{x\in\mathbb{X},u\in\mathbb{U}\\ x=Ax+Bu}} \ell_\lambda(x,u). \tag{4.1.6}$$

Denote the optimal equilibrium of (4.1.3) by (x^e,u^e). If the complementary slackness conditions hold for (x^e,u^e), then (4.1.6) even holds with equality since then

$$\min_{\substack{x\in\mathbb{X},u\in\mathbb{U}\\ x=Ax+Bu}} \ell(x,u) \leq \ell(x^e,u^e) = \ell_\lambda(x^e,u^e) = \min_{\substack{x\in\mathbb{X},u\in\mathbb{U}\\ x=Ax+Bu}} \ell_\lambda(x,u).$$

Thus, if complementary slackness holds, we have strong duality. Another connection of Theorem 4.1.1 to Lagrange duality theory is detailed in the following. Consider an indefinite quadratically constrained quadratic program (QCQP):

$$\begin{aligned} &\underset{x\in\mathbb{R}^n}{\mathit{minimize}} \begin{pmatrix} x \\ 1 \end{pmatrix}^\top \begin{pmatrix} P_0 & q_0 \\ q_0^\top & c_0 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} \\ &\quad s.t. \;\; g_i(x) = \begin{pmatrix} x \\ 1 \end{pmatrix}^\top \begin{pmatrix} P_i & q_i \\ q_i^\top & c_i \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} \leq 0, \;\; i \in \mathbb{I}_{[1,l]}, \end{aligned} \tag{4.1.7}$$

where $P_i \in \mathbb{R}^{n\times n}, q_i \in \mathbb{R}^n, c_i \in \mathbb{R}, i \in \mathbb{I}_{[0,l]}$, without any assumptions on definiteness of the P_i's. [1] Denote the constraints imposed on x in (4.1.7) by $\mathbb{X}$, i.e., $\mathbb{X} = \left\{x \in \mathbb{R}^n \mid g_i(x) \leq 0,\ i \in \mathbb{I}_{[1,l]}\right\}$. Then, the QCQP can be considered as an optimal steady-state problem for the LQ-problem

$$x(k+1) = x(k), \ell(x,u) = \begin{pmatrix} x \\ 1 \end{pmatrix}^\top \begin{pmatrix} P_0 & q_0 \\ q_0^\top & c_0 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} + u^2, \mathbb{X} \text{ as above}, \mathbb{U} = \mathbb{R}.$$

Theorem (4.1.1) delivers the following sufficient conditions for existence and uniqueness of an optimal steady-state (and therefore for existence and uniqueness of a solution to (4.1.7)): The LMIs (4.1.1) reduce to the existence of multipliers $\lambda_i \geq 0, i \in \mathbb{I}_{[1,l]}$, such that $P_0 + \sum_{i=1}^{l} \lambda_i P_i \succ 0$. [2] This implies that there is a unique solution to the modified optimal steady-state problem with cost

$$\ell_\lambda(x,u) = \begin{pmatrix} x \\ 1 \end{pmatrix}^\top \left(\begin{pmatrix} P_0 & q_0 \\ q_0^\top & c_0 \end{pmatrix} + \sum_{i=1}^{l} \lambda_i \begin{pmatrix} P_i & q_i \\ q_i^\top & c_i \end{pmatrix} \right) \begin{pmatrix} x \\ 1 \end{pmatrix} + u^2.$$

Then, as detailed above, the complementary slackness condition ensures that the solution to the modified optimal steady-state problem is also the optimal steady-state for the cost $\ell(x,u)$. From a Lagrange duality point of view, when neglecting the auxiliary variable u, the modified cost $\ell_\lambda(x,u)$ is nothing but the Lagrangian of the original QCQP (4.1.7) and the λ_i's are the dual variables. $P_0 + \sum_{i=1}^{l} \lambda_i P_i \succ 0$ implies strict convexity of $\ell_\lambda(x,u)$ and thus existence and uniqueness of a dual optimal solution. Then, it is well-known (cf. [4]) that complementary slackness guarantees that the duality gap vanishes and the dual optimal solution is also optimal for the primal problem. Thus, our approach to showing strict dissipativity by solving a sequence of convex optimization problems in Algorithm 4.1.5 can be seen as a relaxation for indefinite QCQPs which are, in general, np-hard problems [21].

[1] In Theorem 4.1.1, we considered only convex constraints with $P_i \succeq 0,\ i \in \mathbb{I}_{[1,l]}$, because all reasonable quadratic constraints are of that form (e.g., for scalar x, constraints with $P_k = -1, q_k = 0, c_k = 1$ for some k impose $x^2 \geq 1$ which yields a non-convex constraint set). Nevertheless, the proof of Theorem 4.1.1 does not fail when we allow for $P_i \not\succeq 0$ which is why the presented approach can also cope with problems of the general form from (4.1.7).

[2] For the illustration of the relation between our approach and QCQPs, the main object of interest is the optimization problem (4.1.3). This is directly related to strict cyclo-dissipativity, thus we are not interested in the conditions in Theorem 4.1.1 that ensure boundedness of the storage function from below.

4.1.2. Towards less conservative conditions

As detailed in Remark 4.1.7, the approach from Theorem 4.1.1 can be seen as a convex relaxation of the problem of finding the optimal steady-state of the indefinite LQ-problem. This relaxation is quite conservative since we make no use of the fact that the dissipation inequality only needs to hold for all (x,u) with $Ax+Bu \in \mathbb{X}$. A straightforward approach to incorporate this constraint lies in the use of a third set of multipliers λ_i^f in the cost functional, i.e., $\ell_\lambda(x,u) = \ell(x,u) + \sum_{i=1}^{l^x}\left(\lambda_i^x g_i^x(x) + \lambda_i^f g_i^x(Ax+Bu)\right) + \sum_{j=1}^{l^u}\lambda_j^u g_j^u(u)$. However, this cost contains couplings between input and state which is why we cannot apply [10, Lemma 4.1]. Nevertheless, if we demand strict convexity of the modified "rotated" cost $\ell_\lambda(x,u) + x^\top Px - (Ax+Bu)^\top P(Ax+Bu)$, we arrive at less conservative LMI-based conditions for strict dissipativity which can be checked efficiently. In this case, also non-zero coupling costs (i.e. stage costs with $S \neq 0$) can be considered. Given any $\lambda_i^x, \lambda_i^f, \lambda_j^u \geq 0$, $(i \in \mathbb{I}_{[1,l^x]}, j \in \mathbb{I}_{[1,l^u]})$, we define

$$
\begin{aligned}
Q_\lambda &= Q + \sum_{i=1}^{l^x}\left(\lambda_i^x P_i^x + \lambda_i^f A^\top P_i^x A\right),\\
S_\lambda &= \sum_{i=1}^{l^x}\lambda_i^f B^\top P_i^x A,\\
R_\lambda &= R + \sum_{i=1}^{l^x}\lambda_i^f B^\top P_i^x B + \sum_{j=1}^{l^u}\lambda_j^u P_j^u.
\end{aligned}
$$

Theorem 4.1.8. Consider the LQ-problem (2.1.1) with constraints of the form (3.3.2). Suppose there exist $\lambda_i^x, \lambda_i^f, \lambda_i^c, \lambda_j^u \geq 0$ $(i \in \mathbb{I}_{[1,l^x]}, j \in \mathbb{I}_{[1,l^u]})$ and a symmetric matrix P such that the LMI

$$
\begin{pmatrix} Q_\lambda + P - A^\top PA & S^\top + S_\lambda^\top - A^\top PB \\ S + S_\lambda - B^\top PA & R_\lambda - B^\top PB \end{pmatrix} \succ 0 \tag{4.1.8}
$$

holds. Assume further that $\mathbb{X}$ is bounded or there exist $\lambda_i^c \geq 0$ $(i \in \mathbb{I}_{[1,l^x]})$ such that the LMI

$$
P + \sum_{i=1}^{l^x}\lambda_i^c P_i^x \succ 0 \tag{4.1.9}
$$

holds. Then, for those $\lambda_i^x, \lambda_i^f, \lambda_j^u$, the modified LQ-problem with cost $\ell_\lambda(x,u) = \ell(x,u) + \sum_{i=1}^{l^x} \left(\lambda_i^x g_i^x(x) + \lambda_i^f g_i^x(Ax+Bu)\right) + \sum_{j=1}^{l^u} \lambda_j^u g_j^u(u)$ is strictly dissipative at the steady-state

$$(x^e, u^e) = \underset{\substack{x \in \mathbb{X}, u \in \mathbb{U} \\ x = Ax+Bu}}{\operatorname{argmin}} \ \ell_\lambda(x,u). \tag{4.1.10}$$

Moreover, if the complementary slackness conditions

$$\lambda_i^x g_i^x(x^e) = 0, \lambda_i^f g_i^x(x^e) = 0, \lambda_j^u g_j^u(u^e) = 0 \tag{4.1.11}$$

hold for all $i \in \mathbb{I}_{[1,l^x]}$, $j \in \mathbb{I}_{[1,l^u]}$, then the original system with stage cost $\ell(x,u)$ is strictly dissipative at (x^e, u^e) and has the near equilibrium turnpike property.

Proof. The proof is a combination of the proofs of Theorem 4.1.1 and [10, Lemma 4.1] with the only difference that strict convexity of the "rotated" cost $\ell_\lambda(x,u) + x^\top Px - (Ax+Bu)^\top P(Ax+Bu)$ is trivial when (4.1.8) holds. Therefore, it is omitted. □

Note that $Q_\lambda \geq 0$ and $R_\lambda > 0$ are not necessary for the LMI (4.1.8). Hence, we cannot draw the same conclusion on the direct relation between negative eigenvalues of the cost and occurrence of turnpikes on the boundary of the constraints, as we did in Remark 4.1.3. The following two examples illustrate that conservatism is indeed reduced by considering the additional constraint $Ax + Bu \in \mathbb{X}$ in Theorem 4.1.8.

Example 4.1.9. Consider the indefinite LQ-problem with

$$x(k+1) = u(k), \ \ell(x,u) = -x^2 - u^2, \ \mathbb{X} = [0,1], \ \mathbb{U} = [-2,2].$$

Clearly, the optimal steady-state is

$$(x^*(k,x_0), u^*(k,x_0)) = (1,1) \ \ \forall\, x_0 \in \mathbb{X}, k \geq 1.$$

This cannot be shown using Theorem 4.1.1 since, in order to render the LMIs feasible, both multipliers λ_1^x and λ_1^u have to be non-zero. However, for equilibria we have $u^e = x^e$ and thus the optimal equilibrium input lies in $[0,1]$, contradicting the complementary slackness condition for $\lambda_1^u g_1^u(u^e)$. Theorem 4.1.8 handles this problem by introducing the additional multiplier λ_1^f which renders the right-lower block of (4.1.8) feasible, although $\lambda_1^u = 0$. Since the optimal equilibrium state lies on $\partial\mathbb{X}$, the complementary slackness condition is fulfilled and thus the near equilibrium turnpike property holds.

Example 4.1.10. Consider the indefinite LQ-problem (2.1.1) with

$$x(k+1) = 2x(k) + u(k),\ \ell(x,u) = -x^2,\ \mathbb{X} = [-5,5],\ \mathbb{U} = [0,1],$$
$$\text{i.e., } g_1^x(x) = x^2 - 25,\ g_1^u(u) = u^2 - u.$$

We see that $\mathbb{X}$ is bounded and the LMIs from 4.1.8 are feasible for $\lambda_1^x = 0, \lambda_1^f = 0, \lambda_1^u = 2, P = -1$. The corresponding optimal steady-state of (4.1.10) can then be computed as $(x^e, u^e) = (-1, 1)$ and thus the complementary slackness conditions hold. Hence, the problem is strictly dissipative and has the near equilibrium turnpike property at (x^e, u^e). This is confirmed by simulations where the optimal control problem was solved numerically for various horizons using the MATLAB-solver *fmincon*, cf. Figure 4.1. It can be seen that not only the occurrence of the turnpike but also the exact location of it is influenced by the shape of the constraints. This is an interesting insight since such phenomena could not be investigated using the machinery from [10]. Note that, again, this example cannot be analyzed using Theorem 4.1.1 since the optimal equilibrium state lies in the interior of $\mathbb{X}$ for any choice of multipliers.

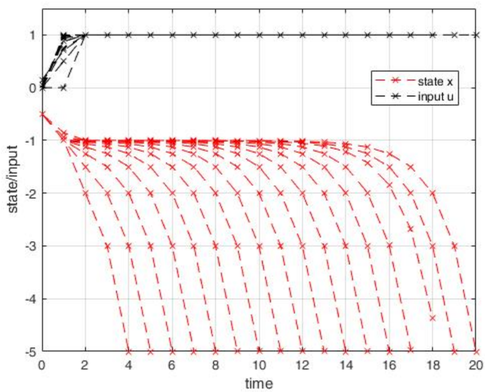

Figure 4.1.: Optimal trajectories for the optimal control problem stated in Example 4.1.10 with $N = 4, \ldots, 20,\ x_0 = -0.5$.

The preceding examples show that Theorem 4.1.8 includes cases that could not be analyzed using Theorem 4.1.1. It is easy to see that, when the coupling cost

S is zero, the LMIs in the latter imply the LMIs in the former for $\lambda_i^f = 0$ and P sufficiently small (cf. the proof of [10, Lemma 4.1]). Thus, Theorem 4.1.8 is *stronger* than Theorem 4.1.1 in the sense that it can handle all cases that could be dealt with using Theorem 4.1.1 and even some more (cf. Examples 4.1.9 and 4.1.10). The only drawback is the loss of geometric intuition.

For the application of Theorem 4.1.8 in practice, we need a systematic procedure to find multipliers such that the LMI (4.1.8) holds. Such a procedure can be formulated in complete analogy to Algorithm 4.1.5 with incorporation of the additional multipliers λ_i^f. Set $\Lambda^x = \{\lambda_1^x, \ldots, \lambda_{l^x}^x\}$, $\Lambda^f = \{\lambda_1^f, \ldots, \lambda_{l^x}^f\}$, $\Lambda^u = \{\lambda_1^u, \ldots, \lambda_{l^u}^u\}$.

Algorithm 4.1.11.

- Step 0: Set $\mathbb{I}^x = \mathbb{I}^f = \mathbb{I}^u = \emptyset$.
- Step 1: Choose $\Lambda^x, \Lambda^f, \Lambda^u$ via the optimization problem

$$\begin{aligned} \underset{\substack{\lambda_i^x, \lambda_k^f, \lambda_j^u, \lambda_i^c \geq 0 \\ P = P^\top \in \mathbb{R}^{n \times n}}}{\textit{minimize}} \quad & \sum_{i=1}^{l^x} \lambda_i^x + \sum_{k=1}^{l^x} \lambda_k^f + \sum_{j=1}^{l^u} \lambda_j^u \\ \textit{s.t.} \quad & \text{(4.1.8) and (if } \mathbb{X} \text{ unbounded) (4.1.9),} \\ & \lambda_i^x = \lambda_k^f = \lambda_j^u = 0 \;\; \left(i \in \mathbb{I}^x, k \in \mathbb{I}^f, j \in \mathbb{I}^u\right). \end{aligned} \tag{4.1.12}$$

  - Case 1.1 - (4.1.12) is feasible: Go to Step 2.
  - Case 1.2 - (4.1.12) is infeasible: Stop the algorithm.
- Step 2: Solve problem (4.1.10) with $\Lambda^x, \Lambda^f, \Lambda^u \to (x^e, u^e)$.
- Step 3: Check conditions (4.1.11) for $\Lambda^x, \Lambda^f, \Lambda^u$ and (x^e, u^e):
  - Case 3.1 - They hold: Stop the algorithm.
  - Case 3.2 - They do not hold for indices $I^x \subseteq \mathbb{I}_{[1,l^x]}$, $I^f \subseteq \mathbb{I}_{[1,l^x]}$, $I^u \subseteq \mathbb{I}_{[1,l^u]}$: Set $\mathbb{I}^x = \mathbb{I}^x \cup I^x, \mathbb{I}^f = \mathbb{I}^f \cup I^f, \mathbb{I}^u = \mathbb{I}^u \cup I^u$ and go back to Step 1.

Exactly as Algorithm 4.1.5, the above algorithm will always terminate, either in Case 1.2 or in Case 3.1. Again, assuming the latter, we have found a steady-state at which the LQ-problem (2.1.1) is strictly dissipative and has the near equilibrium turnpike property. Contrary, in Case 1.2, the algorithm cannot be used to draw conclusions about the properties of interest. The following example illustrates the application of Algorithm (4.1.11) to an LQ-problem involving a permanent magnet synchronous machine (PMSM). The dynamical equations as well as the machine parameters are adopted from [17].

Example 4.1.12. Consider the following model of the PMSM stator current dynamics in rotating coordinates:

$$\dot{x}(t) = \begin{pmatrix} -214.9 & 69.4 \\ -127.9 & -291.7 \end{pmatrix} x(t) + \begin{pmatrix} 17.5 & 0 \\ 0 & 23.8 \end{pmatrix} u(t). \tag{4.1.13}$$

The state x and the input u are the 2-dimensional current and voltage space vectors, respectively. Except from the rated speed, which we assume to be $10\pi \frac{rad}{s}$, all parameters in (4.1.13) as well as the model are adopted from [17, Section 1.8.1] for the scenario of salient pole machines. Due to the low rated speed, we neglected the influence of the counter-electromotive force in the above dynamics. The state and input constraints are given as

$$\mathbb{X} = \left\{x \in \mathbb{R}^2 \mid x^\top x \leq 0.1^2\right\}, \; \mathbb{U} = \left\{u \in \mathbb{R}^2 \mid u^\top u \leq 5^2\right\}.$$

The control objective consists of maximizing the generated torque

$$T = 0.045 x_1 x_2 + 0.94 x_2$$

while keeping the norm of the input voltage vector small. Thus, we consider a stage cost of the form

$$\ell(x,u) = x^\top \begin{pmatrix} 0 & -0.0225 \\ -0.0225 & 0 \end{pmatrix} x + x^\top \begin{pmatrix} 0 \\ -0.94 \end{pmatrix} + r\|u\|^2$$

for some parameter $r > 0$ which allows for an additional trade-off between torque maximization and control effort minimization. Note that the above stage cost is non-convex in the state. Discretization of (4.1.13) with sampling interval $5ms$ yields the following discrete-time system:

$$x(k+1) = \begin{pmatrix} 0.31 & 0.09 \\ -0.17 & 0.2 \end{pmatrix} x(k) + \begin{pmatrix} 0.05 & 0.01 \\ -0.01 & 0.06 \end{pmatrix} u(k).$$

Now, choosing $r = 0.02$, Step 1 in Algorithm 4.1.11 reveals that (4.1.12) is feasible for $\lambda_1^x = 0.0225, \lambda_1^u = \lambda_1^f = 0$. The corresponding optimal steady-state can then be computed as

$$(x^e, u^e) = \left(\begin{pmatrix} 0 \\ 0.1 \end{pmatrix}, \begin{pmatrix} -0.47 \\ 1.19 \end{pmatrix}\right),$$

i.e., $x^e \in \partial\mathbb{X}$ and thus the complementary slackness conditions (4.1.11) are satisfied. Hence, the application of Theorem 4.1.8 allows us to conclude that the above LQ-problem is strictly dissipative and has the near equilibrium turnpike property at (x^e, u^e).

Remark 4.1.13. We conclude the section with a remark on the connection between this section and Chapter 3 on strict dissipativity w.r.t. periodic orbits, in particular between Theorem 4.1.8 and Theorem 3.3.3. In case that $\mathbb{X}$ and $\mathbb{U}$ are compact, the former is a special case of the latter. To see this, assume that all conditions in Theorem 4.1.8 hold, i.e., the LMI (4.1.8) holds for some multipliers and the corresponding optimal steady-state (x^e, u^e) of problem (4.1.10) satisfies the complementary slackness condition. Then, the optimal value $\ell(x^e, u^e)$ as well as the storage function and the multipliers from Theorem 4.1.8 are feasible for the optimization problem (3.3.7) in Algorithm 3.3.2. Due to the maximization in this latter problem, there is always a point for which the constraint holds with equality, i.e., the left-hand-side is zero. Now, from the strictness of (4.1.8), it follows that there can only be one such point and this point is the optimal steady-state computed in (4.1.10). Since the complementary slackness conditions hold at (x^e, u^e), the output of Algorithm 3.3.2 is a set consisting of only this steady-state. Thus, applying Theorem 3.3.3, we conclude strict dissipativity w.r.t. (x^e, u^e). Nevertheless, the theory in this section does not require compactness of $\mathbb{X}$ and $\mathbb{U}$. This allowed us to draw several conclusions about the connection between bounded modes and corresponding eigenvalues of Q and R (cf. Remarks 4.1.2 and 4.1.3), which justifies the approach taken in the present chapter.

4.1.3. Necessary and sufficient conditions with single input constraints

A well-known, non-trivial result is that the S-procedure provides also necessary conditions in case that there is only one constraint [4]. This fact is exploited in the following in order to find necessary and sufficient conditions for strict dissipativity with quadratic storage function and turnpike properties when there is only one ellipsoidal constraint on the input. Since state constraints always induce two constraints on the dissipation inequality due to $Ax + Bu \in \mathbb{X}$, the approach can only be used for problems where the input is constrained and $\mathbb{X} = \mathbb{R}^n$. Therefore, we consider input constraints of the form

$$\mathbb{U} = \left\{ u \in \mathbb{R}^m \mid g^u(u) = u^\top P^u u + 2u^\top q^u + c^u \leq 0 \right\} \qquad (4.1.14)$$

for some $P^u \in \mathbb{R}^{m\times m}$, $q^u \in \mathbb{R}^m$, $c^u \in \mathbb{R}$, with $P^u \geq 0$.

Theorem 4.1.14. Consider the LQ-problem (2.1.1) with $S = 0, Q \geq 0$, no state constraints, i.e., $\mathbb{X} = \mathbb{R}^n$, and input constraints of the form (4.1.14). Then, the following statements hold:

(i) Suppose (A, Q) is detectable and there is some $\lambda^u \geq 0$ such that $R + \lambda^u P^u > 0$ as well as $\lambda^u g^u(u^e) = 0$, where u^e is the optimal equilibrium input corresponding to the optimal steady-state problem with stage cost $\ell_\lambda(x, u) = \ell(x, u) + \lambda^u g^u(u)$, i.e., to

$$(x^e, u^e) = \underset{\substack{x \in \mathbb{X}, u \in \mathbb{U} \\ x = Ax + Bu}}{\operatorname{argmin}} \; \ell_\lambda(x, u). \tag{4.1.15}$$

Then, the problem is strictly dissipative and has the near equilibrium turnpike property at (x^e, u^e).

(ii) Conversely, if the problem is strictly dissipative at some equilibrium (x^e, u^e) with quadratic storage function, then (A, Q) is detectable and there exists $\lambda^u \geq 0$ such that $R + \lambda^u P^u \geq 0$ as well as $\lambda^u g^u(u^e) = 0$.

Proof. **(i):** Since $Q \geq 0, R + \lambda^u P^u > 0$, and (A, Q) is detectable, we can apply [10, Lemma 5.4] to conclude that there exists a positive definite matrix P such that $Q + P - A^\top PA > 0$. Consequently, all conditions of Theorem 4.1.1, including complementary slackness, are met and thus the problem is strictly dissipative and has the near equilibrium turnpike property at (x^e, u^e).

(ii): Suppose the LQ-problem (2.1.1) is strictly dissipative at some equilibrium $(x^e, u^e) \in \mathbb{X} \times \mathbb{U}$ with quadratic storage function $\lambda : \mathbb{R}^n \to \mathbb{R}$. From Proposition 4.3 in [5], it follows that there is a quadratic class $\mathcal{K}_\infty$ function α such that the dissipation inequality (2.2.1) holds for all $(x, u) \in \mathbb{X} \times \mathbb{U}$. Then, by the converse direction of the S-procedure (cf. [4] and recall that $\mathbb{U}$ has non-empty interior), we conclude that there exists some $\lambda^u \geq 0$ such that

$$\ell(x, u) + \lambda^u g^u(u) - \ell(x^e, u^e) + \lambda(x) - \lambda(Ax + Bu) \geq \alpha\left(\|x - x^e\|\right)$$

holds for all $(x, u) \in \mathbb{R}^{n+m}$. By plugging (x^e, u^e) into this inequality, we deduce $\lambda^u g^u(u^e) \geq 0$ and hence, due to $\lambda^u g^u(u^e) \leq 0$, $\lambda^u g^u(u^e) = 0$. Consequently, the modified *unconstrained* LQ-problem with stage cost $\ell_\lambda(x, u)$ is strictly dissipative at (x^e, u^e). Since $\lambda(x) = x^\top Px + q^\top x$ is bounded from below on $\mathbb{R}^n$, it follows that $P \geq 0$. By collecting the quadratic terms in the corresponding dissipation inequality, we conclude that $0 \leq R + \lambda^u P^u - B^\top PB \leq R + \lambda^u P^u$. Moreover, by [10, Theorem 6.1], whose proof does not require positive definiteness of the involved R-matrix, strict dissipativity of the modified LQ-problem with stage cost $\ell_\lambda(x, u)$ implies that (A, Q) is detectable. □

Theorem 4.1.14 provides a powerful tool in the sense that it states not only sufficient but also necessary conditions for strict dissipativity. Of course, this necessity result holds only for strict dissipativity with a quadratic storage function due to the quadratic nature of the S-procedure. The standard result that, for linear quadratic dissipative systems, there is always a quadratic storage function is well-known to hold in the unconstrained case *or* when the cost is convex. In the constrained indefinite case, however, there are no such results to the author's knowledge. From the necessity part (ii) of Theorem 4.1.14, we deduce that, given constraints of the form (4.1.14) and an indefinite input cost matrix R, a necessary condition for strict dissipativity at some equilibrium (x^e, u^e) (with quadratic storage function) is $g^u(u^e) = 0$, i.e., $u^e \in \partial\mathbb{U}$. Thus, assuming everything is quadratic, strict dissipativity (and by [10, Theorem 7.1 (i)] the turnpike property) can only hold when the corresponding optimal equilibrium input lies on the boundary of $\mathbb{U}$. Note, however, that the near equilibrium turnpike property might still hold in this case since strict dissipativity is only known to be necessary for it on interior points of the constraints (cf. [10, Theorem 7.1 (ii)]).

In order to apply the results from [10], we need to assume $Q \geq 0$ for both directions of the proof of Theorem 4.1.14. Furthermore, the existence of some $\lambda^u \geq 0$ with $R + \lambda^u P^u \succ 0$ as well as $\lambda^u g^u(u^e) = 0$ is not necessary for the problem to be strictly dissipative. This is due to the fact that Definition 2.2.1 requires strictness only in the state, but not in the input. In the following, we investigate an example where we not only use sufficient conditions to show strict dissipativity, as it was done for the Theorems 4.1.1 and 4.1.8, but we also use the necessity part of Theorem 4.1.14 to find situations where strict dissipativity does not hold.

Example 4.1.15. Consider the indefinite LQ-problem with

$$x(k+1) = ax(k) + u(k),\ \ell(x,u) = -u^2,\ \mathbb{X} = \mathbb{R},\ \mathbb{U} = [0,1]$$

for some $a \in \mathbb{R}$. Obviously, the optimal behavior consists of applying the maximal input $u(k) = 1$ at all time instances $k \in \mathbb{N}$. Then, the occurrence of strict dissipativity (and thus of turnpikes) depends on the parameter a. Since $Q = 0$, the system is detectable if and only if $|a| < 1$. Consequently, applying Theorem 4.1.14 (ii), we see that the system is not strictly dissipative (with a quadratic storage function) in case that $|a| \geq 1$. Given that $|a| < 1$, on the other hand, and choosing $\lambda^u = 2$, the solution of problem (4.1.15) reveals the optimal steady-state $(x^e, u^e) = (\frac{1}{1-a}, 1)$ which fulfills $\lambda^u g^u(u^e) = 0$. Hence, the problem is strictly dissipative and thus has the near equilibrium turnpike property. These facts can also be explained by looking at the solution of the underlying difference equation $x(k+1) = ax(k) + 1, x(0) = 0$, which can be stated explicitly as $x(k) = \sum_{i=0}^{k-1} a^i = \frac{1-a^k}{1-a}$. As is well known from standard analysis, $x(k)$ converges to $\frac{1}{1-a}$ for $k \to \infty$ if $|a| < 1$ and diverges for $|a| \geq 1$.

4.2. A P-step system approach

The approach that we pursue in this section is, in many aspects, fundamentally different from the rest of this thesis. Instead of employing the S-procedure to verify turnpikes which occur, for indefinite costs, often on the boundary of the constraints, we use the framework of P-step systems. A P-step system can be seen as a P-fold subsequent copy of the original dynamical system and thus, e.g., periodic orbits of the original system can be considered as steady-states of the P-step system. We show that the tools from [10] can be applied in the presence of indefinite state weightings when the stage cost of a suitably defined P-step system is convex. In doing so, we make the assumption that the coupling cost vanishes, i.e., $S = 0$. The section is structured as follows: First, we provide a rigorous definition of P-step systems, involving the matrices of the original LQ-problem. Then, after performing a technical transformation on the P-step system, we state the main results on necessity and sufficiency for strict dissipativity and turnpike properties using P-step systems.

4.2.1. Definition of P-step systems

We define the P-step system corresponding to the LQ-problem (2.1.1) as follows (cf. [19, 20]): Denote the extended state and input vectors by $\tilde{x} = (\tilde{x}_0, \dots, \tilde{x}_{P-1})$ and $\tilde{u} = (\tilde{u}_0, \dots, \tilde{u}_{P-1})$, respectively. The dynamics $\tilde{x}(k+1) = f_P(\tilde{x}(k), \tilde{u}(k))$, $\tilde{x}_{P-1}(0) = x_0$ are defined by repeatedly applying the original vector field $f(x,u) = Ax + Bu$ to the last state $\tilde{x}_{P-1}$, i.e.,

$$f_P(\tilde{x}, \tilde{u}) = \begin{pmatrix} x_{\tilde{u}}(1, \tilde{x}_{P-1}) \\ \vdots \\ x_{\tilde{u}}(P, \tilde{x}_{P-1}) \end{pmatrix} = \begin{pmatrix} f(\tilde{x}_{P-1}, \tilde{u}_0) \\ f(f(\tilde{x}_{P-1}, \tilde{u}_0), \tilde{u}_1) \\ \vdots \end{pmatrix}.$$

Initial conditions for the components $\tilde{x}_0, \dots, \tilde{x}_{P-2}$ are irrelevant due to the nature of the P-step dynamics. Using the linearity of $f(x,u)$, we obtain

$$f_P(\tilde{x}, \tilde{u}) = A_P \tilde{x} + B_P \tilde{u}$$

with

$$A_P = \begin{pmatrix} 0 & \dots & 0 & A \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & A^P \end{pmatrix}, B_P = \begin{pmatrix} B & 0 & \dots & 0 \\ AB & B & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A^{P-1}B & \dots & AB & B \end{pmatrix}.$$

Furthermore, the P-step cost $\ell_P(\tilde{x},\tilde{u}) = \sum_{j=0}^{P-1} \ell(x_{\tilde{u}}(j,\tilde{x}_{P-1}),u_j)$ can be written in quadratic form as

$$\ell_P(\tilde{x},\tilde{u}) = \begin{pmatrix}\tilde{x}\\ \tilde{u}\\ 1\end{pmatrix}^\top \begin{pmatrix} Q_P & S_P^\top & \frac{1}{2}s_P \\ S_P & R_P & \frac{1}{2}v_P \\ \frac{1}{2}s_P^\top & \frac{1}{2}v_P^\top & c_P \end{pmatrix} \begin{pmatrix}\tilde{x}\\ \tilde{u}\\ 1\end{pmatrix}. \tag{4.2.1}$$

The involved matrices are computed as follows: Define

$$Q_{i,j} := \sum_{k=0}^{j} \left(A^k\right)^\top Q A^{k+i} \tag{4.2.2}$$

and $Q_{i,j}^B = B^\top Q_{i,j} B$. Then, we have

$$Q_P = \begin{pmatrix} 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & & 0 & 0 \\ 0 & \dots & 0 & Q_{0,P-1} \end{pmatrix}, S_P = \begin{pmatrix} 0 & \dots & 0 & B^\top Q_{1,P-2} \\ \vdots & & \vdots & \vdots \\ & \dots & & B^\top Q_{P-1,0} \\ 0 & & 0 & 0 \end{pmatrix},$$

$$R_P = \begin{pmatrix} R + Q_{0,P-2}^B & * & \dots & \dots & * \\ Q_{1,P-3}^B & R + Q_{0,P-3}^B & & & \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ Q_{P-2,0}^B & \dots & Q_{1,0}^B & R + Q_{0,0}^B & * \\ 0 & \dots & & 0 & R \end{pmatrix},$$

$$s_P = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sum_{k=0}^{P-1} (A^k)^\top s \end{pmatrix}, v_P = \begin{pmatrix} v + \left(\sum_{k=0}^{P-2} (A^k B)^\top\right) s \\ v + \left(\sum_{k=0}^{P-3} (A^k B)^\top\right) s \\ \vdots \\ v \end{pmatrix}, c_P = P \cdot c.$$

Note that Q_P might be positive semidefinite, even when Q is indefinite. A simple example for this case is

$$Q = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix},\ A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow Q_2 = A + A^\top Q A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \succ 0.$$

Furthermore, from the above definition of R_P, one can see see that $R \succ 0$ is necessary for $R_P \succ 0$ and the latter holds true when the eigenvalues of R are sufficiently large. The optimal control problem for the P-step system corresponding to the LQ-problem (2.1.1) can then be stated as

$$V_N^P(\tilde{x}_0) = \underset{\tilde{u} \in \mathbb{U}^{N \cdot P}(x_0)}{minimize} \sum_{k=0}^{N-1} \ell_P\left(\tilde{x}_{\tilde{u}}(k, x_0), \tilde{u}(k)\right), \tag{4.2.3}$$

where $\tilde{x}_{\tilde{u}}(\cdot, x_0)$ denotes the solution of $\tilde{x}(k+1) = A_P\tilde{x}(k) + B_P\tilde{u}(k)$ with input trajectory $\tilde{u}$ and initial value $\tilde{x}_{P-1}(0) = x_0$.

Remark 4.2.1. The P-step optimal control problem (4.2.3) can be used to analyze the original problem (2.1.1) since $V_N^P(x_0) = V_{N \cdot P}(x_0)$ and the optimal trajectories of the problems coincide. In particular, for any $k \in \mathbb{I}_{[1,N]}$, we have

$$\begin{aligned} \tilde{x}^*(k, x_0) &= \left[x^*((k-1)P+1, x_0)^\top, \dots, x^*\left(kP, x_0\right)^\top\right]^\top, \\ \tilde{u}^*(k, x_0) &= \left[u^*(kP, x_0)^\top, \dots, u^*\left((k+1)P-1, x_0\right)^\top\right]^\top, \end{aligned}$$

where $\tilde{x}^*(\cdot, x_0), \tilde{u}^*(\cdot, x_0)$ denote optimal state and input trajectories, respectively, corresponding to $V_N^P(x_0)$, whereas $x^*(\cdot, x_0), u^*(\cdot, x_0)$ are the optimal trajectories that belong to $V_{N \cdot P}(x_0)$.

4.2.2. Elimination of coupling terms

In the previous section, we derived an explicit formula for the cost matrices of the P-step system based on the original cost. It can be seen that, in general, the P-step cost contains couplings between input and state, i.e., $S_P \neq 0$, although this was not the case for the original cost from (2.1.1). In the following, we perform a transformation on the control input in order to eliminate the coupling matrix S_P. This will allow us to apply the theory from [10], which does not consider such coupling terms. The transformation is adopted from explicit linear quadratic constrained optimal control problems (cf. [2]), and widely used in the condensed quadratic programming formulation of (linear quadratic) model predictive control. The main idea is to define a new input via $\tilde{z} = \tilde{u} + R_P^{-1}S_P\tilde{x}$. Note that $R_P \succ 0$ will be a necessary condition for the application of the methods from [10], hence its inverse exists. Inserting $\tilde{u} = \tilde{z} - R_P^{-1}S_P\tilde{x}$ into the P-step running cost $\ell_P(\tilde{x}, \tilde{u})$ defined in (4.2.1) yields a modified cost function

$$\begin{aligned}
&\tilde{\ell}_P(\tilde{x},\tilde{z}) = \\
&\tilde{x}^\top Q_P \tilde{x} + 2\tilde{x}^\top S_P^\top \left(\tilde{z} - R_P^{-1} S_P \tilde{x}\right) + \left(\tilde{z} - R_P^{-1} S_P \tilde{x}\right)^\top R_P \left(\tilde{z} - R_P^{-1} S_P \tilde{x}\right) \\
&+ s_P^\top \tilde{x} + v_P^\top \left(\tilde{z} - R_P^{-1} S_P \tilde{x}\right) + c_P \\
&= \dots \\
&= \tilde{x}^\top \underbrace{\left(Q_P - S_P^\top R_P^{-1} S_P\right)}_{\tilde{Q}_P :=} \tilde{x} + \tilde{z}^\top R_P \tilde{z} + \underbrace{\left(s_P^\top - v_P^\top R_P^{-1} S_P\right)}_{\tilde{s}_P^\top :=} \tilde{x} + v_P^\top \tilde{z} + c_P.
\end{aligned}$$

Also, the dynamics change accordingly

$$\begin{aligned}
\tilde{x}(k+1) &= A_P \tilde{x}(k) + B_P \left(\tilde{z}(k) - R_P^{-1} S_P \tilde{x}(k)\right) \\
&= \underbrace{\left(A_P - B_P R_P^{-1} S_P\right)}_{\tilde{A}_P :=} \tilde{x}(k) + B_P \tilde{z}(k),
\end{aligned}$$

and the constraint $\tilde{u} \in \mathbb{U}^P$ translates into $\tilde{z} - R_P^{-1} S_P \tilde{x} \in \mathbb{U}^P$. Thus, the decoupling of the cost function causes couplings in the constraints. We define the resulting constraints for the transformed P-step system as

$$\mathbb{V} = \left\{(\tilde{x},\tilde{z}) \in \left(\mathbb{X} \times \mathbb{R}^{mP}\right) \mid \tilde{z} - R_P^{-1} S_P \tilde{x} \in \mathbb{U}^P\right\}.$$

Finally, this yields the following optimal control problem:

$$\begin{aligned}
\tilde{V}_N^P(\tilde{x}_0) = \underset{\tilde{z} \in \mathbb{R}^{mNP}}{minimize} \; & \sum_{k=0}^{N-1} \tilde{\ell}_P\left(\tilde{x}(k), \tilde{z}(k)\right) \\
s.t. \; & \tilde{x}(k+1) = \tilde{A}_P \tilde{x}(k) + B_P \tilde{z}(k), \\
& \tilde{x}(0) = \tilde{x}_0, \\
& (\tilde{x}(k), \tilde{z}(k)) \in \mathbb{V}, \; k = 0, \dots, N.
\end{aligned} \tag{4.2.4}$$

Remark 4.2.2. Clearly, the optimal trajectories of the problems (4.2.3) and (4.2.4) coincide. To be more precise, we have $V_N^P(\tilde{x}_0) = \tilde{V}_N^P(\tilde{x}_0)$ and also the optimal inputs are connected via the transformation formula

$$\tilde{u}^*(\cdot, \tilde{x}_0) = \tilde{z}^*(\cdot, \tilde{x}_0) - R_P^{-1} S_P \tilde{x}^*(\cdot, \tilde{x}_0),$$

where $\tilde{u}^*(\cdot, \tilde{x}_0)$ is the optimal input trajectory for (4.2.3), whereas $\tilde{z}^*(\cdot, \tilde{x}_0)$ denotes the optimal input of (4.2.4).

4.2.3. Characterization of strict dissipativity and turnpike properties

Now, we are in the position to apply the results from [10] to P-step systems. The following result provides sufficient and necessary conditions for strict dissipativity and turnpike properties in indefinite LQ-problems with $\mathbb{X} = \mathbb{R}^n$, given that there is a P-step system whose transformed stage cost satisfies appropriate definiteness assumptions.

Theorem 4.2.3. Consider the LQ-problem (2.1.1) with $\mathbb{X} = \mathbb{R}^n, \mathbb{U} \subseteq \mathbb{R}^m$, and $S = 0$. Suppose there exists a $P \geq 1$ such that $\tilde{Q}_P \geq 0$ and $R_P > 0$. Then, the following statements hold:

(i) If $(\tilde{A}_P, \tilde{Q}_P)$ is detectable, then the LQ-problem (2.1.1) has the near equilibrium turnpike property at some equilibrium $(x^e, u^e) \in \mathbb{X} \times \mathbb{U}$. If, further, $(x^e, u^e) \in \operatorname{int}(\mathbb{X} \times \mathbb{U})$ and (A, B) stabilizable, then the LQ-problem (2.1.1) is strictly dissipative and has the turnpike property at (x^e, u^e).

(ii) If the LQ-problem (2.1.1) has the turnpike property at an equilibrium $(x^e, u^e) \in \mathbb{X} \times \mathbb{U}$, then $(\tilde{A}_P, \tilde{Q}_P)$ is detectable.

Proof. First, we note that most of the results used in this proof were developed for the case that state and input constraints are uncoupled but the respective proofs make no use of this fact. Hence, they also apply for coupled constraints, which are present in (4.2.4).

(i): By applying [10, Theorem 6.1] to the transformed P-step LQ-problem (4.2.4), we conclude that there exists an equilibrium $(\tilde{x}^e, \tilde{z}^e) \in \mathbb{V}$ at which problem (4.2.4) is strictly dissipative, i.e., there exists a storage function $\tilde{\lambda}$ and a class $\mathcal{K}_\infty$ function α such that

$$\tilde{\ell}_P(\tilde{x}, \tilde{z}) - \tilde{\ell}_P(\tilde{x}^e, \tilde{z}^e) + \tilde{\lambda}(\tilde{x}) - \tilde{\lambda}(\tilde{A}_P\tilde{x} + B_P\tilde{z}) \geq \alpha\left(\|\tilde{x} - \tilde{x}^e\|\right)$$

for all $(\tilde{x}, \tilde{z}) \in \mathbb{V}$. Consequently, by re-substituting the original P-step system, we arrive at

$$\ell_P(\tilde{x}, \tilde{u}) - \ell_P(\tilde{x}^e, \tilde{u}^e) + \tilde{\lambda}(\tilde{x}) - \tilde{\lambda}(A_P\tilde{x} + B_P\tilde{u}) \geq \alpha\left(\|\tilde{x} - \tilde{x}^e\|\right) \tag{4.2.5}$$

for all $(\tilde{x}, \tilde{u}) \in \mathbb{X}^P \times \mathbb{U}^P$, i.e., the P-step problem (4.2.3) is strictly dissipative at $(\tilde{x}^e, \tilde{u}^e)$ and thus has the near equilibrium turnpike property at $(\tilde{x}^e, \tilde{u}^e)$ according to [9, Theorem 5.3]. Note that an equilibrium of the P-step system is, in general, a periodic orbit of length P for the original system. From inequality (4.2.5), however, we deduce that the periodic orbit resulting from $(\tilde{x}^e, \tilde{u}^e)$ is in fact P times the steady-state (x^e, u^e), where (x^e, u^e) is equal to any of the (equal) P components of $(\tilde{x}^e, \tilde{u}^e)$. This is due to the fact that, otherwise, for any orbit obtained by phase-shifting $(\tilde{x}^e, \tilde{u}^e)$, the left-hand side of (4.2.5) would be zero while its right-hand side would be greater than zero, contradicting the inequality. The optimal trajectories of the P-step problem and the original LQ-problem coincide and thus problem (2.1.1) has the near equilibrium turnpike property at (x^e, u^e). Moreover, according to [10, Remark 2.2 (iii)], stabilizability of (A, B) as well as $(x^e, u^e) \in \operatorname{int}(\mathbb{X} \times \mathbb{U})$ imply the turnpike property, which in turn implies strict dissipativity by [10, Theorem 7.1 (i)].

(ii): Suppose, for the sake of contradiction, that $(\tilde{A}_P, \tilde{Q}_P)$ is not detectable. Then, the transformed P-step problem (4.2.4) is not strictly dissipative due to [10, Theorem 6.1] and thus the same holds true for the P-step problem (4.2.3). Therefore, according to [10, Theorem 7.1], the latter does not have the turnpike property. Since the optimal trajectories of the problems (2.1.1) and (4.2.3) coincide, we conclude that the original LQ-problem (2.1.1) does not have the turnpike property. □

Remark 4.2.4. Note that the condition $\tilde{Q}_P \geq 0$ in Theorem 4.2.3 is more restrictive than simply demanding $Q_P \geq 0$. Concerning the latter matrix, it is readily verified that, whenever $Q^1 \geq Q^2$, then $Q^1_P \geq Q^2_P$ for any symmetric matrices $Q^1, Q^2 \in \mathbb{R}^{n \times n}$ and any $P \geq 1$. Furthermore, recall that larger eigenvalues of R cause larger eigenvalues of R_P and hence, when $S_P \neq 0$, larger eigenvalues of $\tilde{Q}_P$. Thus, Theorem 4.2.3 allows to extend the class of Q-matrices which are admissible for the techniques from [10] from positive semidefinite to indefinite matrices if the positive eigenvalues of Q and R are sufficiently large and if A is of a particular form that renders Q_P positive semidefinite for some $P \geq 2$.

Example 4.2.5. Consider the indefinite LQ-problem with

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ B = \begin{pmatrix} 2 \\ 1 \end{pmatrix},\ \ell(x, u) = 2x_1^2 - x_2^2 + 10u^2,\ \mathbb{X} = \mathbb{R}^2,\ \mathbb{U} = [-1, 1].$$

For $P = 2$, the matrices corresponding to the transformed P-step LQ-problem (4.2.4) can be computed as

$$\tilde{Q}_2 = \frac{1}{17} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 16 & 4 \\ 0 & 0 & 4 & 1 \end{pmatrix} \geq 0,\ R_2 = \begin{pmatrix} 17 & 0 \\ 0 & 10 \end{pmatrix} \succ 0,$$

$$\tilde{A}_2 = \frac{1}{17}\begin{pmatrix} 0 & 0 & 2 & 9 \\ 0 & 0 & 18 & -4 \\ 0 & 0 & 18 & -4 \\ 0 & 0 & 2 & 9 \end{pmatrix}, \; B_2 = \begin{pmatrix} 2 & 0 \\ 1 & 0 \\ 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

As one readily verifies, $(\tilde{A}_2, \tilde{Q}_2)$ is detectable and thus, by Theorem 4.2.3 (i), the above problem has the near equilibrium turnpike property. Moreover, the corresponding optimal steady-state can be computed as $(x^e, u^e) = (0, 0) \in \operatorname{int}(\mathbb{X} \times \mathbb{U})$ and thus, since (A, B) is stabilizable, the problem is strictly dissipative and has the turnpike property at $(0, 0)$.

In analogy to [10, Theorem 8.3], we can formulate Theorem 4.2.3 also for compact state constraints. In this case, detectability of the matrix pair $(\tilde{A}_P, \tilde{Q}_P)$ is replaced by observability of eigenvalues on the unit circle $\mathbb{C}_{=1}$ in the complex plane.

Theorem 4.2.6. Consider the LQ-problem (2.1.1) with $\mathbb{X} \subset \mathbb{R}^n$ compact and $\mathbb{U} \subseteq \mathbb{R}^m$. Suppose there exists a $P \geq 1$ such that $\tilde{Q}_P \geq 0$ and $R_P > 0$. Then the following statements hold:

(i) If the matrix pair $(\tilde{A}_P, \tilde{Q}_P)$ has no unobservable eigenvalues on $\mathbb{C}_{=1}$, then the LQ-problem (2.1.1) has the near equilibrium turnpike property at some equilibrium $(x^e, u^e) \in \mathbb{X} \times \mathbb{U}$. If, further, $(x^e, u^e) \in \operatorname{int}(\mathbb{X} \times \mathbb{U})$ and (A, B) stabilizable, then the LQ-problem (2.1.1) is strictly dissipative and has the turnpike property at (x^e, u^e).

(ii) If the LQ-problem (2.1.1) has the near equilibrium turnpike property at an equilibrium $(x^e, u^e) \in \operatorname{int}(\mathbb{X} \times \mathbb{U})$, then $(\tilde{A}_P, \tilde{Q}_P)$ has no unobservable eigenvalues on $\mathbb{C}_{=1}$.

Proof. The idea of the proof is identical to the one of Theorem 4.2.3 and therefore it is omitted. □

Remark 4.2.7. Note that, in principle, the results from Section 4.1 can be combined with Theorem 4.2.3 and Theorem 4.2.6. To be more precise, there might exist multipliers as in Section 4.1 such that an LQ-problem with modified stage cost (in the sense of Section 4.1) is strictly convex in the input but non-convex in the state. In this case, given that the dynamics are of a suitable form, the above results for P-step systems could be used to provide sufficient conditions for strict dissipativity of this modified LQ-problem, which in turn leads to strict dissipativity of the original problem of interest.

We conclude the section by applying Theorem 4.2.6 to a specific optimal control problem involving a continuous stirred tank reactor (CSTR). The system dynamics in the following example are adopted from [23].

Example 4.2.8. Consider the indefinite LQ-problem with

$$A = \begin{pmatrix} 0.7776 & -0.0045 \\ 26.6185 & 1.8555 \end{pmatrix}, \; B = \begin{pmatrix} -0.0004 \\ 0.2907 \end{pmatrix},$$
$$\ell(x,u) = -\bar{q}x_1^2 + x_2^2 + u^2,$$
$$\mathbb{X} = [-0.5, 0.5] \times [-5, 5], \; \mathbb{U} = [-15, 15].$$

This setting can be interpreted as maximizing the reactant concentration in a CSTR while keeping the reactor temperature as well as the cooling effort small. The quantity $\bar{q} > 0$ is an additional parameter that allows for a trade-off between the performance objectives. Straightforward algebraic computations reveal that, for $P = 2$, $R_2 \succ 0$ and $\tilde{Q}_2 \succeq 0$ hold for any $\bar{q} \leq 92.88$. Moreover, A_2 has no eigenvalues on $\mathbb{C}_{=1}$. Thus, according to Theorem 4.2.6, the above LQ-problem has the near equilibrium turnpike property for any $\bar{q} \leq 92.88$. When fixing $\bar{q}$, e.g., to $\bar{q} = -50$, the optimal steady-state of the transformed P-step system can be computed explicitly as $(\tilde{x}^e, \tilde{z}^e) = (0,0)$. Reversing the transformation, we conclude that $(x^e, u^e) = (0,0)$ is the optimal steady-state at which the present LQ-problem has the near equilibrium turnpike property. Hence, due to stabilizability of (A,B) as well as $(x^e, u^e) \in \text{int}(\mathbb{X} \times \mathbb{U})$, we conclude that, for $\bar{q} = -50$, the LQ-problem is strictly dissipative and has the turnpike property at $(0,0)$. The above-mentioned upper limit on $\bar{q}$ can be further increased when allowing for a higher P-step system length P. Table 4.1 shows the maximal admissible $\bar{q}$ for which the definiteness assumptions $\tilde{Q}_P \succeq 0, R_P \succ 0$ are satisfied, depending on P.

Table 4.1.: Maximal admissible $\bar{q}$ depending on P (Example 4.2.8)

P	3	4	5	6	7	8	9	10	11
$\bar{q}$	206	313	401	471	524	564	594	617	634

It can be seen that the upper bound on $\bar{q}$ increases with P but the rate of change decreases. Note that these results suggest that there does not seem to be an obvious connection between the minimal P which is sufficient for $\tilde{Q}_P \succeq 0, R_P \succ 0$ and the dimension of the system.

5. Conclusion and Outlook

5.1. Summary

The present thesis dealt with strict dissipativity and turnpike properties w.r.t. steady-states and periodic orbits in discrete-time linear quadratic optimal control problems with indefinite stage cost and constraints on states and inputs. In Chapter 3, for compact constraints, non-strict dissipation inequalities were used to construct a priori unknown optimal periodic orbits. This approach provided necessary as well as sufficient conditions for periodic dissipativity, but was computationally intractable. A convex relaxation of the underlying optimization problem was used to arrive at tractable sufficient conditions for strict dissipativity, where a complementary slackness condition turned out to be a key ingredient. In Section 4.1, similar techniques were used to characterize strict dissipativity w.r.t. steady-states with non-compact constraints. Again, complementary slackness played an important role and it became apparent that negative eigenvalues of the cost, the exact shape of the constraints, and the location of the optimal steady-state are highly intertwined. Section 4.2 provided a fundamentally different approach. The main idea was that, given that the stage cost accumulated over several consecutive time instances satisfies a suitable convexity assumption, necessary and sufficient conditions for strict dissipativity and turnpike properties of the original LQ-problem can be stated in terms of a suitably transformed P-step system. In summary, the present thesis provided novel results on the characterization of strict dissipativity and turnpike properties

- for indefinite LQ-problems,
- w.r.t. steady-states and periodic orbits,
- under explicit consideration of the constraints.

Thereby, it contributes to the theory of classical linear quadratic optimal control as well as to modern economic MPC research, where the above-named properties are commonly made assumptions which need to be verified in practice.

5.2. Future work

There are several promising directions for future research. A large portion of the presented results relied heavily on the S-procedure, which seems to be a very particular approach for analyzing periodic dissipativity. The traditional results on optimal periodic control provide a more fundamental view on this problem. A thorough investigation of the connection between the classical literature on optimal periodic control [3] and periodic dissipativity and turnpike properties is a promising approach for arriving at a more general characterization of periodic optimality.

More directly related to the present thesis is the extension of the results to a continuous-time setting. While this seems to be straightforward for the steady-state case, one would need to employ different technical tools to characterize periodic optimality since, in continuous-time, (non-trivial) periodic orbits are uncountably infinite sets. Furthermore, as already mentioned in Remark 3.2.7, the presented results apply to sets more general than periodic orbits, such as affine subspaces of the state space. The analysis of strict dissipativity (and possibly turnpike properties) w.r.t. subspaces constitutes another interesting direction for future research, which would also be of practical use in tracking MPC (cf. Example 3.2.6).

A. Appendix

A.1. Auxiliary Lemmas

Lemma A.1.1. Let $A \in \mathbb{R}^{n\times n}, C \in \mathbb{R}^{p\times n}$. $\lambda \in \mathbb{C}$ is an unobservable eigenvalue of the matrix pair (A, C) if and only if it is an unobservable eigenvalue of $(A, C^\top C)$.

Proof. **If**:

Suppose λ is an unobservable eigenvalue of $(A, C^\top C)$, i.e., there exists an eigenvector $v \in \mathbb{C}^n$ of A such that $Av = \lambda v$ and $C^\top Cv = 0$. By left-multiplying $v^\top$, the latter equation reads $v^\top C^\top Cv = (Cv)^\top (Cv) = 0$ and thus $Cv = 0$, i.e., λ is an unobservable eigenvalue of (A, C).

Only if:

Suppose λ is an unobservable eigenvalue of (A, C), i.e., there exists an eigenvector $v \in \mathbb{C}^n$ of A such that $Av = \lambda v$ and $Cv = 0$. It directly follows that $C^\top Cv = 0$, which is exactly the desired statement. □

Lemma A.1.2. Consider a finite discrete set of the form $L = \left\{x^1, \dots, x^P\right\}$ with $x^i \in X \subset \mathbb{R}^n$, X compact, and a continuous function $f : X \to \mathbb{R}_{\geq 0}$ with nonnegative image. Suppose that f vanishes exactly on L, i.e.,

$$f(x) > 0 \;\; \forall x \in X \setminus L \quad \text{and} \quad f(x) = 0 \;\; \forall x \in L. \tag{A.1.1}$$

Then, there exists a class $\mathcal{K}_\infty$ function α such that

$$f(x) \geq \alpha\left(|x|_L\right) \;\; \forall x \in X. \tag{A.1.2}$$

Proof. Due to compactness of X, it is sufficient to construct $\alpha \in \mathcal{K}$. The proof is divided into two parts:

(i) For any $x^i \in L$ there exist a neighborhood $\mathcal{N}_i \subset X$ containing x^i and a class $\mathcal{K}$ function α^i such that

$$f(x) \geq \alpha^i\left(\left\|x - x^i\right\|\right) \tag{A.1.3}$$

for all $x \in \mathcal{N}_i$.

(ii) There is a class $\mathcal{K}$ function α such that (A.1.2) holds.

(i): (A.1.1) implies that f has a strict local minimum at x^i, i.e., there exists $\varepsilon > 0$ such that for all $d \in \mathbb{R}^n$ with $\|d\| = 1$ and for all $0 < \delta \leq \varepsilon$ it holds that

$$f\left(x^i\right) < f\left(x^i + \delta d\right).$$

In case that $x^i \in \partial X$, the admissible directions d are only allowed to point inwards the constraint set. Given some $i \in \mathbb{I}_{[1,P]}$ as well as $\varepsilon > 0$ and $d \in \mathbb{R}^n$ with $\|d\| = 1$ from above, define $g_d^i : [0, \varepsilon] \to \mathbb{R}$ as

$$g_d^i(\delta) = f\left(x^i + \delta d\right).$$

Due to the continuity of f and its local minimum at x^i, there must exist a neighborhood $\mathcal{N}_i \subset X$ of x^i such that g_d is strictly increasing for any admissible d. In this neighborhood, we define

$$\alpha^i(r) = \min_{\substack{d \in \mathbb{R}^n \\ \|d\|=1 \\ d \text{ admissible}}} g_d^i(r),$$

noting that α^i is strictly increasing. Then, since for any $x \in \mathcal{N}_i$ there exists a unique $d^* \in \mathbb{R}^n$ with $\|d^*\| = 1$ such that $x = x^i + \|x - x^i\| d^*$, we obtain

$$f(x) = f\left(x^i + \left\|x - x^i\right\| d^*\right) = g_{d^*}^i\left(\left\|x - x^i\right\|\right) \geq \alpha^i\left(\left\|x - x^i\right\|\right).$$

(ii): Outside of the neighborhoods of the x^i's, due to compactness of X, f is bounded from below by a positive constant. Hence, by choosing their slope small enough, the α^i's from (i) can be extended to $X \setminus \mathcal{N}_i$ such that

$$f(x) \geq \alpha^i\left(\left\|x - x^i\right\|\right) \tag{A.1.4}$$

for all $x \in X \setminus \left\{\bigcup_{j=1}^P \mathcal{N}_j\right\}$ and for all $i \in \mathbb{I}_{[1,P]}$. Define

$$\alpha(r) := \min_{i \in \mathbb{I}_{[1,P]}} \alpha^i(r)$$

and note that $\alpha \in \mathcal{K}$. From (A.1.4), we conclude

$$f(x) \geq \alpha\left(|x|_L\right)$$

for any $x \in X \setminus \left\{\bigcup_{i=1}^P \mathcal{N}_i\right\}$. Moreover, according to (i), we have

$$f(x) \geq \alpha\left(|x|_L\right)$$

for all $x \in \mathcal{N}_i$ and all $i \in \mathbb{I}_{[1,P]}$, which concludes the proof. □

Eigenständigkeitserklärung

Ich versichere hiermit, dass ich, Julian Berberich, die vorliegende Arbeit selbstständig angefertigt, keine anderen als die angegebenen Hilfsmittel benutzt und sowohl wörtliche, als auch sinngemäß entlehnte Stellen als solche kenntlich gemacht habe. Die Arbeit hat in gleicher oder ähnlicher Form noch keiner anderen Prüfungsbehörde vorgelegen. Weiterhin bestätige ich, dass das elektronische Exemplar mit den anderen Exemplaren übereinstimmt.

Ort, Datum

Unterschrift

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