Self-Adjointness and Domain of a Class of Generalized Nelson Models

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Abstract

In this thesis, we study the interaction of a single quantum particle and a quantized bosonic field motivated by the general formal expression

$$\Omega(p) + \int\limits_{\mathbb{R}^d} \omega(k) a_k^* a_k dk + \int\limits_{\mathbb{R}^d} \left[\overline{v(k)} e^{ikx} a_k + v(k) e^{-ikx} a_k^* \right] dk \tag{0.1}$$

for the Hamilton operator of the system. Here, $\Omega(p) \geq 0$ is the dispersion relation of the single particle and $\omega(k) \geq m > 0$ the dispersion relation of a boson with mass m. Furthermore, a_k^* and a_k are the creation and annihilation operators of a boson with momentum k; v is the form factor of the interaction.

The first two terms of the sum (0.1) describe the free dynamics of the single particle and the bosonic field, respectively. Their interaction is given by the third summand which is in general not a well-defined operator. We regularize the expression by cutting of the integral at a large value of $|k|$ which leads to the ultraviolet regularized Hamiltonian

$$H_\Lambda := \Omega(p) + d\Gamma(\omega) + \int\limits_{|k| \leq \Lambda} \left[\overline{v(k)} e^{ikx} a_k + v(k) e^{-ikx} a_k^* \right] dk, \tag{0.2}$$

where $\Lambda < \infty$ is the cutoff parameter.

For the cases $\Omega(p) = p^2$ and $\Omega(p) = \sqrt{1 + p^2}$, it is known, for suitable choices of ω and v, that H_Λ has a limit H in the norm-resolvent sense, possibly after subtracting a suitable renormalization energy E_Λ. One of the main goals of this thesis is to generalize these results to a class of Hamiltonians of the general form given by Expression (0.1) and Equation (0.2), respectively. To this end we use and generalize the methods that were developed for the cases $\Omega(p) = p^2$ and $\Omega(p) = \sqrt{1 + p^2}$. Moreover, if the norm-resolvent limit H exists, we then apply these techniques to the study of the domain $D(H)$ of H. Just this last aspect is, to our best knowledge, not yet explored in science literature.

In the case $\Omega(p) = p^2$, we study a generalized model of the Fröhlich Hamiltonian and a generalized model of the Nelson Hamiltonian. The generalized Fröhlich Hamiltonian also contains the large polaron model in two and three space dimensions. The main tool of this discussion is the unitary dressing transform due to Gross. But for a more general $\Omega(p)$, the Gross transform is not applicable and we resort to the more involved resolvent expansion of Hepp and Eckmann. For the generalized Fröhlich Hamiltonian, we are able to give an explicit representation of H and we can show that the intersection

of its domain and the domain of the free Hamiltonian H_0 consists of the zero vector only. We can reproduce this result for a special case of generalized Nelson models. Moreover, we discuss the case, here, that we can construct H only as the self-adjoint operator associated to a quadratic form, call it q. Then, the domain of q intersected with the domain of $H_0^{1/2}$, which is the domain of the form associated to the free Hamiltonian, consists of the zero vector only.

In the case of general Ω in Expression (0.1) or Equation (0.2), respectively, the Gross transform cannot be used. The technique of Hepp and Eckmann that we mentioned above consists of a suitably reorganized resolvent expansion, which allows us to prove existence of the norm-resolvent limit. The existence is shown for a large class of Hamiltonians satisfying assumptions held very generally. The technique is also used to study domain properties. One can prove important set inclusions, say one finds supersets of the domain of H. In this context, the results for the generalized Fröhlich and Nelson Hamiltonians are partially reproduced.

Zusammenfassung

In dieser Arbeit untersuchen wir die Wechselwirkung eines einzelnen, quantenmechanischen Teilchens und eines quantisierten Bosonenfeld motiviert durch den allgemeinen formalen Ausdruck

$$\Omega(p) + \int_{\mathbb{R}^d} \omega(k) a_k^* a_k dk + \int_{\mathbb{R}^d} \left[\overline{v(k)} e^{ikx} a_k + v(k) e^{-ikx} a_k^* \right] dk \tag{0.3}$$

für den Hamiltonoperator des Systems. Dabei ist $\Omega(p) \geq 0$ die Dispersionsrelation des einzelnen Teilchens und $\omega(k) \geq m > 0$ die Dispersionsrelation eines Bosons mit Masse m. Desweiteren stehen a_k^* und a_k für die Erzeugungs- und Vernichtungsoperatoren eines Bosons mit Impuls k; v ist der Formfaktor der Wechselwirkung.

Die ersten beiden Terme der Summe (0.3) beschreiben die freie Dynamik des einzelnen Teilchens beziehungsweise des Bosonenfeldes. Deren Wechselwirkung ist gegeben durch den dritten Summanden, welcher im Allgemeinen kein wohldefinierter Operator ist. Wir regularisieren den Ausdruck, indem wir das Integral für einen großen Wert von $|k|$ abschneiden, was zum ultraviolettregularisierten Hamiltonian

$$H_\Lambda := \Omega(p) + d\Gamma(\omega) + \int_{|k| \leq \Lambda} \left[\overline{v(k)} e^{ikx} a_k + v(k) e^{-ikx} a_k^* \right] dk \tag{0.4}$$

führt. Dabei ist $\Lambda < \infty$ der besagte Cutoffparameter.

In den Fällen $\Omega(p) = p^2$ und $\Omega(p) = \sqrt{1+p^2}$ ist bekannt, dass, unter bestimmter Wahl von ω und v, H_Λ einen Limes H im Normresolventensinn besitzt, eventuell nach Abzug einer geeigneten Renormierungsenergie E_Λ. Eines der Hauptziele dieser Arbeit ist, diese Ergebnisse auf eine Klasse von Hamiltonoperatoren gegeben durch den Ausdruck (0.3) beziehungsweise Gleichung (0.4) zu verallgemeinern. Dazu verwenden und verallgemeinern wir die Methoden, die für die Fälle $\Omega(p) = p^2$ und $\Omega(p) = \sqrt{1+p^2}$ entwickelt wurden. Desweiteren wenden wir diese Techniken an, den Definitionsbereich $D(H)$ von H zu studieren, sofern der Normresolventenlimes H existiert. Gerade dieser letzte Aspekt ist nach unserer Kenntnis in der Wissenschaftsliteratur bislang nicht erforscht.

Im Fall $\Omega(p) = p^2$ untersuchen wir ein verallgemeinertes Modell des Fröhlich und ein verallgemeinertes Modell des Nelson Hamiltonians. Der verallgemeinerte Fröhlich Hamiltonian enthält auch das Modell des großen Polarons in zwei und drei Dimensionen.

Das Hauptwerkzeug in dieser Diskussion ist die unitäre Grosstransformation. Für ein allgemeineneres $\Omega(p)$ ist die Grosstransformation jedoch nicht zielführend und wir machen Gebrauch von der etwas umständlicheren Resolventenentwicklung nach Hepp und Eckmann. Für den verallgemeinerten Fröhlich Hamiltonian ist es uns möglich, eine explizite Darstellung von H herzuleiten, und wir können zeigen, dass der Schnitt seines Definitionsbereiches mit dem des freien Hamiltonians H_0 nur den Nullvektor enthält. Dieses Ergebnis können wir in einem Spezialfall des verallgemeinerten Nelson Modells reproduzieren. Darüberhinaus diskutieren wir hier den Fall, in dem H nur noch als selbstadjungierter Operator zu einer quadratischen Form konstruiert werden kann, wir nennen diese Form q. Es ist dann der Schnitt des Definitionsbereiches von q mit dem Definitionsbereich von $H_0^{1/2}$, also dem Definitionsbereich der Form des freien Hamiltonians, der nur noch den Nullvektor enthält.

Im Falle eines allgemeinen Ω in Ausdruck (0.3) beziehungsweise Gleichung (0.4), kann die Grosstransformation, nicht verwendet werden. Die bereits erwähnte Technik von Hepp und Eckmann basiert auf einer in passender Weise umgeordneten Resolventenentwicklung, welche es uns ermöglicht, die Existenz des Normresolventenlimes zu beweisen. Die Existenz wird für eine große Klasse an Operatoren gezeigt, welche sehr allgemein gehaltene Voraussetzungen erfüllen. Die Technik wird auch zum Studium von Eigenschaften des Definitionsbereiches benutzt. Es werden wichtige Mengeninklusionen bewiesen, das heißt, für den Definitionsbereich von H werden Obermengen gefunden. In diesem Zusammenhang werden auch die Ergebnisse zu den verallgemeinerten Fröhlich und Nelson Hamiltonians teilweise reproduziert.

Contents

1. Introduction

1.1. Models of particles coupled to a bosonic field

In 1964, Edward Nelson published a famous article about the interaction of a conserved number of non-relativistic particles with a quantized bosonic field [1]. This system can be used for an effective description of the strong interaction, one of the four fundamental forces is physics. Figure 1.1 shows the Feynman diagram of a relevant binding mechanism. In the article [1], Nelson introduced some new methods which today, over half a century later, are considered as standard tools of modern mathematical physics. An example is the so-called KLMN-theorem which stands for the names Kato, Lions, Lax-Milgram, and Nelson and which is an important instrument of perturbation theory.

In the case of a single Schrödinger particle, the system Nelson studied is motivated by the formal expression

$$-\Delta + \int\limits_{\mathbb{R}^3} \omega(k) a_k^* a_k dk + \sqrt{\alpha} \int\limits_{\mathbb{R}^3} \frac{1}{\sqrt{\omega(k)}} \left[e^{ikx} a_k + e^{-ikx} a_k^* \right] dk. \tag{1.1}$$

for the Hamiltonian. The operator $-\Delta$ denotes the negative Laplacian in $L^2(\mathbb{R}^3)$, and describes the kinetic energy of the free Schrödinger particle. The second term is an operator in the symmetric Fock space $\mathcal{F}$ and stands for the free energy of the bosonic field with the dispersion relation $\omega(k) = \sqrt{m^2 + k^2}$, where $m > 0$ stands for the mass of a boson. The operators a_k^* and a_k are the creation and annihilation operators of a boson with momentum k. The parameter α is a positive coupling constant and can be set to 1 for our purposes. The interaction of the two subsystems is given in the third term of the sum (1.1).

The problem with the expression above is that this third term is not well-defined as an operator in $L^2(\mathbb{R}^3) \otimes \mathcal{F}$. For that, the factor $\omega(k)^{-1/2}$ would have to be square-integrable with respect to k.

The first step in the analysis of (1.1) is to cut off the integral for a large absolute value of k - this ultraviolet cutoff will be called Λ - and then to study the limit of the operator as $\Lambda \to \infty$. Let H_Λ be this regularized Hamiltonian. Nelson showed that, after a suitable renormalization, the unitary group generated by H_Λ is strongly convergent towards a unitary group generated by some self-adjoint operator H, which we call Nelson Hamiltonian. That is,

$$e^{-it(H_\Lambda + E_\Lambda)} \to e^{-itH} \qquad (\Lambda \to \infty) \tag{1.2}$$

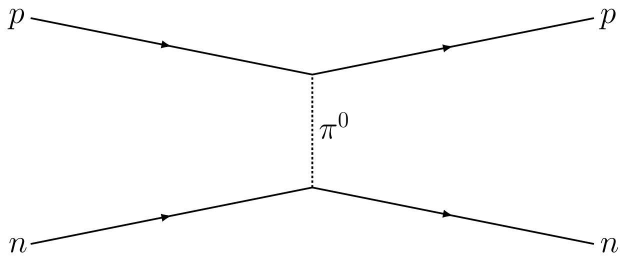

Figure 1.1.: A proton and a neutron in an atomic nucleus interact by exchanging a π^0-boson. Such a system can be effectively described by the Nelson Hamiltonian [1], what makes it interesting in the context of the strong interaction.

strongly in the Hilbert space, where E_Λ is the renormalization energy. This result is equivalent to the statement that

$$H_\Lambda + E_\Lambda \to H \qquad (\Lambda \to \infty) \tag{1.3}$$

in the strong-resolvent sense. It can be shown that this convergence also holds in the norm-resolvent sense, see for example Ammari [2].

Nelson used a unitary dressing transform U_Λ, the Gross transform, to transform the operator $H_\Lambda + E_\Lambda$. Regarding the Gross transform, see [3]. Associated to the transformed operator, a quadratic form can be defined, which is also well-defined in the case $\Lambda \to \infty$. In this case, the quadratic form is also associated to a self-adjoint operator H'. Then, H is obtained by backtransforming H'.

Until today, many papers about the Nelson Hamiltonian as well as the regularized Nelson Hamiltonian were published. A very good collection of these works is given in the introduction of the recently published article of Matte and Møller [4]. But until now, there are no works that treat the domain of the Nelson Hamiltonian. This is still an open question and something we discuss in this thesis. These studies are mainly motivated by our earlier results to analogue questions regarding the polaron model.

The polaron system describes an electron running through a polar crystal, see also Figure 1.2, and is based on the formal expression

$$-\Delta + \int\limits_{\mathbb{R}^3} a_k^* a_k \ dk + \sqrt{\alpha} \int\limits_{\mathbb{R}^3} \frac{1}{|k|} [e^{ikx} a_k + e^{-ikx} a_k^*] \ dk, \tag{1.4}$$

which goes back to Fröhlich, Pelzer and Zienau [5]. The symbols occurring in (1.4) are the same as in Expression (1.1) and, with respect to the well-definition, we have the same problems such that we also introduce an ultraviolet cutoff Λ. Here, we talk about the regularized Fröhlich Hamiltonian H_Λ. In contrast, the norm-resolvent limit of H_Λ is called Fröhlich Hamiltonian.

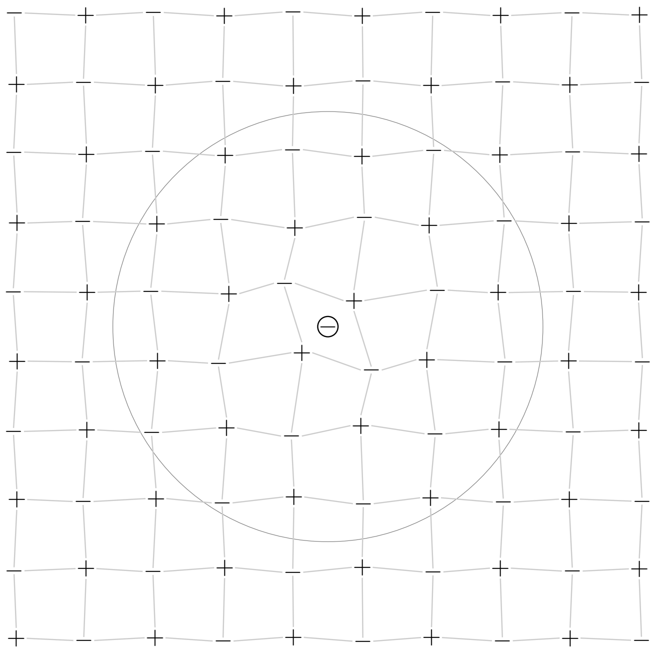

Figure 1.2: An electron in an ionic crystal polarizes its surroundings by Coulomb interaction. The electron and the induced lattice vibrations (phonons) together constitute a quasi-particle, the so-called polaron.

The ultraviolet regularization of the so-called Fröhlich Hamiltonian is well-understood, also because of its similarity to the Nelson model. In this context, the works of Ammari [2], Fröhlich [7], and Löwen [8] shall be mentioned. There are also two central articles of Lieb and Thomas [9] and Lieb and Yamazaki [10] studying the ground state energy of the polaron. Multi-polaron systems were studied for example by Frank, Lieb, Seiringer, and Thomas [11] and by Griesemer and Møller [12]

A central question also concerns the dynamics of a polaron. There are papers of Frank and Schlein [13], Frank and Gang [14], and Griesemer [15] discussing this topic. Furthermore, Marcel Griesemer discussed polaron models with additional electric and magnetic fields in collaboration with Hantsch and Wellig [16, 17] and with Anapolitanos [18].

But it is still an open question to study the domain of the Fröhlich Hamiltonian. This is also one of the main issues of this thesis. Using the techniques of Nelson [1] analogously for the Expression (1.4), we are moreover able to give an explicit representation of the Fröhlich Hamiltonian, say the norm-resolvent limit of H_Λ, where H_Λ is the ultraviolet regularized operator to the Expression (1.4). Here, an energy renormalization is not necessary. This representation is a new and central result. It is given in terms of the unitary Gross transform [3], which also plays an important role in studying the domains of the Fröhlich and the Nelson Hamiltonians. Our results are possible thanks to a new operator bound recently published by Frank and Schlein [13]. In the context of the Nelson Hamiltonian, we prove a generalization of this estimate.

In the two Expressions (1.1) and (1.4), that is in the Nelson and the Fröhlich models, a non-relativistic Schrödinger particle is considered. In quantum field theory, pseudo-relativistic versions of these models are also of interest. Such models are treated, for example, in the articles of Fröhlich [7, 19] and of Sloan [20]. The pseudo-relativistic Nelson model is also among the models studied in these papers. Taking these models into account leads to the most general ansatz for a model of a quantized particle interacting with a bosonic field that we consider in this thesis. We also motivate this sytem by a formal expression:

$$\Omega(p) + \int_{\mathbb{R}^d} \omega(k) a_k^* a_k \; dk + \alpha \int_{\mathbb{R}^d} [\overline{v(k)} e^{ikx} a_k + v(k) e^{-ikx} a_k^*] \; dk. \tag{1.5}$$

But now, we discuss a general function Ω of the particle momentum p with $\Omega(p) \geq 0$, a general massive bosonic dispersion relation ω with $\omega(k) \geq m > 0$, and a general form factor v. Moreover, we consider this model in arbitrary space dimensions $d \in \mathbb{N}$. Since p is the momentum operator of the particle, say $p = -i\nabla$, $\Omega(p)$ now is in general a pseudodifferential operator.

In Expression (1.5), for $d = 3$, we find again the Nelson model from (1.1), set $\Omega(p) = p^2$, $\omega(k) = \sqrt{m^2 + k^2}$, and $v(k) = \omega(k)^{-1/2}$, and the polaron model from (1.4), set $\Omega(p) = p^2$, $\omega(k) = 1$, and $v(k) = |k|^{-1}$. But by choosing $\Omega(p) = \sqrt{\mu^2 + p^2}$ with the particle mass $\mu \geq 0$, one can for example study pseudo-relativistic versions of these models.

Again, we start with an ultraviolet regularization of the interaction part of Expression (1.5), call the associated operator H_Λ. There are also operators included that have to be renormalized. Let E_Λ be the renormalization constant. We study under which conditions on the functions Ω, ω, and v, for $H_\Lambda + E_\Lambda$ exists a limit in the norm-resolvent sense as $\Lambda \to \infty$. The existence is well-known for the examples given above.

The main goal of this part of the thesis is to establish general conditions for the existence of the norm-resolvent limit of $H_\Lambda + E_\Lambda$ in the general case of Expression (1.5), such that the known examples are contained. For that, we cannot use the Gross transform since it does not work for a general particle dispersion relation Ω. One could say that the Gross transform as it is used in [1] was built to study only the case of the quadratic dispersion relation $\Omega(p) = p^2$. Nevertheless, we are able to use a resolvent expansion similarly to Hepp [21] and Eckmann [22] to treat this general model here. Hepp used this expansion to study ultraviolet divergent Lee models and Eckmann used it to study the pseudo-relativistic massive Nelson model in second quantization. This expansion also allows us to make statements about the domain of the operator.

1.2. Structure of the thesis

This thesis is divided in three chapters corresponding to three main projects. The order of the chapters reflects the chronological order in which the work on the projects was done. Hereby, it is remarkable that with proceeding section number the systems we discuss become more and more general.

The first model treated in Chapter 2 describes a class of Hamiltonians where especially the Fröhlich Hamiltonian, see (1.4), is contained. We study its self-adjointness, find an explicit representation of the operator, and also give a strong characterization of its domain. Some important calculations of Chapter 2 are divested in the Appendix A. The subject matter of Chapter 2 is already published in the *Journal of Mathematical Physics*, see [23]. Therefore, this Chapter 2 is a reprint of our published article up to slight changes with respect to formatting, numbers of sections, equations, and theorems, and so on. In Theorem A.1.1 of Appendix A.1, an additional statement is added compared to the published Theorem A.1 from [23].

Chapter 3 and Appendix B treat a more general class of operators motivated by the study of self-adjointness and domain properties of the Nelson Hamiltonian. This system was motivated above in Expression (1.1). It turned out that one can implement the Hamiltonian so generally that the Fröhlich like models from Chapter 2 are also included. Chapter 3 is complete and the main results will be submitted to a trade journal soon.

Finally, in Section 4 and Appendix C, the most general Hamiltonian, see Expression (1.5) is studied. Using the resolvent expansion mentioned above, the conditions are studied under which the operator is renormalizable. In other words, under which conditions does a norm-resolvent limit of the ultraviolet regularized Hamiltonian exist, maybe after a suitable renormalization. The expansion is also used to study domain properties of the operator in the norm-resolvent limit. Here, we can reproduce some statements of the Chapters 2 and 3, but not all. We will try to improve the methods in future work. We will also try to include massless boson models, say to replace the assumption $m > 0$ in the dispersion relation ω in Expression (1.5) by $m \geq 0$.

2. Self-Adjointness and Domain of the Fröhlich Hamiltonian

Marcel Griesemer and Andreas Wünsch

Abstract

In the large polaron model of H. Fröhlich, the electron-phonon interaction is a small perturbation in form sense, but a large perturbation in operator sense. This means that the form-domain of the Hamiltonian is not affected by the interaction but the domain of self-adjointness is. In the particular case of the Fröhlich model, we are nevertheless able, thanks to a recently published new operator bound, to give an explicit characterization of the domain in terms of a suitable dressing transform. Using the mapping properties of this dressing transform, we analyse the smoothness of vectors in the domain of the Hamiltonian with respect to the position of the electron. Up to slight changes, this chapter is already published, see [23].

2.1. Introduction

A popular model for the description of an electron in a polar crystal due to Fröhlich, Pelzer and Zienau is based on the formal expression

$$-\Delta + N + \sqrt{\alpha} \int_{\mathbb{R}^3} [e^{ikx} a(k) + e^{-ikx} a^*(k)] \, \frac{dk}{|k|} \tag{2.1}$$

for the Hamiltonian of the system [5]. Here, Δ denotes the Laplace operator in $L^2(\mathbb{R}^3)$, N is the number operator in the symmetric Fock space over $L^2(\mathbb{R}^3)$, and α is a coupling constant. The third term of (2.1), which accounts for the electron-phonon interaction, is not an operator in the Hilbert space because the form factor is not square integrable. Therefore, expression (2.1), as it stands, is not a densely defined operator and hence cannot readily be adopted as the Hamiltonian of the system. Expression (2.1) does, however, define a closed, semi-bounded quadratic form with domain $D(H_0^{1/2})$, where $H_0 = -\Delta + N$. Indeed, by a simple argument of Lieb and Thomas, the interaction is infinitesimally form bounded with respect to H_0 [9, 10]. There is therefore a unique self-adjoint operator H, the Fröhlich Hamiltonian, associated with the quadratic form defined by (2.1). If H_Λ, for $\Lambda > 0$, is defined in terms of (2.1) with ultraviolet cutoff

$|k| \leq \Lambda$ in the interaction, then it follows, by general arguments, that $H_\Lambda \to H$ in the norm resolvent sense as $\Lambda \to \infty$.

The main purpose of this paper is to describe the domain $D(H)$ of H as explicitly as possible. To this end, we follow Nelson and determine UHU^*, where U is a dressing transform given by Gross [1]. Using a recently published new variant of the Lieb-Thomas bound, we are able to show that UHU^* is self-adjoint on $D(H_0)$ and hence that

$$D(H) = U^* D(H_0). \tag{2.2}$$

This result allows us to determine a core of H in terms of coherent states and to describe the action of H on this core explicitly. Moreover, we show that

$$D(H) \subset \Big(\bigcap_{0<s<3/4} D\big((-\Delta)^s\big)\Big) \cap D(N), \tag{2.3}$$

and that

$$D(H) \cap D\left((-\Delta)^{3/4}\right) = \{0\}. \tag{2.4}$$

The identity (2.4) implies in particular that $D(H) \cap D(-\Delta) = \{0\}$, which has the following simple explanation: when H is applied to a vector $\Psi \in D(H)\backslash\{0\}$, then the interaction part in (2.1), we call it $\sqrt{\alpha}W$, creates a vector $\sqrt{\alpha}W\Psi$ outside of the Hilbert space. In fact, $\sqrt{\alpha}W\Psi$ belongs to the dual of $D(H_0^{1/2})$ equipped with the form norm of H_0. This vector must be canceled by some part of $H_0\Psi$ that is not in the Hilbert space either. This means that $\Psi \notin D(H_0)$ and, since $\Psi \in D(N)$, by (2.3), we conclude that $\Psi \notin D(-\Delta)$. The mechanism of this cancellation of non-Hilbert space parts is illustrated in the appendix by a formal computation of (2.1) applied to vectors Ψ from a core of H where we know the action of H explicitly. Of course, these remarks equally apply to other Hamiltonians describing quantum particles interacting with a quantized field of bosons. Indeed, we prove (2.2) and suitable generalizations of (2.3) and (2.4) for a large class of form factors $v(k)$ including $v(k) = |k|^{-(d-1)/2}$, $k \in \mathbb{R}^d$, describing the polaron in $d = 2$ and $d = 3$ space dimensions, respectively. In this more general case, the admissible exponents in (2.3) and (2.4) are determined by the rate of decay of the form factor as $|k| \to \infty$. Our results could be further generalized to include N-polaron systems or external magnetic fields, but we refrain from such generalizations in order to keep the paper short and the notation simple.

For the massive Nelson model where $H_0 = -\Delta + d\Gamma(\omega)$, $\omega(k) = \sqrt{k^2 + m^2}$ and $v(k) = \omega(k)^{-1/2}$, we expect results analogue to (2.3) and (2.4). In that case, however, the role of the number operator N is played by the field energy $d\Gamma(\omega)$. Its domain is not left invariant by the Gross transform, which complicates matters. We plan to return to this case in a future publication.

The UV renormalization of the Nelson and the Fröhlich models in terms of the Gross transform is well-understood and well-documented in the literature [1, 2, 7, 8]. The much more direct and straightforward characterizations of H based on the Lieb-Thomas argument have not yet been properly described in the literature, and we therefore elaborate on them in Section 2.2. Our main objectives are, however, characterization (2.2) of the domain, see Section 2.3, and the proofs of (2.3) and (2.4) in Section 2.4. In the appendices, we prove an abstract result on resolvent convergence based on form bounds, Appendix A.1, we collect background on annihilation and creation operators, Appendix A.2, and we describe the action of H on vectors from a suitable core of H, Appendix A.3.

2.2. The construction of the Fröhlich Hamiltonian

In this section, we describe the class of Hamiltonians whose domains will be studied in Sections 2.3 and 2.4. These Hamiltonians describe a quantum particle (called electron) in $\mathbb{R}^d$ that is coupled linearly to a quantized field of scalar bosons (called phonons). We begin with notations and hypotheses on the form factors.

Let $\mathscr{H} := L^2(\mathbb{R}^d, dx) \otimes \mathcal{F}$, where $\mathcal{F}$ denotes the symmetric Fock space over $L^2(\mathbb{R}^d, dk)$ with arbitrary $d \in \mathbb{N}$. We may identify $\mathscr{H}$ with $L^2(\mathbb{R}^d, \mathcal{F})$ through the isomorphism given by $\varphi \otimes \eta \mapsto \varphi(x)\eta$. Let $H_0 := -\Delta + N$, where Δ is the (self-adjoint) Laplacian in $L^2(\mathbb{R}^d)$ and N denotes the number operator in $\mathcal{F}$. Let

$$\|\Psi\|_0 := \left\|(H_0 + 1)^{1/2}\Psi\right\| \tag{2.5}$$

for $\Psi \in D(H_0^{1/2})$. The Hamiltonian H_0 is self-adjoint on $D(H_0) = D(-\Delta \otimes \mathbb{1}) \cap D(\mathbb{1} \otimes N)$ and essentially self-adjoint on $D(H_0) \cap \mathscr{H}_0$, where

$$\mathscr{H}_0 := \bigcup_{n \geq 0} \chi(N \leq n)\mathscr{H}. \tag{2.6}$$

The electron-phonon interaction occurs in terms of annihilation and creation of phonons. The usual annihilation and creation operators in Fock space associated with some vector $f \in L^2(\mathbb{R}^d)$ will be denoted by $a(f)$ and $a^*(f)$, respectively. They are closed, adjoint to each other with $D(a(f)) = D(a^*(f)) \supset D(\sqrt{N})$, and they obey the canonical commutation relations $[a(f), a^*(g)] = \langle f, g\rangle$ on $D(N)$. The symmetric field operators

$$\phi(f) := a(f) + a^*(f), \qquad \pi(f) := \phi(if) \tag{2.7}$$

are essentially self-adjoint on $D(N)$, and they obey the commutation relations

$$[\phi(f), \phi(g)] = 2i \operatorname{Im} \langle f, g\rangle, \qquad [\phi(f), \pi(g)] = 2i \operatorname{Re} \langle f, g\rangle. \tag{2.8}$$

The (self-adjoint) closures of the operators $\phi(f)$ and $\pi(f)$ will be denoted by the same symbols.

We will have occasion to work with generalized annihilation and creation operators $a(F)$ and $a^*(F)$ that are operators in $\mathscr{H}$ rather than $\mathcal{F}$. Here $F : L^2(\mathbb{R}^d, dx) \to L^2(\mathbb{R}^d, dx) \otimes L^2(\mathbb{R}^d, dk)$ is a linear operator. In the simplest case, $F\varphi = \varphi \otimes f$ for some $f \in L^2(\mathbb{R}^d, dk)$ and then $a^\#(F) = \mathbb{1} \otimes a^\#(f)$ is the usual annihilation or creation operator in $\mathcal{F}$. Often, but not always, the operator F will be defined in terms of some function $(x, k) \mapsto F_x(k)$, denoted by F as well, through the equation $(F\varphi)(x, k) = \varphi(x)F_x(k)$. In this case, $(a^\#(F)\Psi)(x) = a^\#(F_x)\Psi(x)$. Typically, $F_x(k) = e^{-ikx}f(k)$, where $f \in L^2(\mathbb{R}^d)$ and then the operator norm of F equals the norm of f in $L^2(\mathbb{R}^d)$. See Appendix A.2 for the definition of $a^\#(F)$ in the general case.

For $\Lambda < \infty$, we define $H_\Lambda : D(H_0) \subset \mathscr{H} \to \mathscr{H}$ by

$$H_\Lambda := H_0 + \phi(G_\Lambda), \tag{2.9}$$

where

$$G_{\Lambda,x}(k) := e^{-ikx}v(k)\chi_\Lambda(k). \tag{2.10}$$

Here, χ_Λ denotes the characteristic function of the set $\{k \in \mathbb{R}^d | \ |k| \le \Lambda\}$. On the form factor $v : \mathbb{R}^d \to \mathbb{C}$, we impose the following assumptions:

$$(v1) \qquad v \in L^2_{loc}(\mathbb{R}^d) \quad \text{and} \quad v(k) = v(-k),$$

$$(v2) \qquad \int \frac{|v(k)|^2}{1+k^2}\, dk < \infty.$$

These assumptions are sufficient for the results of the present section. Later, we will replace $(v2)$ by the slightly stronger assumption

$$(v3) \qquad \sup_{q\in\mathbb{R}^d} \int\limits_{|k|\ge K} \frac{|v(k)|^2}{1+(q-k)^2} dk \longrightarrow 0 \qquad (K \to \infty).$$

An example of a form-factor v satisfying these conditions is the function

$$v(k) = |k|^{-(d-1)/2}, \quad d \ge 2, \tag{2.11}$$

which includes the form factors $v(k) = |k|^{-1/2}$ and $v(k) = |k|^{-1}$ of the large polaron models in $d = 2$ and $d = 3$ dimensions, respectively. Use Hölder's inequality with exponents $(d+1)/(d-1)$ and $(d+1)/2$ to see that (2.11) satisfies $(v3)$.

From Corollary A.2.2, it follows that $\phi(G_\Lambda)(N+1)^{-1/2}$ is bounded, and hence, $\phi(G_\Lambda)$ is infinitesimally H_0-bounded. We conclude, by Kato-Rellich, that H_Λ is self-adjoint on $D(H_0)$ and, moreover, that the quadratic form

$$(\Phi, \Psi) \mapsto \langle \Phi, \phi(G_\Lambda)\Psi \rangle \tag{2.12}$$

defined on $D(H_0^{1/2})$ satisfies the Hypothesis (a) of Theorem A.1.1. Let

$$D_K := \int_{|k|\geq K} \frac{1}{k^2} |v(k)|^2 dk. \tag{2.13}$$

Then, $D_K \to 0$ as $K \to \infty$ by Assumption $(v2)$. Therefore, the following lemma establishes the Hypothesis (b) of Theorem A.1.1:

Lemma 2.2.1. *Assume $(v1)$ and $(v2)$. Then, for all $\Lambda_1, \Lambda_2 \in \mathbb{R}_+$ and all $\Phi, \Psi \in D(H_0^{1/2})$, we have*

$$|\langle \Phi, \phi(G_{\Lambda_1})\Psi\rangle - \langle \Phi, \phi(G_{\Lambda_2})\Psi\rangle| \leq |D_{\Lambda_1} - D_{\Lambda_2}|^{1/2} \|\Phi\|_0 \|\Psi\|_0.$$

Proof. Since $\phi(G_\Lambda)$ is symmetric, it suffices to establish the desired bound for the case $\Phi = \Psi$. To this end, fix $\Lambda_1, \Lambda_2 > 0$ and let

$$A_x(k) := \frac{k}{|k|^2}(G_{\Lambda_1,x}(k) - G_{\Lambda_2,x}(k)) \tag{2.14}$$

with components $A_{x,\ell}(k)$, $\ell = 1 \ldots d$. Then, $i\nabla_x \cdot A_x = G_{\Lambda_1,x} - G_{\Lambda_2,x}$, and hence,

$$[p, a(A)] := \sum_{\ell=1}^{d} [p_\ell, a(A_\ell)] = a(G_{\Lambda_1} - G_{\Lambda_2}), \tag{2.15}$$

where $p = -i\nabla$. It follows that

$$\begin{aligned} |\langle \Psi, \phi(G_{\Lambda_1})\Psi\rangle - \langle \Psi, \phi(G_{\Lambda_2})\Psi\rangle| &= |2\,\mathrm{Re}\,\langle \Psi, a(G_{\Lambda_1} - G_{\Lambda_2})\Psi\rangle| \\ &= |2\,\mathrm{Re}\,\langle \Psi, [p, a(A)]\Psi\rangle| \\ &\leq 2\Big(|\langle p\Psi, a(A)\Psi\rangle| \; + |\langle a^*(A)\Psi, p\Psi\rangle|\Big) \\ &\leq 4\|p\Psi\| \big\|\sqrt{N+1}\Psi\big\| \|A\| \\ &\leq 2\|\Psi\|_0^2 \|A\|, \end{aligned} \tag{2.16}$$

where $\|A\|^2 = |D_{\Lambda_1} - D_{\Lambda_2}|$. □

We are now ready to prove the main result of this section.

Theorem 2.2.2. *Assume (v1) and (v2). Then, the following statements hold true:*

(i) The limit $W_\infty(\Phi,\Psi) := \lim_{\Lambda\to\infty} \langle \Phi, \phi(G_\Lambda)\Psi\rangle$ *exists for all* $\Phi, \Psi \in D(H_0^{1/2})$.

(ii) The quadratic form on $D(H_0^{1/2})$ *given by* $\left\langle H_0^{1/2}\Phi, H_0^{1/2}\Psi\right\rangle + W_\infty(\Phi,\Psi)$ *is closed and bounded from below.*

(iii) If H *denotes the (unique) self-adjoint operator associated with the quadratic form from (ii), then* $H_\Lambda \to H$ *in the norm-resolvent sense as* $\Lambda \to \infty$.

Proof. We apply Theorem A.1.1 to the Hamiltonian H_0 and the quadratic form defined by (2.12). We already pointed out that Hypothesis (a) of Theorem A.1.1 is satisfied and Hypothesis (b) follows from Lemma 2.2.1. Now the statements (i), (ii), and (iii) follow from Theorem A.1.1. □

The convergence $H_\Lambda \to H$ in the norm-resolvent sense implies convergence in the strong resolvent sense, which is equivalent to

$$e^{-iH_\Lambda t}\Psi \longrightarrow e^{-iHt}\Psi \qquad (\Lambda\to\infty) \tag{2.17}$$

for all $t\in\mathbb{R}$ and all $\Psi \in \mathscr{H}$. Alternatively, the existence of limit (2.17) can be derived directly from Lemma 2.2.1 and its Corollary 2.2.3, below. Hence, with the help of Stone's theorem, a further and very straightforward characterization of H as the generator of unitary group (2.17) is achieved. This is the content of the Theorem 2.2.4 and its proof.

Corollary 2.2.3. *Assume (v1) and (v2). Then, for every* $\varepsilon > 0$, *there exists* C_ε *such that for all* $\Lambda > 0$,

(a) $\pm\phi(G_\Lambda) \le \varepsilon H_0 + C_\varepsilon$,

(b) $(1-\varepsilon)H_0 - C_\varepsilon \le H_\Lambda \le (1+\varepsilon)H_0 + C_\varepsilon$.

Proof. (b) follows immediately from (a). To prove (a), note that the asserted inequality is true for any fixed $\Lambda = \Lambda_0$. Then, choose Λ_0 sufficiently large and use Lemma 2.2.1. □

Theorem 2.2.4. *Assume (v1) and (v2). Then, for all* $t\in\mathbb{R}$ *and* $\Psi\in\mathscr{H}$, *the limit*

$$U(t)\Psi := \lim_{\Lambda\to\infty} e^{-iH_\Lambda t}\Psi$$

exists and defines a strongly continuous unitary group $U(t)$.

Proof. Let $U_\Lambda(t) = \exp(-iH_\Lambda t)$. Then, for all $\Psi \in D(H_0)$,

$$\begin{aligned}
\|U_{\Lambda_1}(t)\Psi - U_{\Lambda_2}(t)\Psi\|^2 &= 2\|\Psi\|^2 - 2\,\mathrm{Re}\,\langle U_{\Lambda_1}(t)\Psi, U_{\Lambda_2}(t)\Psi\rangle \\
&= -2\,\mathrm{Re}\left\langle \Psi, (U^*_{\Lambda_1}(t)U_{\Lambda_2}(t) - 1)\Psi\right\rangle \\
&= -2\,\mathrm{Re}\, i\int_0^t \left\langle \Psi, U^*_{\Lambda_1}(s)(H_{\Lambda_1} - H_{\Lambda_2})U_{\Lambda_2}(s)\Psi\right\rangle ds \\
&\le C|t|\;|D_{\Lambda_2} - D_{\Lambda_1}|^{1/2}\|\Psi\|_0^2. \qquad (2.18)
\end{aligned}$$

In the last inequality, we used Lemma 2.2.1 and Corollary 2.2.3 (b), which implies that

$$\|U_\Lambda(t)\Psi\|_0 \le C\|\Psi\|_0 \qquad (2.19)$$

with a constant C that is independent of Λ and t. The bound (2.18) implies that $U(t)\Psi$ exists for all $\Psi \in D(H_0)$ and that $U_\Lambda(t)\Psi \to U(t)\Psi$ uniformly for t from compact intervals. Hence, $t \mapsto U(t)\Psi$ is continuous for $\Psi \in D(H_0)$. Since $\|U_\Lambda(t)\| = 1$ and since $D(H_0)$ is dense, it follows that $U(t)$ exists on $\mathscr{H}$, that $\|U(t)\| = 1$, and that $t \mapsto U(t)\Psi$ is continuous. The group properties

$$U(0) = 1 \quad \text{and} \quad U(t+s) = U(t)U(s) \qquad (2.20)$$

follow from the corresponding properties of $U_\Lambda(t)$. They imply that $U(-t) = U(t)^{-1}$ and hence that $U(t)$ is unitary. □

2.3. The Gross transform and the domain of H

In this section, we prove Equation (2.2) in the Introduction in the more general form given in Theorem 2.3.7, below. To this end, we first need to recall, from [1], the dressing transform of Gross and its effect on H_Λ.

The Gross transform $U_\Lambda : \mathscr{H} \to \mathscr{H}$ is a unitary linear operator depending on the parameters $K, \Lambda \ge 0$, where K is fixed most of the time and, therefore, often suppressed in our notation. For given K, Λ with $0 < K < \Lambda \le \infty$, we define

$$U_\Lambda := e^{i\pi(B_\Lambda)}, \qquad (2.21)$$

where

$$B_{\Lambda,x}(k) := -\frac{1}{1+k^2}G_{\Lambda,x}(k)(1 - \chi_K(k)). \qquad (2.22)$$

We will use kB_Λ and k^2B_Λ to denote the functions $kB_{\Lambda,x}(k)$ and $k^2B_{\Lambda,x}(k)$, respectively. Note that, by $(v2)$, $|B_{\Lambda,x}| \le \sup_x |B_{\infty,x}| \in L^2(\mathbb{R}^d)$ and that $\sup_x \|B_{\Lambda,x} - B_{\infty,x}\| \to 0$ as $\Lambda \to \infty$. It follows, by a generalization of Lemma A.2.5, that

$$U_\Lambda \longrightarrow U_\infty \quad (\Lambda \to \infty) \qquad (2.23)$$

strongly in $\mathscr{H}$. To compute $U_\Lambda H_\Lambda U^*_\Lambda$ we need the following lemma. From now on, p and p^2 often denote $-i\nabla$ and $-\Delta$, respectively.

Lemma 2.3.1. *Assume* $(v1)$ *and* $(v2)$*. Then:*

(a) $U_\Lambda D(H_0^{1/2}) = D(H_0^{1/2})$ *for* $\Lambda \le \infty$ *and*

$$U_\Lambda p U_\Lambda^* = p - \phi(kB_\Lambda) \quad \text{on } D(H_0^{1/2}).$$

(b) $U_\Lambda D(H_0) = D(H_0)$ *for* $\Lambda < \infty$ *and*

$$U_\Lambda p^2 U_\Lambda^* = (p - \phi(kB_\Lambda))^2 \quad \text{on } D(H_0).$$

(c) $U_\infty D(H_0)$ *is a form core of* H_0*.*

Since the components of p are essentially self-adjoint on $D(H_0^{1/2})$, part (a) implies that $U_\Lambda p_j U_\Lambda^* = p_j - \phi(k_j B_\Lambda)$ as an equality between self-adjoint operators on their respective domains.

Proof. (a) Let $\mathcal{D} = D(H_0) \cap \mathscr{H}_0$. Then, $\mathcal{D}$ is an operator core and hence a form core of H_0. Moreover, for $\Psi \in \mathcal{D}$, one shows that

$$pU_\Lambda^*\Psi = U_\Lambda^*(p - \phi(kB_\Lambda))\Psi \tag{2.24}$$

by expanding U_Λ^* in its exponential series. Here we used that $[\phi(kB_\Lambda), \pi(B_\Lambda)] = 0$ by assumption $(v1)$ on v. Since $\mathcal{D}$ is a form core of H_0 and since $(p - \phi(kB_\Lambda))$ is bounded w.r.t. $H_0^{1/2}$ Equation (2.24) extends to all $\Psi \in D(H_0^{1/2})$ and we see that $U_\Lambda D(H_0^{1/2}) \subset D(|p|)$. Since $D(H_0^{1/2}) = D(|p|) \cap D(\sqrt{N})$ and since $D(\sqrt{N})$ is left invariant by U_Λ^*, see Lemma A.2.4, we conclude that $U_\Lambda^* D(H_0^{1/2}) \subset D(H_0^{1/2})$. Likewise, $U_\Lambda D(H_0^{1/2}) \subset D(H_0^{1/2})$ by changing the sign of v and part (a) is proved.

(b) Let $\Psi \in D(H_0)$. Then $U_\Lambda^*\Psi \in D(H_0^{1/2})$ by part (a) and $pU_\Lambda^*\Psi$ is given by Equation (2.24). For $\Lambda < \infty$, $(p - \phi(kB_\Lambda))\Psi \in D(|p|) \cap D(\sqrt{N}) = D(H_0^{1/2})$. It follows, by part (a) again, that $U_\Lambda^*(p - \phi(kB_\Lambda))\Psi \in D(H_0^{1/2})$. Hence, in view of Equation (2.24), $pU_\Lambda^*\Psi \in D(|p|)$ and $p^2 U_\Lambda^*\Psi = U_\Lambda^*(p - \phi(kB_\Lambda))^2\Psi$. Since $U_\Lambda^* D(N) \subset D(N)$ by Lemma A.2.4, part (b) follows from $D(H_0) = D(p^2) \cap D(N)$.

(c) Let $\mathscr{H}_1 = D(H_0^{1/2})$ equipped with the form norm of H_0. By part (a), $U_\infty^* : \mathscr{H}_1 \to \mathscr{H}_1$ and this operator is closed which is easy to see from the continuity of U_∞^* in $\mathscr{H}$. Therefore, U_∞^* is bounded in $\mathscr{H}_1$ by the closed graph theorem. Since $D(H_0)$ is dense in $\mathscr{H}_1$, it follows that $U_\infty^* D(H_0)$ is dense in $\mathscr{H}_1$ as well. □

The results from Lemma 2.3.1 (b), the identity

$$p \cdot a^*(kB_\Lambda) + a(kB_\Lambda) \cdot p = a^*(kB_\Lambda) \cdot p + p \cdot a(kB_\Lambda) - \phi(k^2 B_\Lambda), \tag{2.25}$$

and Lemma A.2.4 yield the operator identities

$$\begin{aligned} U_\Lambda p^2 U_\Lambda^* &= p^2 - 2a^*(kB_\Lambda)\cdot p - 2p\cdot a(kB_\Lambda) \\ &\quad + \phi(kB_\Lambda)^2 + \phi(k^2 B_\Lambda) \end{aligned} \tag{2.26}$$

$$U_\Lambda N U_\Lambda^* = N + \phi(B_\Lambda) + \|B_\Lambda\|^2 \tag{2.27}$$

$$U_\Lambda \phi(G_\Lambda) U_\Lambda^* = \phi(G_\Lambda) + 2\,\mathrm{Re}\,\langle B_\Lambda, G_\Lambda\rangle \tag{2.28}$$

on $D(H_0)$ for $\Lambda < \infty$. In the above equations, we introduced various dot-products such as $p\cdot a(kB_\Lambda) = \sum_{j=1}^d p_j a(k_j B_\Lambda)$. In view of the fact that $(1+k^2)B_\Lambda = G_K - G_\Lambda$, by definition (2.22) of B_Λ, we arrive at:

Proposition 2.3.2. *Assume* $(v1)$ *and* $(v2)$*. Then for all* $\Lambda < \infty$*, we have* $U_\Lambda H_\Lambda U_\Lambda^* = H_K + V_{K,\Lambda}$ *on* $D(H_0)$*, where*

$$\begin{aligned} V_{K,\Lambda} &:= -2a^*(kB_\Lambda)\cdot p - 2p\cdot a(kB_\Lambda) + \phi(kB_\Lambda)^2 + C_{K,\Lambda}, \\ C_{K,\Lambda} &:= \|B_\Lambda\|^2 + 2\langle G_\Lambda, B_\Lambda\rangle = \int\limits_{K\le|k|\le\Lambda} |v(k)|^2\big((1+k^2)^{-2} - 2(1+k^2)^{-1}\big)\,dk. \end{aligned}$$

In particular, the operator $H'_{K,\Lambda} := H_K + V_{K,\Lambda}$ *is self-adjoint on* $D(H_0)$*.*

The assumption $(v2)$ implies that kB_Λ is square integrable even for $\Lambda = \infty$, and hence, the creation and annihilation operators $a^*(kB_\Lambda)$ and $a(kB_\Lambda)$ in $V_{K,\Lambda}$ are well-defined for $\Lambda = \infty$. Therefore, the first and the third operators in the sum defining $V_{K,\Lambda}$ are well-defined on $D(H_0)$ for $\Lambda = \infty$. This is not obvious for the second term, $2p\cdot a(kB_\Lambda)$, because $p\cdot a(kB_\Lambda) = a(kB_\Lambda)\cdot p + a(k^2B_\Lambda)$, where the norm of k^2B_Λ may diverge as $\Lambda\to\infty$. By imposing, $(v3)$ this problem can be controlled with the help of Lemma A.2.6 and we arrive at the following:

Lemma 2.3.3. *Assume* $(v1)$ *and* $(v3)$*. Then, for all* $K \le \Lambda \le \infty$*, the operator* $p\cdot a(kB_{\Lambda,x})$ *satisfies* $D(p\cdot a(kB_{\Lambda,x})) \supset D(H_0)$ *and*

$$\sup_{\Lambda\le\infty} \big\|p\cdot a(kB_\Lambda)(H_0+1)^{-1}\big\| \longrightarrow 0 \qquad (K\to\infty).$$

Proof. The operator $p\cdot a(kB_\Lambda) = \sum\limits_{j=1}^d p_j a(k_j B_\Lambda)$ is defined on $\bigcap\limits_{j=1}^d D(p_j a(k_j B_\Lambda))$. We, therefore, need to show that $a(k_jB_\Lambda)\Psi \in D(p_j)$ for all $\Psi\in D(H_0)$ and all $j = 1,2,...,d$. We omit the proof for $\Lambda<\infty$ and only note that for all $\Psi\in D(H_0)$ and $\Lambda<\infty$,

$$p_j a(k_j B_\Lambda)\Psi = a(k_j^2 B_\Lambda)\Psi + a(k_j B_\Lambda)p_j\Psi. \tag{2.29}$$

The right-hand side is convergent in the limit $\Lambda \to \infty$ by Lemma A.2.6 and Corollary A.2.2. Hence, so is the left hand side. Since, moreover,

$$\lim_{\Lambda\to\infty} a(k_j B_\Lambda)\Psi = a(k_j B_\infty)\Psi \tag{2.30}$$

and since p_j is a closed operator, it follows that $a(k_j B_\infty)\Psi \in D(p_j)$ and that $p_j a(k_j B_\infty)\Psi$ is given by the limit of (2.29). This proves that $D(p\cdot a(kB_\Lambda)) \supset D(H_0)$ and that

$$\|p_j a(k_j B_\Lambda)\Psi\| \leq \sup_{\Lambda\leq\infty} \left(\|a(k_j^2 B_\Lambda)\Psi\| + \|a(k_j B_\Lambda)p_j\Psi\|\right). \tag{2.31}$$

Using Corollary A.2.2 and Lemma A.2.6, it is easy to see that

$$\|a(k_j B_\infty)p_j\Psi\| \leq \left(\int\limits_{|k|\geq K} k^2 \frac{|v(k)|^2}{(1+k^2)^2} dk\right)^{1/2} \|H_0\Psi\| \tag{2.32}$$

and

$$\left\|a(k_j^2 B_\infty)\Psi\right\| \leq \left(\sup_{q\in\mathbb{R}^d} \int\limits_{|k|\geq K} \frac{|v(k)|^2}{1+(q-k)^2} dk\right)^{1/2} \|(H_0+1)\Psi\|. \tag{2.33}$$

Upon combining Inequalities (2.31), (2.32) and (2.33), the second assertion of the lemma follows. $\square$

Theorem 2.3.4. *Assume (v1) and (v3). Then, for every $\varepsilon > 0$, there exist $K > 0$ and $C_\varepsilon \in \mathbb{R}$ such that for all $\Lambda \leq \infty$ and all $\Psi \in D(H_0)$,*

$$\|V_{K,\Lambda}\Psi\| \leq \varepsilon\|H_0\Psi\| + C_\varepsilon\|\Psi\|. \tag{2.34}$$

The operator $H'_{K,\infty} = H_K + V_{K,\infty}$ is self-adjoint on $D(H_0)$ provided K is large enough.

Proof. It suffices to establish the desired estimate for each term in the sum

$$V_{K,\Lambda} = \phi(kB_\Lambda)^2 - 2a^*(kB_\Lambda)\cdot p - 2p\cdot a(kB_\Lambda) + C_{K,\Lambda}. \tag{2.35}$$

By Corollary A.2.2 and Lemma A.2.3, for all $\Psi \in D(H_0)$,

$$\|a^*(kB_\Lambda)\cdot p\Psi\| \leq \|kB_\Lambda\| \left\|\sqrt{N+1}p\Psi\right\|, \tag{2.36}$$

$$\left\|\phi(kB_\Lambda)^2\Psi\right\| \leq 4\sqrt{2}\|kB_\Lambda\|^2 \|(N+1)\Psi\|, \tag{2.37}$$

where $N+1$ and $\sqrt{N+1}p$ are H_0-bounded and

$$\|kB_\Lambda\|^2 \leq \int\limits_{|k|\geq K} k^2 \frac{|v(k)|^2}{(1+k^2)^2} dk \to 0 \qquad (K\to\infty). \tag{2.38}$$

This proves Inequality (2.34) as far as the first two terms in (2.35) are concerned. For the operator $p\cdot a(kB_\Lambda)$, the desired estimate follows from Lemma 2.3.3 and $C_{K,\Lambda}$ is bounded uniformly in Λ.

In view of (2.34), the self-adjointness follows from Kato-Rellich because $H_K = H_0 + \phi(G_K)$ where $\phi(G_K)$ is infinitesimally H_0-bounded for every given fixed K. $\square$

Proposition 2.3.5. *Assume (v1) and (v3). Then, for K sufficiently large, $H'_{K,\Lambda} \to H'_{K,\infty}$ in the norm resolvent sense as $\Lambda\to\infty$.*

Proof. For short, we set $H'_\Lambda := H'_{K,\Lambda}$ in this proof. By Theorem 2.3.4, H'_Λ is self-adjoint on $D(H_0)$ for all $\Lambda \leq \infty$ if K is sufficiently large. In view of Theorem VIII.25(b) from [24], it therefore suffices to prove that

$$(H'_\Lambda - H'_\infty)(H'_\infty + i)^{-1} \to 0 \qquad (\Lambda\to\infty) \tag{2.39}$$

which is equivalent to

$$(H'_\Lambda - H'_\infty)(H_0 + i)^{-1} \to 0 \qquad (\Lambda\to\infty) \tag{2.40}$$

due to the boundedness of $(H_0+i)(H'_\infty+i)^{-1}$. By definition of H'_Λ, see Proposition 2.3.2, we have

$$\begin{aligned} H'_\Lambda - H'_\infty &= V_{K,\Lambda} - V_{K,\infty} \\ &= -\phi(kB_\Lambda)\phi(kB_\infty\overline{\chi}_\Lambda) - \phi(kB_\infty\overline{\chi}_\Lambda)\phi(kB_\infty) \\ &\quad + 2a^*(kB_\infty\overline{\chi}_\Lambda)\cdot p + 2p\cdot a(kB_\infty\overline{\chi}_\Lambda) + C_{\Lambda,\infty}. \end{aligned} \tag{2.41}$$

Here, $\overline{\chi}_\Lambda := 1-\chi_\Lambda$, and we used $B_{\Lambda,x} - B_{\infty,x} = -B_{\infty,x}\overline{\chi}_\Lambda$. Convergence (2.40) now follows from (2.41) by the same estimates that were used in the proof of Theorem 2.3.4. $\square$

Corollary 2.3.6. *Assume (v1) and (v3). Then for K sufficiently large, there exists a constant C such that for all $\Lambda\leq\infty$,*

$$\frac{1}{2}H_0 - C \leq H'_{K,\Lambda} \leq \frac{3}{2}H_0 + C.$$

Proof. By Theorem 2.3.4, there exist K and C such that

$$\|(V_{K,\Lambda} + \phi(G_K))\Psi\| \leq \frac{1}{4}\|H_0\Psi\| + \frac{C}{2}\|\Psi\| \tag{2.42}$$

for all $\Psi \in D(H_0)$ and $\Lambda \leq \infty$. Using the lower bound given by the Kato-Rellich-theorem, Theorem X.12 from [25], we conclude that

$$\frac{1}{2}H_0 \pm (\phi(G_K) + V_{K,\Lambda}) \geq -C, \tag{2.43}$$

which implies the desired inequalities. □

Theorem 2.3.7. *Assume* (*v*1) *and* (*v*3)*. Then, there exists a self-adjoint operator* H *such that* $H_\Lambda \to H$ *as* $\Lambda \to \infty$ *in the norm resolvent sense. This operator has the representation*

$$H = U_{K,\infty}^* H'_{K,\infty} U_{K,\infty}, \quad D(H) = U_{K,\infty}^* D(H_0),$$

which is valid for K *sufficiently large. If* $D \subset D(H_0)$ *is a core of* H_0*, then* $U_{K,\infty}^* D$ *is a core of* H*.*

In this theorem, $U_{K,\infty} = U_\infty$ to exhibit the dependence of U_∞ on K. The theorem implies, in particular, that $H := U_{K,\infty}^* H'_{K,\infty} U_{K,\infty}$ is independent of K for K sufficiently large. Because of the convergence $H_\Lambda \to H$, this operator coincides with the operator constructed in Section 2.2.

Proof. Choose K so large that $H'_{K,\Lambda} \to H'_{K,\infty}$ in the norm resolvent sense by Proposition 2.3.5. In the following, K is fixed and suppressed. Let $R'_\Lambda(z) := (H'_\Lambda - z)^{-1}$ and $H := U_\infty^* H'_\infty U_\infty$ in this proof. By Proposition 2.3.2, $H_\Lambda = U_\Lambda^* H'_\Lambda U_\Lambda$ for all $\Lambda < \infty$ and, therefore,

$$\begin{aligned}(H_\Lambda - z)^{-1} - (H - z)^{-1} &= U_\Lambda^* R'_\Lambda(z) U_\Lambda - U_\infty^* R'_\infty(z) U_\infty \\ &= (U_\Lambda^* - U_\infty^*) R'_\Lambda(z) U_\Lambda + U_\infty^* (R'_\Lambda(z) - R'_\infty(z)) U_\Lambda \\ &\quad + U_\infty^* R'_\infty(z)(U_\Lambda - U_\infty).\end{aligned} \tag{2.44}$$

It remains to show that these three terms vanish in the limit $\Lambda \to \infty$. For the second term, this follows from Proposition 2.3.5. For the first and third terms, we have

$$\|(U_\Lambda^* - U_\infty^*) R'_\Lambda(z) U_\Lambda\| \leq \left\|(U_\Lambda^* - U_\infty^*)(H_0 + 1)^{-1/2}\right\| \cdot \left\|(H_0 + 1)^{1/2} R'_\Lambda(z)\right\| \tag{2.45}$$

and

$$\begin{aligned}\|U_\infty^* R'_\infty(z)(U_\Lambda - U_\infty)\| &= \|(U_\Lambda^* - U_\infty^*) R'_\infty(\overline{z})\| \\ &\leq \left\|(U_\Lambda^* - U_\infty^*)(H_0 + 1)^{-1/2}\right\| \cdot \left\|(H_0 + 1)^{1/2} R'_\infty(\overline{z})\right\|.\end{aligned} \tag{2.46}$$

Lemma A.2.5 implies that

$$\left\|(U_\Lambda^* - U_\infty^*)(H_0+1)^{-1/2}\right\| \to 0 \qquad (\Lambda \to \infty) \tag{2.47}$$

because $\sup_{x\in\mathbb{R}^d} \|B_{\Lambda,x} - B_{\infty,x}\| \to 0$ as $\Lambda \to \infty$, and the Corollary 2.3.6 shows that

$$\sup_{\Lambda\le\infty} \left\|(H_0+1)^{1/2} R'_\Lambda(z)\right\| < \infty. \tag{2.48}$$

Combining Properties (2.45), (2.46), (2.47), and (2.48) we see that the first and third terms from (2.44) vanish as $\Lambda \to \infty$. The statement about $D(H)$ follows from Theorem 2.3.4.

Now, if $D \subset D(H_0)$ is a core of H_0, then, by Theorem 2.3.4, D is a core of H'_∞, and hence, $U_\infty^* D$ is a core of $H = U_\infty^* H'_\infty U_\infty$. □

2.4. Regularity of domain vectors

In this section, we prove Equations (2.3) and (2.4) of the Introduction. As a preparation we need the following lemma, which generalizes the statement of Lemma 2.3.1 (a), on the invariance of $D(H_0^{1/2}) = D(|p|) \cap D(N^{1/2})$ under the transformation U_∞.

Lemma 2.4.1. *Assume (v1) and (v2). Then, for $\sigma \in [0,1]$, the subspaces $D(|p|^\sigma) \cap D(N^{\sigma/2})$ are left invariant by U_∞ and U_∞^*.*

Proof. Let $\gamma \in D(|p|^\sigma) \cap D(N^{\sigma/2})$. Then, $U_\infty\gamma \in D(N^{\sigma/2})$, by Lemma A.2.4, and it remains to prove that $U_\infty\gamma \in D(|p|^\sigma)$. Since $U_\infty^* p U_\infty = p + \phi(kB_\infty)$, by Lemma 2.3.1 and the remark thereafter, $U_\infty\gamma \in D(|p|^\sigma)$ is equivalent to $\gamma \in D(|p+\phi(kB_\infty)|^\sigma)$. To prove the latter, we first observe that $\phi(kB_\infty)^2 \le C(2N+1)$, where $C = 2\|kB_\infty\|^2$, and hence,

$$(p+\phi(kB_\infty))^2 \le 2p^2 + 2\phi(kB_\infty)^2 \le 2(p^2 + C(2N+1)). \tag{2.49}$$

This means, in particular, that the form domain of $(p+\phi(kB_\infty))^2$ contains the form domain of p^2+N. From the operator monotonicity of the fractional power σ (see [26], Proposition 10.14), it follows that

$$|p+\phi(kB_\infty)|^{2\sigma} \le 2^\sigma \left(|p|^{2\sigma} + C^\sigma(2N+1)^\sigma\right). \tag{2.50}$$

Inequality (2.50) again includes a statement about form domains. It implies that

$$D(|p+\phi(kB_\infty)|^\sigma) \supset D(|p|^\sigma) \cap D(N^{\sigma/2}). \tag{2.51}$$

In view of the assumption on γ, this is exactly what we needed to show. □

Theorem 2.4.2. *Assume (v1) and (v3). If* $\int |v(k)|^2(1+k^2)^{s-2}dk = \infty$ *for some* $s \in (1,2]$*, then*

$$U_\infty^* D(H_0) \cap D((-\Delta)^{s/2}) = \{0\}.$$

In the case of the Fröhlich Hamiltonian where $v(k) = |k|^{-(d-1)/2}$, choose $s = 3/2$ in Theorem 2.4.2 to prove the Assertion (2.4) in the Introduction.

Proof. Let $\Psi \in D(H_0)$, and suppose that $U_\infty^*\Psi \in D((-\Delta)^{s/2}) = D(|p|^s)$. Then, $U_\infty^*\Psi \in D(|p|^s) \cap D(N)$ by Lemma A.2.4. In view of the inequality

$$N^{(s-1)/2}|p| \le \frac{s-1}{s}N^{s/2} + \frac{1}{s}|p|^s, \tag{2.52}$$

and the assumption $1 < s \le 2$, we conclude that $pU_\infty^*\Psi \in D(|p|^{s-1}) \cap D(N^{(s-1)/2})$. This implies, by Lemma 2.4.1, that $U_\infty pU_\infty^*\Psi \in D(|p|^{s-1})$, where

$$U_\infty pU_\infty^*\Psi = (p - \phi(kB_\infty))\,\Psi. \tag{2.53}$$

The first term on the right-hand side, $p\Psi$, belongs to $D(|p|^{s-1})$ as well, because $s \le 2$ and $\Psi \in D(p^2)$. We now compute $\left\| |p|^{s-1}\phi(kB_\infty)\Psi \right\|$, and we show that this number is infinite unless $\Psi = 0$. To this end, we define the functions

$$|p|_\varepsilon^{s-1} := \frac{|p|^{s-1}}{1+\varepsilon|p|^{s-1}}, \tag{2.54}$$

$$D_\varepsilon(p,k) := |p+k|_\varepsilon^{s-1} - |k|_\varepsilon^{s-1} \tag{2.55}$$

for $p, k \in \mathbb{R}^d$ and $\varepsilon > 0$. Using that $(s-1) \in (0,1]$, it is straightforward to verify that

$$|D_\varepsilon(p,k)| \le |p|^{s-1} \tag{2.56}$$

for all $p, k \in \mathbb{R}^d$ and $\varepsilon > 0$. For $p = -i\nabla_x$, we have

$$|p|_\varepsilon^{s-1}e^{ikx} = e^{ikx}|p+k|_\varepsilon^{s-1} \tag{2.57}$$

which, in view of (A.14), implies

$$\begin{aligned} |p|_\varepsilon^{s-1}a(kB_\infty)\Psi &= a(kB_\infty|p|_\varepsilon^{s-1})\Psi \\ &= a(|k|_\varepsilon^{s-1}kB_\infty)\Psi + a(D_\varepsilon(p,k)kB_\infty)\Psi \end{aligned} \tag{2.58}$$

and

$$|p|_\varepsilon^{s-1}a^*(kB_\infty)\Psi = a^*(kB_\infty|k|_\varepsilon^{s-1})\Psi + a^*(kB_\infty D_\varepsilon(p,-k))\Psi. \tag{2.59}$$

By Inequality (2.56),

$$\|a(D_\varepsilon(p,k)kB_\infty)\Psi\| \le \|kB_\infty\| \cdot \left\| |p|^{s-1}\sqrt{N}\Psi \right\|, \tag{2.60}$$

$$\|a^*(kB_\infty D_\varepsilon(p,-k))\Psi\| \le \|kB_\infty\| \cdot \left\| |p|^{s-1}\sqrt{N+1}\Psi \right\|, \tag{2.61}$$

and by Lemma A.2.6,

$$\left\| a(kB_\infty |k|_\varepsilon^{s-1})\Psi \right\| \le C_s \left\| \sqrt{N}(1+p^2)^{1/2}\Psi \right\|, \tag{2.62}$$

where

$$C_s := \sup_{q\in\mathbb{R}^d} \left(\int \frac{|kB_\infty(k)|^2 |k|^{2(s-1)}}{1+(q-k)^2} dk \right) < \infty \tag{2.63}$$

because $s \le 2$. Note that the bounds (2.60), (2.61) and (2.62) are uniform in $\varepsilon > 0$. Therefore, there exists a constant c such that

$$\begin{aligned} \left\| |p|^{s-1}\phi(kB_\infty)\Psi \right\| &= \lim_{\varepsilon\to 0} \left\| |p|_\varepsilon^{s-1}\phi(kB_\infty)\Psi \right\| \\ &\ge \liminf_{\varepsilon\to 0} \left\| a^*(kB_\infty |k|_\varepsilon^{s-1})\Psi \right\| - c \\ &\ge \lim_{\varepsilon\to 0} \left\| kB_\infty |k|_\varepsilon^{s-1} \right\| \cdot \|\Psi\| - c \end{aligned} \tag{2.64}$$

which is infinite unless $\Psi = 0$. This completes the proof. □

Theorem 2.4.3. *Assume (v1). If $\int |v(k)|^2(1+k^2)^{s-2}dk < \infty$ for some $s \in [1,2]$, then*

$$U_\infty^* D(H_0) \subset D((-\Delta)^{s/2}).$$

In the case of the Fröhlich Hamiltonian where $v(k) = |k|^{-(d-1)/2}$, the assumption of Theorem 2.4.3 is satisfied for all $s \in [1, 3/2)$ and $U_\infty^* D(H_0) = D(H)$ by Theorem 2.3.7. This proves Assertion (2.3) in the Introduction.

Proof. Let $\Psi \in D(H_0)$. From Lemma 2.3.1 we know that $U_\infty^*\Psi \in D(H_0^{1/2}) \subset D(|p|)$ and that $p_j U_\infty^*\Psi = U_\infty^*(p_j - \phi(k_j B_\infty))\Psi =: U_\infty^*\gamma_j$. It follows that

$$\||p|^s U_\infty^*\Psi\|^2 = \sum_{j=1}^d \left\| |p|^{s-1} p_j U_\infty^*\Psi \right\|^2 = \sum_{j=1}^d \left\| |p|^{s-1} U_\infty^*\gamma_j \right\|^2, \tag{2.65}$$

which is finite (and thus proves the theorem) provided we can show that $U_\infty^*\gamma_j \in D(|p|^{s-1})$ for all j. To prove this, it suffices, by Lemma 2.4.1, to show that $\gamma_j \in D(|p|^{s-1}) \cap D(N^{(s-1)/2})$. From $\gamma_j \in D(\sqrt{N})$ it follows that $\gamma_j \in D(N^{(s-1)/2})$ because $s \in [1,2]$. It remains to show that $\||p|^{s-1}\gamma_j\| < \infty$. The first term of $\gamma_j =$

$p_j\Psi - \phi(k_j B_\infty)\Psi$ belongs to $D(|p|^{s-1})$ because $s \leq 2$ and because $\Psi \in D(-\Delta)$. To prove that $\||p|^{s-1}\phi(k_j B_\infty)\Psi\|$ is finite, we recall the estimates in the proof of Theorem 2.4.2 which imply that $\||p|^{s-1}a(k_j B_\infty)\Psi\|$ is finite and that

$$\begin{aligned}\left\||p|^{s-1}a^*(k_j B_\infty)\Psi\right\| &\leq \left\|a^*(k_j B_\infty |k|^{s-1})\Psi\right\| + \|a^*(k_j B_\infty D_\varepsilon(p,-k))\Psi\| \\ &\leq C_s\left(\|\sqrt{N+1}\Psi\| + \left\|\sqrt{N+1}|p|^{s-1}\Psi\right\|\right), \end{aligned} \tag{2.66}$$

where

$$C_s^2 := \int |k|^{2s}\frac{|v(k)|^2}{(1+k^2)^2}dk < \infty. \tag{2.67}$$

In the last inequality, we used the hypothesis on v, and in (2.66), we used $s \geq 1$. $\square$

3. Self-Adjointness and Domain of the Nelson Hamiltonian

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Abstract

It is well-known that the Nelson Hamiltonian can be realized as norm-resolvent limit of operators where an ultraviolet cutoff is imposed and after subtracting a suitable renormalization constant. Edward Nelson was the first to show this by using the unitary dressing transform due to Gross. From the usual computations, one obtains an abstract existence statement about the operator, but no statement about its domain. On the occasion of our recent results about the domain of the Fröhlich Hamiltonian, we also want to give a characterization of the domain of the renormalized Nelson Hamiltonian in this work. The classical Nelson Hamiltonian is considered in three space dimensions. Because of the stronger decay for large absolute Boson momenta in less dimensions, we can use the methods from our previous paper about the Fröhlich Hamiltonian, see also the previous chapter, to treat the operator in one and two dimensions. This also leads to a generalization of our results to the Fröhlich Hamiltonian. The subject matter of this chapter will be published soon.

3.1. Introduction

The Nelson model [1] describes the interaction of a non-relativistic quantum particle with a quantized bosonic field. A formal expression for the Hamiltonian of the system is given by

$$p^2 + d\Gamma(\omega) + \alpha \int_{\mathbb{R}^3} dk \ \frac{1}{\sqrt{\omega(k)}} \left(e^{ikx} a_k + e^{-ikx} a_k^* \right), \tag{3.1}$$

with the momentum operator $p := -i\nabla$ for the particle such that p^2 coincides with the negative Laplacian $-\Delta$ in $L^2(\mathbb{R}^3)$. The term $d\Gamma(\omega)$ stands for the energy of the bosonic field with the dispersion relation $\omega(k) = \sqrt{m^2 + k^2}$ and $m > 0$ and acts in the symmetric Fock space over $L^2(\mathbb{R}^3)$. The parameter α is a real coupling constant and can be set to $\alpha = 1$ for our purposes. The first summand stands for the free Hamiltonian of the particle and the second summand for the free Hamiltonian of the bosonic field. The interaction of these two subsystems is given in the third summand which is not, as it stands, a well-defined operator because of the ultraviolet divergence of the form-factor $\omega^{-1/2}$.

We regularize the operator by cutting off the integral for large absolute values of k, we call the cutoff parameter Λ and the associated Hamiltonian H_Λ. Then, $H_0 := p^2 + d\Gamma(\omega)$ is the free part of (3.1). This operator is non-negative and self-adjoint on $D(H_0) := D(p^2) \cap D(d\Gamma(\omega))$. It goes back to Nelson [1] to show that there exists a self-adjoint operator H which is the strong resolvent limit of $H_\Lambda + E_\Lambda$ as $\Lambda \to \infty$ where E_Λ is a suitable renormalization constant. Then, it was Ammari [2] who showed that this limit also exists in the norm resolvent sense. Thus, the UV-renormalization of the Nelson Hamiltonian and also of other similar models is well understood, see also [7] and [8]. But usually, no statements about the domain of H are made. Thus, on the occasion of our previous work [23], where we studied the domain of the Fröhlich Hamiltonian which describes the large Polaron model (see also Chapter 2), the main task of this chapter is the investigation of the domain of the Nelson Hamiltonian in $d \in \{1, 2, 3\}$ space dimensions.

At the beginning in Section 3.2, we fix our notation and define the class of operators we want to study. We do not restrict ourselves only to the Nelson Hamiltonian, but implement some general assumptions under which our arguments will hold. Therefore, our results are valid for a large class of Hamiltonains. In Section 3.2, we will also introduce the important unitary Gross transform (see [3] or also [1]).

With respect to the decay of the occurring form-factor with respect to large absolute Phonon momenta in the interaction part of the Hamiltonian, there is a big difference in the construction of the norm-resolvent limit H. Under certain conditions, we are able to give an explicit representation of this limit H in terms of the transform due to Gross and also of its domain $D(H)$. The strategy for that is the same as in our paper [23] and Chapter 2, respectively. We execute it in Section 3.3. Then, this section, in fact, is a generalization of Sections III and IV of [23] or Sections 2.3 and 2.4 of Chapter 2, respectively. This generalization is possible due to our new estimate which generalizes an important estimate from Frank and Schlein [13]. Thus, our results to the Fröhlich Hamiltonian from [23] are also included here. Section 3.3 also contains the Nelson Hamiltonian in $d = 1$ and $d = 2$ dimensions as special cases. For them, we obtain the results

$$D(H) \subset \bigcap_{1 \leq s < \frac{5-d}{2}} D(|p|^s) \cap D(d\Gamma(\omega)), \tag{3.2}$$

$$D(H) \cap D(|p|^{\frac{5-d}{2}}) = \{0\}. \tag{3.3}$$

Then, in Section 3.4, the case is studied where we can construct H only over a quadratic form, we call it q. This construction works analogously to that one of Nelson [1]. But for the discussion of the domain of H or q, respectively, we have to find new methods.

The Nelson Hamiltonian in $d = 3$ dimensions is part of this regime and we obtain for it

$$D(q) \subset \bigcap_{0 \le s < 1} D(|p|^s) \cap D(d\Gamma(\omega)^{1/2}), \tag{3.4}$$

$$D(q) \cap D(|p|) = \{0\}. \tag{3.5}$$

Note that the domain of the quadratic form associated to the free Hamiltonian H_0 is given by $D(H_0^{1/2}) = D(|p|) \cap D(d\Gamma(\omega)^{1/2})$. Since $D(H)$ trivially is a subset of its form domain $D(q)$, we see especially for all $d \in \{1, 2, 3\}$ that

$$D(H) \cap D(H_0) = \{0\}, \tag{3.6}$$

and moreover for $d = 1$ and $d = 2$

$$D(H) \subset D(H_0^{1/2}), \tag{3.7}$$

but for $d = 3$

$$D(H) \cap D(H_0^{1/2}) = \{0\}. \tag{3.8}$$

Comparing Equations (3.2) to (3.5), one can see how the regularity of domain vectors of H with respect to the momentum part gets lost with increasing dimension.

In Appendix B.1, we collect background of ladder operators and prove some important estimates for this work. Appendix B.2 contains an important theorem in the framework of interpolation theory that is needed in Appendix B.3 which consists of general transformation properties of Weyl operators which are sufficient handling the Gross transform.

3.2. Hamiltonian with cutoff and Gross Transform

In this section, we fix our notation and define the Nelson Hamiltonian with ultraviolet cutoff. The Hamiltonian describes a quantum particle in $\mathbb{R}^d$ that is coupled to a quantized field of scalar bosons with a positive dispersion relation.

Let $\mathscr{H} := L^2(\mathbb{R}^d, dx) \otimes \mathcal{F}$ where $\mathcal{F}$ denotes the symmetric Fock space over $L^2(\mathbb{R}^d, dk)$. We may identify $\mathscr{H}$ with $L^2(\mathbb{R}^d, \mathcal{F})$ through the isomorphism given by $\varphi \otimes \eta \mapsto \varphi(x)\eta$. We define the finite bosonic particle number spaces

$$\mathcal{F}_0 := \bigcup_{n \ge 0} \chi(N \le n)\mathcal{F} \quad \text{and} \quad \mathscr{H}_0 := \bigcup_{n \ge 0} \chi(N \le n)\mathscr{H}. \tag{3.9}$$

We furthermore define the free Hamiltonian

$$H_0 := p^2 \otimes \mathbb{1} + \mathbb{1} \otimes d\Gamma(\omega), \tag{3.10}$$

where $p := -i\nabla$ stands for the momentum operator of the free particle, that means that p^2 coincides with the negative (self-adjoint) Laplacian $-\Delta$ in $L^2(\mathbb{R}^d)$. The operator $d\Gamma(\omega)$ is the second quantized field energy in $\mathcal{F}$ with the dispersion relation $\omega(k)$ on which we impose the properties

$$(\omega) \qquad \omega \in L^\infty_{loc}(\mathbb{R}^d), \quad \omega(k) = \omega(-k), \quad \omega(k) \geq m > 0, \\ \text{and} \quad \omega(k) = O(k^2) \quad \text{as} \quad |k| \to \infty.$$

Examples for dispersion relations satisfying these conditions are $\omega(k) = 1$ and $\omega(k) = \sqrt{m^2 + k^2}$, which lead to the Fröhlich Hamiltonian [23] and to the massive Nelson model [1], respectively.

The Hamiltonian H_0 is positive and self-adjoint on $D(H_0) = D(p^2 \otimes \mathbb{1}) \cap D(\mathbb{1} \otimes d\Gamma(\omega))$. Furthermore, its form domain is given by the set $D(H_0^{1/2}) = D(|p| \otimes \mathbb{1}) \cap D(\mathbb{1} \otimes d\Gamma(\omega)^{1/2})$. From now on, we will omit the symbol $\mathbb{1}$.

The interaction of the particle and the bosonic field is given in terms of annihilation and creation of bosons. The usual annihilation and creation operators in Fock space associated with some vector $f \in L^2(\mathbb{R}^d)$ will be denoted by $a(f)$ and $a^*(f)$. They are closed, adjoint to each other with $D(a(f)) = D(a^*(f)) \supset D(N^{1/2})$, where N denotes the number operator in $\mathcal{F}$, and they obey the canonical commutation relations $[a(f), a^*(g)] = \langle f, g\rangle$ (others vanish) on $D(N)$. Note, that $D(d\Gamma(\omega)) \subset D(N)$, since $m > 0$. The symmetric field operators

$$\phi(f) := a(f) + a^*(f), \qquad \pi(f) := \phi(if) \tag{3.11}$$

are essentially self-adjoint on $D(N)$ and they satisfy the commutation relations

$$[\phi(f), \phi(g)] = 2i \operatorname{Im} \langle f, g\rangle, \qquad [\phi(f), \pi(g)] = 2i \operatorname{Re} \langle f, g\rangle. \tag{3.12}$$

The (self-adjoint) closures of the operators $\phi(f)$ and $\pi(f)$ will be denoted by the same symbols.

We will have occasion to work with generalized annihilation and creation operators $a(F)$ and $a^*(F)$ that are operators in $\mathscr{H}$ rather than $\mathcal{F}$. Here $F : L^2(\mathbb{R}^d, dx) \to L^2(\mathbb{R}^d, dx) \otimes L^2(\mathbb{R}^d, dk)$ is a linear operator. In the simplest case $F\varphi = \varphi \otimes f$ for some $f \in L^2(\mathbb{R}^d, dk)$ and then $a^\#(F) = \mathbb{1} \otimes a^\#(f)$ is the usual annihilation or creation operator in $\mathcal{F}$. Often, but not always, the operator F will be defined in terms of some function $(x, k) \mapsto F_x(k)$, denoted by F as well, through the equation $(F\varphi)(x, k) = \varphi(x)F_x(k)$. In this case $(a^\#(F)\Psi)(x) = a^\#(F_x)\Psi(x)$. Typically $F_x(k) = e^{-ikx}f(k)$ where $f \in L^2(\mathbb{R}^d)$ and then the operator norm of F equals the norm of f in $L^2(\mathbb{R}^d)$. See Appendix B.1 for the definition of $a^\#(F)$ in the general case.

At some points, the application of operator valued distributions a_k and a_k^* is useful. Using them, the creation and annihilation operators read

$$a(F)\Psi = \int dk\ F^*\ a_k\Psi, \quad a^*(F)\Psi = \int dk\ F\ a_k^*\Psi, \tag{3.13}$$

where

$$(a_k\Psi)^{(n)}(x,k_1,...,k_n) = (n+1)^{1/2}\ \Psi^{(n+1)}(x,k,k_1,...,k_n), \tag{3.14}$$

$$(a_k^*\Psi)^{(n)}(x,k_1,...,k_n) = n^{-1/2}\sum_{j=1}^{n}\delta(k-k_j)\ \Psi^{(n-1)}(x,k_1,...,k_{j-1},k_{j+1},...,k_n). \tag{3.15}$$

The canonical commutation relation then reads $[a_k, a_{k'}^*] = \delta(k-k')$, others vanish. Furthermore, the following representations hold:

$$d\Gamma(\omega) = \int dk\ \omega(k)\ a_k^* a_k, \tag{3.16}$$

$$\left\| d\Gamma(\omega)^{1/2}\Psi \right\| = \left(\int dk\ \omega(k)\ \|a_k\Psi\|^2\right)^{1/2}. \tag{3.17}$$

For a positive $\Lambda < \infty$, we define $H_\Lambda : D(H_0) \subset \mathscr{H} \to \mathscr{H}$ by

$$H_\Lambda := H_0 + \phi(G_\Lambda), \tag{3.18}$$

where

$$G_{\Lambda,x}(k) := v(k)e^{-ikx}\chi_\Lambda(k). \tag{3.19}$$

Here χ_Λ denotes the characteristic function of the set $\{k \in \mathbb{R}^d |\ |k| \leq \Lambda\}$. On the form factor $v : \mathbb{R}^d \mapsto \mathbb{C}$, we impose the assumptions

$$(v) \quad v \in L^2_{loc}(\mathbb{R}^d) \quad \text{and} \quad v(k) = v(-k).$$

In this chapter, three regimes $(R1)$, $(R2)$, and $(R3)$ with respect to the decay of v for large $|k|$ are discussed. These are

$$\begin{aligned} &(R1) \quad \|v\|^2 = \int dk\ |v(k)|^2 < \infty, \\ &(R2) \quad \left\| \tfrac{v}{(1+k^2)^{\frac{1}{2}}} \right\|^2 = \int dk\ \frac{|v(k)|^2}{1+k^2} < \infty, \\ &(R3) \quad \left\| \tfrac{v}{(1+k^2)} \right\|^2 = \int dk\ \frac{|v(k)|^2}{(1+k^2)^2} < \infty. \end{aligned}$$

The assumptions of these regimes become weaker from $(R1)$ to $(R3)$. The strongest assumption $(R1)$ is not very interesting and is included here for reasons of completeness. We can end this case with Lemma 3.2.1. The main focus of this work concerns the regimes $(R2)$ and $(R3)$ that are discussed in Sections 3.3 and 3.4. Important examples of form factors v satisfying $(R2)$ are the Nelson Hamiltonian [1] in two space dimensions, where

$$v(k) = \frac{1}{\sqrt{\omega(k)}} \quad \text{with} \quad \omega(k) = \sqrt{m^2 + k^2} \quad \text{and} \quad m > 0, \tag{3.20}$$

or the Fröhlich Hamiltonian [23] in $d \geq 2$ space dimensions, where

$$v(k) = |k|^{-\frac{d-1}{2}} \quad \text{and} \quad \omega(k) = 1. \tag{3.21}$$

An important example of a form factor v satisfying only $(R3)$ is the Nelson Hamiltonian in three space dimensions.

Lemma 3.2.1. *Assume* (ω), (v), *and* $(R1)$. *Then, the operator* H_Λ *is self-adjoint on* $D(H_0)$ *for all* $\Lambda \leq \infty$.

Proof. From Corollary A.2.2, it follows that $\phi(G_\Lambda)(N+1)^{-1/2}$ is bounded and hence $\phi(G_\Lambda)$ is infinitesimally H_0-bounded for all $\Lambda \leq \infty$. Thus, by Kato-Rellich, H_Λ is self-adjoint on $D(H_0)$ for all $\Lambda \leq \infty$. □

Without property $(R1)$, the proof of Lemma 3.2.1 holds for all $\Lambda < \infty$, nevertheless, because of $v \in L^2_{loc}(\mathbb{R}^d)$. Therefore, H_Λ is self-adjoint on $D(H_0)$ for all $\Lambda < \infty$. We, moreover, define for $\Lambda \leq \infty$

$$E_\Lambda := \int\limits_{|k| \leq \Lambda} dk \, \frac{|v(k)|^2}{\omega(k) + k^2} \leq \infty. \tag{3.22}$$

Note, that $E_\Lambda < \infty$, as long as $\Lambda < \infty$. Because of that, $H_\Lambda + E_\Lambda$ is also self-adjoint on $D(H_0)$ for all $\Lambda < \infty$. In the case without cutoff, E_∞ can be finite or infinite. In the second case, E_∞ plays the role of a renormalization constant.

Now, we introduce the Gross transform and use it to make the unitary dressing transform of $H_\Lambda + E_\Lambda$ from Equations (3.18) and (3.22) as it was done by Nelson in [1].

For $0 < K < \Lambda \leq \infty$, define the unitary operator $U_{K,\Lambda} : \mathscr{H} \to \mathscr{H}$ by

$$U_{K,\Lambda} := e^{i\pi(B_{K,\Lambda})}, \tag{3.23}$$

where

$$\begin{aligned} B_{K,\Lambda,x}(k) &:= -\frac{v(k)}{\omega(k) + k^2} e^{-ikx} \chi_\Lambda(k) \overline{\chi}_K(k) \\ &= -\frac{1}{\omega(k) + k^2} G_{\Lambda,x}(k) \overline{\chi}_K(k) \end{aligned} \tag{3.24}$$

and $\overline{\chi}_K := 1 - \chi_K$. We will use $kB_{K,\Lambda}$ and $|k|^s B_{K,\Lambda}$ to denote the functions $kB_{K,\Lambda,x}(k)$ and $|k|^s B_{K,\Lambda,x}(k)$, respectively. Furthermore, we have

$$U_{K,\Lambda} \to U_{K,\infty} \quad (\Lambda \to \infty) \tag{3.25}$$

strongly in $\mathscr{H}$. Remark, that the assumptions $(R1)$ to $(R3)$ effect the following properties of $B_{K,\Lambda,x}$:

$$(R1) \quad \|v\| < \infty \quad \Leftrightarrow \quad \left\| |k|^2 B_{K,\infty} \right\| < \infty, \tag{3.26}$$

$$(R2) \quad \left\| \frac{v}{(1+k^2)^{\frac{1}{2}}} \right\| < \infty \quad \Leftrightarrow \quad \| |k| B_{K,\infty} \| < \infty, \tag{3.27}$$

$$(R3) \quad \left\| \frac{v}{(1+k^2)} \right\| < \infty \quad \Leftrightarrow \quad \| B_{K,\infty} \| < \infty. \tag{3.28}$$

If we considered a weaker assumption than $(R3)$, $B_{K,\infty,x}(k)$ would not be square-integrable with respect to k anymore and the Gross transform would not be well-defined. Therefore, $(R3)$ is the minimal assumption that can be handled by using the Gross transform.

Now, the results of Lemmas B.3.1, B.3.3, B.3.9, the identity

$$p \cdot a^*(kB_{K,\Lambda}) + a(kB_{K,\Lambda}) \cdot p = a^*(kB_{K,\Lambda}) \cdot p + p \cdot a(kB_{K,\Lambda}) - \phi(k^2 B_{K,\Lambda}), \tag{3.29}$$

and the canonical commutation relation between creation and annihilation operator, yield the operator identities

$$\begin{aligned} U_{K,\Lambda} p^2 U_{K,\Lambda}^* &= p^2 - 2a^*(kB_{K,\Lambda}) \cdot p - 2p \cdot a(kB_{K,\Lambda}) + 2a^*(kB_{K,\Lambda}) a(kB_{K,\Lambda}) \\ &\quad + a(kB_{K,\Lambda})^2 + a^*(kB_{K,\Lambda})^2 + \phi(k^2 B_{K,\Lambda}) + \|kB_{K,\Lambda}\|^2 \end{aligned} \tag{3.30}$$

$$U_{K,\Lambda} d\Gamma(\omega) U_{K,\Lambda}^* = d\Gamma(\omega) + \phi(\omega B_{K,\Lambda}) + \left\|\omega^{1/2} B_{K,\Lambda}\right\|^2 \tag{3.31}$$

$$U_{K,\Lambda} \phi(G_\Lambda) U_{K,\Lambda}^* = \phi(G_\Lambda) + 2\langle B_{K,\Lambda}, G_\Lambda \rangle \tag{3.32}$$

on $D(H_0)$ for $\Lambda < \infty$. From that, and in view of the fact that $(\omega(k)+k^2)B_{K,\Lambda}+G_\Lambda = G_K$, we obtain the transformed Hamiltonian

$$\begin{aligned} H'_{K,\Lambda} :&= U_{K,\Lambda} H_\Lambda U_{K,\Lambda}^* + E_\Lambda \\ &= p^2 + d\Gamma(\omega) + \phi(G_K) - 2a^*(kB_{K,\Lambda}) \cdot p - 2p \cdot a(kB_{K,\Lambda}) \\ &\quad + a(kB_{K,\Lambda})^2 + a^*(kB_{K,\Lambda})^2 + 2a^*(kB_{K,\Lambda}) a(kB_{K,\Lambda}) + E_K \end{aligned} \tag{3.33}$$

which for $\Lambda < \infty$ is a self-adjoint operator on $D(H_0)$. We define $V_{K,\Lambda}$ as the interaction part of $H'_{K,\Lambda}$, that means $H'_{K,\Lambda} =: H_0 + V_{K,\Lambda}$.

3.3. Self-Adjointness and domain via an explicit limit operator

In this section, we discuss Regime ($R2$) which guarantees us enough decay of $B_{K,\Lambda,x}(k)$ for large $|k|$ to construct the norm-resolvent limit of the Hamiltonian $H_\Lambda + E_\Lambda$ analogously as we did in our work about the Fröhlich Hamiltonian in [23] and Chapter 2. Therefore, we are also able to give a strong statement about its domain which proves Equations (3.2) and (3.3) from the Introduction. The results of this section can especially be applied to the Nelson model in one and two space dimensions and to the class of Hamiltonians we discussed in our previous article [23] and Chapter 2, respectively, such that this section can be considered as a generalization of some parts of our work [23] and Chapter 2.

3.3.1. Construction of the operator

For the construction in this section, another important assumption is needed. We impose that

$$(o) \quad \text{there exists a } \gamma \text{ with } \frac{1}{2} \leq \gamma \leq 1 \text{ such that}$$
$$\sup_{p\in\mathbb{R}^d} \int\limits_{|k|\geq K} dk\, \frac{|v(k)|^2}{\omega(k)^{2\gamma}(1+(p-k)^2)^{2(1-\gamma)}} \to 0 \qquad (K\to\infty),$$

which we abbreviate with (o) since it is necessary in the construction where we find an explicit representation of the operator in the norm-resolvent limit. Note that for $\omega(k) = 1$ and $\gamma = \frac{1}{2}$, this leads to the class of operators we discussed in our previous paper [23] and Chapter 2 containing especially the Fröhlich Hamiltonian, we called it ($v3$). Setting $\gamma = 1$, (o) leads to a condition which is satisfied for the Nelson Hamiltonian in one and two space dimensions. Note also that this assumption has got something to do with the important Lemma B.1.2 which we get by generalizing Lemma 10 of Frank and Schlein [13]. This lemma was an important tool in our previous work.

Theorem 3.3.1. *Assume (ω), (v), ($R2$), and (o). Then, for every $\varepsilon > 0$, there exist $K > 0$ and $C_\varepsilon \in \mathbb{R}$ such that for all $\Lambda \leq \infty$ and all $\Psi \in D(H_0)$*

$$\|V_{K,\Lambda}\Psi\| \leq \varepsilon\|H_0\Psi\| + C_\varepsilon\|\Psi\|.$$

Furthermore, the operator $H'_{K,\infty} = H_0 + V_{K,\infty}$ is self-adjoint on $D(H_0)$ if K is large enough.

Proof. If we can show the desired estimate for each summand in

$$\begin{aligned} V_{K,\Lambda} = \phi(G_K) &- 2a^*(kB_{K,\Lambda})\cdot p - 2p\cdot a(kB_{K,\Lambda}) \\ &+ a(kB_{K,\Lambda})^2 + a^*(kB_{K,\Lambda})^2 + 2a^*(kB_{K,\Lambda})a(kB_{K,\Lambda}) + E_K, \end{aligned} \tag{3.34}$$

the first statement of the theorem is proven. The estimate is uniformly in $\Lambda \leq \infty$ and the self-adjointness of $H'_{K,\infty}$ then follows from the theorem of Kato-Rellich.

The operator $\phi(G_K)$ is infinitessimally H_0-bounded for every $K < \infty$ and the finite constant E_K is independant of Λ.

The squared terms of the ladder operators can be handled by using Lemma A.2.3 to get

$$\left\|a^{\#}(kB_{K,\Lambda})a^{\#}(kB_{K,\Lambda})\Psi\right\| \leq \sqrt{2}\,(1+1/m)\,\|kB_{K,\Lambda}\|^2\,(\|H_0\Psi\| + \|\Psi\|)\,, \tag{3.35}$$

where $\|kB_{K,\Lambda}\|^2$ is bounded uniformly in $\Lambda \leq \infty$ and becomes arbitrarily small for K large enough because of $(R2)$.

Next, we treat the term $a^*(kB_{K,\Lambda}) \cdot p$ with the help of Lemma A.2.2 . We obtain

$$\begin{aligned}\|a^*(kB_{K,\Lambda}) \cdot p\Psi\| &\leq \|kB_{K,\Lambda}\|\left\|(N+1)^{1/2}p\Psi\right\| \\ &\leq \sqrt{2}\,(1+1/m)\,\|kB_{K,\Lambda}\|\,(\|H_0\Psi\| + \|\Psi\|)\,.\end{aligned} \tag{3.36}$$

For the remaining term, we calculate

$$p \cdot a(kB_{K,\Lambda})\Psi = a(kB_{K,\Lambda}) \cdot p\Psi + a(k^2B_{K,\Lambda})\Psi, \tag{3.37}$$

where the first summand will be handled analogously to $a^*(kB_{K,\Lambda}) \cdot p$ to get

$$\|a(kB_{K,\Lambda}) \cdot p\Psi\| \leq \sqrt{2}\,(1+1/m)\,\|kB_{K,\Lambda}\|\|H_0\Psi\|. \tag{3.38}$$

For the second summand, we use Lemma B.1.2 and obtain

$$\left\|a(k^2B_{K,\Lambda})\Psi\right\| \leq C_{K,\Lambda,\gamma}\left\|d\Gamma(\omega)^\gamma(1+p^2)^{1-\gamma}\Psi\right\|, \tag{3.39}$$

with

$$C^2_{K,\Lambda,\gamma} := \sup_{p\in\mathbb{R}^d} \int\limits_{K\leq|k|\leq\Lambda} dk\, \frac{|v(k)|^2}{\omega(k)^{2\gamma}(1+(p-k)^2)^{2(1-\gamma)}}. \tag{3.40}$$

Because of (o), there is a $\gamma \in [\frac{1}{2}, 1]$ such that $C_{K,\Lambda,\gamma}$ is bounded uniformly in $\Lambda \leq \infty$ and becomes arbitrarily small for K large enough. In view of the inequality

$$d\Gamma(\omega)^\gamma(1+p^2)^{1-\gamma} \leq \gamma d\Gamma(\omega) + (1-\gamma)(1+p^2) \tag{3.41}$$

which follows for $\frac{1}{2} \leq \gamma \leq 1$ from Young's Inequality, we arrive at

$$\left\|a(k^2B_{K,\Lambda})\Psi\right\| \leq C_{K,\Lambda,\gamma}\|(H_0+1)\Psi\|, \tag{3.42}$$

which completes the proof. $\square$

Proposition 3.3.2. *Assume* (ω), (v), ($R2$) *and* (o). *Then, for K sufficiently large, $H'_{K,\Lambda} \to H'_{K,\infty}$ in the norm resolvent sense as $\Lambda \to \infty$.*

Proof. At first, in view of Theorem 3.3.1, choose K so large that $H'_{K,\infty}$ is self-adjoint on $D(H_0)$. Thus, $H'_{K,\Lambda}$ is self-adjoint on $D(H_0)$ for all $\Lambda \leq \infty$. Since

$$(H'_{K,\infty} + i)^{-1} - (H'_{K,\Lambda} + i)^{-1} = (H'_{K,\Lambda} + i)^{-1}(H'_{K,\Lambda} - H'_{K,\infty})(H'_{K,\infty} + i)^{-1} \tag{3.43}$$

and since $(H'_{K,\Lambda} + i)^{-1}$ is a bounded operator for all $\Lambda \leq \infty$, it suffices to show that

$$(H'_{K,\Lambda} - H'_{K,\infty})(H'_{K,\infty} + i)^{-1} \to 0 \qquad (\Lambda \to \infty) \tag{3.44}$$

which is equivalent to

$$(H'_{K,\Lambda} - H'_{K,\infty})(H_0 + 1)^{-1} \to 0 \qquad (\Lambda \to \infty) \tag{3.45}$$

due to the boundedness of $(H_0 + 1)(H'_{K,\infty} + i)^{-1}$. By definition of $H'_{K,\Lambda}$, see Equation (3.33), we have

$$\begin{aligned} H'_{K,\Lambda} - H'_{K,\infty} = {} & 2a^*(kB_{K,\infty}\overline{\chi}_\Lambda) \cdot p + 2p \cdot a(kB_{K,\infty}\overline{\chi}_\Lambda) \\ & - 2a^*(kB_{K,\infty}\overline{\chi}_\Lambda)a(kB_{K,\Lambda}) - 2a^*(kB_{K,\infty})a(kB_{K,\infty}\overline{\chi}_\Lambda) \\ & - a^*(kB_{K,\infty}\overline{\chi}_\Lambda)a^*(kB_{K,\Lambda}) - a^*(kB_{K,\infty})a^*(kB_{K,\infty}\overline{\chi}_\Lambda) \\ & - a(kB_{K,\infty}\overline{\chi}_\Lambda)a(kB_{K,\Lambda}) - a(kB_{K,\infty})a(kB_{K,\infty}\overline{\chi}_\Lambda), \end{aligned} \tag{3.46}$$

where we used that $B_{K,\infty} - B_{K,\Lambda} = B_{K,\infty}\overline{\chi}_\Lambda$ with $\overline{\chi}_\Lambda := 1 - \chi_\Lambda$. Using Lemma A.2.3 and the same estimates that were used in the proof of Theorem 3.3.1 completes the proof of this proposition. □

Corollary 3.3.3. *Assume* (ω), (v), ($R2$) *and* (o). *Then, for K sufficiently large, there exists a constant C such that for all $\Lambda \leq \infty$*

$$\frac{1}{2}H_0 - C \leq H'_{K,\Lambda} \leq \frac{3}{2}H_0 + C.$$

Proof. By Theorem 3.3.1 there exist K and C such that

$$\|V_{K,\Lambda}\Psi\| \leq \frac{1}{4}\|H_0\Psi\| + \frac{C}{2}\|\Psi\| \tag{3.47}$$

for all $\Psi \in D(H_0)$ and $\Lambda \leq \infty$. Reading this as perturbation of the free Hamiltonian $\frac{1}{2}H_0$ and using the lower bound given by the Kato-Rellich-theorem, Theorem X.12 from [25], we conclude that

$$\frac{1}{2}H_0 \pm V_{K,\Lambda} \geq -C, \tag{3.48}$$

which implies the desired inequalities. □

Theorem 3.3.4. *Assume* (ω), (v), ($R2$), *and* (o). *Then, there exists a self-adjoint operator* H *such that* $H_\Lambda + E_\Lambda \to H$ *as* $\Lambda \to \infty$ *in the norm resolvent sense. This operator has the representation*

$$H = U^*_{K,\infty} H'_{K,\infty} U_{K,\infty}, \quad D(H) = U^*_{K,\infty} D(H_0),$$

which is valid for K *sufficiently large. If* $D \subset D(H_0)$ *is a core of* H_0 *then* $U^*_{K,\infty} D$ *is a core of* H.

Proof. Choose K so large that $H'_{K,\Lambda} \to H'_{K,\infty}$ in the norm resolvent sense by Proposition 3.3.2. Let $R'_{K,\Lambda}(z) := (H'_{K,\Lambda} - z)^{-1}$ and $H := U^*_{K,\infty} H'_{K,\infty} U_{K,\infty}$. By Equation (3.33), $H_\Lambda + E_\Lambda = U^*_{K,\Lambda} H'_{K,\Lambda} U_{K,\Lambda}$ for all $\Lambda < \infty$ and therefore

$$\begin{aligned} &(H_\Lambda + E_\Lambda - z)^{-1} - (H - z)^{-1} \\ &= U^*_{K,\Lambda} R'_{K,\Lambda}(z) U_{K,\Lambda} - U^*_{K,\infty} R'_{K,\infty}(z) U_{K,\infty} \\ &= (U^*_{K,\Lambda} - U^*_{K,\infty}) R'_{K,\Lambda}(z) U_{K,\Lambda} + U^*_{K,\infty} (R'_{K,\Lambda}(z) - R'_{K,\infty}(z)) U_{K,\Lambda} \\ &\quad + U^*_{K,\infty} R'_{K,\infty}(z) (U_{K,\Lambda} - U_{K,\infty}). \end{aligned} \tag{3.49}$$

It remains to show that these three terms vanish in the limit $\Lambda \to \infty$. For the second term this follows from Proposition 3.3.2. For the first and third terms we have

$$\begin{aligned} &\left\| (U^*_{K,\Lambda} - U^*_{K,\infty}) R'_{K,\Lambda}(z) U_{K,\Lambda} \right\| \\ &\quad \le \left\| (U^*_{K,\Lambda} - U^*_{K,\infty})(H_0 + 1)^{-1/2} \right\| \cdot \left\| (H_0 + 1)^{1/2} R'_{K,\Lambda}(z) \right\| \end{aligned} \tag{3.50}$$

and

$$\begin{aligned} &\left\| U^*_{K,\infty} R'_{K,\infty}(z) (U_{K,\Lambda} - U_{K,\infty}) \right\| = \left\| (U^*_{K,\Lambda} - U^*_{K,\infty}) R'_{K,\infty}(\overline{z}) \right\| \\ &\quad \le \left\| (U^*_{K,\Lambda} - U^*_{K,\infty})(H_0 + 1)^{-1/2} \right\| \cdot \left\| (H_0 + 1)^{1/2} R'_{K,\infty}(\overline{z}) \right\|. \end{aligned} \tag{3.51}$$

The Lemma A.2.5 implies that

$$\left\| (U^*_{K,\Lambda} - U^*_{K,\infty})(H_0 + 1)^{-1/2} \right\| \to 0 \qquad (\Lambda \to \infty) \tag{3.52}$$

because $\sup_{x \in \mathbb{R}^d} \| B_{K,\Lambda,x} - B_{K,\infty,x} \| \to 0$ as $\Lambda \to \infty$, and the Corollary 3.3.3 shows that

$$\sup_{\Lambda \le \infty} \left\| (H_0 + 1)^{1/2} R'_{K,\Lambda}(z) \right\| < \infty. \tag{3.53}$$

Combining Properties (3.50), (3.51), (3.52), and (3.53) we see that the first and third term from (3.49) vanish as $\Lambda \to \infty$. The statement about $D(H)$ follows from Theorem 3.3.1.

Now, if $D \subset D(H_0)$ is a core of H_0, then, by Theorem 3.3.1, D is a core of $H'_{K,\infty}$ and hence $U^*_{K,\infty} D$ is a core of $H = U^*_{K,\infty} H'_{K,\infty} U_{K,\infty}$. $\square$

3.3.2. Regularity of domain vectors

The following lemma generalizes the statement of Lemma B.3.8, on the invariance of $D(H_0^{1/2}) = D(|p|) \cap D(d\Gamma(\omega)^{1/2})$ under the transformation $U_{K,\infty}$ in the regime $(R2)$:

Lemma 3.3.5. *Assume (ω), (v), and $(R2)$. Then, for $\sigma \in [0,1]$, the subspaces $D(|p|^\sigma) \cap D(d\Gamma(\omega)^{\sigma/2})$ are left invariant by $U_{K,\infty}$ and $U^*_{K,\infty}$.*

Proof. Let $\gamma \in D(|p|^\sigma) \cap D(d\Gamma(\omega)^{\sigma/2})$. Since $(R2)$ implies $\left\|\omega^{1/2} B_{K,\infty}\right\| < \infty$, we can apply Corollary B.3.7 and get that $U_{K,\infty}\gamma \in D(d\Gamma(\omega)^{\sigma/2})$, and it remains to prove that $U_{K,\infty}\gamma \in D(|p|^\sigma)$. Since $U^*_{K,\infty} p U_{K,\infty} = p + \phi(kB_{K,\infty})$, by Lemma B.3.8 (and the remark thereafter), the statement $U_{K,\infty}\gamma \in D(|p|^\sigma)$ is equivalent to $\gamma \in D(|p + \phi(kB_{K,\infty})|^\sigma)$. To prove the latter, we first observe that $\phi(kB_{K,\infty})^2 \le 2\|kB_{K,\infty}\|^2(2N+1)$, and hence,

$$\begin{aligned}
(p + \phi(kB_{K,\infty}))^2 &\le 2p^2 + 2\phi(kB_{K,\infty})^2 \\
&\le 2(p^2 + 2\|kB_{K,\infty}\|^2(2N+1)) \\
&\le 2(p^2 + 2\,(1 + 1/m)\,\|kB_{K,\infty}\|^2(2d\Gamma(\omega) + 1)). \qquad (3.54)
\end{aligned}$$

This means, in particular, that the form domain of $(p + \phi(kB_{K,\infty}))^2$ contains the form domain of $p^2 + d\Gamma(\omega)$. From the operator monotonicity of the fractional power $\sigma \in [0,1]$ (see [26], Proposition 10.14), it follows that

$$\begin{aligned}
|p + \phi(kB_{K,\infty})|^{2\sigma} &\le 2^\sigma \left(|p|^{2\sigma} + 2^\sigma \|kB_{K,\infty}\|^{2\sigma}(2N+1)^\sigma\right) \\
&\le 2^\sigma \left(|p|^{2\sigma} + 2^\sigma\,(1 + 1/m)^\sigma\,\|kB_{K,\infty}\|^{2\sigma}(2d\Gamma(\omega)+1)^\sigma\right). \qquad (3.55)
\end{aligned}$$

Again, this inequality includes a statement about form domains. It implies that

$$D(|p + \phi(kB_{K,\infty})|^\sigma) \supset D(|p|^\sigma) \cap D(d\Gamma(\omega)^{\sigma/2}). \qquad (3.56)$$

Since $\gamma \in D(|p|^\sigma) \cap D(d\Gamma(\omega)^{\sigma/2})$ from the assumptions, the proof is finished. □

Theorem 3.3.6. *Assume (ω), (v), $(R2)$, (o), and let $\left\|\omega v(1+k^2)^{-1}\right\| < \infty$, but $\left\||k|^s v(1+k^2)^{-1}\right\| = \infty$ for some $s \in (1,2]$. Then*

$$U^*_{K,\infty} D(H_0) \cap D(|p|^s) = \{0\}.$$

Proof. Let $\Psi \in D(H_0) = D(p^2) \cap D(d\Gamma(\omega))$, and suppose that $U^*_{K,\infty}\Psi \in D(|p|^s)$. The assumption $\left\|\omega v(1+k^2)^{-1}\right\| < \infty$ implies that $\|\omega B_{K,\infty}\| < \infty$, such that we can apply Lemma B.3.3 to get $U^*_{K,\infty}\Psi \in D(|p|^s) \cap D(d\Gamma(\omega))$. In view of the inequality

$$d\Gamma(\omega)^{(s-1)/2}|p| \le \frac{s-1}{s} d\Gamma(\omega)^{s/2} + \frac{1}{s}|p|^s, \qquad (3.57)$$

and the assumption $1 < s \leq 2$, we conclude that $pU_{K,\infty}^*\Psi \in D(|p|^{s-1}) \cap D(d\Gamma(\omega)^{(s-1)/2})$. This implies, by Lemma 3.3.5, that $U_{K,\infty}pU_{K,\infty}^*\Psi \in D(|p|^{s-1}) \cap D(d\Gamma(\omega)^{(s-1)/2})$, where

$$U_{K,\infty}pU_{K,\infty}^*\Psi = (p - \phi(kB_{K,\infty}))\,\Psi. \tag{3.58}$$

The first term on the right-hand side, $p\Psi$, belongs to $D(|p|^{s-1})$, because $s \leq 2$ and $\Psi \in D(p^2)$. We now compute $\left\| |p|^{s-1}\phi(kB_{K,\infty})\Psi \right\|$, and we show that this number is infinite unless $\Psi = 0$. To this end, we define the functions

$$|p|_\varepsilon^{s-1} := \frac{|p|^{s-1}}{1 + \varepsilon|p|^{s-1}}, \tag{3.59}$$

$$D_\varepsilon(p,k) := |p+k|_\varepsilon^{s-1} - |k|_\varepsilon^{s-1} \tag{3.60}$$

for $p, k \in \mathbb{R}^d$ and $\varepsilon > 0$. Using that $(s-1) \in (0,1]$, it is straightforward to verify that

$$|D_\varepsilon(p,k)| \leq |p|^{s-1} \tag{3.61}$$

for all $p, k \in \mathbb{R}^d$ and $\varepsilon > 0$. For $p = -i\nabla_x$, we have

$$|p|_\varepsilon^{s-1} e^{ikx} = e^{ikx}|p+k|_\varepsilon^{s-1} \tag{3.62}$$

which, in view of (A.14) and (A.13), implies

$$\begin{aligned} |p|_\varepsilon^{s-1} a(kB_{K,\infty})\Psi &= a(kB_{K,\infty}|p|_\varepsilon^{s-1})\Psi \\ &= a(|k|_\varepsilon^{s-1}kB_{K,\infty})\Psi + a(D_\varepsilon(p,k)kB_{K,\infty})\Psi \end{aligned} \tag{3.63}$$

and

$$|p|_\varepsilon^{s-1} a^*(kB_{K,\infty})\Psi = a^*(kB_{K,\infty}|k|_\varepsilon^{s-1})\Psi + a^*(kB_{K,\infty}D_\varepsilon(p,-k))\Psi. \tag{3.64}$$

By Inequality (3.61),

$$\|a(D_\varepsilon(p,k)kB_{K,\infty})\Psi\| \leq \|kB_{K,\infty}\| \cdot \left\| |p|^{s-1}\sqrt{N}\Psi \right\|, \tag{3.65}$$

$$\|a^*(kB_{K,\infty}D_\varepsilon(p,-k))\Psi\| \leq \|kB_{K,\infty}\| \cdot \left\| |p|^{s-1}\sqrt{N+1}\Psi \right\|, \tag{3.66}$$

which are finite by $(R2)$. By Lemma B.1.2, we get for $\frac{1}{2} \leq \gamma \leq 1$

$$\begin{aligned} &\left\| a(kB_{K,\infty}|k|_\varepsilon^{s-1})\Psi \right\|^2 \\ &\leq \sup_{q\in\mathbb{R}^d} \int\limits_{|k|\geq K} dk\, \frac{|k|^{2s}|v(k)|^2}{(\omega(k)+k^2)^2\omega(k)^{2\gamma}(1+(p-k)^2)^{2(1-\gamma)}(1+\varepsilon|k|^{s-1})^2} \\ &\qquad \left\| d\Gamma(\omega)^\gamma(1+p^2)^{1-\gamma}\Psi \right\|^2 \\ &\leq \frac{1}{m^{2-s}} \sup_{q\in\mathbb{R}^d} \int\limits_{|k|\geq K} dk\, \frac{|v(k)|^2}{\omega(k)^{2\gamma}(1+(p-k)^2)^{2(1-\gamma)}}\ \|(H_0+1)\Psi\|^2 \end{aligned} \tag{3.67}$$

which is also finite by (o). In the last step, we used that $s \in (1,2]$ and Inequality (3.41). Note that the bounds (3.65), (3.66) and (3.67) are uniform in $\varepsilon > 0$. Therefore, there exists a constant c such that

$$\begin{aligned} \left\| |p|^{s-1} \phi(kB_{K,\infty})\Psi \right\| &= \lim_{\varepsilon\to 0} \left\| |p|_\varepsilon^{s-1} \phi(kB_{K,\infty})\Psi \right\| \\ &\geq \liminf_{\varepsilon\to 0} \left\| a^*(kB_{K,\infty}|k|_\varepsilon^{s-1})\Psi \right\| - c \\ &\geq \lim_{\varepsilon\to 0} \left\| kB_{K,\infty}|k|_\varepsilon^{s-1} \right\| \cdot \|\Psi\| - c \end{aligned} \tag{3.68}$$

which is infinite unless $\Psi = 0$. This completes the proof. □

Theorem 3.3.7. *Assume* (ω), (v), *and let* $\left\| |k|^s v(1+k^2)^{-1} \right\| < \infty$ *for some* $s \in [1,2]$. *Then*

$$U_{K,\infty}^* D(H_0) \subset D(|p|^s).$$

Proof. Note that $\left\| |k|^s v(1+k^2)^{-1} \right\| < \infty$ implies $(R2)$ and $\||k|^s B_{K,\infty}\| < \infty$. Let $\Psi \in D(H_0)$. From Lemma B.3.8, we know that $U_{K,\infty}^*\Psi \in D(H_0^{1/2}) \subset D(|p|)$ and that $p_j U_{K,\infty}^*\Psi = U_{K,\infty}^*(p_j - \phi(k_j B_{K,\infty}))\Psi =: U_{K,\infty}^*\gamma_j$ for all $j = 1, ..., d$. It follows that

$$\left\| |p|^s U_{K,\infty}^*\Psi \right\|^2 = \sum_{j=1}^d \left\| |p|^{s-1} p_j U_{K,\infty}^*\Psi \right\|^2 = \sum_{j=1}^d \left\| |p|^{s-1} U_{K,\infty}^*\gamma_j \right\|^2, \tag{3.69}$$

which is finite (and thus proves the theorem) provided we can show that $U_{K,\infty}^*\gamma_j \in D(|p|^{s-1})$ for all j. In view of Lemma B.3.8, this is equivalent to show that $\gamma_j \in D(|p - \phi(k_j B_{K,\infty})|^{s-1})$, and by the argument given in the proof of Lemma 3.3.5 by Inequality (3.55), it suffices to show that $\gamma_j \in D(|p|^{s-1}) \cap D(N^{(s-1)/2})$.

Since $\Psi \in D(d\Gamma(\omega))$ and $s \in [1,2]$, $\gamma_j \in D(\sqrt{N}) \in D(N^{(s-1)/2})$, and it remains to show that $\gamma_j \in D(|p|^{s-1})$. The first term of $\gamma_j = p_j\Psi - \phi(k_j B_{K,\infty})\Psi$ belongs to $D(|p|^{s-1})$ because $s \leq 2$ and because $\Psi \in D(p^2)$. To prove that $\left\| |p|^{s-1}\phi(k_j B_{K,\infty})\Psi \right\|$ is finite, we recall the estimates in the proof of Theorem 3.3.6 which leads to

$$\begin{aligned} \left\| |p|^{s-1}\phi(k_j B_{K,\infty})\Psi \right\| &\leq 2\left(\||k|^s B_{K,\infty}\| \left\| \sqrt{N+1}\Psi \right\| + \||k|B_{K,\infty}\| \left\| \sqrt{N+1}|p|^{s-1}\Psi \right\| \right) \\ &\leq 2\sqrt{1+1/m}\ \||k|^s B_{K,\infty}\|\ \left\| (d\Gamma(\omega)+1)^{1/2}\Psi \right\| \\ &\quad + \frac{1}{\sqrt{2}} \||k|B_{K,\infty}\| \left(\left\| p^2\Psi \right\| + (1+1/m)\|(d\Gamma(\omega)+1)\Psi\| \right), \end{aligned} \tag{3.70}$$

which is finite, since $\|kB_{K,\infty}\|$ and $\||k|^s B_{K,\infty}\|$ are finite by assumptions. □

3.3.3. Applications

Now, we apply our results to the Nelson Hamiltonian. We have

$$v(k) = \frac{1}{\sqrt{\omega(k)}} \quad \text{with} \quad \omega(k) = \sqrt{m^2 + k^2} \quad \text{and} \quad m > 0, \tag{3.71}$$

such that, for one and two space dimensions, we are in regime $(R2)$. By Theorem 3.3.4, there exists the norm-resolvent limit H of $H_\Lambda + E_\Lambda$ with domain $D(H) = U^*_{K,\infty} D(H_0)$ for K sufficiently large. From the mapping properties of $U^*_{K,\infty}$, see Lemma B.3.3, we know that $D(H) \subset D(d\Gamma(\omega))$. Furthermore, the condition (o) is satisfied, choose $\gamma = 1$.

In one space dimension, we still have $\left\| |k|^s v(1+k^2)^{-1} \right\| < \infty$ for $s \in [1, 2)$, but $\left\| k^2 v(1+k^2)^{-1} \right\| = \infty$. In view of Theorems 3.3.6 and 3.3.7, we obtain that

$$D(H) \subset \bigcap_{1 \leq s < 2} D(|p|^s) \cap D(d\Gamma(\omega)), \qquad D(H) \cap D(p^2) = \{0\}. \tag{3.72}$$

In two space dimensions, we see that $\left\| |k|^s v(1+k^2)^{-1} \right\| < \infty$ only for $s \in [1, 3/2)$ and $\left\| |k|^s v(1+k^2)^{-1} \right\| = \infty$ for $s \in [3/2, 2]$. Here, Theorems 3.3.6 and 3.3.7 lead to

$$D(H) \subset \bigcap_{1 \leq s < 3/2} D(|p|^s) \cap D(d\Gamma(\omega)), \qquad D(H) \cap D(|p|^{3/2}) = \{0\}. \tag{3.73}$$

Therefore, we proved Equations (3.2) and (3.3) from the Introduction. We, moreover, remark that the class of operators we treated in our article [23] and Chapter 2 is also included here such that this Section 3.3 in fact is a generalization of the previous case. In the special case of the Hamiltonians with

$$v(k) = |k|^{-\frac{d-1}{2}}, \quad d \geq 2, \qquad \omega(k) = 1, \tag{3.74}$$

which also contain the Fröhlich Hamiltonian in two and three space dimensions, we can reproduce the results

$$D(H) \subset \bigcap_{1 \leq s < 3/2} D(|p|^s) \cap D(N), \qquad D(H) \cap D(|p|^{3/2}) = \{0\}, \tag{3.75}$$

and realize that, with respect to the regularity of the momentum of the quantized particle, the two dimensional Nelson Hamiltonian behaves like the Fröhlich Hamiltonian.

In all cases, we see especially that

$$D(H) \subset D(H_0^{1/2}), \qquad D(H) \cap D(H_0) = \{0\}. \tag{3.76}$$

3.4. Self-Adjointness and domain via quadratic form

Now, we take the weakest regime $(R3)$ into account. Because of the resulting weaker decay of $B_{K,\Lambda,x}(k)$ for large $|k|$, the construction via an explicit operator for the norm-resolvent limit of $H_\Lambda + E_\Lambda$ from the previous section does not work anymore. Nevertheless, following Nelson [1], we can define an operator associated to a quadratic form. So the first part of this section is the generalization of Nelson's method to the class of Hamiltonians we study here. The main focus then shall be directed to the investigation of their domains, which leads, in particular for the three-dimensional Nelson Hamiltonian, to the proof of Equations (3.4) and (3.5) from the Introduction.

3.4.1. Construction of the operator

In view of Equation (3.33), we define on the domain $D(H_0^{1/2}) = D(|p|) \cap D(d\Gamma(\omega)^{1/2})$, the quadratic form

$$\begin{aligned} W_{K,\Lambda}(\Psi) := 2\,\mathrm{Re}\Big\{ &\langle \Psi, a(G_K)\Psi\rangle - 2\langle a(kB_{K,\Lambda})\Psi, p\Psi\rangle + \|a(kB_{K,\Lambda})\Psi\|^2 \\ &+ \big\langle (N+1)^{1/2}\Psi, (N+1)^{-1/2}a(kB_{K,\Lambda})^2\Psi\big\rangle \Big\} + E_K\|\Psi\|^2, \end{aligned} \tag{3.77}$$

which, for $\Lambda < \infty$, is the quadratic form associated to $V_{K,\Lambda}$. The main result of this subsection is Theorem 3.4.3. For the proof of this theorem, we at first establish the two Lemmas 3.4.1 and 3.4.2. These lemmas essentially are a generalization of Lemma 5 of [1] and the proofs work analogously.

Lemma 3.4.1. *Assume (ω), (v), $(R3)$, and let $\left\|v(1+k^2)^{-1/2}\omega^{-1/4}\right\| < \infty$. Then, for all $\varepsilon > 0$ there is a $K < \infty$ and a $b < \infty$, such that*

$$|W_{K,\Lambda}(\Psi)| \le \varepsilon \left\|H_0^{1/2}\Psi\right\|^2 + b\|\Psi\|^2$$

for all $\Psi \in D(H_0^{1/2})$ and all $\Lambda < \infty$.

Proof. Note that $\left\|v(1+k^2)^{-1/2}\omega^{-1/4}\right\| < \infty$ implies $\left\|kB_{K,\infty}\omega^{-1/4}\right\| < \infty$, what also implies $\left\|kB_{K,\infty}\omega^{-1/2}\right\| < \infty$. We estimate each term of (3.77) by the right-hand side of the estimate in Lemma 3.4.1. For the first term, we find for every $\varepsilon > 0$, with the help of Corollary B.1.3,

$$|\langle \Psi, a(G_K)\Psi\rangle| \le \|\Psi\| \left\|\tfrac{G_K}{\sqrt{\omega}}\right\| \left\|d\Gamma(\omega)^{1/2}\Psi\right\| \le \varepsilon \left\|H_0^{1/2}\Psi\right\|^2 + \frac{1}{4\varepsilon}\left\|\tfrac{G_K}{\sqrt{\omega}}\right\|^2 \|\Psi\|^2. \tag{3.78}$$

Also using Corollary B.1.3, the second term leads to

$$|\langle p\Psi, a(kB_{K,\Lambda})\Psi\rangle| \le \|p\Psi\| \left\|\tfrac{kB_{K,\Lambda}}{\sqrt{\omega}}\right\| \left\|d\Gamma(\omega)^{1/2}\Psi\right\| \le \frac{1}{2}\left\|\tfrac{kB_{K,\infty}}{\sqrt{\omega}}\right\| \left\|H_0^{1/2}\Psi\right\|^2, \tag{3.79}$$

where $\left\|kB_{K,\infty}\omega^{-1/2}\right\|$ becomes arbitrarily small for K sufficiently large. Similarily, we get

$$\|a(kB_{K,\Lambda})\Psi\|^2 \leq \left\|\tfrac{kB_{K,\infty}}{\sqrt{\omega}}\right\|^2 \left\|H_0^{1/2}\Psi\right\|^2. \tag{3.80}$$

For the next term, we obtain with the help of Lemma B.1.1

$$\begin{aligned}
&\left|\left\langle (N+1)^{1/2}\Psi, (N+1)^{-1/2} a(kB_{K,\Lambda})^2\Psi\right\rangle\right| \\
&\leq \left(1+\frac{1}{m}\right)^{1/2} \left\|\tfrac{kB_{K,\infty}}{\sqrt[4]{\omega}}\right\|^2 \left(\left\|H_0^{1/2}\Psi\right\|^2 + \|\Psi\|^2\right),
\end{aligned} \tag{3.81}$$

where $\left\|kB_{K,\infty}\omega^{-1/4}\right\|$ becomes arbitrarily small for K sufficiently large. For the last term, $|E_K|\ \|\Psi\|^2$, there is nothing to do. ☐

Lemma 3.4.2. *Assume* (ω), (v), $(R3)$, *and let* $\left\|v(1+k^2)^{-1/2}\omega^{-1/4}\right\| < \infty$. *Then, for all* $\Psi \in D(H_0^{1/2})$, *we have*

$$|W_{K,\Lambda_1}(\Psi) - W_{K,\Lambda_2}(\Psi)| \leq C_{K,\Lambda_1,\Lambda_2}\left\|(H_0+1)^{1/2}\Psi\right\|^2,$$

where $C_{K,\Lambda_1,\Lambda_2} \to 0$ *as* $\Lambda_1, \Lambda_2 \to \infty$.

Proof. Note that $\left\|v(1+k^2)^{-1/2}\omega^{-1/4}\right\| < \infty$ implies $\left\|kB_{K,\infty}\omega^{-1/4}\right\| < \infty$, what also implies $\left\|kB_{K,\infty}\omega^{-1/2}\right\| < \infty$. It is

$$\begin{aligned}
&W_{K,\Lambda_1}(\Psi) - W_{K,\Lambda_2}(\Psi) \\
&= 2\,\mathrm{Re}\Big\{ -2\langle p\Psi, a(kB_{\Lambda_2,\Lambda_1})\Psi\rangle + \|a(kB_{K,\Lambda_1})\Psi\|^2 - \|a(kB_{K,\Lambda_2})\Psi\|^2 \\
&\quad + \left\langle (N+1)^{1/2}\Psi, (N+1)^{-1/2}\left(a(kB_{K,\Lambda_1})^2 - a(kB_{K,\Lambda_2})^2\right)\Psi\right\rangle\Big\}.
\end{aligned} \tag{3.82}$$

Using Corollary B.1.3, we see that

$$|\langle p\Psi, a(kB_{\Lambda_2,\Lambda_1})\Psi\rangle| \leq \frac{1}{2}\left\|\tfrac{kB_{\Lambda_2,\Lambda_1}}{\sqrt{\omega}}\right\| \left\|H_0^{1/2}\Psi\right\|^2, \tag{3.83}$$

where $\left\|kB_{\Lambda_2,\Lambda_1}\omega^{-1/2}\right\| \to 0$ as $\Lambda_1, \Lambda_2 \to \infty$. Furthermore,

$$\begin{aligned}
&\left|\|a(kB_{K,\Lambda_1})\Psi\|^2 - \|a(kB_{K,\Lambda_2})\Psi\|^2\right| \\
&= |\|a(kB_{K,\Lambda_1})\Psi\| - \|a(kB_{K,\Lambda_2})\Psi\||\left(\|a(kB_{K,\Lambda_1})\Psi\| + \|a(kB_{K,\Lambda_2})\Psi\|\right) \\
&\leq \|a(kB_{\Lambda_2,\Lambda_1})\Psi\|\left(\|a(kB_{K,\Lambda_1})\Psi\| + \|a(kB_{K,\Lambda_2})\Psi\|\right) \\
&\leq 2\left\|\tfrac{kB_{K,\infty}}{\sqrt{\omega}}\right\| \left\|\tfrac{kB_{\Lambda_2,\Lambda_1}}{\sqrt{\omega}}\right\| \left\|H_0^{1/2}\Psi\right\|^2,
\end{aligned} \tag{3.84}$$

where we used Corollary B.1.3 again. Finally, using Lemma B.1.1,

$$\begin{aligned}
&\left|\left\langle (N+1)^{1/2}\Psi, (N+1)^{-1/2}\left(a(kB_{K,\Lambda_1})^2 - a(kB_{K,\Lambda_2})^2\right)\Psi\right\rangle\right| \\
&\leq \left\|(N+1)^{1/2}\Psi\right\| \left\|(N+1)^{-1/2}a(kB_{\Lambda_2,\Lambda_1})a(kB_{K,\Lambda_1}+kB_{K,\Lambda_2})\Psi\right\| \\
&\leq \left\|(N+1)^{1/2}\Psi\right\| \left\|\tfrac{kB_{\Lambda_2,\Lambda_1}}{\sqrt[4]{\omega}}\right\| \left\|\tfrac{k(B_{K,\Lambda_1}+B_{K,\Lambda_2})}{\sqrt[4]{\omega}}\right\| \left\|d\Gamma(\omega)^{1/2}\Psi\right\| \\
&\leq 2\left\|\tfrac{kB_{K,\infty}}{\sqrt[4]{\omega}}\right\| \left\|\tfrac{kB_{\Lambda_2,\Lambda_1}}{\sqrt[4]{\omega}}\right\| \left\|(H_0+1)^{1/2}\Psi\right\|^2, \qquad (3.85)
\end{aligned}$$

where $\left\|kB_{\Lambda_2,\Lambda_1}\omega^{-1/4}\right\| \to 0$ as $\Lambda_1, \Lambda_2 \to \infty$. □

From the two lemmas above, we see that, for K large enough, the form $W_{K,\Lambda}$ satisfies the hypotheses of Theorem A.1.1. We obtain the following

Theorem 3.4.3. *Assume* (ω), (v), $(R3)$, *and let* $\left\|v(1+k^2)^{-1/2}\omega^{-1/4}\right\| < \infty$. *Then, for* K *large enough, there exists a unique, self-adjoint, semi-bounded operator* $H'_{K,\infty}$ *with* $D(H'_{K,\infty}) \subset D(H_0^{1/2})$ *associated to the quadratic form*

$$\left\langle H_0^{1/2}\Psi, H_0^{1/2}\Psi\right\rangle + \lim_{\Lambda\to\infty} W_{K,\Lambda}(\Psi)$$

which is well-defined for all $\Psi \in D(H_0^{1/2})$. *Moreover, we have*

$$\begin{aligned}
H'_{K,\Lambda} &\to H'_{K,\infty} \qquad (\Lambda\to\infty), \\
H_\Lambda + E_\Lambda &\to H := U^*_{K,\infty}H'_{K,\infty}U_{K,\infty} \qquad (\Lambda\to\infty)
\end{aligned}$$

in the norm-resolvent sense and $D(H) \subset U^*_{K,\infty}D(H_0^{1/2})$.

In contrast to the regime $(R2)$ that we discussed in Section 3.3, we have no explicit representation of H and $D(H)$ since we have no explicit representation of the operator $H'_{K,\infty}$. What we can say about its domain, nevertheless, is the subject of the next subsection.

3.4.2. Regularity of domain vectors

From Theorem 3.4.3, we only know the form domain of the operator H which is given by $U^*_{K,\infty}D(H_0^{1/2})$. Because of that, in this subsection, we study the effect of $U^*_{K,\infty}$ on the set $D(H_0^{1/2})$. The main results are Theorems 3.4.4 and 3.4.7. Since $D(H)$ is a subset of $U^*_{K,\infty}D(H_0^{1/2})$, we are also able to make statements about $D(H)$.

Theorem 3.4.4. *Assume* (ω), (v), *and let* $\left\|v(1+k^2)^{\frac{s}{2}-1}\right\| < \infty$ *for some* $s \in [0,1]$, *then*

$$U^*_{K,\infty}D(H_0^{1/2}) \subset D(|p|^s).$$

Proof. Note that $\left\| v(1+k^2)^{\frac{s}{2}-1} \right\| < \infty$ implies $\| |k|^s B_{K,\infty} \| < \infty$ and also that we are in Regime $(R3)$. As a preparation, we record the following three facts:

1. From Corollary B.1.5, we know that

$$\begin{aligned} \left\| [a^{\#}(B_{K,\infty}), |p|^s](N+1)^{-1/2} \right\| &= \left\| a^{\#}([B_{K,\infty}, |p|^s])(N+1)^{-1/2} \right\| \\ &\leq \| |k|^s B_{K,\infty} \|. \end{aligned} \tag{3.86}$$

2. $D(H_0^{1/2}) \cap \mathscr{H}_0$ is a core of $H_0^{1/2}$:

   If $\Psi \in D(H_0^{1/2})$ then $\Psi_n := \chi(N \leq n)\Psi \in D(H_0^{1/2}) \cap \mathscr{H}_0$ and it is straightforward to show that $\left\| H_0^{1/2}(\Psi - \Psi_n) \right\| \to 0$ as $n \to \infty$.

3. If $\Psi \in D(H_0^{1/2}) \cap \mathscr{H}_0$ then $e^{-i\pi(B_{K,\infty})t}\Psi \in D(|p|^s)$ and $t \mapsto |p|^s e^{-i\pi(B_{K,\infty})t}\Psi$ is real-analytic:

   It is well-known that for $\Psi \in \mathscr{H}_0$

$$e^{-i\pi(B_{K,\infty})t}\Psi = \sum_{n \geq 0} \frac{(-it)^n}{n!} \pi(B_{K,\infty})^n \Psi, \tag{3.87}$$

   where this series is absolutely convergent for all $t \in \mathbb{R}$. The proof of this fact combined with Fact 1 shows that

$$\begin{aligned} \sum_{n \geq 0} \frac{(-it)^n}{n!} |p|^s \pi(B_{K,\infty})^n \Psi = &\sum_{n \geq 0} \frac{(-it)^n}{n!} \pi(B_{K,\infty})^n |p|^s \Psi \\ &+ \sum_{n \geq 0} \frac{(-it)^n}{n!} [|p|^s, \pi(B_{K,\infty})^n]\Psi \end{aligned} \tag{3.88}$$

   is absolutely convergent for all $t \in \mathbb{R}$. By the closedness of the operator $|p|^s$, it follows that $e^{-i\pi(B_{K,\infty})t}\Psi \in D(|p|^s)$ and that $|p|^s e^{-i\pi(B_{K,\infty})t}\Psi$ is given by (3.88).

We are now ready to prove the theorem. For all $\Phi \in D(|p|)$ and all $\Psi \in D(H_0^{1/2}) \cap \mathscr{H}_0$ it follows from Facts 1 and 3 that

$$\begin{aligned} \left\langle \Phi, |p|^s e^{-i\pi(B_{K,\infty})}\Psi - e^{-i\pi(B_{K,\infty})}|p|^s\Psi \right\rangle &= \left\langle \Phi, e^{-i\pi(B_{K,\infty})(1-t)}|p|^s e^{-i\pi(B_{K,\infty})t}\Psi \right\rangle \Big|_{t=0}^{t=1} \\ &= \int_0^1 \left\langle \Phi, e^{-i\pi(B_{K,\infty})(1-t)}[i\pi(B_{K,\infty}), |p|^s] e^{-i\pi(B_{K,\infty})t}\Psi \right\rangle dt. \end{aligned} \tag{3.89}$$

We write this in the form

$$\begin{aligned}&\left\langle |p|^s\Phi, e^{-i\pi(B_{K,\infty})}\Psi\right\rangle \\ &= \left\langle \Phi, e^{-i\pi(B_{K,\infty})}|p|^s\Psi\right\rangle + \int_0^1 \left\langle \Phi, e^{-i\pi(B_{K,\infty})(1-t)}[i\pi(B_{K,\infty}), |p|^s]e^{-i\pi(B_{K,\infty})t}\Psi\right\rangle dt. \quad (3.90)\end{aligned}$$

For given $\Phi \in D(|p|)$ this equation extends to all $\Psi \in D(H_0^{1/2})$. This follows from Fact 2 and the boundedness of $|p|^s(H_0+1)^{-1/2}$ and $[i\pi(B_{K,\infty}), |p|^s]e^{-i\pi(B_{K,\infty})t}(H_0+1)^{-1/2}$ which follows from Fact 1 and Lemma A.2.4. Equation (3.90) now shows that for all $\Psi \in D(H_0^{1/2})$

$$\Phi \mapsto \left\langle |p|^s\Phi, e^{-i\pi(B_{K,\infty})}\Psi\right\rangle \qquad (3.91)$$

is a bounded anti-linear functional defined on $D(|p|)$, which is a core of $|p|^s$. Since the operator $|p|^s$ is self-adjoint, it follows that $e^{-i\pi(B_{K,\infty})}\Psi \in D(|p|^s)$. □

The following two statements, Lemma 3.4.5 and Corollary 3.4.6, are the main tools for proving the important Theorem 3.4.7.

Lemma 3.4.5. *Assume* (ω), (v), *and let* $\left\|v\omega^{1/2}(1+k^2)^{-1}\right\| < \infty$, $\left\|v(1+k^2)^{\frac{s}{4}-1}\right\| < \infty$, *and* $\left\|v(1+k^2)^{\frac{s}{2}-1}\omega^{-1/2}\right\| < \infty$ *for some* $s \in [0,1]$. *Then, the limit*

$$\lim_{\Lambda\to\infty} (1+d\Gamma(\omega))^{-1/2}\,|p|^s U_{K,\Lambda}^*\Psi$$

exists for all $\Psi \in D(H_0^{1/2})$.

Proof. In this proof, we will suppress the index K. Note that $\left\|v\omega^{1/2}(1+k^2)^{-1}\right\| < \infty$ implies $\left\|\omega^{1/2}B_\infty\right\| < \infty$, $\left\|v(1+k^2)^{\frac{s}{4}-1}\right\| < \infty$ implies $\left\||k|^{s/2}B_{K,\infty}\right\| < \infty$, and $\left\|v(1+k^2)^{\frac{s}{2}-1}\omega^{-1/2}\right\| < \infty$ implies $\left\||k|^sB_{K,\infty}\omega^{-1/2}\right\| < \infty$. Furthermore, the property $\left\|v(1+k^2)^{\frac{s}{4}-1}\right\| < \infty$ implies that we are in Regime $(R3)$.

Let $s \in [0,1]$, $\Psi \in D(H_0^{1/2})$ and remark that we can write

$$U_{\Lambda_1}^* - U_{\Lambda_2}^* = \int_0^1 dt\; U_{\Lambda_1}^*(t)\; i\pi(B_{\Lambda_2} - B_{\Lambda_1})\; U_{\Lambda_2}^*(1-t), \qquad (3.92)$$

$$[|p|^s, U_\Lambda^*(t)] = \int_0^1 dr\; U_\Lambda^*(t(1-r))\; t[i\pi(B_\Lambda), |p|^s]\; U_\Lambda^*(tr), \qquad (3.93)$$

where $U^*_\Lambda(t) := e^{-i\pi(B_\Lambda)t}$. Note that $U^*_\Lambda(1) = U^*_\Lambda$. To prove the Lemma, it suffices to show that

$$(1 + d\Gamma(\omega))^{-1/2} |p|^s \left(U^*_{\Lambda_1} - U^*_{\Lambda_2}\right) (1 + H_0)^{-1/2} \tag{3.94}$$

goes to zero as $\Lambda_1, \Lambda_2 \to \infty$. Now, we write $U^*_{\Lambda_1} - U^*_{\Lambda_2}$ in the representation of Equation (3.92) and commute $|p|^s$ with the operators after it to obtain

$$\begin{aligned}
&\left\|(1 + d\Gamma(\omega))^{-1/2} |p|^s \left(U^*_{\Lambda_1} - U^*_{\Lambda_2}\right) (1 + H_0)^{-1/2}\right\| \\
&\le \left\|(1 + d\Gamma(\omega))^{-1/2} \left(U^*_{\Lambda_1} - U^*_{\Lambda_2}\right) |p|^s (1 + H_0)^{-1/2}\right\| \\
&\quad + \left\|\int_0^1 dt\ \ (1 + d\Gamma(\omega))^{-1/2} [|p|^s, U^*_{\Lambda_1}(t)] i\pi(B_{\Lambda_2} - B_{\Lambda_1}) U^*_{\Lambda_2}(1-t)(1 + H_0)^{-1/2}\right\| \\
&\quad + \left\|\int_0^1 dt\ \ (1 + d\Gamma(\omega))^{-1/2} U^*_{\Lambda_1}(t)[|p|^s, i\pi(B_{\Lambda_2} - B_{\Lambda_1})] U^*_{\Lambda_2}(1-t)(1 + H_0)^{-1/2}\right\| \\
&\quad + \left\|\int_0^1 dt\ \ (1 + d\Gamma(\omega))^{-1/2} U^*_{\Lambda_1}(t) i\pi(B_{\Lambda_2} - B_{\Lambda_1})[|p|^s, U^*_{\Lambda_2}(1-t)](1 + H_0)^{-1/2}\right\|.
\end{aligned} \tag{3.95}$$

For the first term, we use Lemma A.2.5 to get

$$\begin{aligned}
&\left\|(1 + d\Gamma(\omega))^{-1/2} \left(U^*_{\Lambda_1} - U^*_{\Lambda_2}\right) |p|^s (1 + H_0)^{-1/2}\right\| \\
&\le \left\|\left(U_{\Lambda_1} - U_{\Lambda_2}\right) (1 + d\Gamma(\omega))^{-1/2}\right\| \left\||p|^s (1 + H_0)^{-1/2}\right\| \\
&\le 2\left(1 + \frac{1}{m}\right) \|B_{\Lambda_1} - B_{\Lambda_2}\|,
\end{aligned} \tag{3.96}$$

which goes to zero as $\Lambda_1, \Lambda_2 \to \infty$.

Since $i\pi(f) = a(f) - a^*(f)$, the third part of (3.95) consists of two terms. For the term with the annihilation operator, we use Lemma B.1.4 and Lemma B.3.5 to estimate

$$\begin{aligned}
&\left\|\int_0^1 dt\ \ (1 + d\Gamma(\omega))^{-1/2} U^*_{\Lambda_1}(t) a([|p|^s, B_{\Lambda_2} - B_{\Lambda_1}]) U^*_{\Lambda_2}(1-t)(1 + H_0)^{-1/2}\right\| \\
&\le \int_0^1 dt\ \left\|\tfrac{|k|^s (B_{\Lambda_2} - B_{\Lambda_1})}{\sqrt{\omega}}\right\| \left\|d\Gamma(\omega)^{1/2} U^*_{\Lambda_2}(1-t)(1 + H_0)^{-1/2}\right\| \\
&\le 2\left(1 + \left\|\omega^{1/2} B_\infty\right\|^2\right) \left\|\tfrac{|k|^s (B_{\Lambda_2} - B_{\Lambda_1})}{\sqrt{\omega}}\right\|,
\end{aligned} \tag{3.97}$$

which goes to zero as $\Lambda_1, \Lambda_2 \to \infty$. In this calculation, we also took the supremum over all $t \in [0, 1]$. The term with the creation operator yields, using Corollary B.3.6 and again

Lemma B.1.4,

$$\begin{aligned}
&\left\| \int_0^1 dt \ \left(1 + d\Gamma(\omega)\right)^{-1/2} U^*_{\Lambda_1}(t) a^*([|p|^s, B_{\Lambda_2} - B_{\Lambda_1}]) U^*_{\Lambda_2}(1-t)(1+H_0)^{-1/2} \right\| \\
&\leq \int_0^1 dt \ \left\| \left(1 + d\Gamma(\omega)\right)^{-1/2} U^*_{\Lambda_1}(t) \left(1 + d\Gamma(\omega)\right)^{1/2} \right\| \\
&\qquad \cdot \left\| \left(1 + d\Gamma(\omega)\right)^{-1/2} a^*([|p|^s, B_{\Lambda_2} - B_{\Lambda_1}]) U^*_{\Lambda_2}(1-t)(1+H_0)^{-1/2} \right\| \\
&\leq \left(1 + \|\omega^{1/2} B_\infty\|\right) \int_0^1 dt \ \left\| a([|p|^s, B_{\Lambda_2} - B_{\Lambda_1}]) \left(1 + d\Gamma(\omega)\right)^{-1/2} \right\| \\
&\leq \left(1 + \|\omega^{1/2} B_\infty\|\right) \left\| \frac{|k|^s (B_{\Lambda_2} - B_{\Lambda_1})}{\sqrt{\omega}} \right\|,
\end{aligned} \tag{3.98}$$

which goes to zero as $\Lambda_1, \Lambda_2 \to \infty$. Again, we took the supremum over all $t \in [0,1]$.

Using (3.93), we get for the second term of (3.95)

$$\begin{aligned}
&\Big\| \int_0^1 dt \int_0^1 dr \ t \ \left(1 + d\Gamma(\omega)\right)^{-1/2} U^*_{\Lambda_1}(t(1-r)) [i\pi(B_{\Lambda_1}), |p|^s] U^*_{\Lambda_1}(tr) \\
&\qquad\qquad\qquad i\pi(B_{\Lambda_2} - B_{\Lambda_1}) U^*_{\Lambda_2}(1-t)(1+H_0)^{-1/2} \Big\|
\end{aligned} \tag{3.99}$$

We split up the first field operator and get for the term with the creation operator

$$\begin{aligned}
&\Big\| \int_0^1 dt \int_0^1 dr \ t \ \left(1 + d\Gamma(\omega)\right)^{-1/2} U^*_{\Lambda_1}(t(1-r)) a^*([|p|^s, B_{\Lambda_1}]) U^*_{\Lambda_1}(tr) \\
&\qquad\qquad\qquad i\pi(B_{\Lambda_2} - B_{\Lambda_1}) U^*_{\Lambda_2}(1-t)(1+H_0)^{-1/2} \Big\| \\
&\leq \int_0^1 dt \int_0^1 dr \ t \ \left\| \left(1 + d\Gamma(\omega)\right)^{-1/2} U^*_{\Lambda_1}(t(1-r)) \left(1 + d\Gamma(\omega)\right)^{1/2} \right\| \\
&\qquad \cdot \left\| a([|p|^s, B_{\Lambda_1}]) \left(1 + d\Gamma(\omega)\right)^{-1/2} \right\| \cdot \left\| i\pi(B_{\Lambda_2} - B_{\Lambda_1}) U^*_{\Lambda_2}(1-t)(1+H_0)^{-1/2} \right\| \\
&\leq \left(1 + \|\omega^{1/2} B_\infty\|\right) \ \left\| \frac{|k|^s B_\infty}{\sqrt{\omega}} \right\| \ 2\|B_{\Lambda_2} - B_{\Lambda_1}\| \ \left(1 + \|B_\infty\|\right),
\end{aligned} \tag{3.100}$$

which goes to zero as $\Lambda_1, \Lambda_2 \to \infty$. Here, we used Lemma B.1.4, Corollary A.2.2 and two times Corollary B.3.6. We also took the supremum over all $t, r \in [0,1]$. Using the identity

$$U^*_{\Lambda_1}(tr)\pi(B_{\Lambda_2} - B_{\Lambda_1}) = \left(\pi(B_{\Lambda_2} - B_{\Lambda_1}) - 2tr \,\mathrm{Im}\, \langle B_{\Lambda_2} - B_{\Lambda_1}, B_{\Lambda_1} \rangle\right) U^*_{\Lambda_1}(tr) \tag{3.101}$$

in the term of (3.99) with the annihilation operator, we obtain for it

$$\begin{aligned}\Big\| \int_0^1 dt \int_0^1 dr\ t\ \ (1+d\Gamma(\omega))^{-1/2}\, U^*_{\Lambda_1}(t(1-r))\Big\{ a([|p|^s, B_{\Lambda_1}])a(B_{\Lambda_2}-B_{\Lambda_1}) \\ - a([|p|^s, B_{\Lambda_1}])a^*(B_{\Lambda_2}-B_{\Lambda_1}) - 2itr\,\mathrm{Im}\,\langle B_{\Lambda_2}-B_{\Lambda_1}, B_{\Lambda_1}\rangle a([|p|^s, B_{\Lambda_1}])\Big\} \\ U^*_{\Lambda_1}(tr)U^*_{\Lambda_2}(1-t)(1+H_0)^{-1/2}\Big\|\end{aligned} \tag{3.102}$$

what leads to three remaining terms to estimate. The third term of (3.102) can be estimated by

$$\begin{aligned} & 2|\langle B_{\Lambda_2}-B_{\Lambda_1}, B_{\Lambda_1}\rangle| \int_0^1 dt \int_0^1 dr\ t^2 r \\ & \quad \Big\|(1+d\Gamma(\omega))^{-1/2}\, U^*_{\Lambda_1}(t(1-r))a([|p|^s, B_{\Lambda_1}])U^*_{\Lambda_1}(tr)U^*_{\Lambda_2}(1-t)(1+H_0)^{-1/2}\Big\| \\ & \le 2\|B_{\Lambda_2}-B_{\Lambda_1}\|\ \|B_\infty\| \int_0^1 dt \int_0^1 dr\ t^2 r\ \left\|\frac{|k|^s B_\infty}{\sqrt{\omega}}\right\|\ \Big\|d\Gamma(\omega)^{1/2}U^*_{\Lambda_1}(tr)\,(1+d\Gamma(\omega))^{-1/2}\Big\| \\ & \qquad \Big\|(1+d\Gamma(\omega))^{1/2}\, U^*_{\Lambda_2}(1-t)(1+H_0)^{-1/2}\Big\| \\ & \le 2\|B_{\Lambda_2}-B_{\Lambda_1}\|\ \|B_\infty\|\ \left\|\frac{|k|^s B_\infty}{\sqrt{\omega}}\right\|\ (1+\|\omega^{1/2}B_\infty\|)^2, \end{aligned} \tag{3.103}$$

which goes to zero as $\Lambda_1, \Lambda_2 \to \infty$. We used Lemma B.1.4 again and two times Corollary B.3.6 and also took the supremum over all $t, r \in [0,1]$. The first term of (3.102) can be estimated by

$$\begin{aligned} & \int_0^1 dt \int_0^1 dr\ t\ \Big\|\ (1+d\Gamma(\omega))^{-1/2}\, U^*_{\Lambda_1}(t(1-r))a([|p|^s, B_{\Lambda_1}])a(B_{\Lambda_2}-B_{\Lambda_1}) \\ & \qquad U^*_{\Lambda_1}(tr)U^*_{\Lambda_2}(1-t)(1+H_0)^{-1/2}\Big\| \\ & \le \int_0^1 dt \int_0^1 dr\ t\ \Big\|(1+d\Gamma(\omega))^{-1/2}\, U^*_{\Lambda_1}(t(1-r))\,(1+d\Gamma(\omega))^{1/2}\Big\| \\ & \quad \Big\|(1+d\Gamma(\omega))^{-1/2}\, a([|p|^s, B_{\Lambda_1}])a(B_{\Lambda_2}-B_{\Lambda_1})\,(1+d\Gamma(\omega))^{-1/2}\Big\| \\ & \quad \Big\|(1+d\Gamma(\omega))^{1/2}\, U^*_{\Lambda_1}(tr)\,(1+d\Gamma(\omega))^{-1/2}\Big\|\ \Big\|(1+d\Gamma(\omega))^{1/2}\, U^*_{\Lambda_2}(1-t)(1+H_0)^{-1/2}\Big\| \\ & \le \sqrt{1+\frac{1}{m}}\ \Big(1+\|\omega^{1/2}B_\infty\|\Big)^3 \left\|\frac{|k|^s B_\infty}{\sqrt{\omega}}\right\|\ \|B_{\Lambda_2}-B_{\Lambda_1}\|, \end{aligned} \tag{3.104}$$

which goes to zero as $\Lambda_1, \Lambda_2 \to \infty$. We used Lemma B.1.8 and three times Corollary B.3.6 and also took the supremum over all $t, r \in [0, 1]$. The second term of (3.102) can be estimated by

$$\begin{aligned}
&\int_0^1 dt \int_0^1 dr\ t \left\| (1 + d\Gamma(\omega))^{-1/2}\, U_{\Lambda_1}^*(t(1-r)) a([|p|^s, B_{\Lambda_1}]) a^*(B_{\Lambda_2} - B_{\Lambda_1}) \right. \\
&\qquad\qquad\qquad \left. U_{\Lambda_1}^*(tr) U_{\Lambda_2}^*(1-t)(1+H_0)^{-1/2} \right\| \\
&\leq \int_0^1 dt \int_0^1 dr\ t \left\| (1 + d\Gamma(\omega))^{-1/2}\, U_{\Lambda_1}^*(t(1-r))\, (1 + d\Gamma(\omega))^{1/2} \right\| \\
&\quad \left\| (1 + d\Gamma(\omega))^{-1/2}\, a([|p|^s, B_{\Lambda_1}]) a^*(B_{\Lambda_2} - B_{\Lambda_1})\, (1 + d\Gamma(\omega))^{-1/2} \right\| \\
&\quad \left\| (1 + d\Gamma(\omega))^{1/2}\, U_{\Lambda_1}^*(tr)\, (1 + d\Gamma(\omega))^{-1/2} \right\| \left\| (1 + d\Gamma(\omega))^{1/2}\, U_{\Lambda_2}^*(1-t)(1+H_0)^{-1/2} \right\| \\
&\leq \frac{1}{\sqrt{m}}\, (1 + \|\omega^{1/2} B_\infty\|)^3 \left\| \frac{|k|^s B_\infty}{\sqrt{\omega}} \right\| \|B_{\Lambda_2} - B_{\Lambda_1}\| \\
&\quad + (1 + \|\omega^{1/2} B_\infty\|)^3 \left\| |k|^{s/2} B_\infty \right\| \left\| |k|^{s/2} (B_{\Lambda_2} - B_{\Lambda_1}) \right\|,
\end{aligned} \tag{3.105}$$

which goes to zero as $\Lambda_1, \Lambda_2 \to \infty$. Here, we used Lemma B.1.9, three times Corollary B.3.6 and took the supremum over all $t, r \in [0, 1]$ again. This concludes that the second term of (3.95) goes to zero as $\Lambda_1, \Lambda_2 \to \infty$.

It remains to consider the fourth term of (3.95). But here, we can also consider the adjoint operator and show that it goes to zero as $\Lambda_1, \Lambda_2 \to \infty$ by the same estimates that were used for the second term of (3.95). This completes the proof of this lemma. ☐

Corollary 3.4.6. *Assume* (ω), (v), *and let* $\|v\omega^{1/2}(1+k^2)^{-1}\| < \infty$, $\left\| v(1+k^2)^{\frac{s}{4}-1} \right\| < \infty$, *and* $\left\| v(1+k^2)^{\frac{s}{2}-1}\omega^{-1/2} \right\| < \infty$ *for some* $s \in [0, 1]$. *Furthermore, suppose that* $U_{K,\infty}^* \Psi \in D(|p|^s)$ *for a* $\Psi \in D(H_0^{1/2})$. *Then,*

$$(1 + d\Gamma(\omega))^{-1/2}\, |p|^s U_{K,\Lambda}^* \Psi \to (1 + d\Gamma(\omega))^{-1/2}\, |p|^s U_{K,\infty}^* \Psi \qquad (\Lambda \to \infty).$$

Proof. Let $R := (1 + d\Gamma(\omega))^{-1/2}$ and note that $R|p|^s \subset |p|^s R$. We know that $U_{K,\Lambda}^* \Psi \to U_{K,\infty}^* \Psi$ as $\Lambda \to \infty$ and that $U_{K,\Lambda}^* \Psi \in D(H_0^{1/2}) \subset D(|p|^s)$ by Lemma B.3.8. It follows that

$$RU_{K,\Lambda}^* \Psi \to RU_{K,\infty}^* \Psi \qquad (\Lambda \to \infty), \tag{3.106}$$

where $\lim_{\Lambda\to\infty} |p|^s RU^*_{K,\Lambda}\Psi$ exists, by Lemma 3.4.5. Since $|p|^s$ is a closed operator, we conclude that $RU^*_{K,\infty}\Psi \in D(|p|^s)$ and that

$$\lim_{\Lambda\to\infty} |p|^s RU^*_{K,\Lambda}\Psi = |p|^s RU^*_{K,\infty}\Psi. \tag{3.107}$$

The corollary now follows from $R|p|^s \subset |p|^s R$ and from the assumptions that $U^*_{K,\infty}\Psi \in D(|p|^s)$. □

Theorem 3.4.7. *Assume* (ω), (v), *and let* $\left\|v\omega^{1/2}(1+k^2)^{-1}\right\| < \infty$, $\left\|v(1+k^2)^{\frac{s}{4}-1}\right\| < \infty$, *and* $\left\|v(1+k^2)^{\frac{s}{2}-1}\omega^{-1/2}\right\| < \infty$, *but* $\left\|v(1+k^2)^{\frac{s}{2}-1}\right\| = \infty$ *for some* $s \in [0,1]$. *Then*

$$U^*_{K,\infty} D(H_0^{1/2}) \cap D(|p|^s) = \{0\}.$$

Proof. In this proof, we will suppress the index K. Note that $\left\|v\omega^{1/2}(1+k^2)^{-1}\right\| < \infty$ implies $\left\|\omega^{1/2}B_\infty\right\| < \infty$, $\left\|v(1+k^2)^{\frac{s}{4}-1}\right\| < \infty$ implies $\left\||k|^{s/2}B_{K,\infty}\right\| < \infty$, and $\left\|v(1+k^2)^{\frac{s}{2}-1}\omega^{-1/2}\right\| < \infty$ implies $\left\||k|^s B_{K,\infty}\omega^{-1/2}\right\| < \infty$. Furthermore, the property $\left\|v(1+k^2)^{\frac{s}{4}-1}\right\| < \infty$ implies ($R3$). The assumption that $\left\|v(1+k^2)^{\frac{s}{2}-1}\right\| = \infty$ is equivalent to $\left\||k|^s B_\infty\right\| = \infty$.

Let $\Psi \in D(H_0^{1/2})$ and suppose that $U^*_\infty\Psi \in D(|p|^s)$ for some $s \in [0,1]$ for which the hypotheses of the theorem are satisfied. Then, of course, $\left\||p|^s U^*_\infty\Psi\right\| < \infty$. We will show that this is not true unless $\Psi = 0$. Our proof is based on the identity

$$\left\||p|^s U^*_\infty\Psi\right\| = \lim_{\varepsilon\to 0}\lim_{\Lambda\to\infty} \left\|(1+\varepsilon d\Gamma(\omega))^{-1/2}|p|^s U^*_\Lambda\Psi\right\|, \tag{3.108}$$

which follows from Corollary 3.4.6 and monotone convergence. We are now going to write the vector $(1+\varepsilon d\Gamma(\omega))^{-1/2}|p|^s U^*_\Lambda\Psi$ as sum of terms, where all but one have a norm that is uniformly bounded in ε and Λ. We call them "good terms". The norm of the remaining term diverges unless $\Psi = 0$. This will complete the proof.

Using

$$|p|^s U^*_\Lambda = U^*_\Lambda |p|^s + \int_0^1 dt\; U^*_\Lambda(1-t)\;[i\pi(B_\Lambda), |p|^s]\; U^*_\Lambda(t), \tag{3.109}$$

where $U^*_\Lambda(t) := e^{-i\pi(B_\Lambda)t}$ (note that $U^*_\Lambda(1) = U^*_\Lambda$ and $i\pi(B_\Lambda) = a(B_\Lambda) - a^*(B_\Lambda)$), we obtain three summands:

$$
\begin{aligned}
&(1+\varepsilon d\Gamma(\omega))^{-1/2}|p|^s U_\Lambda^*\Psi \\
&= (1+\varepsilon d\Gamma(\omega))^{-1/2}U_\Lambda^*|p|^s\Psi + (1+\varepsilon d\Gamma(\omega))^{-1/2}\int_0^1 dt\ U_\Lambda^*(1-t)a([B_\Lambda,|p|^s])U_\Lambda^*(t)\Psi \\
&\qquad - (1+\varepsilon d\Gamma(\omega))^{-1/2}\int_0^1 dt\ U_\Lambda^*(1-t)a^*([B_\Lambda,|p|^s])U_\Lambda^*(t)\Psi. \qquad (3.110)
\end{aligned}
$$

The first two terms are good terms. Indeed, the norm of the first summand can directly be estimated by $\||p|^s\Psi\|$, which is finite. Using Lemmas B.1.4 and B.3.5 the norm of the second term can be estimated as follows

$$
\begin{aligned}
&\left\|(1+\varepsilon d\Gamma(\omega))^{-1/2}\int_0^1 dt\ U_\Lambda^*(1-t)a([B_\Lambda,|p|^s])U_\Lambda^*(t)\Psi\right\| \\
&\leq \int_0^1 dt\ \|a([B_\Lambda,|p|^s])U_\Lambda^*(t)\Psi\| \leq \int_0^1 dt\ \left\|\frac{|k|^s B_\Lambda}{\sqrt{\omega}}\right\| \left\|d\Gamma(\omega)^{1/2}U_\Lambda^*(t)\Psi\right\| \\
&\leq \left\|\frac{|k|^s B_\infty}{\sqrt{\omega}}\right\| \cdot 2\left(1+\left\|\omega^{1/2}B_\infty\right\|^2\right)\ \left\|(1+d\Gamma(\omega))^{1/2}\Psi\right\|, \qquad (3.111)
\end{aligned}
$$

where the supremum with respect to $t\in[0,1]$ and $\Lambda>0$ was taken in the last step.

It remains to show the divergence of the norm of the third term of (3.110). Using that $e^{ikx}|p|^s e^{-ikx} = |p-k|^s$, this term reads

$$
(1+\varepsilon d\Gamma(\omega))^{-\frac{1}{2}}\int_0^1 dt\ \int dk\ B_{\Lambda,x}(k)U_\Lambda^*(1-t)a_k^* A_{p,k}U_\Lambda^*(t)\Psi, \qquad (3.112)
$$

where we defined

$$
A_{p,k} := |p-k|^s - |p|^s \qquad (3.113)
$$

for short. Recall, that

$$
|A_{p,k}| = ||p-k|^s - |p|^s| \leq |k|^s \qquad (3.114)
$$

for $p,k\in\mathbb{R}^d$ and $s\in[0,1]$, and that

$$
U_\Lambda^*(1-t)a_k^*U_\Lambda(1-t) = a_k^* - (1-t)\overline{B_{\Lambda,x}(k)}. \qquad (3.115)
$$

Identity (3.115) is used now to commute $U_\Lambda^*(1-t)$ and a_k^*. Then, (3.112) becomes

$$\begin{aligned}&(1+\varepsilon d\Gamma(\omega))^{-\frac{1}{2}}\int_0^1 dt\int dk\ B_{\Lambda,x}(k)a_k^*U_\Lambda^*(1-t)A_{p,k}U_\Lambda^*(t)\Psi\\&-(1+\varepsilon d\Gamma(\omega))^{-\frac{1}{2}}\int_0^1 dt\int dk(1-t)|B_\Lambda(k)|^2U_\Lambda^*(1-t)A_{p,k}U_\Lambda^*(t)\Psi.\end{aligned}\tag{3.116}$$

The second term of (3.116) is another good term, due to (3.114), its norm is bounded by

$$\begin{aligned}&\int_0^1 dt\ (1-t)\int dk\ |B_\Lambda(k)|^2\|A_{p,k}U_\Lambda^*(t)\Psi\|\\&\leq\int_0^1 dt\ (1-t)\int dk\ |k|^s|B_\Lambda(k)|^2\|U_\Lambda^*(t)\Psi\|\leq\left\||k|^{s/2}B_\infty\right\|^2\|\Psi\|,\end{aligned}\tag{3.117}$$

which is finite by the assumptions of the theorem.

We continue to analyze the first term of (3.116) and show its divergence. For short, define $\eta_{\Lambda,p,k}(t):=U_\Lambda^*(1-t)A_{p,k}U_\Lambda^*(t)\Psi$. We consider the squared norm of this first term of (3.116), write it as an inner product, commute the ladder oprators, and use the pull-through formulas for $d\Gamma(\omega)$. This calculation reads

$$\begin{aligned}&\left\|(1+\varepsilon d\Gamma(\omega))^{-1/2}\int_0^1 dt\ \int dk\ B_{\Lambda,x}(k)a_k^*\eta_{\Lambda,p,k}(t)\right\|^2\\&=\int_0^1 dt\int_0^1 dt'\int dk\int dk'\ \overline{B_{\Lambda,x}(k)}B_{\Lambda,x}(k')\big\langle a_k^*\eta_{\Lambda,p,k}(t),(1+\varepsilon d\Gamma(\omega))^{-1}a_{k'}^*\eta_{\Lambda,p,k'}(t')\big\rangle\\&=\int_0^1 dt\int_0^1 dt'\int dk\int dk'\ \overline{B_{\Lambda,x}(k)}B_{\Lambda,x}(k')\\&\qquad\big\langle a_{k'}\eta_{\Lambda,p,k}(t),(1+\varepsilon(d\Gamma(\omega)+\omega(k)+\omega(k')))^{-1}a_k\eta_{\Lambda,p,k'}(t')\big\rangle\\&\quad+\int_0^1 dt\int_0^1 dt'\int dk\ |B_\Lambda(k)|^2\big\langle\eta_{\Lambda,p,k}(t),(1+\varepsilon(d\Gamma(\omega)+\omega(k)))^{-1}\eta_{\Lambda,p,k}(t')\big\rangle\\&=:\alpha+\beta,\end{aligned}\tag{3.118}$$

where α and β denotes the two resulting summands.

The term α is a good term as we now show: by Cauchy-Schwarz

$$|\alpha| \leq \int_0^1 dt \int_0^1 dt' \int dk \int dk' \; |B_\Lambda(k)| \; |B_\Lambda(k')| \; \|a_{k'}\eta_{\Lambda,p,k}(t)\| \; \left\|a_k \eta_{\Lambda,p,k'}(t')\right\|. \tag{3.119}$$

By definition of $\eta_{\Lambda,p,k}(t)$, by (3.114) and by the identity

$$U_\Lambda(t) a_k U_\Lambda^*(t) = a_k + t \cdot B_{\Lambda,x}(k), \tag{3.120}$$

used twice, we obtain

$$\begin{aligned} \|a_{k'}\eta_{\Lambda,p,k}(t)\| &\leq |k|^s \|a_{k'} U_\Lambda^*(t)\Psi\| + |k|^s |B_\Lambda(k')| \; \|\Psi\| \\ &\leq |k|^s \|a_{k'}\Psi\| + |k|^s |B_\Lambda(k')| \; \|\Psi\|. \end{aligned} \tag{3.121}$$

We now multiply and divide $\|a_{k'}\Psi\|$ by $\sqrt{\omega(k')}$, analogously for the momentum k, and insert (3.121) into (3.119). Using Cauchy-Schwarz we obtain

$$\begin{aligned} |\alpha| \leq &\left\| \frac{|k|^s B_\Lambda}{\sqrt{\omega}} \right\|^2 \left\| d\Gamma(\omega)^{1/2}\Psi \right\|^2 + \left\| |k|^{s/2} B_\Lambda \right\|^4 \|\Psi\|^2 \\ &+ 2 \left\| \frac{|k|^s B_\Lambda}{\sqrt{\omega}} \right\| \; \left\| |k|^{s/2} B_\Lambda \right\|^2 \; \left\| d\Gamma(\omega)^{1/2}\Psi \right\| \; \|\Psi\|. \end{aligned} \tag{3.122}$$

This is finite by the assumptions of the theorem and the remarks of the beginning of the proof. Thus, α from Equation (3.118) is a good term and the divergence has to be in second term of (3.118), we called it β.

It remains to show that β diverges. To this end, let

$$R_\varepsilon(k) := \left(1 + \varepsilon \left(d\Gamma(\omega) + \omega(k)\right)\right)^{-1/2} \tag{3.123}$$

for short. Then,

$$\begin{aligned} \beta &= \int dk \; |B_\Lambda(k)|^2 \left\| R_\varepsilon(k) \int_0^1 dt \; U_\Lambda^*(1-t) A_{p,k} U_\Lambda^*(t)\Psi \right\|^2 \\ &= \int dk \; |B_\Lambda(k)|^2 \left\| R_\varepsilon(k) A_{p,k} U_\Lambda^*\Psi + R_\varepsilon(k) \int_0^1 dt \; \left[U_\Lambda^*(1-t), A_{p,k} \right] U_\Lambda^*(t)\Psi \right\|^2. \end{aligned} \tag{3.124}$$

Note, that $U_\Lambda^*(1-t)U_\Lambda^*(t) = U_\Lambda^*$ and, therefore, the first summand in the norm does not depend on t anymore. If we define

$$\beta_1 := \int dk \; |B_\Lambda(k)|^2 \|R_\varepsilon(k) A_{p,k} U_\Lambda^*\Psi\|^2, \tag{3.125}$$

$$\beta_2 := \int dk \; |B_\Lambda(k)|^2 \left\| R_\varepsilon(k) \int_0^1 dt \; \left[U_\Lambda^*(1-t), A_{p,k} \right] U_\Lambda^*(t)\Psi \right\|^2, \tag{3.126}$$

then $\beta \geq \frac{1}{2}\beta_1 - 2\beta_2$. To complete the proof, we show that β_2 is a good term, while β_1 diverges.

For β_2, using Lemma B.1.6, Corollary B.3.6, and a representation of the commutator analogously to (3.109), we find

$$\begin{aligned}
\beta_2 &= \int d\ell\ |B_\Lambda(\ell)|^2 \left\| R_\varepsilon(\ell) \int_0^1 dt \int_0^1 dr\ (1-t) U_\Lambda^*(1-t) \Big[A_{p,\ell}, i\pi(B_\Lambda) \Big] U_\Lambda^*(1-r+tr)\Psi \right\|^2 \\
&\leq \int d\ell\ |B_\Lambda(\ell)|^2 \int_0^1 dt \int_0^1 dr\ (1-t) \|\pi([A_{p,\ell}, B_\Lambda]) U_\Lambda^*(1-r+tr)\Psi\| \\
&\leq 16 \int d\ell\ |B_\Lambda(\ell)|^2 |\ell|^s \left\| |k|^{\frac{s}{2}} B_\Lambda \right\|^2 \left(\int_0^1 dt \int_0^1 dr (1-t) \left\| (N+1)^{\frac{1}{2}} U_\Lambda^*(1-r+tr)\Psi \right\| \right)^2 \\
&\leq 8 \left\| |k|^{\frac{s}{2}} B_\Lambda \right\|^4 \sup_{q\in[0,1]} \left\| (N+1)^{\frac{1}{2}} U_\Lambda^*(q) (N+1)^{-\frac{1}{2}} \right\|^2 \left\| (N+1)^{\frac{1}{2}} \Psi \right\|^2 \\
&\leq 8 \left\| |k|^{\frac{s}{2}} B_\Lambda \right\|^4 (1 + \|B_\infty\|)^2 \left\| (N+1)^{\frac{1}{2}} \Psi \right\|^2. \qquad (3.127)
\end{aligned}$$

This is finite by the hypotheses on B_Λ and by Assumption (ω). In the last calculation, we used the integration variable ℓ instead of k for avoiding misunderstandings in view of the application of Lemma B.1.6.

It remains to analyze β_1 from Equation (3.125). Using

$$(|p-k|^s - |p|^s)^2 \geq (|k|^{2s} - 4|p|^s|k|^s) \qquad (s \in [0,1]), \qquad (3.128)$$

we obtain

$$\begin{aligned}
\beta_1 &= \int dk |B_\Lambda(k)|^2 \Big\langle R_\varepsilon(k) U_\Lambda^*\Psi, A_{p,k}^2 R_\varepsilon(k) U_\Lambda^*\Psi \Big\rangle \\
&\geq \int dk |k|^{2s} |B_\Lambda(k)|^2 \|R_\varepsilon(k) U_\Lambda^*\Psi\|^2 - 4 \int dk |k|^s |B_\Lambda(k)|^2 \left\| R_\varepsilon(k) |p|^{\frac{s}{2}} U_\Lambda^*\Psi \right\|^2. \qquad (3.129)
\end{aligned}$$

The second term of (3.129) is a good term, because, by Fatou, Corollary 3.4.6, and monotonic convergence

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \limsup_{\Lambda\to\infty} \int dk\ |k|^s |B_\Lambda(k)|^2 \left\| R_\varepsilon(k) |p|^{s/2} U_\Lambda^*\Psi \right\|^2 \\
&\leq \limsup_{\varepsilon\to 0} \int dk\ |k|^s |B_\infty(k)|^2 \left\| R_\varepsilon(k) |p|^{s/2} U_\infty^*\Psi \right\|^2 \\
&\leq \left\| |k|^{s/2} B_\infty \right\|^2 \left\| |p|^{s/2} U_\infty^*\Psi \right\|^2, \qquad (3.130)
\end{aligned}$$

which is finite by our assumptions.

Finally, we look at the first term of (3.129). For any $\Lambda_0 \leq \Lambda$, by Fatou, Corollary 3.4.6, and monotonic convergence

$$\begin{aligned}
&\liminf_{\varepsilon\to 0}\ \liminf_{\Lambda\to\infty} \int dk|k|^{2s}|B_\Lambda(k)|^2\|R_\varepsilon(k)U_\Lambda^*\Psi\|^2 \\
&\geq \liminf_{\varepsilon\to 0}\ \liminf_{\Lambda\to\infty} \int\limits_{|k|\leq\Lambda_0} dk\ |k|^{2s}|B_\infty(k)|^2\|R_\varepsilon(k)U_\Lambda^*\Psi\|^2 \\
&\geq \liminf_{\varepsilon\to 0} \int\limits_{|k|\leq\Lambda_0} dk\ |k|^{2s}|B_\infty(k)|^2\|R_\varepsilon(k)U_\infty^*\Psi\|^2 \\
&= \||k|^s B_{\Lambda_0}\|^2\|\Psi\|^2.
\end{aligned} \tag{3.131}$$

Since $\||k|^s B_{\Lambda_0}\|$ diverges as $\Lambda_0 \to \infty$, we conclude that $\|\Psi\| = 0$ and the proof is complete. □

3.4.3. Applications

Finally, we apply the results of this section to the Nelson Hamiltonian. We have

$$v(k) = \frac{1}{\sqrt{\omega(k)}} \quad \text{with} \quad \omega(k) = \sqrt{m^2 + k^2} \quad \text{and} \quad m > 0, \tag{3.132}$$

such that we are for three space dimensions in the regime $(R3)$. It is $\left\|kB_{K,\infty}\omega^{-1/4}\right\| < \infty$ such that, from Theorem 3.4.3, there exists the norm-resolvent limit H of $H_\Lambda + E_\Lambda$ with form domain $U_{K,\infty}^* D(H_0^{1/2})$ for K sufficiently large. Here, E_Λ is a renormalization constant. From the mapping properties of $U_{K,\infty}^*$, see Lemma B.3.5, we know that $U_{K,\infty}^* D(d\Gamma(\omega)^{1/2}) = D(d\Gamma(\omega)^{1/2})$ such that $U_{K,\infty}^* D(H_0^{1/2}) \subset D(d\Gamma(\omega)^{1/2})$.

We see that $\left\|\omega^{1/2}B_{K,\infty}\right\| < \infty$ and $\left\||k|^{s/2}B_{K,\infty}\right\| < \infty$ and $\left\||k|^s B_{K,\infty}\omega^{-1/2}\right\| < \infty$ for $s \in [0,1]$. But $\||k|^s B_{K,\infty}\| < \infty$ only for $s \in [0,1)$, $\||k|B_{K,\infty}\| = \infty$. In view of Theorems 3.4.4 and 3.4.7, this leads to

$$U_{K,\infty}^* D(H_0^{1/2}) \subset \bigcap_{0\leq s<1} D(|p|^s) \cap D(d\Gamma(\omega)^{1/2}), \tag{3.133}$$

$$U_{K,\infty}^* D(H_0^{1/2}) \cap D(|p|) = \{0\}. \tag{3.134}$$

Since $D(H)$ is trivially a subset of its form domain $U_{K,\infty}^* D(H_0^{1/2})$, we see especially that

$$D(H) \cap D(H_0^{1/2}) = \{0\}. \tag{3.135}$$

4. Renormalization of a class of generalized Nelson Hamiltonians

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Abstract

In this chapter we consider a system of a quantized particle coupled to a bosonic field with general dispersion relations in both subsystems, which allow us to discuss a large class of Hamiltonians. This class contains, for example, the Nelson and the Fröhlich Hamiltonians, but also a pseudorelativistic version of the Nelson model. Our main goal is to establish a criterion for existence of the norm resolvent limit of the ultraviolet regularized Hamiltonian, possibly after a suitable renormalization. The method by using a Born series also allows us to make statements about the domain of this norm resolvent limit.

4.1. Introduction

In this chapter, we study a very general class of operators which describe the interaction of a single particle with a quantized bosonic field based on the formal expression

$$\Omega(p) + d\Gamma(\omega) + \int_{\mathbb{R}^d} \left[\overline{v(k)}e^{ikx}a_k + v(k)e^{-ikx}a_k^*\right] dk. \tag{4.1}$$

The term $d\Gamma(\omega)$, stands for the free energy of the bosonic field with the dispersion relation $\omega(k) \geq m$, $m > 0$ is the mass of a boson. This operator acts in the symmetric Fock space over $L^2(\mathbb{R}^d)$. The free Hamiltonian of the single particle is given by an arbitrary dispersion relation $\Omega(p) \geq 0$, where $p := -i\nabla$ stands for the momentum operator of the particle.

The interaction of the two subsystems is given by the third summand which consists of the creation and annihilation operators and the form-factor v which is, in general, not square-integrable. This leads to the problem that the expression (4.1), as it stands, is not a well-defined operator. To regularize this expression and to define the operator sensibly, we introduce an ultraviolet cutoff, which we call Λ.

Popular examples of models of the form (4.1) are the Polaron model by Fröhlich, Pelzer and Zienau [5], where $d = 3$, $\Omega(p) = p^2$, $\omega(k) = 1$, and $v(k) = |k|^{-1}$ and the Nelson

Hamiltonian [1], where $d = 3$, $\Omega(p) = p^2$, $\omega(k) = \sqrt{m^2 + k^2}$, and $v(k) = \omega(k)^{-1/2}$. But furthermore, choosing for example $\Omega(p) = \sqrt{\mu^2 + p^2}$ with a mass $\mu \geq 0$, we are also able to study the pseudorelativistic version of the Nelson Hamiltonian as it was studied by Fröhlich [7] and Sloan [20].

We denote the ultraviolet regularized Hamiltonian by H_Λ and we will add a suitable renormalization constant E_Λ. Then, $H_0 := \Omega(p) + d\Gamma(\omega)$ is the free part of the Hamiltonian. The operator H_0 is non-negative and self-adjoint on $D(H_0) := D(\Omega(p)) \cap D(d\Gamma(\omega))$.

The first question we study in this chapter is under which assumptions on Ω, ω, and v the operator $H_\Lambda + E_\Lambda$ has a norm-resolvent limit as $\Lambda \to \infty$. If such a norm-resolvent limit exists, we also want to study properties of its domain.

We already treated questions like these before with respect to the Fröhlich Hamiltonian, see Chapter 2 and the article [23], or the Nelson model, see Chapter 3. In these projects, we could use the Gross transform to study these problems. But this unitary transform is only helpful in the case of the quadratic dispersion $\Omega(p) = p^2$. In this chapter, we use a resolvent expansion following Hepp [21], who studied ultraviolet divergent Lee models, and Eckmann [22], who studied the pseudo-relativistic massive Nelson model in second quantization. With this procedure, we arrive at very general conditions for the renormalization that also include some popular models of the interaction of a particle with a bosonic field.

Using this resolvent expansion, we are also able to study some domain properties of the norm-resolvent limit of $H_\Lambda + E_\Lambda$, we call it H. Applying our results to the Fröhlich Hamiltonian, we can reproduce the inclusion

$$D(H) \subset \bigcap_{1 \leq s < 3/2} D(|p|^s) \cap D(N), \tag{4.2}$$

which we already know from our work [23] and Chapter 2. Applied to the Nelson model in three dimensions, it leads to

$$D(H) \subset \bigcap_{0 \leq s < 1} D(|p|^s) \cap \bigcap_{0 \leq s < \frac{1}{2}} D(d\Gamma(\omega)^s), \tag{4.3}$$

which is weaker than our result from Chapter 3. Because of that, there is a need to improve the used methods which will be done in future work.

For other models, the actual results, however, can be used to get a principal idea of the set where $D(H)$ is included. For example, in the case of the pseudorelativistic Nelson Hamiltonian in two dimensions, we obtain the result

$$D(H) \subset \bigcap_{0 \leq s < \frac{1}{2}} \left(D(|p|^s) \cap D(d\Gamma(\omega)^s) \right). \tag{4.4}$$

Until now, we considered only massive models with $m > 0$. Thus, the next step will also be to try to expand our method to massless models, say $m \geq 0$.

This chapter is organized as follows: In Section 4.2, we fix our notation and define the necessary operators and other objects. We also implement the central assumptions for the remaining chapter. In Section 4.3, we show that under the assumptions made in the section before, we can construct the norm-resolvent limit of the ultraviolet regularized Hamiltonian by using the resolvent expansion. This expansion is also used to study domain properties in Section 4.4. Finally, some important examples of operators that are included in our general model are discussed in Section 4.5.

In Appendix C.1, we prove a very general theorem about the existence of a resolvent. For our resolvent expansion, we have to reorder the occurring Born series. This reordering is justified by a theorem we prove in Appendix C.2. Further tools and very important estimates that have a central meaning for this work are contained in Appendix C.3.

4.2. Definition of the Hamiltonian

Let $d \in \mathbb{N}$ and $\mathscr{H} := L^2(\mathbb{R}^d) \otimes \mathcal{F}(L^2(\mathbb{R}^d))$ with the symmetric Fock space. Sometimes we make use of the isomorphism $L^2(\mathbb{R}^d) \otimes \mathcal{F} \cong L^2(\mathbb{R}^d, \mathcal{F})$. We denote the coordinates of the single particle by x and its momentum by p. The momenta of the bosons will be denoted by $k_1, k_2, \ldots$. Let $\Psi \in \mathscr{H}$, then for fixed $x \in \mathbb{R}^d$, $\Psi(x) \in \mathcal{F}$ and we denote the Fourier transformed by $\Psi(p)$.

Let Ω be the dispersion relation of the particle and ω be the dispersion relation of the bosonic field. The interaction of these two systems will later be given in terms of a form factor v. We impose the assumptions

$$(A0) \quad v \in L^2_{loc}(\mathbb{R}^d), \quad \Omega, \omega \in L^\infty_{loc}(\mathbb{R}^d), \quad \Omega(p) \geq 0, \quad \omega(k) \geq m > 0, \tag{4.5}$$

and define the free Hamiltonian of the system by

$$H_0 := \Omega(p) + d\Gamma(\omega) \tag{4.6}$$

on $D(H_0) := D(\Omega(p)) \cap D(d\Gamma(\omega))$ with

$$D(\Omega(p)) := \left\{ \Psi \in L^2(\mathbb{R}^d, \mathcal{F}) \middle| \int dp \; |\Omega(p)|^2 \|\Psi(p)\|^2_{\mathcal{F}} < \infty \right\}, \tag{4.7}$$

$$D(d\Gamma(\omega)) := \left\{ \Psi \in \mathscr{H} \middle| \int dx \sum_{n \geq 0} \int dk_1 .. \int dk_n \left| \sum_{j=1}^n \omega(k_j) \Psi^{(n)}(x, k_1, .., k_n) \right|^2 < \infty \right\}. \tag{4.8}$$

Here, p is the momentum operator of the single particle in d dimensions, thus $\Omega(p)$ is in general a pseudodifferential operator. It is clear that $\Omega(p)$ acts only on the quantized particle and $d\Gamma(\omega)$ only on the bosons. We will omit symbols like $\mathbb{1}$. The free Hamiltonian H_0 is positive and self-adjoint on $D(H_0)$.

The interaction of the particle and the bosons is given in terms of creation and annihilation operators. For some function $f \in L^2(\mathbb{R}^d)$, these operators in the Fock space will be denoted by $a(f)$ and $a^*(f)$ which are defined on $D(\sqrt{N})$, where N is the number operator. On $D(N)$, they obey the canonical commutation relations

$$[a(f), a^*(g)] = \langle f, g\rangle \qquad \text{(others vanish)}. \tag{4.9}$$

Note that $D(d\Gamma(\omega)) \subset D(N)$, since $m > 0$. From the ladder operators a and a^*, we define the symmetric field operators

$$\phi(f) := a(f) + a^*(f), \qquad \pi(f) := \phi(if), \tag{4.10}$$

which are essentially self-adjoint on $D(N)$. They satisfy

$$[\phi(f), \phi(g)] = 2i \operatorname{Im} \langle f, g\rangle, \qquad [\phi(f), \pi(g)] = 2i \operatorname{Re} \langle f, g\rangle. \tag{4.11}$$

The self-adjoint closures of these field operators will be denoted by the same symbols.

It will be necessary to work with generalized ladder operators $a(F)$ and $a^*(F)$, where F is a linear operator with

$$F: \; L^2(\mathbb{R}^d, dx) \to L^2(\mathbb{R}^d, dx) \otimes L^2(\mathbb{R}^d, dk). \tag{4.12}$$

For the general definition of these operators, see Appendix B of the paper [23] or Appendix A.2 of this thesis. One can also find some remarks in the introduction of [23] or Section 2.1 of this thesis.

At some points, the application of operator valued distributions a_k and a_k^* is useful. Using them, the creation and annihilation operators read

$$a(F)\Psi = \int dk \; F^*(k) \; a_k \Psi, \quad a^*(F)\Psi = \int dk \; F(k) \; a_k^* \Psi, \tag{4.13}$$

where

$$(a_k \Psi)^{(n)}(x, k_1, ..., k_n) = (n+1)^{1/2} \; \Psi^{(n+1)}(x, k, k_1, ..., k_n), \tag{4.14}$$

$$(a_k^* \Psi)^{(n)}(x, k_1, ..., k_n) = n^{-1/2} \sum_{j=1}^{n} \delta(k - k_j) \; \Psi^{(n-1)}(x, k_1, ..., k_{j-1}, k_{j+1}, ..., k_n). \tag{4.15}$$

The canonical commutation relation then reads $[a_k, a_{k'}^*] = \delta(k - k')$, others vanish. Furthermore, the following representations hold:

$$d\Gamma(\omega) = \int dk \; \omega(k) \; a_k^* a_k, \tag{4.16}$$

$$\left\| d\Gamma(\omega)^{1/2} \Psi \right\| = \left(\int dk \; \omega(k) \; \|a_k \Psi\|^2 \right)^{1/2}. \tag{4.17}$$

For $0 \leq \Lambda < \infty$, we define the Hamiltonian H_Λ with interaction on $D(H_0)$ by

$$H_\Lambda := \Omega(p) + d\Gamma(\omega) + \phi(G_\Lambda), \tag{4.18}$$

where

$$G_{\Lambda,x}(k) := v(k)e^{-ikx}\chi_\Lambda(k), \tag{4.19}$$

and recall that $v \in L^2_{loc}(\mathbb{R}^d)$ by assumptions $(A0)$. χ_Λ is the characteristic function of the set $\{k \in \mathbb{R}^d \mid |k| \leq \Lambda\}$. An important starting point of our arguments is the self-adjointness of H_Λ for finite Λ. This is shown in the next Lemma.

Lemma 4.2.1. *Assume $(A0)$. Then, for all $\varepsilon > 0$ and all $\Lambda < \infty$, there is a $C_{\varepsilon,\Lambda} \in \mathbb{R}$ such that for all $\Psi \in D(H_0)$*

$$\|\phi(G_\Lambda)\Psi\| \leq \varepsilon\|H_0\Psi\| + C_{\varepsilon,\Lambda}\|\Psi\|$$

Then, for all $\Lambda < \infty$, H_Λ is self-adjoint on $D(H_0)$.

Proof. Using Corollary A.15, Cauchy-Schwarz, and an inequality of the form $2ab \leq \varepsilon^2 a^2 + \frac{b^2}{\varepsilon^2}$, we get

$$\begin{aligned}\|\phi(G_\Lambda)\Psi\| &\leq 2\|G_\Lambda\| \left\|(N+1)^{1/2}\Psi\right\| \\ &\leq \sqrt{\frac{2}{m}}\varepsilon\|G_\Lambda\|\|d\Gamma(\omega)\Psi\| + \sqrt{2}\|G_\Lambda\|\left(\varepsilon + \frac{1}{\varepsilon}\right)\|\Psi\|,\end{aligned} \tag{4.20}$$

which proves the estimate of the lemma. Note that the norm of G_Λ is finite from $(A0)$, as long as $\Lambda < \infty$. The self-adjointness now follows from Kato-Rellich. □

We continue with further definitions. For functions $F_x(k) := f(k)e^{-ikx}$ and $G_x(k) := g(k)e^{-ikx}$ with $f, g \in L^2_{loc}(\mathbb{R}^d)$, in general, we define

$$E(F,G) := \int dk \frac{\overline{f(k)}g(k)}{\omega(k) + \Omega(k)}, \tag{4.21}$$

and especially

$$E_\Lambda := E(G_\Lambda, G_\Lambda) = \int\limits_{|k|\leq\Lambda} dk \frac{|v(k)|^2}{\omega(k) + \Omega(k)}. \tag{4.22}$$

Note that by the Assumptions (A0), $E_\Lambda < \infty$ if $\Lambda < \infty$. But E_Λ may diverge as $\Lambda \to \infty$. In that case it will have the role of a renormalization constant.

If we define for $\Lambda < \infty$

$$\tilde{H}_\Lambda := H_\Lambda + E_\Lambda, \tag{4.23}$$

then this is also a self-adjoint operator on $D(H_0)$. Let $z \in \rho(\tilde{H}_\Lambda) \supset \mathbb{C}\backslash\mathbb{R}$ and $R_\Lambda(z) := (z - \tilde{H}_\Lambda)^{-1}$ be the resolvent of $\tilde{H}_\Lambda$. Then, by using the Neumann series,

$$\begin{aligned} R_\Lambda(z) &= (z - H_0 - \phi(G_\Lambda) - E_\Lambda)^{-1} \\ &= R_0(z)\,[1 - (\phi(G_\Lambda) + E_\Lambda)\,R_0(z)]^{-1} \\ &= R_0(z) \sum_{n=0}^{\infty} [(\phi(G_\Lambda) + E_\Lambda)\,R_0(z)]^n \\ &= R_0(z) + \sum_{n=1}^{\infty} \sum_{j_1,\ldots,j_n=1}^{3} R_0(z) b_{j_1,\Lambda} R_0(z) \cdots R_0(z) b_{j_n,\Lambda} R_0(z), \end{aligned} \tag{4.24}$$

where $b_{1,\Lambda} := a^*(G_\Lambda)$, $b_{2,\Lambda} := a(G_\Lambda)$, and $b_{3,\Lambda} := E_\Lambda$. The step where we used the Neumann series is only correct if the series (4.24) converges. This is shown in the next Lemma.

Lemma 4.2.2. *Assume* $(A0)$. *Then, for all* $\Lambda < \infty$, *there is a* $z_\Lambda \in \mathbb{C}$ *with* $\operatorname{Re} z_\Lambda \leq -1$ *such that the series* (4.24) *converges absolutely for all* $z \in \mathbb{C}$ *with* $\operatorname{Re} z < \operatorname{Re} z_\Lambda$.

Proof. Consider the norm of the series (4.24). It can be estimated by

$$\|R_0(z)\| + \sum_{n=1}^{\infty} \sum_{j_1,\ldots,j_n=1}^{3} \|R_0(z) b_{j_1,\Lambda} R_0(z) \cdots R_0(z) b_{j_n,\Lambda} R_0(z)\|. \tag{4.25}$$

Remind that $b_{1,\Lambda} := a^*(G_\Lambda)$, $b_{2,\Lambda} := a(G_\Lambda)$, and $b_{3,\Lambda} := E_\Lambda$. From Lemma C.3.1, we find the estimates

$$\left\|a(G_\Lambda) R_0(z)^{1/2}\right\|^2 = \left\|R_0(z)^{1/2} a^*(G_\Lambda)\right\|^2 \leq \frac{4}{m} \int\limits_{|k|\leq\Lambda} dk\ |v(k)|^2 = \frac{4}{m} \|G_\Lambda\|^2, \tag{4.26}$$

where the bound is finite for $\Lambda < \infty$ from $(A0)$. The number E_Λ is also finite in this case. That means, all the factors of the series (4.25) can be controlled by adopting a $R_0(z)^{1/2}$. Let moreover $C_\Lambda := \max\{(4/m)\|G_\Lambda\|^2, E_\Lambda\}$, then (4.25) can be estimated by

$$\begin{aligned} &\|R_0(z)\| + \sum_{n=1}^{\infty} \sum_{j_1,\ldots,j_n=1}^{3} \left\|R_0(z)^{1/2}\right\|^{2+n} C_\Lambda^n \leq \|R_0(z)\| \sum_{n=0}^{\infty} 3^n C_\Lambda^n \|R_0(z)\|^{n/2} \\ &\leq \|R_0(z)\| \sum_{n=0}^{\infty} 3^n C_\Lambda^n |\operatorname{Re} z|^{-n/2}, \end{aligned} \tag{4.27}$$

which converges if $\operatorname{Re} z < -9C_\Lambda^2$. ∎

That means, for $z \in \mathbb{C}$ with $\operatorname{Re} z < \operatorname{Re} z_\Lambda$, the expression (4.24) is a first useful series representation of the resolvent $R_\Lambda(z)$.

The series (4.24) consists of normal-ordered terms of ladder operators, not-normal-ordered terms and terms with E_Λ. There are suitable estimates to control the normal-ordered terms, see Lemma C.3.1. But terms with the combination $a(G_\Lambda)R_0(z)a^*(G_\Lambda)$ diverge as $\Lambda \to \infty$. However, following Eckmann [22], one can normal-order this term to separate the divergence which then is compensated by a corresponding term with an E_Λ which also diverges as $\Lambda \to \infty$.

For a better understanding of these cancellations, let $G_{\Lambda,x}(k) := \kappa \tilde{G}_{\Lambda,x}(k)$ with a positive perturbation parameter κ. Then, $a(G_\Lambda)$ and $a^*(G_\Lambda)$ are linear in κ and E_Λ is quadratic. Ordering the series (4.24) by powers of κ leads to

$$R_\Lambda(z) = R_0(z) + \sum_{n=1}^{\infty} \kappa^n B_\Lambda^{(n)}(z), \tag{4.28}$$

where

$$B_\Lambda^{(1)}(z) = R_0(z)\left(a^*(\tilde{G}_\Lambda) + a(\tilde{G}_\Lambda)\right)R_0(z), \tag{4.29}$$

$$\begin{aligned} B_\Lambda^{(2)}(z) = R_0(z)\Big(&a^*(\tilde{G}_\Lambda)R_0(z)a^*(\tilde{G}_\Lambda) + a^*(\tilde{G}_\Lambda)R_0(z)a(\tilde{G}_\Lambda) \\ &+ a(\tilde{G}_\Lambda)R_0(z)a(\tilde{G}_\Lambda) + a(\tilde{G}_\Lambda)R_0(z)a^*(\tilde{G}_\Lambda) + E(\tilde{G}_\Lambda,\tilde{G}_\Lambda)\Big)R_0(z), \end{aligned} \tag{4.30}$$

$$\begin{aligned} B_\Lambda^{(3)}(z) = R_0\Big(&a^*R_0a^*R_0a^* + a^*R_0a^*R_0a + a^*R_0aR_0a^* + a^*R_0aR_0a \\ &+ aR_0a^*R_0a^* + aR_0a^*R_0a + aR_0aR_0a^* + aR_0aR_0a \\ &+ a^*R_0E + aR_0E + ER_0a^* + ER_0a\Big)R_0, \end{aligned} \tag{4.31}$$

and so on, where we set

$$a = a(\tilde{G}_\Lambda), \quad a^* = a^*(\tilde{G}_\Lambda), \quad E = E(\tilde{G}_\Lambda,\tilde{G}_\Lambda), \quad \text{and} \quad R_0 = R_0(z) \tag{4.32}$$

on the right-hand side of $B_\Lambda^{(3)}(z)$. Considering $B_\Lambda^{(2)}(z)$, we see that the divergent term $a(\tilde{G}_\Lambda)R_0(z)a^*(\tilde{G}_\Lambda)$ which is a second order term of the series (4.24) has to be compensated by $E(\tilde{G}_\Lambda,\tilde{G}_\Lambda)$ which is a first order term of (4.24). In $B_\Lambda^{(3)}(z)$, we have the divergent terms $a^*R_0aR_0a^*$, $aR_0a^*R_0a^*$, $aR_0a^*R_0a$, and $aR_0aR_0a^*$. They are compensated by a^*R_0E, ER_0a^*, ER_0a, and aR_0E. For our purposes, we can set $\kappa = 1$.

To take these compensations into account, it is necessary to reorder the series (4.24). From Lemma 4.2.2, we know that for $z \in \mathbb{C}$ with $\operatorname{Re} z < \operatorname{Re} z_\Lambda$, the series satisfies the requirements of Theorem C.2.1. Applying this theorem leads to a reordered series representation of the resolvent $R_\Lambda(z)$ for these z-values, namely

$$R_\Lambda(z) = R_0(z) + \sum_{n=1}^{\infty} \sum_{\ell=0}^{[\frac{n}{2}]} \sum_{(j_1,\dots,j_{n-\ell}) \in S(n,\ell)} R_0(z) \left(\prod_{i=1}^{n-\ell} a_{j_i,\Lambda} R_0(z) \right), \tag{4.33}$$

where

$$\begin{aligned} S(n,\ell) := \Big\{ (j_1, \dots, j_{n-\ell}) \in \{1,2,3\}^{n-\ell} \,\Big|\, &\text{ the number of } j\text{'s equal 3 is } \ell; \\ &\text{and if } j_i = 2 \text{ for an } i = 1, \dots, n-\ell-1, \text{ then } j_{i+1} \neq 1 \Big\}, \end{aligned} \tag{4.34}$$

and

$$a_{1,\Lambda} := a^*(G_\Lambda), \quad a_{2,\Lambda} := a(G_\Lambda), \quad a_{3,\Lambda} := a(G_\Lambda) R_0(z) a^*(G_\Lambda) + E_\Lambda. \tag{4.35}$$

Note that only for $z \in \mathbb{C}$ with $\operatorname{Re} z < \operatorname{Re} z_\Lambda$, the series (4.24) and (4.33) coincide, and the $\operatorname{Re} z_\Lambda$ depends on Λ and goes to $-\infty$ for $\Lambda \to \infty$. But for the reordered series (4.33), we will show the convergence on a z-domain that is bounded uniformly in Λ which makes the reordered series (4.33) to a unique continuation of the series (4.24). To show this is one of the main issues of the next section. From now, we define $R_\Lambda(z)$ to be the reordered series from (4.33).

We furthermore impose further assumptions on the dispersion relations Ω, ω, and the form factor v:

Assume that there exist $\alpha, \beta, \gamma \in \mathbb{R}$ with $0 < \alpha, \gamma \leq \frac{1}{2} \leq \beta \leq 1$ and $\alpha + \beta \leq 1$ and $\beta + \gamma \leq 1$ such that

$$(A1) \quad \sup_{p \in \mathbb{R}^d} \int\limits_{|k| \geq K} dk \frac{|v(k)|^2}{\omega(k)^{1-2\alpha} [1 + \Omega(p-k) + \omega(k)]^{2(\alpha+\beta)-1}} \to 0 \quad (K \to \infty),$$

$$(A2) \quad \sup_{p \in \mathbb{R}^d} \int\limits_{|k| \geq K} dk \frac{|v(k)|^2}{\omega(k) [1 + \Omega(p-k) + \omega(k)]^{2\gamma}} \to 0 \quad (K \to \infty),$$

$$(A3) \quad \sup_{p \in \mathbb{R}^d} \int\limits_{|k| \geq K} dk \frac{|v(k)|^2 |\Omega(p-k) - \Omega(k)|}{[\omega(k) + \Omega(k)][1 + \Omega(p)]^{2\gamma} [1 + \Omega(p-k) + \omega(k)]} \to 0 \quad (K \to \infty).$$

These are the most general assumptions for our arguments and they allow us to study a large class of Hamiltonians. Systems that satisfy these assumptions are, for example, the Fröhlich and the Nelson Hamiltonians. We will later discuss some examples.

Note also, that these assumptions have a direct connection to the estimates given in Lemma C.3.1 and Corollary C.3.3. In view of that, we define for abbreviation for

$0 < \alpha, \gamma \leq \frac{1}{2} \leq \beta \leq 1$, $z \in \mathbb{C}$ with $\operatorname{Re} z \leq -1$, and $f \in L^2_{loc}(\mathbb{R}^d)$ the following quantities:

$$\mathcal{A}_{\alpha,\beta,z}(f) := \left(1 + m^{-\alpha}\right) \cdot \left(\sup_{p\in\mathbb{R}^d} \int dk \frac{|f(k)|^2}{\omega(k)^{1-2\alpha}[|\operatorname{Re} z| + \Omega(p-k) + \omega(k)]^{2(\alpha+\beta)-1}} \right)^{\frac{1}{2}}, \tag{4.36}$$

$$\mathcal{B}_{\gamma,z}(f) := 2 \sup_{p\in\mathbb{R}^d} \int dk \frac{|f(k)|^2}{\omega(k)[|\operatorname{Re} z| + \Omega(p-k) + \omega(k)]^{2\gamma}}, \tag{4.37}$$

$$\mathscr{C}_{\gamma,z}(f) := \sup_{p\in\mathbb{R}^d} \int dk \frac{|f(k)|^2|\Omega(p-k) - \Omega(k)|}{[\omega(k) + \Omega(k)][|\operatorname{Re} z| + \Omega(p)]^{2\gamma}[|\operatorname{Re} z| + \Omega(p-k) + \omega(k)]}, \tag{4.38}$$

$$\mathcal{E}_{\alpha,\beta,\gamma,z}(f) := \mathcal{A}_{\alpha,\beta,z}(f) + \mathcal{B}_{\gamma,z}(f) + \mathscr{C}_{\gamma,z}(f). \tag{4.39}$$

4.3. Norm-resolvent limit of the (renormalized) Hamiltonian

In this section, we show that, under the assumptions made in Section 4.2, $H_\Lambda + E_\Lambda$ converges in the norm-resolvent sense as $\Lambda \to \infty$. For that, let from now $R_\Lambda(z)$ be the reordered series from (4.33), say

$$R_\Lambda(z) = R_0(z) + \sum_{n=1}^{\infty} \sum_{\ell=0}^{[\frac{n}{2}]} \sum_{(j_1,\dots,j_{n-\ell})\in S(n,\ell)} R_0(z) \left(\prod_{i=1}^{n-\ell} a_{j_i,\Lambda} R_0(z) \right), \tag{4.40}$$

which for suitable $z \in \mathbb{C}$ coincide with the resolvent of $H_\Lambda + E_\Lambda$. Our main tools are the statements in the appendices and the following important

Lemma 4.3.1. *Assume* (A0), (A1), (A2), *and* (A3). *Then, for all* $\varepsilon > 0$, *there is a* $z \in \mathbb{C}$ *with* $\operatorname{Re} z \leq -1$ *such that* $\mathcal{A}_{\alpha,\beta,z}(v) < \varepsilon$, $\mathcal{B}_{\gamma,z}(v) < \varepsilon$, $\mathscr{C}_{\gamma,z}(v) < \varepsilon$, *and* $\mathcal{E}_{\alpha,\beta,\gamma,z}(v) < \varepsilon$.

Proof. Consider for $K > 0$

$$\begin{aligned}
\mathcal{A}_{\alpha,\beta,z}(v)^2 &= \left(1 + \frac{1}{m^\alpha}\right)^2 \sup_{p\in\mathbb{R}^d} \int\limits_{0\leq|k|\leq K} dk \; \frac{|v(k)|^2}{\omega(k)^{1-2\alpha}[|\operatorname{Re} z| + \Omega(p-k) + \omega(k)]^{2(\alpha+\beta)-1}} \\
&\quad + \left(1 + \frac{1}{m^\alpha}\right)^2 \sup_{p\in\mathbb{R}^d} \int\limits_{|k|\geq K} dk \; \frac{|v(k)|^2}{\omega(k)^{1-2\alpha}[|\operatorname{Re} z| + \Omega(p-k) + \omega(k)]^{2(\alpha+\beta)-1}} \\
&\leq \frac{(1 + 1/m^\alpha)^2}{m^{1-2\alpha}|\operatorname{Re} z|^{2(\alpha+\beta)-1}} \int\limits_{0\leq|k|\leq K} dk \; |v(k)|^2 \\
&\quad + \left(1 + m^{-\alpha}\right)^2 \sup_{p\in\mathbb{R}^d} \int\limits_{|k|\geq K} dk \frac{|v(k)|^2}{\omega(k)^{1-2\alpha}[1 + \Omega(p-k) + \omega(k)]^{2(\alpha+\beta)-1}}.
\end{aligned} \tag{4.41}$$

Now, choose first K so large that the second summand becomes smaller than $\varepsilon^2/2$ which is possible because of assumption $(A1)$. Then, choose $|\operatorname{Re} z|$ so large that the first term becomes smaller than $\varepsilon^2/2$. Recall that $v \in L^2_{loc}(\mathbb{R}^d)$ and that $2(\alpha+\beta)-1>0$ because of $\alpha+\beta>\frac{1}{2}$. Thus, the property for $\mathcal{A}_{\alpha,\beta,z}(v)$ was shown.

Analogously, we get

$$\begin{aligned}\mathcal{B}_{\gamma,z}(v) \leq \frac{2}{m|\operatorname{Re} z|^{2\gamma}} & \int\limits_{0\leq|k|\leq K} dk\ |v(k)|^2 \\ & + 2\sup_{p\in\mathbb{R}^d} \int\limits_{|k|\geq K} dk\ \frac{|v(k)|^2}{\omega(k)[1+\Omega(p-k)+\omega(k)]^{2\gamma}},\end{aligned} \tag{4.42}$$

and the same argumentation as for $\mathcal{A}_{\alpha,\beta,z}(v)$ holds, where we need $(A2)$ and that $\gamma>0$.

For $\mathscr{C}_{\gamma,z}(v)$, using the triangle inequality, we obtain

$$\begin{aligned}\mathscr{C}_{\gamma,z}(v) \leq & \sup_{p\in\mathbb{R}^d} \int\limits_{0\leq|k|\leq K} dk \frac{|v(k)|^2\Omega(p-k)}{[\omega(k)+\Omega(k)][|\operatorname{Re} z|+\Omega(p)]^{2\gamma}[|\operatorname{Re} z|+\Omega(p-k)+\omega(k)]} \\ & + \sup_{p\in\mathbb{R}^d} \int\limits_{0\leq|k|\leq K} dk \frac{|v(k)|^2\Omega(k)}{[\omega(k)+\Omega(k)][|\operatorname{Re} z|+\Omega(p)]^{2\gamma}[|\operatorname{Re} z|+\Omega(p-k)+\omega(k)]} \\ & + \sup_{p\in\mathbb{R}^d} \int\limits_{|k|\geq K} dk \frac{|v(k)|^2|\Omega(p-k)-\Omega(k)|}{[\omega(k)+\Omega(k)][|\operatorname{Re} z|+\Omega(p)]^{2\gamma}[|\operatorname{Re} z|+\Omega(p-k)+\omega(k)]} \\ \leq & \left(\frac{1}{m|\operatorname{Re} z|^{2\gamma}}+\frac{1}{|\operatorname{Re} z|^{1+2\gamma}}\right) \int\limits_{0\leq|k|\leq K} dk\ |v(k)|^2 \\ & + \sup_{p\in\mathbb{R}^d} \int\limits_{|k|\geq K} dk \frac{|v(k)|^2|\Omega(p-k)-\Omega(k)|}{[\omega(k)+\Omega(k)][1+\Omega(p)]^{2\gamma}[1+\Omega(p-k)+\omega(k)]},\end{aligned} \tag{4.43}$$

and the same argumentation holds again, where we need $(A3)$ and that $\gamma>0$ again.

Now, if we choose $|\operatorname{Re} z|$ so large that $\mathcal{A}_{\alpha,\beta,z}(v)<\frac{\varepsilon}{3}$, $\mathcal{B}_{\gamma,z}(v)<\frac{\varepsilon}{3}$, and $\mathscr{C}_{\gamma,z}(v)<\frac{\varepsilon}{3}$, we end at $\mathcal{E}_{\alpha,\beta,\gamma,z}(v)<\varepsilon$. □

As preparation for the proof of Theorem 4.3.4, we establish the following two lemmas:

Lemma 4.3.2. *Assume $(A0)$, $(A1)$, $(A2)$, and $(A3)$. Then, there is a $z_0\in\mathbb{C}$ with $\operatorname{Re} z_0\leq -1$ such that the series $R_\Lambda(z)$ converges uniformly in Λ for all $z\in\mathbb{C}$ with $\operatorname{Re} z<\operatorname{Re} z_0$.*

Proof. Note that by Assumptions ($A1$), ($A2$), and ($A3$), there are fixed exponents $\alpha, \beta, \gamma \in \mathbb{R}$ with $0 < \alpha, \gamma \leq \frac{1}{2} \leq \beta \leq 1$, $\alpha + \beta \leq 1$, and $\beta + \gamma \leq 1$. Let $\Lambda < \infty$ and consider the reordered series (4.33)

$$\begin{aligned} R_\Lambda(z) &= R_0(z) + \sum_{n=1}^{\infty} \sum_{\ell=0}^{[\frac{n}{2}]} \sum_{(j_1,...,j_{n-\ell}) \in S(n,\ell)} R_0(z) \left(\prod_{i=1}^{n-\ell} a_{j_i,\Lambda} R_0(z) \right) \\ &= R_0(z) + \sum_{n=1}^{\infty} \sum_{\ell=0}^{[\frac{n}{2}]} \sum_{(j_1,...,j_{n-\ell}) \in S(n,\ell)} R_0(z)^{1-\eta_{j_1}-\varphi_{j_0}} \\ &\qquad \left(\prod_{i=1}^{n-\ell} R_0(z)^{\eta_{j_i}} a_{j_i,\Lambda} R_0(z)^{\varphi_{j_i}} R_0(z)^{1-\eta_{j_{i+1}}-\varphi_{j_i}} \right) \end{aligned} \tag{4.44}$$

with suitable exponents $\eta_{j_i} \geq 0$ and $\varphi_{j_i} \geq 0$ that satisfy $1 - \eta_{j_{i+1}} - \varphi_{j_i} \geq 0$ for all $i = 0, ..., n-\ell$. We define $\varphi_{j_0} := 0 =: \eta_{j_{n-\ell+1}}$.

Now, the strategy is to choose these exponents in such a way, that we can use Lemma C.3.1 and Corollary C.3.3 to compensate the operators $a_{j_i,\Lambda}$. We choose

$$\eta_k := \begin{cases} \beta & : \quad k = 1 \\ \alpha & : \quad k = 2 \\ \gamma & : \quad k = 3 \end{cases}, \qquad \varphi_k := \begin{cases} \alpha & : \quad k = 1 \\ \beta & : \quad k = 2 \\ \gamma & : \quad k = 3 \end{cases}, \tag{4.45}$$

and we remark that $1 - \eta_{j_{i+1}} - \varphi_{j_i} \geq 0$ for all $i = 0, ..., n - \ell$. This is made sure since, by definition of $S(n, \ell)$, an $a_{1,\Lambda}$ cannot follow after an $a_{2,\Lambda}$. Therefore, the case that $1 - \eta_{j_{i+1}} - \varphi_{j_i} = 1 - 2\beta$ which can be smaller than 0 cannot occur. Note that the number of $a_{3,\Lambda}$-operators in the product of Equation (4.44) is given by $\ell = 0, 1, ..., [\frac{n}{2}]$. Let moreover $m = 1, 2, ..., n - \ell$ be the number of creation and annihilation operators $a_{1,\Lambda}$ and $a_{2,\Lambda}$ in this product. Then, $m + \ell = n - \ell$.

With respect to this product in Equation (4.44), we obtain

$$\begin{aligned} &\left\| R_0(z)^{1-\eta_{j_1}-\varphi_{j_0}} \left(\prod_{i=1}^{n-\ell} R_0(z)^{\eta_{j_i}} a_{j_i,\Lambda} R_0(z)^{\varphi_{j_i}} R_0(z)^{1-\eta_{j_{i+1}}-\varphi_{j_i}} \right) \right\| \\ &\leq \left\| R_0(z)^{1-\eta_{j_1}-\varphi_{j_0}} \right\| \left(\prod_{i=1}^{n-\ell} \| R_0(z)^{\eta_{j_i}} a_{j_i,\Lambda} R_0(z)^{\varphi_{j_i}} \| \left\| R_0(z)^{1-\eta_{j_{i+1}}-\varphi_{j_i}} \right\| \right) \\ &\leq \| R_0(z) \|^{\sum_{i=0}^{n-\ell} (1-\eta_{j_{i+1}}-\varphi_{j_i})} \prod_{i=1}^{n-\ell} \| R_0(z)^{\eta_{j_i}} a_{j_i,\Lambda} R_0(z)^{\varphi_{j_i}} \|. \end{aligned} \tag{4.46}$$

It is

$$\sum_{i=0}^{n-\ell} (1 - \eta_{j_{i+1}} - \varphi_{j_i}) = 1 + m(1 - \alpha - \beta) + \ell(1 - 2\gamma) \geq 1, \tag{4.47}$$

such that

$$\|R_0(z)\|^{\sum_{i=0}^{n-\ell}(1-\eta_{j_{i+1}}-\varphi_{j_i})} \leq \|R_0(z)\|. \tag{4.48}$$

Using Lemma C.3.1 and Corollary C.3.3 and considering Assumptions $(A1)$, $(A2)$, and $(A3)$, we can estimate $\|R_0(z)^{\eta_{j_i}} a_{j_i,\Lambda} R_0(z)^{\varphi_{j_i}}\|$ either by $\mathcal{A}_{\alpha,\beta,z}(v)$ or by $\mathcal{B}_{\gamma,z}(v)+\mathscr{C}_{\gamma,z}(v)$. That is, we can estimate all terms $\|R_0(z)^{\eta_{j_i}} a_{j_i,\Lambda} R_0(z)^{\varphi_{j_i}}\|$ by $\mathcal{E}_{\alpha,\beta,\gamma,z}(v)$.

Note that $|S(n,\ell)| \leq 3^{n-\ell}$, such that for the norm of $R_\Lambda(z)$ from Equation (4.44), we arrive at

$$\|R_\Lambda(z)\| \leq \|R_0(z)\| \sum_{n=0}^{\infty} \sum_{\ell=0}^{[\frac{n}{2}]} 3^{n-\ell} \mathcal{E}_{\alpha,\beta,\gamma,z}(v)^{n-\ell}. \tag{4.49}$$

Now, choose $\operatorname{Re} z_0$ so small that $3\mathcal{E}_{\alpha,\beta,\gamma,z_0}(v) \leq \frac{1}{2}$, which is possible because of Lemma 4.3.1. Then, for all $z \in \mathbb{C}$ with $\operatorname{Re} z < \operatorname{Re} z_0$, $3\mathcal{E}_{\alpha,\beta,\gamma,z}(v) < \frac{1}{2}$ and $\frac{1}{1-3\mathcal{E}_{\alpha,\beta,\gamma,z}(v)} < 2$, and we can moreover use the geometric sum to get

$$\begin{aligned}
\sum_{n=0}^{\infty} \sum_{\ell=0}^{[\frac{n}{2}]} 3^{n-\ell} \mathcal{E}_{\alpha,\beta,\gamma,z}(v)^{n-\ell} &= \sum_{n=0}^{\infty} 3^n \mathcal{E}_{\alpha,\beta,\gamma,z}(v)^n \sum_{\ell=0}^{[\frac{n}{2}]} 3^{-\ell} \mathcal{E}_{\alpha,\beta,\gamma,z}(v)^{-\ell} \\
&= \sum_{n=0}^{\infty} (3\mathcal{E}_{\alpha,\beta,\gamma,z}(v))^n \frac{1-(3\mathcal{E}_{\alpha,\beta,\gamma,z}(v))^{[\frac{n}{2}]+1}}{(1-3\mathcal{E}_{\alpha,\beta,\gamma,z}(v))(3\mathcal{E}_{\alpha,\beta,\gamma,z}(v))^{[\frac{n}{2}]}} \\
&\leq 2\sum_{n=0}^{\infty} \left((3\mathcal{E}_{\alpha,\beta,\gamma,z}(v))^{n-[\frac{n}{2}]} + (3\mathcal{E}_{\alpha,\beta,\gamma,z}(v))^{n+1} \right).
\end{aligned} \tag{4.50}$$

Using that

$$\sum_{n=0}^{\infty} (3\mathcal{E}_{\alpha,\beta,\gamma,z}(v))^{n-[\frac{n}{2}]} = \sum_{n=0}^{\infty} \left((3\mathcal{E}_{\alpha,\beta,\gamma,z}(v))^n + (3\mathcal{E}_{\alpha,\beta,\gamma,z}(v))^{n+1} \right), \tag{4.51}$$

we end at

$$\|R_\Lambda(z)\| \leq 4\|R_0(z)\| \sum_{n=0}^{\infty} 3^n \mathcal{E}_{\alpha,\beta,\gamma,z}(v)^n, \tag{4.52}$$

which is finite since, from our choice of z_0, the series converges for all $z \in \mathbb{C}$ with $\operatorname{Re} z < \operatorname{Re} z_0$ uniformly in Λ. □

From this Lemma, the reordered series $R_\Lambda(z)$ converges on a z-domain that is independent of Λ in contrast to the series (4.24). Thus, we are now able to show the existence of the limit of $R_\Lambda(z)$ as $\Lambda \to \infty$ on a non-vanishing subset of $\mathbb{C}$.

Lemma 4.3.3. *Assume (A0), (A1), (A2), and (A3). Then, there is a* $z_0 \in \mathbb{C}$ *with* $\operatorname{Re} z_0 \leq -1$ *such that the limit* $\lim_{\Lambda\to\infty} R_\Lambda(z)$ *exists for all* $z \in \mathbb{C}$ *with* $\operatorname{Re} z < \operatorname{Re} z_0$.

Proof. We have to show that $R_\Lambda(z)$ is a Cauchy sequence with respect to Λ for $\operatorname{Re} z$ sufficiently small. Let without loss of generality $\Lambda_2 < \Lambda_1$ and consider $R_{\Lambda_1}(z) - R_{\Lambda_2}(z)$. By successively adding and subtracting the relevant operators, we get

$$
\begin{aligned}
R_{\Lambda_1}(z) - R_{\Lambda_2}(z) = &\sum_{n=1}^{\infty} \sum_{\ell=0}^{[\frac{n}{2}]} \sum_{(j_1,\dots,j_{n-\ell})\in S(n,\ell)} \sum_{i=1}^{n-\ell} \\
&\left(\prod_{k=1}^{i-1} R_0(z) a_{j_k,\Lambda_2}\right) R_0(z)(a_{j_i,\Lambda_1} - a_{j_i,\Lambda_2}) R_0(z) \left(\prod_{k=i+1}^{n-\ell} a_{j_k,\Lambda_2} R_0(z)\right),
\end{aligned}
\tag{4.53}
$$

where we define $\prod_{k=i}^{i-1} c_k := 1$ for an arbitrary operator sequence c_k. The products of the single operators $a_{1,\Lambda}$, $a_{2,\Lambda}$, and $a_{3,\Lambda}$ can be estimated as in the proof of Lemma 4.3.2, such that we still have to estimate the occurring differences.

Using Lemma C.3.1, we obtain

$$
\begin{aligned}
\left\|R_0(z)^\beta (a_{1,\Lambda_1} - a_{1,\Lambda_2}) R_0(z)^\alpha\right\| &= \left\|R_0(z)^\beta a^*(G_{\Lambda_1} - G_{\Lambda_2}) R_0(z)^\alpha\right\| \\
&\leq \mathcal{A}_{\alpha,\beta,z}(v(\chi_{\Lambda_1} - \chi_{\Lambda_2}))
\end{aligned}
\tag{4.54}
$$

and analogously

$$
\begin{aligned}
\left\|R_0(z)^\alpha (a_{2,\Lambda_1} - a_{2,\Lambda_2}) R_0(z)^\beta\right\| &= \left\|R_0(z)^\alpha a(G_{\Lambda_1} - G_{\Lambda_2}) R_0(z)^\beta\right\| \\
&\leq \mathcal{A}_{\alpha,\beta,z}(v(\chi_{\Lambda_1} - \chi_{\Lambda_2})).
\end{aligned}
\tag{4.55}
$$

From reasons that will become clear soon, we estimate further

$$
\begin{aligned}
\mathcal{A}_{\alpha,\beta,z}(v(\chi_{\Lambda_1} - \chi_{\Lambda_2})) &\leq \mathcal{A}_{\alpha,\beta,-1}(v(1-\chi_{\Lambda_2}))^{1/2} \mathcal{A}_{\alpha,\beta,z}(v)^{1/2} \\
&\leq \mathcal{E}_{\alpha,\beta,\gamma,-1}(v(1-\chi_{\Lambda_2}))^{1/2} \mathcal{E}_{\alpha,\beta,\gamma,z}(v)^{1/2}.
\end{aligned}
\tag{4.56}
$$

In the third case, we can write

$$
\begin{aligned}
a_{3,\Lambda_1} - a_{3,\Lambda_2} &= a(G_{\Lambda_1}) R_0(z) a^*(G_{\Lambda_1}) + E_{\Lambda_1} - a(G_{\Lambda_2}) R_0(z) a^*(G_{\Lambda_2}) - E_{\Lambda_2} \\
&= a(G_{\Lambda_1} - G_{\Lambda_2}) R_0(z) a^*(G_{\Lambda_1}) + E(G_{\Lambda_1} - G_{\Lambda_2}, G_{\Lambda_1}) \\
&\quad + a(G_{\Lambda_2}) R_0(z) a^*(G_{\Lambda_1} - G_{\Lambda_2}) + E(G_{\Lambda_2}, G_{\Lambda_1} - G_{\Lambda_2}).
\end{aligned}
\tag{4.57}
$$

Using Lemma C.3.2 leads to

$$\begin{aligned}
&\|R_0(z)^\gamma(a_{3,\Lambda_1} - a_{3,\Lambda_2})R_0(z)^\gamma\| \\
&\le \|R_0(z)^\gamma(a(G_{\Lambda_1} - G_{\Lambda_2})R_0(z)a^*(G_{\Lambda_1}) + E(G_{\Lambda_1} - G_{\Lambda_2}, G_{\Lambda_1}))R_0(z)^\gamma\| \\
&\quad + \|R_0(z)^\gamma(a(G_{\Lambda_2})R_0(z)a^*(G_{\Lambda_1} - G_{\Lambda_2}) + E(G_{\Lambda_2}, G_{\Lambda_1} - G_{\Lambda_2}))R_0(z)^\gamma\| \\
&\le \mathcal{B}_{\gamma,z}(v\chi_{\Lambda_1})^{1/2}\mathcal{B}_{\gamma,z}(v(\chi_{\Lambda_1} - \chi_{\Lambda_2}))^{1/2} + \mathscr{C}_{\gamma,z}(v(\chi_{\Lambda_1} - \chi_{\Lambda_2})) \\
&\quad + \mathcal{B}_{\gamma,z}(v(\chi_{\Lambda_1} - \chi_{\Lambda_2}))^{1/2}\mathcal{B}_{\gamma,z}(v\chi_{\Lambda_2})^{1/2} \\
&\le 2\mathcal{B}_{\gamma,-1}(v(1 - \chi_{\Lambda_2}))^{1/2}\mathcal{B}_{\gamma,z}(v)^{1/2} + \mathscr{C}_{\gamma,-1}(v(1 - \chi_{\Lambda_2}))^{1/2}\mathscr{C}_{\gamma,z}(v)^{1/2} \\
&\le 3\mathcal{E}_{\alpha,\beta,\gamma,-1}(v(1 - \chi_{\Lambda_2}))^{1/2}\mathcal{E}_{\alpha,\beta,\gamma,z}(v)^{1/2}. \qquad (4.58)
\end{aligned}$$

These estimates and those in the proof of Lemma 4.3.2 before lead to

$$\begin{aligned}
&\|R_{\Lambda_1}(z) - R_{\Lambda_2}(z)\| \\
&\le \sum_{n=1}^{\infty}\sum_{\ell=0}^{[\frac{n}{2}]}\sum_{(j_1,\dots,j_{n-\ell})\in S(n,\ell)}\sum_{i=1}^{n-\ell}\|R_0(z)\|\ \mathcal{E}_{\alpha,\beta,\gamma,z}(v)^{n-\ell-\frac{1}{2}}3\mathcal{E}_{\alpha,\beta,\gamma,-1}(v(1 - \chi_{\Lambda_2}))^{1/2} \qquad (4.59)
\end{aligned}$$

The sum over $i = 1, ..., n - \ell$ leads to a factor $n - \ell$, which is smaller that $2^{n-\ell}$ here. Furthermore, $|S(n,\ell)| \le 3^{n-\ell}$ and we arrive at

$$\begin{aligned}
&\|R_{\Lambda_1}(z) - R_{\Lambda_2}(z)\| \\
&\le 3\mathcal{E}_{\alpha,\beta,\gamma,-1}(v(1 - \chi_{\Lambda_2}))^{1/2}\|R_0(z)\|\ \sum_{n=1}^{\infty}\sum_{\ell=0}^{[\frac{n}{2}]}6^{n-\ell}\mathcal{E}_{\alpha,\beta,\gamma,z}(v)^{n-\ell-\frac{1}{2}}. \qquad (4.60)
\end{aligned}$$

Now, choose $\operatorname{Re} z_0$ so small that $6\mathcal{E}_{\alpha,\beta,\gamma,z_0}(v) \le \frac{1}{2}$, which is possible because of Lemma 4.3.1. Then, for all $z \in \mathbb{C}$ with $\operatorname{Re} z < \operatorname{Re} z_0$, $6\mathcal{E}_{\alpha,\beta,\gamma,z}(v) < \frac{1}{2}$. Similarily to the proof of Lemma 4.3.2, we can show that

$$\sum_{n=1}^{\infty}\sum_{\ell=0}^{[\frac{n}{2}]}6^{n-\ell}\mathcal{E}_{\alpha,\beta,\gamma,z}(v)^{n-\ell-\frac{1}{2}} \le 6\sqrt{6}\sum_{n=0}^{\infty}(6\mathcal{E}_{\alpha,\beta,\gamma,z}(v))^n, \qquad (4.61)$$

which converges for our choice of z_0 for all $z \in \mathbb{C}$ with $\operatorname{Re} z \le \operatorname{Re} z_0$. We obtain

$$\|R_{\Lambda_1}(z) - R_{\Lambda_2}(z)\| \le 18\sqrt{6}\mathcal{E}_{\alpha,\beta,\gamma,-1}(v(1 - \chi_{\Lambda_2}))^{1/2}\|R_0(z)\|\ \sum_{n=0}^{\infty}(6\mathcal{E}_{\alpha,\beta,\gamma,z}(v))^n. \qquad (4.62)$$

The right-hand side of this estimate becomes arbitrarily small for Λ_2 sufficiently large. □

Theorem 4.3.4. *Assume (A0), (A1), (A2), and (A3). Then, there is a self-adjoint operator $H : D(H) \to \mathscr{H}$ such that $H_\Lambda + E_\Lambda \to H$ in the norm-resolvent sense as $\Lambda \to \infty$. The operator H is bounded from below.*

Proof. The two lemmas before allow us to define the limit operator of $R_\Lambda(z)$ as $\Lambda \to \infty$. Then, we will show that this operator satisfies the assumptions of Theorem C.1.1.

At first, let $z_0 \in \mathbb{C}$ with $\operatorname{Re} z_0 \leq -1$ and choose $\operatorname{Re} z_0$ so small that the series $R_\Lambda(z)$ converges uniformly in Λ for all $z \in \mathbb{C}$ with $\operatorname{Re} z < \operatorname{Re} z_0$ by Lemma 4.3.2 and that $\lim_{\Lambda\to\infty} R_\Lambda(z)$ exists by Lemma 4.3.3. Then, for all these z, $R_\Lambda(z)$ is the resolvent of the operator $H_\Lambda + E_\Lambda$ and we can define

$$R_\infty(z) := \lim_{\Lambda\to\infty} R_\Lambda(z), \tag{4.63}$$

which is a bounded linear operator in $\mathscr{H}$.

Now, let $z, z_1, z_2 \in \rho_{z_0} := \{z \in \mathbb{C} \mid \operatorname{Re} z < \operatorname{Re} z_0\}$ and $\Phi, \Psi \in \mathscr{H}$. Then,

$$\langle \Phi, R_\infty(z)\Psi\rangle = \lim_{\Lambda\to\infty} \langle \Phi, R_\Lambda(z)\Psi\rangle = \lim_{\Lambda\to\infty} \langle R_\Lambda(\overline{z})\Phi, \Psi\rangle = \langle R_\infty(\overline{z})\Phi, \Psi\rangle, \tag{4.64}$$

which shows that $R_\infty(z)^* = R_\infty(\overline{z})$. Furthermore, we find

$$\begin{aligned} R_\infty(z_1) - R_\infty(z_2) &= \lim_{\Lambda\to\infty} (R_\Lambda(z_1) - R_\Lambda(z_2)) = \lim_{\Lambda\to\infty} (z_2 - z_1) R_\Lambda(z_1) R_\Lambda(z_2) \\ &= (z_2 - z_1) R_\infty(z_1) R_\infty(z_2), \end{aligned} \tag{4.65}$$

where we used a resolvent identity. Finally, using the series (4.33) and the same estimates as in the proof of Lemma 4.3.2 again, we see

$$\begin{aligned} \|z R_\infty(z)\Psi - \Psi\| &= \left\| z \lim_{\Lambda\to\infty} R_\Lambda(z)\Psi - \Psi \right\| \\ &\leq \|z R_0(z)\Psi - \Psi\| + |z| \|R_0(z)\| \sum_{n=1}^{\infty} \sum_{\ell=0}^{[\frac{n}{2}]} 3^{n-\ell} \mathcal{E}_{\alpha,\beta,\gamma,z}(v)^{n-\ell} \\ &\leq \|z R_0(z)\Psi - \Psi\| + 4 \sum_{n=1}^{\infty} (3\mathcal{E}_{\alpha,\beta,\gamma,z}(v))^n, \end{aligned} \tag{4.66}$$

where we also used $\|R_0(z)\| \leq |z|^{-1}$. The first summand of this inequality vanishes as $\operatorname{Re} z \to -\infty$ by a resolvent property, the second summand vanishes as $\operatorname{Re} z \to -\infty$ by dominated convergence. Therefore, all assumptions of Theorem C.1.1 are satisfied and it follows that, for all $z \in \rho_{z_0}$, $R_\infty(z)$ is the resolvent of a self-adjoint operator H which is bounded from below by $-\operatorname{Re} z_0$.

We have shown that $R_\Lambda(z) \to R_\infty(z)$ for all $z \in \rho_{z_0}$. Since $H_\Lambda + E_\Lambda$ and H are self-adjoint, from Theorem VIII.19(a) of [24] follows that $R_\Lambda(z) \to R_\infty(z)$ for all $z \in \mathbb{C}\backslash\mathbb{R}$. Thus, $H_\Lambda + E_\Lambda \to H$ as $\Lambda \to \infty$ in the norm-resolvent sense. $\square$

4.4. Domain of the norm-resolvent limit

We can use the expansion of the resolvent made in the last section to study some properties of the domain of definition of the norm-resolvent limit of $H_\Lambda + E_\Lambda$, which will be done in this section. For that, we fix the particle dispersion relation to $\Omega(p) := (\mu^2 + p^2)^{r/2}$ with $\mu, r \geq 0$. The main results of this section are Theorems 4.4.2 and 4.4.3. We start with the following

Lemma 4.4.1. *Let $p := -i\nabla_x$, $\mu \geq 0$, $r \geq 0$, and $\Omega(p) := (\mu^2 + p^2)^{r/2}$. Then $D(\Omega(p)) = D(|p|^r)$.*

Proof. Recall that

$$D(\Omega(p)) = \left\{ \Psi \in L^2(\mathbb{R}^d, \mathcal{F}) \middle| \int dp \; |\Omega(p)|^2 \|\Psi(p)\|_{\mathcal{F}}^2 < \infty \right\}. \tag{4.67}$$

For suitable Ψ, one finds the inequality

$$\|\Omega(p)\Psi\|^2 \leq C_s \mu^{2s} \|\Psi\|^2 + C_s \||p|^s \Psi\|^2 \leq C_s \mu^{2s} \|\Psi\|^2 + C_s \|\Omega(p)\Psi\|^2 \tag{4.68}$$

with the finite constant C_s, what proves the lemma. □

Theorem 4.4.2. *Let $\Omega(p) := (\mu^2 + p^2)^{r/2}$ with $\mu \geq 0$ and $r \geq 0$ and assume (A0), (A1), (A2), and (A3). Let $H : D(H) \to \mathscr{H}$ be the norm-resolvent limit of $H_\Lambda + E_\Lambda$ as $\Lambda \to \infty$ and assume that $\mathcal{A}_{\alpha,\beta,-1}((1 + |k|)^{s - r(1-\beta)} v) < \infty$ and $\mathcal{B}_{\gamma,-1}((1 + |k|)^{s - r(1-\gamma)} v) < \infty$ for some $s \in [0, \min\{1 + r(1 - \beta), r\}]$. Then,*

$$D(H) \subset D(|p|^s) = D(\Omega(p)^{\frac{s}{r}}).$$

Proof. From Theorem 4.3.4 the norm-resolvent limit H exists. Let $R_\infty(z)$ be the resolvent of H for $z \in \rho(H)$ and let $\Psi \in \mathscr{H}$. Note that $R_\infty(z)\Psi \in D(H)$ for all $z \in \rho(H)$. We know that, for $\operatorname{Re} z \leq -1$ small enough, $R_\infty(z) = \lim_{\Lambda \to \infty} R_\Lambda(z)$, where $R_\Lambda(z)$ is the resolvent of $H_\Lambda + E_\Lambda$, and that we can expand $R_\Lambda(z)$ in its (reordered) series from Equation (4.33).

Let $A : D(A) \to \mathscr{H}$ be a self-adjoint operator. If we can show that for all $\Phi \in D(A)$

$$|\langle A\Phi, R_\infty(z)\Psi \rangle| \leq C_{z,\Psi} \|\Phi\| \qquad (C_{z,\Psi} < \infty) \tag{4.69}$$

(that means $\Phi \mapsto \langle A\Phi, R_\infty(z)\Psi \rangle$ is a bounded linear functional), then $R_\infty(z)\Psi \in D(A^*) = D(A)$ and therefore $D(H) \subset D(A)$.

Here, we choose $A = |p|^s$ with $s \in [0, \min\{1 + r(1-\beta), r\}]$ and consider

$$\begin{aligned}
|\langle |p|^s \Phi, R_\infty(z)\Psi\rangle| &= \left|\left\langle |p|^s\Phi, \lim_{\Lambda\to\infty} R_\Lambda(z)\Psi\right\rangle\right| = \lim_{\Lambda\to\infty} |\langle \Phi, |p|^s R_\Lambda(z)\Psi\rangle| \\
&\leq \|\Phi\| \limsup_{\Lambda\to\infty} \Bigg\| |p|^s R_0(z)\Psi \\
&\qquad + |p|^s \sum_{n=1}^{\infty} \sum_{\ell=0}^{[\frac{n}{2}]} \sum_{(j_1,..,j_{n-\ell})\in S(n,\ell)} R_0(z) \left(\prod_{i=1}^{n-\ell} a_{j_i,\Lambda} R_0(z)\right)\Psi \Bigg\|,
\end{aligned} \tag{4.70}$$

where we used the reordered series from Equation (4.33). If we can show that the operator in front of Ψ is a bounded operator uniformly in Λ, we are done. Then, $D(H) \subset D(|p|^s)$ and $D(|p|^s) = D(\Omega(p)^{s/r})$ by Lemma 4.4.1.

For the first summand, we easily get $\||p|^s R_0(z)\| \leq 1$ since $s \leq r$. Thus, the bigger second term remains. For that, we consider

$$\begin{aligned}
&|p|^s R_0(z) \left(\prod_{i=1}^{n-\ell} a_{j_i,\Lambda} R_0(z)\right) \\
&= |p|^s R_0(z)^{1-\eta_{j_1}-\varphi_{j_0}} \left(\prod_{i=1}^{n-\ell} R_0(z)^{\eta_{j_i}} a_{j_i,\Lambda} R_0(z)^{\varphi_{j_i}} R_0(z)^{1-\eta_{j_{i+1}}-\varphi_{j_i}}\right),
\end{aligned} \tag{4.71}$$

where $\eta_{j_i} \geq 0$, $\varphi_{j_i} \geq 0$ with $1 - \eta_{j_{i+1}} - \varphi_{j_i} \geq 0$ for all $i = 0, ..., n-\ell$ and we define $\varphi_{j_0} := 0 =: \eta_{j_{n-\ell+1}}$. As in the proof of Lemma 4.3.2, these numbers will later be chosen such that the terms $\|R_0(z)^{\eta_{j_i}} a_{j_i,\Lambda} R_0(z)^{\varphi_{j_i}}\|$ are bounded uniformly in Λ in view of Lemma C.3.1 and Corollary C.3.3.

The strategy now is to permute $|p|^s$ with the a_{j_i} and to use the resolvents that are not needed to estimate the a_{j_i} (say the $R_0(z)^{1-\eta_{j_{i+1}}-\varphi_{j_i}}$) to compensate the whole power s of $|p|$. It follows a value of s, that can maximally be compensated, namely

$$s \leq r \sum_{i=0}^{n-\ell} (1 - \eta_{j_{i+1}} - \varphi_{j_i}). \tag{4.72}$$

At some point, the whole power s will be used up and no further permutations will be needed. Also to resist this case, we define for $i = 0, ..., n-\ell$ the numbers

$$\tau_i := \begin{cases} 0 & : \; s \leq r \sum\limits_{k=0}^{i-1} (1 - \eta_{j_{k+1}} - \varphi_{j_k}) \\ s - r \sum\limits_{k=0}^{i-1} (1 - \eta_{j_{k+1}} - \varphi_{j_k}) & : \; \sum\limits_{k=0}^{i-1} (1 - \eta_{j_{k+1}} - \varphi_{j_k}) \leq \frac{s}{r} \leq \sum\limits_{k=0}^{i} (1 - \eta_{j_{k+1}} - \varphi_{j_k}) \\ r(1 - \eta_{j_{i+1}} - \varphi_{j_i}) & : \; s > r \sum\limits_{k=0}^{i} (1 - \eta_{j_{k+1}} - \varphi_{j_k}) \end{cases}, \tag{4.73}$$

where we use the notation $\sum\limits_{k=l}^{l-1} a_k = 0$ for some sequence $(a_k)_{k\in\mathbb{N}}$. We remark that $0 \leq \tau_i \leq r(1-\eta_{j_{i+1}} - \varphi_{j_i})$ for all $i = 0, ..., n-\ell$ and that $\sum\limits_{i=0}^{n-\ell} \tau_i = s$.

Using these numbers τ_i, we obtain by permutation the powers of $|p|$

$$
\begin{aligned}
&|p|^s R_0(z) \left(\prod_{i=1}^{n-\ell} a_{j_i,\Lambda} R_0(z) \right) \\
&= R_0(z)^{1-\eta_{j_1}-\varphi_{j_0}} |p|^{\tau_0} \left(\prod_{i=1}^{n-\ell} R_0(z)^{\eta_{j_i}} a_{j_i,\Lambda} R_0(z)^{\varphi_{j_i}} R_0(z)^{1-\eta_{j_{i+1}}-\varphi_{j_i}} |p|^{\tau_i} \right) \\
&\quad + R_0(z)^{1-\eta_{j_1}} |p|^{\tau_0} \sum_{k=1}^{n-\ell} \left(\prod_{i=1}^{k-1} R_0(z)^{\eta_{j_i}} a_{j_i,\Lambda} R_0(z)^{\varphi_{j_i}} R_0(z)^{1-\eta_{j_{i+1}}-\varphi_{j_i}} \right) \\
&\qquad R_0(z)^{\eta_{j_k}} \left[|p|^{s-\sum\limits_{i=0}^{k-1}\tau_i}, a_{j_k,\Lambda} \right] R_0(z)^{\varphi_{j_k}} \\
&\qquad \left(\prod_{i=k+1}^{n-\ell} R_0(z)^{1-\eta_{j_i}-\varphi_{j_{i-1}}} R_0(z)^{\eta_{j_i}} a_{j_i,\Lambda} R_0(z)^{\varphi_{j_i}} \right) R_0(z)^{1-\varphi_{j_{n-\ell}}},
\end{aligned}
\tag{4.74}
$$

where we use the notation $\prod\limits_{k=l}^{l-1} a_k = 1$ for some operator sequence $(a_k)_{k\in\mathbb{N}}$.

Since $\tau_i \leq r(1-\eta_{j_{i+1}} - \varphi_{j_i})$ for all $i = 0, ..., n-\ell$, we get

$$
\left\| R_0(z)^{1-\eta_{j_{i+1}}-\varphi_{j_i}} |p|^{\tau_i} \right\| \leq 1 \tag{4.75}
$$

for all $i = 0, ..., n-\ell$. With the right choice of η_{j_i} and φ_{j_i}, we can also bound the terms

$$
\| R_0(z)^{\eta_{j_i}} a_{j_i,\Lambda} R_0(z)^{\varphi_{j_i}} \| \tag{4.76}
$$

uniformly in Λ in view of Lemma C.3.1 and Corollary C.3.3, what will be done soon. Finally, we have to bound the terms

$$
\left\| R_0(z)^{\eta_{j_k}} \left[|p|^{s-\sum\limits_{i=0}^{k-1}\tau_i}, a_{j_k,\Lambda} \right] R_0(z)^{\varphi_{j_k}} \right\|, \tag{4.77}
$$

which will be done using Lemmas C.3.4 and C.3.5. In view of these lemmas, we see that we can control all these terms by controlling the one with the highest power of $|p|$. Since $s - \sum\limits_{i=0}^{k-1} \tau_i$ is a decreasing sequence with respect to k, this highest exponent is $s - \tau_0$ and from the definition of τ_0, we see that $s - \tau_0 = s - r(1-\eta_{j_1})$.

As in the proof of Lemma 4.3.2, we choose

$$\eta_k := \begin{cases} \beta & : \quad k=1 \\ \alpha & : \quad k=2\,, \\ \gamma & : \quad k=3 \end{cases} \qquad \varphi_k := \begin{cases} \alpha & : \quad k=1 \\ \beta & : \quad k=2\,, \\ \gamma & : \quad k=3 \end{cases} \tag{4.78}$$

and we remark that $1-\eta_{j_{i+1}}-\varphi_{j_i}\geq 0$ for all $i=0,...,n-\ell$ since occurring products of $a_{1,\Lambda}$ and $a_{2,\Lambda}$ in Equation (4.74) are normal-ordered. Recall that $\ell\in[0,[\frac{n}{2}]]$ is the number of $a_{3,\Lambda}$-operators in these products. Moreover, let $m\in[1,n-\ell]$ be the number of creation and annihilation operators in these products. Then, it is $m+\ell=n-\ell$ and we see that

$$s-r\sum_{i=0}^{n-\ell}(1-\eta_{j_{i+1}}-\varphi_{j_i})=s-r-rm(1-\alpha-\beta)-r\ell(1-2\gamma)\leq 0, \tag{4.79}$$

which has to be fulfilled, see Equation (4.72). In view of Equation (4.74) and remarks thereafter, we arrive at

$$\begin{aligned}
&\left\|\sum_{n=1}^{\infty}\sum_{\ell=0}^{[\frac{n}{2}]}\sum_{(j_1,...,j_{n-\ell})\in S(n,\ell)}|p|^s R_0(z)\left(\prod_{i=1}^{n-\ell}a_{j_i,\Lambda}R_0(z)\right)\right\| \\
&\leq \sum_{n=1}^{\infty}\sum_{\ell=0}^{[\frac{n}{2}]}\sum_{(j_1,...,j_{n-\ell})\in S(n,\ell)}\left\{\left(\prod_{i=1}^{n-\ell}\|R_0(z)^{\eta_{j_i}}a_{j_i,\Lambda}R_0(z)^{\varphi_{j_i}}\|\right)\right. \\
&\qquad+\sum_{k=1}^{n-\ell}\left(\prod_{i=1}^{k-1}\|R_0(z)^{\eta_{j_i}}a_{j_i,\Lambda}R_0(z)^{\varphi_{j_i}}\|\right)\left\|R_0(z)^{\eta_{j_k}}\left[|p|^{s-\sum_{i=0}^{k-1}\tau_i},a_{j_k,\Lambda}\right]R_0(z)^{\varphi_{j_k}}\right\| \\
&\qquad\left.\left(\prod_{i=k+1}^{n-\ell}\|R_0(z)^{\eta_{j_i}}a_{j_i,\Lambda}R_0(z)^{\varphi_{j_i}}\|\right)\right\}.
\end{aligned} \tag{4.80}$$

In view of Lemma C.3.1 and Corollary C.3.3, all terms of the form $\|R_0(z)^{\eta_{j_i}}a_{j_i,\Lambda}R_0(z)^{\varphi_{j_i}}\|$ can be estimated by $\mathcal{E}_{\alpha,\beta,\gamma,z}(v)$ uniformly in Λ. The terms with the commutators can be controlled by the one with the highest power, say

$$\left\|R_0(z)^{\eta_{j_1}}\left[|p|^{s-r(1-\eta_{j_1})},a_{j_1,\Lambda}\right]R_0(z)^{\varphi_{j_1}}\right\|. \tag{4.81}$$

Thereby, two terms have to be controlled, namely for $j_1=1$ and $j_1=3$. The case $j_1=1$ leads to

$$\left\|R_0(z)^{\beta}a^*\left(\left[|p|^{s-r(1-\beta)},G_\Lambda\right]\right)R_0(z)^{\alpha}\right\|\leq\mathcal{A}_{\alpha,\beta,-1}((1+|k|)^{s-r(1-\beta)}v), \tag{4.82}$$

where we could use Lemma C.3.4, since from assumptions $s - r(1-\beta) \leq 1$. For $j_1 = 3$, we have

$$\left\| R_0(z)^\gamma \left[|p|^{s-r(1-\gamma)}, a_{3,\Lambda} \right] R_0(z)^\gamma \right\| \leq \mathcal{B}_{\gamma,-1}((1+|k|)^{s-r(1-\gamma)} v) \tag{4.83}$$

by Lemma C.3.5, which we can use since $s - r(1-\gamma) \leq 1$ from the assumptions.

Define $M := \max\{\mathcal{A}_{\alpha,\beta,-1}((1+|k|)^{s-r(1-\beta)} v), \mathcal{B}_{\gamma,-1}((1+|k|)^{s-r(1-\gamma)} v)\}$ and note that $M < \infty$ from assumptions. Using also that $|S(n,\ell)| \leq 3^{n-\ell}$, we get as upper boundary for (4.80)

$$\sum_{n=1}^{\infty} \sum_{\ell=0}^{[\frac{n}{2}]} 3^{n-\ell} \mathcal{E}_{\alpha,\beta,\gamma,z}(v)^{n-\ell} + M \sum_{n=1}^{\infty} \sum_{\ell=0}^{[\frac{n}{2}]} 3^{n-\ell} (n-\ell) \mathcal{E}_{\alpha,\beta,\gamma,z}(v)^{n-\ell-1}. \tag{4.84}$$

The first series converges for $\operatorname{Re} z$ small enough, see the proof of Lemma 4.3.2. In the second series, we can estimate $n - \ell \leq 2^{n-\ell}$ and get

$$\sum_{n=1}^{\infty} \sum_{\ell=0}^{[\frac{n}{2}]} 6^{n-\ell} \mathcal{E}_{\alpha,\beta,\gamma,z}(v)^{n-\ell-1} \leq 36 \sum_{n=0}^{\infty} 6^n \mathcal{E}_{\alpha,\beta,\gamma,z}(v)^n \tag{4.85}$$

analogously to the proofs of Lemma 4.3.2 or Lemma 4.3.3, respectively. This series also converges for $\operatorname{Re} z$ small enough.

Therefore, we have shown that Expression (4.70) is bounded uniformly in Λ which completes the proof. $\square$

Theorem 4.4.3. *Assume (A0), (A1), (A2), and (A3). Let $H : D(H) \to \mathscr{H}$ be the norm-resolvent limit of $H_\Lambda + E_\Lambda$ as $\Lambda \to \infty$ and assume that $\mathcal{A}_{\alpha,\beta,-1}((1+\omega)^{s-(1-\beta)} v) < \infty$ and $\mathcal{B}_{\gamma,-1}((1+\omega)^{s-(1-\gamma)} v) < \infty$ for some $s \in [0,1]$. Then,*

$$D(H) \subset D(d\Gamma(\omega)^s).$$

Proof. From the same reasons as in the proof of Theorem 4.4.2, we show that $\Phi \mapsto \langle d\Gamma(\omega)^s \Phi, R_\infty(z)\Psi \rangle$ is a bounded linear functional for all $\Phi \in D(d\Gamma(\omega)^s)$. The proof is analogue to that one of Theorem 4.4.2.

We have

$$\begin{aligned} |\langle d\Gamma(\omega)^s \Phi, R_\infty(z)\Psi \rangle| &= \left| \left\langle d\Gamma(\omega)^s \Phi, \lim_{\Lambda\to\infty} R_\Lambda(z)\Psi \right\rangle \right| = \lim_{\Lambda\to\infty} |\langle \Phi, d\Gamma(\omega)^s R_\Lambda(z)\Psi \rangle| \\ &\leq \|\Phi\| \limsup_{\Lambda\to\infty} \Bigg\| d\Gamma(\omega)^s R_0(z)\Psi \\ &\qquad + d\Gamma(\omega)^s \sum_{n=1}^{\infty} \sum_{\ell=0}^{[\frac{n}{2}]} \sum_{(j_1,\dots,j_{n-\ell}) \in S(n,\ell)} R_0(z) \left(\prod_{i=1}^{n-\ell} a_{j_i,\Lambda} R_0(z) \right) \Psi \Bigg\| \end{aligned} \tag{4.86}$$

and for the first operator in the big norm we get $\|d\Gamma(\omega)^s R_0(z)\| \leq 1$ since $s \leq 1$.

Analogously to the proof before, we introduce the exponents η and φ and with them τ (but here with $r = 1$) and obtain

$$\begin{aligned}
&d\Gamma(\omega)^s R_0(z) \left(\prod_{i=1}^{n-\ell} a_{j_i,\Lambda} R_0(z) \right) \\
&= R_0(z)^{1-\eta_{j_1}-\varphi_{j_0}} d\Gamma(\omega)^{\tau_0} \left(\prod_{i=1}^{n-\ell} R_0(z)^{\eta_{j_i}} a_{j_i,\Lambda} R_0(z)^{\varphi_{j_i}} R_0(z)^{1-\eta_{j_{i+1}}-\varphi_{j_i}} d\Gamma(\omega)^{\tau_i} \right) \\
&\quad + R_0(z)^{1-\eta_{j_1}} d\Gamma(\omega)^{\tau_0} \sum_{k=1}^{n-\ell} \left(\prod_{i=1}^{k-1} R_0(z)^{\eta_{j_i}} a_{j_i,\Lambda} R_0(z)^{\varphi_{j_i}} R_0(z)^{1-\eta_{j_{i+1}}-\varphi_{j_i}} \right) \\
&\qquad R_0(z)^{\eta_{j_k}} \left[d\Gamma(\omega)^{s-\sum_{i=0}^{k-1} \tau_i}, a_{j_k,\Lambda} \right] R_0(z)^{\varphi_{j_k}} \\
&\qquad \left(\prod_{i=k+1}^{n-\ell} R_0(z)^{1-\eta_{j_i}-\varphi_{j_{i-1}}} R_0(z)^{\eta_{j_i}} a_{j_i,\Lambda} R_0(z)^{\varphi_{j_i}} \right) R_0(z)^{1-\varphi_{j_{n-\ell}}}.
\end{aligned} \tag{4.87}$$

Analogously to the proof of Theorem 4.4.2, $\left\| R_0(z)^{1-\eta_{j_{i+1}}-\varphi_{j_i}} d\Gamma(\omega)^{\tau_i} \right\| \leq 1$, and analogously, to control the terms of the form

$$\left\| R_0(z)^{\eta_{j_k}} \left[d\Gamma(\omega)^{s-\sum_{i=0}^{k-1} \tau_i}, a_{j_k,\Lambda} \right] R_0(z)^{\varphi_{j_k}} \right\|, \tag{4.88}$$

it is enough to restrict to the one with the biggest exponent, say $s - (1 - \eta_{j_1})$.

We choose η_k and φ_k as in the proof of Theorem 4.4.2. Then, terms of (4.87) of the form $\|R_0(z)^{\eta_{j_i}} a_{j_i,\Lambda} R_0(z)^{\varphi_{j_i}}\|$ can be estimated by $\mathcal{E}_{\alpha,\beta,\gamma,z}(v)$ by using Lemma C.3.1 and Corollary C.3.3. We get

$$\begin{aligned}
&\left\| \sum_{n=1}^{\infty} \sum_{\ell=0}^{[\frac{n}{2}]} \sum_{(j_1,\dots,j_{n-\ell}) \in S(n,\ell)} d\Gamma(\omega)^s R_0(z) \left(\prod_{i=1}^{n-\ell} a_{j_i,\Lambda} R_0(z) \right) \right\| \\
&\leq \sum_{n=1}^{\infty} \sum_{\ell=0}^{[\frac{n}{2}]} \sum_{(j_1,\dots,j_{n-\ell}) \in S(n,\ell)} \Big\{ \mathcal{E}_{\alpha,\beta,\gamma,z}(v)^{n-\ell} \\
&\qquad + (n-\ell) \mathcal{E}_{\alpha,\beta,\gamma,z}(v)^{n-\ell-1} \left\| R_0(z)^{\eta_{j_1}} \left[d\Gamma(\omega)^{s-(1-\eta_{j_1})}, a_{j_1,\Lambda} \right] R_0(z)^{\varphi_{j_1}} \right\| \Big\}.
\end{aligned} \tag{4.89}$$

Again, we use $|S(n,\ell)| \leq 3^{n-\ell}$, the norm with the commutator can be controlled by using Lemma C.3.6 and Lemma C.3.7 by

$$M := \max\{\mathcal{A}_{\alpha,\beta,-1}((1+\omega)^{s-(1-\beta)} v), \mathcal{B}_{\gamma,-1}((1+\omega)^{s-(1-\gamma)} v)\}, \tag{4.90}$$

which is finite by assumptions. We arrive at

$$\sum_{n=1}^{\infty}\sum_{\ell=0}^{[\frac{n}{2}]} 3^{n-\ell}\mathcal{E}_{\alpha,\beta,\gamma,z}(v)^{n-\ell} + M\sum_{n=1}^{\infty}\sum_{\ell=0}^{[\frac{n}{2}]} 3^{n-\ell}(n-\ell)\mathcal{E}_{\alpha,\beta,\gamma,z}(v)^{n-\ell-1}, \tag{4.91}$$

which is bounded uniformly in Λ for $\operatorname{Re} z$ sufficiently small, see the proof of Theorem 4.4.2. This completes the proof. □

4.5. Applications and Examples

Finally, we will apply our results of the previous section to some famous systems that are included in our general assumptions made at the beginning. Such models are for example the Fröhlich Hamiltonian, the Nelson Hamiltonian, and the relativistic Nelson model as it was used by Sloan [20].

4.5.1. Fröhlich Hamiltonian

We start with the Fröhlich-like models we already studied in the article [23] and Chapter 2, respectively. These are included in the operator (4.18) by

$$\Omega(p) = p^2, \quad \omega(k) = 1 \tag{4.92}$$

and for the form factor $v \in L^2_{loc}(\mathbb{R}^d)$ we assume

$$\sup_{p\in\mathbb{R}^d} \int\limits_{|k|\geq K} dk\, \frac{|v(k)|^2}{1+(p-k)^2} \to 0 \quad (K\to\infty). \tag{4.93}$$

The Hamiltonian with ultraviolet cutoff then reads

$$H_\Lambda = p^2 + N + \phi(G_\Lambda). \tag{4.94}$$

Note that Assumption (4.93) implies $E_\infty < \infty$. This means that for these models no renormalization is needed. Here, the constant E_∞ is just a constant energy shift.

At first, we have to check, if this operator satisfies the assumptions $(A1)$, $(A2)$, and $(A3)$. For that, we choose $\alpha = \beta = \gamma = \frac{1}{2}$, which fulfills $0 < \alpha, \gamma \leq \frac{1}{2} \leq \beta \leq 1$, $\alpha + \beta \leq 1$, and $\beta + \gamma \leq 1$. We see that the assumptions $(A1)$ and $(A2)$ reduces to

$$\sup_{p\in\mathbb{R}^d} \int\limits_{|k|\geq K} dk\, \frac{|v(k)|^2}{2+(p-k)^2} \to 0 \quad (K\to\infty) \tag{4.95}$$

which is implied by (4.93). For $(A3)$, we consider

$$\frac{|v(k)|^2\big|(p-k)^2-k^2\big|}{[1+k^2][1+p^2][2+(p-k)^2]} \le \frac{|v(k)|^2}{1+(p-k)^2}\,\frac{2p^2+k^2}{[1+k^2][1+p^2]} \le \frac{3|v(k)|^2}{1+(p-k)^2}, \tag{4.96}$$

such that (4.93) also implies

$$\sup_{p\in\mathbb{R}^d}\int\limits_{|k|\ge K} dk\,\frac{|v(k)|^2\big|(p-k)^2-k^2\big|}{[1+k^2][1+p^2][2+(p-k)^2]} \to 0 \quad (K\to\infty), \tag{4.97}$$

which is assumption $(A3)$.

Now, from Theorem 4.3.4, we obtain that $H_\Lambda + E_\Lambda$ has a norm-resolvent limit as $\Lambda\to\infty$, we call it H, and this H is bounded from below.

Now, set

$$v(k) = |k|^{-(d-1)/2}, \quad d\ge 2 \tag{4.98}$$

which satisfies (4.93) and includes the large polaron models in $d=2$ and $d=3$ dimensions. In view of Theorem 4.4.2, we consider

$$\mathcal{A}_{\alpha,\beta,-1}((1+|k|)^{s-1}v)^2 = \left(1+\frac{1}{\sqrt{m}}\right)^2 \sup_{p\in\mathbb{R}^d}\int dk\,\frac{(1+|k|)^{2(s-1)}}{|k|^{d-1}[2+(p-k)^2]}, \tag{4.99}$$

$$\mathcal{B}_{\gamma,-1}((1+|k|)^{s-1}v) = 2\sup_{p\in\mathbb{R}^d}\int dk\,\frac{(1+|k|)^{2(s-1)}}{|k|^{d-1}[2+(p-k)^2]}, \tag{4.100}$$

which are both finite for $s<\frac{3}{2}$. Use Hölder's inequality to see this.

In view of Theorem 4.4.3, we see that the conditions $\mathcal{A}_{\alpha,\beta,-1}(2^{s-1/2}v)<\infty$ and $\mathcal{B}_{\gamma,-1}(2^{s-1/2}v)<\infty$ are already satisfied by $(A0)$, $(A1)$, and $(A2)$. Therefore, we conclude

$$D(H) \subset \bigcap_{1\le s<\frac{3}{2}} D(|p|^s)\cap D(N), \tag{4.101}$$

such that we reproduced one of our results from [23] and Chapter 2.

4.5.2. Nelson Hamiltonian

If we choose

$$\Omega(p)=p^2, \quad \omega(k)=\sqrt{m^2+k^2}, \quad m>0, \quad v(k)=\frac{1}{\sqrt{\omega(k)}}, \tag{4.102}$$

the Operator (4.18) yields the Nelson Hamiltonian with ultraviolet cutoff

$$H_\Lambda := p^2 + d\Gamma(\omega) + \phi(G_\Lambda), \tag{4.103}$$

which we consider in $d = 2$ and $d = 3$ dimensions. Note that in $d = 2$ dimensions, E_∞ is finite. In $d = 3$ dimensions, $E_\infty = \infty$ such that it has the role of a renormalization constant in this case.

We choose $\alpha = \gamma = \frac{1+\delta}{4}$ and $\beta = \frac{3-\delta}{4}$ with a $\delta \in (-1, 1]$. Then, Assumption $(A1)$ leads to the condition

$$\sup_{p\in\mathbb{R}^d} \int\limits_{|k|\geq K} dk \, \frac{1}{\omega(k)^{2-\frac{1+\delta}{2}}[1+(p-k)^2+\omega(k)]} \to 0 \quad (K\to\infty), \tag{4.104}$$

which is satisfied if $\delta < 7 - 2d$, use Hölder's inequality to see this. Assumption $(A2)$ leads to

$$\sup_{p\in\mathbb{R}^d} \int\limits_{|k|\geq K} dk \, \frac{1}{\omega(k)^2[1+(p-k)^2+\omega(k)]^{\frac{1+\delta}{2}}} \to 0 \quad (K\to\infty), \tag{4.105}$$

which is satisfied if $\delta > d - 3$, use also Hölder's inequality. For $(A3)$, we consider

$$\begin{aligned}
&\frac{\left|(p-k)^2-k^2\right|}{\omega(k)[\omega(k)+k^2][1+p^2]^{\frac{1+\delta}{2}}[1+(p-k)^2+\omega(k)]} \\
&\leq \frac{p^2+2|p|\;|k|}{\omega(k)[\omega(k)+k^2][1+p^2]^{\frac{1+\delta}{2}}[1+(p-k)^2+\omega(k)]} \\
&\leq \frac{2}{[\omega(k)+k^2][1+(p-k)^2]} + \frac{1}{\omega(k)[\omega(k)+k^2][1+(p-k)^2]^{\frac{1+\delta}{2}}}\left(\frac{p^2}{1+(p-k)^2}\right)^{\frac{1-\delta}{2}} \\
&\leq \frac{2}{[\omega(k)+k^2][1+(p-k)^2]} + \frac{(2(1+k^2))^{\frac{1-\delta}{2}}}{\omega(k)[\omega(k)+k^2][1+(p-k)^2]^{\frac{1+\delta}{2}}},
\end{aligned} \tag{4.106}$$

where we used $\frac{p^2}{1+(p-k)^2} \leq 2(1+k^2)$ in the last step. For $(A3)$, this leads to

$$\begin{aligned}
&\sup_{p\in\mathbb{R}^d} \int\limits_{|k|\geq K} dk \, \frac{\left|(p-k)^2-k^2\right|}{\omega(k)[\omega(k)+k^2][1+p^2]^{\frac{1+\delta}{2}}[1+(p-k)^2+\omega(k)]} \\
&\leq \sup_{p\in\mathbb{R}^d} \int\limits_{|k|\geq K} dk \, \frac{2}{[\omega(k)+k^2][1+(p-k)^2]} \\
&\quad + \sup_{p\in\mathbb{R}^d} \int\limits_{|k|\geq K} dk \, \frac{2^{\frac{1-\delta}{2}}(1+k^2)^{\frac{1-\delta}{2}}}{\omega(k)[\omega(k)+k^2][1+(p-k)^2]^{\frac{1+\delta}{2}}}.
\end{aligned} \tag{4.107}$$

Using Cauchy-Schwarz for the first summand and Hölder's inequality with $\delta > \frac{d-3}{2}$ for the second, shows that this term vanishes as $K \to \infty$ such that $(A3)$ is also satisfied.

Now again, from Theorem 4.3.4, $H_\Lambda + E_\Lambda$ has a norm-resolvent limit as $\Lambda \to \infty$, we call it H, and this H is bounded from below. In $d = 2$ dimensions, we can choose all $\delta \in (-\frac{1}{2}, 1]$ and in $d = 3$ dimensions all $\delta \in (0, 1)$.

In view of Theorem 4.4.2, we consider

$$\begin{aligned} \mathcal{A}_{\alpha,\beta,-1}((1+|k|)^{s-\frac{1+\delta}{2}} v)^2 &= \left(1+m^{-\alpha}\right)^2 \\ &\quad \cdot \sup_{p\in\mathbb{R}^d} \int dk \frac{(1+|k|)^{2s-1-\delta}}{\omega(k)^{\frac{3-\delta}{2}}[1+(p-k)^2+\omega(k)]}, \qquad (4.108) \\ \mathcal{B}_{\gamma,-1}((1+|k|)^{s-\frac{3-\delta}{2}} v) &= 2 \sup_{p\in\mathbb{R}^d} \int dk \, \frac{(1+|k|)^{2s-3+\delta}}{\omega(k)^2[1+(p-k)^2+\omega(k)]^{\frac{1+\delta}{2}}}, \qquad (4.109) \end{aligned}$$

which are both finite for $s < \frac{9+\delta}{4} - \frac{d}{2}$. Use Hölder's inequality to see this.

In view of Theorem 4.4.3, we consider

$$\begin{aligned} \mathcal{A}_{\alpha,\beta,-1}((1+\omega)^{s-\frac{1+\delta}{4}} v)^2 &= \left(1+m^{-\alpha}\right)^2 \\ &\quad \cdot \sup_{p\in\mathbb{R}^d} \int dk \, \frac{(1+\omega(k))^{2s-\frac{1+\delta}{2}}}{\omega(k)^{\frac{3-\delta}{2}}[1+(p-k)^2+\omega(k)]}, \qquad (4.110) \\ \mathcal{B}_{\gamma,-1}((1+\omega)^{s-\frac{3-\delta}{4}} v) &= 2 \sup_{p\in\mathbb{R}^d} \int dk \, \frac{(1+\omega(k))^{2s-\frac{3-\delta}{2}}}{\omega(k)^2[1+(p-k)^2+\omega(k)]^{\frac{1+\delta}{2}}}, \qquad (4.111) \end{aligned}$$

which are both finite for $s < 2 - \frac{d}{2}$. Use also Hölder's inequality to see this.

We see that in view of both Theorems, we can maximize the range of allowed s-values by maximizing δ. This leads for $d = 2$ to the result

$$D(H) \subset \bigcap_{0\le s<\frac{3}{2}} D(|p|^s) \cap \bigcap_{0\le s<1} D(d\Gamma(\omega)^s) \qquad (4.112)$$

and for $d = 3$ to

$$D(H) \subset \bigcap_{0\le s<1} D(|p|^s) \cap \bigcap_{0\le s<\frac{1}{2}} D(d\Gamma(\omega)^s). \qquad (4.113)$$

With respect to the momentum operator, we reproduced the property we have also shown in Chapter 3. But our inclusions with respect to the field energy operator are not as good as in that work.

4.5.3. Pseudorelativistic Nelson Hamiltonian

In this example, we choose in $d = 2$ dimensions

$$\Omega(p) = \sqrt{\mu^2 + p^2}, \quad \mu \geq 0, \quad \omega(k) = \sqrt{m^2 + k^2}, \quad m > 0, \quad v(k) = \frac{1}{\sqrt{\omega(k)}}, \tag{4.114}$$

and obtain the pseudorelativstic Nelson Hamiltonian with ultraviolet cutoff

$$H_\Lambda := \sqrt{\mu^2 + p^2} + d\Gamma(\omega) + \phi(G_\Lambda) \tag{4.115}$$

as it was studied by Fröhlich [7] and Sloan [20]. Note that in $d = 2$ dimensions, $E_\infty = \infty$ such that it has the role of a renormalization constant.

Now, we check if the Assumptions $(A1)$, $(A2)$, and $(A3)$ are satisfied. For Assumption $(A1)$, we get

$$\sup_{p \in \mathbb{R}^2} \int\limits_{|k| \geq K} dk \, \frac{1}{\omega(k)^{2-2\alpha}[1 + \Omega(p-k) + \omega(k)]^{2(\alpha+\beta)-1}} \leq \int\limits_{|k| \geq K} dk \, \frac{1}{\omega(k)^{1+2\beta}}, \tag{4.116}$$

which goes to 0 as $K \to \infty$ for $\beta > \frac{1}{2}$. Assumption $(A2)$ leads to

$$\sup_{p \in \mathbb{R}^2} \int\limits_{|k| \geq K} dk \, \frac{1}{\omega(k)^2[1 + \Omega(p-k) + \omega(k)]^{2\gamma}} \leq \int\limits_{|k| \geq K} dk \, \frac{1}{\omega(k)^{2+2\gamma}}, \tag{4.117}$$

which goes to 0 as $K \to \infty$ for $\gamma > 0$. For $(A3)$, note that $|\Omega(p-k) - \Omega(k)| \leq \Omega(p)$. We consider

$$\begin{aligned}
&\frac{|\Omega(p-k) - \Omega(k)|}{\omega(k)[\omega(k) + \Omega(k)][1 + \Omega(p)]^{2\gamma}[1 + \Omega(p-k) + \omega(k)]} \\
&\leq \frac{\Omega(p)^{1-2\gamma}}{\omega(k)[\omega(k) + \Omega(k)][1 + \Omega(p-k) + \omega(k)]} \\
&\leq \frac{1}{\omega(k)[\omega(k) + \Omega(k)][1 + \Omega(p-k) + \omega(k)]^{2\gamma}} \left(\frac{\Omega(p)}{1 + |p|} \right)^{1-2\gamma} \\
&\leq (1 + \mu)^{1-2\gamma} \frac{1}{\omega(k)[\omega(k) + \Omega(k)][1 + \Omega(p-k) + \omega(k)]^{2\gamma}},
\end{aligned} \tag{4.118}$$

which leads for $\gamma > 0$ to

$$\begin{aligned}
&\sup_{p \in \mathbb{R}^2} \int\limits_{|k| \geq K} dk \, \frac{|\Omega(p-k) - \Omega(k)|}{\omega(k)[\omega(k) + \Omega(k)][1 + \Omega(p)]^{2\gamma}[1 + \Omega(p-k) + \omega(k)]} \\
&\leq (1 + \mu)^{1-2\gamma} \int\limits_{|k| \geq K} dk \, \frac{1}{\omega(k)^{3+2\gamma}} \to 0 \quad (K \to \infty).
\end{aligned} \tag{4.119}$$

From Theorem 4.3.4, $H_\Lambda + E_\Lambda$ has a norm-resolvent limit as $\Lambda \to \infty$, we call it H, and this H is bounded from below. We can choose $\alpha = \gamma = \frac{\delta}{2}$ and $\beta = \frac{2-\delta}{2}$ for a $\delta \in (0,1)$.

In view of Theorem 4.4.2, we consider

$$\mathcal{A}_{\alpha,\beta,-1}((1+|k|)^{s-\frac{\delta}{2}}v)^2 \leq \left(1+\frac{1}{m^{\frac{\delta}{2}}}\right)^2 \int dk\ \frac{(1+|k|)^{2s-\delta}}{\omega(k)^{3-\delta}}, \tag{4.120}$$

$$\mathcal{B}_{\gamma,-1}((1+|k|)^{s-\frac{2-\delta}{2}}v) \leq 2\int dk\ \frac{(1+|k|)^{2s-2+\delta}}{\omega(k)^{2+\delta}}, \tag{4.121}$$

which are both finite for $s < \frac{1}{2}$.

In view of Theorem 4.4.3, we consider

$$\mathcal{A}_{\alpha,\beta,-1}((1+\omega)^{s-\frac{\delta}{2}}v)^2 \leq \left(1+\frac{1}{m^{\frac{\delta}{2}}}\right)^2 \int dk\ \frac{(1+\omega(k))^{2s-\delta}}{\omega(k)^{3-\delta}}, \tag{4.122}$$

$$\mathcal{B}_{\gamma,-1}((1+\omega)^{s-\frac{2-\delta}{2}}v) \leq 2\int dk\ \frac{(1+\omega(k))^{2s-2+\delta}}{\omega(k)^{2+\delta}}, \tag{4.123}$$

which are both finite for $s < \frac{1}{2}$.

Finally, in view of the Theorems 4.4.2 and 4.4.3, we obtain for this pseudorelativistic Nelson model in $d = 2$ the result

$$D(H) \subset \bigcap_{0\leq s<\frac{1}{2}} \big(D(|p|^s) \cap D(d\Gamma(\omega)^s)\big). \tag{4.124}$$

A. Appendix for Chapter 2: Self-Adjointness and Domain of the Fröhlich Hamiltonian

This chapter is the appendix for Chapter 2. Up to slight changes, it corresponds to the appendix of the article [23].

A.1. Quadratic forms and resolvent convergence

The following theorem is our main tool for the proof of Theorem 2.2.2. It is essentially due to Nelson [1]. A similar theorem, without proof, is given in the Appendix of the thesis of Ammari [2]. In view of the Theorem A.1 of the article [23], an additional statement is added that is needed for the proof of Theorem 3.4.3 in Section 3.4.1.

Theorem A.1.1. *Let $H_0 \geq 0$ be a self-adjoint operator in the Hilbert space $\mathscr{H}$ and let $\|\Psi\|_0 := \|(H_0+1)^{1/2}\Psi\|$ for $\Psi \in D(H_0^{1/2})$. For each $\Lambda < \infty$ let W_Λ be a quadratic form defined on $D(H_0^{1/2})$ such that*

(a) for all $\Psi \in D(H_0^{1/2})$ and all $\Lambda < \infty$,

$$|W_\Lambda(\Psi)| \leq a\|\Psi\|_0^2 + b_\Lambda\|\Psi\|^2,$$

where $a < 1$,

(b) for all $\Psi \in D(H_0^{1/2})$,

$$|W_\Lambda(\Psi) - W_{\Lambda'}(\Psi)| \leq C_{\Lambda,\Lambda'}\|\Psi\|_0^2,$$

where $C_{\Lambda,\Lambda'} \to 0$ as $\Lambda, \Lambda' \to \infty$.

Let $W_\infty(\Phi,\Psi) := \lim_{\Lambda\to\infty} W_\Lambda(\Phi,\Psi)$. Then, (a) extends to $\Lambda = \infty$ with some finite b_∞, and for each $\Lambda \leq \infty$, there exists a self-adjoint, semibounded operator H_Λ with $D(H_\Lambda) \subset D(H_0^{1/2})$ and

$$\langle \Phi, H_\Lambda\Psi\rangle = \left\langle H_0^{1/2}\Phi, H_0^{1/2}\Psi\right\rangle + W_\Lambda(\Phi,\Psi) \tag{A.1}$$

for all $\Phi \in D(H_0^{1/2})$ and $\Psi \in D(H_\Lambda)$. Furthermore, for all $z \in \mathbb{C}\backslash\mathbb{R}$,

$$(H_\Lambda - z)^{-1} \to (H_\infty - z)^{-1} \qquad (\Lambda \to \infty)$$

in the operator norm.
Let moreover U_Λ, $0 < \Lambda \leq \infty$, be a one-parameter family of unitary operators in $\mathscr{H}$ with

$$(H_0 + 1)^{-1/2} (U_\Lambda - U_\infty) \to 0 \qquad (\Lambda \to \infty)$$

in the operator norm. Then, for all $z \in \mathbb{C}\backslash\mathbb{R}$,

$$(U_\Lambda^* H_\Lambda U_\Lambda - z)^{-1} \to (U_\infty^* H_\infty U_\infty - z)^{-1} \qquad (\Lambda \to \infty)$$

in the operator norm.

Proof. Choose $\Lambda_0 > 0$ so large, that $C_{\Lambda,\Lambda'} \leq (1-a)/2$ for all $\Lambda, \Lambda' \geq \Lambda_0$. Then, for $\Lambda \geq \Lambda_0$

$$\begin{aligned} |W_\Lambda(\Psi)| &\leq |W_\Lambda(\Psi) - W_{\Lambda_0}(\Psi)| + |W_{\Lambda_0}(\Psi)| \\ &\leq C_{\Lambda,\Lambda_0} \|\Psi\|_0^2 + a\|\Psi\|_0^2 + b_{\Lambda_0}\|\Psi\|^2 \\ &\leq \frac{1}{2}(1+a)\|\Psi\|_0^2 + b_{\Lambda_0}\|\Psi\|^2. \end{aligned} \tag{A.2}$$

In the limit $\Lambda \to \infty$, it follows that

$$|W_\infty(\Psi)| \leq \frac{1}{2}(1+a)\|\Psi\|_0^2 + b_{\Lambda_0}\|\Psi\|^2. \tag{A.3}$$

From assumption (a) and from (A.2), it follows that for each $\Lambda \leq \infty$ the quadratic form

$$\left\langle H_0^{1/2}\Phi, H_0^{1/2}\Psi \right\rangle + W_\Lambda(\Phi, \Psi) \tag{A.4}$$

with $\Phi, \Psi \in D(H_0^{1/2})$ is closed, bounded from below and hence associated with a unique self-adjoint operator H_Λ such that (A.1) holds (see [27]). The Inequalities (A.2) and (A.3) imply that

$$H_0 \leq \frac{2}{1-a}(H_\Lambda + M), \quad \Lambda_0 \leq \Lambda \leq \infty, \tag{A.5}$$

where $M := b_{\Lambda_0} + 1$. By assumption (b), $C_\Lambda := \limsup_{\Lambda' \to \infty} C_{\Lambda,\Lambda'} \xrightarrow{\Lambda \to \infty} 0$ and

$$|W_\Lambda(\Psi) - W_\infty(\Psi)| = \lim_{\Lambda' \to \infty} |W_\Lambda(\Psi) - W_{\Lambda'}(\Psi)| \leq C_\Lambda \|\Psi\|_0^2. \tag{A.6}$$

Using (A.6), we conclude that for all $\Lambda \geq \Lambda_0$ and all $\Phi, \Psi \in \mathscr{H}$,

$$\begin{aligned}
&|\langle \Phi, (R_\Lambda(z) - R_\infty(z))\, \Psi\rangle| \\
&= |\langle R_\Lambda(\overline{z})\Phi, (H_\infty - z)R_\infty(z)\Psi\rangle - \langle (H_\Lambda - \overline{z})R_\Lambda(\overline{z})\Phi, R_\infty(z)\Psi\rangle| \\
&= |W_\infty(R_\Lambda(\overline{z})\Phi, R_\infty(z)\Psi) - W_\Lambda(R_\Lambda(\overline{z})\Phi, R_\infty(z)\Psi)| \\
&\leq C_\Lambda \|R_\Lambda(\overline{z})\Phi\|_0 \|R_\infty(z)\Psi\|_0
\end{aligned} \tag{A.7}$$

where, by (A.5), $\|R_\Lambda(\overline{z})\Phi\|_0 \leq C_z\|\Phi\|$ and $\|R_\infty(z)\Psi\|_0 \leq C_z\|\Psi\|$ with C_z independent of Λ for $\Lambda_0 \leq \Lambda \leq \infty$.

Now, let $\tilde{R}_\Lambda(z)$ be the resolvent of $U_\Lambda^* H_\Lambda U_\Lambda$ for all $\Lambda \leq \infty$. We obtain

$$\begin{aligned}
&\left|\left\langle \Phi, \left(\tilde{R}_\Lambda(z) - \tilde{R}_\infty(z)\right) \Psi\right\rangle\right| = |\langle \Phi, (U_\Lambda^* R_\Lambda(z) U_\Lambda - U_\infty^* R_\infty(z) U_\infty)\, \Psi\rangle| \\
&\leq |\langle (U_\Lambda - U_\infty)\Phi, R_\Lambda(z)U_\Lambda\Psi\rangle| + |\langle (R_\Lambda(\overline{z}) - R_\infty(\overline{z}))U_\infty\Phi, U_\Lambda\Psi\rangle| \\
&\quad + |\langle R_\infty(\overline{z})U_\infty\Phi, (U_\Lambda - U_\infty)\Psi\rangle|.
\end{aligned} \tag{A.8}$$

Because of calculation (A.7) and because of the unitarity of U_Λ, we get for the second summand

$$\begin{aligned}
|\langle (R_\Lambda(\overline{z}) - R_\infty(\overline{z}))U_\infty\Phi, U_\Lambda\Psi\rangle| &\leq C_\Lambda \|R_\Lambda(\overline{z})U_\infty\Phi\|_0 \|R_\infty(z)U_\Lambda\Psi\|_0 \\
&\leq C_\Lambda C_z^2 \|\Phi\|\, \|\Psi\|.
\end{aligned} \tag{A.9}$$

For the first summand of Equation (A.8), we estimate

$$\begin{aligned}
&\left|\left\langle (H_0+1)^{-1/2}(U_\Lambda - U_\infty)\Phi, (H_0+1)^{1/2}R_\Lambda(z)U_\Lambda\Psi\right\rangle\right| \\
&\leq \left\|(H_0+1)^{-1/2}(U_\Lambda - U_\infty)\right\| \, \|\Phi\| \, \left\|(H_0+1)^{1/2}R_\Lambda(z)U_\Lambda\Psi\right\|
\end{aligned} \tag{A.10}$$

The first factor goes to zero for $\Lambda \to \infty$ from assumptions, the last factor is bounded uniformly in Λ since from Equation (A.5)

$$\left\|(H_0+1)^{1/2}R_\Lambda(z)U_\Lambda\Psi\right\| \leq \sqrt{\frac{2}{1-a}}\left\|(H_\Lambda+M)^{1/2}R_\Lambda(z)U_\Lambda\Psi\right\| \leq \sqrt{\frac{2}{1-a}}\|\Psi\|. \tag{A.11}$$

The third summand of Equation (A.8) can be estimated analogously, such that each summand of (A.8) goes to zero as $\Lambda \to \infty$. □

A.2. Creation and annihilation operators

Let $\mathcal{F} = \bigoplus_{n\geq 0} \mathcal{F}_n$ be the symmetric Fock space over some Hilbert space $\mathfrak{h}$, let $\mathscr{H} = L^2(\mathbb{R}^d) \otimes \mathcal{F}$, and let $\mathscr{H}_0 = \bigcup_{n\geq 0} \chi(N \leq n)\mathscr{H}$. Suppose

$$B : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \otimes \mathfrak{h} \tag{A.12}$$

is a bounded linear operator. Then, we define the operator $a^*(B)$ in $\mathscr{H}$ on vectors $\Psi = (\Psi^{(n)})_{n\geq 0} \in \mathscr{H}$ by

$$a^*(B)\Psi^{(n)} = \sqrt{n+1}\ S_{n+1}(B \otimes \mathbb{1})\Psi^{(n)}, \tag{A.13}$$

where S_{n+1} denotes the orthogonal projection from $\bigotimes^{n+1}\mathfrak{h}$ onto $\mathcal{F}_{n+1}$. We define the annihilation operator $a(B)$ in such a way that $\langle a(B)\Phi, \Psi\rangle = \langle \Phi, a^*(B)\Psi\rangle$ for all $\Phi, \Psi \in \mathscr{H}_0$. This is achieved by setting $a(B) = 0$ on $L^2(\mathbb{R}^d) \otimes \mathcal{F}_{n=0}$ and

$$a(B)\Psi^{(n)} = \sqrt{n}\ (B^* \otimes \mathbb{1})\Psi^{(n)}. \tag{A.14}$$

Since $\mathscr{H}_0$ is dense in $\mathscr{H}$, it follows that both $a(B)$ and $a^*(B)$ are closable, and we denote the closures by $a(B)$ and $a^*(B)$ as well. It is straightforward to show that $a^*(B)$ is the adjoint of $a(B)$, see for example [28].

The following lemma easily follows from (A.13) and (A.14):

Lemma A.2.1. *Let $B : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \otimes \mathfrak{h}$ be a bounded linear operator with norm $\|B\|$. Let $a^\#$ stand for a or a^*. Then, $D(a^\#(B)) \supset D(\sqrt{N})$, and for all $\Psi \in D(\sqrt{N})$,*

$$\|a(B)\Psi\| \leq \|B\| \left\|\sqrt{N}\ \Psi\right\|, \qquad \|a^*(B)\Psi\| \leq \|B\| \left\|\sqrt{N+1}\ \Psi\right\|.$$

Creation and annihilation operators $a^*(f)$ and $a(f)$ for $f \in \mathfrak{h}$ are defined in terms of the linear operator from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d) \otimes \mathfrak{h}$ which maps Ψ to $\Psi \otimes f$. The norm of this operator is $\|f\|$. Lemma A.2.1, therefore, implies

Corollary A.2.2. *For all $f \in \mathfrak{h}$ and all $\Psi \in D(\sqrt{N})$*

$$\|a(f)\Psi\| \leq \|f\| \left\|\sqrt{N}\ \Psi\right\|, \qquad \|a^*(f)\Psi\| \leq \|f\| \left\|\sqrt{N+1}\ \Psi\right\|.$$

From these Estimates and from the pull-through formulas $a(f)N = (N+1)a(f)$ and $Na^*(f) = a^*(f)(N+1)$ the next two lemmas follow easily.

Lemma A.2.3. *Let $a^\#$ stand for a or a^*. For all $f, g \in L^2(\mathbb{R}^d)$,*

$$\begin{aligned} \left\|a^\#(f)a^\#(g)(N+1)^{-1}\right\| &\leq \sqrt{2}\|f\|\ \|g\|, \\ \left\|\phi(f)^2(N+1)^{-1}\right\| &\leq 4\sqrt{2}\|f\|^2. \end{aligned}$$

Lemma A.2.4. *Let $f, g \in L^2(\mathbb{R}^d)$. Then, the domains of $\phi(g)$ and N^σ, for $\sigma \in [0,1]$, are left invariant by $e^{-i\pi(f)}$ and*

$$\begin{aligned} e^{i\pi(f)}\phi(g)e^{-i\pi(f)} &= \phi(g) + 2\operatorname{Re}\langle f, g\rangle \quad \text{on } D(\phi(g)), \\ e^{i\pi(f)}Ne^{-i\pi(f)} &= N + \phi(f) + \|f\|^2 \quad \text{on } D(N). \end{aligned}$$

Proof. For the first equation including the statement on the domain of $\phi(g)$, see Proposition 5.2.4 of [28]. The method of proof of this proposition in [28] can be generalized to prove the invariance of $D(N)$ and the second equation. The invariance of $D(N^\sigma)$ for $\sigma \in (0,1)$ now follows by a simple interpolation argument based on the Hadamard three-lines theorem. $\square$

Lemma A.2.5. *Let $f, g \in L^2(\mathbb{R}^d)$. Then*

$$\left\|(e^{i\pi(f)} - e^{i\pi(g)})(N+1)^{-1/2}\right\| \leq 2\|f-g\| + |\mathrm{Im}\,\langle f, g\rangle|.$$

Proof. For any $\Psi \in D(N^{1/2})$, we have

$$\begin{aligned}
\left\|(e^{i\pi(f)} - e^{i\pi(g)})\Psi\right\| = \left\|e^{-i\pi(g)}e^{i\pi(f)}\Psi - \Psi\right\| &= \left\|\int_0^1 e^{-i\pi(g)t}\pi(f-g)e^{i\pi(f)t}\Psi dt\right\| \\
&\leq \int_0^1 \left\|e^{-i\pi(f)t}\pi(f-g)e^{i\pi(f)t}\Psi\right\|dt \\
&= \int_0^1 \|(\pi(f-g) + 2t\ \mathrm{Im}\,\langle f-g, f\rangle)\Psi\|dt \\
&\leq \|\pi(f-g)\Psi\| + |\mathrm{Im}\,\langle g, f\rangle|\ \|\Psi\|. \qquad \text{(A.15)}
\end{aligned}$$

The lemma now follows from Corollary A.2.2. $\square$

Lemma A.2.6. *Let $f \in L^2(\mathbb{R}^d)$ and let $F : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d)$ be defined by $(F\varphi)(x,k) = \varphi(x)e^{-ikx}f(k)$. Then, for all $\Psi \in D(H_0)$,*

$$\|a(F)\Psi\| \leq C_f\left\|\sqrt{N}(1-\Delta)^{1/2}\Psi\right\|,$$

where

$$C_f := \left(\sup_{h\in\mathbb{R}^d}\int \frac{|f(k)|^2}{1+(h-k)^2}dk\right)^{1/2}.$$

This Lemma is due to Frank and Schlein, see Lemma 10 in [13]. For completeness of the present paper, we give a short proof. It is based on Lemma A.2.1 with $B = (1-\Delta)^{-1/2}F$.

Proof. Let $L = (1-\Delta)^{1/2}$ and note that, by (A.14),

$$a(F)\Psi = a(L^{-1}F)L\Psi. \qquad \text{(A.16)}$$

From Lemma A.2.1, it thus follows that

$$\|a(F)\Psi\| \leq \|L^{-1}F\|\|\sqrt{N}L\Psi\|, \qquad \text{(A.17)}$$

which is the desired estimate provided that $\|L^{-1}F\| \leq C_f$. To prove this, let $\varphi \in L^2(\mathbb{R}^d)$. Then, by definition of F and by Fourier transform,

$$\begin{aligned}
\|L^{-1}F\varphi\|^2 &= \int \left|(L^{-1}F\varphi)(x,k)\right|^2 \, dxdk \\
&= \int \frac{1}{1+p^2}|\hat{\varphi}(p+k)|^2|f(k)|^2 \, dp\, dk \\
&= \int \left(\int \frac{1}{1+(p-k)^2}|f(k)|^2 \, dk\right) |\hat{\varphi}(p)|^2 \, dp \leq C_f \|\varphi\|^2. \qquad \text{(A.18)}
\end{aligned}$$

□

A.3. An operator core in terms of coherent states

In this appendix, we apply the formal Expression (2.1) to vectors from $D(H)$. By means of formal manipulations, we illustrate the argument given in the Introduction concerning cancellation of ”vectors” outside the Hilbert space.

Let $\Omega \in \mathcal{F}$ denote the vacuum vector. Then, the space

$$D := \{\gamma \otimes e^{-i\pi(f)}\Omega \mid \gamma, f \in C_0^\infty(\mathbb{R}^d)\} \qquad \text{(A.19)}$$

is a core of H_0, and hence, $U_\infty^* D$ is a core of H by Theorem 2.3.7. The elements $\Psi \in U_\infty^* D$ have the form

$$\begin{aligned}
\Psi(x) = U_\infty^*\left(\gamma \otimes e^{-i\pi(f)}\Omega\right)(x) &= \gamma(x) e^{-i\pi(B_{\infty,x}+f)}\Omega \; e^{-i\,\mathrm{Im}\,\langle B_{\infty,x}, f\rangle} \\
&= \varphi(x)\eta(x), \qquad \text{(A.20)}
\end{aligned}$$

where

$$\varphi(x) := \gamma(x) e^{-i\,\mathrm{Im}\,\langle B_{\infty,x}, f\rangle} e^{-\frac{1}{2}\|B_{\infty,x}+f\|^2} \qquad \text{(A.21)}$$

belongs to $C_0^\infty(\mathbb{R}^d)$ and

$$\eta(x) := \sum_{n\geq 0} \frac{1}{n!} a^*(B_{\infty,x}+f)^n \Omega. \qquad \text{(A.22)}$$

We now formally apply $-\Delta + N + a(G_\infty) + a^*(G_\infty)$ to (A.20). Using the Leibniz rule to compute $\Delta\Psi$, we obtain

$$\begin{aligned}
-\Delta\Psi &= (-\Delta\varphi)\eta - 2\nabla\varphi \cdot \nabla\eta + \varphi(-\Delta\eta), \qquad \text{(A.23)} \\
N\Psi(x) &= a^*(B_{\infty,x}+f)\Psi(x), \qquad \text{(A.24)} \\
a(G_{\infty,x})\Psi(x) &= \langle G_{\infty,x}, B_{\infty,x}+f\rangle \Psi(x), \qquad \text{(A.25)} \\
a^*(G_{\infty,x})\Psi(x) &= \varphi(x) a^*(G_{\infty,x})\eta(x), \qquad \text{(A.26)} \\
\varphi(x)(-\Delta\eta)(x) &= \varphi(x) a^*(k^2 B_{\infty,x})\eta(x) + \varphi(x) a^*(kB_{\infty,x})^2\eta(x). \qquad \text{(A.27)}
\end{aligned}$$

All terms on the right-hand side of these five equations are Hilbert space vectors, with the exception of $\varphi a^*(k^2 B_\infty)\eta$ and $\varphi a^*(G_\infty)\eta$. The sum of these two terms, however, is

$$\varphi a^*(k^2 B_\infty + G_\infty)\eta = \varphi a^*(G_K - B_\infty)\eta, \tag{A.28}$$

which is a Hilbert space vector again. Altogether, we get the formal result

$$\begin{aligned}
&(-\Delta + N + a(G_{\infty,x}) + a^*(G_{\infty,x}))\Psi(x) \\
&= (-\Delta\varphi)(x)\eta(x) + 2(i\nabla\varphi)(x)\cdot a^*(kB_{\infty,x})\eta(x) + a^*(kB_{\infty,x})^2\Psi(x) \\
&\qquad + a^*(f)\Psi(x) + a^*(G_{K,x})\Psi(x) + \langle G_{\infty,x}, B_{\infty,x} + f\rangle\Psi(x),
\end{aligned} \tag{A.29}$$

which is a Hilbert space vector. A rigorous application of the operator $H = U_\infty^* H_\infty' U_\infty$ (see Theorem 2.3.7) on the vector Ψ from Equation (A.20), which is a long straightforward calculation, leads to exactly the same result (A.29).

B. Appendix for Chapter 3: Self-Adjointness and Domain of the Nelson Hamiltonian

This chapter is the appendix for Chapter 3. Like Chapter 3, it will also be part of an upcoming publication.

B.1. Creation and annihilation operators

In this section, we collect more background to creation and annihilation operators which is needed in Chapter 3. For the general definiton and other standard estimates of these operators, see Section A.2. Section B.1 here can be seen as a continuation of the relevant Section A.2 in Appendix A.

Lemma B.1.1. *Let* $\omega(k) \geq 0$*, and* $f, g \in L^2(\mathbb{R}^d)$*. Then,*

$$\left\|(N+1)^{-1/2} a(f) a(g) (1 + d\Gamma(\omega))^{-1/2}\right\| \leq \left\|\tfrac{f}{\omega^{1/4}}\right\| \left\|\tfrac{g}{\omega^{1/4}}\right\|.$$

Proof. For the proof of this Lemma, we follow the proof of Lemma 5 in [1]. Let $\Psi \in D(d\Gamma(\omega)^{1/2})$, then

$$\begin{aligned}
&\left\|(N+1)^{-1/2} a(f) a(g) \Psi\right\|^2 \qquad\qquad \text{(B.1)}\\
&= \sum_{n \geq 0} \int dk_1 .. \int dk_n \frac{1}{n+1} \left| \sqrt{n+1} \textstyle\int dk \overline{f(k)} \sqrt{n+2} \int d\tilde{k} \overline{g(\tilde{k})} \Psi^{(n+2)}(k, \tilde{k}, k_1, .., k_n) \right|^2 \\
&= \sum_{n \geq 0} \int dk_1 .. \int dk_n (n+2) \left| \textstyle\int dk \int d\tilde{k} \frac{\overline{f(k)}}{\omega(k)^{1/4}} \frac{\overline{g(\tilde{k})}}{\omega(\tilde{k})^{1/4}} (\omega(k)\omega(\tilde{k}))^{1/4} \Psi^{(n+2)}(k, \tilde{k}, k_1, .., k_n) \right|^2 \\
&\leq \left\|\tfrac{f}{\omega^{1/4}}\right\|^2 \left\|\tfrac{g}{\omega^{1/4}}\right\|^2 \sum_{n \geq 0} \int dk_1 .. \int dk_{n+2} (n+2) (\omega(k_{n+1})\omega(k_{n+2}))^{1/2} \left|\Psi^{(n+2)}(k_1, .., k_{n+2})\right|^2,
\end{aligned}$$

where we applied Cauchy-Schwarz. Using $\omega(k_{n+1})^{1/2}\omega(k_{n+2})^{1/2} \leq \frac{1}{2}(\omega(k_{n+1}) + \omega(k_{n+2}))$

and the bosonic symmetry of Ψ, we can estimate this expression by

$$\begin{aligned}
&\left\|\tfrac{f}{\omega^{1/4}}\right\|^2\left\|\tfrac{g}{\omega^{1/4}}\right\|^2\sum_{n\geq 0}\int dk_1\ ...\int dk_{n+2}\sum_{j=1}^{n+2}\omega(k_j)\left|\Psi^{(n+2)}(k_1,...,k_{n+2})\right|^2\\
&=\left\|\tfrac{f}{\omega^{1/4}}\right\|^2\left\|\tfrac{g}{\omega^{1/4}}\right\|^2\sum_{n\geq 0}\int dk_1\ ...\int dk_{n+2}\left|(d\Gamma(\omega)^{1/2}\Psi)^{(n+2)}(k_1,...,k_{n+2})\right|^2\\
&\leq\left\|\tfrac{f}{\omega^{1/4}}\right\|^2\left\|\tfrac{g}{\omega^{1/4}}\right\|^2\left\|d\Gamma(\omega)^{1/2}\Psi\right\|^2, \qquad \text{(B.2)}
\end{aligned}$$

which completes the proof. ☐

Lemma B.1.2. *Let $\alpha \geq \frac{1}{2}$, $\beta \geq 0$, $\omega(k) \geq 0$, $p := -i\nabla_x$, and $F_x(k) := f(k)e^{-ikx}$ with $f \in L^2(\mathbb{R}^d)$. Then, for all $\Psi \in D(d\Gamma(\omega)^\alpha(1+p^2)^\beta)$,*

$$\|a(F)\Psi\| \leq \left(\sup_{q\in\mathbb{R}^d}\int dk\ \frac{|f(k)|^2}{\omega(k)^{2\alpha}(1+(q-k)^2)^{2\beta}}\right)^{\frac{1}{2}}\left\|d\Gamma(\omega)^\alpha(1+p^2)^\beta\Psi\right\|.$$

Proof. We consider

$$\begin{aligned}
&\|a(F)\Psi\|^2\\
&=\sum_{n\geq 0}\int dp\int dk_1...\int dk_n\ (n+1)\left|\int dk\ \tfrac{\overline{f(k)}\omega(k)^\alpha(1+(p-k)^2)^\beta}{\omega(k)^\alpha(1+(p-k)^2)^\beta}\ \Psi^{(n+1)}(p-k,k,k_1,...,k_n)\right|^2\\
&\leq\sum_{n\geq 0}\int dp\int dk_1...\int dk_n\ \left(\int dk\ \frac{|f(k)|^2}{\omega(k)^{2\alpha}(1+(p-k)^2)^{2\beta}}\right)\\
&\quad\left(\int dk_{n+1}\ (n+1)\omega(k_{n+1})^{2\alpha}(1+(p-k_{n+1})^2)^{2\beta}\left|\Psi^{(n+1)}(p-k_{n+1},k_1,...,k_{n+1})\right|^2\right)\\
&\leq\sup_{q\in\mathbb{R}^d}\left(\int dk\ \frac{|f(k)|^2}{\omega(k)^{2\alpha}(1+(q-k)^2)^{2\beta}}\right)\\
&\qquad\cdot\sum_{n\geq 0}\int dp\int dk_1...\int dk_{n+1} \qquad \text{(B.3)}\\
&\qquad\quad\sum_{j=1}^{n+1}\omega(k_j)^{2\alpha}(1+(p-k_{n+1})^2)^{2\beta}\left|\Psi^{(n+1)}(p-k_{n+1},k_1,...,k_{n+1})\right|^2,
\end{aligned}$$

where we used Cauchy-Schwarz, the bosonic symmetry, and renamed p to q in the first expression. Now, substituting $p-k_{n+1}$ to p and using that $\sum_{j=1}^{n+1}\omega^{2\alpha}(k_j) \leq (\sum_{j=1}^{n+1}\omega(k_j))^{2\alpha}$

for $\alpha \geq \frac{1}{2}$, leads to

$$\|a(F)\Psi\|^2 \leq \sup_{q\in\mathbb{R}^d} \left(\int dk \, \frac{|f(k)|^2}{\omega(k)^{2\alpha}(1+(q-k)^2)^{2\beta}} \right) \left\| d\Gamma(\omega)^\alpha (1+p^2)^\beta \Psi \right\|^2, \tag{B.4}$$

which we had to show. □

Remark. This Lemma is a generalization of Lemma 10 of Frank and Schlein [13] which was an important tool in the paper [23] and Chapter 2, respectively. The main advantage of this lemma is that we can use the dispersion relation of the bosonic field and the dispersion relation of the quantized particle as well to get better decay for large absolute values of k. The lemma also contains the following Corollary B.1.3 as a special case, set $\beta = 0$ in Equation (B.4). Especially for $\alpha = \frac{1}{2}$, this is a well-known estimate.

Corollary B.1.3. *Let $\alpha \geq \frac{1}{2}$, $\omega(k) \geq 0$, and $f \in L^2(\mathbb{R}^d)$. Then, for all $\Psi \in D(d\Gamma(\omega)^\alpha)$,*

$$\|a(f)\Psi\| \leq \left\| \tfrac{f}{\omega^\alpha} \right\| \, \|d\Gamma(\omega)^\alpha \Psi\|.$$

Lemma B.1.4. *Let $\omega(k) \geq 0$, $p := -i\nabla_x$, and $F_x(k) := f(k)e^{-ikx}$ with $|k|^s f \in L^2(\mathbb{R}^d)$ for some $s \in [0,1]$. Then, for all $\Psi \in D(d\Gamma(\omega)^{1/2})$,*

$$\begin{aligned}
\|a([|p|^s, F])\Psi\| &\leq \left\| \tfrac{|k|^s f}{\sqrt{\omega}} \right\| \, \left\| d\Gamma(\omega)^{1/2}\Psi \right\|, \\
\|a^*([|p|^s, F])\Psi\| &\leq \max\left\{ \||k|^s f\|, \left\| \tfrac{|k|^s f}{\sqrt{\omega}} \right\| \right\} \, \left\| (1+d\Gamma(\omega))^{1/2}\Psi \right\|.
\end{aligned}$$

Proof. Let $\Psi \in D(d\Gamma(\omega)^{1/2})$. Remark that $e^{ikx}|p|^s = |p-k|^s e^{ikx}$ and that for all $p, k \in \mathbb{R}^d$ and $s \in [0,1]$,

$$|\, |p \pm k|^s - |p|^s| \leq |k|^s. \tag{B.5}$$

We arrive at

$$\begin{aligned}
&\|a([|p|^s, F])\Psi\|^2 \\
&= \sum_{n\geq 0} \int dp \int dk_1 ... \int dk_n \left| \int dk \; \overline{f(k)} \, (|p-k|^s - |p|^s) \, (a_k\Psi)^{(n)}(p-k, k_1, ..., k_n) \right|^2 \\
&\leq \sum_{n\geq 0} \int dp \int dk_1 ... \int dk_n \left(\int dk \; \frac{|f(k)|^2 \, (|p-k|^s - |p|^s)^2}{\omega(k)} \right) \\
&\qquad\qquad \left(\int dk \; \omega(k) \left| (a_k\Psi)^{(n)}(p-k, k_1, ..., k_n) \right|^2 \right) \\
&\leq \left\| \tfrac{|k|^s f}{\sqrt{\omega}} \right\|^2 \int dk \; \omega(k) \; \|a_k\Psi\|^2 = \left\| \tfrac{|k|^s f}{\sqrt{\omega}} \right\|^2 \left\| d\Gamma(\omega)^{1/2}\Psi \right\|^2.
\end{aligned} \tag{B.6}$$

For the creation operator, we obtain

$$\begin{aligned}
&\|a^*([|p|^s, F])\Psi\|^2 \\
&= \int dk \int dk' \; \overline{f(k)} f(k') \Big\langle \Psi, (|p-k|^s - |p|^s) e^{ikx} (|p|^s - |p+k'|^s) e^{-ik'x} a_k a^*_{k'} \Psi \Big\rangle \\
&= \int dk \; |f(k)|^2 \; \big\langle \Psi, (|p-k|^s - |p|^s)^2 \Psi \big\rangle^2 \\
&\quad + \int dk \int dk' \; \overline{f(k)} f(k') \Big\langle \Psi, e^{-ik'x} (|p-k-k'|^s - |p-k'|^s) \\
&\qquad\qquad\qquad\qquad\qquad\qquad (|p-k-k'|^s - |p-k|^s) e^{ikx} a^*_{k'} a_k \Psi \Big\rangle \\
&= \int dk \; |f(k)|^2 \; \|(|p-k|^s - |p|^s)\Psi\|^2 \\
&\quad + \int dk \int dk' \; \overline{f(k)} f(k') \Big\langle (|p-k-k'|^s - |p-k'|^s) e^{ik'x} a_{k'} \Psi, \\
&\qquad\qquad\qquad\qquad\qquad (|p-k-k'|^s - |p-k|^s) e^{ikx} a_k \Psi \Big\rangle,
\end{aligned} \tag{B.7}$$

where we used that $[a_k, a^*_{k'}] = \delta(k-k')$. We can further estimate to get

$$\begin{aligned}
&\|a^*([|p|^s, F])\Psi\|^2 \\
&\leq \int dk \; |f(k)|^2 \; \|(|p-k|^s - |p|^s)\Psi\|^2 \\
&\quad + \int dk \int dk' \; |f(k)| \; |f(k')| \; \Big\|(|p-k-k'|^s - |p-k'|^s) e^{ik'x} a_{k'} \Psi\Big\| \\
&\qquad\qquad\qquad\qquad\qquad \Big\|(|p-k-k'|^s - |p-k|^s) e^{ikx} a_k \Psi\Big\| \\
&\leq \int dk \; |k|^{2s} |f(k)|^2 \|\Psi\|^2 + \left(\int dk \; |k|^s |f(k)| \; \|a_k \Psi\| \right)^2 \\
&\leq \int dk \; |k|^{2s} |f(k)|^2 \|\Psi\|^2 + \left(\int dk \; \frac{|k|^{2s} |f(k)|^2}{\omega(k)} \right) \left(\int dk \; \omega(k) \|a_k \Psi\|^2 \right) \\
&= \||k|^s f\|^2 \|\Psi\|^2 + \left\| \frac{|k|^s f}{\sqrt{\omega}} \right\|^2 \big\| d\Gamma(\omega)^{1/2} \Psi \big\|^2 \\
&\leq \max \left\{ \||k|^s f\|^2, \left\| \frac{|k|^s f}{\sqrt{\omega}} \right\|^2 \right\} \; \big\|(d\Gamma(\omega) + 1)^{1/2} \Psi\big\|.
\end{aligned} \tag{B.8}$$

This is the desired estimate. □

Lemma B.1.4 contains for $\omega(k) = 1$ the following important special case:

Corollary B.1.5. *Let $p := -i\nabla_x$ and $F_x(k) := f(k)e^{-ikx}$ with $|k|^s f \in L^2(\mathbb{R}^d)$ for some $s \in [0,1]$. Then, for all $\Psi \in D(\sqrt{N})$,*

$$\|a([|p|^s, F])\Psi\| \leq \||k|^s f\| \left\|\sqrt{N}\Psi\right\|,$$
$$\|a^*([|p|^s, F])\Psi\| \leq \||k|^s f\| \left\|\sqrt{N+1}\Psi\right\|.$$

Lemma B.1.6. *Let $\omega(k) \geq 0$, $p := -i\nabla_x$, $\ell \in \mathbb{R}^d$, and $F_x(k) := f(k)e^{-ikx}$ with $|k|^{s/2} f \in L^2(\mathbb{R}^d)$ for some $s \in [0,1]$. Then, for all $\Psi \in D(d\Gamma(\omega)^{1/2})$,*

$$\|a([|p \pm \ell|^s - |p|^s, F])\Psi\| \leq 2|\ell|^{s/2} \left\| \frac{|k|^{s/2} f}{\sqrt{\omega}} \right\| \left\|d\Gamma(\omega)^{1/2}\Psi\right\|,$$
$$\|a^*([|p \pm \ell|^s - |p|^s, F])\Psi\| \leq 2|\ell|^{s/2} \max\left\{ \left\||k|^{s/2} f\right\|, \left\| \frac{|k|^{s/2} f}{\sqrt{\omega}} \right\| \right\} \left\|(d\Gamma(\omega) + 1)^{1/2}\,\Psi\right\|.$$

Proof. Let $\Psi \in D(d\Gamma(\omega)^{1/2})$. Similarly to the proof of Lemma B.1.4, we get

$$\|a([|p \pm \ell|^s - |p|^s, F])\Psi\| = \left\| \int dk\ \overline{f(k)} D_s(p,k,\ell) e^{ikx} a_k \Psi \right\|, \tag{B.9}$$

where

$$D_s(p,k,\ell) := |p - k \pm \ell|^s - |p-k|^s - |p \pm \ell|^s + |p|^s. \tag{B.10}$$

Note that, for $p, k, \ell \in \mathbb{R}^d$ and $s \in [0,1]$,

$$|D_s(p,k,\ell)| \leq 2|k|^s \quad \text{and} \quad |D_s(p,k,\ell)| \leq 2|\ell|^s. \tag{B.11}$$

That means

$$|D_s(p,k,\ell)| \leq 2\min\{|k|^s, |\ell|^s\} = 2\min\{|k|^{s/2}, |\ell|^{s/2}\}^2 \leq 2|k|^{s/2}|\ell|^{s/2}. \tag{B.12}$$

Especially, this upper boundary is independent of the first argument p. Therefore, we arrive at

$$\begin{aligned}
&\|a([|p \pm \ell|^s - |p|^s, F])\Psi\|^2 \\
&\leq \sum_{n \geq 0} \int dp \int dk_1 ... \int dk_n \left(\int dk\ |f(k)||D_s(p,k,\ell)| \left|(a_k\Psi)^{(n)}(p-k, k_1, ..., k_n)\right| \right)^2, \\
&\leq 4|\ell|^s \left\||k|^{s/2} f\right\|^2 \sum_{n \geq 0} \int dp \int dk_1 ... \int dk_n \int dk\ \left|(a_k\Psi)^{(n)}(p-k, k_1, ..., k_n)\right|^2 \\
&= 4|\ell|^s \left\||k|^{s/2} f\right\|^2 \int dk\ \|a_k\Psi\|^2 = 4|\ell|^s \left\||k|^{s/2} f\right\|^2 \left\|\sqrt{N}\Psi\right\|^2.
\end{aligned} \tag{B.13}$$

For the creation operator, we find

$$
\begin{aligned}
\|a^*([|p\pm\ell|^s-|p|^s,F])\Psi\|^2 &= \left\|\int dk\ f(k)D_s(p,-k,\ell)e^{-ikx}a_k^*\Psi\right\|^2\\
&=\int dk\int dk'\ \overline{f(k)}f(k')\Big\langle D_s(p,-k,\ell)e^{-ikx}a_k^*\Psi, D_s(p,-k',\ell)e^{-ik'x}a_{k'}^*\Psi\Big\rangle\\
&=\int dk\int dk'\ \overline{f(k)}f(k')\Big\langle \Psi, D_s(p,k,\ell)e^{i(k-k')x}D_s(p,k',\ell)a_k a_{k'}^*\Psi\Big\rangle,
\end{aligned}
\tag{B.14}
$$

where we permuted the functions D_s with e^{ikx} or $e^{-ik'x}$, respectively. Using $a_k a_{k'}^* = a_{k'}^* a_k + \delta(k-k')$, further permuting of the functions D_s with e^{ikx} or $e^{-ik'x}$, and Estimate (B.12), we obtain

$$
\begin{aligned}
&\|a^*([|p\pm\ell|^s-|p|^s,F])\Psi\|^2\\
&=\int dk\ |f(k)|^2\|D_s(p,k,\ell)\Psi\|^2\\
&\quad+\int dk\int dk'\ \overline{f(k)}f(k')\Big\langle D_s(p-k,k',\ell)e^{ik'x}a_{k'}\Psi, D_s(p-k',k,\ell)e^{ikx}a_k\Psi\Big\rangle\\
&\le 4|\ell|^s\int dk\ |k|^s|f(k)|^2\|\Psi\|^2+4|\ell|^s\left(\int dk\ |k|^{s/2}|f(k)|\ \|a_k\Psi\|\right)^2\\
&\le 4|\ell|^s\int dk\ |k|^s|f(k)|^2\|\Psi\|^2+4|\ell|^s\left(\int dk\ \frac{|k|^s|f(k)|^2}{\omega(k)}\right)\left(\int dk\ \omega(k)\|a_k\Psi\|^2\right)\\
&\le 4|\ell|^s\left(\left\||k|^{s/2}f\right\|^2\|\Psi\|^2+\left\|\frac{|k|^{s/2}f}{\sqrt{\omega}}\right\|^2\left\|d\Gamma(\omega)^{1/2}\Psi\right\|^2\right)\\
&\le 4|\ell|^s\max\left\{\left\||k|^{s/2}f\right\|,\left\|\frac{|k|^{s/2}f}{\sqrt{\omega}}\right\|\right\}^2\left\|(d\Gamma(\omega)+1)^{1/2}\,\Psi\right\|^2.
\end{aligned}
\tag{B.15}
$$

This completes the proof. ☐

Lemma B.1.6 contains for $\omega(k)=1$ the following important special case:

Corollary B.1.7. *Let $p:=-i\nabla_x$, $\ell\in\mathbb{R}^d$ and $F_x(k):=f(k)e^{-ikx}$ with $|k|^{s/2}f\in L^2(\mathbb{R}^d)$ for some $s\in[0,1]$. Then, for all $\Psi\in D(\sqrt{N})$,*

$$
\begin{aligned}
\|a([|p\pm\ell|^s-|p|^s,F])\Psi\| &\le 2|\ell|^{s/2}\left\||k|^{s/2}f\right\|\ \left\|\sqrt{N}\Psi\right\|,\\
\|a^*([|p\pm\ell|^s-|p|^s,F])\Psi\| &\le 2|\ell|^{s/2}\left\||k|^{s/2}f\right\|\ \left\|\sqrt{N+1}\Psi\right\|.
\end{aligned}
$$

Lemma B.1.8. *Let $\omega(k)\ge m>0$, $p:=-i\nabla_x$, $F_x(k):=f(k)e^{-ikx}$ with $|k|^s f\in L^2(\mathbb{R}^d)$ for some $s\in[0,1]$, and $G_x(k):=g(k)e^{-ikx}$ with $g\in L^2(\mathbb{R}^d)$. Then,*

$$
\left\|(1+d\Gamma(\omega))^{-1/2}a([|p|^s,F])a(G)(1+d\Gamma(\omega))^{-1/2}\right\|\le\sqrt{1+\frac{1}{m}}\left\|\frac{|k|^s f}{\sqrt{\omega}}\right\|\ \|g\|.
$$

Remark. For estimating the operator norm in this lemma, we could also try to use Corollary B.1.3 and Lemma B.1.4. Then, the boundary would be $\sqrt{1+1/m}\,\||k|^s f\|\,\|g/\sqrt{\omega}\|$. But, in Chapter 3, typically $\||k|^s f\| = \infty$ such that we need the additional estimate given by Lemma B.1.8.

Proof. Let $\Psi \in D(d\Gamma(\omega)^{1/2})$. Note that for $p, k \in \mathbb{R}^d$, we have $e^{ikx}|p|^s e^{-ikx} = |p-k|^s$ and $||p-k|^s - |p|^s| \leq |k|^s$ for $s \in [0,1]$. Therefore, we obtain

$$\begin{aligned}
&\left\|(1+d\Gamma(\omega))^{-1/2} a([|p|^s, F]) a(G)\Psi\right\|^2 \\
&= \left\|(1+d\Gamma(\omega))^{-1/2} \textstyle\int dk \int dk'\, \overline{f(k)g(k')}\,(|p-k|^s - |p|^s)\, e^{i(k+k')x} a_k a_{k'}\Psi\right\|^2 \\
&= \sum_{n\geq 0} \int dp \int dk_1 ... \int dk_n \left(1 + \sum_{j=1}^n \omega(k_j)\right)^{-1} (n+1)(n+2) \\
&\quad \left|\textstyle\int dk \int dk'\, \overline{f(k)g(k')}\,(|p-k|^s - |p|^s)\, \Psi^{(n+2)}(p-k-k', k, k', k_1, ..., k_n)\right|^2 \\
&\leq \left(1+\frac{1}{m}\right) \sum_{n\geq 0} \int dp \int dk_1 ... \int dk_n (n+2) \qquad\qquad (B.16)\\
&\quad \left(\int dk \int dk' \frac{|f(k)||g(k')|}{\sqrt{\omega(k)}} ||p-k|^s - |p|^s| \sqrt{\omega(k)} \left|\Psi^{(n+2)}(p-k-k', k, k', k_1, ..., k_n)\right|\right)^2.
\end{aligned}$$

Using Cauchy-Schwarz with respect to the k- and k'-integration and that $||p-k|^s - |p|^s| \leq |k|^s$ for $s \in [0,1]$, we arrive at

$$\begin{aligned}
&\left\|(1+d\Gamma(\omega))^{-1/2} a([|p|^s, F]) a(G)\Psi\right\|^2 \\
&\leq \left(1+\frac{1}{m}\right) \left\|\tfrac{|k|^s f}{\sqrt{\omega}}\right\|^2 \|g\|^2 \\
&\quad \cdot \sum_{n\geq 0} \int dp \int dk_1 ... \int dk_{n+2} (n+2)\omega(k_{n+2}) \left|\Psi^{(n+2)}(p, k_1, ..., k_{n+2})\right|^2 \\
&\leq \left(1+\frac{1}{m}\right) \left\|\tfrac{|k|^s f}{\sqrt{\omega}}\right\|^2 \|g\|^2 \left\|d\Gamma(\omega)^{1/2}\Psi\right\|^2, \qquad\qquad (B.17)
\end{aligned}$$

which proves the statement. $\square$

Lemma B.1.9. *Let $\omega(k) \geq m > 0$, $p := -i\nabla_x$, $F_x(k) := f(k)e^{-ikx}$ with $|k|^s f \in L^2(\mathbb{R}^d)$, and $G_x(k) := g(k)e^{-ikx}$ with $g \in L^2(\mathbb{R}^d)$ for some $s \in [0,1]$. Then,*

$$\begin{aligned}
&\left\|(1+d\Gamma(\omega))^{-1/2} a([|p|^s, F]) a^*(G)(1+d\Gamma(\omega))^{-1/2}\right\| \\
&\leq \frac{1}{\sqrt{m}} \left\|\tfrac{|k|^s f}{\sqrt{\omega}}\right\| \|g\| + \left\||k|^{s/2} f\right\| \left\||k|^{s/2} g\right\|.
\end{aligned}$$

Remark. Here, we could also use Corollary B.1.3 and Lemma B.1.4 for estimating the operator norm. But the resulting boundary would not be useful for our purposes in Chapter 3 such that we need another estimate.

Proof. Let $\Psi \in D(d\Gamma(\omega)^{1/2})$ and analogously to the proof of Lemma B.1.8, we get

$$\begin{aligned}
&\left\|(1+d\Gamma(\omega))^{-1/2} a([|p|^s, F]) a^*(G)\Psi\right\| \\
&= \left\|(1+d\Gamma(\omega))^{-1/2} \int dk \ \int dk' \ \overline{f(k)} g(k') \left(|p-k|^s - |p|^s\right) e^{i(k-k')x} a_k a^*_{k'} \Psi\right\| \\
&\leq \left\|(1+d\Gamma(\omega))^{-1/2} \int dk \ \int dk' \ \overline{f(k)} g(k') \left(|p-k|^s - |p|^s\right) e^{i(k-k')x} a^*_{k'} a_k \Psi\right\| \\
&\qquad + \left\|(1+d\Gamma(\omega))^{-1/2} \int dk \ \overline{f(k)} g(k) \left(|p-k|^s - |p|^s\right) \Psi\right\|,
\end{aligned} \tag{B.18}$$

where we interchanged a_k and $a^*_{k'}$ by using their commutation relation.

Since $\left||p-k|^s - |p|^s\right| \leq |k|^s$ for $s \in [0,1]$, the second summand can be estimated by

$$\left(\int dk \ |f(k)| \ |g(k)| \ |k|^s\right) \ \left\|(1+d\Gamma(\omega))^{-1/2}\Psi\right\| \leq \left\||k|^{s/2} f\right\| \ \left\||k|^{s/2} g\right\| \ \|\Psi\| \tag{B.19}$$

by using Cauchy-Schwarz. For $\eta \in \mathscr{H}$, define $\tilde{\eta} := (1+d\Gamma(\omega))^{-1/2}\eta$ and note that $\tilde{\eta} \in D(d\Gamma(\omega)^{1/2})$. For the first summand of (B.18), we write

$$\begin{aligned}
&\left\|(1+d\Gamma(\omega))^{-1/2} \int dk \ \int dk' \ \overline{f(k)} g(k') \left(|p-k|^s - |p|^s\right) e^{i(k-k')x} a^*_{k'} a_k \Psi\right\| \\
&= \sup_{\|\eta\|=1} \left|\left\langle \eta, (1+d\Gamma(\omega))^{-1/2} \int dk \ \int dk' \ \overline{f(k)} g(k') \left(|p-k|^s - |p|^s\right) e^{i(k-k')x} a^*_{k'} a_k \Psi\right\rangle\right| \\
&\leq \sup_{\|\eta\|=1} \int dk \ \int dk' \ |f(k)| \ |g(k')| \ \left|\left\langle a_{k'}\tilde{\eta}, \left(|p-k|^s - |p|^s\right) e^{i(k-k')x} a_k \Psi\right\rangle\right|, \\
&\leq \sup_{\|\eta\|=1} \int dk \ \int dk' \ \frac{|k|^s \ |f(k)|}{\sqrt{\omega(k)}} \ |g(k')| \ \|a_{k'}\tilde{\eta}\| \ \sqrt{\omega(k)} \|a_k \Psi\| \\
&\leq \sup_{\|\eta\|=1} \left\|\frac{|k|^s f}{\sqrt{\omega}}\right\| \ \|g\| \left(\int dk' \ \|a_{k'}\tilde{\eta}\|^2\right)^{1/2} \left(\int dk \ \omega(k) \|a_k \Psi\|^2\right)^{1/2} \\
&= \sup_{\|\eta\|=1} \left\|\frac{|k|^s f}{\sqrt{\omega}}\right\| \ \|g\| \ \left\|\sqrt{N}(1+d\Gamma(\omega))^{-1/2}\eta\right\| \ \left\|d\Gamma(\omega)^{1/2}\Psi\right\| \\
&= \frac{1}{\sqrt{m}} \left\|\frac{|k|^s f}{\sqrt{\omega}}\right\| \ \|g\| \ \left\|d\Gamma(\omega)^{1/2}\Psi\right\|.
\end{aligned} \tag{B.20}$$

This and Equation (B.19) proves the lemma. □

B.2. Interpolation theory

One of the most important aspects of Section B.3 is the behavior of the domain of self-adjoint operators under a unitary transform. If this domain is invariant under the transform, we can use interpolation theory to show this invariance behavior for the domain of fractional powers of this self-adjoint operator. This is done in the following

Theorem B.2.1. *Let $\mathscr{H}$ be a Hilbert space, $A = A^* \geq 0$, $U \in \mathscr{L}(\mathscr{H})$ with $U:\ D(A) \to D(A)$. Then $U:\ D(A^s) \to D(A^s)$ for all $s \in [0,1]$ and*

$$\|(A+1)^s U(A+1)^{-s}\| \leq \|U\|^{1-s}\big\|(A+1)U(A+1)^{-1}\big\|^s.$$

Proof. The main tool for this proof is Hadamard's three lines theorem, see for example Reed and Simon II [25].

Let $\mathcal{S} := \{z \in \mathbb{C} \mid 0 \leq \operatorname{Re} z \leq 1\}$ and

$$\mathcal{D} := \bigcup_{n\in\mathbb{N}} \operatorname{Ran}\, \chi_{[-n,n]}(A). \tag{B.21}$$

Note that $\mathcal{D}$ is dense in $D(A^s)$ with respect to the graph norm of A^s for all $s \in [0,1]$. For $\Phi, \Psi \in \mathcal{D}$, let moreover

$$\varphi : \mathcal{S} \to \mathbb{C},\ z \mapsto \big\langle (A+1)^{\overline{z}}\Phi, U(A+1)^{-z}\Psi\big\rangle. \tag{B.22}$$

Then,

$$\begin{aligned}|\varphi(z)| &= \big|\big\langle (A+1)^{\overline{z}}\Phi, U(A+1)^{-z}\Psi\big\rangle\big| \leq \big\|(A+1)^{\operatorname{Re} z - i\operatorname{Im} z}\Phi\big\|\big\|U(A+1)^{-\operatorname{Re} z - i\operatorname{Im} z}\Psi\big\| \\ &\leq \big\|(A+1)^{\operatorname{Re} z}\Phi\big\|\|U\|\|\Psi\| \leq \|(A+1)\Phi\|\|U\|\|\Psi\|,\end{aligned} \tag{B.23}$$

where we used the unitarity of $(A+1)^{\pm i\operatorname{Im} z}$, the boundedness of U and $(A+1)^{-\operatorname{Re} z}$ and $(A+1)^{\operatorname{Re} z} \leq (A+1)$. Therefore, φ is bounded on $\mathcal{S}$.

Since $\Phi, \Psi \in \mathcal{D}$ are analytic vectors, we can make the expansion

$$\begin{aligned}\varphi(z) &= \big\langle (A+1)^{\overline{z}}\Phi, U(A+1)^{-z}\Psi\big\rangle = \big\langle e^{\ln(A+1)\cdot\overline{z}}\Phi, Ue^{-\ln(A+1)\cdot z}\Psi\big\rangle \\ &= \sum_{j,k=0}^{\infty} \frac{(-1)^k z^{j+k}}{j!\ k!}\big\langle (\ln(A+1))^j\Phi, U(\ln(A+1))^k\Psi\big\rangle.\end{aligned} \tag{B.24}$$

This is a series in z which converges absolutely for all $z \in \mathbb{C}$. Thus, we have found a power series for $\varphi(z)$ which shows, that φ is analytic on $\mathcal{S}$.

Furthermore, the following properties hold:

$$|\varphi(i\operatorname{Im} z)| \leq \|U\|\|\Phi\|\|\Psi\|, \tag{B.25}$$

$$|\varphi(1 + i\operatorname{Im} z)| \leq \big\|(A+1)U(A+1)^{-1}\big\|\|\Phi\|\|\Psi\|. \tag{B.26}$$

The term $\left\|(A+1)U(A+1)^{-1}\right\|$ is finite by the closed graph theorem. Now, we can use Hadamard's three lines theorem and get for all $z \in \mathcal{S}$ and all $\Phi, \Psi \in \mathcal{D}$

$$|\varphi(z)| \leq \|U\|^{1-\operatorname{Re} z} \left\|(A+1)U(A+1)^{-1}\right\|^{\operatorname{Re} z} \|\Phi\| \|\Psi\| \tag{B.27}$$

and especially for all $s \in [0,1]$

$$|\langle (A+1)^s \Phi, U(A+1)^{-s}\Psi\rangle| \leq \|U\|^{1-s} \left\|(A+1)U(A+1)^{-1}\right\|^{s} \|\Phi\| \|\Psi\|. \tag{B.28}$$

On the left-hand side, we have a bounded functional, linear in Ψ and anti-linear in Φ. Since $\mathcal{D}$ is dense in $\mathscr{H}$ and since $\mathcal{D}$ is dense in $D(A^s)$ with respect to the graph norm of A^s, the Inequality (B.28) extends to all $\Phi \in D(A^s)$ and all $\Psi \in \mathscr{H}$.

Since $(A+1)^s$ is self-adjoint, we can conclude that $U(A+1)^{-s}\Psi \in D(A^s)$ for all $\Psi \in D(A^s)$, which shows the first statement of the theorem. Moreover, we get that for all $s \in [0,1]$ and all $\Phi, \Psi \in \mathscr{H}$

$$|\langle \Phi, (A+1)^s U(A+1)^{-s}\Psi\rangle| \leq \|U\|^{1-s} \left\|(A+1)U(A+1)^{-1}\right\|^{s} \|\Phi\| \|\Psi\|, \tag{B.29}$$

which proves the estimate of the theorem. $\square$

B.3. Mapping properties of Weyl operators

For the computations and arguments in Chapter 3, we need the Gross transform, which of course is a Weyl operator. Therefore, we collect some properties of Weyl operators here. For a given $f \in L^2(\mathbb{R}^d)$, the Weyl operator is defined by $e^{i\pi(f)}$. Note that Lemma B.3.1 and parts of Lemma B.3.2 are already included in Lemma A.2.4. Nevertheless, we recall them here from reasons of completeness.

Lemma B.3.1. *Let $f, g \in L^2(\mathbb{R}^d)$. Then $e^{i\pi(f)}D(\phi(g)) = D(\phi(g))$ and*

$$e^{i\pi(f)}\phi(g)e^{-i\pi(f)} = \phi(g) + 2\operatorname{Re}\langle g, f\rangle \qquad \text{on } D(\phi(g)).$$

Proof. For the proof of this Lemma, see Proposition 5.2.4.(1) in Bratteli and Robinson [28]. $\square$

Lemma B.3.2. *Let $f \in L^2(\mathbb{R}^d)$. Then $e^{i\pi(f)}D(N^s) = D(N^s)$ for all $s \in [0,1]$ and*

$$e^{i\pi(f)}Ne^{-i\pi(f)} = N + \phi(f) + \|f\|^2 \qquad \text{on } D(N),$$

and for all $\Psi \in D(\sqrt{N})$

$$\left\|\sqrt{N}e^{-i\pi(f)}\Psi\right\|^2 = \left\|\sqrt{N}\Psi\right\|^2 + \langle \Psi, \phi(f)\Psi\rangle + \|f\|^2\|\Psi\|^2.$$

Furthermore, for all $s \in [0,1]$ and all $\Psi \in D(N^s)$

$$\left\|N^s e^{-i\pi(f)}\Psi\right\| \leq (1+\|f\|)^{2s}\,\|(N+1)^s\Psi\|.$$

Proof. The method of the proof of Proposition 5.2.4.(1) in Bratteli and Robinson [28] can also be used to prove $e^{i\pi(f)}D(N) = D(N)$ and

$$e^{i\pi(f)}Ne^{-i\pi(f)}\Psi = \left(N + \phi(f) + \|f\|^2\right)\Psi \tag{B.30}$$

for all $\Psi \in D(N)$. The second equation of the lemma is a direct consequence of that.

Now, that $e^{i\pi(f)}D(N^s) = D(N^s)$ for all $s \in [0,1]$, and the associated estimate follow by Theorem B.2.1. □

Lemma B.3.3. *Let $\omega(k) > 0$ and $f, \omega^{1/2}f \in L^2(\mathbb{R}^d)$. Then*

$$e^{i\pi(f)}D(d\Gamma(\omega)) = D(d\Gamma(\omega)) \quad \Leftrightarrow \quad \omega f \in L^2(\mathbb{R}^d).$$

If $\omega f \in L^2(\mathbb{R}^d)$ then

$$e^{i\pi(f)}d\Gamma(\omega)e^{-i\pi(f)} = d\Gamma(\omega) + \phi(\omega f) + \left\|\omega^{1/2}f\right\|^2 \qquad \text{on } D(d\Gamma(\omega)).$$

Proof. Let be $\Psi \in D(d\Gamma(\omega)) \cap \mathcal{F}_0$ such that we can expand

$$e^{i\pi(f)}\Psi = \sum_{j\geq 0}\frac{1}{j!}(i\pi(f))^j\Psi =: \lim_{L\to\infty}\Phi_L \tag{B.31}$$

with the L-th partial sum Φ_L. Since $\Phi_L \in D(d\Gamma(\omega)) \cap \mathcal{F}_0$ for all $L \in \mathbb{N}$ and if we choose ω such that $\omega f \in L^2(\mathbb{R}^d)$, we can apply $d\Gamma(\omega)$ to Φ_L and obtain

$$\begin{aligned} d\Gamma(\omega)\Phi_L = &\sum_{j=0}^{L}\frac{1}{j!}(i\pi(f))^j d\Gamma(\omega)\Psi + \sum_{j=0}^{L-1}\frac{1}{j!}(i\pi(f))^j(-\phi(\omega f))\Psi \\ &+ \sum_{j=0}^{L-2}\frac{1}{j!}(i\pi(f))^j\left\|\omega^{1/2}f\right\|^2\Psi \end{aligned} \tag{B.32}$$

which converges as $L \to \infty$ to

$$e^{i\pi(f)}\left(d\Gamma(\omega) - \phi(\omega f) + \left\|\omega^{1/2}f\right\|^2\right)\Psi. \tag{B.33}$$

Thus, for all $\Psi \in D(d\Gamma(\omega)) \cap \mathcal{F}_0$, we have a sequence $(\Phi_L)_{L\in\mathbb{N}}$ in $D(d\Gamma(\omega)) \cap \mathcal{F}_0$ with $\Phi_L \to e^{i\pi(f)}\Psi$ and $d\Gamma(\omega)\Phi_L$ converges as $L \to \infty$. Since $d\Gamma(\omega)$ is a closed operator, we get $e^{i\pi(f)}\Psi \in D(d\Gamma(\omega))$ and

$$d\Gamma(\omega)e^{i\pi(f)} = e^{i\pi(f)}\left(d\Gamma(\omega) - \phi(\omega f) + \left\|\omega^{1/2}f\right\|^2\right) \qquad \text{on } D(d\Gamma(\omega)) \cap \mathcal{F}_0. \tag{B.34}$$

From that equality, we obtain for $\Psi \in D(d\Gamma(\omega)) \cap \mathcal{F}_0$

$$\left\|d\Gamma(\omega)e^{i\pi(f)}\Psi\right\| \leq C_{\omega,f}\left(\|d\Gamma(\omega)\Psi\| + \|\Psi\|\right) \tag{B.35}$$

where $C_{\omega,f}$ is a positive, finite constant if $\omega f, \omega^{1/2} f \in L^2(\mathbb{R}^d)$. In this last estimate, it is also important that $\omega(k) > 0$. Therefore, we can extend Identity (B.34) to all $\Psi \in D(d\Gamma(\omega))$ and get $e^{i\pi(f)} D(d\Gamma(\omega)) \subset D(d\Gamma(\omega))$. Since the whole argumentation also holds for $e^{-i\pi(f)}$, we obtain $D(d\Gamma(\omega)) \subset e^{i\pi(f)} D(d\Gamma(\omega))$.

It remains to show the "$\Rightarrow$" of the equivalence in the lemma. From assumptions, $e^{i\pi(f)}\Psi \in D(d\Gamma(\omega))$ for all $\Psi \in D(d\Gamma(\omega))$ and thus

$$\infty > \left\| d\Gamma(\omega) e^{i\pi(f)} \Psi \right\| = \lim_{\varepsilon \to 0} \left\| d\Gamma(\omega_\varepsilon) e^{i\pi(f)} \Psi \right\| \tag{B.36}$$

for all $\Psi \in D(d\Gamma(\omega))$, where we used the regularization $\omega_\varepsilon := \omega \cdot (1 + \varepsilon\omega)^{-1}$ and monotonic convergence. Since $\omega_\varepsilon f, \omega_\varepsilon^{1/2} f \in L^2(\mathbb{R}^d)$, for all $\varepsilon > 0$, we can use the showed transformation to get

$$\begin{aligned} \lim_{\varepsilon \to 0} \left\| d\Gamma(\omega_\varepsilon) e^{i\pi(f)} \Psi \right\| &= \lim_{\varepsilon \to 0} \left\| \left(d\Gamma(\omega_\varepsilon) - \phi(\omega_\varepsilon f) + \left\| \omega_\varepsilon^{1/2} f \right\|^2 \right) \Psi \right\| \\ &\geq \liminf_{\varepsilon \to 0} \left(\| a^*(\omega_\varepsilon f) \Psi \| - \left(\| a(\omega_\varepsilon f) \Psi \| + \| d\Gamma(\omega_\varepsilon) \Psi \| + \left\| \omega_\varepsilon^{1/2} f \right\|^2 \| \Psi \| \right) \right). \end{aligned} \tag{B.37}$$

Since $\| a^*(\omega_\varepsilon f) \Psi \| \geq \| \omega_\varepsilon f \| \| \Psi \|$ and the other terms converge as $\varepsilon \to 0$, we obtain for all $\Psi \in D(d\Gamma(\omega))$

$$\infty > \left\| d\Gamma(\omega) e^{i\pi(f)} \Psi \right\| \geq \liminf_{\varepsilon \to 0} (\| \omega_\varepsilon f \| \| \Psi \|) - C \geq \| \omega f \| \| \Psi \| - C \tag{B.38}$$

with a positive, finite constant C. In the last step, we used the lemma of Fatou. Thus, we have $\| \omega f \| < \infty$. ☐

The next corollary follows directly from Lemma B.3.3 by using Theorem B.2.1.

Corollary B.3.4. *Let $\omega(k) > 0$ and $f, \omega^{1/2} f, \omega f \in L^2(\mathbb{R}^d)$. Then $e^{i\pi(f)} D(d\Gamma(\omega)^s) = D(d\Gamma(\omega)^s)$ for all $s \in [0, 1]$ and all $\Psi \in D(d\Gamma(\omega)^s)$*

$$\left\| d\Gamma(\omega)^s e^{i\pi(f)} \Psi \right\| \leq \left(1 + 2\| \omega f \| + \left\| \omega^{1/2} f \right\|^2 \right)^s \| (1 + d\Gamma(\omega))^s \Psi \|.$$

Lemma B.3.5. *Let $\omega(k) > 0$ and $f, \omega^{1/2} f \in L^2(\mathbb{R}^d)$. Then $e^{i\pi(f)} D(d\Gamma(\omega)^{1/2}) = D(d\Gamma(\omega)^{1/2})$ and for all $\Psi \in D(d\Gamma(\omega)^{1/2})$*

$$\left\| d\Gamma(\omega)^{1/2} e^{i\pi(f)} \Psi \right\| \leq \left(1 + \left\| \omega^{1/2} f \right\| \right) \left(\left\| d\Gamma(\omega)^{1/2} \Psi \right\| + \| \Psi \| \right).$$

Proof. Use

$$\Psi \in D(d\Gamma(\omega)^{1/2}) :\Leftrightarrow \sum_{n \geq 1} \int dk_1 ... \int dk_n \left(\sum_{j=1}^n \omega(k_j) \right) \left| \Psi^{(n)}(k_1, ..., k_n) \right|^2 < \infty. \tag{B.39}$$

Let $\Psi \in D(d\Gamma(\omega))$ and define $\omega_\varepsilon := \omega(1+\varepsilon\omega)^{-1}$. We need to know, if $e^{i\pi(f)}\Psi \in D(d\Gamma(\omega)^{1/2})$. For that

$$\begin{aligned}
&\sum_{n\geq 1}\int dk_1 ... \int dk_n \left(\sum_{j=1}^n \omega(k_j)\right) \left|(e^{i\pi(f)}\Psi)^{(n)}(k_1,...,k_n)\right|^2 \\
&= \lim_{\varepsilon\to 0}\sum_{n\geq 1}\int dk_1 ... \int dk_n \left(\sum_{j=1}^n \omega_\varepsilon(k_j)\right) \left|(e^{i\pi(f)}\Psi)^{(n)}(k_1,...,k_n)\right|^2 \\
&= \lim_{\varepsilon\to 0}\left\|d\Gamma(\omega_\varepsilon)^{1/2}e^{i\pi(f)}\Psi\right\|^2 \\
&= \lim_{\varepsilon\to 0}\left\langle \Psi, \left(d\Gamma(\omega_\varepsilon) - \phi(\omega_\varepsilon f) + \left\|\omega_\varepsilon^{1/2} f\right\|^2\right)\Psi\right\rangle \\
&= \lim_{\varepsilon\to 0}\left(\left\|d\Gamma(\omega_\varepsilon)^{1/2}\Psi\right\|^2 + \left\|\omega_\varepsilon^{1/2} f\right\|^2\|\Psi\|^2 - 2\,\mathrm{Re}\,\langle\Psi, a(\omega_\varepsilon f)\Psi\rangle\right) \\
&\leq \lim_{\varepsilon\to 0}\left(\left\|d\Gamma(\omega_\varepsilon)^{1/2}\Psi\right\|^2 + \left\|\omega_\varepsilon^{1/2} f\right\|^2\|\Psi\|^2 + 2\|\Psi\|\left\|\omega_\varepsilon^{1/2} f\right\|\left\|d\Gamma(\omega_\varepsilon)^{1/2}\Psi\right\|\right) \\
&= \lim_{\varepsilon\to 0}\left(\left\|d\Gamma(\omega_\varepsilon)^{1/2}\Psi\right\| + \left\|\omega_\varepsilon^{1/2} f\right\|\|\Psi\|\right)^2 \\
&\leq \left(1+\left\|\omega^{1/2}f\right\|\right)^2\left(\left\|d\Gamma(\omega)^{1/2}\Psi\right\| + \|\Psi\|\right)^2 < \infty.
\end{aligned} \tag{B.40}$$

In the first step, we used monotonic convergence, and then Lemma B.3.3. In the last steps, we used Corollary B.1.3 and monotonic convergence again. Thus, $e^{i\pi(f)}D(d\Gamma(\omega)) \subset D(d\Gamma(\omega)^{1/2})$, and for $\Psi \in D(d\Gamma(\omega))$, we get from this calculation

$$\left\|d\Gamma(\omega)^{1/2}e^{i\pi(f)}\Psi\right\| \leq \left(1+\left\|\omega^{1/2}f\right\|\right)\left(\left\|d\Gamma(\omega)^{1/2}\Psi\right\| + \|\Psi\|\right). \tag{B.41}$$

Since $D(d\Gamma(\omega))$ is a form core for $d\Gamma(\omega)$ and $d\Gamma(\omega)^{1/2}$ is a closed operator, this inequality extends to all $\Psi \in D(d\Gamma(\omega)^{1/2})$ and $e^{i\pi(f)}D(d\Gamma(\omega)^{1/2}) \subset D(d\Gamma(\omega)^{1/2})$. The other inclusion follows since the whole argumentation also holds for $e^{-i\pi(f)}$. □

Note that, for the invariance of the domain $D(d\Gamma(\omega)^{1/2})$ in Lemma B.3.5, less requirements are necessary compared to the invariance of the domain $D(d\Gamma(\omega))$ in Lemma B.3.3. We obtain the following corollaries.

Corollary B.3.6. *Let $\omega(k) > 0$ and $f, \omega^{1/2}f \in L^2(\mathbb{R}^d)$. Then*

$$\begin{aligned}
&\left\|(1+d\Gamma(\omega))^{1/2}e^{i\pi(f)}(1+d\Gamma(\omega))^{-1/2}\right\| \leq 1+\left\|\omega^{1/2}f\right\|, \\
&\left\|(1+d\Gamma(\omega))^{-1/2}e^{i\pi(f)}(1+d\Gamma(\omega))^{1/2}\right\| \leq 1+\left\|\omega^{1/2}f\right\|.
\end{aligned}$$

Proof. From Lemma B.3.5, we know that $(1+d\Gamma(\omega))^{1/2}e^{i\pi(f)}(1+d\Gamma(\omega))^{-1/2}$ is bounded. Let $\Psi \in \mathscr{H}$ and consider

$$\begin{aligned}
&\left\|(1+d\Gamma(\omega))^{1/2}e^{i\pi(f)}(1+d\Gamma(\omega))^{-1/2}\Psi\right\|^2 \\
&= \left\langle (1+d\Gamma(\omega))^{-1/2}\Psi, \left(1+d\Gamma(\omega)-\phi(\omega f)+\left\|\omega^{1/2}f\right\|^2\right)(1+d\Gamma(\omega))^{-1/2}\Psi \right\rangle \\
&\leq \left(\|\Psi\|^2+\left\|\omega^{1/2}f\right\|^2\|\Psi\|^2+2\left|\left\langle (1+d\Gamma(\omega))^{-1/2}\Psi, a(\omega f)(1+d\Gamma(\omega))^{-1/2}\Psi\right\rangle\right|\right) \\
&\leq \|\Psi\|^2\left(1+\left\|\omega^{1/2}f\right\|^2\right)^2, \qquad \text{(B.42)}
\end{aligned}$$

where we used Lemma B.3.3 and Corollary B.1.3. The second estimate of the Corollary follows by considering the adjoint operator of the first one and the unessential exchange of f and $-f$. □

Corollary B.3.7. *Let $\omega(k) > 0$ and $f, \omega^{1/2}f \in L^2(\mathbb{R}^d)$. Then $e^{i\pi(f)}D(d\Gamma(\omega)^{s/2}) = D(d\Gamma(\omega)^{s/2})$ for all $s \in [0,1]$, and for all $\Psi \in D(d\Gamma(\omega)^{s/2})$*

$$\left\|d\Gamma(\omega)^{s/2}e^{i\pi(f)}\Psi\right\| \leq (1+\left\|\omega^{1/2}f\right\|)^s\left\|(1+d\Gamma(\omega))^{s/2}\Psi\right\|.$$

Proof. That $e^{i\pi(f)}D(d\Gamma(\omega)^{s/2}) = D(d\Gamma(\omega)^{s/2})$ for all $s \in [0,1]$ follows from Lemma B.3.5 with the help of Theorem B.2.1. We also obtain

$$\begin{aligned}
\left\|(1+d\Gamma(\omega))^{s/2}e^{i\pi(f)}(1+d\Gamma(\omega))^{-s/2}\right\| &\leq \left\|(1+d\Gamma(\omega))^{1/2}e^{i\pi(f)}(1+d\Gamma(\omega))^{-1/2}\right\|^s \\
&\leq (1+\left\|\omega^{1/2}f\right\|)^s, \qquad \text{(B.43)}
\end{aligned}$$

where we use Corollary B.3.6. This completes the proof. □

Next, we collect properties of Weyl operators on the product space $L^2(\mathbb{R}^d)\otimes\mathcal{F}$. Recall, that $D(H_0) = D(p^2) \cap D(d\Gamma(\omega))$ and $D(H_0^{1/2}) = D(|p|) \cap D(d\Gamma(\omega)^{1/2})$ in Chapter 3.

Lemma B.3.8. *Let $\omega(k) > 0$, $p := -i\nabla_x$, and $F_x(k) := f(k)e^{-ikx}$ with $f, \omega^{1/2}f, k_jf \in L^2(\mathbb{R}^d)$ and $\mathrm{Re}\langle f, k_jf\rangle = 0$ for all $j \in \{1,...,d\}$. Then $e^{i\pi(F)}D(H_0^{1/2}) = D(H_0^{1/2})$ and*

$$e^{i\pi(F)}pe^{-i\pi(F)} = p - \phi(kF) \qquad \text{on } D(H_0^{1/2}).$$

Remark. Since the components of p are essentially self-adjoint on $D(H_0^{1/2})$, this lemma implies that $e^{i\pi(F)}p_je^{-i\pi(F)} = p_j - \phi(k_jF)$ is an equality between self-adjoint operators on their respective domains.

Proof. Analogously to our article [23] and the proof of Lemma 2.3.1 in this thesis, define $\mathcal{D} := D(H_0) \cap \mathscr{H}_0$, which is an operator core and hence a form core of H_0. Moreover, for $\Psi \in \mathcal{D}$, one shows that

$$pe^{-i\pi(F)}\Psi = e^{-i\pi(F)}(p-\phi(kF))\Psi \qquad \text{(B.44)}$$

by expanding $e^{-i\pi(F)}$ in its exponential series. We used that $[\phi(kF), \pi(F)]$ vanishes because of the assumption $\mathrm{Re}\langle f, k_j f\rangle = 0$ for all $j \in \{1, ..., d\}$. Since $\mathcal{D}$ is a form core of H_0 and since $(p - \phi(kF))$ is bounded with respect to $H_0^{1/2}$, Equation (B.44) extends to all $\Psi \in D(H_0^{1/2})$ and we see that $e^{-i\pi(F)}D(H_0^{1/2}) \subset D(|p|)$. Since $D(H_0^{1/2}) = D(|p|) \cap D(d\Gamma(\omega)^{1/2})$ and since $D(d\Gamma(\omega)^{1/2})$ is left invariant by $e^{-i\pi(F)}$ because of the assumption $\omega^{1/2} f \in L^2(\mathbb{R}^d)$, see Lemma B.3.5, we conclude that $e^{-i\pi(F)}D(H_0^{1/2}) \subset D(H_0^{1/2})$. Likewise, $e^{i\pi(F)}D(H_0^{1/2}) \subset D(H_0^{1/2})$ by changing the sign of f and the lemma is proved.

□

Lemma B.3.9. *Let* $\omega(k) > 0$, $p := -i\nabla_x$, *and* $F_x(k) := f(k)e^{-ikx}$ *with* $f, \omega^{1/2} f, \omega f, k_j f$, $k^2 f \in L^2(\mathbb{R}^d)$ *and* $\mathrm{Re}\langle f, k_j f\rangle = 0$ *for all* $j \in \{1, ..., d\}$. *Then* $e^{i\pi(F)}D(H_0) = D(H_0)$ *and*

$$e^{i\pi(F)} p^2 e^{-i\pi(F)} = (p - \phi(kF))^2 \qquad \text{on } D(H_0).$$

Proof. Let $\Psi \in D(H_0)$. Then $e^{-i\pi(F)}\Psi \in D(H_0^{1/2})$ by Lemma B.3.8 and $pe^{-i\pi(F)}\Psi$ is given by

$$pe^{-i\pi(F)}\Psi = e^{-i\pi(F)}\,(p - \phi(kF))\,\Psi. \tag{B.45}$$

Since $k^2 f \in L^2(\mathbb{R}^d)$, we have $(p - \phi(kF))\Psi \in D(|p|) \cap D(\sqrt{N})$. If we use Lemma B.3.8 with $\omega(k) = 1$, we get $e^{-i\pi(F)}D(|p|) \cap D(\sqrt{N}) = D(|p|) \cap D(\sqrt{N})$, and from that $e^{-i\pi(F)}(p - \phi(kF))\Psi \in D(|p|) \cap D(\sqrt{N})$. In view of Equation (B.45), we see especially that $e^{-i\pi(F)}\Psi \in D(p^2)$ and

$$p^2 e^{-i\pi(F)}\Psi = e^{-i\pi(F)}(p - \phi(kF))^2\Psi. \tag{B.46}$$

Because of the assumption $\omega f \in L^2(\mathbb{R}^d)$, we can also use Lemma B.3.3 to get

$$e^{-i\pi(F)}D(d\Gamma(\omega)) = D(d\Gamma(\omega)). \tag{B.47}$$

Since $\Psi \in D(H_0) = D(p^2) \cap D(d\Gamma(\omega))$, this shows that $e^{-i\pi(F)}\Psi \in D(H_0)$, and moreover $e^{-i\pi(F)}D(H_0) \subset D(H_0)$. That $e^{-i\pi(F)}D(H_0) = D(H_0)$ follows again by changing the sign of f. □

C. Appendix for Chapter 4: Renormalization of a class of generalized Nelson Hamiltonians

This chapter is the appendix for Chapter 4. It contains very important theorems about the existence of resolvents and about reordering. Some central estimates that are needed in Chapter 4 are also divested here.

C.1. Construction theorem

The following theorem is one of the important tools for constructing the resolvent in the proof of Theorem 4.3.4. It essentially is the last part of the proof of Theorem 3.5 in Hepp [21].

Theorem C.1.1. *Let $\mathscr{H}$ be a Hilbert space, $\delta > 0$, and $\rho_\delta := \{z \in \mathbb{C} \mid \operatorname{Re} z < -\delta\}$. Let $\{R(z)\}_{z\in\rho_\delta}$ be a family of bounded linear operators in $\mathscr{H}$ with*

(a) $R(z)^* = R(\overline{z})$ *for all* $z \in \rho_\delta$,

(b) $R(z_1) - R(z_2) = (z_2 - z_1)R(z_1)R(z_2)$ *for all* $z_1, z_2 \in \rho_\delta$,

(c) $zR(z)\Psi \to \Psi$ *as* $\operatorname{Re} z \to -\infty$ *for all* $\Psi \in \mathscr{H}$.

Then, there exists a self-adjoint operator $H : D(H) \subset \mathscr{H} \to \mathscr{H}$ and, for all $z \in \rho_\delta$, it holds $\sigma(H) \subset [-\delta, \infty)$ and $(z - H)^{-1} = R(z)$.

Proof. At first, we show that $R(z)$ is an injective operator. Then, one can define the densely defined operator $z - R(z)^{-1}$ on the range of $R(z)$. This operator will be self-adjoint and its resolvent will coincide with $R(z)$.

Let $z_1, z_2 \in \rho_\delta$ with $z_1 \neq z_2$. From Assumption (b), we know

$$\frac{1}{z_2 - z_1}\left(R(z_1) - R(z_2)\right) = R(z_1)R(z_2). \tag{C.1}$$

Since the left-hand side does not change by interchanging z_1 and z_2, the operators $R(z_1)$ and $R(z_2)$ commute for all $z_1, z_2 \in \rho_\delta$.

Let $K(z) := \{\Psi \in \mathscr{H} \mid R(z)\Psi = 0\}$ and $\Psi \in K(z)$. Then, it follows with (b) that

$$0 = R(z_1)\Psi = R(z_2)\Psi + (z_2 - z_1)R(z_2)R(z_1)\Psi = R(z_2)\Psi, \tag{C.2}$$

thus $\Psi \in K(z_2)$ and $K(z)$ is independent of $z \in \rho_\delta$.

Let $\Psi \in K(z_0)$ for some $z_0 \in \rho_\delta$, then

$$\Psi \in \bigcap_{z\in\rho_\delta} K(z) \qquad \text{and} \qquad zR(z)\Psi = 0 \quad \text{for all } z \in \rho_\delta, \tag{C.3}$$

which, using (c), goes to $\Psi = 0$ in the limit $\operatorname{Re} z \to -\infty$. Therefore, we arrive at $K(z) = \{0\}$ such that $R(z)$ is injective for all $z \in \rho_\delta$. Denote the range of $R(z)$ by $D(H(z))$, such that $R(z) : \mathscr{H} \to D(H(z))$ is a bijective mapping, and define

$$H(z) := z - R(z)^{-1} \qquad \text{on } D(H(z)) := R(z)\mathscr{H}. \tag{C.4}$$

Now, let $\Phi \in D(H(z))^\perp$. Then, for all $\Psi \in \mathscr{H}$

$$0 = \langle \Phi, R(z)\Psi\rangle = \langle R(\overline{z})\Phi, \Psi\rangle, \tag{C.5}$$

which implies that $R(\overline{z})\Phi = 0$ and since $R(\overline{z})$ is injective, we get $\Phi = 0$. This means that $D(H(z))^\perp = \{0\}$, which implies $\overline{D(H(z))} = \mathscr{H}$. Therefore, $H(z)$ is densely defined.

Let be $\Psi \in \mathscr{H}$ and therefore $R(z_1)\Psi$ any vector in $D(H(z_1))$. From (b), we obtain

$$R(z_1)\Psi = R(z_2)\Psi + (z_2 - z_1)R(z_2)R(z_1)\Psi, \tag{C.6}$$

which shows that it is also a vector in $D(H(z_2))$. We can conclude that $D(H(z))$ is also independent of $z \in \rho_\delta$. Applying $H(z_2)$ on the right-hand side of (C.6) and using (b), leads to

$$\begin{aligned}
&\left(z_2 - R(z_2)^{-1}\right) R(z_2)\Psi + (z_2 - z_1)\left(z_2 - R(z_2)^{-1}\right) R(z_2)R(z_1)\Psi \\
&= z_2R(z_2)\Psi - \Psi + z_2^2R(z_2)R(z_1)\Psi - z_1z_2R(z_2)R(z_1)\Psi - z_2R(z_1)\Psi + z_1R(z_1)\Psi \\
&= z_2\left(R(z_2) - R(z_1)\right)\Psi + \left(z_2^2 - z_1z_2\right) R(z_2)R(z_1)\Psi + \left(z_1R(z_1) - 1\right)\Psi \\
&= z_2\left(z_1 - z_2\right) R(z_2)R(z_1)\Psi + \left(z_2^2 - z_1z_2\right) R(z_2)R(z_1)\Psi + \left(z_1 - R(z_1)^{-1}\right) R(z_1)\Psi \\
&= H(z_1)R(z_1)\Psi,
\end{aligned} \tag{C.7}$$

which shows $H(z_2)R(z_1)\Psi = H(z_1)R(z_1)\Psi$ such that $H(z)$ is also independent of $z \in \rho_\delta$. We call this operator H and its domain $D(H)$. We can choose $\lambda \in \rho_\delta \cap \mathbb{R}$ and get $H = \lambda - R(\lambda)^{-1}$. Recall that $R(\lambda)^{-1}$ is a bijective map from $D(H)$ to $\mathscr{H}$ and remark that from (a) $R(\lambda) = R(\lambda)^*$. The operator H is symmetric since for $\Phi, \Psi \in D(H)$

$$\begin{aligned}
\langle \Phi, H\Psi\rangle &= \left\langle R(\lambda)R(\lambda)^{-1}\Phi, \left(\lambda - R(\lambda)^{-1}\right) R(\lambda)R(\lambda)^{-1}\Psi\right\rangle \\
&= \left\langle R(\lambda)R(\lambda)^{-1}\Phi, \left(\lambda R(\lambda) - 1\right) R(\lambda)^{-1}\Psi\right\rangle \\
&= \left\langle \left(\lambda R(\lambda) - 1\right) R(\lambda)^{-1}\Phi, R(\lambda)R(\lambda)^{-1}\Psi\right\rangle \\
&= \left\langle \left(\lambda - R(\lambda)^{-1}\right)\Phi, \Psi\right\rangle \\
&= \langle H\Phi, \Psi\rangle.
\end{aligned} \tag{C.8}$$

Now, $R(\lambda)^{-1} = -\big(\lambda - R(\lambda)^{-1} - \lambda\big) = \lambda - H$. That means that the range of $\lambda - H$ is also $\mathscr{H}$ and thus $H = H^*$.

Let be $\rho(H)$ the resolvent set of H and for all $z \in \rho(H)$ let $\tilde{R}(z) := (z-H)^{-1}$ be the resolvent of H. For $z \in \rho_\delta$, we know from our construction of H that $z - H = R(z)^{-1}$ is bijective and $(z-H)^{-1} = R(z)$ is bounded from assumptions. Thus, $\rho_\delta \subset \rho(H)$ and $\tilde{R}(z) = R(z)$ for $z \in \rho_\delta$.

Moreover, since $H = H^*$, we know that $\{z \in \mathbb{C} \mid \operatorname{Im} z \neq 0\} \subset \rho(H)$, and can conclude that $\sigma(H) \subset [-\delta, \infty)$. Therefore, H is bounded from below by $-\delta$. □

C.2. Reordering theorem

The following theorem allows us to change the order of summation of the sum (4.24).

Theorem C.2.1. *Let $\mathscr{H}$ be a Hilbert space and c_1, c_2, c_3 be bounded operators in $\mathscr{H}$ such that the series*

$$\sum_{n=1}^{\infty} \sum_{j_1,\dots,j_n=1}^{3} \|c_{j_1}\| \, \|c_{j_2}\| \cdots \|c_{j_n}\|$$

converges. Then, the operator series

$$\sum_{n=1}^{\infty} \sum_{j_1,\dots,j_n=1}^{3} c_{j_1} c_{j_2} \cdots c_{j_n}$$

converges absolutely and can be reordered to the likewise absolutely convergent series

$$\sum_{n=1}^{\infty} \sum_{\ell=0}^{[\frac{n}{2}]} \sum_{(j_1,\dots,j_{n-\ell}) \in S(n,\ell)} a_{j_1} a_{j_2} \cdots a_{j_{n-\ell}},$$

where

$$S(n,\ell) := \Big\{ (j_1, \dots, j_{n-\ell}) \in \{1,2,3\}^{n-\ell} \;\Big|\; \text{the number of } j\text{'s equal } 3 \text{ is } \ell; \text{ and if } j_i = 2 \text{ for an } i = 1, \dots, n-\ell-1, \text{ then } j_{i+1} \neq 1 \Big\},$$

and $a_1 = c_1$, $a_2 = c_2$, and $a_3 = c_2 c_1 + c_3$.

Remark: For better understanding, we give a more detailed description of the set $S(n,\ell)$: This set consists of tuples $(j_1, \dots, j_{n-\ell})$, where ℓ of the j's are equal to 3. Between these j's that are equal 3, the j's are ordered, say a 1 must not follow after a 2. Note that $0 \leq \ell \leq [\frac{n}{2}]$.

We list the sets of the first orders of n:

$$
\begin{aligned}
n=1\colon\ & S(1,0)=\{(1),(2)\}\\[1ex]
n=2\colon\ & S(2,0)=\{(1,1),(1,2),(2,2)\}\\
& S(2,1)=\{(3)\}\\[1ex]
n=3\colon\ & S(3,0)=\{(1,1,1),(1,1,2),(1,2,2),(2,2,2)\}\\
& S(3,1)=\{(1,3),(2,3),(3,1),(3,2)\}\\[1ex]
n=4\colon\ & S(4,0)=\{(1,1,1,1),(1,1,1,2),(1,1,2,2),(1,2,2,2),(2,2,2,2)\}\\
& S(4,1)=\{(1,1,3),(1,2,3),(2,2,3),(1,3,1),(1,3,2),\\
& \qquad\qquad (2,3,1),(2,3,2),(3,1,1),(3,1,2),(3,2,2)\}\\
& S(4,2)=\{(3,3)\}
\end{aligned}
$$

Compare this with the terms $B_\Lambda^{(n)}(z)$ in Equation (4.28). Now, we prove Theorem C.2.1.

Proof. At first, we define the two series occurring in the theorem as

$$R_1 := \sum_{n=1}^{\infty} \sum_{j_1,\dots,j_n=1}^{3} c_{j_1} c_{j_2} \cdots c_{j_n}, \tag{C.9}$$

$$R_2 := \sum_{n=1}^{\infty} \sum_{\ell=0}^{[\frac{n}{2}]} \sum_{(j_1,\dots,j_{n-\ell})\in S(n,\ell)} a_{j_1} a_{j_2} \cdots a_{j_{n-\ell}}, \tag{C.10}$$

and $R_1^{(N)}$ and $R_2^{(N)}$ for $N \in \mathbb{N}$ as the corresponding partial sums. The absolute convergence of R_1 follows directly from the assumption concerning the series over the operator norms. Comparing the two series, one determines that some of the summands of R_1 are combined to one summand of R_2, see the definition of a_3. Taking this into account, we have to show at first that each summand of R_1 is contained in R_2 exactly one time.

For that, we consider an arbitrary summand of the series R_1, that is

$$\sum_{n=1}^{\infty} \sum_{j_1,\dots,j_n=1}^{3} c_{j_1} \cdots c_{j_n}. \tag{C.11}$$

We determine such a summand as a product of ordered sequences of c_1's and c_2's that are interrupted by c_3 or c_2c_1. Let $m \in \mathbb{N}_0$ be the number of such interruptions and $b_0 := c_2c_1$ and $b_3 := c_3$. Then, with $(s_1, s_2, \dots, s_m) \in \{0,3\}^m$, an arbitrary summand can be written in the form

$$(\dots b_{s_1} \dots b_{s_2} \dots b_{s_3} \dots \;\dots b_{s_m} \dots), \tag{C.12}$$

where ... stands for an ordered sequence of c_1's and c_2's. Note that m can be zero, that means a summand without interruptions. But there is also the possibility to have two or more successive interruptions.

Summing over all $(s_1, s_2, ..., s_m) \in \{0, 3\}^m$ leads to

$$\begin{aligned} &\sum_{(s_1,s_2,...,s_m)\in\{0,3\}^m} (...b_{s_1}...b_{s_2}...b_{s_3}... \;...b_{s_m}...) \\ &\qquad = (...(c_2c_1 + c_3)...(c_2c_1 + c_3)...(c_2c_1 + c_3)... \;...(c_2c_1 + c_3)...) \\ &\qquad = (...a_3...a_3...a_3... \;...a_3...), \end{aligned} \tag{C.13}$$

which is an arbitrary summand of the series R_2, given by

$$\sum_{n=1}^{\infty}\sum_{\ell=0}^{[\frac{n}{2}]} \sum_{(j_1,...,j_{n-\ell})\in S(n,\ell)} a_{j_1}\cdots a_{j_{n-\ell}}. \tag{C.14}$$

The ordered sequences ... stay unchanged. That means indeed, each summand of R_1 is contained in R_2, after expanding the terms with an a_3. Because of that, the absolute convergence of R_2 follows by the assumptions concerning the series over the operator norms as well.

Now, it remains to show that the two series converge to the same limit. For that, we consider the difference of the partial sums $R_1^{(2N)}$ and $R_2^{(2N)}$. At first, we determine that $R_1^{(2N)}$ contains all summands of $R_2^{(2N)}$. On the other hand, $R_2^{(2N)}$ contains all summands of $R_1^{(N)}$. Because of that, we can write

$$R_1^{(2N)} - R_2^{(2N)} = \overline{\sum_{n=N}^{2N}} \; \overline{\sum_{j_1,...,j_n=1}^{3}} c_{j_1}c_{j_2}\cdots c_{j_n}, \tag{C.15}$$

where the lines over the sums indicate that some of the summands in view of the complete sums are missing. We estimate

$$\begin{aligned} \left\| R_1^{(2N)} - R_2^{(2N)} \right\| &\leq \overline{\sum_{n=N}^{2N}} \; \overline{\sum_{j_1,...,j_n=1}^{3}} \|c_{j_1}\| \, \|c_{j_2}\| \;\cdots \|c_{j_n}\| \\ &\leq \sum_{n=N}^{\infty} \sum_{j_1,...,j_n=1}^{3} \|c_{j_1}\| \, \|c_{j_2}\| \;\cdots \|c_{j_n}\|, \end{aligned} \tag{C.16}$$

which goes to zero for $N \to \infty$ from assumptions. That means, $R_1^{(N)}$ and $R_2^{(N)}$ have the same limit as $N \to \infty$ and the proof is done. □

C.3. Important estimates

The following estimates are important tools for the main theorems in the Sections 4.3 and 4.4. As in that sections, let be $H_0 = \Omega(p) + d\Gamma(\omega)$ the free Hamiltonian in $\mathscr{H} = L^2(\mathbb{R}^d) \otimes \mathcal{F}(L^2(\mathbb{R}^d))$ with Ω and ω according to $(A0)$, $R_0(z) = (z - H_0)^{-1}$ for suitable $z \in \mathbb{C}$, and a^* and a the creation and annihilation operators. Moreover, recall that for $F_x(k) := f(k)e^{-ikx}$ and $G_x(k) := g(k)e^{-ikx}$

$$E(F,G) = \int dk \frac{\overline{f(k)}g(k)}{\omega(k) + \Omega(k)}. \tag{C.17}$$

At some points, we need the general definition of creation and annihilation operators that we give in Appendix A.2. The estimates given in Appendix A.2 and Appendix B.1 are important for our purposes here as well. Therefore, Section C.3 can be seen as a continuation of these appendices.

Lemma C.3.1. *Let $F_x(k) := f(k)e^{-ikx}$ with $f \in L^2(\mathbb{R}^d)$, $z \in \mathbb{C}$ with $\operatorname{Re} z \leq -1$ and $0 \leq \alpha \leq \frac{1}{2}$, $0 \leq \beta \leq 1$. Then,*

$$\begin{aligned}
&\left\| R_0(z)^\alpha a(F) R_0(z)^\beta \right\| \\
&\qquad \leq \left(1 + \frac{1}{m^\alpha}\right) \left\{ \sup_{p \in \mathbb{R}^d} \int dk \frac{|f(k)|^2}{\omega(k)^{1-2\alpha}[|\operatorname{Re} z| + \Omega(p-k) + \omega(k)]^{2(\alpha+\beta)-1}} \right\}^{1/2}, \\
&\left\| R_0(z)^\beta a^*(F) R_0(z)^\alpha \right\| \\
&\qquad \leq \left(1 + \frac{1}{m^\alpha}\right) \left\{ \sup_{p \in \mathbb{R}^d} \int dk \frac{|f(k)|^2}{\omega(k)^{1-2\alpha}[|\operatorname{Re} z| + \Omega(p-k) + \omega(k)]^{2(\alpha+\beta)-1}} \right\}^{1/2}.
\end{aligned}$$

Proof. Let $\Psi \in \mathscr{H}$ and consider in momentum space

$$\begin{aligned}
&\left\| R_0(z)^\alpha a(F) R_0(z)^\beta \Psi \right\|^2 \\
&= \sum_{n \geq 0} \int dp \int dk_1 \dots \int dk_n \frac{n+1}{\left| z - \Omega(p) - \sum_{\ell=1}^n \omega(k_\ell) \right|^{2\alpha}} \left| \int dk \, \frac{\overline{f(k)} \Psi^{(n+1)}(p-k,k,k_1,\dots,k_n)}{[z - \Omega(p-k) - \sum_{\ell=1}^n \omega(k_\ell) - \omega(k)]^\beta} \right|^2.
\end{aligned} \tag{C.18}$$

Using $\operatorname{Re} z \leq -1$, $\Omega(p) \geq 0$, and $\omega(k) \geq m$, yields

$$\frac{n+1}{\left| z - \Omega(p) - \sum_{\ell=1}^n \omega(k_\ell) \right|^{2\alpha}} \leq \left(1 + \frac{1}{m^{2\alpha}}\right) (n+1)^{1-2\alpha} \tag{C.19}$$

and

$$\left\| R_0(z)^\alpha a(F) R_0(z)^\beta \Psi \right\|^2 \tag{C.20}$$
$$\le \left(1 + \frac{1}{m^{2\alpha}}\right) \sum_{n\ge 0} \int dp \int dk_1 \ldots \int dk_n \left| \int dk \, \frac{\overline{f(k)} (n+1)^{\frac{1}{2}-\alpha} \omega(k)^{\frac{1}{2}-\alpha} \Psi^{(n+1)}(p-k, k, k_1, \ldots, k_n)}{\omega(k)^{\frac{1}{2}-\alpha} [z - \Omega(p-k) - \sum_{\ell=1}^{n} \omega(k_\ell) - \omega(k)]^\beta} \right|^2,$$

where we inserted a suitable power of $\omega(k)$. Now, we split the power β in a suitable way, use Cauchy-Schwarz, and get

$$\left\| R_0(z)^\alpha a(F) R_0(z)^\beta \Psi \right\|^2$$
$$\le \left(1 + \frac{1}{m^{2\alpha}}\right) \sum_{n\ge 0} \int dp \int dk_1 \ldots \int dk_n$$
$$\int dk \, \frac{|f(k)|^2}{\omega(k)^{1-2\alpha} \left| z - \Omega(p-k) - \sum_{\ell=1}^{n} \omega(k_\ell) - \omega(k) \right|^{2(\alpha+\beta)-1}}$$
$$\int dk_{n+1} \, \frac{(n+1)^{1-2\alpha} \omega(k_{n+1})^{1-2\alpha} \left| \Psi^{(n+1)}(p - k_{n+1}, k_1, \ldots, k_{n+1}) \right|^2}{\left| z - \Omega(p - k_{n+1}) - \sum_{\ell=1}^{n+1} \omega(k_\ell) \right|^{1-2\alpha}}$$
$$\le \left(1 + \frac{1}{m^{2\alpha}}\right) \sup_{p\in\mathbb{R}^d} \int dk \, \frac{|f(k)|^2}{\omega(k)^{1-2\alpha} [|\mathrm{Re}\, z| + \Omega(p-k) + \omega(k)]^{2(\alpha+\beta)-1}} \tag{C.21}$$
$$\sum_{n\ge 0} \int dp \int dk_1 \ldots \int dk_{n+1} \, \frac{[(n+1)\omega(k_{n+1})]^{1-2\alpha}}{[|\mathrm{Re}\, z| + \Omega(p) + \sum_{\ell=1}^{n+1} \omega(k_\ell)]^{1-2\alpha}} \left| \Psi^{(n+1)}(p, k_1, \ldots, k_{n+1}) \right|^2.$$

Since $\Psi^{(n+1)}(p, k_1, \ldots, k_{n+1})$ and $\sum_{\ell=1}^{n+1} \omega(k_\ell)$ are symmetric with respect to permutations of the k-momenta, we can make the following substitution in the integrand:

$$[(n+1)\omega(k_{n+1})]^{1-2\alpha} = (n+1)^{-2\alpha} \sum_{j=1}^{n+1} \omega(k_{n+1})^{1-2\alpha} \to (n+1)^{-2\alpha} \sum_{j=1}^{n+1} \omega(k_j)^{1-2\alpha}. \tag{C.22}$$

Since $0 \le \alpha \le \frac{1}{2}$, we can use

$$\sum_{j=1}^{n+1} \omega(k_j)^{1-2\alpha} \le (n+1)^{2\alpha} \left(\sum_{j=1}^{n+1} \omega(k_j) \right)^{1-2\alpha} \tag{C.23}$$

(see Lemma C.3.8), and obtain

$$\begin{aligned}
&\left\|R_0(z)^\alpha a(F) R_0(z)^\beta \Psi\right\|^2 \\
&\leq \left(1+\frac{1}{m^{2\alpha}}\right) \sup_{p\in\mathbb{R}^d} \int dk \; \frac{|f(k)|^2}{\omega(k)^{1-2\alpha}[|\mathrm{Re}\, z| + \Omega(p-k) + \omega(k)]^{2(\alpha+\beta)-1}} \\
&\qquad \sum_{n\geq 0} \int dp \int dk_1 ... \int dk_{n+1} \frac{\left[\sum_{j=1}^{n+1} \omega(k_j)\right]^{1-2\alpha}}{\left[|\mathrm{Re}\, z| + \Omega(p) + \sum_{\ell=1}^{n+1} \omega(k_\ell)\right]^{1-2\alpha}} \left|\Psi^{(n+1)}(p, k_1, ..., k_{n+1})\right|^2 \\
&\leq \left(1+\frac{1}{m^{2\alpha}}\right) \sup_{p\in\mathbb{R}^d} \int dk \; \frac{|f(k)|^2}{\omega(k)^{1-2\alpha}[|\mathrm{Re}\, z| + \Omega(p-k) + \omega(k)]^{2(\alpha+\beta)-1}} \; \|\Psi\|^2, \qquad \text{(C.24)}
\end{aligned}$$

which shows the first estimate of the lemma. The second estimate follows by adjoining the first one. ☐

Lemma C.3.2. *Let* $F_x(k) := f(k)e^{-ikx}$ *and* $G_x(k) := g(k)e^{-ikx}$ *with* $f, g \in L^2(\mathbb{R}^d)$, $z \in \mathbb{C}$ *with* $\mathrm{Re}\, z \leq -1$, *and* $0 \leq \alpha, \beta \leq 1$. *Then,*

$$\begin{aligned}
&\left\|R_0(z)^\alpha \left[a(F)R_0(z)a^*(G) + E(F,G)\right] R_0(z)^\beta\right\| \\
&\leq 2 \sup_{p\in\mathbb{R}^d} \left\{ \left(\int dk \frac{|g(k)|^2}{\omega(k)[|\mathrm{Re}\, z| + \Omega(p-k) + \omega(k)]^{2\alpha}}\right) \right. \\
&\qquad \left. \left(\int dk \frac{|f(k)|^2}{\omega(k)[|\mathrm{Re}\, z| + \Omega(p-k) + \omega(k)]^{2\beta}}\right) \right\}^{1/2} \\
&\quad + \sup_{p\in\mathbb{R}^d} \int dk \frac{|f(k)|\cdot|g(k)|\cdot|\Omega(p-k) - \Omega(k)|}{[\omega(k) + \Omega(k)][|\mathrm{Re}\, z| + \Omega(p)]^{\alpha+\beta}[|\mathrm{Re}\, z| + \Omega(p-k) + \omega(k)]}.
\end{aligned}$$

Proof. Let $\Phi, \Psi \in \mathscr{H}$ and consider

$$\begin{aligned}
&\left\langle \Phi, R_0(z)^\alpha \left[a(F)R_0(z)a^*(G) + E(F,G)\right] R_0(z)^\beta \Psi \right\rangle \\
&= \int dk \; \int d\tilde{k} \; \overline{f(k)} g(\tilde{k}) \left\langle R_0(\overline{z})^\alpha \Phi, e^{ikx} a_k R_0(z) e^{-i\tilde{k}x} a^*_{\tilde{k}} R_0(z)^\beta \Psi \right\rangle \\
&\quad + \int dk \; \frac{\overline{f(k)} g(k)}{\omega(k) + \Omega(k)} \left\langle R_0(\overline{z})^\alpha \Phi, R_0(z)^\beta \Psi \right\rangle.
\end{aligned} \qquad \text{(C.25)}$$

In the inner product of the first integral, we use the pull-through formulas and the commutation relation $[a_k, a^*_{\tilde{k}}] = \delta(k-\tilde{k})$:

$$\begin{aligned}
&\Big\langle R_0(\overline{z})^\alpha \Phi, e^{ikx} a_k R_0(z) e^{-i\tilde{k}x} a^*_{\tilde{k}} R_0(z)^\beta \Psi \Big\rangle \\
&= \Big\langle R_0(\overline{z})^\alpha \Phi, [z - \Omega(p-k) - d\Gamma(\omega) - \omega(k)]^{-1} e^{i(k-\tilde{k})x} a_k a^*_{\tilde{k}} R_0(z)^\beta \Psi \Big\rangle \\
&= \Big\langle R_0(\overline{z})^\alpha \Phi, e^{-i\tilde{k}x} a^*_{\tilde{k}} [z - \Omega(p-k-\tilde{k}) - d\Gamma(\omega) - \omega(k) - \omega(\tilde{k})]^{-1} e^{ikx} a_k R_0(z)^\beta \Psi \Big\rangle \\
&\quad + \delta(k-\tilde{k}) \Big\langle R_0(\overline{z})^\alpha \Phi, [z - \Omega(p-k) - d\Gamma(\omega) - \omega(k)]^{-1} e^{i(k-\tilde{k})x} R_0(z)^\beta \Psi \Big\rangle \\
&= \Big\langle [\overline{z} - \Omega(p-k-\tilde{k}) - d\Gamma(\omega) - \omega(k) - \omega(\tilde{k})]^{-1/2} e^{i\tilde{k}x} a_{\tilde{k}} R_0(\overline{z})^\alpha \Phi, \\
&\quad [z - \Omega(p-k-\tilde{k}) - d\Gamma(\omega) - \omega(k) - \omega(\tilde{k})]^{-1/2} e^{ikx} a_k R_0(z)^\beta \Psi \Big\rangle \\
&\quad + \delta(k-\tilde{k}) \Big\langle R_0(\overline{z})^\alpha \Phi, [z - \Omega(p-k) - d\Gamma(\omega) - \omega(k)]^{-1} e^{i(k-\tilde{k})x} R_0(z)^\beta \Psi \Big\rangle.
\end{aligned} \tag{C.26}$$

Therefore, we obtain

$$\begin{aligned}
&\left| \left\langle \Phi, R_0(z)^\alpha \left[a(F) R_0(z) a^*(G) + E(F,G) \right] R_0(z)^\beta \Psi \right\rangle \right| \\
&\leq \Big| \int dk \int d\tilde{k} \int dp\, \overline{f(k)} g(\tilde{k}) \\
&\qquad \Big\langle [\overline{z} - \Omega(p-k-\tilde{k}) - d\Gamma(\omega) - \omega(k) - \omega(\tilde{k})]^{-1/2} \Big(e^{i\tilde{k}x} a_{\tilde{k}} R_0(\overline{z})^\alpha \Phi \Big)(p), \\
&\qquad\quad [z - \Omega(p-k-\tilde{k}) - d\Gamma(\omega) - \omega(k) - \omega(\tilde{k})]^{-1/2} \Big(e^{ikx} a_k R_0(z)^\beta \Psi \Big)(p) \Big\rangle \Big| \\
&\quad + \Big| \int dk\, \overline{f(k)} g(k) \Big\langle R_0(\overline{z})^\alpha \Phi, \\
&\qquad \big([z - \Omega(p-k) - d\Gamma(\omega) - \omega(k)]^{-1} + [\omega(k) + \Omega(k)]^{-1} \big) R_0(z)^\beta \Psi \Big\rangle \Big|.
\end{aligned} \tag{C.27}$$

The first summand of (C.27), we estimate further to

$$\begin{aligned}
&\int dk \int d\tilde{k} \int dp\, |f(k)| \left| g(\tilde{k}) \right| \\
&\qquad \Big\| [\overline{z} - \Omega(p-k-\tilde{k}) - d\Gamma(\omega) - \omega(k) - \omega(\tilde{k})]^{-1/2} \Big(e^{i\tilde{k}x} a_{\tilde{k}} R_0(\overline{z})^\alpha \Phi \Big)(p) \Big\| \\
&\qquad \Big\| [z - \Omega(p-k-\tilde{k}) - d\Gamma(\omega) - \omega(k) - \omega(\tilde{k})]^{-1/2} \big(e^{ikx} a_k R_0(z)^\beta \Psi \big)(p) \Big\| \\
&\leq \int dk \int d\tilde{k} \int dp\, |f(k)| \left| g(\tilde{k}) \right| \\
&\qquad \Big\| [1 + d\Gamma(\omega) + \omega(\tilde{k})]^{-1/2} \Big(e^{i\tilde{k}x} a_{\tilde{k}} R_0(\overline{z})^\alpha \Phi \Big)(p) \Big\| \\
&\qquad \Big\| [1 + d\Gamma(\omega) + \omega(k)]^{-1/2} \big(e^{ikx} a_k R_0(z)^\beta \Psi \big)(p) \Big\|.
\end{aligned} \tag{C.28}$$

Using the pull-through formulas again leads to

$$\begin{aligned}
&\int dk \int d\tilde{k} \int dp \; |f(k)| \left|g(\tilde{k})\right| \\
&\qquad \left\| [\overline{z} - \Omega(p-\tilde{k}) - d\Gamma(\omega) - \omega(\tilde{k})]^{-\alpha} \left(e^{i\tilde{k}x} a_{\tilde{k}} [1 + d\Gamma(\omega)]^{-1/2} \Phi \right)(p) \right\| \\
&\qquad \left\| [z - \Omega(p-k) - d\Gamma(\omega) - \omega(k)]^{-\beta} \left(e^{ikx} a_k [1 + d\Gamma(\omega)]^{-1/2} \Psi \right)(p) \right\| \\
&\leq \int dk \int d\tilde{k} \int dp \; |f(k)| \left|g(\tilde{k})\right| \\
&\qquad \left\| [|\mathrm{Re}\, z| + \Omega(p-\tilde{k}) + \omega(\tilde{k})]^{-\alpha} \left(e^{i\tilde{k}x} a_{\tilde{k}} [1 + d\Gamma(\omega)]^{-1/2} \Phi \right)(p) \right\| \\
&\qquad \left\| [|\mathrm{Re}\, z| + \Omega(p-k) + \omega(k)]^{-\beta} \left(e^{ikx} a_k [1 + d\Gamma(\omega)]^{-1/2} \Psi \right)(p) \right\| \\
&= \int dp \int d\tilde{k} \; \frac{\left|g(\tilde{k})\right|}{[|\mathrm{Re}\, z| + \Omega(p-\tilde{k}) + \omega(\tilde{k})]^{\alpha}} \cdot \frac{\omega(\tilde{k})^{1/2}}{\omega(\tilde{k})^{1/2}} \cdot \left\| a_{\tilde{k}} [1 + d\Gamma(\omega)]^{-1/2} \Phi(p) \right\| \\
&\qquad \int dk \; \frac{|f(k)|}{[|\mathrm{Re}\, z| + \Omega(p-k) + \omega(k)]^{\beta}} \cdot \frac{\omega(k)^{1/2}}{\omega(k)^{1/2}} \cdot \left\| a_k [1 + d\Gamma(\omega)]^{-1/2} \Psi(p) \right\|.
\end{aligned} \tag{C.29}$$

Next, we use Cauchy-Schwarz and rename $\tilde{k}$ also to k to obtain

$$\begin{aligned}
&\int dp \Bigg[\left(\int dk \frac{|g(k)|^2}{\omega(k)[|\mathrm{Re}\, z| + \Omega(p-k) + \omega(k)]^{2\alpha}} \right) \left(\int dk \; \omega(k) \left\| a_k [1 + d\Gamma(\omega)]^{-1/2} \Phi(p) \right\|^2 \right) \\
&\quad \left(\int dk \; \frac{|f(k)|^2}{\omega(k)[|\mathrm{Re}\, z| + \Omega(p-k) + \omega(k)]^{2\beta}} \right) \left(\int dk \; \omega(k) \left\| a_k [1 + d\Gamma(\omega)]^{-1/2} \Psi(p) \right\|^2 \right) \Bigg]^{\frac{1}{2}}.
\end{aligned} \tag{C.30}$$

Taking the supremum over all $p \in \mathbb{R}^d$ of the front integrals and using the inequality $a \cdot b \leq \frac{1}{2}(a^2 + b^2)$ for non-negative numbers a and b, leads to

$$\begin{aligned}
&\sup_{p \in \mathbb{R}^d} \left[\left(\int dk \frac{|g(k)|^2}{\omega(k)[|\mathrm{Re}\, z| + \Omega(p-k) + \omega(k)]^{2\alpha}} \right) \left(\int dk \frac{|f(k)|^2}{\omega(k)[|\mathrm{Re}\, z| + \Omega(p-k) + \omega(k)]^{2\beta}} \right) \right]^{\frac{1}{2}} \\
&\cdot \frac{1}{2} \int dp \; \left(\int dk \; \omega(k) \left\| a_k [1 + d\Gamma(\omega)]^{-1/2} \Phi(p) \right\|^2 + \int dk \; \omega(k) \left\| a_k [1 + d\Gamma(\omega)]^{-1/2} \Psi(p) \right\|^2 \right) \\
&\leq \sup_{p \in \mathbb{R}^d} \left[\left(\int dk \frac{|g(k)|^2}{\omega(k)[|\mathrm{Re}\, z| + \Omega(p-k) + \omega(k)]^{2\alpha}} \right) \right. \\
&\qquad \left. \left(\int dk \frac{|f(k)|^2}{\omega(k)[|\mathrm{Re}\, z| + \Omega(p-k) + \omega(k)]^{2\beta}} \right) \right]^{\frac{1}{2}} \cdot \frac{1}{2} \left(\|\Phi\|^2 + \|\Psi\|^2 \right),
\end{aligned} \tag{C.31}$$

where we used the representation (4.17) in the last step, and we are done with this first term of (C.27) for the moment.

We can estimate the second integral of (C.27) to

$$\sum_{n\geq 0}\int dp\ \int dk_1\ ...\int dk_n\ \left|\Phi^{(n)}(p,k_1,...,k_n)\right|\ \left|\Psi^{(n)}(p,k_1,...,k_n)\right|$$

$$\int dk\ \frac{|f(k)|\ |g(k)|\ \left|\Omega(p-k)-\Omega(k)+\sum_{\ell=1}^{n}\omega(k_\ell)-z\right|}{\left|\Omega(p)+\sum_{\ell=1}^{n}\omega(k_\ell)-z\right|^{\alpha+\beta}[\omega(k)+\Omega(k)]\left|\Omega(p-k)+\sum_{\ell=1}^{n}\omega(k_\ell)+\omega(k)-z\right|}$$

$$\leq \frac{1}{2}\left(\|\Phi\|^2+\|\Psi\|^2\right)\cdot\sup_{n\in\mathbb{N}}\sup_{p\in\mathbb{R}^d}\sup_{k_1,...,k_n\in\mathbb{R}^d} \tag{C.32}$$

$$\int dk\ \frac{|f(k)|\ |g(k)|\ \left|\Omega(p-k)-\Omega(k)+\sum_{\ell=1}^{n}\omega(k_\ell)-z\right|}{\left|\Omega(p)+\sum_{\ell=1}^{n}\omega(k_\ell)-z\right|^{\alpha+\beta}[\omega(k)+\Omega(k)]\left|\Omega(p-k)+\sum_{\ell=1}^{n}\omega(k_\ell)+\omega(k)-z\right|}.$$

Now, define $\tilde{z} := z-\sum_{\ell=1}^{n}\omega(k_\ell)$. Then, from $\operatorname{Re} z\leq -1$, $\operatorname{Re}\tilde{z}\leq -1-\sum_{\ell=1}^{n}\omega(k_\ell)$ and $|\operatorname{Re}\tilde{z}|\geq|\operatorname{Re} z|$. For parts of the fraction in the term above, we get

$$\begin{aligned}
&\frac{|\Omega(p-k)-\Omega(k)-\tilde{z}|}{|\Omega(p)-\tilde{z}|^{\alpha+\beta}|\Omega(p-k)+\omega(k)-\tilde{z}|}\leq\frac{|\Omega(p-k)-\Omega(k)|+|\tilde{z}|}{|\Omega(p)-\tilde{z}|^{\alpha+\beta}|\Omega(p-k)+\omega(k)-\tilde{z}|}\\
&\leq\frac{|\Omega(p-k)-\Omega(k)|}{|\Omega(p)-\tilde{z}|^{\alpha+\beta}|\Omega(p-k)+\omega(k)-\tilde{z}|}+\frac{|\tilde{z}|^{1-(\alpha+\beta)}}{|\Omega(p-k)+\omega(k)-\tilde{z}|}\\
&\leq\frac{|\Omega(p-k)-\Omega(k)|}{[\Omega(p)+|\operatorname{Re} z|]^{\alpha+\beta}[\Omega(p-k)+\omega(k)+|\operatorname{Re} z|]}+\frac{1}{[\Omega(p-k)+\omega(k)+|\operatorname{Re} z|]^{\alpha+\beta}},
\end{aligned} \tag{C.33}$$

such that (C.32) can be estimated by

$$\begin{aligned}
&\frac{1}{2}\Big(\sup_{p\in\mathbb{R}^d}\int dk\frac{|f(k)|\ |g(k)|\ |\Omega(p-k)-\Omega(k)|}{[\omega(k)+\Omega(k)][\Omega(p)+|\operatorname{Re} z|]^{\alpha+\beta}[\Omega(p-k)+\omega(k)+|\operatorname{Re} z|]}\\
&\quad+\sup_{p\in\mathbb{R}^d}\int dk\frac{|f(k)|\ |g(k)|}{[\omega(k)+\Omega(k)][\Omega(p-k)+\omega(k)+|\operatorname{Re} z|]^{\alpha+\beta}}\Big)\left(\|\Phi\|^2+\|\Psi\|^2\right).
\end{aligned} \tag{C.34}$$

In the second summand of this term, we can omit the $\Omega(k)$ in the denominator and use Cauchy-Schwarz to get

$$\int dk\ \frac{|f(k)|\ |g(k)|}{[\omega(k)+\Omega(k)][\Omega(p-k)+\omega(k)+|\operatorname{Re} z|]^{\alpha+\beta}} \tag{C.35}$$

$$\leq\left(\int dk\ \frac{|g(k)|^2}{\omega(k)[\Omega(p-k)+\omega(k)+|\operatorname{Re} z|]^{2\alpha}}\right)^{\frac{1}{2}}\left(\int dk\ \frac{|f(k)|^2}{\omega(k)[\Omega(p-k)+\omega(k)+|\operatorname{Re} z|]^{2\beta}}\right)^{\frac{1}{2}},$$

which now corresponds to the right-hand side of (C.31). Using (C.31), (C.34), and (C.35), we arrive at

$$
\begin{aligned}
&\left|\left\langle \Phi, R_0(z)^\alpha \left[a(F)R_0(z)a^*(G) + E(F,G)\right] R_0(z)^\beta \Psi\right\rangle\right| \\
&\leq \Bigg(2\sup_{p\in\mathbb{R}^d} \Bigg[\left(\int dk \frac{|g(k)|^2}{\omega(k)[|\mathrm{Re}\, z| + \Omega(p-k) + \omega(k)]^{2\alpha}} \right) \\
&\qquad\qquad \left(\int dk \frac{|f(k)|^2}{\omega(k)[|\mathrm{Re}\, z| + \Omega(p-k) + \omega(k)]^{2\beta}} \right) \Bigg]^{\frac{1}{2}} \\
&\quad + \sup_{p\in\mathbb{R}^d} \int dk \; \frac{|f(k)|\;|g(k)|\;|\Omega(p-k) - \Omega(k)|}{[\omega(k)+\Omega(k)][|\mathrm{Re}\, z| + \Omega(p)]^{\alpha+\beta}[|\mathrm{Re}\, z| + \Omega(p-k) + \omega(k)]} \Bigg) \\
&\quad \cdot \frac{1}{2}\left(\|\Phi\|^2 + \|\Psi\|^2 \right).
\end{aligned}
\tag{C.36}
$$

Taking the supremum over all $\Phi, \Psi \in \mathscr{H}$ with $\|\Phi\| = 1$ and $\|\Psi\| = 1$ completes the proof. ☐

Corollary C.3.3. *Let $F_x(k) := f(k)e^{-ikx}$ with $f \in L^2(\mathbb{R}^d)$, $z \in \mathbb{C}$ with $\mathrm{Re}\, z \leq -1$, and $0 \leq \gamma \leq 1$. Then,*

$$
\begin{aligned}
&\|R_0(z)^\gamma \left[a(F)R_0(z)a^*(F) + E(F,F)\right] R_0(z)^\gamma\| \\
&\leq 2\sup_{p\in\mathbb{R}^d} \int dk \frac{|f(k)|^2}{\omega(k)[|\mathrm{Re}\, z| + \Omega(p-k) + \omega(k)]^{2\gamma}} \\
&\quad + \sup_{p\in\mathbb{R}^d} \int dk \frac{|f(k)|^2 |\Omega(p-k) - \Omega(k)|}{[\omega(k)+\Omega(k)][|\mathrm{Re}\, z| + \Omega(p)]^{2\gamma}[|\mathrm{Re}\, z| + \Omega(p-k) + \omega(k)]}.
\end{aligned}
$$

Proof. Follows directly from Lemma C.3.2 with $f = g$ and $\alpha = \beta = \gamma$. ☐

Lemma C.3.4. *Let $p := -i\nabla_x$, $F_x(k) := f(k)e^{-ikx}$ with $|k|^\sigma f \in L^2(\mathbb{R}^d)$ for some $\sigma \in [0,1]$, $z \in \mathbb{C}$ with $\mathrm{Re}\, z \leq -1$, and $\alpha, \beta \in \mathbb{R}$ with $0 \leq \alpha \leq \frac{1}{2}$ and $0 \leq \beta \leq 1$. Then,*

$$
\begin{aligned}
&\left\|R_0(z)^\alpha a([|p|^\sigma, F]) R_0(z)^\beta\right\| \\
&\leq \left(1 + \frac{1}{m^\alpha}\right) \left\{ \sup_{p\in\mathbb{R}^d} \int dk \frac{(1+|k|)^{2\sigma}|f(k)|^2}{\omega(k)^{1-2\alpha}[|\mathrm{Re}\, z| + \Omega(p-k) + \omega(k)]^{2(\alpha+\beta)-1}} \right\}^{1/2}, \\
&\left\|R_0(z)^\beta a^*([|p|^\sigma, F]) R_0(z)^\alpha\right\| \\
&\leq \left(1 + \frac{1}{m^\alpha}\right) \left\{ \sup_{p\in\mathbb{R}^d} \int dk \frac{(1+|k|)^{2\sigma}|f(k)|^2}{\omega(k)^{1-2\alpha}[|\mathrm{Re}\, z| + \Omega(p-k) + \omega(k)]^{2(\alpha+\beta)-1}} \right\}^{1/2}.
\end{aligned}
$$

Proof. Note that

$$e^{ikx}|p|^\sigma - |p|^\sigma e^{ikx} = (|p-k|^\sigma - |p|^\sigma)\, e^{ikx}. \tag{C.37}$$

and that for $p, k \in \mathbb{R}^d$ and $\sigma \in [0,1]$

$$|\,|p-k|^\sigma - |p|^\sigma| \leq |k|^\sigma. \tag{C.38}$$

Thus, we get for $\Psi \in \mathscr{H}$ in momentum space

$$\begin{aligned}
&\left\| R_0(z)^\alpha a([|p|^\sigma, F]) R_0(z)^\beta \Psi \right\|^2 \\
&= \sum_{n\geq 0} \int dp \int dk_1 \ldots \int dk_n \; \frac{1}{\left| z - \Omega(p) - \sum\limits_{\ell=1}^n \omega(k_\ell) \right|^{2\alpha}} \\
&\qquad \left| \int dk\; \overline{f(k)}\, (|p-k|^\sigma - |p|^\sigma) \left(a_k R_0(z)^\beta \Psi \right)^{(n)} (p-k, k_1, \ldots, k_n) \right|^2 \\
&\leq \sum_{n\geq 0} \int dp \int dk_1 \ldots \int dk_n \; \frac{n+1}{\left| z - \Omega(p) - \sum\limits_{\ell=1}^n \omega(k_\ell) \right|^{2\alpha}} \\
&\qquad \left(\int dk\; \frac{|k|^\sigma |f(k)| \left| \Psi^{(n+1)}(p-k, k, k_1, \ldots, k_n) \right|}{\left| z - \Omega(p-k) - \sum\limits_{\ell=1}^n \omega(k_\ell) - \omega(k) \right|^\beta} \right)^2 .
\end{aligned} \tag{C.39}$$

Now, the proof works analogously to the proof of Lemma C.3.1, we easily have to replace $f(k)$ by $|k|^\sigma f(k)$. We arrive at

$$\begin{aligned}
&\left\| R_0(z)^\alpha a([|p|^\sigma, F]) R_0(z)^\beta \Psi \right\|^2 \\
&\leq \left(1 + \frac{1}{m^{2\alpha}} \right) \sup_{p\in\mathbb{R}^d} \int dk\; \frac{|k|^{2\sigma} |f(k)|^2}{\omega(k)^{1-2\alpha} [|\mathrm{Re}\, z| + \Omega(p-k) + \omega(k)]^{2(\alpha+\beta)-1}} \; \|\Psi\|^2.
\end{aligned} \tag{C.40}$$

In the calculations made in Chapter 4, it is necessary that this upper boundary increases if σ increases. For that, we estimate $|k|^{2\sigma} \leq (1+|k|)^{2\sigma}$ which completes the proof of the first estimate of the lemma. Since $\omega(k) > 0$, this last estimate does not change the regularity behavior of the expression.

The second estimate of the lemma follows by adjoining the first one. □

Remark. The right-hand sides of the estimates in Lemma C.3.4 are just given by $\mathcal{A}_{\alpha,\beta,z}((1+|k|)^\sigma f)$. This quantity is defined in Equation (4.36).

Lemma C.3.5. *Let* $p := -i\nabla_x$, $F_x(k) := f(k)e^{-ikx}$ *with* $|k|^\sigma f \in L^2(\mathbb{R}^d)$ *for some* $\sigma \in [0,1]$, $z \in \mathbb{C}$ *with* $\operatorname{Re} z \leq -1$, *and* $\gamma \in \mathbb{R}$ *with* $0 \leq \gamma \leq 1$. *Then,*

$$\begin{aligned}&\|R_0(z)^\gamma \left[|p|^\sigma, a(F)R_0(z)a^*(F) + E(F,F)\right] R_0(z)^\gamma\| \\ &\leq 2 \sup_{p\in\mathbb{R}^d} \int dk \frac{(1+|k|)^{2\sigma}|f(k)|^2}{\omega(k)[|\operatorname{Re} z| + \Omega(p-k) + \omega(k)]^{2\gamma}}\end{aligned}$$

Proof. We start with the equality

$$\begin{aligned}&[|p|^\sigma, a(F)R_0(z)a^*(F) + E(F,F)] \\ &= a([F,|p|^\sigma])R_0(z)a^*(F) + a(F)R_0(z)a^*([|p|^\sigma, F])\end{aligned} \tag{C.41}$$

and the remarks that

$$e^{\pm ikx}|p|^\sigma = |p \mp k|^\sigma e^{\pm ikx}, \tag{C.42}$$

$$e^{ikx}\left(|p|^\sigma - \left|p+\tilde{k}\right|^\sigma\right) = \left(|p-k|^\sigma - \left|p-k+\tilde{k}\right|^\sigma\right)e^{ikx}. \tag{C.43}$$

For $\Phi, \Psi \in \mathscr{H}$, this leads to

$$\begin{aligned}&\langle \Phi, R_0(z)^\gamma \left\{a([F,|p|^\sigma])R_0(z)a^*(F) + a(F)R_0(z)a^*([|p|^\sigma, F])\right\} R_0(z)^\gamma\Psi\rangle \\ &= \Big\langle R_0(\overline{z})^\gamma\Phi, \int dk\ \overline{f(k)}\left(|p|^\sigma e^{ikx} - e^{ikx}|p|^\sigma\right) a_k R_0(z)\int d\tilde{k}\ f(\tilde{k})e^{-i\tilde{k}x}a^*_{\tilde{k}}R_0(z)^\gamma\Psi \\ &\qquad + \int dk\ \overline{f(k)}e^{ikx}a_kR_0(z)\int d\tilde{k}\ f(\tilde{k})\left(|p|^\sigma e^{-i\tilde{k}x} - e^{-i\tilde{k}x}|p|^\sigma\right)a^*_{\tilde{k}}R_0(z)^\gamma\Psi\Big\rangle \\ &= \int dk \int d\tilde{k}\int dp\ \overline{f(k)}f(\tilde{k})\left(|p|^\sigma - \left|p-k+\tilde{k}\right|^\sigma\right) \\ &\qquad \Big\langle \left(R_0(\overline{z})^\gamma\Phi\right)(p), \left(e^{ikx}a_kR_0(z)e^{-i\tilde{k}x}a^*_{\tilde{k}}R_0(z)^\gamma\Psi\right)(p)\Big\rangle.\end{aligned} \tag{C.44}$$

Analogously to the proof of Lemma C.3.2, we get

$$\begin{aligned}&\Big\langle \left(R_0(\overline{z})^\gamma\Phi\right)(p), \left(e^{ikx}a_kR_0(z)e^{-i\tilde{k}x}a^*_{\tilde{k}}R_0(z)^\gamma\Psi\right)(p)\Big\rangle \\ &= \Big\langle [\overline{z} - \Omega(p-k-\tilde{k}) - d\Gamma(\omega) - \omega(k) - \omega(\tilde{k})]^{-1/2}\left(e^{i\tilde{k}x}a_{\tilde{k}}R_0(\overline{z})^\gamma\Phi\right)(p), \\ &\qquad [z - \Omega(p-k-\tilde{k}) - d\Gamma(\omega) - \omega(k) - \omega(\tilde{k})]^{-1/2}\left(e^{ikx}a_kR_0(z)^\gamma\Psi\right)(p)\Big\rangle \\ &\quad + \delta(k-\tilde{k})\Big\langle \left(R_0(\overline{z})^\gamma\Phi\right)(p), [z - \Omega(p-k) - d\Gamma(\omega) - \omega(k)]^{-1}\left(e^{i(k-\tilde{k})x}R_0(z)^\gamma\Psi\right)(p)\Big\rangle.\end{aligned} \tag{C.45}$$

Now, we insert this term in Equation (C.44). Note that because of $|p|^\sigma - \left|p-k+\tilde{k}\right|^\sigma$, the term with the δ-distribution vanishes. Therefore, we obtain

$$\begin{aligned}
&|\langle\Phi, R_0(z)^\gamma \left\{a([F,|p|^\sigma])R_0(z)a^*(F) + a(F)R_0(z)a^*([|p|^\sigma, F])\right\} R_0(z)^\gamma\Psi\rangle| \\
&= \Bigg| \int dk \int d\tilde{k} \int dp\, \overline{f(k)}f(\tilde{k}) \left(|p|^\sigma - \left|p-k+\tilde{k}\right|^\sigma\right) \\
&\quad \Big\langle [\overline{z} - \Omega(p-k-\tilde{k}) - d\Gamma(\omega) - \omega(k) - \omega(\tilde{k})]^{-1/2} \left(e^{i\tilde{k}x}a_{\tilde{k}}R_0(\overline{z})^\gamma\Phi\right)(p), \\
&\qquad [z - \Omega(p-k-\tilde{k}) - d\Gamma(\omega) - \omega(k) - \omega(\tilde{k})]^{-1/2} \left(e^{ikx}a_k R_0(z)^\gamma\Psi\right)(p)\Big\rangle\Bigg|.
\end{aligned} \tag{C.46}$$

This expression can be further estimated, using that for $p, k, \tilde{k} \in \mathbb{R}^d$ and $\sigma \in [0,1]$

$$\left|\, |p|^\sigma - \left|p-k+\tilde{k}\right|^\sigma \right| \le \left|k-\tilde{k}\right|^\sigma \le 2(1+|k|)^\sigma \left(1+\left|\tilde{k}\right|\right)^\sigma, \tag{C.47}$$

to obtain

$$\begin{aligned}
&\int dk \int d\tilde{k} \int dp\, 2(1+|k|)^\sigma |f(k)| \left(1+\left|\tilde{k}\right|\right)^\sigma \left|f(\tilde{k})\right| \\
&\quad \left\|[\overline{z} - \Omega(p-k-\tilde{k}) - d\Gamma(\omega) - \omega(k) - \omega(\tilde{k})]^{-1/2} \left(e^{i\tilde{k}x}a_{\tilde{k}}R_0(\overline{z})^\gamma\Phi\right)(p)\right\| \\
&\quad \left\|[z - \Omega(p-k-\tilde{k}) - d\Gamma(\omega) - \omega(k) - \omega(\tilde{k})]^{-1/2} \left(e^{ikx}a_k R_0(z)^\gamma\Psi\right)(p)\right\|.
\end{aligned} \tag{C.48}$$

Now, the rest of the proof works analogously to the relevant part of the proof of Lemma C.3.2. One only has to set $\alpha = \beta = \gamma$ and to replace $f(k)$ and $g(k)$ by $\sqrt{2}(1+|k|)^\sigma f(k)$, each. We arrive at the upper boundary

$$\left(\|\Phi\|^2 + \|\Psi\|^2\right) \sup_{p\in\mathbb{R}^d} \int dk\, \frac{(1+|k|)^{2\sigma}|f(k)|^2}{\omega(k)[|\mathrm{Re}\, z| + \Omega(p-k) + \omega(k)]^{2\gamma}} \tag{C.49}$$

Taking the supremum over all $\Phi, \Psi \in \mathscr{H}$ with $\|\Phi\| = 1$ and $\|\Psi\| = 1$ completes the proof. $\square$

Remark. The right-hand side of the estimate in Lemma C.3.5 is just given by $\mathcal{B}_{\gamma,z}((1+|k|)^\sigma f)$. This quantity is defined in Equation (4.37).

Lemma C.3.6. *Let $F_x(k) := f(k)e^{-ikx}$ with $f \in L^2(\mathbb{R}^d)$, $\sigma \in [0,1]$, $z \in \mathbb{C}$ with $\mathrm{Re}\, z \le -1$, and $\alpha, \beta \in \mathbb{R}$ with $0 \le \alpha \le \frac{1}{2}$ and $0 \le \beta \le 1$. Then,*

$$\begin{aligned}
&\left\|R_0(z)^\alpha[d\Gamma(\omega)^\sigma, a(F)]R_0(z)^\beta\right\| \\
&\le \left(1+\frac{1}{m^\alpha}\right)\left\{\sup_{p\in\mathbb{R}^d}\int dk \frac{(1+\omega(k))^{2\sigma}|f(k)|^2}{\omega(k)^{1-2\alpha}[|\mathrm{Re}\, z| + \Omega(p-k) + \omega(k)]^{2(\alpha+\beta)-1}}\right\}^{1/2}, \\
&\left\|R_0(z)^\beta[d\Gamma(\omega)^\sigma, a^*(F)]R_0(z)^\alpha\right\| \\
&\le \left(1+\frac{1}{m^\alpha}\right)\left\{\sup_{p\in\mathbb{R}^d}\int dk \frac{(1+\omega(k))^{2\sigma}|f(k)|^2}{\omega(k)^{1-2\alpha}[|\mathrm{Re}\, z| + \Omega(p-k) + \omega(k)]^{2(\alpha+\beta)-1}}\right\}^{1/2}.
\end{aligned}$$

Proof. Note that for suitable Φ

$$
\begin{aligned}
&\left([d\Gamma(\omega)^\sigma, a(F)]\Phi\right)^{(n)}(p, k_1, ..., k_n) \\
&= \sqrt{n+1} \int dk \, \overline{f(k)} \left(\left(\sum_{\ell=1}^{n} \omega(k_\ell) \right)^\sigma - \left(\sum_{\ell=1}^{n} \omega(k_\ell) + \omega(k) \right)^\sigma \right) \\
&\qquad \Phi^{(n+1)}(p-k, k, k_1, ..., k_n)
\end{aligned}
\tag{C.50}
$$

and it will be important that for $\sigma \in [0,1]$

$$
\left| \left(\sum_{\ell=1}^{n} \omega(k_\ell) \right)^\sigma - \left(\sum_{\ell=1}^{n} \omega(k_\ell) + \omega(k) \right)^\sigma \right| \leq \omega(k)^\sigma. \tag{C.51}
$$

Thus, we get for $\Psi \in \mathscr{H}$ in momentum space

$$
\begin{aligned}
&\left\| R_0(z)^\alpha [d\Gamma(\omega)^\sigma, a(F)] R_0(z)^\beta \Psi \right\|^2 \\
&= \sum_{n \geq 0} \int dp \int dk_1 \, ... \int dk_n \, \frac{n+1}{\left| z - \Omega(p) - \sum_{\ell=1}^{n} \omega(k_\ell) \right|^{2\alpha}} \\
&\quad \left| \int dk \overline{f(k)} \left(\left(\sum_{\ell=1}^{n} \omega(k_\ell) \right)^\sigma - \left(\sum_{\ell=1}^{n} \omega(k_\ell) + \omega(k) \right)^\sigma \right) \left(R_0(z)^\beta \Psi \right)^{(n+1)} (p-k, k, k_1, .., k_n) \right|^2 \\
&\leq \sum_{n \geq 0} \int dp \int dk_1 \, ... \int dk_n \, \frac{n+1}{\left| z - \Omega(p) - \sum_{\ell=1}^{n} \omega(k_\ell) \right|^{2\alpha}} \\
&\qquad \left(\int dk \, \frac{\omega(k)^\sigma |f(k)| \left| \Psi^{(n+1)}(p-k, k, k_1, ..., k_n) \right|}{\left| z - \Omega(p-k) - \sum_{\ell=1}^{n} \omega(k_\ell) - \omega(k) \right|^\beta} \right)^2
\end{aligned}
\tag{C.52}
$$

Now, the proof works analogously to the proof of Lemma C.3.1, we easily have to replace $f(k)$ by $\omega(k)^\sigma f(k)$. We arrive at

$$
\begin{aligned}
&\left\| R_0(z)^\alpha [d\Gamma(\omega)^\sigma, a(F)] R_0(z)^\beta \Psi \right\|^2 \\
&\leq \left(1 + \frac{1}{m^{2\alpha}} \right) \sup_{p \in \mathbb{R}^d} \int dk \, \frac{\omega(k)^{2\sigma} |f(k)|^2}{\omega(k)^{1-2\alpha} [|\mathrm{Re}\, z| + \Omega(p-k) + \omega(k)]^{2(\alpha+\beta)-1}} \, \|\Psi\|^2.
\end{aligned}
\tag{C.53}
$$

In the calculations made in Chapter 4, it is necessary that this upper boundary increases if σ increases. For that, we again estimate $\omega(k)^{2\sigma} \leq (1+\omega(k))^{2\sigma}$ which completes the proof of the first estimate of the lemma. Since $\omega(k) > 0$, this last estimate does not change the regularity behavior of the expression.

The second estimate of the lemma follows by adjoining the first one. □

Remark. The right-hand sides of the estimates in Lemma C.3.6 are just given by $\mathcal{A}_{\alpha,\beta,z}((1+\omega)^\sigma f)$. This quantity is defined in Equation (4.36).

Lemma C.3.7. *Let $F_x(k) := f(k)e^{-ikx}$ with $f \in L^2(\mathbb{R}^d)$, $\sigma \in [0,1]$, $z \in \mathbb{C}$ with $\operatorname{Re} z \leq -1$, and $\gamma \in \mathbb{R}$ with $0 \leq \gamma \leq 1$. Then,*

$$\begin{aligned}
&\|R_0(z)^\gamma \left[d\Gamma(\omega)^\sigma, a(F)R_0(z)a^*(F) + E(F,F)\right] R_0(z)^\gamma\| \\
&\leq 2 \sup_{p\in\mathbb{R}^d} \int dk \frac{(1+\omega(k))^{2\sigma}|f(k)|^2}{\omega(k)[|\operatorname{Re} z| + \Omega(p-k) + \omega(k)]^{2\gamma}}
\end{aligned}$$

Proof. We start with the equality

$$\begin{aligned}
&[d\Gamma(\omega)^\sigma, a(F)R_0(z)a^*(F) + E(F,F)] \\
&= [d\Gamma(\omega)^\sigma, a(F)]R_0(z)a^*(F) + a(F)R_0(z)[d\Gamma(\omega)^\sigma, a^*(F)]
\end{aligned} \tag{C.54}$$

and consider for $\Phi, \Psi \in \mathscr{H}$ the inner product

$$\begin{aligned}
&\langle \Phi, R_0(z)^\gamma \left\{[d\Gamma(\omega)^\sigma, a(F)]R_0(z)a^*(F) + a(F)R_0(z)[d\Gamma(\omega)^\sigma, a^*(F)]\right\} R_0(z)^\gamma \Psi\rangle \\
&= \Big\langle R_0(\overline{z})^\gamma \Phi, \int dk\ \overline{f(k)}e^{ikx} \left(d\Gamma(\omega)^\sigma a_k - a_k d\Gamma(\omega)^\sigma\right) R_0(z) \int d\tilde{k}\ f(\tilde{k})e^{-i\tilde{k}x} a^*_{\tilde{k}} R_0(z)^\gamma \Psi \\
&\qquad + \int dk\ \overline{f(k)}e^{ikx} a_k R_0(z) \int d\tilde{k}\ f(\tilde{k})e^{-i\tilde{k}x} \left(d\Gamma(\omega)^\sigma a^*_{\tilde{k}} - a^*_{\tilde{k}} d\Gamma(\omega)^\sigma\right) R_0(z)^\gamma \Psi \Big\rangle \\
&= \int dk \int d\tilde{k}\ \overline{f(k)} f(\tilde{k}) \\
&\quad \Big\langle R_0(\overline{z})^\gamma \Phi, \left(d\Gamma(\omega)^\sigma - \left(d\Gamma(\omega) + \omega(k) - \omega(\tilde{k})\right)^\sigma\right) e^{ikx} a_k R_0(z) e^{-i\tilde{k}x} a^*_{\tilde{k}} R_0(z)^\gamma \Psi \Big\rangle,
\end{aligned} \tag{C.55}$$

where we used the pull-through formulas in the last step. Using these again and the commutation relation $[a_k, a^*_{\tilde{k}}] = \delta(k - \tilde{k})$, we obtain

$$\begin{aligned}
&e^{ikx} a_k R_0(z) e^{-i\tilde{k}x} a^*_{\tilde{k}} \\
&= [z - \Omega(p-k) - d\Gamma(\omega) - \omega(k)]^{-1} e^{i(k-\tilde{k})x} \left(a^*_{\tilde{k}} a_k + \delta(k-\tilde{k})\right).
\end{aligned} \tag{C.56}$$

Inserting this in Equation (C.55), we see, that because of $d\Gamma(\omega)^\sigma - (d\Gamma(\omega) + \omega(k) - \omega(\tilde{k}))^\sigma$, the term with the δ-distribution vanishes. Equation (C.55) then yields

$$\begin{aligned}
&\int dk \int d\tilde{k}\ \overline{f(k)} f(\tilde{k}) \Big\langle [\overline{z} - \Omega(p-k-\tilde{k}) - d\Gamma(\omega) - \omega(k) - \omega(\tilde{k})]^{-1/2} e^{i\tilde{k}x} a_{\tilde{k}} R_0(\overline{z})^\gamma \Phi, \\
&\qquad [z - \Omega(p-k-\tilde{k}) - d\Gamma(\omega) - \omega(k) - \omega(\tilde{k})]^{-1/2} \\
&\qquad \left(d\Gamma(\omega)^\sigma - \left(d\Gamma(\omega) + \omega(k) - \omega(\tilde{k})\right)^\sigma\right) e^{ikx} a_k R_0(z)^\gamma \Psi \Big\rangle.
\end{aligned} \tag{C.57}$$

Now, note that

$$\left|d\Gamma(\omega)^\sigma - \left(d\Gamma(\omega) + \omega(k) - \omega(\tilde{k})\right)^\sigma\right| \leq \left|\omega(k) - \omega(\tilde{k})\right|^\sigma$$
$$\leq 2(1+\omega(k))^\sigma(1+\omega(\tilde{k}))^\sigma, \tag{C.58}$$

such that we get for the absolute value of the inner product from the beginning

$$|\langle \Phi, R_0(z)^\gamma \{[d\Gamma(\omega)^\sigma, a(F)]R_0(z)a^*(F) + a(F)R_0(z)[d\Gamma(\omega)^\sigma, a^*(F)]\} R_0(z)^\gamma \Psi\rangle|$$
$$\leq 2\int dk \int d\tilde{k}\ (1+\omega(k))^\sigma |f(k)| (1+\omega(\tilde{k}))^\sigma \left|f(\tilde{k})\right| \tag{C.59}$$
$$\left\|[1+d\Gamma(\omega)+\omega(\tilde{k})]^{-1/2} e^{i\tilde{k}x} a_{\tilde{k}} R_0(\overline{z})^\gamma \Phi\right\| \left\|[1+d\Gamma(\omega)+\omega(k)]^{-1/2} e^{ikx} a_k R_0(z)^\gamma \Psi\right\|$$

Now, the rest of the proof works again analogously to the the relevant part of the proof of Lemma C.3.2. One only has to set $\alpha = \beta = \gamma$ and to replace $f(k)$ and $g(k)$ by $\sqrt{2}(1+\omega(k))^\sigma f(k)$, each. We arrive at the upper boundary

$$\left(\|\Phi\|^2 + \|\Psi\|^2\right) \sup_{p\in\mathbb{R}^d} \int dk\ \frac{(1+\omega(k))^{2\sigma}|f(k)|^2}{\omega(k)[|\mathrm{Re}\, z| + \Omega(p-k) + \omega(k)]^{2\gamma}} \tag{C.60}$$

Taking the supremum over all $\Phi, \Psi \in \mathscr{H}$ with $\|\Phi\| = 1$ and $\|\Psi\| = 1$ completes the proof. $\square$

Remark. The right-hand side of the estimate in Lemma C.3.7 is just given by $\mathcal{B}_{\gamma,z}((1+\omega)^\sigma f)$. This quantity is defined in Equation (4.37).

The next Lemma is needed for the proof of Lemma C.3.1.

Lemma C.3.8. *Let $n \in \mathbb{N}$, $0 \leq \tau \leq 1$, $x_j \geq 0$ for all $j = 1, ..., n$. Then*

$$\sum_{j=1}^n x_j^\tau \leq n^{1-\tau} \left(\sum_{j=1}^n x_j\right)^\tau .$$

Proof. Use the concavity of the function $f(x) = x^\tau$ for $0 \leq \tau \leq 1$ and Jensen's inequality. $\square$

Bibliography

[1] Edward Nelson. Interaction of nonrelativistic particles with a quantized scalar field. *J. Mathematical Phys.* 5, 1190–1197 (1964).

[2] Zied Ammari. Asymptotic completeness for a renormalized nonrelativistic Hamiltonian in quantum field theory: the Nelson model. *Math. Phys. Anal. Geom.* 3, 217–285 (2000).

[3] Eugene P. Gross. Particle-like solutions in field theory. *Annals of Physics* 19, 219–233 (1962).

[4] Oliver Matte and Jacob Schach-Møller. Feynman-Kac Formulas for the Ultraviolet Renormalized Nelson Model. *https://arxiv.org/pdf/1701.02600.pdf (preprint)* (2016).

[5] Herbert Fröhlich, H. Pelzer, and Sigurd Zienau. XX. Properties of slow electrons in polar materials. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 41, 221–242 (1950).

[6] Otfried Madelung. *Introduction to solid state theory: [revised and partly rewritten translation of Festkörpertheorie].* Springer, Berlin; Heidelberg [u.a.], study ed., 3. print. edition (1996).

[7] Jürg Fröhlich. Existence of Dressed One Electron States in a Class of Persistent Models. *Fortschritte der Physik* 22, 159–198 (1974).

[8] Hartmut Löwen. Spectral properties of an optical polaron in a magnetic field. *J. Math. Phys.* 29, 1498–1504 (1988).

[9] Elliott H. Lieb and Lawrence E. Thomas. Exact ground state energy of the strong-coupling polaron. *Comm. Math. Phys.* 183, 511–519 (1997).

[10] Elliott H. Lieb and Kazuo Yamazaki. Ground-State Energy and Effective Mass of the Polaron. *Phys. Rev.* 111, 728–733 (1958).

[11] Rupert L. Frank, Elliott H. Lieb, Robert Seiringer, and Lawrence E. Thomas. Bipolaron and N-Polaron Binding Energies. *Phys. Rev. Lett.* 104, 210402 (2010).

[12] Marcel Griesemer and Jacob Schach Møller. Bounds on the Minimal Energy of Translation Invariant N-Polaron Systems. *Communications in Mathematical Physics* 297, 283–297 (2010).

[13] Rupert L. Frank and Benjamin Schlein. Dynamics of a strongly coupled polaron. *Lett. Math. Phys.* 104, 911–929 (2014).

[14] Rupert Frank and Zhou Gang. Derivation of an Effective Evolution Equation for a Strongly Coupled Polaron. *https://arxiv.org/pdf/1505.03059v1.pdf (preprint)* (2015).

[15] Marcel Griesemer. On the dynamics of polarons in the strong-coupling limit. *https://arxiv.org/pdf/1612.00395.pdf (preprint)* (2016).

[16] Marcel Griesemer, Fabian Hantsch, and David Wellig. On the magnetic Pekar functional and the existence of bipolarons. *Rev. Math. Phys.* 24, 1250014, 13 (2012).

[17] Marcel Griesemer and David Wellig. The strong-coupling polaron in static electric and magnetic fields. *Journal of Physics A: Mathematical and Theoretical* 46, 425202 (2013).

[18] Ioannis Anapolitanos and Marcel Griesemer. Multipolarons in a Constant Magnetic Field. *Annales Henri Poincaré* 15, 1037–1059 (2014).

[19] Jürg Fröhlich. On the infrared problem in a model of scalar electrons and massless, scalar bosons. *Ann. Inst. H. Poincaré Sect. A (N.S.)* 19, 1–103 (1973).

[20] Alan D. Sloan. The polaron without cutoffs in two space dimensions. *J. Mathematical Phys.* 15, 190–201 (1974).

[21] Klaus Hepp. *Théorie de la renormalisation.* Cours donné à l'École Polytechnique, Paris. Lecture Notes in Physics, Vol. 2. Springer-Verlag, Berlin-New York (1969).

[22] Jean-Pierre Eckmann. A model with persistent vacuum. *Comm. Math. Phys.* 18, 247–264 (1970).

[23] Marcel Griesemer and Andreas Wünsch. Self-adjointness and domain of the Fröhlich Hamiltonian. *J. Math. Phys.* 57, 021902, 15 (2016).

[24] Michael Reed and Barry Simon. *Methods of modern mathematical physics. I. Functional analysis.* Academic Press, New York-London (1972).

[25] Michael Reed and Barry Simon. *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness.* Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London (1975).

[26] Konrad Schmüdgen. *Unbounded self-adjoint operators on Hilbert space*, volume 265 of *Graduate Texts in Mathematics*. Springer, Dordrecht (2012).

[27] Tosio Kato. *Perturbation theory for linear operators*. Springer-Verlag, Berlin-New York, second edition (1976). Grundlehren der Mathematischen Wissenschaften, Band 132.

[28] Ola Bratteli and Derek W. Robinson. *Operator algebras and quantum statistical mechanics. 2*. Texts and Monographs in Physics. Springer-Verlag, Berlin, second edition (1997). Equilibrium states. Models in quantum statistical mechanics.

[29] Nelson Dunford and Jacob T. Schwartz. *Linear operators. Part I*. Wiley Classics Library. John Wiley & Sons, Inc., New York (1988). General theory, With the assistance of William G. Bade and Robert G. Bartle, Reprint of the 1958 original, A Wiley-Interscience Publication.

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Ehrenwörtliche Erklärung

Ich erkläre, dass ich diese Dissertation, abgesehen von den ausdrücklich bezeichneten Hilfsmitteln, selbständig verfasst habe.

Stuttgart, den 15. November 2017 *Andreas Wünsch*