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Mathematical Justification of a Baer–Nunziato Model for a Compressible Viscous Fluid with Phase Transition

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Abstract. In this work, we justify a Baer–Nunziato system including appropriate closure terms as the macroscopic description of a compressible viscous fluid that can occur in a liquid or a vapor phase in the isothermal framework. As a mathematical model for the two-phase fluid on the detailed scale we chose a non-local version of the Navier–Stokes–Korteweg equations in the one-dimensional and periodic setting. Our justification relies on anticipating the macroscopic description of the two-phase fluid as the limit system for a sequence of solutions with highly oscillating initial densities. Interpreting the density as a parametrized measure, we extract a limit system consisting of a kinetic equation for the parametrized measure and a momentum equation for the velocity. Under the assumption that the initial density distributions converge in the limit to a convex combination of Dirac-measures, we show by a uniqueness result that the parametrized measure also has to be a convex combination of Dirac-measures and, that the limit system reduces to the Baer–Nunziato system. This work extends existing results concerning the justification of Baer–Nunziato models as the macroscopic description of multi-fluid models in the sense, that we allow for phase transition effects on the detailed scale. This work also includes a new global-in-time well-posedness result for the Cauchy problem of the non-local Navier–Stokes–Korteweg equations.

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Keywords. Compressible two-phase flow, Navier–Stokes–Korteweg equations, homogenization, parametrized measure.

1. Introduction

We consider a homogeneous compressible viscous fluid that can occur in a liquid and a vapor phase. A widely accepted mathematical description of such a two-phase fluid is given by the Navier–Stokes–Korteweg (NSK) equations ([1]). These equations model the two-phase fluid with a diffuse interface, i.e. the interface between the liquid and the vapor is assumed to have small but positive Lebesgue-measure and the density varies rapidly but smoothly over the interface. Let us assume that we are given some positive time $T > 0$. In the isothermal and periodic framework, the NSK model describes then the dynamics of the two-phase fluid via the fluid's density $\rho\colon [0,T) \times \mathbb{T} \to \mathbb{R}_{>0}$ and the fluid's velocity $u\colon [0,T) \times \mathbb{T} \to \mathbb{R}$ that obey the system of equations

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 & \text{in } (0,T) \times \mathbb{T}, \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x P(\rho) - \mu \partial_{xx} u + \kappa \rho \partial_{xxx} \rho = 0 & \text{in } (0,T) \times \mathbb{T}, \end{cases} \tag{1}$$

with initial conditions

$$\rho(0,\cdot) = \rho_0, \quad u(0,\cdot) = u_0 \quad \text{in } \mathbb{T}. \tag{2}$$

Here, $\mu > 0$ denotes the constant viscosity coefficient and $\kappa > 0$ denotes the constant capillarity coefficient. In order to account for phase transition effects, the equation of state $P\colon [0,\infty) \to [0,\infty)$ is assumed to be of Van-der-Waals type. More precisely, we shall assume that there exist some constants $0 < B_1 < B_2 < \infty$, such that P is monotonically increasing on $[0,B_1] \cup [B_2,\infty)$ and monotonically decreasing on $[B_1,B_2]$. Accordingly, we call then the fluid's state liquid (spinodal, vapor), if the fluid's density satisfies $\rho \in [0,B_1)$

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$([B_1, B_2), [B_2, \infty))$. An illustration of a pressure function of Van-der-Waals type is given in Figure 1 in Section 2.

To find effective equations for the NSK model (1), we start from a sequence of initial data $(\rho_n^0, u_n^0)_{n\in\mathbb{N}}$, where $n \in \mathbb{N}$ should display the number of phase transitions that we have initially. Due to the high number of phase transitions, we expect the initial density sequence $(\rho_n^0)_{n\in\mathbb{N}}$ to be highly oscillating between the vapor density and the liquid density. As a simplification of the problem, we assume that such oscillations do not occur for the velocity sequence. After constructing an appropriate corresponding sequence of solutions $(\rho_n, u_n)_{n\in\mathbb{N}}$, it is then reasonable to assume that the macroscopic equations are found in the limit $n \to \infty$ (i.e. in the limit where the number of phase transitions tends to infinity). Thus, the derivation of effective equations reduces to study the propagation of initial density oscillations for system (1). One method to analyze the propagation of initial density oscillations relies on interpreting the density as a parametrized measure ([2–4]). However, system (1) seems hardly accessible for such an investigation due to the capillarity term $\kappa\rho\partial_{xxx}\rho$. To overcome this remedy, we choose a non-local approximation of (1), that was proposed in [5]. In this system, that we call the non-local NSK system from now on, the capillarity term $\kappa\rho\partial_{xxx}\rho$ is substituted by a term of lower order by introducing an additional unknown that satisfies an elliptic equation. More precisely, in the non-local NSK model, the dynamics of the two-phase fluid are described by the fluid's density $\rho\colon [0,T)\times\mathbb{T} \to \mathbb{R}_{\geq 0}$, the fluid's velocity $u\colon [0,T)\times\mathbb{T}\to\mathbb{R}$ and the order parameter $c\colon [0,T)\times\mathbb{T}\to\mathbb{R}$ that satisfy the system of equations

$$\begin{cases} \partial_t\rho + \partial_x(\rho u) = 0 & \text{in } (0,T)\times\mathbb{T}, \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x P(\rho) - \mu\partial_{xx}u - \gamma\rho\partial_x(c-\rho) = 0 & \text{in } (0,T)\times\mathbb{T}, \\ -\kappa\partial_{xx}c + \gamma(c-\rho) = 0 & \text{in } (0,T)\times\mathbb{T}, \end{cases} \tag{3}$$

with initial conditions (2). Here, the quantities μ, κ, P are defined as before and $\gamma > 0$ denotes the constant coupling coefficient. Formally, we notice that for $\gamma \to \infty$ we recover the compressible NSK system (1). At this point we refer to [6] for a numerical solution of (3) and to [7] for a rigorous mathematical result concerning the convergence of (3) to (1) via a relative entropy approach.

The momentum equation $(3)_2$ can be rewritten as

$$\partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x P_\gamma(\rho) - \mu\partial_{xx}u - \gamma\rho\partial_x c = 0, \tag{4}$$

with an artificial pressure function

$$P_\gamma(r) := P(r) + \frac{\gamma}{2}r^2 \tag{5}$$

being monotone for a pressure function P of Van-der-Waals type provided γ is chosen large enough. In this paper, we justify the following system as a macroscopic description for a compressible liquid-vapor flow that is modeled with the non-local NSK equations on the detailed scale:

$$\begin{cases} \partial_t\alpha_+ + u\partial_x\alpha_+ = \dfrac{\alpha_+\alpha_-}{\mu}\Big(P(\rho_+) - P(\rho_-) + \dfrac{\gamma}{2}(\rho_+^2 - \rho_-^2)\Big) & \text{in } (0,T)\times\mathbb{T}, \\ \partial_t\alpha_- + u\partial_x\alpha_- = \dfrac{\alpha_-\alpha_+}{\mu}\Big(P(\rho_-) - P(\rho_+) + \dfrac{\gamma}{2}(\rho_-^2 - \rho_+^2)\Big) & \text{in } (0,T)\times\mathbb{T}, \\ \partial_t\rho_+ + \partial_x(\rho_+ u) = \dfrac{\rho_+\alpha_-}{\mu}\Big(P(\rho_-) - P(\rho_+) + \dfrac{\gamma}{2}(\rho_-^2 - \rho_+^2)\Big) & \text{in } (0,T)\times\mathbb{T}, \\ \partial_t\rho_- + \partial_x(\rho_- u) = \dfrac{\rho_-\alpha_+}{\mu}\Big(P(\rho_+) - P(\rho_-) + \dfrac{\gamma}{2}(\rho_+^2 - \rho_-^2)\Big) & \text{in } (0,T)\times\mathbb{T}, \\ \partial_t(\rho u) + \partial_x(\rho u^2) = \mu\partial_{xx}u - \partial_x\overline{P} + \gamma\rho\partial_x c & \text{in } (0,T)\times\mathbb{T}, \\ -\kappa\partial_{xx}c + \gamma c = \gamma\rho & \text{in } (0,T)\times\mathbb{T}, \end{cases} \tag{6}$$

where

$$\rho = \alpha_+\rho_+ + \alpha_-\rho_-, \quad \overline{P} = \alpha_+P(\rho_+) + \alpha_-P(\rho_-) + \frac{\gamma}{2}(\alpha_+\rho_+^2 + \alpha_-\rho_-^2).$$

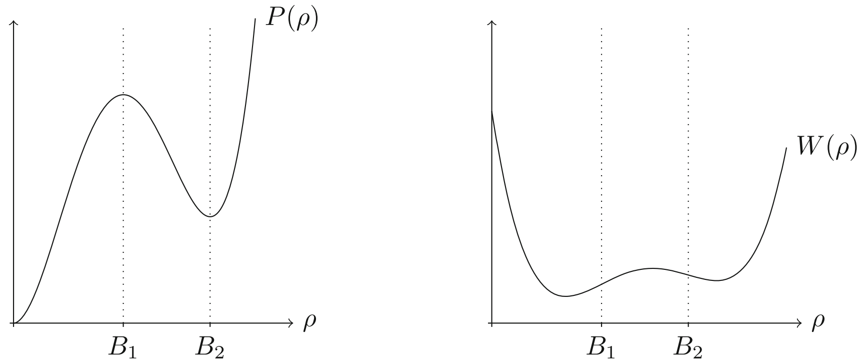

Fig. 1. Left: Example for a pressure function $P(\rho)$ of Van-der-Waals type. Right: A corresponding pressure potential $W(\rho)$.

Here, $\alpha_+, \alpha_-, \rho_+, \rho_- \colon [0,T)\times\mathbb{T} \to \mathbb{R}$ denote the volume fraction of the liquid phase, the volume fraction of the vapor phase, the partial density of the liquid phase, the partial density of the vapor phase, respectively. The effective model (6) falls into the class of one-velocity Baer–Nunziato (BN) multi-fluid models. For a detailed background on this class of multi-fluid models see e.g. [8–11]. For a discussion and interpretation of the macroscopic model (6), we refer to the end of Section 2. Our justification follows the homogenization methodology in [12], where the authors justify a one-velocity multi-fluid BN model for the compressible Navier-Stokes equations with density-dependent viscosity in the one-dimensional periodic framework. There, the authors analyzed the propagation of initial density oscillations by interpreting the density as a parameterized measure and obtained effective equations in the form of a kinetic equation for the parametrized measure and a momentum equation for the velocity. After characterizing the parameterized measure as a convex combination of Dirac-measures these equations then reduce to a one-velocity multi-fluid BN system. Hence, the key point to justify such a multi-fluid model is to prove the propagation of the special structure for the parametrized measure in time. This methodology was also used in [4,13,14] to justify rigorously a one-velocity BN model for the compressible isentropic Navier–Stokes equations with constant viscosity in the 3-D framework. See also [15] for the derivation of the kinetic equation in terms of a probability density function without the characterization of the parametrized measure. At this point, we also refer to [16], where the authors extended the methodology for a two-component flow with two different pressure laws for each phase and to [17] for a semi-discrete approach and a numerical investigation of the problem setting in [16]. However, all these results do not incorporate phase transition phenomena in the description of the fluid on the detailed scale. Thus, this work can be seen as a generalization of the results in [12] to a setting where phase transition effects are included on the detailed scale. Mathematically, this generalization requires to deal with a non-monotone pressure law and an additional term in the momentum equation. The difficulties resulting from the additional term in the momentum equation are twofold. On the one hand, the a priori estimates in [12] rely on the momentum equation, so that it is not clear, whether these carry over to the non-local NSK system. On the other hand, we have to prove convergence for the additional term in the homogenization process. We overcome the non-monotonicity of the pressure function by rewriting the momentum equation in terms of the artificial pressure function P_γ (see (4)) that we assume to be monotone (c.f. Section 2). As a novelty, we verify a priori estimates corresponding to them in [12] for the non-local NSK system. In particular an entropy inequality due to Bresch and Desjardin (c.f. Section 3) is derived. Then, we solve the convergence issue by using a compensated compactness lemma from [12]. As we shall see, this lemma can be applied here, since the a priori estimates for the order parameter are strong enough. Our main tool to control the order parameter is an elliptic regularity estimate.

This paper is organized as follows. In Section 2, we precisely formulate the assumptions that we impose on the pressure function P and the coupling parameter γ, and state our main results Theorem 2.4 and

Theorem 2.5. These are given by a global-in-time well-posedness result (Theorem 2.4) for the Cauchy problem (3), (2) and a homogenization result (Theorem 2.5). In Section 3, we give a full proof of the global-in-time well-posedness result. By providing the proof of this result, we also recover crucial a priori bounds that are used in Section 4. In Section 4, we collect refined a priori estimates concerning the effective viscous flux that we need for the homogenization procedure. In Section 5, we perform the homogenization procedure following the methodology in [12] and conclude the proof of the homogenization result. In Section 6 we give some conclusions. In the Appendix, we provide some technical results on the existence of solutions to a specific BN system and a uniqueness result for measure-valued solutions to a kinetic equation.

Notations

We denote the one-dimensional torus of period 1 by $\mathbb{T}$, i.e. $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. For a domain D and $k \in \mathbb{N} \cup \{\infty\}$ we denote by $C^k(D)$ the space of k-times continuously differentiable functions and by $C^k_c(D)$ the set all functions in $C^k(D)$ having compact support in D. Also, we write in the context of distributions $\mathcal{D}(D) := C^\infty_c(D)$ for the test function space and $\mathcal{D}'(D)$ for the space of distributions defined on $\mathcal{D}(D)$. In particular, $C^k(\mathbb{T})$ denotes the space of all k-times continuously differentiable functions on $\mathbb{T}$ that can be identified with the space of all k-times continuously differentiable 1-periodic functions defined on $\mathbb{R}$. We denote for a bounded function f defined on D its supremum-norm on D as $||f||_{L^\infty(D)}$ and the closure of $C^\infty_c(D)$ under the supremum-norm as $C^0_0(D)$. Since $\mathbb{T}$ is compact, we have $C^k_c(\mathbb{T}) = C^k(\mathbb{T})$. For $p \in [1, \infty]$, we denote the Lebesgue space on D as $L^p(D)$ and the $L^p(D)$-norm as $|| \cdot ||_{L^p(D)}$. Thereby, we denote by $W^{k,p}(D)$ the Sobolev space of order k on D and, if $p = 2$, we shortly write $H^k(D)$. We denote the norm on $W^{k,p}(D)$ as $|| \cdot ||_{W^{k,p}(D)}$. Specifically for $D = \mathbb{T}$, we have

$$||f||_{L^p(\mathbb{T})} = \left(\int_{\mathbb{T}} |f(x)|^p \,\mathrm{d}x \right)^{\frac{1}{p}} = \left(\int_0^1 |f(x)|^p \,\mathrm{d}x \right)^{\frac{1}{p}} \quad \text{for } f \in L^p(\mathbb{T}),\ p \in [1, \infty),$$

and we define the mean operator on $L^1(\mathbb{T})$ as

$$\mathbb{E}\colon L^1(\mathbb{T}) \to \mathbb{R}, \quad f \mapsto \mathbb{E}(f) := \int_{\mathbb{T}} f(x) \,\mathrm{d}x.$$

Then, we denote the k-th Sobolev space of mean free functions on $\mathbb{T}$ as

$$\dot{H}^k(\mathbb{T}) := \{f \in H^k(\mathbb{T}) \mid \mathbb{E}(f) = 0\} \subseteq H^k(\mathbb{T}),$$

and, for $k \geq 1$, we denote its dual space as

$$\dot{H}^{-k}(\mathbb{T}) := (\dot{H}^k(\mathbb{T}))^\star.$$

It is easy to see that there exists a mean free primitive operator on $\dot{H}^k(\mathbb{T})$ for $k \geq 0$ and we denote this operator as

$$\partial_x^{-1}\colon \dot{H}^k(\mathbb{T}) \to \dot{H}^{k+1}(\mathbb{T}), \quad f \mapsto \partial_x^{-1} f,$$

where $\partial_x \partial_x^{-1} f = f$. This operator straightforwardly extends to negative Sobolev spaces, i.e. to exponents $k < 0$, via

$$\langle \partial_x^{-1} f, \phi \rangle := \langle f, \partial_x^{-1} \phi \rangle$$

for any $f \in \dot{H}^k(\mathbb{T})$ and any $\phi \in \dot{H}^{k+1}(\mathbb{T})$. For some Banach space B and some time $T > 0$, we denote Bochner space of time dependent functions with values in B as $L^p(0, T; B)$ and the Bochner norm on $L^p(0, T; B)$ as $|| \cdot ||_{L^p(0,T;B)}$. Also, for some interval $I \subseteq \mathbb{R}$, we denote by $C(I; B)$ the space of continuous functions from I into the Banach space B. The space of weakly continuous functions from I into B, i.e. the space of all functions from I into B that are continuous with respect to the weak topology on B is denoted by $C_{\mathrm{w}}(I;B)$. Note that $L \in C_{\mathrm{w}}(I;B)$ if and only if for any $\phi \in B^\star$, where $B^\star$ denotes the dual space of B, the map

$$I \ni t \mapsto \langle \phi, L_t \rangle_{B^\star,B} \in \mathbb{R}$$

is a continuous function. Finally, for a locally compact Hausdorff space X, we denote the space of signed finite Radon measures on X as $\mathcal{M}(X)$. This space can be naturally identified with the dual space of $C_0^0(X)$. For some $\mu \in \mathcal{M}(X)$, we denote its total variation norm as

$$||\mu||_{\mathcal{M}(X)} := \sup_{\phi \in C_0^0(X),\, |\phi| \le 1} \langle \mu, \phi \rangle.$$

The space of positive finite Radon measures on X is denoted by $\mathcal{M}^+(X)$, the space of compactly supported finite Radon measures on X is defined by $\mathcal{M}_c(X)$ and the space of positive compactly supported finite Radon measures on X is denoted by $\mathcal{M}_c^+(X)$.

2. The Main Result

The first result of this paper is the global-in-time existence of strong solutions for the Cauchy problem (3), (2). This result requires the artificial pressure function P_γ (see (5)) to be monotone (c.f. Section 3). This will impose a certain relation between the pressure function P and the coupling parameter γ. In the following definition, we make precise what kind of assumptions on P and γ we need in this paper. See [18] for a corresponding definition.

Definition 2.1. (Admissible Pressure Function) Let $P\colon [0,\infty) \to [0,\infty)$ and $\gamma > 0$ satisfy

1. $P \in C^1([0,\infty))$, $P(0) = 0$,
2. there exist two constants $\beta \in [2,\infty)$ and $a \in (0,\infty)$, such that $\lim_{r\to\infty} \frac{P'(r)}{r^{\beta-1}} = a > 0$,
3. the artificial pressure function P_γ defined in (5) is monotonically increasing on $[0,\infty)$.

Then we call the pair (P,γ) *admissible.*

Remark 2.2. With Definition 2.1 we describe various types of monotone and non-monotone pressure functions. For instance, any isentropic pressure law $P(r) := r^\beta$, where $\beta \in [2,\infty)$, satisfies that (P,γ) is admissible for any $\gamma \in (0,\infty)$. Moreover, any pressure function of Van-der-Waals type $P_{\mathrm{VdW}} \in C^1([0,\infty))$ (cf. Section 1) satisfying $P(0) = 0$ and the growth condition 2 gives rise to some $\gamma_0 \in (0,\infty)$, such that $(P_{\mathrm{VdW}},\gamma)$ is admissible for any $\gamma \in [\gamma_0,\infty)$. This is due to the fact that the non-monotone region of P_{VdW} is a compact subset of $(0,\infty)$ (cf. Figure 1). In particular, we account in the subsequent theory for the physically relevant case, when the non-local NSK equations are used as a phase transition model.

Let us assume that we are given some admissible pair (P,γ). Then we associate to $P\colon [0,\infty) \to [0,\infty)$ some pressure potential $W\colon [0,\infty) \to [0,\infty)$ through the relation

$$P'(r) = W''(r)r. \tag{7}$$

We notice that monotonically increasing/decreasing regions of P correspond to convex/concave regions of W.

Remark 2.3. Condition 2 in Definition 2.1 implies that there exists some positive constant C_ρ, such that

$$r^2 \le C_\rho + C_\rho W(r) \quad \forall\, r \in [0,\infty). \tag{8}$$

For the rest of this paper, we assume that the capillarity and viscosity coefficients are fixed positive constants $\kappa, \mu > 0$. Moreover, let us assume that the coupling parameter $\gamma > 0$, the pressure function P, and the initial conditions are given such that

$$(P,\gamma) \quad \text{is admissible}, \quad \rho_0, u_0 \in H^1(\mathbb{T}), \quad M_0^{-1} \le \rho_0(x) \le M_0 \quad \forall\, x \in \mathbb{T}, \tag{9}$$

for some $M_0 > 0$. Under these assumptions, our first result states that the non-local NSK equations admit a unique global-in-time strong solution and reads as follows:

Theorem 2.4. *Assume that the assumptions* (9) *hold true. Then there exist unique functions*

$$\rho \in C([0,\infty); H^1(\mathbb{T})), \quad \rho > 0 \quad \text{on } [0,\infty)\times\mathbb{T}, \quad \partial_t\rho \in C([0,\infty); L^2(\mathbb{T})),$$
$$u \in C([0,\infty); H^1(\mathbb{T})) \cap L^2_{\mathrm{loc}}(0,\infty; H^2(\mathbb{T})), \qquad \partial_t u \in L^2_{\mathrm{loc}}(0,\infty; L^2(\mathbb{T})),$$
$$c \in C([0,\infty); H^3(\mathbb{T})),$$

that satisfy (3) *a.e. on* $(0,\infty)\times\mathbb{T}$ *and the initial conditions* (2) *a.e. on* $\mathbb{T}$.

The global-in-time result Theorem 2.4 is then used to formulate our second result, which is the main result of this paper. Before we state this result, let us recall how a $L^\infty(\mathbb{T})$-function gives rise to a positive finite Radon measure on $\mathbb{T}\times\mathbb{R}$. Let $f \in L^\infty(\mathbb{T})$. Then we can define

$$\Theta\colon C^0_0(\mathbb{T}\times\mathbb{R}) \to \mathbb{R}, \quad b \mapsto \langle\Theta, b\rangle := \int_{\mathbb{T}} b(x, f(x))\,\mathrm{d}x.$$

It is an easy matter to check that this defines a positive linear functional on $C^0_0(\mathbb{T}\times\mathbb{R})$, so that indeed $\Theta \in \mathcal{M}^+(\mathbb{T}\times\mathbb{R})$. If we assume furthermore that f is bounded from above and below, i.e.

$$C_1 \le f(x) \le C_2 \quad \text{for a.e. } x \in \mathbb{T},$$

where $C_1 < C_2$ are two constants, then we obtain that Θ has compact support, more precisely

$$\mathrm{spt}\,(\Theta) \subseteq \mathbb{T}\times[C_1, C_2].$$

This implies $\Theta \in \mathcal{M}^+_c(\mathbb{T}\times\mathbb{R})$. We will use these notions in the following main result, which justifies a one-velocity BN system as a macroscopic description for a compressible liquid-vapor flow that is described on the detailed scale by the non-local NSK system (3).

Theorem 2.5. *(Main Result) Let* $\gamma, M_0 > 0$ *and assume that* (P,γ) *is admissible. Suppose that* $\rho^0_n, u^0_n \in H^1(\mathbb{T})$ *satisfy*

$$M_0^{-1} \le \rho^0_n(x) \le M_0 \quad \forall\, x \in \mathbb{T}, \quad \forall\, n \in \mathbb{N}, \qquad \sup_{n\in\mathbb{N}}\big\{\|u^0_n\|_{H^1(\mathbb{T})}\big\} < \infty,$$

for some positive constant M_0 *and, that* $u^0_n \rightharpoonup u_0$ *weakly in* $H^1(\mathbb{T})$*. For* $n \in \mathbb{N}$*, let* (ρ_n, u_n, c_n) *denote the global-in-time strong solution of* (3), (2) *with initial data* (ρ^0_n, u^0_n) *(c.f. Theorem 2.4). Assume that there exists* $\alpha^0_+, \alpha^0_-, \rho^0_+, \rho^0_- \in L^\infty(\mathbb{T})$ *with*

$$0 \le \alpha^0_\pm \le 1, \quad \alpha^0_+ + \alpha^0_- = 1, \quad M_0^{-1} \le \rho^0_\pm \le M_0 \quad \text{a.e. on } \mathbb{T},$$

such that $\Theta^0_n \in \mathcal{M}^+_c(\mathbb{T}\times\mathbb{R})$*, defined through*

$$\langle\Theta^0_n, b\rangle := \int_{\mathbb{T}} b(x, \rho^0_n(x))\,\mathrm{d}x \quad \forall\, b \in C^0_0(\mathbb{T}\times\mathbb{R}),$$

satisfies

$$\langle\Theta^0_n, b\rangle \longrightarrow \int_{\mathbb{T}} \alpha^0_+(x) b(x, \rho^0_+(x)) + \alpha^0_-(x) b(x, \rho^0_-(x))\,\mathrm{d}x \quad \forall\, b \in C^0_0(\mathbb{T}\times\mathbb{R}). \tag{10}$$

Then there exist some time $T_1 > 0$, $\alpha_+, \alpha_-, \rho_+, \rho_- \in L^\infty(0,T_1; L^\infty(\mathbb{T})) \cap C([0,T_1]; L^1(\mathbb{T}))$ *and* $u \in L^\infty(0,T_1; H^1(\mathbb{T})) \cap C([0,T_1]; C(\mathbb{T}))$*, such that (up to a subsequence)* $\Theta_n \in C_{\mathrm{w}}([0,T_1]; \mathcal{M}^+(\mathbb{T}\times\mathbb{R}))$*, defined for any* $t \in [0,T_1]$ *through*

$$\langle\Theta_n(t), b\rangle := \int_{\mathbb{T}} b(x, \rho_n(t,x))\,\mathrm{d}x \quad \forall\, b \in C^0_0(\mathbb{T}\times\mathbb{R}),$$

converges in $C_{\mathrm{w}}([0,T_1]; \mathcal{M}^+(\mathbb{T}\times\mathbb{R}))$ *to* Θ*, where* Θ *is given for any* $t \in [0,T_1]$ *through*

$$\langle\Theta(t), b\rangle = \int_{\mathbb{T}} \alpha_+(t,x) b(x, \rho_+(t,x)) + \alpha_-(t,x) b(x, \rho_-(t,x))\,\mathrm{d}x \quad \forall\, b \in C^0_0(\mathbb{T}\times\mathbb{R}).$$

Moreover, we have

$$u_n \to u \quad \text{in } C([0,T_1]; C(\mathbb{T})), \quad c_n \rightharpoonup c \quad \text{in } L^2(0,T_1; H^2(\mathbb{T})).$$

The functions $\alpha_+, \alpha_-, \rho_+, \rho_-, u, c$ satisfy the BN system (6) in $\mathcal{D}'((0,T_1)\times\mathbb{T})$ and initial conditions

$$\alpha_+(0,\cdot)=\alpha_+^0,\quad \alpha_-(0,\cdot)=\alpha_-^0,\quad \rho_+(0,\cdot)=\rho_+^0,\quad \rho_-(0,\cdot)=\rho_-^0\quad u(0,\cdot)=u_0$$

a.e. in $\mathbb{T}$.

To the end of this section let us discuss how our results justify the BN system (6) as the macroscopic description of a two-phase fluid modeled by the non-local NSK system on the detailed scale. We start with a sequence of initial data $(\rho_n^0, u_n^0)_{n\in\mathbb{N}}$ and construct the corresponding sequence of solutions $(\rho_n, u_n, c_n)_{n\in\mathbb{N}}$ according to Theorem 2.4. We anticipate to find the macroscopic description of the two-phase fluid in the limit $n\to\infty$, when the initial density sequence highly oscillates between the liquid and the vapor density. To detail an example of such a sequence of initial densities, let us assume that the two-phase fluid is described initially on the macroscopic scale via smooth functions

$$0<\rho_+^0,\rho_-^0,\alpha_+^0,\alpha_-^0\in C^\infty(\mathbb{T}),\quad \alpha_+^0+\alpha_-^0=1,\quad \rho_+^0>\rho_-^0\quad \text{on } \mathbb{T},$$

where $\alpha_+^0,\alpha_-^0,\rho_+^0,\rho_-^0$ represent the volume fraction of the liquid phase, the volume fraction of the vapor phase, the partial density of the liquid phase and the partial density of the vapor phase, respectively. A sequence of initial densities $(\rho_n^0)_{n\in\mathbb{N}}$ corresponding to this macroscopic initial configuration could then look as follows:

For $n\in\mathbb{N}$ and $i\in\{0,1,\cdots,n-1\}$, we decompose the unit interval $[0,1]$ into

$$[0,1]:=\bigcup_{i=0}^{n-1} I_i,\quad I_i:=\left[\frac{i}{n},\frac{i+1}{n}\right],$$

and denote the center of I_i as $x_i:=\frac{2i+1}{2n}$. For $i\in\{0,1,\cdots,n-1\}$, we further decompose I_i as

$$I_i=\mathcal{D}_{i,+}^n\dot\cup\mathcal{D}_{i,-}^n\dot\cup\mathcal{S}_i^n,\quad \mathcal{D}_{i,+}^n:=\left[x_i-\frac{\alpha_+^0(x_i)}{2n}+\frac{C_\alpha}{4n^\lambda},x_i+\frac{\alpha_+^0(x_i)}{2n}-\frac{C_\alpha}{4n^\lambda}\right],$$

$$\mathcal{D}_{i,-}^n:=\left[\frac{i}{n},\frac{i}{n}+\frac{\alpha_-^0(x_i)}{2n}-\frac{C_\alpha}{4n^\lambda}\right]\cup\left[\frac{i+1}{n}-\frac{\alpha_-^0(x_i)}{2n}+\frac{C_\alpha}{4n^\lambda},\frac{i+1}{n}\right],$$

$$\mathcal{S}_i^n:=\left(\frac{i}{n}+\frac{\alpha_-^0(x_i)}{2n}-\frac{C_\alpha}{4n^\lambda},x_i-\frac{\alpha_+^0(x_i)}{2n}+\frac{C_\alpha}{4n^\lambda}\right)$$
$$\cup\left(x_i+\frac{\alpha_+^0(x_i)}{2n}-\frac{C_\alpha}{4n^\lambda},\frac{i+1}{n}-\frac{\alpha_-^0(x_i)}{2n}+\frac{C_\alpha}{4n^\lambda}\right),$$

where $\lambda\in(1,\infty)$ and $C_\alpha:=\frac{1}{2}\min_{x\in\mathbb{T}}\{\alpha_+^0(x),\alpha_-^0(x)\}>0$. Note that

$$|\mathcal{D}_{i,+}^n|=\frac{\alpha_+^0(x_i)}{n}-\frac{C_\alpha}{2n^\lambda},\quad |\mathcal{D}_{i,-}^n|=\frac{\alpha_-^0(x_i)}{n}-\frac{C_\alpha}{2n^\lambda},\quad |\mathcal{S}_i^n|=\frac{C_\alpha}{n^\lambda}.\tag{11}$$

Here, $\mathcal{D}_{i,+}^n$, $\mathcal{D}_{i,-}^n$ and $\mathcal{S}_i^n$ shall represent the regions on the detailed scale, where the fluid's state is liquid, vapor and spinodal, respectively. Accordingly, we take for $n\in\mathbb{N}$ an initial density $\rho_n^0\in C^\infty(\mathbb{T})$ satisfying

$$\min_{x\in\mathbb{T}}\rho_-^0(x)\le\rho_n^0\le\max_{x\in\mathbb{T}}\rho_+^0(x),\quad \rho_n^0(x)=\begin{cases}\rho_+^0(x_i) & x\in\mathcal{D}_{i,+}^n,\ i\in\{0,\cdots,n-1\},\\ \rho_-^0(x_i) & x\in\mathcal{D}_{i,-}^n,\ i\in\{1,\cdots,n-2\},\\ \rho_-^0(0) & x\in\mathcal{D}_{i,-}^n,\ i\in\{0,n-1\},\\ \text{smooth} & \text{elsewhere.}\end{cases}$$

Note that the sequence of initial densities $(\rho_n^0)_n\in\mathbb{N}$ is highly oscillating between the liquid's and the vapor's density. Moreover, we have in view of (11) and due to the continuity of $\rho_+^0,\rho_-^0,\alpha_+^0,\alpha_-^0$, that $(\rho_n^0)_{n\in\mathbb{N}}$ satisfies (10). Note that in this example, it is crucial to have $\lambda\in(1,\infty)$, since for $\lambda=1$, the limit density distribution will in general not consist only of Dirac-measures. Theorem 2 tells us then that for a small time, after identifying the density sequence $(\rho_n)_{n\in\mathbb{N}}$ as a sequence of parametrized measures $(\Theta_n)_{n\in\mathbb{N}}$, we find a limit velocity u and a limit density distribution Θ, such that $(\Theta_n,u_n)_{n\in\mathbb{N}}$ converges to

(Θ, u) in an appropriate way and moreover, the structure of the initial density distribution is preserved with functions $\alpha_+, \alpha_-, \rho_+, \rho_-$ that solve the BN system (6). Agreeing that (Θ, u) describes our two-phase fluid on the macroscopic scale, this justifies the BN system (6) as the macroscopic description. Since on the detailed scale, the NSK system (1) approximates (3) for $\gamma \to \infty$, it would be interesting, whether one could find a limit system for $\gamma \to \infty$ in (6) and whether the approximation $\gamma \to \infty$ commutes with the homogenization limit. The limit BN system for $\gamma \to \infty$ could then be interpreted as a macroscopic model for the NSK system (1).

3. Construction of Global-in-Time Strong Solutions

In this section, we provide the proof of Theorem 2.4. The uniqueness of strong solutions for the Cauchy problem (3), (2) can be proven by classical arguments (see e.g. in [19]). Therefore, we omit the proof for the uniqueness part here and focus on the existence part. For this part, we follow the lines in [20] and provide a continuation argument for the local-in-time strong solution. This continuation argument relies on suitable a priori estimates for the local-in-time strong solution. In comparison to the compressible Navier–Stokes equations, we have to deal with the additional term $\gamma\rho\partial_x c$ in the momentum equation. Exploiting elliptic regularity results, we are able to control this term in such a way, that the a priori estimates from [20] carry over. Our proof relies on the following local-in-time existence result for the Cauchy problem (3), (2).

Proposition 3.1. *Assume that the assumptions* (9) *hold true.*
Then there exist a time $T_0 > 0$ *that only depends on*

$$||u_0||_{H^1(\mathbb{T})}, ||\rho_0||_{H^1(\mathbb{T})}, M_0, \gamma, \kappa, \mu,$$

and unique functions

$$\begin{aligned} &\rho \in C([0,T_0); H^1(\mathbb{T})), \quad \rho > 0 \quad \text{on } [0,T_0)\times\mathbb{T}, \quad \partial_t\rho \in C([0,T_0); L^2(\mathbb{T})),\\ &u \in C([0,T_0); H^1(\mathbb{T})) \times L^2(0,T_0; H^2(\mathbb{T})), \qquad \partial_t u \in L^2(0,T_0; L^2(\mathbb{T})),\\ &c \in C([0,T_0); H^3(\mathbb{T})) \end{aligned}$$

that satisfy (3) *a.e. on* $(0,T_0)\times\mathbb{T}$ *and the initial conditions* (2) *a.e. in* $\mathbb{T}$.

The proof of Proposition 3.1 can be obtained via the method of successive approximation as in [21] or via a fixed point argument as in [19]. We omit a proof here.

Let us denote by (ρ, u, c) the local-in-time strong solution existing on $[0, T_0)$. If we show that

$$\begin{aligned} &C(T_0)^{-1} \le \rho(t,x) \le C(T_0) \quad \forall\,(t,x) \in [0,T_0)\times\mathbb{T},\\ &||\rho||_{L^\infty(0,T_0;H^1(\mathbb{T}))} + ||u||_{L^\infty(0,T_0;H^1(\mathbb{T}))} \le C(T_0) \end{aligned} \tag{12}$$

for some $C(T_0) > 0$ only depending on the initial data, a continuation argument yields that the local solution (ρ, u, c) must be global, i.e. $T_0 = \infty$. The goal for the rest of this section is the verification of the bounds in (12). Throughout this whole section $C(T_0) > 0$ denotes a generic constant that may vary from line to line but only depends on $||\rho_0||_{H^1(\mathbb{T})}, ||u_0||_{H^1(\mathbb{T})}, M_0, \mu, \kappa, \gamma, T_0$. To verify the bounds (12), we exploit the classical energy estimate and an entropy inequality (BD-entropy inequality) that was first derived by the authors in [22] for capillary fluids of Korteweg type and then verified by the authors in [20,23] for the compressible Navier–Stokes system with density dependent viscosity. Before we come to these inequalities, let us state the following auxiliary lemma that exploits the fact that c satisfies an elliptic equation.

Lemma 3.2. *Assume that the assumptions* (9) *hold true and let* (ρ, u, c) *denote the local-in-time strong solution of* (3), (2) *that exists on* $[0, T_0)$.

Then there exists a constant $C_{\text{ell}} > 0$ that does only depend on κ, γ, such that for all $t \in [0, T_0)$, we have the inequality

$$||c(t)||_{H^2(\mathbb{T})} \le C_{\text{ell}} ||\rho(t)||_{L^2(\mathbb{T})}.$$

Proof. For any $t \in [0, T_0)$, we have that $c(t)$ satisfies the elliptic equation

$$-\kappa \partial_{xx} c(t) + \gamma c(t) = \gamma \rho(t) \quad \text{a.e. on } \mathbb{T}.$$

Thus, Lemma 3.2 follows from standard interior regularity estimates, see e.g. [24]. □

Now we come to the classical energy estimate.

Proposition 3.3. *(Energy dissipation) Assume that the assumptions (9) hold true and let (ρ, u, c) denote the local-in-time strong solution of (3), (2) that exists on $[0, T_0)$.*

Then we have for all $t \in [0, T_0)$ the inequality

$$\int_{\mathbb{T}} \frac{1}{2}\rho u^2 + W(\rho) + \frac{\gamma}{2}|\rho - c|^2 + \frac{\kappa}{2}|\partial_x c|^2 \,\mathrm{d}x + \int_0^t \int_{\mathbb{T}} \mu |\partial_x u|^2 \,\mathrm{d}x \,\mathrm{d}\tau \le E_0,$$

where

$$E_0 := \frac{1}{2} M_0 ||u_0||^2_{L^2(\mathbb{T})} + \max_{\lambda \in [M_0^{-1}, M_0]} \{|W(\lambda)|\} + \gamma M_0^2 + \gamma C_{\text{ell}}^2 M_0^2 + \frac{\kappa}{2} C_{\text{ell}}^2 M_0^2.$$

Proof. Since we are in the strong solution framework, the following calculations are justified. First, using the continuity equation $(3)_1$, we may rewrite the momentum equation $(3)_2$ as

$$\rho(\partial_t u + u \partial_x u) + P'(\rho)\partial_x \rho - \mu \partial_{xx} u + \gamma \rho \partial_x(\rho - c) = 0.$$

Multiplying this equation by u, and integrating over the spatial domain $\mathbb{T}$ yields then for almost all $t \in [0, T_0)$ the relation

$$\int_{\mathbb{T}} \rho(\partial_t u + u\partial_x u) u \,\mathrm{d}x + \int_{\mathbb{T}} P'(\rho)(\partial_x \rho) u \,\mathrm{d}x - \mu \int_{\mathbb{T}} (\partial_{xx} u) u \,\mathrm{d}x + \gamma \int_{\mathbb{T}} \rho\big(\partial_x(\rho - c)\big) u \,\mathrm{d}x = 0. \tag{13}$$

Now we analyze each term in equation (13) separately. For the first term, we obtain by virtue of the continuity equation $(3)_1$ for almost all $t \in [0, \infty)$

$$\int_{\mathbb{T}} \rho(\partial_t u + u \partial_x u) u \,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}} \frac{1}{2} \rho |u|^2 \,\mathrm{d}x. \tag{14}$$

For the second term, we obtain again by virtue of the continuity equation $(3)_1$ and relation (7) using integration by parts

$$\begin{aligned} \int_{\mathbb{T}} P'(\rho)(\partial_x \rho) u \,\mathrm{d}x &= \int_{\mathbb{T}} W''(\rho)(\partial_x \rho)\rho u \,\mathrm{d}x = \int_{\mathbb{T}} \partial_x(W'(\rho))\rho u \,\mathrm{d}x \\ &= -\int_{\mathbb{T}} W'(\rho)\partial_x(\rho u) \,\mathrm{d}x = \int_{\mathbb{T}} W'(\rho)\partial_t \rho \,\mathrm{d}x \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}} W(\rho) \,\mathrm{d}x. \end{aligned} \tag{15}$$

For the third term in equation (13), integration by parts yields

$$-\mu \int_{\mathbb{T}} u \partial_{xx} u \,\mathrm{d}x = \mu \int_{\mathbb{T}} |\partial_x u|^2 \,\mathrm{d}x. \tag{16}$$

For the last term, we use again the continuity equation $(3)_1$ and integration by parts to obtain

$$\int_{\mathbb{T}} \gamma\big(\partial_x(\rho - c)\big)\rho u \,\mathrm{d}x = \int_{\mathbb{T}} (\partial_t \rho)\gamma(\rho - c) \,\mathrm{d}x. \tag{17}$$

Multiplying the elliptic equation $(3)_3$ by $\partial_t c$ and integrating over the spatial domain $\mathbb{T}$ yields for almost all $t \in [0, T_0)$

$$0 = \int_{\mathbb{T}} \kappa(\partial_t c)\partial_{xx} c \,\mathrm{d}x + \int_{\mathbb{T}} \gamma(\partial_t c)(\rho - c)\,\mathrm{d}x = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} \frac{\kappa}{2}|\partial_x c|^2 \,\mathrm{d}x + \int_{\mathbb{T}} \gamma(\partial_t c)(\rho - c)\,\mathrm{d}x.$$

Thus, by using (17),

$$\begin{aligned}\int_{\mathbb{T}} \gamma\big(\partial_x(\rho - c)\big)\rho u \,\mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} \frac{\gamma}{2}|\rho - c|^2 \mathrm{d}x + \int_{\mathbb{T}} \gamma(\partial_t c)(\rho - c)\,\mathrm{d}x \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} \frac{\gamma}{2}|\rho - c|^2 + \int_{\mathbb{T}} \frac{\kappa}{2}|\partial_x c|^2 \,\mathrm{d}x. \end{aligned} \tag{18}$$

Together, equations (13), (14), (15), (16) and (18) yield for almost all $t \in [0, T_0)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} \frac{1}{2}\rho|u|^2 + W(\rho) + \frac{\gamma}{2}|\rho - c|^2 + \frac{\kappa}{2}|\partial_x c|^2 \,\mathrm{d}x + \int_{\mathbb{T}} \mu|\partial_x u|^2 \,\mathrm{d}x = 0.$$

After integration in time we obtain for any $t \in [0, T_0)$

$$\begin{aligned}&\int_{\mathbb{T}} \frac{1}{2}\rho|u|^2 + W(\rho) + \frac{\gamma}{2}|\rho - c|^2 + \frac{\kappa}{2}|\partial_x c|^2 \,\mathrm{d}x + \int_0^t \int_{\mathbb{T}} \mu|\partial_x u| \,\mathrm{d}x \,\mathrm{d}\tau \\ &= \int_{\mathbb{T}} \rho_0(x)|u_0(x)|^2 + W(\rho_0(x)) + \frac{\gamma}{2}|\rho_0(x) - c(0,x)|^2 + \frac{\kappa}{2}|\partial_x c(0,x)|^2 \,\mathrm{d}x \\ &\leq E_0,\end{aligned}$$

where in the last inequality we have used Hölder's and Young's inequality and Lemma 3.2 for $t = 0$. □

Combining Proposition 3.3 and relation (8), we obtain some control on the density.

Proposition 3.4. *Assume that the assumptions* (9) *hold true and let* (ρ, u, c) *denote the local-in-time strong solution of* (3), (2) *that exists on* $[0, T_0)$.

Then there exists some constant $E_1 \geq 0$ *that only depends on* $M_0, ||u_0||_{L^2(\mathbb{T})}, \mu, \gamma, \kappa$, *such that for any* $t \in [0, T_0)$, *we have*

$$||\rho(t)||_{L^2(\mathbb{T})} + ||c(t)||_{W^{1,\infty}(\mathbb{T})} + ||c(t)||_{H^2(\mathbb{T})} \leq E_1.$$

Proof. With relation (8) and Proposition 3.3 we have

$$\int_{\mathbb{T}} \rho^2 \,\mathrm{d}x \leq C_\rho + C_\rho \int_{\mathbb{T}} W(\rho)\,\mathrm{d}x \leq C_\rho + C_\rho E_0.$$

Using the Sobolev embedding $W^{1,2}(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$ and the elliptic estimate provided by Lemma 3.2, we obtain

$$||c(t)||_{W^{1,\infty}(\mathbb{T})} \leq C||c(t)||_{H^2(\mathbb{T})} \leq CC_{\mathrm{ell}}||\rho(t)||_{L^2(\mathbb{T})} \leq CC_{\mathrm{ell}}\sqrt{C_\rho + C_\rho E_0},$$

where C denotes the embedding constant of $W^{1,2}(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$. Setting

$$E_1 := 3\max\Big\{\sqrt{C_\rho + C_\rho E_0}, CC_{\mathrm{ell}}\sqrt{C_\rho + C_\rho E_0}, C_{\mathrm{ell}}\sqrt{C_\rho + C_\rho E_0}\Big\}$$

yields the claim. □

As a next step, we derive the BD-entropy inequality. From this inequality we gain some control on the derivative of the density. Compared to the situation for the compressible Navier–Stokes equations in [20], we have to treat an additional term involving the order parameter in the momentum equation. Fortunately, the a priori estimates for c derived in Proposition 3.4 are strong enough to control this additional term. We emphasize that the monotonicity of P_γ is crucial in order to use the following result.

Proposition 3.5. *Assume that the assumptions* (9) *hold true and let* (ρ, u, c) *denote the local-in-time strong solution of* (3), (2) *that exists on* $[0, T_0)$.

Then we have for all $t \in [0, T_0)$ *that*

$$\int_0^t \int_{\mathbb{T}} (\partial_x \varphi(\rho)) \partial_x P_\gamma(\rho) \,\mathrm{d}x \,\mathrm{d}t \geq 0,$$

and moreover,

$$\int_{\mathbb{T}} \frac{1}{2}\rho|u + \partial_x\varphi(\rho)|^2 + W(\rho) + \frac{\gamma}{2}|\rho - c|^2 + \frac{\kappa}{2}|\partial_x c|^2 \,\mathrm{d}x + \int_0^t \int_{\mathbb{T}} (\partial_x\varphi(\rho))\partial_x P_\gamma(\rho)\,\mathrm{d}x\,\mathrm{d}t \leq C(T_0), \tag{19}$$

where $\varphi(r) := \int_1^r \frac{\mu}{s^2}\,\mathrm{d}s$. *In particular, we have*

$$\left\|\partial_x\left(\frac{1}{\sqrt{\rho}}\right)\right\|_{L^\infty(0,T_0;L^2(\mathbb{T}))} \leq C(T_0). \tag{20}$$

Proof. For the proof, we follow the lines in [20]. In the proof of Proposition 3.3, we have already verified the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} \frac{1}{2}\rho|u|^2 + W(\rho) + \frac{\gamma}{2}|\rho - c|^2 + \frac{\kappa}{2}|\partial_x c|^2 \,\mathrm{d}x = -\mu\int_{\mathbb{T}} |\partial_x u|^2\,\mathrm{d}x. \tag{21}$$

We infer with a mollification argument for the density and integration by parts the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} \rho u \partial_x\varphi(\rho)\,\mathrm{d}x = \int_{\mathbb{T}} \partial_t(\rho u)\partial_x\varphi(\rho)\,\mathrm{d}x - \int_{\mathbb{T}} \partial_x(\rho u)\varphi'(\rho)\partial_t\rho\,\mathrm{d}x.$$

On the one hand, the momentum equation $(3)_2$ yields then

$$\int_{\mathbb{T}} \partial_t(\rho u)\partial_x\varphi(\rho)\,\mathrm{d}x = \int_{\mathbb{T}} \mu(\partial_x\varphi(\rho))\partial_{xx}u + \gamma\rho(\partial_x c)\partial_x\varphi(\rho) - \partial_x(\rho u^2)\partial_x\varphi(\rho) - (\partial_x P_\gamma(\rho))\partial_x\varphi(\rho)\,\mathrm{d}x$$

and on the other hand the continuity equation $(3)_1$ yields

$$-\int_{\mathbb{T}} \partial_x(\rho u)\varphi'(\rho)\partial_t\rho\,\mathrm{d}x = \int_{\mathbb{T}} |\partial_x(\rho u)|^2\varphi'(\rho)\,\mathrm{d}x,$$

so that together we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} \rho u\partial_x\varphi(\rho)\,\mathrm{d}x = \int_{\mathbb{T}} \mu(\partial_x\varphi(\rho))\partial_{xx}u + \gamma\rho(\partial_x c)\partial_x\varphi(\rho) + |\partial_x(\rho u)|^2\varphi'(\rho) - (\partial_x(\rho u^2))\partial_x\varphi(\rho) - (\partial_x P_\gamma(\rho))\partial_x\varphi(\rho)\,\mathrm{d}x. \tag{22}$$

Exploiting the regularity of the strong solution (ρ, u, c), we deduce from the continuity equation by a regularization argument that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} \frac{1}{2}\rho|\partial_x\varphi(\rho)|^2\,\mathrm{d}x = -\int_{\mathbb{T}} \mu(\partial_{xx}u)\partial_x\varphi(\rho)\,\mathrm{d}x. \tag{23}$$

Note that the proof of this identity is delicate, since on the one hand, we have to regularize the density in order to justify taking the time derivative of the integrand on the left hand side, and on the other hand, we need the continuity equation for the regularized density. For the technical details and the regularization argument we refer to Appendix A in [20], where this relation was proven in a more complicated situation, where the viscosity depends on the density. Combining (22) and (23), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} \rho u\partial_x\varphi(\rho) + \frac{1}{2}\rho|\partial_x\varphi(\rho)|^2\,\mathrm{d}x + \int_{\mathbb{T}} (\partial_x P_\gamma(\rho))\partial_x\varphi(\rho)\,\mathrm{d}x$$

$$
\begin{aligned}
&= \int_{\mathbb{T}} \gamma\rho(\partial_x c)\partial_x\varphi(\rho) + |\partial_x(\rho u)|^2\varphi'(\rho) - \partial_x(\rho u^2)\partial_x\varphi(\rho)\,\mathrm{d}x \\
&= \int_{\mathbb{T}} \gamma\rho(\partial_x c)\partial_x\varphi(\rho) + \varphi'(\rho)\big(|\partial_x(\rho u)|^2 - \partial_x(\rho u^2)\partial_x\rho\big)\,\mathrm{d}x \\
&= \int_{\mathbb{T}} \gamma\rho(\partial_x c)\partial_x\varphi(\rho) + \varphi'(\rho)\rho^2|\partial_x u|^2\,\mathrm{d}x \\
&= \int_{\mathbb{T}} \gamma\rho(\partial_x c)\partial_x\varphi(\rho) + \mu|\partial_x u|^2\,\mathrm{d}x, \qquad (24)
\end{aligned}
$$

where we have used the relation

$$|\partial_x(\rho u)|^2 - \partial_x(\rho u^2)\partial_x\rho = \rho^2|\partial_x u|^2.$$

Combining (21) and (24) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta(t) + \int_{\mathbb{T}} (\partial_x P_\gamma(\rho))\partial_x\varphi(\rho)\,\mathrm{d}x = \int_{\mathbb{T}} \gamma\rho(\partial_x c)\partial_x\varphi(\rho)\,\mathrm{d}x, \qquad (25)$$

where

$$\eta(t) := \int_{\mathbb{T}} \frac{1}{2}\rho|u + \partial_x\varphi(\rho)|^2 + W(\rho) + \frac{\gamma}{2}|\rho - c|^2 + \frac{\kappa}{2}|\partial_x c|^2\,\mathrm{d}x.$$

Since (P, γ) is admissible, we have that P_γ is monotone and therefore

$$(\partial_x P_\gamma(\rho))\partial_x\varphi(\rho) = P_\gamma'(\rho)\varphi'(\rho)|\partial_x\rho|^2 \geq 0,$$

so that the second term on the left-hand side in (25) is non-negative. With Proposition 3.3, Proposition 3.4, the conservation of mass

$$M_0^{-1} \leq \int_{\mathbb{T}} \rho\,\mathrm{d}x = \int_{\mathbb{T}} \rho_0\,\mathrm{d}x \leq M_0, \qquad (26)$$

and Young's inequality, we estimate

$$
\begin{aligned}
\int_{\mathbb{T}} \gamma\rho\partial_x c\partial_x\varphi(\rho)\,\mathrm{d}x &\leq \gamma\|\partial_x c(t)\|_{L^\infty(\mathbb{T})} \int_{\mathbb{T}} \sqrt{\rho}\sqrt{\rho}|\partial_x\varphi(\rho)|\,\mathrm{d}x \\
&\leq \frac{\gamma}{2}||\partial_x c(t)||_{L^\infty(\mathbb{T})}\left(\int_{\mathbb{T}} \rho + \int_{\mathbb{T}} \rho|\partial_x\varphi(\rho)|^2\,\mathrm{d}x\right) \\
&\leq C(T_0)\left(1 + \int_{\mathbb{T}} \rho|\partial_x\varphi(\rho)|^2\,\mathrm{d}x\right) \\
&\leq C(T_0)\left(1 + \int_{\mathbb{T}} \rho|u + \partial_x\varphi(\rho)|^2\,\mathrm{d}x\right) \\
&\leq C(T_0)\Big(1 + \eta(t)\Big),
\end{aligned}
$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta(t) + \int_{\mathbb{T}} \partial_x P_\gamma(\rho)\partial_x\varphi(\rho)\,\mathrm{d}x \leq C(T_0)\Big(1 + \eta(t)\Big).$$

An application of Gronwall's inequality yields then the relation (19). Relation (19) and Proposition 3.3 imply then

$$\int_{\mathbb{T}} \rho|\partial_x\varphi(\rho)|^2\,\mathrm{d}x \leq C(T_0).$$

Since

$$\rho|\partial_x\varphi(\rho)|^2 = 4\mu^2\left|\partial_x\left(\frac{1}{\sqrt{\rho}}\right)\right|^2,$$

we conclude (20). □

From the BD-entropy inequality, we deduce L^∞-bounds for the density.

Lemma 3.6. *Assume that the assumptions* (9) *hold true and let* (ρ, u, c) *denote the local-in-time strong solution of* (3), (2) *that exists on* $[0, T_0)$.

Then we have

$$C(T_0)^{-1} \le \rho(t,x) \le C(T_0) \quad \forall\, (t,x) \in [0,T_0) \times \mathbb{T}.$$

Proof. We first prove the lower bound. To do this, we use the Sobolev embedding $H^1(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$. According to Proposition 3.5, we only have to show that for any $t \in [0, T_0)$

$$\left|\left| \frac{1}{\sqrt{\rho}}(t) \right|\right|_{L^2(\mathbb{T})} \le C(T_0).$$

Using the Poincaré inequality and Proposition 3.5, it suffices to prove

$$\mathbb{E}\left[\frac{1}{\sqrt{\rho}}(t)\right] \le C(T_0).$$

In the following calculations we omit the argument for t. Conservation of mass (see (26)) implies with Hölder's inequality

$$\int_{\mathbb{T}} \rho \frac{1}{\sqrt{\rho}}\, \mathrm{d}x = \int_{\mathbb{T}} \sqrt{\rho}\, \mathrm{d}x \le \sqrt{M_0} \le C(T_0). \tag{27}$$

Together with Proposition 3.4, Poincaré's inequality and Proposition 3.5, this implies

$$\int_{\mathbb{T}} \rho \left(\frac{1}{\sqrt{\rho}} - \mathbb{E}\left[\frac{1}{\sqrt{\rho}}\right] \right) \mathrm{d}x \le ||\rho(t)||_{L^2(\mathbb{T})} \left|\left| \frac{1}{\sqrt{\rho}} - \mathbb{E}\left[\frac{1}{\sqrt{\rho}}\right] \right|\right|_{L^2(\mathbb{T})} \le C(T_0).$$

Finally, we obtain from the above relations that

$$M_0^{-1} \mathbb{E}\left[\frac{1}{\sqrt{\rho}}\right] \le \int_{\mathbb{T}} \rho \mathbb{E}\left[\frac{1}{\sqrt{\rho}}\right] \mathrm{d}x = \int_{\mathbb{T}} \rho \frac{1}{\sqrt{\rho}}\, \mathrm{d}x - \int_{\mathbb{T}} \rho \left(\frac{1}{\sqrt{\rho}} - \mathbb{E}\left[\frac{1}{\sqrt{\rho}}\right] \right) \mathrm{d}x \le C(T_0),$$

which implies

$$\mathbb{E}\left[\frac{1}{\sqrt{\rho}}\right] \le C(T_0).$$

The proof for the lower bound is complete. For the upper bound, we use the Sobolev embedding $W^{1,1}(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$, so that we only have to prove

$$||\sqrt{\rho}||_{L^1(\mathbb{T})} + ||\partial_x \sqrt{\rho}||_{L^1(\mathbb{T})} \le C(T_0).$$

We have already estimated the first term in (27). For the second term, we use Proposition 3.4 and Proposition 3.5 to estimate

$$||\partial_x \sqrt{\rho}||_{L^1(\mathbb{T})} = \int_{\mathbb{T}} \rho \left| \partial_x \left(\frac{1}{\sqrt{\rho}} \right) \right| \mathrm{d}x \le ||\rho(t)||_{L^2(\mathbb{T})} \left|\left| \partial_x \left(\frac{1}{\sqrt{\rho}} \right) \right|\right|_{L^2(\mathbb{T})} \le C(T_0).$$

Thus, we have

$$\sqrt{\rho} \le C(T_0) \quad \text{on } [0, T_0] \times \mathbb{T},$$

which yields, in particular, the required upper bound for ρ. □

With the L^∞-bounds for the density, we come back to the BD-entropy inequality and obtain an uniform-in-time H^1-bound for ρ.

Lemma 3.7. *Assume that the assumptions* (9) *hold true and let* (ρ, u, c) *denote the local-in-time strong solution of* (3), (2) *that exists on* $[0, T_0)$.

Then we have

$$||\rho||_{L^\infty(0,T_0;H^1(\mathbb{T}))} \leq C(T_0).$$

Proof. With Proposition 3.4 we only have to show that

$$||\partial_x \rho||_{L^\infty(0,T_0;L^2(\mathbb{T}))} \leq C(T_0).$$

This can be done by using Proposition 3.5 and Lemma 3.6

$$\int_{\mathbb{T}} |\partial_x \rho|^2 = \int_{\mathbb{T}} \rho^3 \left| \frac{\partial_x \rho}{\rho^{\frac{3}{2}}} \right|^2 \mathrm{d}x \leq C(T_0) \left|\left| \partial_x \left(\frac{1}{\sqrt{\rho}} \right) \right|\right|^2_{L^2(\mathbb{T})} \leq C(T_0).$$

□

Finally, we prove an uniform-in-time H^1-bound for the velocity u.

Proposition 3.8. *Assume that the assumptions* (9) *hold true and let* (ρ, u, c) *denote the local-in-time strong solution* (3), (2) *that exists on* $[0, T_0)$.

Then we have

$$||u||_{L^\infty(0,T_0;H^1(\mathbb{T}))} \leq C(T_0).$$

Proof. We have

$$\rho(\partial_t u + u \partial_x u) = \mu \partial_{xx} u - \partial_x P_\gamma(\rho) + \gamma \rho \partial_x c \quad \text{a.e. in } (0, T_0) \times \mathbb{T}.$$

Both sides are in $L^2(0, T_0; L^2(\mathbb{T}))$. Hence, we may multiply both sides by $\partial_{xx} u \in L^2(0, T_0; L^2(\mathbb{T}))$ and integrate in space and time to obtain

$$\int_0^t \int_{\mathbb{T}} \rho(\partial_t u + u \partial_x u) \partial_{xx} u \,\mathrm{d}x \,\mathrm{d}\tau = \int_0^t \int_{\mathbb{T}} \mu |\partial_{xx} u|^2 - (\partial_x P_\gamma(\rho)) \partial_{xx} u + \gamma \rho (\partial_x c) \partial_{xx} u \,\mathrm{d}x \,\mathrm{d}\tau. \tag{28}$$

After an approximation argument using integration by parts, the left hand side can be rewritten as

$$\begin{aligned}
&\int_0^t \int_{\mathbb{T}} \rho(\partial_t u + u \partial_x u) \partial_{xx} u \,\mathrm{d}x \,\mathrm{d}\tau \\
&= -\frac{1}{2} \int_{\mathbb{T}} \rho(t,x) |\partial_x u(t,x)|^2 \,\mathrm{d}x + \frac{1}{2} \int_{\mathbb{T}} \rho_0 |\partial_x u_0|^2 \,\mathrm{d}x \\
&\quad - \int_0^t \int_{\mathbb{T}} (\partial_x \rho)(\partial_t u + u \partial_x u) \partial_x u - \rho |\partial_x u|^2 \partial_x u \,\mathrm{d}x \,\mathrm{d}\tau.
\end{aligned}$$

Plugging this relation into (28) yields

$$\begin{aligned}
&\frac{1}{2} \int_{\mathbb{T}} \rho(t,x) |\partial_x u(t,x)|^2 \,\mathrm{d}x + \int_0^t \int_{\mathbb{T}} \mu |\partial_{xx} u|^2 \,\mathrm{d}x \,\mathrm{d}\tau \\
&= \frac{1}{2} \int_{\mathbb{T}} \rho_0 |\partial_x u_0|^2 \,\mathrm{d}x + \int_0^t \int_{\mathbb{T}} (\partial_x P_\gamma(\rho)) \partial_{xx} u - \int_0^t \int_{\mathbb{T}} (\partial_x \rho) \big(\partial_t u + u \partial_x u\big) \partial_x u \,\mathrm{d}x \,\mathrm{d}\tau \\
&\quad - \int_0^t \int_{\mathbb{T}} \rho |\partial_x u|^2 \partial_x u \,\mathrm{d}x \,\mathrm{d}\tau - \int_0^t \int_{\mathbb{T}} \gamma \rho (\partial_x c) \partial_{xx} u \,\mathrm{d}x \,\mathrm{d}\tau
\end{aligned}$$

We estimate the right hand side with the help of Proposition 3.7. For $\varepsilon > 0$, let us denote by $C(\varepsilon) > 0$ a generic positive constant that may vary from line to line but only depends on ε. First we obtain with Young's inequality and Lemma 3.7

$$\int_0^t\int_{\mathbb{T}}(\partial_x P_\gamma(\rho))\partial_{xx}u\,\mathrm{d}x\,\mathrm{d}\tau \le \frac{1}{4\mu\varepsilon}\int_0^t ||\partial_x P_\gamma(\rho)||^2_{L^2(\mathbb{T})}\,\mathrm{d}\tau + \varepsilon\int_0^t\int_{\mathbb{T}}\mu|\partial_{xx}u|^2\,\mathrm{d}x\,\mathrm{d}\tau$$
$$\le C(T_0)C(\varepsilon)\int_0^t 1\,\mathrm{d}\tau + \varepsilon\int_0^t\int_{\mathbb{T}}\mu|\partial_{xx}u|^2\,\mathrm{d}x\,\mathrm{d}\tau.$$

Next, we use the relation

$$(\partial_x\rho)(\partial_t u + u\partial_x u) = \frac{\partial_x\rho}{\rho}(\mu\partial_{xx}u - \partial_x P_\gamma(\rho) + \gamma\rho\partial_x c),$$

to obtain with Young's inequality

$$\left|\int_{\mathbb{T}}(\partial_x\rho)(\partial_t u + u\partial_x u)\partial_x u\,\mathrm{d}x\right| \le \sqrt{\mu}\left|\left|\frac{1}{\rho}\right|\right|_{L^\infty(\mathbb{T})}||\rho||_{H^1(\mathbb{T})}||\partial_x u||_{L^\infty(\mathbb{T})}\left(\int_{\mathbb{T}}\mu|\partial_{xx}u|^2\,\mathrm{d}x\right)^{\frac{1}{2}}$$
$$+ \left|\left|\frac{1}{\rho}\right|\right|_{L^\infty(\mathbb{T})}||\rho||_{H^1(\mathbb{T})}||\partial_x P_\gamma(\rho)||_{L^2(\mathbb{T})}||\partial_x u||_{L^\infty(\mathbb{T})}$$
$$+ \gamma||\rho||_{H^1(\mathbb{T})}||\partial_x c||_{L^\infty(\mathbb{T})}||\partial_x u||_{L^2(\mathbb{T})}$$
$$\le C(T_0)C(\varepsilon)\left(1 + ||\partial_x u||^2_{L^2(\mathbb{T})}\right) + \varepsilon\int_{\mathbb{T}}\mu|\partial_{xx}u|^2\,\mathrm{d}x,$$

where we have used Proposition 3.7 to bound the H^1-norms of ρ, the control on $\partial_x c$ that is provided by Proposition 3.4 and the interpolation inequality

$$||\partial_x u||_{L^\infty(\mathbb{T})} \le C||\partial_x u||^{\frac{1}{2}}_{L^2(\mathbb{T})}||\partial_{xx}u||^{\frac{1}{2}}_{L^2(\mathbb{T})},$$

for some constant $C > 0$. This relation yields

$$\left|\int_0^t\int_{\mathbb{T}}\partial_x\rho(\partial_t u + u\partial_x u)\partial_x u\,\mathrm{d}x\,\mathrm{d}\tau\right|$$
$$\le C(T_0)C(\varepsilon)\int_0^t\left(1 + ||\partial_x u||^2_{L^2(\mathbb{T})}\right)\mathrm{d}\tau + \varepsilon\int_0^t\int_{\mathbb{T}}\mu|\partial_{xx}u|^2\,\mathrm{d}x\,\mathrm{d}\tau.$$

By similar arguments, we find

$$\left|\int_0^t\int_{\mathbb{T}}\rho|\partial_x u|^2\partial_x u\,\mathrm{d}x\,\mathrm{d}\tau\right| \le C(T_0)C(\varepsilon)\int_0^t\left(1 + ||\partial_x u||^4_{L^2(\mathbb{T})}\right)\mathrm{d}\tau + \varepsilon\int_0^t\int_{\mathbb{T}}\mu|\partial_{xx}u|^2\,\mathrm{d}x\,\mathrm{d}\tau.$$

Finally, we estimate with Proposition 3.4

$$\left|\int_0^t\int_{\mathbb{T}}\gamma\rho\partial_x c\partial_{xx}u\,\mathrm{d}x\,\mathrm{d}\tau\right| \le C(T_0)C(\varepsilon)\int_0^t\int_{\mathbb{T}}|\partial_x c|^2\,\mathrm{d}x\,\mathrm{d}\tau + \varepsilon\int_0^t\int_{\mathbb{T}}\mu|\partial_{xx}u|^2\,\mathrm{d}x\,\mathrm{d}\tau$$
$$\le C(T_0)C(\varepsilon)\int_0^t 1\,\mathrm{d}\tau + \varepsilon\int_0^t\int_{\mathbb{T}}\mu|\partial_{xx}u|^2\,\mathrm{d}x\,\mathrm{d}\tau.$$

Choosing $\varepsilon = \frac{1}{8}$, we obtain from the preceding estimates the relation

$$\frac{1}{2}\int_{\mathbb{T}}\rho(t,x)|\partial_x u(t,x)|^2\,\mathrm{d}x + \frac{1}{2}\int_0^t\int_{\mathbb{T}}\mu|\partial_{xx}u|^2\,\mathrm{d}x\,\mathrm{d}\tau$$
$$\le \frac{1}{2}\int_{\mathbb{T}}\rho_0|\partial_x u_0|^2\,\mathrm{d}x + C(T_0)\int_0^t\left(1 + ||\partial_x u||^4_{L^2(\mathbb{T})}\right)\mathrm{d}\tau$$
$$\le C(T_0) + C(T_0)\int_0^t\left(1 + ||\partial_x u||^4_{L^2(\mathbb{T})}\right)\mathrm{d}\tau.$$

Using

$$\int_0^t ||\partial_x u||^2_{L^2(\mathbb{T})}\,\mathrm{d}\tau \le C(T_0),$$

which holds due to Proposition 3.3, we conclude with Gronwall's inequality that

$$||\partial_x u||_{L^\infty(0,T_0;L^2(\mathbb{T}))} \leq C(T_0).$$

Together with Proposition 3.3, this yields the claim. □

4. Refined A Priori Estimates

In this section, we prove an a priori estimate concerning the effective viscous flux that is necessary to perform the homogenization procedure in Section 5. Since we will assume strong oscillations in the initial densities during the homogenization process, it is crucial that this a priori estimate does only depend on $M_0, ||u_0||_{H^1(\mathbb{T})}, \mu, \gamma, \kappa$, but not on quantities that involve the derivative of the density. Let us recall that we have already proven two such a priori estimates in Section 3, namely Proposition 3.3 and Proposition 3.4. However, these a priori estimates are not strong enough to control the homogenization procedure. Following [12], we introduce for a global-in-time strong solution (ρ, u, c) of (3), (2) the effective viscous flux as

$$\Sigma := \mu\partial_x u - P_\gamma(\rho)$$

and accordingly the initial effective viscous flux as

$$\Sigma_0 := \mu\partial_x u_0 - P_\gamma(\rho_0).$$

We then prove the following estimate.

Theorem 4.1. *Assume that the hypotheses of Theorem 2.4 hold true and let (ρ, u, c) denote the global-in-time strong solution of (3), (2).*

Then there exist some time $T_0 > 0$ and some constant $C_0 > 0$, both only depending on $M_0, ||u_0||_{H^1(\mathbb{T})}, \gamma$, κ, μ, such that we have

$$(2M_0)^{-1} \leq \rho(t,x) \leq 2M_0 \quad \forall\,(t,x) \in [0,T_0] \times \mathbb{T}, \tag{29}$$

and

$$||u||^2_{L^\infty(0,T_0;H^1(\mathbb{T}))} + ||\partial_x\Sigma||^2_{L^2(0,T_0;L^2(\mathbb{T}))} \leq C_0. \tag{30}$$

The proof of Theorem 4.1 follows the lines in [12] via two lemmata. To shorten the notation, let us introduce the following quantities:

$$\begin{aligned} K^0_{P_\gamma} &:= \max_{\lambda\in[(2M_0)^{-1},2M_0]} \big\{|P_\gamma(\lambda)|\big\}, \\ K^0_u &:= \frac{8}{\mu^2}\Big(2 + 5M_0\mu\Big)\Big(||\Sigma_0||^2_{L^2(\mathbb{T})} + 1 + |K^0_{P_\gamma}|^2\Big), \\ K^0_d &:= \frac{1}{\mu}\bigg[C_{\text{Sob}}\sqrt{2\mu E_0 + K^0_u + 2|K^0_{P_\gamma}|^2} + K^0_{P_\gamma}\bigg]. \end{aligned}$$

Here, $C_{\text{Sob}} > 0$ denotes the constant of the Sobolev embedding $H^1(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$. We emphasize, that $K^0_{P_\gamma}, K^0_u$ and K^0_d only depend on $M_0, ||u_0||_{H^1(\mathbb{T})}, \mu, \gamma$ and κ. Being in the strong solution framework, we may apply a continuity argument to find some small time $\tilde{T}_0 \in (0,\infty)$, such that we have

$$(2M_0)^{-1} \leq \rho(t,x) \leq 2M_0 \quad \forall\,(t,x) \in [0,\tilde{T}_0] \times \mathbb{T}, \tag{31}$$

and

$$||\partial_x u||^2_{L^\infty(0,\tilde{T}_0;L^2(\mathbb{T}))} + ||\partial_x\Sigma||^2_{L^2(0,\tilde{T}_0,L^2(\mathbb{T})} \leq K^0_u. \tag{32}$$

In order to conclude the proof of Theorem 4.1, we have to show that $\tilde{T}_0$ in fact only depends on the allowed quantities. This will be done as follows: We show that there exists some time $T_0 \in (0,\infty)$, only depending on the allowed quantities, such that if $T_0 < \tilde{T}_0$, we have in fact sharper versions of the inequalities (31) and (32) on $[0, T_0]$. Then, via a connectedness argument we prove that this already implies $\tilde{T}_0 = T_0$. To prove the sharper estimates, we proceed as in [12] via two lemmata. For the first lemma, we can use the arguments demonstrated in [12], since these only rely on the continuity equation $(3)_1$ and an a priori estimate for the derivative of the velocity, that is provided by Proposition 3.3. Therefore, we will refer for details concerning the proof to [12]. However, the second lemma relies on the momentum equation. Therefore, we have to modify the proof given in [12]. The crucial ingredient making our modification work is the control on the order parameter provided by Proposition 3.4.

Lemma 4.2. *Assume that the hypotheses of Theorem 2.4 hold true and let (ρ, u, c) denote the global-in-time strong solution of (3), (2). Suppose that for some $\tilde{T}_0 \in (0, 1)$, the inequalities (31) and (32) hold on $[0, \tilde{T}_0]$.*

Then there exists some $T_\rho \in (0, \infty)$, only depending on $M_0, ||u_0||_{H^1(\mathbb{T})}, \mu, \gamma, \kappa$, and the pressure law P, such that if $\tilde{T}_0 < T_\rho$, we have

$$||\partial_x u||_{L^1(0,\tilde{T}_0;L^\infty(\mathbb{T}))} \leq \sqrt{\tilde{T}_0} K_d^0,$$

and

$$\frac{2}{3M_0} \leq \rho(t, x) \leq \frac{3}{2} M_0 \quad \forall\, (t, x) \in [0, \tilde{T}_0] \times \mathbb{T}. \tag{33}$$

Proof. Both estimates follow from Proposition 3.3 by the arguments presented in the proof of Proposition 6 and Proposition 7 in [12]. □

For the second lemma, we also follow the proof in [12]. This proof relies on the momentum equation of the compressible Navier–Stokes equations. Using the artificial pressure function, we enter the framework of [12], but we have to deal with the additional term $\gamma\rho\partial_x c$ in the momentum equation. However, using the control provided by Proposition 3.4, the method of proof given in [12] also applies for our situation.

Lemma 4.3. *Assume that the hypotheses of Theorem 2.4 hold true and let (ρ, u, c) denote the global-in-time strong solution of (3), (2). Suppose that for some $\tilde{T}_0 \in (0, 1)$, the inequalities (31) and (32) hold on $[0, \tilde{T}_0]$.*

Then there exists some time $T_u \in (0, \infty)$ only depending on $M_0, ||u_0||_{H^1(\mathbb{T})}, \mu, \gamma, \kappa$, such that if $\tilde{T}_0 < T_u$, we have

$$||\partial_x u||^2_{L^\infty(0,\tilde{T}_0;L^2(\mathbb{T}))} + ||\partial_x \Sigma||^2_{L^2(0,\tilde{T}_0;L^2(\mathbb{T}))} \leq \frac{K_u^0}{2}. \tag{34}$$

Proof. Throughout this proof, we denote by $\mathcal{C} > 0$ a generic positive constant that may vary from line to line but only depends on $M_0, ||u_0||_{H^1(\mathbb{T})}, \mu, \gamma, \kappa$. Being in the strong solution framework, it is easy to verify that ρ satisfies the continuity equation $(3)_1$ in the renormalized sense, that is, we have for any $b \in C^1([0, \infty))$, the equation

$$\partial_t b(\rho) + \partial_x(b(\rho)u) - (b'(\rho)\rho - b(\rho))\partial_x u = 0 \quad \text{a.e. on } (0, \tilde{T}_0) \times \mathbb{T}. \tag{35}$$

We fix $T \in (0, \tilde{T}_0)$ and define on $(0, T)$ a multiplier via

$$m := \mu\partial_t u - \partial_x^{-1}\left[\partial_t P_\gamma(\rho) - \mathbb{E}[\partial_t P_\gamma(\rho)]\right] \in L^2(0, T; L^2(\mathbb{T})).$$

Note that $m \in L^2(0, T; L^2(\mathbb{T}))$, since $\partial_t u$, $\partial_t P_\gamma(\rho) = P'_\gamma(\rho)\partial_t\rho \in L^2(0, T; L^2(\mathbb{T}))$. In the sequel, we omit the argument of $P_\gamma(\rho)$ for the sake of brevity. We rewrite the momentum equation as

$$\rho(\partial_t u + u\partial_x u - \gamma\partial_x c) = \partial_x \Sigma.$$

Then, we multiply this equation by m and integrate over space and time to obtain

$$\int_0^T \int_{\mathbb{T}} \rho(\partial_t u + u\partial_x u - \gamma\partial_x c) m \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\mathbb{T}} (\partial_x \Sigma) m \,\mathrm{d}x\,\mathrm{d}t. \tag{36}$$

We notice, that formally $\partial_x m = \partial_t \Sigma + \mathbb{E}[\partial_t P_\gamma]$. We are not allowed to integrate by parts on the right hand side due to the lack of spatial regularity for $\partial_t u$. However, we may perform a Fourier series approximation of u. By the regularity of u, this approximation is strong enough to show that the right hand side of equation (36) can be rewritten as

$$\int_0^T \int_{\mathbb{T}} (\partial_x \Sigma) m \,\mathrm{d}x\,\mathrm{d}t = -\frac{1}{2}\int_{\mathbb{T}} |\Sigma(T,\cdot)|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}} |\Sigma_0|^2 \,\mathrm{d}x - \int_0^T \mathbb{E}(\Sigma) \int_{\mathbb{T}} \partial_t P_\gamma \,\mathrm{d}x\,\mathrm{d}t. \tag{37}$$

For details concerning the approximation argument we refer to [12]. Concerning the left hand side of equation (36) a straightforward application of Young's inequality yields

$$\begin{aligned}
&\int_0^T \int_{\mathbb{T}} \rho(\partial_t u + u\partial_x u - \gamma \partial_x c)\partial_x^{-1}\big[\partial_t P_\gamma - \mathbb{E}[\partial_t P_\gamma]\big] \\
&\le \frac{\mu}{2}\int_0^T \int_{\mathbb{T}} \rho(|\partial_t u|^2 + |u\partial_x u|^2 + |\gamma\partial_x c|^2) + \frac{3}{2\mu}\int_0^T \int_{\mathbb{T}} \rho\Big|\partial_x^{-1}\big[\partial_t P_\gamma - \mathbb{E}[\partial_t P_\gamma]\big]\Big|^2,
\end{aligned}$$

which, by virtue of the elementary calculation

$$\begin{aligned}
\mu\rho(\partial_t u + u\partial_x u - \gamma\partial_x c)\partial_t u =& \frac{\mu}{2}\rho|\partial_t u + u\partial_x u - \gamma\partial_x c|^2 + \frac{\mu}{2}\rho(|\partial_t u|^2 - |u\partial_x u|^2 - |\gamma\partial_x c|^2) \\
&+ \mu\gamma\rho u\partial_x u \partial_x c,
\end{aligned}$$

results in

$$\begin{aligned}
&\int_0^T \int_{\mathbb{T}} \rho(\partial_t u + u\partial_x u - \gamma\partial_x c) m \;\,\mathrm{d}x\,\mathrm{d}t \\
&\ge \frac{\mu}{2}\int_0^T \int_{\mathbb{T}} \rho|\partial_t u + u\partial_x u - \gamma\partial_x c|^2 \,\mathrm{d}x\,\mathrm{d}t - \mu\int_0^T \int_{\mathbb{T}} \rho|u\partial_x u|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \mu\gamma^2 \int_0^T \int_{\mathbb{T}} \rho|\partial_x c|^2 \,\mathrm{d}x\,\mathrm{d}t + \mu\gamma\int_0^T \int_{\mathbb{T}} \rho u(\partial_x u)\partial_x c \,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \frac{3}{2\mu}\int_0^T \int_{\mathbb{T}} \rho\Big|\partial_x^{-1}\big[\partial_t P_\gamma - \mathbb{E}[\partial_t P_\gamma]\big]\Big|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\ge \frac{\mu}{4M_0}\int_0^T \int_{\mathbb{T}} |\partial_x \Sigma|^2 \,\mathrm{d}x\,\mathrm{d}t - \mu\int_0^T \int_{\mathbb{T}} \rho|u\partial_x u|^2 \,\mathrm{d}x\,\mathrm{d}t - \mu\gamma^2\int_0^T \int_{\mathbb{T}} \rho|\partial_x c|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \mu\gamma\int_0^T \int_{\mathbb{T}} \rho u(\partial_x u)\partial_x c \,\mathrm{d}x\,\mathrm{d}t - \frac{3}{2\mu}\int_0^T \int_{\mathbb{T}} \rho\Big|\partial_x^{-1}\big[\partial_t P_\gamma - \mathbb{E}[\partial_t P_\gamma]\big]\Big|^2 \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{38}$$

In the last inequality, we have used the bound (31) for the density ρ and the momentum equation $(3)_2$. Combining the equations (36), (37) and (38) yields

$$\begin{aligned}
&\frac{1}{2}||\Sigma(T,\cdot)||^2_{L^2(\mathbb{T})} + \frac{\mu}{2M_0}||\partial_x\Sigma||^2_{L^2(0,T;L^2(\mathbb{T}))} \\
&\le \frac{1}{2}||\Sigma_0||^2_{L^2(\mathbb{T})} - \int_0^T \mathbb{E}(\Sigma)\Big(\int_{\mathbb{T}} \partial_t P_\gamma \,\mathrm{d}x\Big)\,\mathrm{d}t + \mu\int_0^T \int_{\mathbb{T}} \rho|u\partial_x u|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \mu\gamma^2\int_0^T \int_{\mathbb{T}} \rho|\partial_x c|^2 \,\mathrm{d}x\,\mathrm{d}t - \mu\gamma\int_0^T \int_{\mathbb{T}} \rho u(\partial_x u)\partial_x c \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{3}{2\mu}\int_0^T \int_{\mathbb{T}} \rho\Big|\partial_x^{-1}\big[\partial_t P_\gamma - \mathbb{E}[\partial_t P_\gamma]\big]\Big|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&=: \frac{1}{2}||\Sigma_0||^2_{L^2(\mathbb{T})} + \sum_{i=1}^{5} I_i.
\end{aligned} \tag{39}$$

We estimate each term on the right hand side of inequality (39) separately. For I_1, we use the renormalized continuity equation (35), to rewrite I_1 as

$$I_1 = \int_0^T \mathbb{E}[\Sigma] \int_{\mathbb{T}} \Big(P_\gamma'(\rho)\rho - P_\gamma(\rho)\Big)\partial_x u \,\mathrm{d}x\,\mathrm{d}t.$$

Young's inequality and Jensen's inequality provide us

$$|I_1| \leq \mathcal{C}\int_0^T ||\Sigma(t,\cdot)||^2_{L^2(\mathbb{T})}\,\mathrm{d}t + \mathcal{C}\int_0^T ||\partial_x u(t,\cdot)||^2_{L^2(\mathbb{T})}\,\mathrm{d}t.$$

For I_2, we use (31) and the Sobolev embedding $H^1(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$ together with Proposition 3.3 to estimate

$$\begin{aligned}|I_2| &\leq \mathcal{C}\int_0^T ||u(t,\cdot)||^2_{L^\infty(\mathbb{T})}||\partial_x u(t,\cdot)||^2_{L^2(\mathbb{T})}\,\mathrm{d}t\\ &\leq \mathcal{C}\int_0^T \big(1+||\partial_x u(t,\cdot)||^2_{L^2(\mathbb{T})}\big)||\partial_x u(t,\cdot)||^2_{L^2(\mathbb{T})}\,\mathrm{d}t\end{aligned}$$

By virtue of $\partial_x u = \frac{\Sigma + P_\gamma}{\mu}$ and since ρ is bounded via (31), we may estimate $|I_2|$ further as

$$|I_2| \leq \mathcal{C}\int_0^T \big(1+||\partial_x u(t,\cdot)||^2_{L^2(\mathbb{T})}\big)\,\mathrm{d}t + \mathcal{C}\int_0^T \big(1+||\partial_x u(t,\cdot)||^2_{L^2(\mathbb{T})}\big)||\Sigma(t,\cdot)||^2_{L^2(\mathbb{T})}\,\mathrm{d}t.$$

Regarding I_3 and I_4 we use Proposition 3.3 and Proposition 3.4 to estimate

$$|I_3| \leq \mathcal{C}\int_0^T 1\,\mathrm{d}t,$$

and

$$\begin{aligned}|I_4| &\leq \mathcal{C}\int_0^T\int_{\mathbb{T}} |u\partial_x u|\,\mathrm{d}x\,\mathrm{d}t \leq \mathcal{C}\int_0^T\int_{\mathbb{T}} |u|^2\,\mathrm{d}x\,\mathrm{d}t + \mathcal{C}\int_0^T\int_{\mathbb{T}} |\partial_x u|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\leq \mathcal{C}\int_0^T \big(1+||\partial_x u(t,\cdot)||^2_{L^2(\mathbb{T})}\big)\,\mathrm{d}t.\end{aligned}$$

Finally, for I_5, we use the renormalized continuity equation (35) to obtain

$$\partial_t P_\gamma - \mathbb{E}[\partial_t P_\gamma] = -\partial_x(P_\gamma u) - \Big((P_\gamma'\rho - P_\gamma)\partial_x u - \mathbb{E}[(P_\gamma'\rho - P_\gamma)\partial_x u]\Big),$$

so that its mean free primitive computes to

$$\partial_x^{-1}\big[\partial_t P_\gamma - \mathbb{E}[\partial_t P_\gamma]\big] = -\big(P_\gamma u - \mathbb{E}[P_\gamma u]\big) - \partial_x^{-1}\big[(P_\gamma'\rho - P_\gamma)\partial_x u - \mathbb{E}[(P_\gamma'\rho - P_\gamma)\partial_x u]\big].$$

Then, using Poincaré's inequality, Jensen's inequality, Proposition 3.3 and the bound (31), we deduce

$$|I_5| \leq \mathcal{C}\int_0^T \big(1+||\partial_x u(t,\cdot)||^2_{L^2(\mathbb{T})}\big)\,\mathrm{d}t.$$

In total, our estimates for I_1, I_2, I_3, I_4, I_5 together with inequality (39) imply that

$$\begin{aligned}&||\Sigma(T,\cdot)||^2_{L^2(\mathbb{T})} + \frac{\mu}{4M_0}||\partial_x \Sigma||^2_{L^2(0,T;L^2(\mathbb{T}))}\\ &\leq ||\Sigma_0||^2_{L^2(\mathbb{T})} + \int_0^T f(t)\,\mathrm{d}t + \int_0^T f(t)||\Sigma(t,\cdot)||^2_{L^2(\mathbb{T})}\,\mathrm{d}t,\end{aligned} \tag{40}$$

with

$$f(t) := \mathcal{C}\big(1+||\partial_x u(t,\cdot)||^2_{L^2(\mathbb{T})}\big).$$

By virtue of the inequality (32), we have

$$\int_0^T f(t)\mathrm{d}t = \int_0^T \mathcal{C}(1+||\partial_x u(t,\cdot)||^2_{L^2(\mathbb{T})})\,\mathrm{d}t \leq \mathcal{C}(1+K_u^0)T.$$

Since K_u^0 only depends on $M_0, ||u_0||_{H^1(\mathbb{T})}, \mu, \gamma$ and κ, we find some small time $T_u \in (0,\infty)$ only depending on these quantities, such that if $\tilde{T}_0 \leq T_u$, we have for any $T \in (0,\tilde{T}_0)$

$$\int_0^T f(t)\,\mathrm{d}t \leq \min\Big\{||\Sigma_0||^2_{L^2(\mathbb{T})} + 1, \ln(2)\Big\}. \tag{41}$$

Let us assume $\tilde{T}_0 < T_u$. Then relation (41) in combination with Gronwall's inequality yields for any $T \in (0,\tilde{T}_0)$

$$\begin{aligned} ||\Sigma(T,\cdot)||^2_{L^2(\mathbb{T})} &\leq ||\Sigma_0||^2_{L^2(\mathbb{T})} \exp\Big(\int_0^T f(t)\,\mathrm{d}t\Big) + \int_0^T f(t)\exp\Big(\int_t^T f(s)\,\mathrm{d}s\Big)\,\mathrm{d}t \\ &\leq 4\Big(||\Sigma_0||^2_{L^2(\mathbb{T})} + 1\Big), \end{aligned} \tag{42}$$

so that, using $\partial_x u = \frac{\Sigma + P_\gamma}{\mu}$,

$$||\partial_x u(T,\cdot)||^2_{L^2(\mathbb{T})} \leq \frac{2}{\mu^2}||\Sigma(T,\cdot)||^2_{L^2(\mathbb{T})} + \frac{2}{\mu^2}||P_\gamma||^2_{L^2(\mathbb{T})} \leq \frac{8}{\mu^2}\Big(||\Sigma_0||^2_{L^2(\mathbb{T})} + 1 + |K^0_{P_\gamma}|^2\Big). \tag{43}$$

Using (41) and (42) in (40), we conclude

$$\frac{\mu}{2M_0}||\partial_x\Sigma||^2_{L^2(0,\tilde{T}_0;L^2(\mathbb{T})} \leq 5\Big(||\Sigma_0||^2_{L^2(\mathbb{T})} + 1\Big).$$

In total we obtain the inequality

$$||\partial_x u||^2_{L^\infty(0,\tilde{T}_0;L^2(\mathbb{T}))} + ||\partial_x\Sigma||^2_{L^2(0,\tilde{T}_0;L^2(\mathbb{T}))} \leq \frac{K_u^0}{2},$$

provided $\tilde{T}_0 < T_u$. This yields the claim. □

We finish this section with the proof of Theorem 4.1.

Proof of Theorem 4.1. Using the strong regularity of (ρ, u, c) there exists some $0 < \tilde{T}_0 < 1$, such that the inequalities (31) and (32) hold on $[0,\tilde{T}_0]$. We define

$$T_0 := \min\big(1, T_\rho, T_u\big),$$

where $T_\rho, T_u \in (0,\infty)$ are the positive times provided by Lemma 4.2 and by Lemma 4.3, respectively. First, we notice that Proposition 3.3 yields

$$||u||^2_{L^\infty(0,\tilde{T}_0;L^2(\mathbb{T}))} \leq 4M_0E_0.$$

Hence, by virtue of the bounds (31) and (32), we are done if we conclude $T_0 \leq \tilde{T}_0$. Let us suppose the contrary, i.e. $T_0 > \tilde{T}_0$. Then, we may assume without loss of generality that

$$\tilde{T}_0 = \sup\{T \in [0,T_0] \mid (31), (32) \text{ hold on } [0,T]\} < T_0. \tag{44}$$

Since $\tilde{T}_0 < T_0$, we have by Lemma 4.2 and by Lemma 4.3 that the sharper bounds (33) and (34) hold on $[0,\tilde{T}_0]$. Then, by continuity, we find some small $\varepsilon > 0$, such that the bounds (31) and (32) hold on $[0,\tilde{T}_0+\varepsilon]$. This contradicts (44). □

5. Homogenization

In this section we perform the homogenization procedure and investigate the propagation of initial density oscillations for the system (3) with the usage of parametrized measures as in [12]. To do so, let us fix initial data $\rho_n^0, u_n^0 \in H^1(\mathbb{T})$ for $n \in \mathbb{N}$ satisfying the uniform bounds

$$M_0^{-1} \leq \rho_n^0(x) \leq M_0, \quad \forall x \in \mathbb{T}, \quad \forall n \in \mathbb{N}$$

and

$$\sup_{n\in\mathbb{N}} ||u_n^0||_{H^1(\mathbb{T})} < \infty,$$

where M_0 denotes some positive constant. This setting allows for strong oscillations in the initial densities, for instance, we could choose the initial densities as described in Section 2 after Theorem 2.5. In accordance with the hypotheses of Theorem 2.5, we may use the Banach–Alaoglu theorem, to obtain some $u \in H^1(\mathbb{T})$, such that after passing to a subsequence

$$u_n^0 \rightharpoonup u_0 \quad \text{in } H^1(\mathbb{T}).$$

Then, according to the well-posedness result Theorem 2.4, we construct for $n \in \mathbb{N}$ the global-in-time strong solution (ρ_n, u_n, c_n) of (3), (2) with initial data (ρ_n^0, u_n^0). Let us denote as in Section 4 the corresponding effective viscous flux as

$$\Sigma_n := \mu\partial_x u_n + P_\gamma(\rho_n) \quad \forall\, n \in \mathbb{N}.$$

The a priori bounds provided by Proposition 3.3, Proposition 3.4 and Theorem 4.1 then imply the following uniform bounds and convergences.

Lemma 5.1. *Let the hypotheses of Theorem 2.5 hold true.*

Then there exists some $T_0 > 0$, such that we have the following uniform bounds:

1. $\rho_n, \frac{1}{\rho_n}, P_\gamma(\rho_n)$ *are uniformly bounded in* $L^\infty(0, T_0; L^\infty(\mathbb{T}))$*, in particular*
$$(2M_0)^{-1} \le \rho_n(t,x) \le 2M_0 \quad \forall\,(t,x) \in [0,T_0]\times\mathbb{T}, \quad \forall\, n \in \mathbb{N},$$
2. u_n *is uniformly bounded in* $L^\infty(0, T_0; H^1(\mathbb{T}))$,
3. $\partial_x u_n$ *is uniformly bounded in* $L^2(0, T_0; L^\infty(\mathbb{T}))$,
4. Σ_n *is uniformly bounded in* $L^2(0, T_0; H^1(\mathbb{T}))$,
5. c_n *is uniformly bounded in* $L^\infty(0, T_0; H^2(\mathbb{T}))$.

Moreover, there exists $\rho, \overline{P} \in L^\infty(0, T_0; L^\infty(\mathbb{T}))$, $u \in L^\infty(0, T_0; H^1(\mathbb{T}))$ *with* $\partial_x u \in L^2(0, T_0; L^\infty(\mathbb{T}))$, $\Sigma \in L^2(0, T_0; H^1(\mathbb{T}))$ *and* $c \in L^\infty(0, T_0; H^2(\mathbb{T}))$*, such that, after passing to a subsequence, we have the following convergences:*

6. $\rho_n, P_\gamma(\rho_n) \overset{*}{\rightharpoonup} \rho, \overline{P}$ *in* $L^\infty(0, T_0; L^\infty(\mathbb{T}))$,
7. $u_n \overset{*}{\rightharpoonup} u$ *in* $L^\infty(0, T_0; H^1(\mathbb{T}))$ *and* $\partial_x u \in L^2(0, T_0; L^\infty(\mathbb{T}))$,
8. $\Sigma_n \rightharpoonup \Sigma$ *in* $L^2(0, T_0; H^1(\mathbb{T}))$,
9. $c_n \overset{*}{\rightharpoonup} c$ *in* $L^\infty(0, T_0; H^2(\mathbb{T}))$.

Furthermore, we have

$$(2M_0)^{-1} \le \rho(t,x) \le 2M_0 \quad \forall\,(t,x) \in [0,T_0]\times\mathbb{T}, \tag{45}$$

and

$$||u||^2_{L^\infty(0,T_0;H^1(\mathbb{T}))} + ||\Sigma||^2_{L^2(0,T_0;H^1(\mathbb{T}))} \le C_1, \tag{46}$$

where C_1 is a positive constant that only depends on $M_0, ||u_0||_{H^1(\mathbb{T})}, \mu, \gamma, \kappa$.

Proof. The uniform bounds 1, 2, 3, 4 and 5 follow from Proposition 3.3, Proposition 3.4 and Theorem 4.1 by using the uniform bounds on ρ_n^0 and u_n^0 according to the hypotheses of Theorem 2.5. These bounds imply via the Banach–Alaoglu theorem, after passing to a subsequence, the convergences 6, 7, 8 and 9. The regularity $\partial_x u \in L^2(0, T_0; L^\infty(\mathbb{T}))$ and the inequalities (45) and (46) follow from semi-continuity results on weak convergence. □

Using the Aubin–Lions lemma (see for instance [25]), we obtain stronger convergence for the velocity.

Lemma 5.2. *(Improved Convergences) Under the hypotheses and notation of Lemma 5.1, we have that*

1. $u_n \to u$ *in* $C([0,T_0]; C(\mathbb{T}))$,
2. $|u_n|^2 \to |u|^2$ *in* $L^2(0, T_0; L^2(\mathbb{T}))$,

3. $\rho_n u_n \rightharpoonup \rho u$ *in* $L^2(0,T_0;L^2(\mathbb{T}))$,
4. $\rho_n |u_n|^2 \rightharpoonup \rho |u|^2$ *in* $L^2(0,T_0;L^2(\mathbb{T}))$.

Proof. With 1 in Lemma 5.1, we may rewrite the momentum equation $(3)_2$ using the continuity equation $(3)_1$ as

$$\partial_t u_n = -u_n \partial_x u_n + \frac{1}{\rho_n}\partial_x \Sigma_n + \gamma \partial_x c_n.$$

This yields

$$\begin{aligned} &||\partial_t u_n||_{L^2(0,T_0;L^2(\mathbb{T}))} \\ &\leq ||u_n||_{L^\infty(0,T_0;L^\infty(\mathbb{T}))}||u_n||_{L^2(0,T_0;H^1(\mathbb{T}))} + \left|\left|\frac{1}{\rho_n}\right|\right|_{L^\infty(0,T_0;L^\infty(\mathbb{T}))}||\Sigma_n||_{L^2(0,T_0;H^1(\mathbb{T}))} \\ &\quad + \gamma ||c_n||_{L^\infty(0,T_0;H^2(\mathbb{T}))}. \end{aligned}$$

By virtue of the uniform bounds in Lemma 5.1 the above inequality implies that $\partial_t u_n$ is uniformly bounded in $L^2(0,T_0;L^2(\mathbb{T}))$. Together with 2 in Lemma 5.1 and the fact that the Sobolev embedding $H^1(\mathbb{T}) \hookrightarrow C(\mathbb{T})$ is compact, we may apply the Aubin–Lion lemma (see for instance [25]) to conclude that

$$u_n \to u \quad \text{in } C([0,T_0];C(\mathbb{T})). \tag{47}$$

The convergence 1 implies clearly 2. Combining the strong convergence of u_n and $|u_n|^2$ with the weak convergence of ρ_n yields the convergences 3 and 4. □

Before passing to the limit in system (3), we need one more compactness result concerning the effective viscous flux, that was one of the key ingredients in the existence proof of global finite energy weak solutions ([26,27]). This compactness result was transferred to the one-dimensional periodic framework in [12].

Lemma 5.3. *Let the hypotheses and notation of Lemma 5.1 hold true and let* $\beta \in C^1(\mathbb{R})$. *Then there exists a function* $\overline{\beta} \in L^\infty(0,T_0;L^\infty(\mathbb{T}))$ *such that, after passing to a subsequence, we have*

1. $\beta(\rho_n) \overset{*}{\rightharpoonup} \overline{\beta}$ *in* $L^\infty(0,T_0;L^\infty(\mathbb{T}))$,
2. $\beta(\rho_n)\Sigma_n \rightharpoonup \overline{\beta}\Sigma$ *in* $L^2(0,T_0;L^2(\mathbb{T}))$,
3. $\beta(\rho_n)\partial_x c_n \rightharpoonup \overline{\beta}\partial_x c$ *in* $L^2(0,T_0;L^2(\mathbb{T}))$.

Proof. We fix $\beta \in C^1(\mathbb{R})$ and denote for $n \in \mathbb{N}$

$$\beta_n := \beta(\rho_n), \qquad w_n := \partial_x^{-1}(\beta_n - \mathbb{E}[\beta_n]).$$

In view of Lemma 5.1 and Poincaré's inequality, we have that

$$\begin{aligned} &\beta_n \quad \text{is uniformly bounded in } L^\infty(0,T_0;L^\infty(\mathbb{T})), \\ &w_n \quad \text{is uniformly bounded in } L^\infty(0,T_0;W^{1,\infty}(\mathbb{T})). \end{aligned}$$

By the Banach–Alaoglu theorem we conclude that there exists a function $\overline{\beta} \in L^\infty(0,T_0;L^\infty(\mathbb{T}))$, such that, after passing to a subsequence that we do not relabel,

$$\begin{aligned} &\beta_n \overset{*}{\rightharpoonup} \overline{\beta} \quad \text{in } L^\infty(0,T_0;L^\infty(\mathbb{T})), \quad \mathbb{E}[\beta_n] \overset{*}{\rightharpoonup} \mathbb{E}[\overline{\beta}] \quad \text{in } L^\infty((0,T_0)), \\ &w_n \overset{*}{\rightharpoonup} \partial_x^{-1}(\overline{\beta} - \mathbb{E}[\overline{\beta}]) =: \overline{w} \quad \text{in } L^\infty(0,T_0;W^{1,\infty}(\mathbb{T})). \end{aligned}$$

From the continuity equation, we have that

$$\partial_t \beta_n = -\partial_x(\beta_n u_n) - (\beta'(\rho_n)\rho_n - \beta_n)\partial_x u_n,$$

which yields

$$\partial_t w_n = -\beta_n u_n - \partial_x^{-1}\big[(\beta'(\rho_n)\rho_n - \beta_n)\partial_x u_n - \mathbb{E}[(\beta'(\rho_n)\rho_n - \beta(\rho_n))\partial_x u_n]\big],$$

and

$$\partial_t \mathbb{E}[\beta_n] = -\mathbb{E}[(\beta'(\rho_n)\rho_n - \beta_n)\partial_x u_n].$$

By virtue of the uniform bounds in Lemma 5.1, we conclude from these relations by using Poincaré's inequality that

$$\begin{aligned} &\partial_t w_n \quad \text{is uniformly bounded in } L^\infty(0,T_0;L^2(\mathbb{T})), \\ &\partial_t \mathbb{E}[w_n] \quad \text{is uniformly bounded in } L^\infty((0,T_0)). \end{aligned}$$

In view of the Aubin–Lions lemma, we thus have

$$w_n \to \overline{w} \quad \text{in } C([0,T_0];L^2(\mathbb{T})), \quad \mathbb{E}[w_n] \to \mathbb{E}[\overline{w}] \quad \text{in } C([0,T_0]). \tag{48}$$

Let us fix $\varphi \in C_c^\infty((0,T_0)\times\mathbb{T})$. Then we have by using integration by parts

$$\begin{aligned} &\int_0^{T_0}\int_{\mathbb{T}} \beta_n(\partial_x c_n)\varphi \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^{T_0}\int_{\mathbb{T}} (\partial_x w_n)(\partial_x c_n)\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_0^{T_0}\int_{\mathbb{T}} \mathbb{E}[\beta_n](\partial_x c_n)\varphi \,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_0^{T_0}\int_{\mathbb{T}} w_n(\partial_{xx} c_n)\varphi + w_n(\partial_x c_n)\partial_x\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_0^{T_0}\int_{\mathbb{T}} \mathbb{E}[\beta_n](\partial_x c_n)\varphi \,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

In view of (48) and convergence 9 in Lemma 5.1, we conclude from this relation

$$\begin{aligned} &\lim_{n\to\infty}\int_0^{T_0}\int_{\mathbb{T}} \beta_n(\partial_x c_n)\varphi \,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_0^{T_0}\int_{\mathbb{T}} \overline{w}(\partial_{xx} c)\varphi + \overline{w}(\partial_x c)(\partial_x\varphi) \,\mathrm{d}x\,\mathrm{d}t + \int_0^{T_0}\int_{\mathbb{T}} \mathbb{E}[\overline{\beta}](\partial_x c)\varphi \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^{T_0}\int_{\mathbb{T}} \overline{\beta}(\partial_x c)\varphi \,\mathrm{d}x\,\mathrm{d}t, \end{aligned}$$

where we have used again integration by parts in the third line. From this relation and the fact that

$$\beta_n \partial_x c_n \quad \text{is uniformly bounded in } L^2(0,T_0;L^2(\mathbb{T})),$$

we conclude the convergence 3. The proof of 2 follows from analogous arguments (see Lemma 10 in [12]). □

Finally, we pass to the limit in system (3). We obtain the following system for the limit quantities (ρ,u,c):

Proposition 5.4. *(Limit system) Under the hypotheses and notation of Lemma 5.1, we have that (ρ,u,c) solves the system*

$$\begin{cases} \partial_t\rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) - \partial_x\Sigma - \gamma\rho\partial_x c = 0, \\ -\kappa\partial_{xx}c + \gamma(c-\rho) = 0 \end{cases} \tag{49}$$

in $\mathcal{D}'((0,T_0)\times\mathbb{T})$, where

$$\Sigma = \mu\partial_x u - \overline{P} \quad \text{a.e. in } (0,T_0)\times\mathbb{T}. \tag{50}$$

Moreover, we have

$$u(0,\cdot) = u_0 \quad \text{a.e. in } \mathbb{T}. \tag{51}$$

Proof. The fact that (ρ, u, c) solves (49) in $\mathcal{D}'((0,T_0)\times\mathbb{T})$ follows from Lemma 5.1, Lemma 5.2 and Lemma 5.3 with $\beta = \mathrm{Id}_{\mathbb{R}}$. In particular, Lemma 5.1 implies that both sides of

$$\Sigma_n = \mu\partial_x u_n - P_\gamma(\rho_n)$$

converge weakly in $L^2(0,T_0;L^2(\mathbb{T}))$ to their corresponding limits. This implies (50). Assertion (51) follows from the convergence 1 in Lemma 5.2. □

We notice that we have obtained an unclosed quantity $\overline{P}$. Due to the strong oscillations in the density, we cannot expect any strong convergence of the density. In particular, we cannot hope for a relation $\overline{P} = P_\gamma(\rho)$ due to the nonlinear nature of the pressure function P_γ. In fact, this is not what we want to achieve, since this would imply that all oscillations in the density sequence $(\rho_n)_{n\in\mathbb{N}}$ would have been disappeared. In order to close the system mathematically, we interpret the density sequence $(\rho_n)_{n\in\mathbb{N}}$ as a sequence of parametrized measures $(\Theta_n)_{n\in\mathbb{N}}$ and investigate the limit of this sequence. More precisely, we define for each $t\in[0,T_0]$ the map

$$C_c^0(\mathbb{T}\times\mathbb{R}) \to \mathbb{R}, \quad \langle\Theta_n(t),\varphi\rangle := \int_{\mathbb{T}} \varphi(x,\rho_n(t,x))\,\mathrm{d}x \tag{52}$$

It is not hard to see, that for each $t\in[0,T_0]$ we have $\Theta_n(t)\in\mathcal{M}^+(\mathbb{T}\times\mathbb{R})$. In the following proposition, we collect some properties of this map, including the fact that we can extract a converging subsequence of $(\Theta_n)_{n\in\mathbb{N}}$ and identify a kinetic equation for the limit measure. This equation can be seen as a closure for the system (49).

Proposition 5.5. *(Properties of Θ_n) Under the hypotheses and notation of Lemma 5.1, let Θ_n be defined via (52) for $n\in\mathbb{N}$.*

Then we have $\Theta_n \in C_{\mathrm{w}}([0,T_0];\mathcal{M}^+(\mathbb{T}\times\mathbb{R}))$. Furthermore, we have for any $t\in[0,T_0]$

$$\mathrm{spt}\,\big(\Theta_n(t)\big) \subseteq \mathbb{T}\times\big[(2M_0)^{-1},2M_0\big] \tag{53}$$

and

$$||\Theta_n(t)||_{\mathcal{M}^+(\mathbb{T}\times\mathbb{R})} = 1. \tag{54}$$

Moreover, there exists some $\Theta\in C_{\mathrm{w}}([0,T_0];\mathcal{M}^+(\mathbb{R}\times\mathbb{T}))$, such that, after passing to a subsequence,

$$\Theta_n \to \Theta \quad \text{in } C_{\mathrm{w}}([0,T_0];\mathcal{M}^+(\mathbb{T}\times\mathbb{R})), \tag{55}$$

that is,

$$\sup_{t\in[0,T_0]}\big|\langle\Theta_n(t)-\Theta(t),\varphi\rangle\big| \to 0 \quad \forall\,\varphi\in C_0^0(\mathbb{T}\times\mathbb{R}).$$

Furthermore, Θ satisfies

$$\mathrm{spt}\,\big(\Theta(t)\big) \subseteq \mathbb{T}\times\big[(2M_0)^{-1},2M_0\big] \tag{56}$$

and

$$\partial_t\Theta + \partial_x(\Theta u) - \frac{1}{\mu}\partial_\xi\Big([\xi\Sigma + \xi P_\gamma(\xi)]\Theta\Big) - \frac{1}{\mu}\Big([\Sigma + P_\gamma(\xi)]\Theta\Big) = 0 \tag{57}$$

in $\mathcal{D}'((0,T_0)\times\mathbb{T}\times\mathbb{R})$.

Proof. Taking $\varphi\in C_c^0(\mathbb{T}\times\mathbb{R})$ with $\varphi = 1$ on $\mathbb{T}\times[(2M_0)^{-1},(2M_0)]$ yields in view of Lemma 5.1 that

$$\langle\Theta_n(t),\varphi\rangle = \int_{\mathbb{T}}\varphi(x,\rho_n(t,x))\,\mathrm{d}x = 1,$$

and thus $\|\Theta(t)\|_{\mathcal{M}^+(\mathbb{T}\times\mathbb{R})} \geq 1$. On the other hand, we have for any $\varphi\in C_c^\infty(\mathbb{T}\times\mathbb{R})$ that

$$\langle\Theta_n(t),\varphi\rangle \leq \|\varphi\|_{L^\infty(\mathbb{T}\times\mathbb{R})},$$

and thus $\|\Theta_n(t)\|_{\mathcal{M}^+(\mathbb{T}\times\mathbb{R})} \leq 1$. Together, (54) follows. For the weak continuity, we fix $t,s\in[0,T_0]$ and verify with the mean value theorem for $\varphi\in C_c^1(\mathbb{T}\times\mathbb{R})$

$$\langle \Theta_n(t) - \Theta_n(s), \varphi \rangle \le ||\partial_2 \varphi||_{L^\infty(\mathbb{T}\times\mathbb{R})} ||\rho_n(t,\cdot) - \rho_n(s,\cdot)||_{L^1(\mathbb{T})}.$$

Here ∂_2 denotes the partial derivative with respect to the second variable. Recalling that $\rho_n \in C([0,T_0]; H^1(\mathbb{T}))$ and using the fact that $C_c^\infty(\mathbb{T}\times\mathbb{R})$ lies dense in $C_0^0(\mathbb{T}\times\mathbb{R})$ with respect to $||\cdot||_{L^\infty(\mathbb{T}\times\mathbb{R})}$ yields the continuity of the map

$$[0,T_0] \ni t \mapsto \langle \Theta_n(t), \varphi \rangle \in \mathbb{R}$$

for any $\varphi \in C_c^0(\mathbb{T}\times\mathbb{R})$. Hence $\Theta_n \in C_{\mathrm{w}}([0,T_0]; \mathcal{M}^+(\mathbb{T}\times\mathbb{R}))$. To verify (53), we take $\varphi \in C_c^0(\mathbb{T}\times\mathbb{R})$ with $\operatorname{spt}(\varphi) \subseteq \mathbb{T} \times \big(\mathbb{R} \setminus [(2M_0)^{-1}, 2M_0]\big)$. Then we have for any $t \in [0,T_0]$

$$\langle \Theta_n(t), \varphi \rangle = \int_{\mathbb{T}} \varphi(x, \rho_n(t,x)) \mathrm{d}x = 0,$$

by virtue of 1 in Lemma 5.1. Thus assertion (53) follows. As $C_c^\infty(\mathbb{T}\times\mathbb{R})$ is separable and since $C_c^\infty(\mathbb{T}\times\mathbb{R})$ lies dense in $C_0^0(\mathbb{T}\times\mathbb{R})$ with respect to the norm $||\cdot||_{L^\infty(\mathbb{T}\times\mathbb{R})}$, we find some countable set $\mathcal{S} \subset C_c^\infty(\mathbb{T}\times\mathbb{R})$ that lies dense in $C_0^0(\mathbb{T}\times\mathbb{R})$ with respect to the norm $||\cdot||_{L^\infty(\mathbb{T}\times\mathbb{R})}$. Let us fix some $\varphi \in \mathcal{S}$. Without loss of generality, we can assume that φ can be written as $\varphi = \phi\beta$ for some $\phi \in C^\infty(\mathbb{T})$ and some $\beta \in C_c^\infty(\mathbb{R})$. Since ρ_n and u_n satisfy the continuity equation, we have that

$$\partial_t \beta(\rho_n) = -\partial_x\big(\beta(\rho_n)u_n\big) - \big(\beta'(\rho_n)\rho_n - \beta(\rho_n)\big)\partial_x u_n \quad \text{a.e. in } (0,T_0)\times\mathbb{T}.$$

Multiplying this equation with ϕ, integrating in space over $\mathbb{T}$ and using integration by parts leads to

$$\partial_t \langle \Theta_n, \varphi \rangle = \int_{\mathbb{T}} \beta(\rho_n) u_n \partial_x \phi - \big(\beta'(\rho_n)\rho_n - \beta(\rho_n)\big)(\partial_x u_n)\phi \, \mathrm{d}x \quad \text{a.e. in } (0,T_0).$$

In view of Lemma 5.1, we deduce from this relation by using Hölder's inequality that

$$\big(\partial_t \langle \Theta_n, \varphi \rangle\big)_{n\in\mathbb{N}} \quad \text{is bounded in } L^2((0,T_0)).$$

Moreover, we have due to (54) that

$$\big(\langle \Theta_n, \varphi \rangle\big)_{n\in\mathbb{N}} \quad \text{is bounded in } L^\infty((0,T_0)).$$

Since the embedding $W^{1,2}((0,T_0)) \hookrightarrow C([0,T_0])$ is compact, we find some function $F_\varphi \in C([0,T_0])$, such that, after passing to a non-relabeled subsequence,

$$\langle \Theta_n, \varphi \rangle \to F_\varphi \quad \text{in } C([0,T_0]).$$

Since $\mathcal{S}$ is countable, we obtain after a diagonal argument that for any $\varphi \in \mathcal{S}$ there exists some function $F_\varphi \in C([0,T_0])$, such that we have, after passing to a non-relabeled subsequence,

$$\langle \Theta, \varphi \rangle \to F_\varphi \quad \text{in } C([0,T_0]) \quad \forall\, \varphi \in \mathcal{S}. \tag{58}$$

For $\varphi \in C_0^0(\mathbb{T}\times\mathbb{R})$ we have by the density of $\mathcal{S} \subseteq C_0^0(\mathbb{T}\times\mathbb{R})$ that there exists some sequence $(\varphi_k)_{k\in\mathbb{N}} \subseteq \mathcal{S}$ that satisfies

$$\varphi_k \to \varphi \quad \text{in } L^\infty(\mathbb{T}\times\mathbb{R}).$$

In view of (54), we have for any $k,l \in \mathbb{N}$ that

$$||F_{\varphi_k} - F_{\varphi_l}||_{C([0,T_0])} \le \limsup_{n\to\infty} |\langle \Theta_n, \varphi_k - \varphi_l \rangle| \le ||\varphi_k - \varphi_l||_{L^\infty(\mathbb{T}\times\mathbb{R})}.$$

In particular, we have that $(F_{\varphi_k})_{k\in\mathbb{N}}$ is a Cauchy sequence and thus converges in $C([0,T_0])$. We then define

$$F_\varphi := \lim_{k\to\infty} F_{\varphi_k}, \tag{59}$$

where the limit is meant with respect to the norm $||\cdot||_{C([0,T_0])}$. Note that this definition does not depend on the specific choice of the approximating sequence. Moreover, we infer by (54), (58) and (59) that

$$\langle \Theta_n, \varphi \rangle \to F_\varphi \quad \text{in } C([0,T_0]) \quad \forall\, \varphi \in C_0^0(\mathbb{T}\times\mathbb{R}). \tag{60}$$

Now we fix $t \in [0,T_0]$ and consider the map

$$C_0^0(\mathbb{T}\times\mathbb{R})\to\mathbb{R},\quad \varphi\mapsto F_\varphi(t).$$

In view of (54) and (60), we have

$$|F_\varphi(t)|\le\limsup_{n\to\infty}|\langle\Theta_n(t),\varphi\rangle|\le||\varphi||_{L^\infty(\mathbb{T}\times\mathbb{R})}\quad\forall\,\varphi\in C_0^0(\mathbb{T}\times\mathbb{R}),$$
$$F_\varphi(t)=\lim_{n\to\infty}\langle\Theta_n(t),\varphi\rangle\ge 0\quad\forall\varphi\in C_0^0(\mathbb{T}\times\mathbb{R}),\ \ \varphi\ge 0.$$

In particular, we have that $\varphi\mapsto F_\varphi(t)$ defines a bounded positive linear functional on $(C_0^0(\mathbb{T}\times\mathbb{R}),||\cdot||_{L^\infty(\mathbb{T}\times\mathbb{R})})$. By Riesz' representation theorem, we thus find some $\Theta(t)\in\mathcal{M}^+(\mathbb{T}\times\mathbb{R})$, such that

$$F_\varphi(t)=\langle\Theta(t),\varphi\rangle\quad\forall\,\varphi\in C_0^0(\mathbb{T}\times\mathbb{R}).$$

In this way, we obtain a map $\Theta\colon[0,T_0]\to\mathcal{M}^+(\mathbb{T}\times\mathbb{R})$, which satisfies

$$\langle\Theta,\varphi\rangle=F_\varphi\in C([0,T_0]),\quad\langle\Theta_n,\varphi\rangle\to\langle\Theta,\varphi\rangle\quad\text{in } C([0,T_0])\quad\forall\,\varphi\in C_0^0(\mathbb{T}\times\mathbb{R}).$$

This is precisely $\Theta\in C_{\mathrm{w}}([0,T_0];\mathcal{M}^+(\mathbb{T}\times\mathbb{R}))$ and (55). The fact that the limit measure Θ satisfies (57) follows from Lemma 5.1, Lemma 5.2, Lemma 5.3 and (55). For a detailed proof we refer the reader to the proof of Proposition 12 in [12]. Relation (56) is a consequence of the convergence (55) and (53). □

Following [14], we construct now an appropriate solution to the target BN system (6) subject to the initial conditions $\alpha_+^0,\alpha_-^0,\rho_+^0,\rho_-^0$ given by the hypothesis of Theorem 2.5. This solution gives then rise to a parametrized measure that satisfy the same equation (57) as Θ with the same initial condition (see hypothesis of Theorem 2.5). Then, a uniqueness result concludes the proof of Theorem 2.5. Since the techniques used to prove the following theorem follow techniques demonstrated in [14], we give the proof of the following result in the Appendix.

Theorem 5.6. *Let the hypotheses and notation of Lemma 5.1 hold true. Then there exist some $T_1\in(0,T_0]$ and a weak solution*

$$\rho_\pm,\alpha_\pm\in L^\infty(0,T_1;L^\infty(\mathbb{T}))\cap C([0,T_1];L^1(\mathbb{T}))$$

of the BN system

$$\begin{cases}\partial_t\alpha_\pm+u\partial_x\alpha_\pm=\dfrac{\alpha_\pm}{\mu}(P_\gamma(\rho_\pm)-\overline{P}) & \text{in } (0,T_1)\times\mathbb{T},\\ \partial_t\rho_\pm+\partial_x(\rho_\pm u)=\dfrac{\rho_\pm}{\mu}(\overline{P}-P_\gamma(\rho_\pm)) & \text{in } (0,T_1)\times\mathbb{T},\\ \rho_\pm(0,\cdot)=\rho_\pm^0,\quad\alpha_\pm(0,\cdot)=\alpha_\pm^0 & \text{in } \mathbb{T},\end{cases}\tag{61}$$

where $u\in L^\infty(0,T_0;H^1(\mathbb{T}))$ with $\partial_x u\in L^2(0,T_0;L^\infty(\mathbb{T}))$ denotes the limit velocity in Lemma 5.1.

Moreover, we have

$$\alpha_\pm\ge 0,\quad(2M_0)^{-1}\le\rho_\pm\le 2M_0\quad\text{a.e. on } [0,T_1]\times\mathbb{T}.\tag{62}$$

Proof. This result follows directly from Theorem A.1. □

The constructed solutions to the BN system gives rise to a parametrized measure $\overline{\Theta}$ as follows: For $t\in[0,T_1]$, we define $\overline{\Theta}(t)$ via

$$\langle\overline{\Theta}(t),\varphi\rangle:=\int_{\mathbb{T}}\alpha_+(t,x)\varphi(x,\rho_+(t,x))+\alpha_-(t,x)\varphi(x,\rho_-(t,x))\,\mathrm{d}x\tag{63}$$

for $\varphi\in C_c^0(\mathbb{T}\times\mathbb{R})$. It is straightforward to verify that $\overline{\Theta}(t)\in\mathcal{M}^+(\mathbb{T}\times\mathbb{R})$. In the following proposition, we collect some properties of $\overline{\Theta}$, including the crucial fact that $\overline{\Theta}$ satisfies the same equation as Θ.

Proposition 5.7. *(Properties of $\overline{\Theta}$) Under the hypothesis and notation of Theorem 5.6, let us define $\overline{\Theta}$ via (63).*

Then we have $\overline{\Theta}\in C_{\mathrm{w}}([0,T_1];\mathcal{M}^+(\mathbb{T}\times\mathbb{R}))$. Furthermore, for any $t\in[0,T_1]$, we have

$$\operatorname{spt}\left(\overline{\Theta}(t)\right)\subseteq\mathbb{T}\times\left[(2M_0)^{-1},2M_0\right]\tag{64}$$

and

$$||\overline{\Theta}(t)||_{\mathcal{M}(\mathbb{T}\times\mathbb{R})} \leq ||\alpha_+||_{C([0,T_1];L^1(\mathbb{T}))} + ||\alpha_-||_{C([0,T_1];L^1(\mathbb{T}))}. \tag{65}$$

Moreover, $\overline{\Theta}$ *satisfies*

$$\partial_t\overline{\Theta} + \partial_x(\overline{\Theta}u) - \frac{1}{\mu}\partial_\xi\Big([\xi\Sigma + \xi P_\gamma(\xi)]\overline{\Theta}\Big) - \frac{1}{\mu}\Big([\Sigma + P_\gamma(\xi)]\overline{\Theta}\Big) = 0 \tag{66}$$

in $\mathcal{D}'((0,T_1)\times\mathbb{T}\times\mathbb{R})$.

Proof. The weak continuity of $\overline{\Theta}$ follows from the continuity-in-time of $\alpha_\pm$ and $\rho_\pm$. Assertion (64) follows from (62) and assertion (65) is trivial. Finally, equation (66) follows from a regularization argument (see for instance [28]) and the fact that $\Sigma = \mu\partial_x u - \overline{P}$. □

Finally, using the properties of Θ and $\overline{\Theta}$ and a uniqueness result for measure valued solutions to transport equations (see Theorem B.1), we prove our main result Theorem 2.5.

Proof of Theorem 2.5. We stick to the notation of Proposition 5.5 and Proposition 5.7. We investigate the difference $\nu := \Theta - \overline{\Theta}$. We have $\nu \in C_{\mathrm{w}}([0,T_1];\mathcal{M}(\mathbb{T}\times\mathbb{R}))$ with

$$\sup_{t\in[0,T_0]} ||\nu(t)||_{\mathcal{M}(\mathbb{T}\times\mathbb{R})} < \infty$$

and

$$\mathrm{spt}\,\big(\nu(t)\big) \subseteq \mathbb{T}\times\big[(2M_0)^{-1}, 2M_0\big] \quad \forall\, t\in[0,T_1]. \tag{67}$$

Furthermore, we have by hypotheses of Theorem 2.5 that $\Theta(0) = \overline{\Theta}(0)$, so that

$$\nu(0) = 0.$$

Moreover, ν satisfies

$$\partial_t\nu + \partial_x(\nu u) - \frac{1}{\mu}\partial_\xi\Big((\xi\Sigma - \xi P_\gamma(\xi))\nu\Big) - \frac{1}{\mu}\Big((\Sigma - P_\gamma(\xi))\nu\Big) = 0$$

in $\mathcal{D}'((0,T_0)\times\mathbb{T}\times\mathbb{R})$ since both, Θ and $\overline{\Theta}$ satisfy this equation (see Proposition 5.5 and Proposition 5.7). Now, we choose a cutoff function $\chi \in C_c^\infty(\mathbb{R})$ such that

$$\chi \equiv 1 \quad \text{on } \big[(2M_0)^{-1}, 2M_0\big].$$

By virtue of (67) we conclude with the definition of χ, that ν satisfies in fact

$$\partial_t\nu + \mathrm{div}_{(x,\xi)}(\mathbf{V}\nu) + g\nu = 0 \quad \text{in } \mathcal{D}'((0,T_0)\times\mathbb{T}\times\mathbb{R}),$$

with $\mathbf{V} := (V_1, V_2)$,

$$V_1(t,x,\xi) := u(t,x), \quad V_2(t,x,\xi) := -\frac{1}{\mu}(\xi\Sigma(t,x) - \xi P_\gamma(\xi))\chi(\xi)$$

and

$$g(t,x,\xi) := -\frac{1}{\mu}(\Sigma(t,x) - P_\gamma(\xi))\chi(\xi).$$

We notice that all prerequisites of Theorem B.1 are satisfied, such that we can conclude

$$\nu = 0,$$

i.e. $\Theta = \overline{\Theta}$. By the convergence 6 in Lemma 5.1, we have for any $\eta \in C_c^\infty(0,T_1)$ and any $\phi \in C^\infty(\mathbb{T})$, that

$$\int_0^{T_1}\int_{\mathbb{T}} P_\gamma(\rho_n(t,x))\eta(t)\phi(x)\,\mathrm{d}x\,\mathrm{d}t \longrightarrow \int_0^{T_1}\int_{\mathbb{T}} \overline{P}(t,x)\eta(t)\phi(x)\,\mathrm{d}x\,\mathrm{d}t.$$

On the other hand, we have since $\overline{\Theta} = \Theta$ that

$$
\begin{aligned}
\int_0^{T_1}\int_{\mathbb{T}} P_\gamma(\rho_n(t,x))\eta(t)\phi(x)\,\mathrm{d}x\,\mathrm{d}t &= \int_0^{T_1}\eta(t)\langle\Theta_n(t),P_\gamma(\xi)\phi(x)\rangle\,\mathrm{d}t\\
\longrightarrow \int_0^{T_1}\eta(t)\langle\overline{\Theta}(t),P_\gamma(\xi)\phi(x)\rangle\,\mathrm{d}t\\
&= \int_0^{T_1}\int_{\mathbb{T}}\eta(t)\phi(x)\big(\alpha_+(t,x)P_\gamma(\rho_+(t,x))+\alpha_-(t,x)P_\gamma(\rho_-(t,x))\big)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Since η and ϕ were arbitrary, we conclude

$$
\overline{P} = \alpha_+P_\gamma(\rho_+) + \alpha_-P_\gamma(\rho_-). \tag{68}
$$

By similar arguments, we conclude

$$
\rho = \alpha_+\rho_+ + \alpha_-\rho_-. \tag{69}
$$

Using the relations (68) in $(61)_1$ and $(61)_2$ and adding up both equations yields that $\alpha_+ + \alpha_-$ satisfies

$$
\begin{cases}
\partial_t(\alpha_+ + \alpha_-) + u\partial_x(\alpha_+ + \alpha_-) = \dfrac{\overline{P} - (\alpha_+ + \alpha_-)\overline{P}}{\mu} & \text{in } (0,T_1)\times\mathbb{T},\\
(\alpha_+ + \alpha_-)(0,\cdot) = 1 & \text{in } \mathbb{T}.
\end{cases}
$$

Since $u \in L^1(0,T;W^{1,\infty}(\Omega))$, we have that weak solutions to this Cauchy problem are unique (see e.g. [28]). Thus, we conclude

$$
\alpha_+ + \alpha_- = 1 \quad \text{a.e. on } [0,T_1]\times\mathbb{R}. \tag{70}
$$

Using the relations (68) and (70) in (61) yields that $(\alpha_+,\alpha_-,\rho_+,\rho_-,u,c)$ solves (6) in $\mathcal{D}'((0,T_1)\times\mathbb{T})$. The proof of Theorem 2.5 is now complete.

□

6. Conclusions

In this paper, we have investigated the propagation of initial density oscillations for the non-local NSK system. With the use of parametrized measures, we have derived a closed homogenized system consisting of a momentum equation for the velocity and a kinetic equation for a parametrized measure. After assuming that the parametrized measure is a convex combination of Dirac-measures initially, we have proven that the kinetic equation preserves this structure. With that structure for the parametrized measure, the kinetic equation then reduces to the BN system (6). In that sense, we have justified the BN system (6) rigorously as macroscopic description for a compressible liquid-vapor flow that is modeled with the non-local NSK equations on the detailed scale. It would be interesting to extend this work to the 3D case as in [4] in the framework of finite-energy weak solutions. It seems that the arguments of [4] should apply in order to prove that the effective equations are given by a kinetic equation for a parametrized measure and a momentum equation for the velocity. However, proving the propagation of convex combinations of Dirac-measures seems difficult, since the arguments in [14] rely on an isentropic pressure law. As mentioned by the authors in [14], it is not clear how these arguments carry over to a more general pressure law as for instance a pressure law of Van-der-Waals type. Nevertheless, we could interpret the parametrized measure as a probability density function as done in [15] and investigate the resulting equations numerically.

Appendix A Existence of Solutions to a BN System

Here, we provide a proof of the following result that is concerned with the existence of weak solutions to a BN system. The proof follows the lines in [14]. There, the author proved the analogous result for an isentropic pressure law. Since we are dealing with a different pressure law, we have to adjust the proof. The precise statement reads as follows:

Theorem A.1. *Let $T_0 > 0$ and $\rho_0, \alpha_0 \in L^\infty(\mathbb{T})$ with*

$$\alpha_0(x) \geq 0, \quad M_0^{-1} \leq \rho_0(x) \leq M_0 \quad \text{for a.e. } x \in \mathbb{T}$$

for some positive constant M_0. Assume that (P,γ) is admissible and let $\pi \in L^\infty(0,T_0;L^\infty(\mathbb{T}))$ and $u \in L^\infty(0,T_0;L^2(\mathbb{T})) \cap L^2(0,T_0;H^1(\mathbb{T}))$ with $\partial_x u \in L^1(0,T_0;L^\infty(\mathbb{T}))$.

Then there exist some time $T_1 \in (0,T_0]$ and $\alpha, \rho \in L^\infty(0,T_1;L^\infty(\mathbb{T})) \cap C([0,T_1];L^1(\mathbb{T}))$, such that

$$\begin{cases} \partial_t \alpha + u\partial_x \alpha = \alpha(P_\gamma(\rho) - \pi), \\ \partial_t \rho + \partial_x(\rho u) = \rho(\pi - P_\gamma(\rho)) \end{cases} \tag{A1}$$

hold in $\mathcal{D}'((0,T_1)\times\mathbb{T})$ and

$$\alpha(0,\cdot) = \alpha_0, \quad \rho(0,\cdot) = \rho_0 \quad \text{a.e. in } \mathbb{T}. \tag{A2}$$

Moreover, the solution (α,ρ) satisfies

$$\alpha(t,x) \geq 0, \quad (2M_0)^{-1} \leq \rho(t,x) \leq 2M_0 \quad \text{for a.e. } (t,x) \in [0,T_1]\times\mathbb{T}. \tag{A3}$$

In order to prove this theorem, we first investigate the linearized equation corresponding to (A1). The following result follows from the results in [28].

Lemma A.2. *Let $T_0 > 0$ and $\alpha_0, \rho_0 \in L^\infty(\mathbb{T})$. Let $u \in L^\infty(0,T_0;L^\infty(\mathbb{T})) \cap L^2(0,T_0;H^1(\mathbb{T}))$ with $\partial_x u \in L^1(0,T_0;L^\infty(\mathbb{T}))$.*

Then, for any $f, g \in L^\infty(0,T_0;L^\infty(\mathbb{T}))$, there exist unique functions

$$\alpha, \rho \in L^\infty(0,T_0;L^\infty(\mathbb{T})) \cap C([0,T_0];L^1(\mathbb{T}))$$

that satisfy

$$\begin{cases} \partial_t \alpha + u\partial_x \alpha = \alpha f, \\ \partial_t \rho + \partial_x(\rho u) = \rho g \end{cases} \tag{A4}$$

in $\mathcal{D}'((0,T_0)\times\mathbb{T})$ and

$$\alpha(0,\cdot) = \alpha_0, \quad \rho(0,\cdot) = \rho_0 \quad \text{a.e. in } \mathbb{T}. \tag{A5}$$

Proof. See Proposition II.1 and Corollary II.2 in [28]. □

Lemma A.2 gives rise to a solution operator for any $T \in (0,T_0]$ via

$$\mathcal{L}_T \colon L^\infty(0,T;L^\infty(\mathbb{T}))^2 \to L^\infty(0,T;L^\infty(\mathbb{T}))^2, \quad (f,g) \mapsto (\alpha,\rho),$$

where α, ρ is the unique weak solution to problem (A4), (A5). This solution operator has the following properties:

Lemma A.3. *Under the assumptions of Lemma A.2, there exists some $C_0 > 0$ only depending on*

$$||\alpha_0||_{L^\infty(\mathbb{T})}, ||\rho_0||_{L^\infty(\mathbb{T})}, ||\partial_x u||_{L^1(0,T_0;L^\infty(\mathbb{T}))},$$

such that for any $M > 0$ and any $T \in (0,T_0]$, we have

$$\mathcal{L}_T(B(0,M)) \subseteq B(0, C_0 \exp(TM)), \tag{A6}$$

where $B(0,M) \subseteq L^\infty(0,T;L^\infty(\mathbb{T}))^2$ denotes the ball of radius M in $L^\infty(0,T;L^\infty(\mathbb{T}))^2$.

Moreover, if $T \leq \min(T_0,1)$, we have for any $(\overline{f},\overline{g}), (\hat{f},\hat{g}) \in B(0,M)$ the inequality

$$||\mathcal{L}_T(\overline{f},\overline{g}) - \mathcal{L}_T(\hat{f},\hat{g})||_{L^\infty(0,T;L^1(\mathbb{T}))} \leq L_{\mathcal{L}}(T,M)||(\overline{f},\overline{g}) - (\hat{f},\hat{g})||_{L^\infty(0,T;L^1(\mathbb{T}))}, \tag{A7}$$

where

$$L_{\mathcal{L}}(T,M) := TC_0 \exp(2M + C_0).$$

Proof. The proof follows completely the lines of Lemma 2 in [14] without any further adjustments, therefore we will not repeat it here. □

As a next step, we compute the right hand sides of (A4) via the nonlinear mapping

$$\mathcal{NL}_T\colon L^\infty(0,T;L^\infty(\mathbb{T})) \to L^\infty(0,T;L^\infty(\mathbb{T})), \quad (\alpha,\rho) \mapsto (f,g),$$

where

$$f := P_\gamma(\rho) - \pi, \quad g := \pi - P_\gamma(\rho).$$

Notice that, since (P,γ) is admissible (see condition 2 in Definition 2.1), there exist two constants $C_P \in (0,\infty)$ and $\beta \in [2,\infty)$, such that we have

$$P'(r) \le C_P + C_P r^{\beta-1}, \quad P(r) \le C_P + C_P r^\beta \quad \forall\, r \in [0,\infty). \tag{A8}$$

We will fix these constants for the rest of this section. We obtain the following properties for the mapping $\mathcal{NL}_T$, which are exactly the ones from [14] adjusted to our pressure function:

Lemma A.4. *Under the assumptions of Lemma A.1, there exists some $C_1 > 0$ only depending on*

$$||\pi||_{L^\infty(0,T_0;L^\infty(\mathbb{T}))}, \gamma, C_P,$$

such that for any $T \in (0,T_0]$, and any $R > 0$, we have

$$\mathcal{NL}_T(B(0,R)) \subseteq B(0, C_1(1+R^2+R^\beta)), \tag{A9}$$

where $B(0,R)$ denotes the ball of radius R in $L^\infty(0,T;L^\infty(\mathbb{T}))^2$.

Moreover, we have for any $(\overline{\alpha},\overline{\rho}), (\hat{\alpha},\hat{\rho}) \in B(0,R)$ the inequality

$$||\mathcal{NL}_T(\overline{\alpha},\overline{\rho}) - \mathcal{NL}_T(\hat{\alpha},\hat{\rho})||_{L^\infty(0,T;L^1(\mathbb{T}))} \le L_{\mathcal{NL}}(R)||(\overline{\alpha},\overline{\rho}) - (\hat{\alpha},\hat{\rho})||_{L^\infty(0,T;L^1(\mathbb{T}))}, \tag{A10}$$

where

$$L_{\mathcal{NL}}(R) := C_1(1+R+R^{\beta-1}).$$

Proof. For $(\alpha,\rho) \in B(0,R)$ and $\mathcal{NL}_T(\alpha,\rho) = (f,g)$, i.e.

$$f = \pi - P_\gamma(\rho), \quad g = P_\gamma(\rho) - \pi,$$

we estimate using the relation (A8)

$$\begin{aligned}
||f||_{L^\infty(0,T,L^\infty(\mathbb{T}))} &\le ||\pi||_{L^\infty(0,T_0;L^\infty(\mathbb{T}))} + ||P(\rho)||_{L^\infty(0,T,L^\infty(\mathbb{T}))} + \frac{\gamma}{2}||\rho||^2_{L^\infty(0,T;L^\infty(\mathbb{T}))} \\
&\le ||\pi||_{L^\infty(0,T_0;L^\infty(\mathbb{T}))} + C_P + C_P||\rho||^\beta_{L^\infty(0,T;L^\infty(\mathbb{T}))} \\
&\quad + \frac{\gamma}{2}||\rho||^2_{L^\infty(0,T;L^\infty(\mathbb{T}))} \\
&\le \max\Big\{||\pi||_{L^\infty(0,T_0;L^\infty(\mathbb{T}))} + C_P, \frac{\gamma}{2}\Big\}\big(1+R^2+R^\beta\big).
\end{aligned}$$

The estimate for g is completely the same. To verify assertion (A10), we fix $(\overline{\alpha},\overline{\rho}), (\hat{\alpha},\hat{\rho}) \in B(0,R)$ and denote

$$\mathcal{NL}_T(\overline{\alpha},\overline{\rho}) = (\overline{f},\overline{g}), \quad \mathcal{NL}_T(\hat{\alpha},\hat{\rho}) = (\hat{f},\hat{g}).$$

Then we estimate for a.e. $t \in [0,T]$ using the mean value theorem and relation (A8)

$$\begin{aligned}
\int_{\mathbb{T}} |\overline{f}(t,x) - \hat{f}(t,x)|\,\mathrm{d}x &= \int_{\mathbb{T}} |P_\gamma(\overline{\rho}(t,x)) - P_\gamma(\hat{\rho}(t,x))|\,\mathrm{d}x \\
&\le \max_{\lambda\in[0,R]}\big\{|P'_\gamma(\lambda)|\big\}\int_{\mathbb{T}} |\overline{\rho}(t,x) - \hat{\rho}(t,x)|\,\mathrm{d}x \\
&\le \big(C_P + C_P R^{\beta-1} + \gamma R\big)\int_{\mathbb{T}} |\overline{\rho}(t,x) - \hat{\rho}(t,x)|\,\mathrm{d}x \\
&\le \max\{C_P,\gamma\}\big(1+R+R^{\beta-1}\big)\int_{\mathbb{T}} |\overline{\rho}(t,x) - \hat{\rho}(t,x)|\,\mathrm{d}x.
\end{aligned}$$

The estimate for the difference between $\overline{g}$ and $\hat{g}$ is completely the same. Hence, setting

$$C_1 := \max\{||\pi||_{L^\infty(0,T_0;L^\infty(\mathbb{T}))} + C_p, \gamma\}$$

yields the claim. □

Finally, we provide the proof of Theorem A.1 via a fixed point argument.

Proof of Theorem A.1. For $T \in (0, T_0]$, we define the map

$$\Phi_T := \mathcal{L}_T \circ \mathcal{NL}_T\colon L^\infty(0,T,L^\infty(\mathbb{T}))^2 \to L^\infty(0,T;L^\infty(\mathbb{T}))^2.$$

By definition of $\mathcal{L}_T$ and $\mathcal{NL}_T$, any fixed point $(\alpha,\rho) \in L^\infty(0,T;L^\infty(\mathbb{T}))^2$ of Φ_T satisfies $\alpha, \rho \in C([0,T];L^1(\mathbb{T}))$ and (A1), (A2), so we are done with the first part of Theorem A.1, if we show that there exists some $T_1 \in (0,T_0]$, such that the map Φ_{T_1} admits a fixed point. To do this, we define

$$R_\star := 2C_0, \quad T_\star := \min\left(\frac{\ln(2)}{C_1(1+(2C_0)^\beta + (2C_0)^2)}, 1, T_0\right),$$

where C_0 and C_1 are the constants provided by Lemma A.3 and Lemma A.4, respectively. With (A9) and (A6) we obtain

$$\mathcal{L}_{T_\star}(\mathcal{NL}_{T_\star}(B(0,R_\star))) \subseteq B(0, C_0\exp(T_\star C_1(1+R_\star^\beta + R_\star^2))).$$

Since

$$\exp(T_\star C_1(1+R_\star^\beta+R_\star^2)) \le \exp\left(\frac{C_1(1+(2C_0)^\beta+(2C_0)^2)\ln(2)}{C_1(1+(2C_0)^\beta+(2C_0)^2)}\right) = 2,$$

we have shown that

$$\Phi_{T_\star}(B(0,R_\star)) \subseteq B(0,R_\star),$$

and in particular

$$\Phi_T(B(0,R_\star)) \subseteq B(0,R_\star) \quad \forall T \in (0,T_\star]. \tag{A11}$$

By (A7) and (A10), we have for any $(\overline{\alpha},\overline{\rho}),(\hat{\alpha},\hat{\rho}) \in B(0,R_\star)$ and any $T\in(0,T_\star]$ that

$$\begin{aligned}
&||\Phi_T(\overline{\alpha},\overline{\rho}) - \Phi_T(\hat{\alpha},\hat{\rho})||_{L^\infty(0,T,L^1(\mathbb{T}))}\\
&\le L_{\mathcal{L}}(T, C_1(1+R_\star^2+R_\star^\beta))L_{\mathcal{NL}}(R_\star)||(\overline{\alpha},\overline{\rho}) - (\hat{\alpha},\hat{\rho})||_{L^\infty(0,T,L^1(\mathbb{T}))}\\
&= TC_0\exp(2C_1(1+R_\star^2+R_\star^\beta)+C_0)C_1(1+R_\star^2+R_\star^{\beta-1})||(\overline{\alpha},\overline{\rho}) - (\hat{\alpha},\hat{\rho})||_{L^\infty(0,T;L^1(\mathbb{T}))}.
\end{aligned}$$

Thus, we can choose $T_1 \in (0,T_\star]$ so small, such that

$$||\Phi_{T_1}(\overline{\alpha},\overline{\rho}) - \Phi_{T_1}(\hat{\alpha},\hat{\rho})||_{L^\infty(0,T_1,L^1(\mathbb{T}))} \le \frac{1}{2}||(\overline{\alpha},\overline{\rho}) - (\hat{\alpha},\hat{\rho})||_{L^\infty(0,T_1,L^1(\mathbb{T}))}. \tag{A12}$$

We construct a sequence in $B(0,R_\star)$ via the recursion

$$(\alpha_{n+1},\rho_{n+1}) := \Phi_{T_1}(\alpha_n,\rho_n) \quad \forall n \in \mathbb{N}_0,$$

where we consider for $n=0$ the initial data as functions in $B(0,R_\star)$ that are constant in time, so $(\alpha_0,\rho_0)(t) := (\alpha_0,\rho_0)$ for all $t\in[0,T_1]$. By virtue of (A11), we have

$$(\alpha_n,\rho_n) \in B(0,R_\star) \quad \forall n\in\mathbb{N}. \tag{A13}$$

Due to (A12), the sequence $(\alpha_n,\rho_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty(0,T_1,L^1(\mathbb{T}))$ and therefore there exists some $(\alpha,\rho)\in L^\infty(0,T_1,L^1(\mathbb{T}))$, such that

$$(\alpha_n,\rho_n) \longrightarrow (\alpha,\rho) \quad \text{in } L^\infty(0,T_1,L^1(\mathbb{T})).$$

Relation (A13) implies $(\alpha,\rho)\in L^\infty(0,T_1,L^\infty(\mathbb{T}))$. Finally we verify that (α,ρ) is a fixed point of the map Φ_{T_1}. We estimate for any $n\in\mathbb{N}$

$$\begin{aligned}
&||\Phi_{T_1}(\alpha,\rho) - (\alpha,\rho)||_{L^\infty(0,T_1;L^1(\mathbb{T}))}\\
&\le ||\Phi_{T_1}(\alpha,\rho) - \Phi_{T_1}(\alpha_n,\rho_n)||_{L^\infty(0,T_1;L^1(\mathbb{T}))} + ||\Phi_{T_1}(\alpha_n,\rho_n) - (\alpha,\rho)||_{L^\infty(0,T_1;L^1(\mathbb{T}))}
\end{aligned}$$

$$\leq \frac{1}{2}||(\alpha,\rho)-(\alpha_n,\rho_n)||_{L^\infty(0,T_1;L^1(\mathbb{T}))}+||(\alpha_{n+1},\rho_{n+1})-(\alpha,\rho)||_{L^\infty(0,T_1;L^1(\mathbb{T}))}.$$

For $n\to\infty$, the right-hand side of this inequality converges to zero and therefore

$$||\Phi_{T_1}(\alpha,\rho)-(\alpha,\rho)||_{L^\infty(0,T_1;L^1(\mathbb{T}))}=0,$$

which gives in particular $\Phi(\alpha,\rho)=(\alpha,\rho)$ a.e. in $(0,\tilde{T})\times\mathbb{T}$, so that (α,ρ) is a fixed point of Φ_{T_1}. To prove assertion (A3), we notice that the velocity field has the regularity $u\in L^1(0,T_0;W^{1,\infty}(\mathbb{T}))$. This provides us a Lipschitz continuous flow map

$$X\colon[0,T_0]\times[0,T_0]\times\mathbb{T}\to\mathbb{R}.$$

Then, it is well-known (see for instance [29] in a more general context), that we can represent α and ρ via

$$\alpha(t,x)=\alpha_0(X(0,t,x))\exp\bigg(\int_0^t f(s,X(s,t,x))\,\mathrm{d}s\bigg)$$

and

$$\rho(t,x)=\rho_0(X(0,t,x))\exp\bigg(\int_0^t g(s,X(s,t,x))\,\mathrm{d}s\bigg)$$

for almost all $(t,x)\in[0,T_1]\times\mathbb{T}$, where

$$f:=P_\gamma(\rho)-\pi,\quad g:=-\partial_x u+\pi-P_\gamma(\rho).$$

Since $\alpha_0\geq 0$ almost everywhere on $\mathbb{T}$, we conclude $\alpha\geq 0$ almost everywhere on $[0,T_1]\times\mathbb{T}$. Moreover, we estimate the density using the bound on ρ_0 as

$$M_0^{-1}\exp\bigg(\int_0^t -||g(s,\cdot)||_{L^\infty(\mathbb{T})}\,\mathrm{d}s\bigg)\leq\rho(t,x)\leq M_0\exp\bigg(\int_0^t ||g(s,\cdot)||_{L^\infty(\mathbb{T})}\,\mathrm{d}s\bigg)$$

for almost all $(t,x)\in[0,T_1]\times\mathbb{T}$. Since $g\in L^1(0,T_1;L^\infty(\mathbb{T}))$, we can restrict T_1 if necessary to obtain assertion (A3). □

Appendix B A Uniqueness Result

The following uniqueness result is a variation of the uniqueness result provided in [12]. The proof follows a duality argument that can also be found in [29] for instance. The precise formulation of the uniqueness result reads as follows:

Theorem B.1. *Let $T>0$. Let $\mathbf{V}=(V_1,V_2)\in L^1(0,T;C_b(\mathbb{T}\times\mathbb{R}))^2$ and $g\in L^1(0,T;C_b(\mathbb{T}\times\mathbb{R}))$ with*

$$\partial_1 V_1\in L^1(0,T;L^\infty(\mathbb{T}\times\mathbb{R})),\quad \partial_2 V_1=0 \textit{ a.e. in } (0,T)\times\mathbb{T}\times\mathbb{R},\tag{B14}$$

$$\int_0^T\int_{\mathbb{T}}\sup_{x_2\in\mathbb{R}}|\partial_1 V_2(t,x_1,x_2)|\,\mathrm{d}x_1\,\mathrm{d}t<\infty,\ \partial_2 V_2\in L^1(0,T;L^\infty(\mathbb{T}\times\mathbb{R})),\tag{B15}$$

$$\int_0^T\int_{\mathbb{T}}\sup_{x_2\in\mathbb{R}}|\partial_1 g(t,x_1,x_2)|\,\mathrm{d}x_1\,\mathrm{d}t<\infty,\ \partial_2 g\in L^1(0,T;L^\infty(\mathbb{T}\times\mathbb{R})).\tag{B16}$$

Assume that $\nu\in C_{\mathrm{w}}([0,T];\mathcal{M}(\mathbb{T}\times\mathbb{R}))$ satisfies

$$\sup_{t\in[0,T_0]}||\nu(t)||_{\mathcal{M}(\mathbb{T}\times\mathbb{R})}<\infty,\quad \operatorname{spt}\big(\nu(t)\big)\subseteq\mathbb{T}\times K\quad\forall\,t\in[0,T],\tag{B17}$$

for some compact subset $K\subseteq\mathbb{R}$ and

$$\partial_t\nu+\operatorname{div}_{(x_1,x_2)}(\mathbf{V}\nu)+g\nu=0\quad \textit{in } \mathcal{D}'((0,T)\times\mathbb{T}\times\mathbb{R}),\tag{B18}$$

with $\nu(0)=0$.

Then we have $\nu\equiv 0$.

Proof. We proceed as in [12] with a duality argument. The measure ν satisfying (B18), means that for any $\varphi \in C_c^\infty((0,T)\times\mathbb{T}\times\mathbb{R})$, we have

$$\int_0^T \langle \nu(t), \partial_t\varphi + \mathbf{V}\cdot\nabla\varphi - g\varphi\rangle \,\mathrm{d}t = 0.$$

By a standard regularization argument, this implies that for any $t\in[0,T]$ and any $\varphi \in C_c^1([0,t]\times\mathbb{T}\times\mathbb{R})$, we have

$$\langle \nu(t), \varphi(t,\cdot)\rangle = \int_0^t \langle \nu(s), \partial_t\varphi + \mathbf{V}\cdot\nabla\varphi - g\varphi\rangle \,\mathrm{d}s = 0. \tag{B19}$$

We take a standard mollifier on $\mathbb{R}$, i.e.

$$\omega \in C_c^\infty((0,1)), \quad \int_{\mathbb{R}} \omega = 1, \quad 0\le\omega\le 1,$$
$$\omega_\varepsilon(x) := \frac{1}{\varepsilon}\omega(\frac{x}{\varepsilon}) \quad \text{for } \varepsilon>0,\ x\in\mathbb{R},$$

and mollify $\mathbf{V}$ and g via

$$V_1^\varepsilon(t,x,\xi) = \int_{\mathbb{R}}\int_{\mathbb{R}} \omega_\varepsilon(x-y)\omega_\varepsilon(\xi-\eta)V_1(t,y,\eta)\,\mathrm{d}y\,\mathrm{d}\eta,$$
$$V_2^\varepsilon(t,x,\xi) := \int_{\mathbb{R}}\int_{\mathbb{R}} \omega_{\sqrt{\varepsilon}}(x-y)\omega_{\sqrt{\varepsilon}}(\xi-\eta)V_2(t,y,\eta)\,\mathrm{d}x\,\mathrm{d}\eta,$$
$$g^\varepsilon(t,x,\xi) = \int_{\mathbb{R}}\int_{\mathbb{R}} \omega_{\sqrt{\varepsilon}}(x-y)\omega_{\sqrt{\varepsilon}}(\xi-\eta)g(t,y,\eta)\,\mathrm{d}y\,\mathrm{d}\eta.$$

Then we have for any $k\in\mathbb{N}$ that

$$\mathbf{V}^\varepsilon := (V_1^\varepsilon, V_2^\varepsilon), g^\varepsilon \in L^1(0,T;C_b^k(\mathbb{T}\times\mathbb{R})).$$

Using that $g\in L^1(0,T;C_b(\mathbb{T}\times\mathbb{R}))$, we have for almost all $t\in[0,T]$ the convergence

$$||g^\varepsilon(t) - g(t)||_{L^\infty(\mathbb{T}\times K)} \to 0 \quad \text{for } \varepsilon\to 0.$$

Together with

$$\int_0^T ||g_\varepsilon(t)||_{L^\infty(\mathbb{T}\times\mathbb{R})}\,\mathrm{d}t \le \int_0^T ||g(t)||_{L^\infty(\mathbb{T}\times\mathbb{R})}\,\mathrm{d}t, \tag{B20}$$

this implies using Lebesgue's dominated convergence theorem

$$||g^\varepsilon - g||_{L^1(0,T;L^\infty(\mathbb{T}\times K))} \to 0 \quad \text{for } \varepsilon\to 0. \tag{B21}$$

By the same reasoning, we obtain

$$||\mathbf{V}^\varepsilon - \mathbf{V}||_{L^1(0,T;L^\infty(\mathbb{T}\times K))} \to 0 \quad \text{for } \varepsilon\to 0. \tag{B22}$$

From now on, we denote by $\mathcal{C}$ a generic constant that may vary from line to line but does not depend on ε. By virtue of $\nabla V_1 \in L^1(0,T;L^\infty(\mathbb{T}\times\mathbb{R}))$, we obtain for almost all $t\in[0,T]$ and any $x,\xi\in\mathbb{R}$

$$\begin{aligned}|V_1^\varepsilon(t,x,\xi) - V_1(t,x,\xi)| &\le \int_{B_\varepsilon(x)}\int_{B_\varepsilon(\xi)} \omega_\varepsilon(x-y)\omega_\varepsilon(\xi-\eta)|V_1(t,y,\eta) - V_1(t,x,\xi)|\,\mathrm{d}y\,\mathrm{d}\eta \\ &\le \sqrt{2}\varepsilon||\nabla V_1(t)||_{L^\infty(\mathbb{T}\times\mathbb{R})}\int_{B_\varepsilon(x)}\int_{B_\varepsilon(\xi)} \omega_\varepsilon(x-y)\omega_\varepsilon(\xi-\eta)\,\mathrm{d}y\,\mathrm{d}\eta \\ &= \sqrt{2}||\nabla V_1(t)||_{L^\infty(\mathbb{T}\times\mathbb{R})}\varepsilon.\end{aligned}$$

Thus,

$$||V_1^\varepsilon - V_1||_{L^1(0,T;L^\infty(\mathbb{T}\times\mathbb{R}))} \le \mathcal{C}\varepsilon. \tag{B23}$$

We calculate for almost all $t \in [0,T]$ and any $x, \xi \in \mathbb{T} \times \mathbb{R}$:

$$\begin{aligned} |\partial_2 g^\varepsilon(t,x,\xi)| &= \left| \int_\mathbb{R} \int_\mathbb{R} \omega_{\sqrt{\varepsilon}}(x-y)\omega_{\sqrt{\varepsilon}}(\xi-\eta)\partial_2 g(t,y,\eta)\,\mathrm{d}y\,\mathrm{d}\eta \right| \\ &\le ||\partial_2 g(t)||_{L^\infty(\mathbb{T}\times\mathbb{R})} \int_\mathbb{R}\int_\mathbb{R} \omega_{\sqrt{\varepsilon}}(x-y)\omega_{\sqrt{\varepsilon}}(\xi-\eta)\,\mathrm{d}y\,\mathrm{d}\eta \\ &= ||\partial_2 g||_{L^\infty(\mathbb{T}\times\mathbb{R})}, \end{aligned}$$

so that, using $\partial_2 g \in L^1(0,T;L^\infty(\mathbb{T}\times\mathbb{R}))$,

$$||\partial_2 g^\varepsilon||_{L^1(0,T;L^\infty(\mathbb{T}\times\mathbb{R}))} \le \mathcal{C}. \tag{B24}$$

By the same reasoning, we observe

$$||\partial_2 V_2^\varepsilon||_{L^1(0,T;L^\infty(\mathbb{T}\times\mathbb{R}))} + ||\nabla V_1^\varepsilon||_{L^1(0,T;L^\infty(\mathbb{T}\times\mathbb{R})} \le \mathcal{C}. \tag{B25}$$

Moreover, we have for almost all $t \in [0,T]$ and any $(x,\xi) \in \mathbb{T}\times\mathbb{R}$ that

$$\begin{aligned} |\partial_1 V_2^\varepsilon(t,x,\xi)| &= \left| \int_\mathbb{R}\int_\mathbb{R} \omega_{\sqrt{\varepsilon}}(x-y)\omega_{\sqrt{\varepsilon}}(\xi-\eta)\partial_1 V_2(t,y,\eta)\,\mathrm{d}y\,\mathrm{d}\eta \right| \\ &\le \frac{1}{\sqrt{\varepsilon}} ||\omega||_{L^\infty(\mathbb{R})} \int_\mathbb{R}\int_\mathbb{T} \omega_{\sqrt{\varepsilon}}(\xi-\eta)|\partial_1 V_2(t,y,\eta)|\,\mathrm{d}y\,\mathrm{d}\eta \\ &\le \frac{1}{\sqrt{\varepsilon}} \int_\mathbb{R} \omega_{\sqrt{\varepsilon}}(\xi-\eta)\,\mathrm{d}\eta \int_\mathbb{T} \sup_{\eta\in\mathbb{R}} |\partial_1 V_2(t,y,\eta)|\,\mathrm{d}y \\ &= \frac{1}{\sqrt{\varepsilon}} \int_\mathbb{T} \sup_{\eta\in\mathbb{R}} |\partial_1 V_2(t,y,\eta)|\,\mathrm{d}y. \end{aligned}$$

Thus, by using (B15),

$$||\partial_1 V_2^\varepsilon||_{L^1(0,T;L^\infty(\mathbb{T}\times\mathbb{R}))} \le \frac{1}{\sqrt{\varepsilon}} \int_0^T \int_\mathbb{T} \sup_{\eta\in\mathbb{R}} |\partial_1 V_2(t,y,\eta)|\,\mathrm{d}y\,\mathrm{d}t \le \frac{\mathcal{C}}{\sqrt{\varepsilon}}. \tag{B26}$$

By the same tokens we deduce from (B16) that

$$||\partial_1 g^\varepsilon||_{L^1(0,T;L^\infty(\mathbb{T}\times\mathbb{R}))} \le \frac{\mathcal{C}}{\sqrt{\varepsilon}}. \tag{B27}$$

Now, we fix some $\varphi^\sharp \in C_c^\infty(\mathbb{T}\times\mathbb{R})$ and consider for $t \in [0,T]$ the backward convection problem

$$\begin{cases} \partial_t \varphi^\varepsilon + \mathbf{V}^\varepsilon \cdot \nabla\varphi^\varepsilon = g^\varepsilon \varphi^\varepsilon & \text{in } (0,t)\times\mathbb{T}\times\mathbb{R}, \\ \varphi^\varepsilon(t,\cdot) = \varphi^\sharp & \text{in } \mathbb{T}\times\mathbb{R}. \end{cases} \tag{B28}$$

By well-known results on advection equations, (B28) admits a solution φ^ε with regularity

$$\varphi^\varepsilon \in W^{1,1}([0,t]; C_b^k(\mathbb{T}\times\mathbb{R})) \quad \forall k \in \mathbb{N}.$$

Moreover, since $\varphi^\sharp$ has compact support, there exists some compact set $L \subseteq \mathbb{R}$ such that

$$\operatorname{spt}\varphi^\varepsilon(s,\cdot,\cdot) \subseteq L \quad \forall\, s \in [0,t].$$

By a standard regularization argument via time-mollification, we are allowed to use φ^ε as a test function in (B19). We obtain

$$\begin{aligned} \langle \nu(t), \varphi^\sharp \rangle &= \int_0^t \langle \nu(s), \partial_t\varphi^\varepsilon + \mathbf{V}\cdot\nabla\varphi^\varepsilon - g\varphi^\varepsilon \rangle\,\mathrm{d}s \\ &= \int_0^t \langle \nu(s), (\mathbf{V}-\mathbf{V}^\varepsilon)\cdot\nabla\varphi^\varepsilon - (g-g^\varepsilon)\varphi^\varepsilon \rangle\,\mathrm{d}s, \end{aligned} \tag{B29}$$

where we have used (B28) in the last equality. Using (B17), we estimate

$$\langle \nu(t), \varphi^\sharp \rangle \le \mathcal{C}(I_1 + I_2 + I_3)$$

with

$$I_1 = \int_0^t ||(V_1 - V_1^\varepsilon)\partial_1\varphi^\varepsilon||_{L^\infty(\mathbb{T}\times K)}\,\mathrm{d}s,$$
$$I_2 = \int_0^t ||(V_2 - V_2^\varepsilon)\partial_2\varphi^\varepsilon||_{L^\infty(\mathbb{T}\times K)}\,\mathrm{d}s,$$
$$I_3 = \int_0^t ||(g - g^\varepsilon)\varphi^\varepsilon||_{L^\infty(\mathbb{T}\times K)}\,\mathrm{d}s.$$

Applying the classical maximum principle for the advection equation (B28) yields in combination with (B20) for any $s \in [0,t]$

$$||\varphi^\varepsilon(s)||_{L^\infty(\mathbb{T}\times\mathbb{R})} \le ||\varphi^\sharp||_{L^\infty(\mathbb{T}\times\mathbb{R})} \exp\left(\int_0^t ||g^\varepsilon(s)||_{L^\infty(\mathbb{T}\times\mathbb{R})}\,\mathrm{d}s\right) \le \mathcal{C}, \tag{B30}$$

so that with the convergence (B21)

$$|I_3| \le C\int_0^t ||g(s) - g^\varepsilon(s)||_{L^\infty(\mathbb{T}\times K)}\mathrm{d}s \to 0 \quad \text{for } \varepsilon \to 0.$$

For I_1, I_2, we need estimates for $\varphi_1 := \partial_1\varphi^\varepsilon, \varphi_2 := \partial_2\varphi^\varepsilon$. Differentiating (B28) with respect to the second spatial variable, we notice that φ_2^ε satisfies the following advection problem

$$\begin{cases} \partial_t\varphi_2^\varepsilon + \mathbf{V}^\varepsilon\cdot\nabla\varphi_2^\varepsilon = (\partial_2 g^\varepsilon)\varphi^\varepsilon + (g^\varepsilon - \partial_2 V_2^\varepsilon)\varphi_2^\varepsilon & \text{in } (0,t)\times\mathbb{T}\times\mathbb{R},\\ \varphi_2^\varepsilon(t) = \partial_2\varphi^\sharp & \text{in } \mathbb{T}\times\mathbb{R}. \end{cases}$$

Applying again a classical maximum principle for this advection problem, we obtain for any $s \in [0,t]$

$$\begin{aligned} ||\varphi_2^\varepsilon(s)||_{L^\infty(\mathbb{T}\times\mathbb{R})} &\le \left(||\partial_2\varphi^\sharp||_{L^\infty(\mathbb{T}\times\mathbb{R})} + \int_0^t ||\partial_2 g^\varepsilon(\tau)||_{L^\infty(\mathbb{T}\times\mathbb{R})}||\varphi(\tau)||_{L^\infty(\mathbb{T}\times\mathbb{R})}\,\mathrm{d}\tau\right)\\ &\qquad\times\exp\left(\int_0^t ||g^\varepsilon(\tau)||_{L^\infty(\mathbb{T}\times\mathbb{R})} + ||\partial_2 V_2^\varepsilon(\tau)||_{L^\infty(\mathbb{T}\times\mathbb{R})}\,\mathrm{d}\tau\right)\\ &\le \mathcal{C}. \end{aligned} \tag{B31}$$

Here, we have used (B20), (B24), (B25) and (B30). From relation (B31) we deduce with the convergence (B22)

$$|I_2| \le \mathcal{C}\int_0^t ||V_2(s) - V_2^\varepsilon(s)||_{L^\infty(\mathbb{T}\times K)}\,\mathrm{d}s \to 0 \quad \text{for } \varepsilon \to 0.$$

Differentiating (B28) with respect to the first variable, we infer that φ_1^ε satisfies the following advection problem:

$$\begin{cases} \partial_t\varphi_1^\varepsilon + \mathbf{V}^\varepsilon\cdot\nabla\varphi_1^\varepsilon = (\partial_1 g^\varepsilon)\varphi^\varepsilon - (\partial_1 V_2^\varepsilon)\varphi_2^\varepsilon + (g^\varepsilon - \partial_1 V_1^\varepsilon)\varphi_1^\varepsilon & \text{in } (0,t)\times\mathbb{T}\times\mathbb{R},\\ \varphi_1(t) = \partial_1\varphi^\sharp & \text{in } \mathbb{T}\times\mathbb{R}. \end{cases}$$

Applying again the maximum principle for this advection problem, we infer for any $s \in [0,t]$ that

$$\begin{aligned} &||\varphi_1^\varepsilon(s)||_{L^\infty(\mathbb{T}\times\mathbb{R})}\\ &\le \left(||\partial_1\varphi^\sharp||_{L^\infty(\mathbb{T}\times\mathbb{R})} + \int_0^t ||(\partial_1 g^\varepsilon)\varphi^\varepsilon||_{L^\infty(\mathbb{T}\times\mathbb{R})} + ||(\partial_1 V_2^\varepsilon)\varphi_2^\varepsilon||_{L^\infty(\mathbb{T}\times\mathbb{R})}\,\mathrm{d}\tau\right)\\ &\quad\times\exp\left(\int_0^t (||g^\varepsilon||_{L^\infty(\mathbb{T}\times\mathbb{R})} + ||\partial_1 V_1^\varepsilon||_{L^\infty(\mathbb{T}\times\mathbb{R})})\,\mathrm{d}\tau\right)\\ &\le \frac{\mathcal{C}}{\sqrt{\varepsilon}}, \end{aligned}$$

where we have used (B20), (B26), (B27), (B30) and (B31). Using (B23), we obtain finally

$$|I_1| \leq \frac{\mathcal{C}}{\sqrt{\varepsilon}} \int_0^t ||V_1(s) - V_1^\varepsilon(s)||_{L^\infty(\mathbb{T}\times K)}\, \mathrm{d}s \leq \frac{C}{\sqrt{\varepsilon}}\varepsilon \to 0 \quad \text{for } \varepsilon \to 0.$$

Overall, we have shown that (B29) yields in the limit $\varepsilon \to 0$:

$$\langle \nu(t), \varphi^\sharp \rangle = 0.$$

Since $\varphi^\sharp \in C_c^\infty(\mathbb{T}\times\mathbb{R})$ was arbitrary, we have shown

$$\langle \nu(t), \varphi \rangle = 0 \quad \forall\, \varphi \in C_c^\infty(\mathbb{T}\times\mathbb{R}).$$

Since $C_c^\infty(\mathbb{T}\times\mathbb{R})$ lies dense in $C_0^0(\mathbb{T}\times\mathbb{R})$ with respect to $||\cdot||_{L^\infty(\mathbb{T}\times\mathbb{R})}$, we have shown

$$\langle \nu(t), \varphi \rangle = 0 \quad \forall\, \varphi \in C_0^0(\mathbb{T}\times\mathbb{R}).$$

which means exactly $\nu(t) = 0$. Since $t \in [0, T]$ was arbitrary we conclude $\nu = 0$. □

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Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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