Supercharacter theories

for

Sylow p-subgroups

${}^3D_4^{syl}\left(q^3\right)$, $G_2^{syl}(q)$ and ${}^2G_2^{syl}\left(3^{2m+1}\right)$

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Abstract

Let $p = \operatorname{Char}\mathbb{F}_{q^3} = \operatorname{Char}\mathbb{F}_q \neq 2$. Motivated by N. Yan's, C. A. M. André and A. M. Neto's, and M. Jedlitschky's remarkable work on supercharacter theories for the Sylow p-subgroups of the Chevalley groups of type A, B, C and D (classical, untwisted), we study supercharacter theories for the following Sylow p-subgroups:

${}^3D_4^{syl}(q^3)$:	the Sylow p-subgroups of the Steinberg triality groups of type D_4 (classical, twisted)
$G_2^{syl}(q)$:	the Sylow p-subgroups of the Chevalley groups of type G_2 (exceptional, untwisted)
${}^2G_2^{syl}(3^{2m+1})$:	the Sylow 3-subgroups of the Ree groups of type G_2 (exceptional, twisted)

Embedding Sylow p-subgroups $G_2^{syl}(q)$ into ${}^3D_4^{syl}(q^3)$ of 8×8-matrices, we construct bijective 1-cocycles and non-degenerate bilinear forms (i.e. the monomial linearisations) for ${}^3D_4^{syl}(q^3)$ and for $G_2^{syl}(q)$. Then we obtain the monomial modules and classify the orbit modules. Consequently, we determine the desired supermodules, supercharacters and superclasses for ${}^3D_4^{syl}(q^3)$ and for $G_2^{syl}(q)$.

For the twisted groups ${}^2G_2^{syl}(3^{2m+1})$ which are subgroups of $G_2^{syl}(3^{2m+1})$, we construct monomial linearisations, obtain the monomial modules and classify the orbit modules. Then we determine the supermodules and supercharacter theories (containing supercharacters and superclasses) for ${}^2G_2^{syl}(3^{2m+1})$.

Moreover, we calculate the conjugacy classes, determine the irreducible characters by Clifford theory and establish the character tables of ${}^3D_4^{syl}(q^3)$, $G_2^{syl}(q)$ when $\operatorname{Char}\mathbb{F}_q > 3$ and ${}^2G_2^{syl}(3)$.

Finally, we also construct bijective 1-cocycles, determine monomial linearisations and obtain monomial modules for the following exceptional Sylow p-subgroups:

$E_6^{syl}(q)$:	the Sylow p-subgroups of the Chevalley groups of type E_6 (of 27×27-matrices)
$F_4^{syl}(q)$:	the Sylow p-subgroups of the Chevalley groups of type F_4 (of 27×27-matrices)
${}^2E_6^{syl}(q^2)$:	the Sylow p-subgroups of the Steinberg groups of type E_6 (of 27×27-matrices, twisted)
$E_7^{syl}(q)$:	the Sylow p-subgroups of the Chevalley groups of type E_7 (of 56×56-matrices)

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Zusammenfassung

Sei $p = \mathrm{Char}\,\mathbb{F}_{q^3} = \mathrm{Char}\,\mathbb{F}_q \neq 2$. Motiviert durch N. Yan's, C. A. M. André und A. M. Neto's, und M. Jedlitschky's des bemerkenswerte Arbeit über Supercharaktertheorien für die p-Sylowuntergruppen der Chevalley-Gruppen vom Typ A, B, C und D (klassisch, ungetwistet). Wir untersuchen die Supercharaktertheorien für die folgenden p-Sylowuntergruppen:

${}^3D_4^{syl}(q^3)$:	die p-Sylowuntergruppen der Steinberg-Trialität-Gruppen vom Typ D_4 (klassisch, getwistet)
$G_2^{syl}(q)$:	die p-Sylowuntergruppen der Chevalley-Gruppen vom Typ G_2 (exzeptionell, ungetwistet)
${}^2G_2^{syl}(3^{2m+1})$:	die p-Sylowuntergruppen der Ree-Gruppen vom Typ G_2 (exzeptionell, getwistet)

Indem wir p-Sylowuntergruppen $G_2^{syl}(q)$ in ${}^3D_4^{syl}(q^3)$ von 8×8-Matrizen einbetten, konstruieren wir die bijektiven 1-Kozykeln und die nicht-ausgearteten Bilinearformen (d.h. die monomialen Linearisierungen) für ${}^3D_4^{syl}(q^3)$ und für $G_2^{syl}(q)$. Dadurch erhalten wir die monomialen Moduln und können die Orbitmoduln klassifizieren. Anschließend bestimmen wir die gewünschten Supermoduln, Supercharaktere and Superklassen für ${}^3D_4^{syl}(q^3)$ und für $G_2^{syl}(q)$.

Für die getwisteten Gruppen ${}^2G_2^{syl}(3^{2m+1})$, die Untergruppen von $G_2^{syl}(3^{2m+1})$ sind, konstruieren wir ebenfalls die monomialen Linearisierungen, erhalten die monomialen Moduln und klassifizieren die Orbitmoduln. Dann bestimmen wir die Supermoduln und Supercharakter Theorien (mit Supercharakteren und Superklassen) für ${}^2G_2^{syl}(3^{2m+1})$.

Außerdem berechnen wir die Konjugiertenklassen, bestimmen die irreduziblen Charaktere mittels Clifford Theorie und geben die Charaktertafeln von ${}^3D_4^{syl}(q^3)$, $G_2^{syl}(q)$ wenn $\mathrm{Char}\,\mathbb{F}_q > 3$ und ${}^2G_2^{syl}(3)$ an.

Schließlich konstruieren wir auch die bijektiven 1-Kozykeln, bestimmen die monomialen Linearisierungen und erhalten die monomialen Moduln für die folgenden exzeptionellen p-Sylowuntergruppen:

$E_6^{syl}(q)$:	die p-Sylowuntergruppen der Chevalley-Gruppen vom Typ E_6 (27×27-Matrizen)
$F_4^{syl}(q)$:	die p-Sylowuntergruppen der Chevalley-Gruppen vom Typ F_4 (27×27-Matrizen)
${}^2E_6^{syl}(q^2)$:	die p-Sylowuntergruppen der Steinberg-Gruppen vom Typ E_6 (27×27-Matrizen, getwistet)
$E_7^{syl}(q)$:	die p-Sylowuntergruppen der Chevalley-Gruppen vom Typ E_7 (56×56-Matrizen)

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Introduction

Let p be a fixed prime, $\mathbb{N}^*$ the set $\{1, 2, 3, \dots\}$ of positive integers, $\mathbb{N} = \mathbb{N}^* \cup \{0\}$, $q := p^k$ with $k \in \mathbb{N}^*$, $\mathbb{F}_q$ the finite field with q elements and $A_n(q)$ ($n \in \mathbb{N}^*$) the group of upper unitriangular $n \times n$-matrices (i.e. upper triangular matrices, with 1s on the main diagonal) with entries in $\mathbb{F}_q$. Then $A_n(q)$ is a Sylow p-subgroup of the general linear group $GL_n(q)$ of invertible $n \times n$-matrices over $\mathbb{F}_q$.

It is an open question to determine the conjugacy classes of $A_n(q)$ for all n and q. Higman's conjecture [Hig60] is that for a fixed n, the number of conjugacy classes of $A_n(q)$ is determined by a polynomial in q with integral coefficients depending on n. G. Lehrer [Leh74] refined Higman's conjecture: the number of irreducible characters of $A_n(q)$ of degree q^c ($c \in \mathbb{N}$) is an integer polynomial in q. I. M. Isaacs [Isa95] proved that the degrees of complex irreducible characters of $\mathbb{F}_q$-algebra groups are powers of q. Then I. M. Isaacs [Isa07] gave a strengthened form of Lehrer's conjecture: the number of irreducible characters of degree q^c is some polynomial in $(q-1)$ with non-negative integeral coefficients. A. Vera-López and J. M. Arregi [VLA03] proved Higman's conjecture for $n \leq 13$. A. Evseev [Evs11] computed the number of irreducible characters of $A_n(q)$ for $n \leq 13$ which confirmed Isaacs' conjecture. Recently, I. Pak and A. Soffer [PS15] verified Higman's conjecture for $n \leq 16$.

Based on Kirillov's orbit method, C.A.M. André [And95] and later but independently N. Yan [Yan10] determined the André-Yan supercharacter theory for $A_n(q)$ which is an approximation to the character theory. P. Diaconis and I.M. Isaacs [DI08] introduced the notion of *supercharacter theory* for an arbitrary finite group, and studied supercharacter theories for algebra groups. Roughly, a supercharacter theory replaces irreducible characters by supercharacters, conjugacy classes by superclasses, irreducible modules by supermodules. In such a way, a *supercharacter table* is constructed as a replacement for a character table. E. Marberg [Mar11] constructed a family of orthogonal characters of an algebra group which decomposed the supercharacters in [DI08]. Q. Guo [Guo16] determined the irreducible $A_n(q)$-constituents of the permutation module of $GL_n(q)$ on the cosets of a maximal parabolic subgroup of $GL_n(q)$. R. Dipper and Q. Guo [DG15] obtained a lower bound for the degrees of irreducible constituents of André-Yan supercharacters.

C.A.M. André and A.M. Neto [AN06, AN09a, AN09b] studied the supercharacter theories for the Sylow p-subgroups of untwisted Chevalley groups of types B_n, C_n and D_n (i.e. the classical finite groups: the odd orthogonal groups, the symplectic groups and the even positive orthogonal groups, respectively). M. Jedlitschky introduced the *monomial linearisation* method for a finite group, as a result he decomposed the *André-Neto supercharacters* of Sylow p-subgroups (i.e. the unipotent even positive orthogonal groups) of Lie type D [Jed13]. Recently, C.A.M. André, P.J. Freitas and A.M. Neto [AFN15] extended the construction of [AN06, AN09b] to involutive algebra groups. S. Andrews [And15, And16] constructed supercharacter theories of finite unipotent groups in the orthogonal, symplectic and unitary types (i.e. the Sylow p-groups of untwisted Chevalley groups of types B_n and D_n, of type C_n, and of the twisted Chevalley groups of type ${}^2A_n$, respectively).

It is natural to consider Higman's conjecture, Lehrer's conjecture and Isaacs' conjecture for the Sylow p-subgroups of other finite groups of Lie type. Let $G(q)$ be a finite group of Lie type, $U(q)$ a Sylow p-subgroup of $G(q)$, $k(U(q))$ the number of conjugacy classes, $\#\mathrm{Irr}(U(q))$ the number of all complex irreducible characters, $\#\mathrm{Irr}(U(q), q^c)$ the number of complex irreducible characters of degree q^c, $\#M(U(q))$ the number of pairwise orthogonal irreducible constituents of the regular $U(q)$-module $\mathbb{C}U(q)$. S. M. Goodwin, P. Mosch and G. Röhrle [GR09, GMR14, GMR16] obtained an algorithm and calculated $k(U(q))$ for $U(q)$ of rank at most 8, except E_8.

For the Sylow p-subgroup $D_4^{syl}(q)$ of the Chevalley group $D_4(q)$ of type D_4, F. Himstedt, T. Le and K. Magaard [HLM11] determined the complex irreducible characters and calculated $\#\mathrm{Irr}(D_4^{syl}(q), q^c)$. M. Jedlitschky [Jed13, Appendix A.3] decomposed the regular module $\mathbb{C}D_4^{syl}(q)$ with $p \neq 2$ into irreducible constituents, and obtained $\#\mathrm{Irr}(D_4^{syl}(q), q^c)$ and $\#\mathrm{Irr}(D_4^{syl}(q))$. S. M. Goodwin, T. Le and K. Magaard [GLM15] constructed the *generic character table* of $D_4^{syl}(q)$. Higman's conjecture, Lehrer's conjecture and Isaacs' conjecture are true for $D_4^{syl}(q)$.

For the Sylow p-subgroup $G_2^{syl}(q)$ $(p > 3)$ of the Chevalley group $G_2(q)$ of type G_2, F. Himstedt, T. Le and K. Magaard [HLM16] determined most irreducible characters (except $q^2 - 2q + 2$ linear characters) of $G_2^{syl}(q)$ by parameterizing *midafis* (minimal degree almost faithful irreducible characters).

S. M. Goodwin, T. Le, K. Magaard and A. Paolini [GLMP16] parameterized the irreducible characters of the Sylow p-subgroup $F_4^{syl}(q)$ $(p > 2)$ of the Chevalley group $F_4(q)$ of type F_4.

Let ${}^3D_4^{syl}(q^3)$ be a Sylow p-subgroup of the Steinberg triality group ${}^3D_4^{syl}(q^3)$. T. Le [Le13] constructed and counted all ordinary irreducible characters of ${}^3D_4^{syl}(q^3)$, mainly using Clifford theory.

In this thesis, we firstly determine supercharacter theories for the 3 types of groups as follows: the Sylow p-subgroup ${}^3D_4^{syl}(q^3)$ of the Steinberg triality group ${}^3D_4(q^3)$, the Sylow p-subgroup $G_2^{syl}(q)$ of the Chevalley group $G_2(q)$, and the Sylow 3-subgroup ${}^2G_2^{syl}(3^{2m+1})$ of the Ree group ${}^2G_2(3^{2m+1})$. Then we establish the character tables of ${}^3D_4^{syl}(q^3)$, $G_2^{syl}(q)$ when $\mathrm{Char}\,\mathbb{F}_q > 3$ and ${}^2G_2^{syl}(3)$. After that, we construct the monomial linearisations for Sylow p-subgroups of the Chevalley groups of type E_6, type F_4, the twisted type ${}^2E_6$ and type E_7 with the 1-cocycle approach.

An informal overview

Chapter 1: The construction of monomial modules

We define some notations and recall the construction of the monomial modules.

Chapter 2: Supercharacter theories for ${}^3D_4^{syl}(q^3)$

In *Section 2.1*, we determine the Sylow p-subgroup ${}^3D_4^{syl}(q^3)$ for the Steinberg triality group ${}^3D_4(q^3)$. The construction of the Sylow p-subgroup ${}^3D_4(q^3)$ depends on the structure constants. Thus we construct a Chevalley basis of the Lie algebra of Lie type D_4, and set the suitable structure constants, and obtain the Sylow p-subgroup ${}^3D_4^{syl}(q^3)$. The Sylow p-subgroup of Lie type D_4 is denoted by $D_4^{syl}(q)$ (see §2.1.1). The matrix sizes of $D_4^{syl}(q)$ and ${}^3D_4^{syl}(q^3)$ are 8×8. The relations of the groups are ${}^3D_4^{syl}(q^3) \leqslant D_4^{syl}(q^3) \leqslant A_8(q^3)$.

In *Section 2.2*, we construct a 1-cocycle f and determine a monomial linearisation $(f|_{{}^3D_4^{syl}(q^3)}, \kappa_q|_{V\times V})$ for ${}^3D_4^{syl}(q^3)$. This is one important result of this chapter. This method generalizes Jedlitschky's construction for the Sylow p-subgroup $D_4^{syl}(q)$ of type D_4. Then we obtain a classification of ${}^3D_4^{syl}(q^3)$-orbit modules. By the homomorphisms between the orbit modules, a new classification of ${}^3D_4^{syl}(q^3)$-orbit modules is established in which every ${}^3D_4^{syl}(q^3)$-orbit module is isomorphic to some hook-separated staircase module.

In *Section 2.3*, we determine the supercharacter theory for ${}^3D_4^{syl}(q^3)$ and establish the supercharacter table of ${}^3D_4^{syl}(q^3)$. This is the main result of Chapter 2. T. Le [Le13] constructed and counted all ordinary characters of ${}^3D_4^{syl}(q^3)$. We give more specific constructions of the irreducible characters of ${}^3D_4^{syl}(q^3)$, and establish the almost faithful irreducible characters (i.e. the family $\mathfrak{F}_6$) in a different way. Then we obtain the conjugacy classes of ${}^3D_4^{syl}(q^3)$, calculate the character table of ${}^3D_4^{syl}(q^3)$, and determine the relations between supercharacters and irreducible characters of ${}^3D_4^{syl}(q^3)$. The three statements are true, if we consider the analogue of Higman's conjecture, Lehrer's conjecture and Isaacs' conjecture of $A_n(q)$ for ${}^3D_4^{syl}(q^3)$.

Chapter 3: Supercharacter theories for $G_2^{syl}(q)$

In *Section 3.1*, we construct a Lie algebra $\mathcal{L}_{G_2}$ of type G_2 which is a subalgebra of $\mathcal{L}_{D_4}$, and determine the Sylow p-subgroup $G_2^{syl}(q)$ of the Chevalley group of type $\mathcal{L}_{G_2}$ over the field $\mathbb{F}_q$. The matrix size of $G_2^{syl}(q)$ is 8×8. The construction of the Lie algebra of type G_2 is motivated by the paper [HRT01]. In that paper [HRT01], R. B. Howlett, L. J. Rylands and D. E.Taylor provided matrix generators for exceptional groups of Lie type. From the paper, we also obtain a Sylow p-subgroup of 7×7 matrices, and there are also other constructions of the Lie algebra of type G_2 (e.g. [Hum78, §19.3], [Rum14, §8]). In this thesis, we determine the Sylow p-subgroup $G_2^{syl}(q)$ of 8×8 matrices such that $G_2^{syl}(q) \leqslant {}^3D_4^{syl}(q^3)$. We refer to the results of ${}^3D_4^{syl}(q^3)$, when we study the orbit modules and supercharacter theories of $G_2^{syl}(q)$.

The relations of the groups are obtained as follows:

$$
\begin{tikzcd}
 & {}^3D_4^{syl}(q^3) & & & \\
G_2^{syl}(q) \arrow[ur, dash, "\leqslant"] & & D_4^{syl}(q^3) \arrow[ul, dash, "\leqslant"'] & \leqslant & A_8(q^3) \\
 & D_4^{syl}(q) \arrow[ul, dash, "\leqslant"] \arrow[ur, dash, "\leqslant"'] & & &
\end{tikzcd}
$$

In *Section 3.2*, we construct a 1-cocycle f and determine a monomial linearisation $(f|_{G_2^{syl}(q)}, \kappa|_{V\times V})$ for $G_2^{syl}(q)$, and obtain a monomial $G_8(q)$-module $\mathbb{C}G_2^{syl}(q)$.

In *Section 3.3*, we determine the supercharacter theory for $G_2^{syl}(q)$, and establish the supercharacter table of $G_2^{syl}(q)$. When $p > 3$, we exhibit the relations between supercharacters and irreducible characters of $G_2^{syl}(q)$, and calculate the character table of $G_2^{syl}(q)$. F. Himstedt, T. Le, and K. Magaard [HLM16, §8.3] obtained some irreducible characters of $G_2^{syl}(q)$ by parameterizing *midafis*. S. M. Goodwin, P. Mosch and G. Röhrle [GR09, GMR14, GMR16] obtained the number of conjugacy classes of $G_2^{syl}(q)$ with an algotithm. We construct the irreducible characters by Clifford theory, calculate the conjugacy classes directly, and establish the character table of $G_2^{syl}(q)$ $(p > 3)$. The three statements hold, if we consider the analogue of Higman's conjecture, Lehrer's conjecture and Isaacs' conjecture of $A_n(q)$ for $G_2^{syl}(q)$.

Chapter 4: Supercharacter theories for ${}^2G_2^{syl}(3^{2m+1})$

In *Section 4.1*, we choose the suitable structure constants of extraspecial pairs, and construct a Sylow 3-subgroup ${}^2G_2^{syl}(3^{2m+1})$ of the Ree group ${}^2G_2(3^{2m+1})$ such that ${}^2G_2^{syl}(3^{2m+1}) \leqslant G_2^{syl}(3^{2m+1})$.

In *Section 4.2*, we determine a monomial linearisation $(f|_{{}^2G_2^{syl}(3^{2m+1})}, \kappa|_{V\times V})$ for ${}^2G_2^{syl}(3^{2m+1})$. We get a monomial $A_8(q)$-module $\mathbb{C}\big({}^2G_2^{syl}(3^{2m+1})\big)$, and obtain a classification of ${}^2G_2^{syl}(3^{2m+1})$-orbit modules.

In *Section 4.3*, we calculate the conjugacy classes of ${}^2G_2^{syl}(3^{2m+1})$, establish a supercharacter theory for ${}^2G_2^{syl}(3^{2m+1})$, and determine the character table of ${}^2G_2^{syl}(3)$. The three statements are true, if we consider the analogue of Higman's conjecture, Lehrer's conjecture and Isaacs' conjecture of $A_n(q)$ for ${}^2G_2^{syl}(3)$.

Chapter 5: Monomial $E_6^{syl}(q)$***-,*** $F_4^{syl}(q)$***-,*** ${}^2E_6^{syl}(q^2)$***- and*** $E_7^{syl}(q)$***-modules***

In this chapter, we establish the Sylow p-subgroups $E_6^{syl}(q)$, $F_4^{syl}(q)$, ${}^2E_6^{syl}(q^2)$ and $E_7^{syl}(q)$, and the matrix sizes of them are 27×27, 27×27, 27×27 and 56×56 respectively. We determine monomial linearisations for the Sylow p-subgroups $E_6^{syl}(q)$, $F_4^{syl}(q)$, ${}^2E_6^{syl}(q^2)$ and $E_7^{syl}(q)$, and obtain monomial $E_6^{syl}(q)$-, $F_4^{syl}(q)$-, ${}^2E_6^{syl}(q^2)$- and

$E_7^{syl}(q)$-modules. The relations of the groups are shown as follows:

$$\begin{tikzcd}[column sep=small]
 & {}^2E_6^{syl}(q^2) & & & \\
F_4^{syl}(q) \arrow[ur, dash, "\leqslant"] & & E_6^{syl}(q^2) \arrow[ul, dash, "\leqslant"'] & \leqslant & A_{27}(q^2) \\
 & E_6^{syl}(q) \arrow[ul, dash, "\leqslant"] \arrow[ur, dash, "\leqslant"'] & & &
\end{tikzcd}$$

Our main results of the dissertation are listed in the following table.

The steps and results of the dissertation

Steps	Substeps	D_4	twisted D_4	G_2	twisted G_2	E_6	F_4	twisted E_6	E_7
1. Sylow p-subgroups: U	Chevalley basis	Yes		Yes					
	Sylow p-subgroup		${}^3D_4^{syl}(q^3)$	$G_2^{syl}(q)$	${}^2G_2^{syl}(3^{2m+1})$	$E_6^{syl}(q)$	$F_4^{syl}(q)$	${}^2E_6^{syl}(q^2)$	$E_7^{syl}(q)$
2. Monomial U-modules $\mathbb{C}U$ $\mathrm{Char}\,\mathbb{F}_q = p \neq 2$	Vector space: V		Yes	Yes	Yes	Yes	Yes	Yes	Yes
	Non-degenerate symmetric bilinear form		Yes	Yes	Yes	Yes	Yes	Yes	Yes
	Surj. 1-cocylce: f		$f\|_U$ bij.	$f\|_U$ bij.	$f\|_U$ bij.	f bij.	f bij.	f bij.	f bij.
3. U-orbit modules $p \neq 2$	Classification of orbit modules		Yes	Yes	Yes				
	Two orbit modules either iso. or orth.		No	No	Yes				
	Orbit modules isom. to staircase modules		Yes	Yes					
	Orbit modules isom. to hook-separated staircase modules		Yes	Yes					
4. Supercharacters of U $p \neq 2$	Superclasses, Supercharacters, Supermodules		Yes	Yes	Yes				
	Supercharacter tables		Yes	Yes	Yes				
5. Irreducible characters of U $p \neq 2$	Conjugacy classes		Yes	when $p \notin \{2,3\}$	when $m = 0$				
	Irreducible characters		Yes	$p \notin \{2,3\}$	$m = 0$				
	Relation formulae of super- and irr. characters		Yes	$p \notin \{2,3\}$	$m = 0$				
	Character tables		Yes	$p \notin \{2,3\}$	$m = 0$				

1. The construction of monomial modules

In this chapter, we mainly recall the construction of the monomial linearisations following M. Jedlitschky's approach [Jed13] and the orbit modules. Our main references are R. Dipper's script [Dip13] and M. Jedlitschky's PhD thesis [Jed13].

1.1. The setting

We list some notations which we use frequently.

1.1.1 Notation. *Define the following notations:*

p	:	*a prime number,*
q	:	*a fixed power of* p*,*
K	:	*a field,*
K^*	:	*the multiplicative group* $K\backslash\{0\}$ *of* K*,*
K^+	:	*the additive group of* K*,*
$\mathbb{F}_q$	:	*the finite field with* q *elements,*
$\mathbb{F}_{q^3}$	:	*the finite field with* q^3 *elements,*
$\mathbb{C}$	:	*the complex field,*
$\mathbb{Q}$	:	*the set of all rational numbers,*
$\mathbb{Z}$	:	*the set of all integers,*
$\mathbb{N}$	:	*the set* $\{0, 1, 2, \ldots\}$ *of all non-negative integers,*
$\mathbb{N}^*$	:	*the set* $\{1, 2, \ldots\}$ *of all positive integers,*
I_n	:	$n \times n$ *identity matrix* $I_{n\times n}$*,*
O_n	:	$n \times n$ *zero matrix* $O_{n\times n}$*.*

Let $N \in \mathbb{N}^*$.

1.1.2 Notation. *Define the sets of matrix entry coordinates*

$$\square := \{(i, j) \mid 1 \leq i, j \leq N\}, \quad \textit{and} \quad \diagdown\!\!\mid := \{(i, j) \mid 1 \leq i < j \leq N\}.$$

1.1.3 Notation. *(1) Let* $M, N \in \mathbb{N}^*$*, denote* $\mathrm{Mat}_{M\times N}(K)$ *the set of all* $M \times N$ *matrices with entries in the field* K*. In particular,* $\mathrm{Mat}_N(K) := \mathrm{Mat}_{N\times N}(K)$*.*

(2) *Define the* **general linear group** $GL_N(K)$ *to be the subset of* $\mathrm{Mat}_{N\times N}(K)$ *consisting of all invertible matrices.*

(3) *Let* $m \in \mathrm{Mat}_{N\times N}(K)$. *Then set* $m := (m_{i,j})$, *where* $m_{i,j} \in K$ *denotes the* (i,j)-*entry of* m *(the entry in the* i*-th row and* j*-th column). If* $N < 10$, *set* $m_{ij} := m_{i,j}$.

(4) *Denote* $e_{i,j} \in \mathrm{Mat}_{N\times N}(K)$ *the matrix unit with* 1 *in the* (i,j)*-position and* 0 *elsewhere. If* $N < 10$, *set* $e_{ij} := e_{i,j}$.

(5) *Denote by* $A^\top$ *the* **transpose** *of* A.

1.1.4 Notation. *Set* $-(q) := -(\mathbb{F}_q)$, *for example* $GL_N(q) := GL_N(\mathbb{F}_q)$.

1.1.5 Notation/Lemma. *Set*

$$V_0 :=\mathrm{Mat}_{N\times N}(K).$$

Then V_0 *is a* K*-vector space.*

1.1.6 Definition. *Let* $A_N(K)$ *denote the set of upper unitriangular* $N \times N$*-matrices over the field* K, *and it is called the* **unipotent linear group**.

1.1.7 Notation/Lemma. *Let* $A = (A_{i,j}) \in V_0 = \mathrm{Mat}_{N\times N}(K)$, *then the* **trace** *of* A *is*

$$\mathrm{tr}(A) := \sum_{i=1}^{N} A_{i,i}.$$

In particular, $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ *for all* $A, B \in V_0$.

1.1.8 Definition ([Jed13], 1.2.16). (1) *Let* $A \in \mathrm{Mat}_{N\times N}(K)$, *then define the* **support** *of* A *as follows:*

$$\mathrm{supp}(A) :=\{(i,j) \in \square \mid A_{i,j} \neq 0\}.$$

(2) *Let* $V \subseteq V_0$ *be a vector subspace of* V_0, *then set*

$$\mathrm{supp}(V) := \bigcup_{A\in V} \mathrm{supp}(A).$$

(3) *Let* $g \in A_N(K)$, *then the* **essential support** *of* g *is defined as*

$$\overline{\mathrm{supp}}(g) :=\mathrm{supp}(g - I_N) = \{(i,j) \in \triangledown \mid g_{i,j} \neq 0\}.$$

For $J \in \square$, *let* $V_J := \{A \in V_0 \mid \mathrm{supp}(A) \subseteq J\}$.

(4) Let H be a subgroup of $A_N(K)$. Then set

$$\overline{\mathrm{supp}}(H) := \bigcup_{h \in H} \overline{\mathrm{supp}}(h).$$

1.1.9 Notation. *Let V be a vector subspace of V_0 and $J := \mathrm{supp}(V) \subseteq \triangledown$. Define the subset J^c of $\triangledown$ and the vector subspace V^c of V_0 as follows:*

$$\begin{aligned} J^c :=& J \bigcup \{(i, i+1) \in \triangledown \backslash J \mid (i, i+2) \in J\} \\ & \bigcup \{(i, j) \in \triangledown \backslash J \mid \exists\, j_1 < j < j_2 \text{ and } (i, j_1), (i, j_2) \in J\}, \\ V^c :=& V_{\mathrm{supp}(V)^c} = V_{J^c}. \end{aligned}$$

1.2. The construction of monomial modules

In this section, we recall the construction of the monomial modules. For the construction of monomial linearisations we refer to [Dip13], [Jed13], [DG15], [DG16] and [Guo16]. For the general background of modules, characters and idempotents we refer to [AB95], [JL01], [CR81] and [CR62].

In this thesis, a group action is a right action and a module is a right module. Let G be a finite multiplicative group, $\mathrm{Irr}(G)$ the set of all complex irreducible characters of G, V a finite abelian additive group and K a field. If V is a K-vector space, it is finite dimensional. If X is a set, KX denotes the K-vector space with the K-basis X.

1.2.1 Notation. *Let M be a right KG-module and $-*-$ be the module operation as follows:*

$$-*-\colon M \times G \to M : (m, g) \mapsto m * g.$$

Then the right KG-module M is also denoted by $(M, *)_{KG}$, or by M_{KG} for short, or by M if no ambiguity arises.

1.2.2 Definition (Monomial module)**.** *Let M be a right KG-module with a K-basis $\mathcal{B}$, then $\mathcal{B}$ is called a* ***monomial basis*** *(and M is called* ***monomial****) if there exist*

(a) a group action $-.-\colon \mathcal{B} \times G \to \mathcal{B} : (b, g) \mapsto b.g$,

(b) a function $\alpha\colon \mathcal{B} \times G \to K^*$,

such that the module operation $-*-$ of G on the basis $\mathcal{B}$ of M is given by

$$b * g = \alpha(b, g)b.g \qquad \text{for all } g \in G,\ b \in \mathcal{B}.$$

1.2.3 Definition. *Suppose a group G acts on a set V from the right*

$$-\circ-\colon V \times G \to V : (A, g) \mapsto A \circ g.$$

(1) Let $V := (V, +)$ be an abelian group and

$$(A + B) \circ g = A \circ g + B \circ g \qquad \text{for all } A, B \in V \text{ and } g \in G.$$

We say G ***acts on*** *V* ***as (group) automorphisms****.*

(2) Let K be a field, V a K-vector space and

- *$(A + B) \circ g = A \circ g + B \circ g \qquad$ for all $A, B \in V$ and $g \in G$,*
- *$(kA) \circ g = k(A \circ g) \qquad$ for all $k \in K$, $A \in V$ and $g \in G$.*

*We say that G **acts on V as K-(vector space) automorphisms** or that **the action of G on V is linear**.*

1.2.4 Definition/Lemma. *Let G be a finite group, K a field and*

$$K^G := \{\tau\colon G \to K \mid \tau \text{ is a map}\}.$$

(1) Define addition and scalar multiplication on K^G as follows: for $\tau, \sigma \in K^G$ and $\lambda \in K$, $\tau + \sigma$ and $\lambda\tau$ by $(\tau + \sigma)(g) = \tau(g) + \sigma(g)$ and $(\lambda\tau)(g) = \lambda(\tau(g))$ for all $g \in G$. Then K^G is a K-vector space.

(2) For $g \in G$, set $\tau_g\colon G \to K : h \mapsto \begin{cases} 1, & g = h \\ 0, & g \neq h \end{cases} = \delta_{g,h}$, where $\delta_{g,h}$ is the Kronecker delta. Then $\{\tau_g \mid g \in G\}$ is a K-basis of K^G. In particular, $\tau = \sum_{g\in G} \tau(g)\tau_g$ for all $\tau \in K^G$.

(3) The map $\Phi\colon K^G \to KG$ induced by $\tau_g \mapsto g$ is a K-isomorphism. In particular, $\Phi(\tau) = \sum_{g\in G} \tau(g)g$ for all $\tau \in K^G$.

(4) Let KG be the group algebra with the multiplication

$$(\sum_{g\in G} \alpha_g g)(\sum_{h\in G} \beta_h h) = \sum_{x\in G}\sum_{g\in G} \alpha_g \beta_{g^{-1}x} x.$$

For $\tau, \sigma \in K^G$, the multiplication $\tau\sigma$ is defined by

$$\tau\sigma\colon G \to K : y \mapsto \sum_{g\in G} \tau(g)\sigma(g^{-1}y).$$

Then K^G is an associate K-algebra, and $\Phi\colon K^G \to KG : \tau \mapsto \sum_{g\in G} \tau(g)g$ is an algebra isomorphism. In particular, $\tau_g\tau_h = \tau_{gh}$ for all $g, h \in G$.

1.2.5 Lemma. *Let V be a finite abelian group and $\hat{V} := \mathrm{Hom}(V, \mathbb{C}^*)$. Then*

$$\mathrm{Irr}(V) = \hat{V} \subseteq \mathbb{C}^V$$

is a linearly independent subset of the $\mathbb{C}$-vector space $\mathbb{C}^V$.

1.2.6 Remark. *Let V be abelian and finite, then*

$$\dim_{\mathbb{C}} \mathbb{C}\hat{V} = |\hat{V}| \overset{1.2.5}{=} |V| = \dim_{\mathbb{C}} \mathbb{C}V \overset{1.2.4}{=} \dim_{\mathbb{C}} \mathbb{C}^V.$$

Thus $\mathbb{C}\hat{V} = \mathbb{C}^V$ (as $\mathbb{C}$-vector spaces).

1.2.7 Lemma. *Let* $f\colon G \to V$ *be a map, then*

$$f^*\colon \mathbb{C}^V \to \mathbb{C}^G : \phi \mapsto f^*(\phi) = \phi f$$

defines a $\mathbb{C}$*-linear map, and the following diagram is commutative:*

$$\begin{tikzcd} V \arrow[r, "\phi"] & \mathbb{C} \\ G \arrow[u, "f"] \arrow[ur, "f^*(\phi)"'] & \end{tikzcd}$$

The map f *is surjective (bijective, injective) if and only if* f^* *is injective (bijective, surjective). If* f *is surjective,* $\{\hat{\chi}f \mid \hat{\chi} \in \hat{V}\}$ *is a* $\mathbb{C}$*-basis of* $\operatorname{im} f^* = f^*(\mathbb{C}^V)$.

1.2.8 Corollary. *Let* $f\colon G \to V$ *be a surjective map, and* $U \leqslant G$ *such that* $f|_U$ *is bijective. Then*

$$f|_U^*\colon \mathbb{C}^V \to \mathbb{C}^U : \phi \mapsto f|_U^*(\phi) = \phi f|_U = f^*(\phi)|_U$$

defines a $\mathbb{C}$*-isomorphism, and the following diagram is commutative:*

$$\begin{tikzcd} V \arrow[r, "\phi"] & \mathbb{C} \\ U \arrow[u, "f|_U"] \arrow[ur, "f|_U^*(\phi)=f^*(\phi)|_U"'] & \end{tikzcd}$$

In particular, $\{\hat{\chi}f|_U \mid \hat{\chi} \in \hat{V}\}$ *is a* $\mathbb{C}$*-basis of* $\mathbb{C}^U$.

1.2.9 Definition (1-cocycle)**.** *Let* V *be an abelian group. Suppose* G *acts on* V*,* $(A, g) \mapsto A \circ g$ $(A \in V, g \in G)$*, as automorphisms. Then a map* $f\colon G \to V$ *is called a (right)* ***1-cocycle*** *of* G *in* V*, if it satisfies*

$$f(xg) = f(x) \circ g + f(g) \qquad \text{for all } x, g \in G. \tag{1.2.10}$$

Suppose that $f\colon G \to V$ is a surjective 1-cocycle and U is a subgroup of G such that $f|_U$ is bijective (i.e. $f|_U$ is a bijective 1-cocyle of U in V). Then $\mathbb{C}^V$, $\mathbb{C}V$, $\mathbb{C}^U$, $\mathbb{C}U$ and $\operatorname{im} f^*$ are pairwise $\mathbb{C}$-isomorphic:

$$\begin{tikzcd} \mathbb{C}\{\hat{\chi}f|_U \mid \hat{\chi} \in \hat{V}\} = \operatorname{im} f|_U^* = & \mathbb{C}^U \arrow[r, "\Phi"] & \mathbb{C}U \arrow[d, "f_{\mathbb{C}U}"] \\ \mathbb{C}\{\hat{\chi} \mid \hat{\chi} \in \hat{V}\} = \mathbb{C}\hat{V} = & \mathbb{C}^V \arrow[u, "f|_U^*"] \arrow[r, "\Psi"] \arrow[d, "f^*"] & \mathbb{C}V \\ \mathbb{C}\{\hat{\chi}f \mid \hat{\chi} \in \hat{V}\} = & \operatorname{im} f^* & \end{tikzcd}$$

where $\Psi\colon \hat\chi \mapsto \sum_{B\in V}\hat\chi(B)B$, $\Phi\colon \hat\chi f|_U \mapsto \sum_{u\in U}\hat\chi f(u)u$, $f^*\colon \hat\chi\mapsto\hat\chi f$, $f|_U^*\colon \hat\chi\mapsto\hat\chi f|_U$, and $f_{\mathbb{C}U} : u\mapsto f(u)$ is the extension of $f|_U$ to $\mathbb{C}U$ by linearity. Let $\chi := \hat\chi f$,

$$[\hat\chi] := \sum_{B\in V}\hat\chi(B)B \quad\text{ and }\quad [\hat\chi f|_U] := [\chi|_U] := \sum_{u\in U}\chi(u)u,$$

then

$$\mathbb{C}V = \mathbb{C}\{[\hat\chi] \mid \hat\chi\in\hat V\} \quad\text{ and }\quad \mathbb{C}U = \mathbb{C}\{[\hat\chi f|_U] \mid \hat\chi\in\hat V\}.$$

1.2.11 Lemma. *Let K be a field, V, W K-vector spaces (or abelian groups), G a group and $\varphi\colon V\to W$ a K-isomorphism (or group isomorphism). Suppose that V is a KG-module $(V, .)_{KG}$ and that the elements of G act on V as K-automorphisms (or as group automorphisms). Define a new operation by*

$$-*-\colon W\times G\to W : (w,g)\mapsto w*g := \varphi(\varphi^{-1}(w).g)$$

and for $\sum_{g\in G}\alpha_g g\in KG$ extend the operation by linearity

$$w*\Big(\sum_{g\in G}\alpha_g g\Big) = \sum_{g\in G}\alpha_g(w*g) \quad \textit{for all } w\in W.$$

Then W is a KG-module $(W,*)_{KG}$, the elements of G act on W as K-automorphisms (or as group automorphisms) and φ is a KG-module isomorphism.

1.2.12 Definition/Lemma. *Let K be an arbitrary field and G be a finite group. Define an operation by*

$$-*-\colon K^G\times G\to K^G : (\varphi,g)\mapsto\varphi*g,$$

where $(\varphi*g)(x) = \varphi(xg^{-1})$ for all $x\in G$. Then K^G becomes a right KG-module $(K^G,*)_{KG}\cong KG_{KG}$, where KG_{KG} is the right regular module.

Proof. By 1.2.4 and 1.2.11, it is enough to define a module operation $-*-$of KG on K^G. For $\tau_g\in K^G$ and $h\in G$, we set $\tau_g*h := \Phi^{-1}(\Phi(\tau_g)h) = \tau_{gh}$. Then

$$(\tau_g*h)(x) = \tau_{gh}(x) = \delta_{gh,x} = \delta_{g,xh^{-1}} = \tau_g(xh^{-1}) \qquad \text{for all } x\in G.$$

Thus $(\varphi*g)(x) = \varphi(xg^{-1})$ for all $\varphi\in K^G$ and $x\in G$. □

1.2.13 Definition/Lemma. *Let V be an abelian group on which G acts from the right as automorphisms. Then the group action of G on V induces a group action $-.-$ of G on $\hat V$ given by*

$$-.-\colon \hat V\times G\to\hat V : (\hat\chi,g)\mapsto\hat\chi.g,$$

where $(\hat\chi.g)(A) = \hat\chi(A\circ g^{-1})$ for all $A\in V$.

1.2.14 Lemma. *Let $f\colon G \to V$ be a surjective 1-cocycle. Then for $\hat{\chi} \in \hat{V}$ and $g \in G$,*

$$(\hat{\chi} f) * g = (\hat{\chi} f)(g^{-1}) \cdot (\hat{\chi}.g)f = \chi(g^{-1}) \cdot (\hat{\chi}.g)f \in \operatorname{im} f^*,$$

where $\chi = \hat{\chi} f$ and $- \cdot -$ is the scalar multiplication. By extending the operation $- * -$ linearly, $\operatorname{im} f^*$ becomes a monomial module $(\operatorname{im} f^*, *)_{\mathbb{C}G}$.

Proof. Let $\hat{\chi} \in \hat{V}$ and $g \in G$, then for $x \in G$,

$$\begin{aligned}((\hat{\chi} f) * g)(x) =& (\hat{\chi} f)(xg^{-1}) \overset{1.2.9}{=} \hat{\chi}(f(x) \circ g^{-1} + f(g^{-1})) = (\hat{\chi} f)(g^{-1}) \cdot \hat{\chi}(f(x) \circ g^{-1}) \\ \overset{1.2.13}{=}& (\hat{\chi} f)(g^{-1}) \cdot (\hat{\chi}.g)(f(x)) = (\hat{\chi} f)(g^{-1}) \cdot \big((\hat{\chi}.g)f\big)(x). = \chi(g^{-1}) \cdot (\hat{\chi}.g)f.\end{aligned}$$

Thus $(\hat{\chi} f) * g = (\hat{\chi} f)(g^{-1}) \cdot (\hat{\chi}.g)f$. □

1.2.15 Theorem. *(Monomial $\mathbb{C}G$-modules) [Jedlitschky, [Jed13]] Let $f\colon G \to V$ be a surjective 1-cocycle, and $U \leqslant G$ such that $f|_U$ is bijective. Then the $\mathbb{C}$-vector spaces $\mathbb{C}^V$, $\mathbb{C}^U$, $\mathbb{C}V$ and $\mathbb{C}U$ can be made into monomial $\mathbb{C}G$-modules by extending the following operations linearly: for all $\hat{\chi} \in \hat{V}$ and $g \in G$*

$$\begin{aligned} \hat{\chi} * g :=& \hat{\chi}\left(f(g^{-1})\right) \cdot \hat{\chi}.g, & (\hat{\chi} f|_U) * g :=& \hat{\chi}\left(f(g^{-1})\right) \cdot (\hat{\chi}.g)f|_U, \\ [\hat{\chi}] * g :=& \hat{\chi}\left(f(g^{-1})\right) \cdot [\hat{\chi}.g], & [\hat{\chi} f|_U] * g :=& \hat{\chi}\left(f(g^{-1})\right) \cdot [(\hat{\chi}.g)f|_U],\end{aligned}$$

and the $\mathbb{C}G$-modules $(\mathbb{C}^V, *)_{\mathbb{C}G}$, $(\mathbb{C}V, *)_{\mathbb{C}G}$, $(\mathbb{C}^U, *)_{\mathbb{C}G}$ and $(\mathbb{C}U, *)_{\mathbb{C}G}$ are isomorphic to $(\operatorname{im} f^*, *)_{\mathbb{C}G}$:

$$\begin{tikzcd} (\mathbb{C}^U, *)_{\mathbb{C}G} \arrow[r, "\Phi"] & (\mathbb{C}U, *)_{\mathbb{C}G} \arrow[d, "f_{\mathbb{C}U}"] \\ (\mathbb{C}^V, *)_{\mathbb{C}G} \arrow[u, "f|_U^*"] \arrow[r, "\Psi"] \arrow[d, "f^*"] & (\mathbb{C}V, *)_{\mathbb{C}G} \\ (\operatorname{im} f^*, *)_{\mathbb{C}G} & \end{tikzcd}$$

1.2.16 Corollary (Monomial $\mathbb{C}U$-module)**.** *The vector spaces $\mathbb{C}U$, $\mathbb{C}V$, $\mathbb{C}^U$, $\mathbb{C}^V$, and $\operatorname{im} f^*$ can be made into monomial isomorphic $\mathbb{C}G$-modules by extending the restriction of the operations $- * -$ linearly. In particular, the operation $- * -$ of U on $\mathbb{C}U$ is the usual right operation of U on $\mathbb{C}U$, i.e. for all $\hat{\chi} \in \hat{V}$ and $x \in U$*

$$(\sum_{u \in U} \hat{\chi} f(u)u) * x = \hat{\chi}\left(f(x^{-1})\right) \cdot (\sum_{u \in U} (\hat{\chi}.x)f(u)u) = \sum_{u \in U} \hat{\chi} f(u)ux,$$

so $(\mathbb{C}U, *)_{\mathbb{C}U} = \mathbb{C}U_{\mathbb{C}U}$.

1.2.17 Theorem (2.8, [DG16])**.** *Let $f\colon G \to V$ be a surjective 1-cocycle with kernel $\ker f$. Then $\ker f \leqslant G$ and $(\mathbb{C}V, *)_{\mathbb{C}G} \cong (\mathbb{C}^V, *)_{\mathbb{C}G} \cong \operatorname{Ind}_{\ker f}^G \mathbb{C}_{\ker f}$, where $\mathbb{C}_{\ker f}$ is the trivial $\ker f$-module.*

1.2.18 Remark. $\operatorname{im} f^* = \mathbb{C}\{(\sum_{h\in\ker f} \tau_h) * [\chi|_U] \mid \hat{\chi} \in \hat{V}\} \subseteq \mathbb{C}^G$ *(see A.1).*

If we want to apply the monomial $\mathbb{C}G$ (or $\mathbb{C}U$)-modules, we should solve the two problems: 'describe the set $\hat{V}$ of linear characters of V' and 'describe the action of G on $\hat{V}$ such that it is easy to calculate'.

1.2.19 Assumption. *For the remainder of this subsection, let G be a finite group and V a finite dimensional vector space over some finite field $\mathbb{F}_q$. Suppose that G acts on V as $\mathbb{F}_q$-automorphisms from the right: $V \times G \to V : (A, g) \mapsto A \circ g$.*

1.2.20 Definition ([Jed13], 2.1.19)**.** *Let V be an $\mathbb{F}_q$-vector space on which G acts as $\mathbb{F}_q$-automorphism, $f\colon G \to V$ a surjective 1-cocycle and $\kappa\colon V \times V \to \mathbb{F}_q$ a non-degenerate bilinear form on V. Then (f, κ) is called a **monomial linearisation** for G.*

Let $\mathcal{B} = \{v_1, \dots, v_m\}$ be an $\mathbb{F}_q$-basis of V, then

$$\kappa\colon V \times V \to \mathbb{F}_q : (v_i, v_j) \mapsto \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

is a non-degenerate symmetric bilinear form.

1.2.21 Lemma (c.f. §2.1 of [Jed13]/ XIII §5 of [Lan95])**.** *There exists an unique linear action $-.-$ of G on V:*

$$-.-\colon V \times G \to V : (A, g) \mapsto A.g$$

where $\kappa(A.g, B) = \kappa(A, B \circ g^{-1})$ for all $B \in V$.

1.2.22 Notation. *Let $\vartheta\colon \mathbb{F}_q^+ \to \mathbb{C}^*$ denote a fixed nontrivial linear character of the additive group $\mathbb{F}_q^+$ of $\mathbb{F}_q$ once and for all. In particular, $\sum_{x\in\mathbb{F}_q^+} \vartheta(x) = 0$.*

1.2.23 Lemma. *Let V be a finite dimensional vector space over some finite field $\mathbb{F}_q$ and let κ denote a non-degenerate bilinear form. Then*

$$\hat{V} = \{\hat{\chi}_A \mid A \in V\},$$

where $\hat{\chi}_A\colon V \to \mathbb{C}^* : X \to \vartheta\big(\kappa(A, X)\big)$, and for $A \in V$ the following diagram is commutative:

$$\begin{tikzcd} V \arrow[r, "\kappa(A,-)"] \arrow[dr, "\hat{\chi}_A"'] & \mathbb{F}_q^+ \arrow[d, "\vartheta"] \\ & \mathbb{C}^* \end{tikzcd}$$

In particular $\hat{\chi}_A = \hat{\chi}_B \iff A = B$, and $(\hat{\chi}_A).g = \hat{\chi}_{(A.g)}$ for all $A \in V$ and $g \in G$.

Set $\chi_A := \hat{\chi}_A f$ for all $A \in V$, i.e. the following diagram is commutative:

$$\begin{tikzcd}
V \arrow[r, dotted, "\kappa(A,-)"] \arrow[dr, "\hat{\chi}_A"] & \mathbb{F}_q^+ \arrow[d, dotted, "\vartheta"] \\
G \arrow[u, "f"] \arrow[r, "\chi_A"'] & \mathbb{C}^*
\end{tikzcd}$$

Let $c := a + bi \in \mathbb{C}$ (i the imaginary unit), then $\overline{c} := a - bi$ denotes the complex conjugate of c. Let $A \in V$ and $g \in G$, then $\chi_A(1) = 1$, $\chi_{A.g}(g) = \overline{\chi_A(g^{-1})} = \chi_{(-A)}(g^{-1})$ and

$$\chi_A * g = \overline{\chi_{A.g}(g)}\chi_{A.g}, \qquad \hat{\chi}_A * g := \overline{\chi_{A.g}(g)}\hat{\chi}_{A.g}, \qquad (\chi_A|_U) * g := \overline{\chi_{A.g}(g)}\chi_{A.g}|_U,$$
$$[\hat{\chi}_A] * g := \overline{\chi_{A.g}(g)}[\hat{\chi}_{A.g}], \qquad [\chi_A|_U] * g := \overline{\chi_{A.g}(g)}[\chi_{A.g}|_U].$$

In general, $\overline{\chi_A(g^{-1})} \neq \chi_A(g)$. Let

$$[A] := \frac{1}{|U|} \sum_{u \in U} \overline{\chi_A(u)} u \in \mathbb{C}U, \qquad \text{and} \qquad [A]_V := \frac{1}{|V|} \sum_{B \in V} \overline{\hat{\chi}_A(B)} B \in \mathbb{C}V,$$

then $\{[A]_V \mid A \in V\}$ is the set of the pairwise orthogonal central primitive idempotent elements of $\mathbb{C}V$, and is a $\mathbb{C}$-basis of $\mathbb{C}V$. Thus $\{[A] \mid A \in V\}$ is a $\mathbb{C}$-basis of $\mathbb{C}U$. We have

$$\mathbb{C}^U = \mathbb{C}\{\chi_A|_U \mid A \in V\}, \qquad \mathbb{C}U = \mathbb{C}\{[A] \mid A \in V\},$$
$$\mathbb{C}^V = \mathbb{C}\hat{V} = \mathbb{C}\{\hat{\chi}_A \mid A \in V\}, \qquad \mathbb{C}V = \mathbb{C}\{[A]_V \mid A \in V\}.$$

1.2.24 Corollary ([Jed13]). *Suppose that G has a monomial linearisation (f, κ) and that there exists a subgroup $U \leqslant G$, such that the map $f|_U : U \to V$ is bijective. Then $\mathbb{C}U$ is a monomial $\mathbb{C}G$-module with the module operation*

$$[A] * g = \chi_{A.g}(g)[A.g] \qquad \text{for all } g \in G, A \in V, \tag{1.2.25}$$

The restriction of the $*$-operation to U on $\mathbb{C}U$ is given by the usual right multiplication of U on $\mathbb{C}U$, i.e.

$$[A] * u = [A]u = \frac{1}{|U|} \sum_{y \in U} \overline{\chi_A(y)} yu \qquad \text{for all } u \in U, A \in V.$$

Proof. For all $g \in G, A \in V$,

$$\begin{aligned}
[A] * g = &\left(\frac{1}{|U|} \sum_{y \in U} \overline{\chi_A(y)} y \right) * g = \left(\frac{1}{|U|} \sum_{y \in U} \chi_{-A}(y) y \right) * g \\
= &\chi_{-A}(g^{-1}) \cdot \frac{1}{|U|} \sum_{y \in U} ((\hat{\chi}_{-A}).g) f(y) y = \overline{\chi_A(g^{-1})} \cdot \frac{1}{|U|} \sum_{y \in U} (\hat{\chi}_{(-A.g)}) f(y) y \\
= &\chi_{A.g}(g) \cdot \frac{1}{|U|} \sum_{y \in U} \overline{\hat{\chi}_{A.g} f(y)} y \\
= &\chi_{A.g}(g)[A.g].
\end{aligned}$$

$\square$

We mainly consider the right module $(\mathbb{C}U, *)_{\mathbb{C}U} = \mathbb{C}U_{\mathbb{C}U}$. Let $A \in V$, we define the U**-orbit module** associated to A by

$$\mathbb{C}\mathcal{O}_U([A]) := \mathbb{C}\{[A]u \mid u \in U\} = \mathbb{C}\{[A.u] \mid u \in U\}.$$

Then $\mathbb{C}\mathcal{O}_U([A])$ has a $\mathbb{C}$-basis given by

$$\{[A.u] \mid u \in U\} = \{[C] \mid C \in \mathcal{O}_U(A)\},$$

where $\mathcal{O}_U(A) := \{A.g \in V \mid g \in U\}$ is the **orbit** of A under the operation $-.-$ defined in 2.2.27. By just substituting every U with G, we define the $\mathbb{C}G$-orbit module $\mathbb{C}\mathcal{O}_G([A])$, which has a $\mathbb{C}$-basis $\{[C] \mid C \in \mathcal{O}_G(A)\}$.

The **stabilizer** $\mathrm{Stab}_U(A)$ of A in U is defined to be

$$\mathrm{Stab}_U(A) = \{u \in U \mid A.u = A\},$$

then $\dim_{\mathbb{C}}\mathbb{C}\mathcal{O}_U([A]) = |\mathcal{O}_U(A)| = \frac{|U|}{|\mathrm{Stab}_U(A)|}$. Let $A, B \in V$, then $\mathbb{C}\mathcal{O}_U([A])$ and $\mathbb{C}\mathcal{O}_U([B])$ are identical (if $A.u = B$ for some $u \in U$) or their intersection is $\{0\}$. Set

$$\mathrm{Stab}_U(A, B) := \mathrm{Stab}_U(A) \cap \mathrm{Stab}_U(B).$$

Two $\mathbb{C}U$-modules having no nontrivial $\mathbb{C}U$-homomorphism between them are called **orthogonal**.

2. Supercharacter theories for ${}^{3}D_{4}^{syl}\left(q^{3}\right)$

In this chapter, we determine a Chevalley basis of $\mathcal{L}_{D_4}$ (2.1.1), and a matrix Sylow p-subgroup ${}^{3}D_{4}^{syl}\left(q^{3}\right)$ (2.1.12) for the Steinberg triality group ${}^{3}D_{4}\left(q^{3}\right)$.

After that, we construct, for technical reasons, a larger group $G_8\left(q^{3}\right)$ (2.1.16), determine a monomial linearisation $(f, \kappa_q|_{V\times V})$ for $G_8\left(q^{3}\right)$ (2.2.32) and get a monomial linearisation $(f|_{{}^{3}D_{4}^{syl}(q^{3})}, \kappa_q|_{V\times V})$ for ${}^{3}D_{4}^{syl}\left(q^{3}\right)$ (2.2.33). We obtain a monomial $G_8\left(q^{3}\right)$-module $\mathbb{C}\big({}^{3}D_{4}^{syl}\left(q^{3}\right)\big)$ (2.2.34). Then we give a classification of ${}^{3}D_{4}^{syl}\left(q^{3}\right)$-orbit modules (Table 2.1). Every ${}^{3}D_{4}^{syl}\left(q^{3}\right)$-orbit module is isomorphic to some hook-separated staircase module (2.2.79). Some irreducible modules are determined, and any two orbit modules from different families are orthogonal (2.2.83).

Finally, we determine the supercharacter theory for ${}^{3}D_{4}^{syl}\left(q^{3}\right)$ (2.3.21) and establish the supercharacter table of ${}^{3}D_{4}^{syl}\left(q^{3}\right)$ in Table 2.5. Then we determine the conjugacy classes (2.3.51), construct the complex irreducible characters (2.3.45), calculate the character table of ${}^{3}D_{4}^{syl}\left(q^{3}\right)$ (2.3.52), and obtain the relations between the supercharacters and the irreducible characters (2.3.53).

2.1. Sylow p-subgroup ${}^{3}D_{4}^{syl}\left(q^{3}\right)$ of the Steinberg triality group

Let $p \neq 2$. In this section, we set the suitable structure constants of $\mathcal{L}_{D_4}$(A.5.6), fix a Chevalley basis of $\mathcal{L}_{D_4}$ (2.1.1), and determine the Sylow p-subgroup ${}^{3}D_{4}^{syl}\left(q^{3}\right)$ (2.1.12) of upper unitriangular 8×8-matrices for ${}^{3}D_{4}\left(q^{3}\right)$. We determine a bigger group $G_8\left(q^{3}\right)$ (see 2.1.16).

2.1.1. Sylow p-subgroup ${}^{3}D_{4}^{syl}\left(q^{3}\right)$ of the Steinberg triality group

In this subsection, we establish the matrix Sylow p-subgroup $D_{4}^{syl}(q)$ of the Chevalley group $D_4(q)$ (2.1.4). and determine a Sylow p-subgroup ${}^{3}D_{4}^{syl}\left(q^{3}\right)$ of the Steinberg triality group ${}^{3}D_{4}\left(q^{3}\right)$ (2.1.12). The background and notations of complex Lie algebra can be found in Appendix A.2, more details on root system, Chevalley basis and Lie algebra of type D_4 can be found in Appendix A.3, for the construction of a Sylow

p-subgroup of a Chevalley group $D_4(q)$ details can be found in Appendix A.4. The general construction of the Steinberg triality group ${}^3D_4(q^3)$ can be found in [Car72], [Gec90] and [Gec91].

Let $J_8^+ := \sum_{i=1}^{8} e_{i,9-i} \in GL_8(\mathbb{C})$. By A.3, the set

$$\{A \in \mathrm{Mat}_{8\times 8}(\mathbb{C}) \mid A^\top J_8^+ + J_8^+ A = 0\}$$

forms a simple Lie algebra of type D_4. For $1 \leq i \leq 4$, let $h_i := e_{i,i} - e_{9-i,9-i} \in \mathrm{Mat}_{8\times 8}(\mathbb{C})$, then a Cartan subalgebra of $\mathcal{L}_{D_4}$ is

$$\mathcal{H}_{D_4} = \{\sum_{i=1}^{4} \lambda_i h_i \mid \lambda_i \in \mathbb{C}\}.$$

Let $\mathcal{H}_{D_4}^*$ be the dual space of $\mathcal{H}_{D_4}$, and $h := \sum_{i=1}^{4} \lambda_i h_i$. For $1 \leq i \leq 4$, let $\varepsilon_i \in \mathcal{H}_{D_4}^*$ be defined by $\varepsilon_i(h) = \lambda_i$ for all $i = 1, 2, 3, 4$. Let $\mathcal{V}_4 := \mathcal{V}_{D_4}$ be a $\mathbb{R}$-vector subspace of $\mathcal{H}_{D_4}^*$ spanned by $\{h_i \mid i = 1, 2, 3, 4\}$, then $\mathcal{V}_4$ becomes a Euclidean space (see Appendix A.3). The set

$$\Phi_{D_4} = \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq 4\}$$

is a root system of type D_4. The fundamental system of roots of the root system Φ_{D_4} is

$$\Delta_{D_4} = \{\varepsilon_1 - \varepsilon_2,\ \varepsilon_2 - \varepsilon_3,\ \varepsilon_3 - \varepsilon_4,\ \varepsilon_3 + \varepsilon_4\}.$$

The positive system of roots of Φ_{D_4} is

$$\Phi_{D_4}^+ := \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq 4\}.$$

We choose the following Chevalley basis of $\mathcal{L}_{D_4}$ to keep the description of the Steinberg triality group as simple as possible (see Appendix A.5).

2.1.1 Lemma (Chevalley basis of $\mathcal{L}_{D_4}$). *The Lie algebra $\mathcal{L}_{D_4}$ has a Chevalley basis $\{h_r \mid r \in \Delta_{D_4}\} \cup \{e_{\pm r} \mid r \in \Phi_{D_4}^+\}$ as follows:*

$r \in \Phi_{D_4}^+$	h_r	e_r	e_{-r}
$r_1 := \varepsilon_1 - \varepsilon_2$	$(e_{1,1} - e_{8,8}) - (e_{2,2} - e_{7,7})$	$e_{1,2} - e_{7,8}$	$e_{2,1} - e_{8,7}$
$r_2 := \varepsilon_2 - \varepsilon_3$	$(e_{2,2} - e_{7,7}) - (e_{3,3} - e_{6,6})$	$e_{2,3} - e_{6,7}$	$e_{3,2} - e_{7,6}$
$r_3 := \varepsilon_3 - \varepsilon_4$	$(e_{3,3} - e_{6,6}) - (e_{4,4} - e_{5,5})$	$e_{3,4} - e_{5,6}$	$e_{4,3} - e_{6,5}$
$r_4 := \varepsilon_3 + \varepsilon_4$	$(e_{3,3} - e_{6,6}) + (e_{4,4} - e_{5,5})$	$e_{3,5} - e_{4,6}$	$e_{5,3} - e_{6,4}$
$r_5 := r_1 + r_2$		$-(e_{1,3} - e_{6,8})$	$-(e_{3,1} - e_{8,6})$
$r_6 := r_2 + r_3$		$e_{2,4} - e_{5,7}$	$e_{4,2} - e_{7,5}$
$r_7 := r_2 + r_4$		$e_{2,5} - e_{4,7}$	$e_{5,2} - e_{7,4}$
$r_8 := r_1 + r_2 + r_3$		$e_{1,4} - e_{5,8}$	$e_{4,1} - e_{8,5}$
$r_9 := r_1 + r_2 + r_4$		$e_{1,5} - e_{4,8}$	$e_{5,1} - e_{8,4}$
$r_{10} := r_2 + r_3 + r_4$		$e_{2,6} - e_{3,7}$	$e_{6,2} - e_{7,3}$
$r_{11} := r_1 + r_2 + r_3 + r_4$		$e_{1,6} - e_{3,8}$	$e_{6,1} - e_{8,3}$
$r_{12} := r_1 + 2r_2 + r_3 + r_4$		$e_{1,7} - e_{2,8}$	$e_{7,1} - e_{8,2}$

Define the matrix group

$$\bar{D}_4(q) := \langle\, \exp(t\, e_r) \,|\, r \in \Phi_{D_4},\ t \in \mathbb{F}_q \,\rangle\,.$$

The Chevalley group of type $\mathcal{L}_{D_4}$ over the field $\mathbb{F}_q$ is denoted by

$$D_4(q) := \langle\, \exp(t\, \mathrm{ad}\, e_r) \,|\, r \in \Phi_{D_4},\ t \in \mathbb{F}_q \,\rangle\,.$$

By the adaption of [Car72, Theorem 11.3.2], $\bar{D}_4(q)$ is a commutator subgroup, denoted by $\Omega_8(q, f_D)$, of the (positive) orthogonal group $O_8(q, f_D)$, where f_D is the quadratic form $x_1x_8+x_2x_7+x_3x_7+x_4x_5$. By [Car72, §11.3], $D_4(q) \cong \bar{D}_4(q)/Z(\bar{D}_4(q))$, and $D_4(q)$ is isomorphic to the projective group

$$P\Omega_8(q, f_D) = \Omega_8(q, f_D)/Z(O_8(q, f_D)) \cap \Omega_8(q, f_D).$$

We set

$$D_4^{syl}(q) := \left\langle\, \exp(te_r) \,\middle|\, r \in \Phi^+_{D_4},\ t \in \mathbb{F}_q \,\right\rangle.$$

Since $D_4^{syl}(q) \cap Z(O_8(q, f_D)) = \{I_8\}$, $D_4^{syl}(q)$ is a Sylow p-subgroup of the Chevalley group $D_4(q)$ (see A.4). Compare the orders of $O_8(q, f_D)$ and $P\Omega_8(q, f_D)$ (see [Gro02, Chapter 9]), $D_4^{syl}(q)$ is also a Sylow p-subgroup of $O_8(q, f_D)$.

We set $x_r(t) := \exp(te_r) = I_8 + t \cdot e_r$ for all $r \in \Phi_{D_4}$ and $t \in \mathbb{F}_q$, and the root subgroups $X_r := \{x_r(t) \mid t \in \mathbb{F}_q\}$ for all $r \in \Phi_{D_4}$.

Before we obtain a good expression for the elements of $D_4^{syl}(q)$, we need the following property.

2.1.2 Proposition. $D_4^{syl}(q) = \left\{ \prod_{r \in \Phi^+_{D_4}} x_r(t_r) \,\middle|\, t_r \in \mathbb{F}_q \right\}$, *where the product can be taken in an arbitrary, but fixed, order.*

Proof. c.f. A.4.4. □

2.1.3 Notation. *Define the following sets of matrix entry coordinates*

$$\begin{aligned}
\square :=& \{(i,j) \mid 1 \le i, j \le 8\},\\
\lhd :=& \{(i,j) \mid 1 \le i < j \le 8\},\\
\bigtriangledown :=& \{(i,j) \in \square \mid i < j < 9 - i\}.
\end{aligned}$$

M. Jedlitschky determines $D_4^{syl}(q)$ with the construction (see [Jed13, 1.1.4])

$$D_4^{syl}(q) := A_8(q) \cap O_8(q, f_D).$$

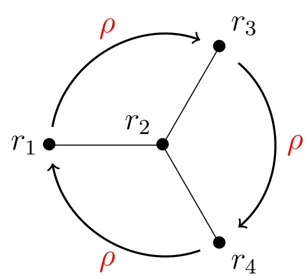
We describe $D_4^{syl}(q)$ as a projective group $P\Omega_8(q, f_D)$. Now we give the explicit expression of $D_4^{syl}(q)$, which shows that every element u of $D_4^{syl}(q)$ is uniquely determined by the entries $\{u_{i,j} \mid (i,j) \in \bigtriangledown\}$ (c.f. [Jed13, 1.1.19]).

2.1.4 Proposition (Sylow p-subgroup $D_4^{syl}(q)$). *A matrix Sylow p-subgroup of the Chevalley group $D_4(q)$ is given as follows:*

$$
\begin{aligned}
&D_4^{syl}(q)\\
&=\left\{ \begin{array}{l} x_{r_4}(t_{r_4})x_{r_3}(t_{r_3})x_{r_{10}}(t_{r_{10}})x_{r_7}(t_{r_7})x_{r_6}(t_{r_6})x_{r_2}(t_{r_2}) \\ \cdot x_{r_{12}}(t_{r_{12}})x_{r_{11}}(t_{r_{11}})x_{r_9}(t_{r_9})x_{r_8}(t_{r_8})x_{r_5}(t_{r_5})x_{r_1}(t_{r_1}) \end{array} \;\middle|\; t_{r_i} \in \mathbb{F}_q,\ i = 1,2,\ldots,12 \right\}\\
&=\left\{ \left(\begin{array}{cccc|cccc}
1 & t_{r_1} & -t_{r_5} & t_{r_8} & t_{r_9} & t_{r_{11}} & t_{r_{12}} & -t_{r_1}t_{r_{12}}+t_{r_5}t_{r_{11}}-t_{r_8}t_{r_9} \\
 & 1 & t_{r_2} & t_{r_6} & t_{r_7} & t_{r_{10}} & -t_{r_2}t_{r_{10}}-t_{r_6}t_{r_7} & \begin{array}{c} t_{r_1}t_{r_2}t_{r_{10}}+t_{r_1}t_{r_6}t_{r_7} \\ -t_{r_2}t_{r_{11}}+t_{r_5}t_{r_{10}}-t_{r_6}t_{r_9} \\ -t_{r_7}t_{r_8}-t_{r_{12}} \end{array} \\
 & & 1 & t_{r_3} & t_{r_4} & -t_{r_3}t_{r_4} & \begin{array}{c} t_{r_2}t_{r_3}t_{r_4}-t_{r_3}t_{r_7} \\ -t_{r_4}t_{r_6}-t_{r_{10}} \end{array} & \begin{array}{c} -t_{r_1}t_{r_2}t_{r_3}t_{r_4}+t_{r_1}t_{r_3}t_{r_7} \\ +t_{r_1}t_{r_4}t_{r_6}-t_{r_3}t_{r_4}t_{r_5} \\ +t_{r_1}t_{r_{10}}-t_{r_3}t_{r_9} \\ -t_{r_4}t_{r_8}-t_{r_{11}} \end{array} \\
 & & & 1 & 0 & -t_{r_4} & t_{r_2}t_{r_4}-t_{r_7} & \begin{array}{c} -t_{r_1}t_{r_2}t_{r_4}+t_{r_1}t_{r_7} \\ -t_{r_4}t_{r_5}-t_{r_9} \end{array} \\
\hline
 & & & & 1 & -t_{r_3} & t_{r_2}t_{r_3}-t_{r_6} & \begin{array}{c} -t_{r_1}t_{r_2}t_{r_3}+t_{r_1}t_{r_6} \\ -t_{r_3}t_{r_5}-t_{r_8} \end{array} \\
 & & & & & 1 & -t_{r_2} & t_{r_1}t_{r_2}+t_{r_5} \\
 & & & & & & 1 & -t_{r_1} \\
 & & & & & & & 1
\end{array} \right) \right.\\
&\qquad \left| \; t_{r_i} \in \mathbb{F}_q,\ i = 1,2,\ldots,12 \right\}.
\end{aligned}
$$

Proof. By §A.4, $D_4^{syl}(q)$ is a Sylow p-subgroup of $D_4(q)$. By 2.1.2 and calculation, we get the matrix form as claimed. $\square$

Now we determine the Sylow p-subgroup ${}^3D_4^{syl}(q^3)$ of the Steinberg triality group. Let ρ be a linear transformation of $\mathcal{V}_4$ into itself arising from a non-trivial symmetry of the Dynkin diagram of $\mathcal{L}_{D_4}$ sending r_1 to r_3, r_3 to r_4, r_4 to r_1, and fixing r_2,



then $\rho^3 = \mathrm{id}_{\mathcal{V}_4}$ and

$$r_1 \overset{\rho}{\mapsto} r_3 \overset{\rho}{\mapsto} r_4, \quad r_2 \overset{\rho}{\mapsto} r_2, \quad r_5 \overset{\rho}{\mapsto} r_6 \overset{\rho}{\mapsto} r_7, \quad r_8 \overset{\rho}{\mapsto} r_{10} \overset{\rho}{\mapsto} r_9, \quad r_{11} \overset{\rho}{\mapsto} r_{11}, \quad r_{12} \overset{\rho}{\mapsto} r_{12}.$$

Thus $\rho(\Phi_{D_4}) = \Phi_{D_4}$ (see [Car72, Proposition 12.2.2]).

Let an automorphism of the Lie algebra $\mathcal{L}_{D_4}$ be determined by

$$h_r \mapsto h_{\rho(r)}, \quad e_r \mapsto e_{\rho(r)}, \quad e_{-r} \mapsto e_{-\rho(r)}, \qquad (r \in \Delta_{D_4})$$

and for every $r \in \Phi_{D_4}$ satisfying $e_r \mapsto \gamma_r e_{\rho(r)}$. We choose the suitable structure constants such that $\gamma_r = 1$ for all $r \in \Phi_{D_4}$ (A.5). If we choose different structure constants, some γ_r may be -1, but the following construction also determine a Sylow p-subgroup of Steinberg triality group, with the different signs of some t_i of the matrix expression of 2.1.12.

The Chevalley group $D_4(q^3)$ has a field automorphism F_q sending $x_r(t)$ to $x_r(t^q)$ which arises from the automorphism of $\mathbb{F}_{q^3}$:

$$\mathbb{F}_{q^3} \to \mathbb{F}_{q^3} : t \mapsto t^q,$$

and a graph automorphism ρ sending $x_r(t)$ to $x_{\rho(r)}(t)$ $(r \in \Phi_{D_4})$ (c.f. [Car72, 12.2.3]). Let $F := \rho F_q = F_q \rho$. For a subgroup X of $D_4(q^3)$, we set $X^F := \{x \in X | F(x) = x\}$. Then $D_4(q^3)^F = {}^3D_4(q^3)$.

Let $r \in \Phi^+_{D_4}$ and $t \in \mathbb{F}_{q^3}$, we write

$$x_{r^1}(t) := \begin{cases} x_r(t) & \text{if } \rho(r) = r,\ t^q = t \\ x_r(t) \cdot x_{\rho(r)}(t^q) \cdot x_{\rho^2(r)}(t^{q^2}) & \text{if } \rho(r) \neq r,\ t^{q^3} = t \end{cases}.$$

Then a Sylow p-subgroup of ${}^3D_4(q^3)$ is determined

$${}^3D_4^{syl}\left(q^3\right) := \left\{ x_{r_2^1}(t_2) x_{r_1^1}(t_1) x_{r_5^1}(t_5) x_{r_8^1}(t_8) x_{r_{11}^1}(t_{11}) x_{r_{12}^1}(t_{12}) \;\middle|\; \begin{cases} t_1, t_5, t_8 \in \mathbb{F}_{q^3}, \\ t_2, t_{11}, t_{12} \in \mathbb{F}_q \end{cases} \right\}.$$

2.1.5 Reminder ([Car72], 9.4.10 and 14.3.2)**.**

$$\begin{aligned} |D_4(q)| &= \frac{1}{4} q^{12}(q^2-1)(q^4-1)(q^6-1)(q^4-1), \\ |{}^3D_4\left(q^3\right)| &= q^{12}(q^2-1)(q^6-1)(q^8+q^4+1). \end{aligned}$$

2.1.6 Corollary. $|{}^3D_4^{syl}(q^3)| = q^{12}$.

2.1.7 Definition/Lemma (in ${}^3D_4^{syl}(q^3)$)**.** *Let* $t \in \mathbb{F}_{q^3}$*, set*

$$\begin{aligned} x_1(t) :&= x_{r_1^1}(t) = x_{r_3^1}(t^q) = x_{r_4^1}(t^{q^2}) \\ &= x_{\varepsilon_1-\varepsilon_2}(t) \cdot x_{\varepsilon_3-\varepsilon_4}(t^q) \cdot x_{\varepsilon_3+\varepsilon_4}(t^{q^2}) \\ &= \begin{pmatrix} 1 & t & & & & & & \cdot \\ & 1 & & & & & \cdot & \\ & & 1 & t^q & t^{q^2} & -t^{q^2+q} & & \\ & & & 1 & \cdot & -t^{q^2} & & \\ & & & & 1 & -t^q & & \\ & & & & & 1 & & \\ & & & & & & 1 & -t \\ & & & & & & & 1 \end{pmatrix}, \ t \in \mathbb{F}_{q^3}, \end{aligned}$$

$$\begin{aligned} x_2(t) :&= x_{r_2^1}(t) = x_{\varepsilon_2-\varepsilon_3}(t) \\ &= \begin{pmatrix} 1 & & & & & & & \cdot \\ & 1 & t & & & & \cdot & \\ & & 1 & & & \cdot & & \\ & & & 1 & \cdot & & & \\ & & & & 1 & & & \\ & & & & & 1 & -t & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{pmatrix}, \ t \in \mathbb{F}_q, \end{aligned}$$

$$\begin{aligned} x_3(t) :&= x_{r_5^1}(t) = x_{r_6^1}(t^q) = x_{r_7^1}(t^{q^2}) \\ &= x_{\varepsilon_1-\varepsilon_3}(t) \cdot x_{\varepsilon_2-\varepsilon_4}(t^q) \cdot x_{\varepsilon_2+\varepsilon_4}(t^{q^2}) \\ &= \begin{pmatrix} 1 & & -t & & & & & \cdot \\ & 1 & & t^q & t^{q^2} & & -t^{q^2+q} & \\ & & 1 & & & \cdot & & \\ & & & 1 & \cdot & & -t^{q^2} & \\ & & & & 1 & & -t^q & \\ & & & & & 1 & & t \\ & & & & & & 1 & \\ & & & & & & & 1 \end{pmatrix}, \ t \in \mathbb{F}_{q^3}, \end{aligned}$$

$$
\begin{aligned}
x_4(t) :&= x_{r_8^1}(t) = x_{r_{10}^1}(t^q) = x_{r_9^1}(t^{q^2}) \\
&= x_{\varepsilon_1-\varepsilon_4}(t) \cdot x_{\varepsilon_2+\varepsilon_3}(t^q) \cdot x_{\varepsilon_1+\varepsilon_4}(t^{q^2}) \\
&= \begin{pmatrix}
1 & & & t & t^{q^2} & & & -t^{q^2+1} \\
 & 1 & & & & t^q & \cdot & \\
 & & 1 & & & \cdot & -t^q & \\
 & & & 1 & \cdot & & & -t^{q^2} \\
 & & & & 1 & & & -t \\
 & & & & & 1 & & \\
 & & & & & & 1 & \\
 & & & & & & & 1
\end{pmatrix}, \ t \in \mathbb{F}_{q^3},
\end{aligned}
$$

$$
\begin{aligned}
x_5(t) :&= x_{r_{11}^1}(t) = x_{\varepsilon_1+\varepsilon_3}(t) \\
&= \begin{pmatrix}
1 & & & & & t & & \cdot \\
 & 1 & & & & & \cdot & \\
 & & 1 & & & \cdot & & -t \\
 & & & 1 & \cdot & & & \\
 & & & & 1 & & & \\
 & & & & & 1 & & \\
 & & & & & & 1 & \\
 & & & & & & & 1
\end{pmatrix}, \ t \in \mathbb{F}_q,
\end{aligned}
$$

$$
\begin{aligned}
x_6(t) :&= x_{r_{12}^1}(t) = x_{\varepsilon_1+\varepsilon_2}(t) \\
&= \begin{pmatrix}
1 & & & & & & t & \cdot \\
 & 1 & & & & & \cdot & -t \\
 & & 1 & & & \cdot & & \\
 & & & 1 & \cdot & & & \\
 & & & & 1 & & & \\
 & & & & & 1 & & \\
 & & & & & & 1 & \\
 & & & & & & & 1
\end{pmatrix}, \ t \in \mathbb{F}_q.
\end{aligned}
$$

The root subgroups of ${}^3D_4(q^3)$ are determined:

$$
\begin{aligned}
X_1 :=& \{x_1(t) \mid t \in \mathbb{F}_{q^3}\}, \\
X_3 :=& \{x_3(t) \mid t \in \mathbb{F}_{q^3}\}, \\
X_4 :=& \{x_4(t) \mid t \in \mathbb{F}_{q^3}\}, \\
X_2 :=& \{x_2(t) \mid t^q = t, t \in \mathbb{F}_{q^3}\} = \{x_2(t) \mid t \in \mathbb{F}_q\}, \\
X_5 :=& \{x_5(t) \mid t^q = t, t \in \mathbb{F}_{q^3}\} = \{x_5(t) \mid t \in \mathbb{F}_q\}, \\
X_6 :=& \{x_6(t) \mid t^q = t, t \in \mathbb{F}_{q^3}\} = \{x_6(t) \mid t \in \mathbb{F}_q\}.
\end{aligned}
$$

2.1.8 Corollary. *$|{}^3D_4^{syl}(q^3)| = q^{12}$, $|D_4^{syl}(q^3)| = q^{36}$, $|A_8(q^3)| = q^{84}$ and*

$$ {}^3D_4^{syl}\left(q^3\right) \leqslant D_4^{syl}\left(q^3\right) \leqslant A_8\left(q^3\right). $$

2.1.9 Definition. *A subgroup $P \leqslant {}^3D_4^{syl}(q^3)$ is a **pattern subgroup**, if it is generated by some root subgroups, i.e. $P := \langle X_i \mid i \in I \subseteq \{1, 2, \ldots, 6\} \rangle \leqslant {}^3D_4^{syl}(q^3)$.*

We get the commutators of ${}^3D_4^{syl}(q^3)$ by calculation.

2.1.10 Lemma. *Let $t_1, t_3, t_4 \in \mathbb{F}_{q^3}$, $t_2, t_5, t_6 \in \mathbb{F}_q$ and define the commutator*

$$ [x_i(t_i), x_j(t_j)] := x_i(t_i)^{-1}x_j(t_j)^{-1}x_i(t_i)x_j(t_j). $$

Then the non-trivial commutators of ${}^3D_4^{syl}(q^3)$ can be determined as follows:

$$ \begin{aligned} [x_1(t_1), x_2(t_2)] =& x_3(-t_2t_1)x_4(t_2t_1^{q+1})x_5(-t_2t_1^{q^2+q+1})x_6(2t_2^2t_1^{q^2+q+1}), \\ [x_2(t_2), x_1(t_1)] =& x_3(t_2t_1)x_4(-t_2t_1^{q+1})x_5(t_2t_1^{q^2+q+1})x_6(t_2^2t_1^{q^2+q+1}), \\ [x_1(t_1), x_3(t_3)] =& x_4(t_1t_3^q + t_1^qt_3)x_5(-t_1^{q+1}t_3^{q^2} - t_1^{q^2+q}t_3 - t_1^{q^2+1}t_3^q) \\ & \cdot x_6(-t_1t_3^{q^2+q} - t_1^qt_3^{q^2+1} - t_1^{q^2}t_3^{q+1}), \\ [x_1(t_1), x_4(t_4)] =& x_5(t_1t_4^q + t_1^qt_4^{q^2} + t_1^{q^2}t_4), \\ [x_3(t_3), x_4(t_4)] =& x_6(t_3t_4^q + t_3^qt_4^{q^2} + t_3^{q^2}t_4), \\ [x_2(t_2), x_5(t_5)] =& x_6(t_2t_5). \end{aligned} $$

2.1.11 Notation. *Set*

$$ x(t_1, t_2, t_3, t_4, t_5, t_6) := x_2(t_2)x_1(t_1)x_3(t_3)x_4(t_4)x_5(t_5)x_6(t_6) \in {}^3D_4^{syl}\left(q^3\right). $$

2.1.12 Proposition (Sylow p-subgroup ${}^3D_4^{syl}(q^3)$)**.** *A matrix Sylow p-subgroup of the Steinberg triality group ${}^3D_4(q^3)$ is ${}^3D_4^{syl}(q^3)$ with*

$$ {}^3D_4^{syl}\left(q^3\right) = \{x(t_1, t_2, t_3, t_4, t_5, t_6) \mid t_1, t_3, t_4 \in \mathbb{F}_{q^3},\ t_2, t_5, t_6 \in \mathbb{F}_q\}, $$

where

$$x(t_1,t_2,t_3,t_4,t_5,t_6)$$
$$= \begin{pmatrix}
1 & t_1 & -t_3 & t_1t_3^q + t_4 & t_1t_3^{q^2} + t_4^{q^2} & t_1t_4^q + t_5 & -t_1t_3^{q^2+q} + t_3t_4^q + t_6 & -t_1t_3^qt_4^{q^2} - t_1t_3^{q^2}t_4 - t_1t_6 + t_3t_5 - t_4^{q^2+1} \\
 & 1 & t_2 & t_1^qt_2 + t_3^q & t_1^{q^2}t_2 + t_3^{q^2} & -t_1^{q^2+q}t_2 + t_4^q & -t_1^qt_2t_3^{q^2} - t_1^{q^2}t_2t_3^q - t_2t_4^q - t_3^{q^2+q} & -t_1^{q^2+q}t_2t_3 - t_1^qt_2t_4^{q^2} - t_1^{q^2}t_2t_4 - t_2t_5 - t_3^qt_4^{q^2} - t_3^{q^2}t_4 - t_6 \\
 & & 1 & t_1^q & t_1^{q^2} & -t_1^{q^2+q} & -t_1^qt_3^{q^2} - t_1^{q^2}t_3^q - t_4^q & -t_1^{q^2+q}t_3 - t_1^qt_4^{q^2} - t_1^{q^2}t_4 - t_5 \\
 & & & 1 & 0 & -t_1^{q^2} & -t_3^{q^2} & -t_1^{q^2}t_3 - t_4^{q^2} \\
 & & & & 1 & -t_1^q & -t_3^q & -t_1^qt_3 - t_4 \\
 & & & & & 1 & -t_2 & t_1t_2 + t_3 \\
 & & & & & & 1 & -t_1 \\
 & & & & & & & 1
\end{pmatrix}$$

Proof. By 2.1.6, ${}^3D_4^{syl}(q^3)$ is a Sylow p-subgroup of ${}^3D_4(q^3)$. By calculation, we obtain the matrix expression as claimed by 2.1.7. $\square$

Let $J := \{(1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(2,3)\} \subseteq \bigtriangledown$. We compare the Sylow p-subgroups $A_n(q)$, $D_n^{syl}(q)$ and ${}^3D_4^{syl}(q^3)$.

2.1.13 Comparison (Sylow p-subgroups). *(1) There exists an order of*

$$\prod_{(i,j)\in \triangledown} \tilde{x}_{i,j}(t_{i,j}),$$

such that for every matrix $u = (u_{i,j}) \in A_n(q)$, we have $u_{k,l} = t_{k,l}$ for all $(k,l) \in \triangledown$.

(2) There exists an order of $\prod_{(i,j)\in\bigtriangledown} x_{i,j}(t_{i,j})$, such that for every matrix $u = (u_{i,j}) \in D_n^{syl}(q)$, we have $u_{k,l} = t_{k,l}$ for all $(k,l) \in \bigtriangledown$ (see 2.1.4).

(3) For ${}^3D_4^{syl}(q^3)$, the property does not hold. In 2.1.12, for instance, we have matrix entries t_1, t_2 and up to sign also t_3 with postitions in J, but t_4, t_5 and t_6 appear in J only in polynomials involving the other parameters.

2.1.2. The group $G_8(q^3)$

In this subsection, we construct a group $G_8(q^3)$ such that ${}^3D_4^{syl}(q^3) \leqslant G_8(q^3) \leqslant A_8(q^3)$. Then we determine a monomial $G_8(q^3)$-module to imitate the D_n case in Section 2.2. In Subsection 2.3.1, we use the group $G_8(q^3)$ to calculate the superclasses of ${}^3D_4^{syl}(q^3)$.

2.1.14 Notation. *For* $t \in \mathbb{F}_{q^3}$, *set*

$$\begin{aligned} \tilde{x}_{ij}(t) &:= 1 + te_{ij} \in A_8\left(q^3\right) && \textit{for all} (i,j) \in \nabla, \\ x_{i,j}(t) &:= 1 + te_{i,j} - te_{\bar{j},\bar{i}} = \tilde{x}_{ij}(t)\tilde{x}_{9-j,9-i}(-t) \in D_4^{syl}\left(q^3\right) && \textit{for all } (i,j) \in \triangledown. \end{aligned}$$

The following is a well known property.

2.1.15 Proposition. *Let* $(i,j) \in \nabla$, *then*

$$A_8(\mathbb{F}_q) = \left\{ \prod_{(i,j)\in\nabla} \tilde{x}_{i,j}(t_{i,j}) \,\middle|\, t_{i,j} \in \mathbb{F}_q \right\},$$

where the product can be taken in an arbitrary, but fixed, order. In particular,

$$\prod_{(i,j)\in\nabla} \tilde{x}_{i,j}(t_{i,j}) = \prod_{(i,j)\in\nabla} \tilde{x}_{i,j}(s_{i,j}) \iff t_{i,j} = s_{i,j} \quad \textit{for all } (i,j) \in \nabla.$$

2.1.16 Definition/Lemma (A bigger group $G_8(q^3)$). *Set*

$$G_8\left(q^3\right) := \left\{ u = (u_{i,j}) \in A_8\left(q^3\right) \,\middle|\, \begin{cases} u_{i,j} = 0 & \textit{if } (i,j) = (4,5) \\ u_{i,j} \in \mathbb{F}_q & \textit{if } (i,j) \in \{(2,3),(6,7)\} \\ u_{i,j+1} = u_{i,j}^q \in \mathbb{F}_{q^3} & \textit{if } (i,j) \in \{(2,4),(3,4)\} \\ u_{i-1,j} = u_{i,j}^q \in \mathbb{F}_{q^3} & \textit{if } (i,j) \in \{(5,6),(5,7)\} \end{cases} \right\}$$

$$= \left\{ \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & u_{12} & u_{13} & u_{14} & u_{15} & u_{16} & u_{17} & u_{18} \\ \hline & 1 & u_{23} & u_{24} & u_{24}^q & u_{26} & u_{27} & u_{28} \\ \hline & & 1 & u_{34} & u_{34}^q & u_{36} & u_{37} & u_{38} \\ \hline & & & 1 & 0 & u_{56}^q & u_{57}^q & u_{48} \\ \hline & & & & 1 & u_{56} & u_{57} & u_{58} \\ \hline & & & & & 1 & u_{67} & u_{68} \\ \hline & & & & & & 1 & u_{78} \\ \hline & & & & & & & 1 \\ \hline \end{array} \,\middle|\, u_{ij} \in \begin{cases} \mathbb{F}_q, & (i,j) \in \{(2,3),(6,7)\} \\ \mathbb{F}_{q^3}, & \textit{otherwise} \end{cases} \right\}.$$

Then $G_8(q^3)$ *is a subgroup of* $A_8(q^3)$ *of order* q^{65}.

Proof. By direct calculation. $\square$

2.1.17 Notation. *Write $\ddot{J} := \triangledown\backslash\{(2,5),(3,5),(4,5),(4,6),(4,7)\}$. For $(i,j) \in \ddot{J}$ and $t \in \mathbb{F}_{q^3}$, set*

$$\dot{x}_{ij}(t) := \begin{cases} \tilde{x}_{ij}(t)\tilde{x}_{i(j+1)}(t^q), & (i,j) \in \{(2,4),(3,4)\} \\ \tilde{x}_{ij}(t)\tilde{x}_{(i-1)j}(t^q), & (i,j) \in \{(5,6),(5,7)\} \\ \tilde{x}_{ij}(t), & \textit{otherwise} \end{cases}.$$

2.1.18 Definition. *For $(i,j) \in \ddot{J}$, define the subgroups of $G_8(q^3)$ as follows:*

$$\dot{X}_{i,j} := \begin{cases} \{\dot{x}_{ij}(t) \mid t \in \mathbb{F}_q\}, & (i,j) \in \{(2,3),(6,7)\} \\ \{\dot{x}_{ij}(t) \mid t \in \mathbb{F}_{q^3}\}, & \textit{otherwise} \end{cases}.$$

2.1.19 Proposition.

$$G_8\left(q^3\right) = \left\{ \prod_{(i,j)\in\ddot{J}} \dot{x}_{ij}(t_{ij}) \,\middle|\, t_{ij} \in \begin{cases} \mathbb{F}_q, & (i,j) \in \{(2,3),(6,7)\} \\ \mathbb{F}_{q^3}, & \textit{otherwise} \end{cases} \right\},$$

where the product can be taken in an arbitrary, but fixed, order.

Proof. Let S denote the right side, then $S \subseteq G_8(q^3)$ since $\dot{x}_{ij}(t_{ij}) \in G_8(q^3)$.

Fix an order of the product. Suppose $\prod_{(i,j)\in\ddot{J}} \dot{x}_{ij}(t_{ij}) = \prod_{(i,j)\in\ddot{J}} \dot{x}_{ij}(s_{ij})$, then $t_{i,j} = s_{i,j}$ for all $(i,j) \in \ddot{J}$ by Proposition 2.1.15 and $|S| = q^{65}$. Then $S = G_8(q^3)$ since $|G_8(q^3)| = q^{65}$. $\square$

2.1.20 Reminder.

$${}^3D_4^{syl}\left(q^3\right) = \{x_2(t_2)x_1(t_1)x_3(t_3)x_4(t_4)x_5(t_5)x_6(t_6) \mid t_1,t_3,t_4 \in \mathbb{F}_{q^3},\ t_2,t_5,t_6 \in \mathbb{F}_q\}$$

$$= \left\{ \begin{array}{l} \tilde{x}_{23}(t_2)\tilde{x}_{67}(-t_2) \\ \cdot\tilde{x}_{12}(t_1)\tilde{x}_{78}(-t_1)\cdot\tilde{x}_{34}(t_1^q)\tilde{x}_{35}(t_1^{q^2}) \\ \cdot\tilde{x}_{56}(-t_1^q)\tilde{x}_{46}(-t_1^{q^2})\cdot\tilde{x}_{36}(t_1^{q^2+q}) \\ \cdot\tilde{x}_{13}(-t_3)\tilde{x}_{68}(t_3)\cdot\tilde{x}_{24}(t_3^q)\tilde{x}_{25}(t_3^{q^2}) \\ \cdot\tilde{x}_{57}(-t_3^q)\tilde{x}_{47}(-t_3^{q^2})\cdot\tilde{x}_{27}(t_3^{q^2+q}) \\ \cdot\tilde{x}_{14}(t_4)\tilde{x}_{58}(-t_4)\cdot\tilde{x}_{26}(t_4^q)\tilde{x}_{37}(-t_4^q) \\ \cdot\tilde{x}_{15}(t_4^{q^2})\tilde{x}_{48}(-t_4^{q^2})\cdot\tilde{x}_{16}(t_5)\tilde{x}_{38}(-t_5) \\ \cdot\tilde{x}_{17}(t_6)\tilde{x}_{28}(-t_6) \end{array} \,\middle|\, t_1,t_3,t_4 \in \mathbb{F}_{q^3},\ t_2,t_5,t_6 \in \mathbb{F}_q \right\}.$$

2.1.21 Corollary.

$$G_8\left(q^3\right)=\left\{\begin{array}{l}\tilde{x}_{23}(t_{23})\cdot\tilde{x}_{67}(t_{67})\\ \cdot\tilde{x}_{12}(t_{12})\cdot\tilde{x}_{78}(t_{78})\cdot\tilde{x}_{34}(t_{34})\tilde{x}_{35}(t_{34}^q)\\ \cdot\tilde{x}_{56}(t_{56})\tilde{x}_{46}(t_{56}^q)\cdot\tilde{x}_{36}(t_{36})\\ \cdot\tilde{x}_{13}(t_{13})\cdot\tilde{x}_{68}(t_{68})\cdot\tilde{x}_{24}(t_{24})\tilde{x}_{25}(t_{24}^q)\\ \cdot\tilde{x}_{57}(t_{57})\tilde{x}_{47}(t_{57}^q)\cdot\tilde{x}_{27}(t_{27})\\ \cdot\tilde{x}_{14}(t_{14})\tilde{x}_{58}(t_{58})\cdot\tilde{x}_{26}(t_{26})\tilde{x}_{37}(t_{37})\\ \cdot\tilde{x}_{15}(t_{15})\tilde{x}_{48}(t_{48})\cdot\tilde{x}_{18}(t_{18})\\ \cdot\tilde{x}_{16}(t_{16})\cdot\tilde{x}_{38}(t_{38})\\ \cdot\tilde{x}_{17}(t_{17})\cdot\tilde{x}_{28}(t_{28})\end{array}\;\middle|\; t_{ij}\in\left\{\begin{array}{ll}\mathbb{F}_q, & (i,j)\in\{(2,3),(6,7)\}\\ \mathbb{F}_{q^3}, & \textit{otherwise}\end{array}\right.\right\}$$

$$=\left\{\left.\begin{array}{|c|c|c|c||c|c|c|c|}\hline 1 & t_{12} & t_{13} & \begin{array}{r}t_{12}t_{24}\\+t_{14}\end{array} & \begin{array}{r}t_{12}t_{24}^q\\+t_{15}\end{array} & t_{12}t_{26}+t_{16} & \begin{array}{r}t_{12}t_{24}t_{57}^q\\+t_{12}t_{24}^qt_{57}\\+t_{12}t_{27}\\+t_{13}t_{37}\\+t_{17}\end{array} & \begin{array}{r}t_{12}t_{24}^qt_{58}\\+t_{12}t_{24}t_{48}\\+t_{14}t_{48}\\+t_{13}t_{38}\\+t_{12}t_{28}\\+t_{18}\end{array}\\ \hline & 1 & t_{23} & \begin{array}{r}t_{23}t_{34}\\+t_{24}\end{array} & \begin{array}{r}t_{23}t_{34}^q\\+t_{24}^q\end{array} & \begin{array}{r}t_{23}t_{34}^qt_{56}\\+t_{23}t_{34}t_{56}^q\\+t_{23}t_{36}\\+t_{26}\end{array} & \begin{array}{r}t_{24}t_{57}^q\\+t_{24}^qt_{57}\\+t_{23}t_{34}t_{57}^q\\+t_{23}t_{34}^qt_{57}\\+t_{23}t_{37}\\+t_{27}\end{array} & \begin{array}{r}t_{23}t_{34}^qt_{56}t_{68}\\+t_{23}t_{34}t_{56}^qt_{68}\\+t_{23}t_{36}t_{68}\\+t_{24}^qt_{58}\\+t_{24}t_{48}\\+t_{23}t_{34}^qt_{58}\\+t_{23}t_{34}t_{48}\\+t_{23}t_{38}\\+t_{28}\end{array}\\ \hline & & 1 & t_{34} & t_{34}^q & \begin{array}{r}t_{34}^qt_{56}\\+t_{34}t_{56}^q\\+t_{36}\end{array} & \begin{array}{r}t_{34}t_{57}^q\\+t_{34}^qt_{57}\\+t_{37}\end{array} & \begin{array}{r}t_{34}^qt_{56}t_{68}\\+t_{34}t_{56}^qt_{68}\\+t_{36}t_{68}\\+t_{34}^qt_{58}\\+t_{34}t_{48}\\+t_{38}\end{array}\\ \hline & & & 1 & 0 & t_{56}^q & t_{57}^q & t_{56}^qt_{68}+t_{48}\\ \hline\hline & & & & 1 & t_{56} & t_{57} & t_{56}t_{68}+t_{58}\\ \hline & & & & & 1 & t_{67} & t_{67}t_{78}+t_{68}\\ \hline & & & & & & 1 & t_{78}\\ \hline & & & & & & & 1\\ \hline\end{array}\right.\right.$$

$$\left| t_{ij}\in\left\{\begin{array}{ll}\mathbb{F}_q, & (i,j)\in\{(2,3),(6,7)\}\\ \mathbb{F}_{q^3}, & \textit{otherwise}\end{array}\right.\right\}.$$

2.1.22 Corollary. *$|{}^3D_4^{syl}\left(q^3\right)| = q^{12}$, $|G_8\left(q^3\right)| = q^{65}$, $|A_8\left(q^3\right)| = q^{84}$ and*

$$ {}^3D_4^{syl}\left(q^3\right) \leqslant G_8\left(q^3\right) \leqslant A_8\left(q^3\right). $$

2.1.23 Comparison (Bigger groups). *(1) The bigger group of $A_n(q)$ is $A_n(q)$ itself.*

(2) The bigger group of $D_n^{syl}(q)$ is $A_{2n}(q)$ (see [Jed13, 3.1.2]).

(3) $G_8\left(q^3\right)$ is one bigger group of ${}^3D_4^{syl}\left(q^3\right)$ (see 2.1.16).

2.2. Monomial ${}^3D_4^{syl}(q^3)$-module

Let $p \neq 2$, $n = 4$, $N = 8$, $U := {}^3D_4^{syl}(q^3)$ and $G := G_8(q^3)$.

In this section, we determine a monomial linearisation $(f, \kappa_q|_{V\times V})$ for $G_8(q^3)$ (2.2.32), for which $f|_U$ is bijective (2.2.31). We obtain a monomial $G_8(q^3)$-module $\mathbb{C}U$ (2.2.34). Then we classify the U-orbit modules (Table 2.1) and the $G_8(q^3)$-orbit modules (2.2.57). We prove that every U-orbit module is isomorphic to some hook-separated staircase module (2.2.79). Finally, some irreducible modules are determined, and show any two orbit modules from different families are orthogonal (2.2.83).

2.2.1. Monomial ${}^3D_4^{syl}(q^3)$-module

Let $G := G_8(q^3)$, $U := {}^3D_4^{syl}(q^3)$ and $V_0 := \mathrm{Mat}_{8\times 8}(q^3)$. In this subsection, we construct an $\mathbb{F}_q$-subspace V of V_0 (2.2.15), a projection $\pi\colon V_0 \to V$ (2.2.16 and 2.2.19) and a non-degenerate bilinear form $\kappa_q|_{V\times V}$ (2.2.18). Then we determine an $\mathbb{F}_q$-linear group action $- \circ -$ (2.2.25) and a surjective 1-cocycle f of G in V (2.2.32). Thus the monomial linearisation $(f, \kappa_q|_{V\times V})$ for G (2.2.32) is established. Since $f|_U$ is a bijective 1-cocycle (2.2.31), $(f|_U, \kappa_q|_{V\times V})$ is a monomial linearisation for U (2.2.33). Then we make $\mathbb{C}U$ into a monomial G-module (2.2.34).

It is essential to choose a suitable vector space V for the 1-cocycle and a suitable non-degenerate bilinear form κ_q on V_0.

2.2.1 Notation/Lemma. *Set $V_0 := \mathrm{Mat}_{8\times 8}(q^3)$, then V_0 is an $\mathbb{F}_{q^3}$-vector space and also an $\mathbb{F}_q$-vector space.*

2.2.2 Notation/Lemma. *The map*

$$\kappa\colon V_0 \times V_0 \to \mathbb{F}_{q^3} : (A, B) \mapsto \mathrm{tr}(A^\top B)$$

is a symmetric $\mathbb{F}_{q^3}$-bilinear form on V_0 which is called the ***trace form****.*

We know $J = \{(1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (2,3)\}$.

2.2.3 Notation/Lemma. *Set*

$$V_J = \bigoplus_{(i,j)\in J} \mathbb{F}_{q^3} e_{ij}$$

$$= \left\{ \begin{pmatrix} \cdot & \boxed{A_{12}} & \boxed{A_{13}} & \boxed{A_{14}} & \boxed{A_{15}} & \boxed{A_{16}} & \boxed{A_{17}} & \cdot \\ & \cdot & \boxed{A_{23}} & & & & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & \cdot & \cdot & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{pmatrix} \in V_0 \;\middle|\; A_{i,j} \in \mathbb{F}_{q^3} \right\}.$$

Then V_J is an $\mathbb{F}_{q^3}$-subspace and also an $\mathbb{F}_q$-subspace of V_0 with $\dim_{\mathbb{F}_q} V_J = 21$. In particular, $V_\square = V_0$.

2.2.4 Lemma. *(a) Let $A = (A_{i,j}) \in V_0$ and $(i,j) \in \square$, then $\kappa(A, e_{i,j}) = A_{i,j}$.*

(b) Let $A, B, C \in V_0$, then $\kappa(B^\top A, C) = \kappa(A, BC) = \kappa(AC^\top, B)$.

Proof. (a) Let $A \in V_0$, then

$$\kappa(A, e_{i,j}) = \operatorname{tr}(A^\top e_{i,j}) = \sum_{k=1}^{N} (A^\top e_{i,j})_{k,k} = \sum_{k=1}^{N}\sum_{l=1}^{N} (A^\top_{k,l}(e_{i,j})_{l,k}) \overset{(l,k)=(i,j)}{=} A^\top_{j,i} = A_{i,j}.$$

(b) Let $A, B, C \in V_0$, then

$$\begin{aligned}\kappa(B^\top A, C) =& \operatorname{tr}((A^\top B)C) = \operatorname{tr}(A^\top(BC)) = \kappa(A, BC),\\ \kappa(AC^\top, B) =& \operatorname{tr}(CA^\top B) = \operatorname{tr}(A^\top BC) = \kappa(A, BC).\end{aligned}$$

□

2.2.5 Corollary. *Let $A, B \in V_0$, then $\kappa(A, B) = \sum_{(i,j)\in\square} A_{i,j}B_{i,j}$.*

2.2.6 Lemma. *Let $V_J^\perp$ denote the orthogonal complement of V_J in V_0 with respect to the trace form κ, i.e.*

$$V_J^\perp := \{B \in V_0 \mid \kappa(A, B) = 0,\ \forall\ A \in V_J\}.$$

Then $V_J^\perp = V_{\square\setminus J}$ and $V_0 = V_J \oplus V_J^\perp$.

Proof. Let $A, B \in V_0$, then $\kappa(A, B) = \sum_{(i,j)\in\square} A_{i,j}B_{i,j}$ by **2.2.5**, so $V_{\square\setminus J} \subseteq V_J^\perp$. On the other hand, let $B \in V_J^\perp$ and $(i,j) \in J$, then $0 = \kappa(B, e_{ij}) = B_{ij}$ and $B \in V_{\square\setminus J}$. Thus $V_J^\perp = V_{\square\setminus J}$. Since $V_J \cap V_{\square\setminus J} = \{0\}$, $V_0 = V_J \oplus V_{\square\setminus J} = V_J \oplus V_J^\perp$. □

2.2.7 Corollary. *$\kappa|_{V_J\times V_J} : V_J \times V_J \to \mathbb{F}_{q^3}$ is a non-degenerate bilinear form. In particular, κ is a non-degenerate symmetric bilinear form.*

Proof. Let $A \in V_J$ such that $\kappa(A, B) = 0$ for all $B \in V_J$, then $A \in V_J^\perp$. Then $A = 0$ since $V_J \cap V_J^\perp = \{0\}$. We know that κ is bilinear, thus $\kappa|_{V_J\times V_J}$ is bilinear. □

2.2.8 Notation/Lemma. *Let*

$$\pi_J : V_0 = V_J \oplus V_J^\perp \to V_J : A \mapsto \sum_{(i,j)\in J} A_{i,j}e_{i,j}.$$

Then π_J is a projection of V_0 to the first component V_J.

2.2.9 Lemma. *Suppose $A, B \in V_0$, such that $\operatorname{supp}(A) \cap \operatorname{supp}(B) \subseteq J$. Then*

$$\begin{aligned}&\kappa(A, B) = \kappa(\pi_J(A), B) = \kappa(A, \pi_J(B))\\ =&\kappa(\pi_J(A), \pi_J(B)) = \kappa|_{V_J\times V_J}(\pi_J(A), \pi_J(B)).\end{aligned}$$

Proof. We know

$$\kappa(A,B) = \sum_{(i,j)\in\square} A_{i,j}B_{i,j} \overset{\mathrm{supp}(A)\cap\mathrm{supp}(B)\subseteq J}{=} \sum_{(i,j)\in J} A_{i,j}B_{i,j} = \kappa(\pi_J(A),\pi_J(B)).$$

Thus the result is proved. □

2.2.10 Lemma. *Let*

$$\phi_0\colon \mathbb{F}_{q^3} \to \mathbb{F}_q : t \mapsto t + t^q + t^{q^2},$$

then ϕ_0 is an $\mathbb{F}_q$-epimorphism and $|\mathrm{ker}\phi_0| = q^2$.

Proof. (1) Let $t, s \in \mathbb{F}_{q^3}$ and $k \in \mathbb{F}_q$, then

$$\begin{aligned}\phi_0(t+s) =& (t+s) + (t+s)^q + (t+s)^{q^2} = (t + t^q + t^{q^2}) + (s + s^q + s^{q^2}) \\ =& \phi_0(t) + \phi_0(s), \\ \phi_0(kt) =& (kt) + (kt)^q + (kt)^{q^2} = k(t + t^q + t^{q^2}) = k\phi_0(t).\end{aligned}$$

Thus ϕ_0 is an $\mathbb{F}_q$-homomorphism.

(2) Observe that $\ker\phi_0 = \{t \in \mathbb{F}_{q^3} \mid t+t^q+t^{q^2} = 0\}$, and that the degree of $t+t^q+t^{q^2}$ is q^2, so $|\ker\phi_0| \le q^2$. On the other hand, $\mathrm{im}\,\phi_0 \subseteq \mathbb{F}_q \implies |\mathrm{im}\,\phi_0| \le q$. We have

$$\mathbb{F}_{q^3}/\ker\phi_0 \cong \mathrm{im}\,\phi_0 \Longrightarrow |\ker\phi_0| = \frac{|\mathbb{F}_{q^3}|}{|\mathrm{im}\,\phi_0|} \ge q^2.$$

Thus, $|\mathrm{ker}\phi_0| = q^2$ and $\mathrm{im}\,\phi_0 = \mathbb{F}_q$.
Therefore, ϕ_0 is an $\mathbb{F}_q$-epimorphism. □

2.2.11 Proposition. *There exists an element $\eta \in \mathbb{F}_{q^3}\backslash\mathbb{F}_q$ (i.e. $\eta \in \mathbb{F}_{q^3}$ but $\eta \notin \mathbb{F}_q$) such that*

$$\eta^{q^2} + \eta^q + \eta = 1.$$

Proof. By Lemma 2.2.10, $\#\{t \in \mathbb{F}_{q^3} \mid t^{q^2} + t^q + t = 1\} = q^2$. But $|\mathbb{F}_q| = q$, so there are at least $q^2 - q\ (>1)$ elements of $\mathbb{F}_{q^3}$ which satisfy the equation $t^{q^2} + t^q + t = 1$. Then the claim is proved. □

From now on in this chapter, we fix an element $\eta \in \mathbb{F}_{q^3}\backslash\mathbb{F}_q$ such that $\eta^{q^2}+\eta^q+\eta = 1$.

2.2.12 Corollary. $1 + \eta^{1-q^2} \neq 0$.

Proof. Suppose that $1+\eta^{1-q^2}=0$. Multiply η^{q^2} $(\neq 0)$ to both sides of the formula, then

$$\eta^{q^2}+\eta=0 \overset{2.2.11}{\Longrightarrow} \eta^q=1 \implies \eta=1\in\mathbb{F}_q.$$

This is a contradiction since $\eta\in\mathbb{F}_{q^3}\backslash\mathbb{F}_q$. □

2.2.13 Notation/Lemma. *Let*

$$\pi_q\colon \mathbb{F}_{q^3}\to\mathbb{F}_q : x\mapsto \phi_0(\eta x)=(\eta x)^{q^2}+(\eta x)^q+\eta x,$$

then π_q is an $\mathbb{F}_q$-epimorphism and $\mathbb{F}_{q^3}=\ker\pi_q\oplus\mathbb{F}_q$. In particular, $\pi_q|_{\mathbb{F}_q}=\mathrm{id}_{\mathbb{F}_q}$ and $\pi_q^2=\pi_q$.

Proof. Since $\eta\neq 0$ and ϕ_0 is an $\mathbb{F}_q$-epimorphism, π_q is an $\mathbb{F}_q$-epimorphism.

Let $x\in\mathbb{F}_q$, then $\pi_q(x)=x\pi_q(1)=x\phi_0(\eta)=x(\eta^{q^2}+\eta^q+\eta)=x$, so $\pi_q|_{\mathbb{F}_q}=\mathrm{id}_{\mathbb{F}_q}$. Then $\pi_q^2=\pi_q$ since the image $\operatorname{im}\pi_q=\mathbb{F}_q$.

The image $\operatorname{im}\pi_q=\mathbb{F}_q$ and the kernel $\ker\pi_q$ are $\mathbb{F}_q$-subspaces of $\mathbb{F}_{q^3}$, so $(\ker\pi_q+\mathbb{F}_q)\subseteq\mathbb{F}_{q^3}$. Let $x\in\ker\pi_q\cap\mathbb{F}_q$, then $x\overset{x\in\mathbb{F}_q}{=}\pi_q(x)\overset{x\in\ker\pi_q}{=}0$, so $\ker\pi_q\cap\mathbb{F}_q=\{0\}$. Let $x\in\mathbb{F}_{q^3}$, then $\pi_q(x-\pi_q(x))=(\pi_q-\pi_q^2)(x)=0$, so $x-\pi_q(x)\in\ker\pi_q$. Hence $x=(x-\pi_q(x))+\pi_q(x)$. Therefore, $\mathbb{F}_{q^3}=\ker\pi_q\oplus\mathbb{F}_q$. □

2.2.14 Proposition. *Let*

$$\kappa_q:=\pi_q\circ\kappa\colon V_0\times V_0\to\mathbb{F}_q,$$

where $\kappa_q(A,B)=\pi_q\circ\kappa(A,B)=\pi_q\big(\mathrm{tr}(A^\top B)\big)$. Then κ_q is a symmetric $\mathbb{F}_q$-bilinear form on V_0.

Proof. We know that κ is a symmetric bilinear map (2.2.2), and that π_q is an $\mathbb{F}_q$-epimorphism (2.2.13), then the claim is proved. □

2.2.15 Notation/Lemma. *Let*

$$\begin{aligned} V:&=\{A=(A_{ij})\in V_0 \mid \mathrm{supp}(A)\in J,\ A_{16},\ A_{17},\ A_{23}\in\mathbb{F}_q,\ A_{15}^q=A_{14}\}\\ &=\left\{\left(\begin{array}{ccccccccc} \cdot & \boxed{A_{12}} & \boxed{A_{13}} & \boxed{A_{15}^q} & \boxed{A_{15}} & \boxed{A_{16}} & \boxed{A_{17}} & \cdot & \\ & \cdot & \boxed{A_{23}} & & & & \cdot & & \\ & & \cdot & & & \cdot & & & \\ & & & \cdot & \cdot & & & & \\ & & & & & & & & \end{array}\right)\in V_0 \;\middle|\; \begin{cases} A_{12},A_{13},A_{15}\in\mathbb{F}_{q^3}\\ A_{16},A_{17},A_{23}\in\mathbb{F}_q\end{cases}\right\}.\end{aligned}$$

Then V is a 12-dimensional subspace of V_J over $\mathbb{F}_q$ and $\mathrm{supp}(V)=J$.

We define the following map π, which plays a crucial role in our later statement.

2.2.16 Notation/Lemma. *Set*

$$\pi\colon V_0 \to V : A \mapsto \begin{array}{l} A_{12}e_{12} + A_{13}e_{13} \\ +(\frac{A_{14}^{q^2}+A_{15}\eta^{1-q^2}}{1+\eta^{1-q^2}})^q e_{14} + \frac{A_{14}^{q^2}+A_{15}\eta^{1-q^2}}{1+\eta^{1-q^2}} e_{15} \\ +\pi_q(A_{16})e_{16} + \pi_q(A_{17})e_{17} + \pi_q(A_{23})e_{23} \end{array},$$

i.e.

$$\pi(A) = \left(\begin{array}{|c|c|c|c|c|c|} \hline A_{12} & A_{13} & (\frac{A_{14}^{q^2}+A_{15}\eta^{1-q^2}}{1+\eta^{1-q^2}})^q & \frac{A_{14}^{q^2}+A_{15}\eta^{1-q^2}}{1+\eta^{1-q^2}} & \pi_q(A_{16}) & \pi_q(A_{17}) \\ \hline \multicolumn{1}{c|}{} & \pi_q(A_{23}) & \multicolumn{4}{l}{} \\ \cline{2-2} \end{array} \right)$$

Then π is an $\mathbb{F}_q$-epimorphism. In particular, $\pi|_V = \mathrm{id}_V$, $\pi^2 = \pi$ and $\pi(I_8) = O_8$.

Proof. Let $A = (A_{ij}), B = (B_{ij}) \in V_0$ and $k \in \mathbb{F}_q$, then

$$\begin{aligned} &\pi(A+B) \\ =&(A_{12}+B_{12})e_{12} + (A_{13}+B_{13})e_{13} \\ &+ (\frac{(A_{14}+B_{14})^{q^2} + (A_{15}+B_{15})\eta^{1-q^2}}{1+\eta^{1-q^2}})^q e_{14} + \frac{(A_{14}+B_{14})^{q^2} + (A_{15}+B_{15})\eta^{1-q^2}}{1+\eta^{1-q^2}} e_{15} \\ &+ \pi_q(A_{16}+B_{16})e_{16} + \pi_q(A_{17}+B_{17})e_{17} + \pi_q(A_{23}+B_{13})e_{23} \\ =&(A_{12}+B_{12})e_{12} + (A_{13}+B_{13})e_{13} + (\frac{A_{14}^{q^2} + A_{15}\eta^{1-q^2}}{1+\eta^{1-q^2}})^q e_{14} + (\frac{B_{14}^{q^2} + B_{15}\eta^{1-q^2}}{1+\eta^{1-q^2}})^q e_{14} \\ &+ \frac{A_{14}^{q^2} + A_{15}\eta^{1-q^2}}{1+\eta^{1-q^2}} e_{15} + \frac{B_{14}^{q^2} + B_{15}\eta^{1-q^2}}{1+\eta^{1-q^2}} e_{15} \\ &+ \pi_q(A_{16}+B_{16})e_{16} + \pi_q(A_{17}+B_{17})e_{17} + \pi_q(A_{23}+B_{13})e_{23} \\ \overset{2.2.13}{=}&\pi(A) + \pi(B). \end{aligned}$$

We obtain $\pi(kA) = k\pi(A)$ similarly. Thus π is an $\mathbb{F}_q$-homomorphism. By 2.2.13 and 2.2.15, $\pi|_V = \mathrm{id}_V$ and $\pi^2 = \pi$. We get that $\pi(I_8) = O_8$ from the definition of π. □

2.2.17 Proposition. *Let $V^\perp$ denote the orthogonal complement of the subspace V of V_0 with respect to κ_q, i.e.*

$$V^\perp := \{B \in V_0 \mid \kappa_q(A,B) = 0 \ \textit{for all}\ A \in V\},$$

and

$$W := \bigoplus_{(i,j)\notin J} \mathbb{F}_{q^3} e_{ij} + \ker \pi_q e_{16} + \ker \pi_q e_{17} + \ker \pi_q e_{23} + \{x e_{15} - x^q \eta^{q-1} e_{14} \mid x \in \mathbb{F}_{q^3}\}$$
$$= \{A = (A_{ij}) \in V_0 \mid A_{12} = A_{13} = 0, A_{14} = -A_{15}^q \eta^{q-1} \in \mathbb{F}_{q^3}, A_{16}, A_{17}, A_{23} \in \ker \pi_q\}$$

$$= \left\{ \begin{array}{|c|c|c|c|c|c|c|c|} \hline A_{11} & 0 & 0 & -A_{15}^q\eta^{q-1} & A_{15} & A_{16} & A_{17} & A_{18} \\ \hline A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} & A_{27} & A_{28} \\ \hline A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & A_{36} & A_{37} & A_{38} \\ \hline A_{41} & A_{42} & A_{43} & A_{44} & A_{45} & A_{46} & A_{47} & A_{48} \\ \hline A_{51} & A_{52} & A_{53} & A_{54} & A_{55} & A_{56} & A_{57} & A_{58} \\ \hline A_{61} & A_{62} & A_{63} & A_{64} & A_{65} & A_{66} & A_{67} & A_{68} \\ \hline A_{71} & A_{72} & A_{73} & A_{74} & A_{75} & A_{76} & A_{77} & A_{78} \\ \hline A_{81} & A_{82} & A_{83} & A_{84} & A_{85} & A_{86} & A_{87} & A_{88} \\ \hline \end{array} \in V_0 \;\middle|\; A_{16}, A_{17}, A_{23} \in \ker \pi_q \right\}.$$

Then $W = V^\perp$.

Proof. (1) *Claim that $V^\perp \supseteq W$.*

For all $B = (B_{i,j}) \in W$ and for all $A = (A_{i,j}) \in V$, we get

$$\begin{aligned} \kappa_q(A,B) &= \pi_q \kappa(A,B) \\ &= \pi_q(A_{16}B_{16} + A_{17}B_{17} + A_{23}B_{23} + A_{15}B_{15} - A_{15}^q B_{15}^q \eta^{q-1}) \\ &\overset{\substack{A_{16},A_{17},A_{23}\in\mathbb{F}_q \\ B_{16},B_{17},B_{23}\in\ker\pi_q}}{=} \pi_q(A_{15}B_{15} - A_{15}^q B_{15}^q \eta^{q-1}) \\ &= (A_{15}B_{15}\eta)^{q^2} + (A_{15}B_{15}\eta)^q + A_{15}B_{15}\eta \\ &\quad - \left((A_{15}^q B_{15}^q \eta^q)^{q^2} + (A_{15}^q B_{15}^q \eta^q)^q + A_{15}^q B_{15}^q \eta^q\right) \\ &= 0. \end{aligned}$$

Then $B \in V^\perp$ for all $B \in W$, so $W \subseteq V^\perp$.

(2) *Claim that $V^\perp \subseteq W$.*

For all $B = (B_{ij}) \in V^\perp \subseteq V_0$ and for all $A = (A_{ij}) \in V$, then

$$\begin{aligned} 0 &= \kappa_q(A,B) = \pi_q \kappa(A,B) \\ &= \pi_q(A_{12}B_{12} + A_{13}B_{13} + A_{15}^q B_{14} + A_{15}B_{15} + A_{16}B_{16} + A_{17}B_{17} + A_{23}B_{23}). \end{aligned}$$

(2.1) *We claim that $B_{12} = B_{13} = 0$.*

Assume that $B_{12} \neq 0$. Let $A = A_{12}e_{12} = B_{12}^{-1}e_{12}$, then

$$0 = \kappa_q(A,B) = \pi_q(A_{12}B_{12}) = \pi_q(1) = 1.$$

This is a contradiction. Thus we get $B_{12} = 0$. Similarly, $B_{13} = 0$.

(2.2) *We claim* $B_{16}, B_{17}, B_{23} \in \ker \pi_q$.
Let $A = A_{16}e_{16} = e_{16}$, then

$$0 = \kappa_q(A, B) = \pi_q(A_{16}B_{16}) = \pi_q(B_{16}) \Longrightarrow B_{16} \in \ker \pi_q$$

Similarly, $B_{17}, B_{23} \in \ker \pi_q$.

(2.3) *We claim that* $B_{14} = -B_{15}^q \eta^{q-1}$.
Assume $B_{14} \neq -B_{15}^q \eta^{q-1}$, i.e. $B_{14}^{q^2}\eta^{q^2} + B_{15}\eta \neq 0$. Let $A = A_{15}^q e_{14} + A_{15}e_{15}$ $(A_{15} \in \mathbb{F}_{q^3})$, then

$$\begin{aligned}
0 &= \kappa_q(A, B) = \pi_q(A_{15}^q B_{14} + A_{15}B_{15}) \\
&= \Big((A_{15}^q B_{14}\eta)^{q^2} + (A_{15}^q B_{14}\eta)^q + (A_{15}^q B_{14}\eta)\Big) \\
&\quad + \Big((A_{15}B_{15}\eta)^{q^2} + (A_{15}B_{15}\eta)^q + (A_{15}B_{15}\eta)\Big) \\
&= A_{15}(B_{14}^{q^2}\eta^{q^2} + B_{15}\eta) + A_{15}^q(B_{14}\eta + B_{15}^q\eta^q) + A_{15}^{q^2}(B_{14}^q\eta^q + B_{15}^{q^2}\eta^{q^2}) \\
&= A_{15}(B_{14}^{q^2}\eta^{q^2} + B_{15}\eta) + (A_{15}(B_{14}^{q^2}\eta^{q^2} + B_{15}\eta))^q + (A_{15}(B_{14}^{q^2}\eta^{q^2} + B_{15}\eta))^{q^2} \\
&= \phi_0\big(A_{15}(B_{14}^{q^2}\eta^{q^2} + B_{15}\eta)\big) \\
&\overset{B_{14}^{q^2}\eta^{q^2}+B_{15}\eta\neq 0}{\Longrightarrow} \ker\phi_0 = \mathbb{F}_{q^3} \\
&\Longrightarrow |\ker\phi_0| = q^3 > q^2.
\end{aligned}$$

This is a contradiction since $|\ker\phi_0| = q^2$. Thus $B_{14} = -B_{15}^q\eta^{q-1}$.

Hence $B \in W$ and $V^\perp \subseteq W$.
Therefore, $V^\perp = W$. □

2.2.18 Lemma. *(1) Let* $x \in \mathbb{F}_{q^3}$*, then*

$$\pi_q(xy) = 0,\ \forall\, y \in \mathbb{F}_{q^3} \iff x = 0.$$

(2) $\kappa_q|_{V\times V}$ *is a non-degenerate* $\mathbb{F}_q$*-bilinear form.*

Proof. (1) Let $x \in \mathbb{F}_{q^3}$.

($\Longleftarrow$) Let $x = 0$, then $\pi_q(xy) = \pi_q(0) = 0$.

($\Longrightarrow$) Assume that $x \neq 0$. Let $y := x^{-1}$, then $0 = \pi_q(xy) = \pi_q(1) = 1$ which is a contradiction. Thus $x = 0$.

(2) It is sufficient to prove that if $A \in V$ and $\kappa_q(A, B) = 0$ $(\forall\, B \in V)$ then $A = 0$.

$$\begin{aligned}
0 =& \kappa_q(A, B) = \pi_q\kappa(A, B) \\
=& \pi_q(A_{12}B_{12} + A_{13}B_{13} + (A_{15}^q B_{15})^q + A_{15}B_{15} + A_{16}B_{16} + A_{17}B_{17} + A_{23}B_{23}).
\end{aligned}$$

(2.1) Let $B := e_{16}$, then $0 = \kappa_q(A, B) = \pi_q(A_{16}) \overset{A_{16}\in\mathbb{F}_q}{=} A_{16}$. Similarly, $A_{16} = A_{17} = A_{23} = 0$.

(2.2) Assume $A_{15} \neq 0$. Let $B := A_{15}^{-q}e_{14} + A_{15}^{-1}e_{15}$, then $0 = \kappa_q(A, B) = \pi_q((A_{15}B_{15})^q + A_{15}B_{15}) = \pi_q(1 + 1) = 2$. This is a contradiction since $\operatorname{Char} \mathbb{F}_{q^3} \neq 2$. Thus $A_{15} = 0$.

(2.3) Assume $A_{12} \neq 0$. Let $B := A_{12}^{-1}e_{12}$, then $0 = \kappa_q(A, B) = \pi_q(A_{12}B_{12}) = \pi_q(1) = 1$. This is a contradiction, thus $A_{12} = 0$. Similarly, $A_{12} = A_{13} = 0$.

Hence $A = 0$.

□

2.2.19 Corollary. *$V_0 = V \oplus V^\perp$, and $\pi\colon V_0 = V \oplus V^\perp \to V$ is the projection to the first component V.*

Proof. We know that $V + V^\perp \subseteq V_0$ since V and $V^\perp$ are $\mathbb{F}_q$-subspaces of V_0. Let $A \in V \cap V^\perp$, then $\kappa_q(A, B) \overset{A\in V^\perp}{=} 0 \overset{A\in V}{=} \kappa_q|_{V\times V}(A, B)$ for all $B \in V$. We get $A = 0$ since $\kappa_q|_{V\times V}$ is non-degenerate by 2.2.18. Thus $V \cap V^\perp = \{0\}$. Let $A \in V_0$, then $A = \pi(A) + (A - \pi(A))$ and $\pi(A) \in V$. It is enough to show that $A - \pi(A) \in V^\perp$. Let $B = (B_{i,j}) = \pi(A) \in V$ and $C = (C_{i,j}) = A - \pi(A)$, by 2.2.13 $\mathbb{F}_{q^3} = \ker \pi_q \oplus \mathbb{F}_q$, thus it is sufficient to prove that $C_{14} = -C_{15}^q\eta^{q-1}$. We have

$$\begin{aligned} C_{15} =& A_{15} - \frac{A_{14}^{q^2} + A_{15}\eta^{1-q^2}}{1 + \eta^{1-q^2}} = \frac{A_{15} - A_{14}^{q^2}}{1 + \eta^{1-q^2}}, \\ C_{14} =& A_{14} - \left(\frac{A_{14}^{q^2} + A_{15}\eta^{1-q^2}}{1 + \eta^{1-q^2}}\right)^q = \frac{A_{14} - A_{15}^q}{1 + \eta^{q-1}}\eta^{q-1} = -C_{15}^q\eta^{q-1}. \end{aligned}$$

Hence $V_0 = V \oplus V^\perp$. □

2.2.20 Lemma. *$\pi = \pi \circ \pi_J$. In particular, let $A \in V_0$ and $\pi_J(A) \in V$, then $\pi(A) = \pi_J(A)$.*

Proof. Let $A \in V_0$, the $\pi(A)$ depends on the entries of $\{A_{i,j} \mid (i, j) \in J\}$ by 2.2.13, thus $\pi(A) = \pi \circ \pi_J(A)$. Let $\pi_J(A) \in V$, then $\pi(A) = \pi \circ \pi_J(A) = \pi(\pi_J(A)) = \pi_J(A)$ since $\pi|_V = \mathrm{id}_V$ by 2.2.13. □

2.2.21 Corollary. *Suppose $A, B \in V_0$, such that $\operatorname{supp}(A) \cap \operatorname{supp}(B) \subseteq J$ and $\pi_J(A) \in V$. Then*

$$\begin{aligned} &\kappa_q(A, B) = \kappa_q(\pi(A), B) = \kappa_q(A, \pi(B)) \\ =&\kappa_q(\pi(A), \pi(B)) = \kappa_q|_{V\times V}(\pi(A), \pi(B)). \end{aligned}$$

Proof. We have

$$\begin{aligned}
&\kappa_q(A,B)=\pi_q\circ\kappa(A,B)\overset{2.2.9}{=}\pi_q\circ\kappa(\pi_J(A),B)=\kappa_q(\pi_J(A),B)\\
\overset{2.2.20}{=}&\kappa_q(\pi(A),B)\overset{2.2.19}{=}\kappa_q(\pi(A),\pi(B))=\kappa_q|_{V\times V}(\pi(A),\pi(B))\\
\overset{\pi(B)\in V}{=}&\kappa_q(A,\pi(B)).
\end{aligned}$$

Then the result is proved. □

2.2.22 Lemma. *Let $A\in V$ and $g\in G$, then*

$$\pi_J(Ag^\top)\in V.$$

In particular, $\pi_J(Ag^\top)=\pi(Ag^\top)$.

Proof. Let $A\in V$ and $g\in G$. It is sufficient to prove that

$$\begin{cases}(Ag^\top)_{15}^q=(Ag^\top)_{14}, & \\ (Ag^\top)_{ij}\in\mathbb{F}_q, & (i,j)\in\{(2,3),(1,6),(1,7)\}.\end{cases}$$

(1)

$$\begin{aligned}
(Ag^\top)_{15}^q=&\Big(\sum_{j=1}^{8}A_{1j}g_{j5}^\top\Big)^q=\Big(\sum_{j=1}^{8}A_{1j}g_{5j}\Big)^q=\Big(\sum_{j=2}^{7}A_{1j}g_{5j}\Big)^q=\Big(\sum_{j=5}^{7}A_{1j}g_{5j}\Big)^q\\
=&(A_{15}+A_{16}g_{56}+A_{17}g_{57})^q=A_{15}^q+A_{16}g_{56}^q+A_{17}g_{57}^q\\
\overset{\begin{cases}g_{46}=g_{56}^q\\ g_{47}=g_{57}^q\end{cases}}{=}&A_{15}^q+A_{16}g_{46}+A_{17}g_{47}\\
\overset{\begin{cases}A_{14}=A_{15}^q\\ g_{45}=0\end{cases}}{=}&A_{14}+A_{15}g_{45}+A_{16}g_{46}+A_{17}g_{47}\\
=&(Ag^\top)_{14}.
\end{aligned}$$

(2)

$$\begin{aligned}
(Ag^\top)_{16}&=\sum_{j=1}^{8}A_{1j}g_{j6}^\top=\sum_{j=6}^{7}A_{1j}g_{6j}=A_{16}+A_{17}g_{67}=A_{16}-A_{17}g_{23}\in\mathbb{F}_q,\\
(Ag^\top)_{17}&=\sum_{j=1}^{8}A_{1j}g_{7j}=A_{17}\in\mathbb{F}_q,\\
(Ag^\top)_{23}&=\sum_{j=1}^{8}A_{2j}g_{3j}=A_{23}\in\mathbb{F}_q.
\end{aligned}$$

Thus $\pi_J(Ag^\top)\in V$. □

2.2.23 Lemma. *Let $A, B \in V$ and $g, h \in A_8(q^3)$, then*

$$\operatorname{supp}(Bh^\top) \cap \triangledown \subseteq J \quad \textit{and} \quad \operatorname{supp}(Bh^\top) \cap \operatorname{supp}(Ag) \subseteq J.$$

Proof. Every $h \in A_8(q^3)$ acts on V from the right by a sequence of elementary column operations from left to right, and $h^\top$ acts on V from the right by a series of elementary column operations from right to left. Then the statements are obtained. □

2.2.24 Corollary. *Let $A, B \in V$ and $g \in G$, then*

$$\kappa_q(A, Bg) = \kappa_q(A, \pi(Bg)) = \kappa_q(Ag^\top, B) = \kappa_q(\pi(Ag^\top), B).$$

Proof. Let $A, B \in V$ and $g \in G$, then

$$\kappa_q(A, Bg) \overset{2.2.21\,\&\,2.2.23}{=} \kappa_q(A, \pi(Bg)),$$
$$\kappa_q(A, Bg) = \kappa_q(Ag^\top, B) \overset{\substack{2.2.21\,\&\,2.2.22\\ \&\,2.2.23}}{=} \kappa_q(\pi(Ag^\top), B).$$

□

2.2.25 Proposition (Group action of G on V)**.** *The map*

$$- \circ -: V \times G \to V : (A, g) \mapsto A \circ g := \pi(Ag)$$

is a group action, and the elements of the group G act as $\mathbb{F}_q$-automorphisms.

Proof. Let $A, B \in V$, $g, h \in G$ and $c \in \mathbb{F}_q$, then

(a) $A \circ I_8 = \pi(A \cdot I_8) = \pi(A) = A$.

(b) *We claim that* $A \circ (gh) = (A \circ g) \circ h$.

$$\begin{aligned}
\kappa_q(B, A \circ (gh)) &= \kappa_q(B, \pi(A(gh))) \overset{2.2.21}{=} \kappa_q(B, A(gh)) \\
&= \kappa_q(B, (Ag)h) = \kappa_q(Bh^\top, Ag) \overset{\substack{2.2.21\,\&\,2.2.22\\ \&\,2.2.23}}{=} \kappa_q(\pi(Bh^\top), Ag) \\
&\overset{2.2.21}{=} \kappa_q(\pi(Bh^\top), \pi(Ag)) = \kappa_q(\pi(Bh^\top), A \circ g) \overset{2.2.21}{=} \kappa_q(Bh^\top, A \circ g) \\
&= \kappa_q(B, (A \circ g)h) = \kappa_q(B, \pi((A \circ g)h)) \overset{2.2.21}{=} \kappa_q(B, (A \circ g) \circ h) \\
&\overset{2.2.18(2)}{\Longrightarrow} A \circ (gh) = (A \circ g) \circ h.
\end{aligned}$$

Thus, $- \circ -$ is a group action.

(c) $(A + B) \circ g = \pi((A + B)g) = \pi(Ag + Bg) = \pi(Ag) + \pi(Bg) = A \circ g + B \circ g$.

(d) $(cA) \circ g = \pi((cA)g) = \pi(c(Ag)) = c\pi(Ag) = c(A \circ g)$.

The proof is completed. $\square$

2.2.26 Corollary. *Let $A, B \in V$ and $g \in G$, then*

$$\kappa_q(A, B \circ g) = \kappa_q(A, Bg) = \kappa_q(Ag^\top, B) = \kappa_q(A \circ g^\top, B).$$

By 1.2.21, we get a new action:

2.2.27 Corollary. *There exists an unique linear action $-.-$ of G on V:*

$$-.-\colon V \times G \to V : (A, g) \mapsto A.g := A \circ g^{-\top}$$

such that $\kappa_q|_{V\times V}(A.g, B) = \kappa_q|_{V\times V}(A, B \circ g^{-1})$ for all $B \in V$. In particular, for all $A, B \in V$ and $g \in G$,

$$\kappa_q|_{V\times V}(A, B \circ g) = \kappa_q|_{V\times V}(A.g^{-1}, B) = \kappa_q|_{V\times V}(A \circ g^\top, B).$$

2.2.28 Notation. *Set $f := \pi|_G \colon G \to V$.*

2.2.29 Lemma. *Let $x, g \in G$, $1 := 1_G = I_8$ and $0 := O_8 = O_{8\times 8}$, then*

$$f(x)g \equiv (x-1)g \mod V^\perp.$$

In particular, $f(x) \equiv x - 1 \mod V^\perp$.

Proof. Let $x, g \in G$.

(a) $f(x) = \pi(x) \overset{\pi(1)=0}{=} \pi(x) - \pi(1) \overset{\pi \text{ linear}}{=} \pi(x-1)$, so $f(x) \equiv x - 1 \mod V^\perp$.

(b) For all $A \in V$,

$$\begin{aligned}
&\kappa_q(A, f(x)g) = \kappa_q(Ag^\top, f(x)) \overset{2.2.21}{=} \kappa_q(\pi(Ag^\top), f(x)) \\
&\overset{\text{proof }(a)}{=} \kappa_q(\pi(Ag^\top), x-1) \overset{\substack{2.2.21\ \&\ 2.2.22\\ \&\ 2.2.23}}{=} \kappa_q(Ag^\top, x-1) = \kappa_q(A, (x-1)g) \\
&\overset{2.2.18(2)}{\Longrightarrow} f(x)g \equiv (x-1)g \mod V^\perp.
\end{aligned}$$

$\square$

2.2.30 Proposition. *Let $x, g \in G$, then*

$$f(xg) = f(x) \circ g + f(g).$$

Proof. Let $x, g \in G$. For all $A \in V$,

$$\begin{aligned}
&\kappa_q(A, f(xg)) \overset{2.2.29}{=} \kappa_q(A, xg - 1) = \kappa_q(A, (x-1)g + (g-1)) \\
&\overset{2.2.29}{=} \kappa_q(A, f(x)g + f(g)) \overset{2.2.21}{=} \kappa_q(A, \pi(f(x)g) + f(g)) \overset{2.2.21}{=} \kappa_q(A, f(x) \circ g + f(g)).
\end{aligned}$$

Thus $f(xg) = f(x) \circ g + f(g)$ by 2.2.18 (2). $\square$

2.2.31 Proposition (Bijective 1-cocycle of ${}^3D_4^{syl}\,(q^3)$). *Let $U = {}^3D_4^{syl}\,(q^3)$, then $f|_U := \pi|_U \colon U \to V$ is a bijection. In particular, $f|_U$ is a bijective 1-cocycle of U in V.*

Proof. Let $x := x(t_1, t_2, t_3, t_4, t_5, t_6) \in U$, then

$$
\begin{aligned}
f|_U(x) &= \pi(x(t_1, t_2, t_3, t_4, t_5, t_6)) \\
&= \left(\begin{array}{|c|c|l|l|l|l|}
\hline
t_1 & -t_3 & \begin{array}{l} (\frac{t_1^{q^2} t_3 + t_1 t_3^{q^2} \eta^{1-q^2}}{1+\eta^{1-q^2}})^q \\ +t_4 \end{array} & \begin{array}{l} \frac{t_1^{q^2} t_3 + t_1 t_3^{q^2} \eta^{1-q^2}}{1+\eta^{1-q^2}} \\ +t_4^{q^2} \end{array} & \begin{array}{l} \pi_q(t_1 t_4^q) \\ +t_5 \end{array} & \begin{array}{l} \pi_q(-t_1 t_3^{q^2+q} \\ \quad + t_3 t_4^q) \\ +t_6 \end{array} \\
\hline
\multicolumn{1}{c|}{} & t_2 & \multicolumn{4}{c}{} \\
\cline{2-2}
\end{array} \right)
\end{aligned}
$$

(1) Since $f \colon G \to V$ is well defined, $f|_U$ is well defined.

(2) *We claim that $f|_U$ is surjective.*

For all $A \in V$, let $A := A_{12}e_{12} + A_{13}e_{13} + A_{15}^q e_{14} + A_{15}e_{15} + A_{16}e_{16} + A_{17}e_{17} + A_{23}e_{23}$. There is an element $x := x(t_1, t_2, t_3, t_4, t_5, t_6) \in U$ such that $f|_U(x) = A$, where

$$
\begin{aligned}
t_1 &= A_{12}, \\
t_2 &= A_{23} \in \mathbb{F}_q, \\
t_3 &= -A_{13}, \\
t_4 &= (A_{15} - \frac{t_1^{q^2} t_3 + t_1 t_3^{q^2} \eta^{1-q^2}}{1 + \eta^{1-q^2}})^q = A_{15}^q - \frac{t_1 t_3^q + t_1^q t_3 \eta^{q-1}}{1 + \eta^{q-1}} \\
&= A_{15}^q + \frac{A_{12} A_{13}^q + A_{12}^q A_{13} \eta^{q-1}}{1 + \eta^{q-1}}, \\
t_5 &= A_{16} - \pi_q(t_1 t_4^q) \\
&= A_{16} - \pi_q(A_{12}(A_{15}^{q^2} + \frac{A_{12}^q A_{13}^{q^2} + A_{12}^{q^2} A_{13}^q \eta^{q^2-q}}{1 + \eta^{q^2-q}})) \in \mathbb{F}_q, \\
t_6 &= A_{17} - \pi_q(-t_1 t_3^{q^2+q} + t_3 t_4^q) \\
&= A_{17} + \pi_q(A_{12} A_{13}^{q^2+q} + A_{13}(A_{15}^{q^2} + \frac{A_{12}^q A_{13}^{q^2} + A_{12}^{q^2} A_{13}^q \eta^{q^2-q}}{1 + \eta^{q^2-q}})) \in \mathbb{F}_q.
\end{aligned}
$$

Thus, $f|_U$ is surjective.

(3) We know that $|U| = q^{12} = |V|$ and that f is surjective, thus $f|_U$ is bijective.

(4) We obtain that $f|_U$ is a bijective 1-cocycle of U in V from 2.2.30.

$\square$

2.2.32 Corollary (Monomial linearisation for $G_8\,(q^3)$). *$f := \pi|_G \colon G \to V$ is a surjective 1-cocycle of G in V, and $(f, \kappa_q|_{V \times V})$ is a monomial linearisation for $G = G_8\,(q^3)$.*

2.2.33 Corollary (Monomial linearisation for ${}^3D_4^{syl}(q^3)$)**.** *$(f|_{{}^3D_4^{syl}(q^3)}, \kappa_q|_{V\times V})$ is a monomial linearisation for ${}^3D_4^{syl}(q^3)$.*

Now we establish the monomial G-module $\mathbb{C}\big({}^3D_4^{syl}(q^3)\big)$, which is essential for the construction of the supercharacter theory for ${}^3D_4^{syl}(q^3)$.

2.2.34 Theorem (Fundamental theorem for ${}^3D_4^{syl}(q^3)$)**.** *Let $U = {}^3D_4^{syl}(q^3)$, $G = G_8(q^3)$ and*

$$[A] = \frac{1}{|U|}\sum_{u\in U}\overline{\chi_A(u)}u \qquad \text{for all } A \in V.$$

Then the set $\{[A] \mid A \in V\}$ forms a $\mathbb{C}$-basis for the complex group algebra $\mathbb{C}U$. Let

$$[A] * g := \chi_{A.g}(g)[A.g] = \vartheta\kappa_q(A.g, f(g))[A.g] \qquad \text{for all } g \in G,\ A \in V,$$

then $\mathbb{C}U$ is a monomial $\mathbb{C}G$-module. The restriction of the $*$-operation to U is given by the usual right multiplication of U on $\mathbb{C}U$, i.e.

$$[A] * u = [A]u = \frac{1}{|U|}\sum_{y\in U}\overline{\chi_A(y)}yu \qquad \text{for all } u \in U,\ A \in V.$$

Proof. By 2.2.32, $(f, \kappa_q|_{V\times V})$ is a monomial linearisation for G, satisfying that $f|_U$ is a bijective map. By 2.2.27, $A.u := \pi(Au^{-\top})$. Thus the whole theorem is proved in view of 1.2.24. $\square$

2.2.35 Comparison (Monomial linearisations)**.** *Let U be $A_n(q)$, $D_n^{syl}(q)$ or ${}^3D_4^{syl}(q^3)$, G be a bigger group of U, $V_0 := V_{\square}$, V a subspace of V_0, $J := \mathrm{supp}(V)$, $f\colon G \to V$ a surjective 1-cocycle of G such that $f|_U$ is injective, $\kappa\colon V\times V \to \mathbb{F}_q$ (or $\mathbb{F}_{q^3}$) a trace form such that $(f, \kappa|_{V\times V})$ is a monomial linearisation for G (i.e. $(f|_U, \kappa|_{V\times V})$ is a monomial linearisation for U), then the corresponding notations for $A_n(q)$ (see [Jed13, 2.2]), $D_n^{syl}(q)$ (see [Jed13, 3.1]) and ${}^3D_4^{syl}(q^3)$ (see §2.2.1) are listed as follows:*

U	G	V_0	J	V	$f\colon G \to V$	$\kappa\|_{V\times V}$
$A_n(q)$	$A_n(q)$	$\mathrm{Mat}_{n\times n}(q)$	$\lhd$	$V = V_{\lhd}$	$f(g) = \pi_{\lhd}(g) = g - I_n$	$\kappa\|_{V\times V}$
$D_n^{syl}(q)$	$A_{2n}(q)$	$\mathrm{Mat}_{2n\times 2n}(q)$	$\triangledown$	$V = V_{\triangledown}$	$f(g) = \pi_{\triangledown}(g)$	$\kappa\|_{V\times V}$
${}^3D_4^{syl}(q^3)$	$G_8(q^3)$	$\mathrm{Mat}_{8\times 8}(q^3)$	J	$V \neq V_J$	$f(g) = \pi(g) \neq \pi_J(g)$	$\kappa_q\|_{V\times V}$

From now on, we mainly consider the regular right module $(\mathbb{C}U, *)_{\mathbb{C}U} = \mathbb{C}U_{\mathbb{C}U}$ in this chapter.

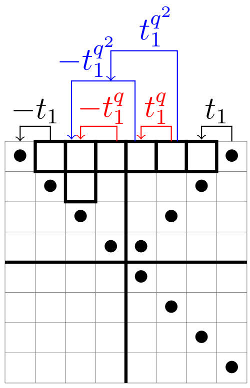
2.2.2. ${}^3D_4^{syl}(q^3)$-orbit modules

Let $U := {}^3D_4^{syl}(q^3)$, $A \in V$ and $x_i(t_i) \in U$ ($i = 1, 2, \ldots, 6$, $t_1, t_3, t_4 \in \mathbb{F}_{q^3}$, $t_2, t_5, t_6 \in \mathbb{F}_q$). In this subsection, we get a classification of U-orbit modules (2.2.50), and obtain the stabilizers $\mathrm{Stab}_U(A)$ for all $A \in V$ (2.2.52).

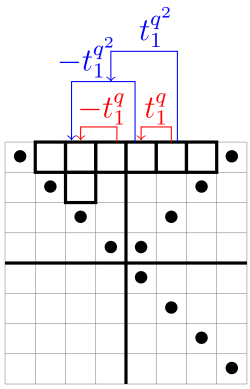
2.2.36 Reminder. *$\mathbb{F}_q^* = \mathbb{F}_q\backslash\{0\}$, $\mathbb{F}_{q^3}^* = \mathbb{F}_{q^3}\backslash\{0\}$ and $[\ ]$ is an idempotent element bracket (e.g. $[A]$).*

2.2.37 Lemma. *Let $A \in V$ and $x_i(t_i) \in U$ with $i \in \{1, 2, \ldots, 6\}$, $t_1, t_3, t_4 \in \mathbb{F}_{q^3}$ and $t_2, t_5, t_6 \in \mathbb{F}_q$. Then $A.x_i(t_i)$ and the corresponding figures of moves are obtained as follows:*

(1) $A.x_1(t_1) = A.\left(x_{12}(t_1)x_{34}(t_1^q)x_{35}(t_1^{q^2})\right) = A.\left(x_{34}(t_1^q)x_{35}(t_1^{q^2})\right)$.

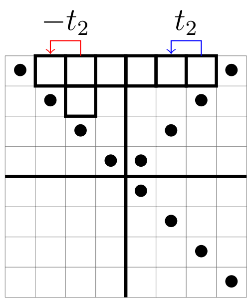


$A.x_1(t_1)$



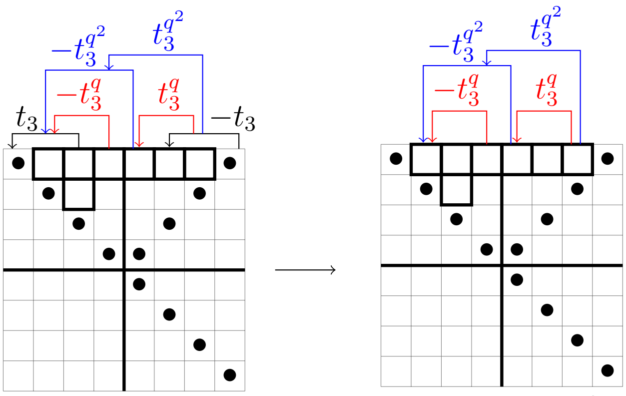
$A.(x_{34}(t_1^q)x_{35}(t_1^{q^2}))$

(2) $A.x_2(t_2) = A.x_{23}(t_2)$.



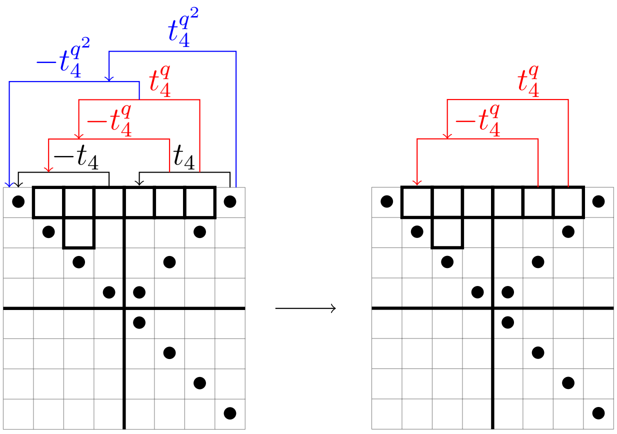
$A.x_2(t_2)$

(3) $A.x_3(t_3) = A.\left(x_{13}(-t_3)x_{24}(t_3^q)x_{25}(t_3^{q^2})\right) = A.\left(x_{24}(t_3^q)x_{25}(t_3^{q^2})\right)$.



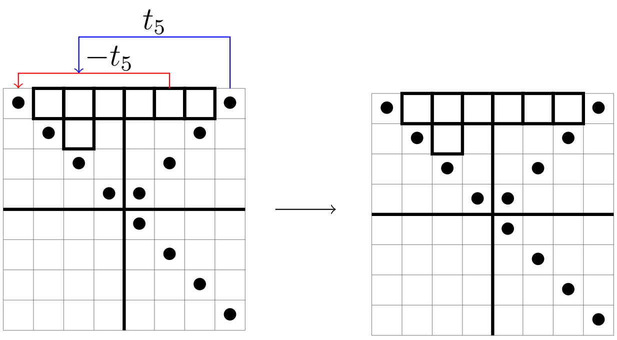
$A.x_3(t_3)$ $\qquad$ $A.(x_{24}(t_3^q)x_{25}(t_3^{q^2}))$

(4) $A.x_4(t_4) = A.\left(x_{14}(t_4)x_{26}(t_4^q)x_{15}(t_4^{q^2})\right) = A.x_{26}(t_4^q)$.



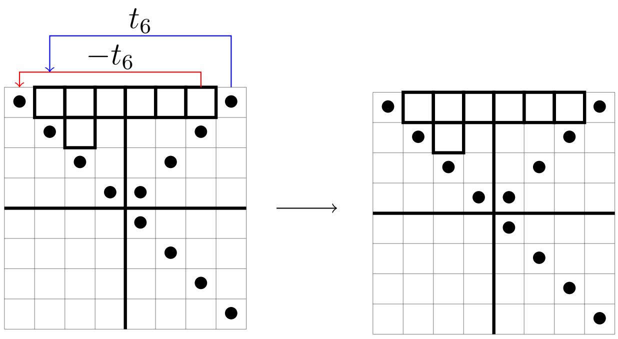
$A.x_4(t_4)$ $\qquad$ $A.x_{26}(t_4^q)$

(5) $A.x_5(t_5) = A.x_{16}(t_5) = A$.



$A.x_5(t_5)$ $\qquad$ A

(6) $A.x_6(t_6) = A.x_{17}(t_6) = A$.



$A.x_6(t_6)$ $\qquad$ A

Let $A \in V$, we often do not draw the whole matrix A but only the entries A_{ij} where $(i,j) \in J$, since $\mathrm{supp}(A) \subseteq J$. By 2.2.37, we calculate $[A].x_i(t_i)$ for all $A \in V$ and $x_i(t_i) \in U$.

2.2.38 Lemma. *Let* $A \in V$ *and* $x_i(t_i) \in U$ *with* $i \in \{1,2,3,4,5,6\}$, $t_1, t_3, t_4 \in \mathbb{F}_{q^3}$ *and* $t_2, t_5, t_6 \in \mathbb{F}_q$. *Then*

(1)

$$
\begin{aligned}
&[A].x_1(t_1) = \vartheta\pi_q(A_{12}t_1)[A.x_1(t_1)] \\
=&\vartheta\pi_q(A_{12}t_1)\left[\begin{array}{|c|c|c|c|c|c|}\hline A_{12} & \begin{matrix} A_{13} - A_{15}^q t_1^q \\ -(A_{15} + A_{16}t_1^q)t_1^{q^2} \end{matrix} & \begin{matrix} A_{16}t_1^{q^2} \\ +A_{15}^q \end{matrix} & \begin{matrix} A_{16}t_1^q \\ +A_{15} \end{matrix} & A_{16} & A_{17} \\ \hline \multicolumn{1}{c|}{} & A_{23} & \multicolumn{4}{c}{} \\ \cline{2-2} \end{array}\right],
\end{aligned}
$$

(2)

$$
\begin{aligned}
&[A].x_2(t_2) = \vartheta(A_{23}t_2)[A.x_2(t_2)] \\
=&\vartheta(A_{23}t_2)\left[\begin{array}{|c|c|c|c|c|c|}\hline A_{12} - A_{13}t_2 & A_{13} & A_{15}^q & A_{15} & A_{16} + A_{17}t_2 & A_{17} \\ \hline \multicolumn{1}{c|}{} & A_{23} & \multicolumn{4}{c}{} \\ \cline{2-2} \end{array}\right],
\end{aligned}
$$

(3)

$$
\begin{aligned}
&[A].x_3(t_3) = \vartheta\pi_q(-A_{13}t_3)[A.x_3(t_3)] \\
=&\vartheta\pi_q(-A_{13}t_3)\left[\begin{array}{|c|c|c|c|c|c|}\hline \begin{matrix} A_{12} - A_{15}^q t_3^q \\ -(A_{15} + A_{17}t_3^q)t_3^{q^2} \end{matrix} & A_{13} & \begin{matrix} A_{17}t_3^{q^2} \\ +A_{15}^q \end{matrix} & \begin{matrix} A_{17}t_3^q \\ +A_{15} \end{matrix} & A_{16} & A_{17} \\ \hline \multicolumn{1}{c|}{} & A_{23} & \multicolumn{4}{c}{} \\ \cline{2-2} \end{array}\right],
\end{aligned}
$$

(4)

$$
\begin{aligned}
&[A].x_4(t_4) = \vartheta\pi_q(A_{15}^q t_4 + A_{15}t^{q^2})[A.x_4(t_4)] \\
=&\vartheta\pi_q(A_{15}^q t_4 + A_{15}t^{q^2})\left[\begin{array}{|c|c|c|c|c|c|}\hline A_{12} - A_{16}t_4^q & A_{13} + A_{17}t_4^q & A_{15}^q & A_{15} & A_{16} & A_{17} \\ \hline \multicolumn{1}{c|}{} & A_{23} & \multicolumn{4}{c}{} \\ \cline{2-2} \end{array}\right],
\end{aligned}
$$

(5) $[A].x_5(t_5) = \vartheta(A_{16}t_5)[A]$,

(6) $[A].x_6(t_6) = \vartheta(A_{17}t_6)[A]$.

2.2.39 Lemma. *Let $u \in \mathbb{F}_{q^3}^*$, and*

$$\zeta_u \colon \mathbb{F}_{q^3} \to \mathbb{F}_{q^3} : t \mapsto ut^{q^2} + u^q t^q$$

then ζ_u is an $\mathbb{F}_q$-automorphism.

Proof. Since taking the q-th power is an $\mathbb{F}_q$-homomorphism, we get that ζ_u is an $\mathbb{F}_q$-homomorphism. We have

$$\begin{aligned} & 0 = ut^{q^2} + u^q t^q = (u^{1-q}t^{q^2-q} + 1)u^q t^q = ((u^{-1}t^q)^{q-1} + 1)u^q t^q \\ \iff & (u^{-1}t^q)^{q-1} = -1 \text{ or } t = 0 \\ \iff & (u^{-1}t^q)^{2(q-1)} = 1 \text{ or } t = 0 \\ \iff & \text{the order of } (u^{-1}t^q)^{q-1} \text{ is } 2 \text{ i.e. } |(u^{-1}t^q)^{q-1}| = 2 \text{ or } t = 0. \end{aligned}$$

Suppose that $t \neq 0$, then $|(u^{-1}t^q)^{q-1}| = 2$ and

$$\begin{aligned} u^{-1}t^q \in \mathbb{F}_{q^3}^* &\implies |u^{-1}t^q| \Big| (q^3 - 1) = (q-1)(q^2+q+1) \\ &\implies |(u^{-1}t^q)^{q-1}| \Big| (q^2+q+1) \\ &\overset{q^2+q+1 \text{ is odd}}{\implies} |(u^{-1}t^q)^{q-1}| \neq 2. \end{aligned}$$

Thus,

$$\begin{aligned} & 0 = ut^{q^2} + u^q t^q \\ \iff & t = 0 \\ \implies & \ker \zeta_u = \{0\} \\ \implies & \operatorname{im} \zeta_u = \mathbb{F}_{q^3} \\ \implies & \zeta_u \text{ is an } \mathbb{F}_q\text{-automorphism.} \end{aligned}$$

□

2.2.40 Corollary. *The map $\mathbb{F}_{q^3} \to \mathbb{F}_{q^3} : t \mapsto t + t^q$ is an $\mathbb{F}_q$-automorphism.*

2.2.41 Lemma (U-orbit modules). *Let $A = (A_{ij}) \in V$, then the U-orbit module $\mathbb{C}\mathcal{O}_U([A])$ $(A \in V)$ is obtained as follows:*

$$
\begin{aligned}
&\mathbb{C}\mathcal{O}_U([A]) \\
&=\mathbb{C}\Bigg\{ \left[\begin{array}{|c|c|c|c|c|c|}
\hline
\begin{array}{l} A_{12} \\ -A_{13}t_2 \\ -A_{15}^q t_3^q \\ -A_{16}t_1^{q^2}t_3^q \\ -A_{17}t_2t_1^{q^2}t_3^q \\ -A_{15}t_3^{q^2} \\ -A_{16}t_1^qt_3^{q^2} \\ -A_{17}t_2t_1^qt_3^{q^2} \\ -A_{17}t_3^{q^2+q} \\ -A_{16}t_4^q \\ -A_{17}t_2t_4^q \end{array} &
\begin{array}{l} A_{13} \\ -A_{15}^qt_1^q \\ -A_{15}t_1^{q^2} \\ -A_{16}t_1^{q^2+q} \\ -A_{17}t_2t_1^{q^2+q} \\ +A_{17}t_4^q \end{array} &
\begin{array}{l} A_{15}^q \\ +A_{16}t_1^{q^2} \\ +A_{17}t_2t_1^{q^2} \\ +A_{17}t_3^{q^2} \end{array} &
\begin{array}{l} A_{15} \\ +A_{16}t_1^q \\ +A_{17}t_2t_1^q \\ +A_{17}t_3^q \end{array} &
\begin{array}{l} A_{16} \\ +A_{17}t_2 \end{array} &
A_{17} \\
\hline
\multicolumn{1}{c|}{} & A_{23} & \multicolumn{4}{c}{} \\
\cline{2-2}
\end{array}\right] \\
&\qquad \Bigg| \; t_1, t_3, t_4 \in \mathbb{F}_{q^3},\ t_2 \in \mathbb{F}_q \Bigg\}.
\end{aligned}
$$

Proof. By 2.2.38, we calculate the general orbit modules directly. □

We define the pattern, the main condition, the verge pattern and the core pattern of $A \in V$.

2.2.42 Definition. *The elements of V are called* ***patterns****. The monomial action of G on $\mathbb{C}U$: $([A], g) \mapsto [A] * g$ (e.g. 2.2.34) and also the corresponding permutation operation on V: $(A, g) \mapsto A.g$ (e.g. 2.2.27) are called* ***truncated column operation****. So the permutation operation $(A, g) \mapsto A.g$ $(A \in V, g \in G)$ may be derived by a sequence of truncated column operations (see 2.2.37) by 2.1.19.*

Now we consider the entries in the same row (in different columns) of $A \in V$.

2.2.43 Definition. *Suppose $A \in V$ is a pattern. Then*

(1) $(i, j) \in J$ is a ***main condition*** *of A if and only if A_{ij} is the rightmost non-zero entry in the i-th row. We set*

$$\mathrm{main}(A) := \{(i, j) \in J \mid (i, j) \text{ is a main condition of } A\}.$$

(2) A main condition $(i, j) \in J$ is a ***left main condition*** *if $j \leq 4$. We set*

$$\mathrm{l.main}(A) := \{\mathrm{main}(A) \mid j \leq 4\}.$$

(3) *A main condition $(i,j) \in J$ is a **right main condition** if $j > 4$. We set*

$$\mathrm{r.main}(A) :=\{\mathrm{main}(A) \mid j > 4\}.$$

(4) *The coordinate (i,j) is called **the i-th main condition** of A, if $(i,j) \in \mathrm{main}(A)$. Set*

$$\mathrm{main}_i(A) := \{(i,j) \in J \mid (i,j) \textit{ is the } i\textit{-th main condition of } A\}.$$

Note that there exists at most one i-th main condition of A.

2.2.44 Definition. *Let $A \in V$, we write*

$$\mathrm{main}(\mathbb{C}\mathcal{O}_U([A])) = \mathrm{main}(\mathcal{O}_U(A)) = \mathrm{main}(\mathbb{C}\mathcal{O}_G([A])) = \mathrm{main}(\mathcal{O}_G(A)) := \mathrm{main}(A).$$

2.2.45 Definition (Staircase pattern). *Let $A \in V$ be a pattern, then A is a **staircase pattern**, if the elements in $\mathrm{main}(A)$ lie in different columns. Analogously, a U-orbit (or G-orbit) module M is called a **staircase U-module** (or **staircase G-orbit**), if the elements in $\mathrm{main}(M)$ lie in different columns.*

2.2.46 Definition. *Let $A \in V$, then the **verge** of A is*

$$\mathrm{verge}(A) := \sum_{(i,j)\in\mathrm{main}(A)} A_{i,j}e_{i,j}.$$

*The i**-th verge** of A is*

$$\mathrm{verge}_i(A) := \sum_{(i,k)\in\mathrm{main}_i(A)} A_{i,k}e_{i,k}.$$

*The (staircase) pattern A is called the **(staircase) verge pattern**, if $A = \mathrm{verge}(A)$.*

2.2.47 Definition. *Let $A \in V$ be a pattern.*

(1) *The coordinate $(i,j) \in J$ is a **minor condition** of A if and only if $(i,\bar{j})$ is a right main condition. Set*

$$\mathrm{minor}(A) := \{(i,j) \in J \mid (i,j) \textit{ is a minor condition of } A\} \subseteq J.$$

Note that $\mathrm{minor}(A)$ is invariant under truncated column operartion, and that if $(i,j) \in \mathrm{minor}(A)$ then $j \leq 4$.

(2) *Define the **core** of A to be*

$$\mathrm{core}(A) := \mathrm{main}(A) \cup \mathrm{minor}(A).$$

Note that the union is disjoint. There is no supplementary condition for $A \in V$, i.e. $\mathrm{suppl}(A) = \emptyset$ (c.f. [Jed13, 3.2.14]).

(3) The (staircase) pattern A is called a ***(staircase) core pattern*** *if* $\mathrm{supp}(A) \subseteq \mathrm{core}(A)$.

2.2.48 Notation. *Define the families of U-orbit modules as follows:*

$$\begin{aligned}
\mathfrak{F}_6 :=& \{\mathbb{C}\mathcal{O}_U(A) \mid A \in V,\ A_{17} \neq 0\},\\
\mathfrak{F}_5 :=& \{\mathbb{C}\mathcal{O}_U(A) \mid A \in V,\ A_{16} \neq 0,\ A_{17} = 0\},\\
\mathfrak{F}_4 :=& \{\mathbb{C}\mathcal{O}_U(A) \mid A \in V,\ A_{15} \neq 0,\ A_{16} = A_{17} = 0\},\\
\mathfrak{F}_3 :=& \{\mathbb{C}\mathcal{O}_U(A) \mid A \in V,\ A_{13} \neq 0,\ A_{15} = A_{16} = A_{17} = 0\},\\
\mathfrak{F}_{1,2} :=& \{\mathbb{C}\mathcal{O}_U(A) \mid A \in V,\ A_{13} = A_{15} = A_{16} = A_{17} = 0\}.
\end{aligned}$$

Let $A \in V$, we also say $A \in \mathfrak{F}_i$, if $\mathbb{C}\mathcal{O}_U([A]) \in \mathfrak{F}_i$.

2.2.49 Notation. *Let $a^* \in \mathbb{F}_{q^3}^* = \mathbb{F}_{q^3}\backslash\{0\}$, then denote by T^{a^*} a complete set of coset representatives (i.e. a transversal) of $(a^*\mathbb{F}_q^+)$ in $\mathbb{F}_{q^3}^+$. Thus $|T^{a^*}| = q^2$. If $\bar{t}_0 \in T^{a^*}$ and $\bar{t}_0 \in a^*\mathbb{F}_q^+$, we set $\bar{t}_0 = 0$.*

2.2.50 Proposition (Classification of U-orbit modules)**.** *Every U-orbit module is one of the following forms in Table 2.1. In particular, U-orbit module of families $\mathfrak{F}_{1,2}$, $\mathfrak{F}_4$, $\mathfrak{F}_5$ and $\mathfrak{F}_6$ contains one and only one staircase core pattern.*

Table 2.1.: Classification of ${}^3D_4^{syl}(q^3)$-orbit modules

Family	$\mathbb{C}\mathcal{O}_U([A])$ $(A \in V)$	$\dim_{\mathbb{C}}\mathbb{C}\mathcal{O}_U([A])$	$\lvert\mathfrak{F}_i\rvert$
$\mathfrak{F}_6$	$\mathbb{C}\mathcal{O}_U\left(\left[\begin{array}{cccccc} A_{12} & & & & & A_{17}^* \\ & A_{23} & & & & \end{array}\right]\right)$	q^7	$(q-1)q^4$
$\mathfrak{F}_5$	$\mathbb{C}\mathcal{O}_U\left(\left[\begin{array}{cccccc} & A_{13} & & & A_{16}^* & \\ & A_{23} & & & & \end{array}\right]\right)$	q^6	$(q-1)q^4$
$\mathfrak{F}_4$	$\mathbb{C}\mathcal{O}_U\left(\left[\begin{array}{cccccc} & & A_{15}^{*\,q} & A_{15}^* & & \\ & A_{23} & & & & \end{array}\right]\right)$	q^6	$(q^3-1)q$
$\mathfrak{F}_3$	$\mathbb{C}\mathcal{O}_U\left(\left[\begin{array}{cccccc} \bar{A}_{12}^{A_{13}^*} & A_{13}^* & & & & \\ & A_{23} & & & & \end{array}\right]\right)$	q	$(q^3-1)q^3$
$\mathfrak{F}_{1,2}$	$\mathbb{C}\mathcal{O}_U\left(\left[\begin{array}{cccccc} A_{12} & & & & & \\ & A_{23} & & & & \end{array}\right]\right)$	1	q^4

where $A_{13}^*, A_{15}^* \in \mathbb{F}_{q^3}^*$, $A_{16}^*, A_{17}^* \in \mathbb{F}_q^*$, and $\bar{A}_{12}^{A_{13}^*} \in T^{A_{13}^*}$.

Proof. Let $A = (A_{ij}) \in V$ with $A_{17} = A_{17}^* \in \mathbb{F}_q^*$, then

$$\begin{aligned}
&\mathbb{C}\mathcal{O}_U([A]) \\
=&\mathbb{C}\left\{ \left[\begin{array}{|c|c|c|c|c|c|} \hline A_{12} + \frac{A_{13}A_{16}+A_{15}^{q+1}}{A_{17}^*} - \frac{B_{13}B_{16}+B_{15}^{q+1}}{A_{17}^*} & B_{13} & B_{15}^{q^2} & B_{15} & B_{16} & A_{17}^* \\ \hline \multicolumn{1}{c|}{} & A_{23} & \multicolumn{4}{|c}{} \\ \cline{2-2} \end{array} \right] \middle| \, B_{13}, B_{15} \in \mathbb{F}_{q^3}, B_{16} \in \mathbb{F}_q \right\}
\end{aligned}$$

Thus $\dim_{\mathbb{C}} \mathbb{C}\mathcal{O}_U([A]) = q^7$. Let $u := x(t_1, -\frac{A_{16}}{A_{17}^*}, -\frac{A_{15}^{q^2}}{A_{17}^*}, -\frac{A_{13}^{q^2}-(A_{15}t_1+A_{15}^{q^2}t_1^q)}{A_{17}^*}, t_5, t_6)$, then there is a staircase core pattern

$$C := A.u = \begin{array}{|c|c|c|c|c|c|} \hline A_{12} + \frac{A_{13}A_{16}+A_{15}^{q+1}}{A_{17}^*} & \quad & & & & A_{17}^* \\ \hline \multicolumn{1}{c|}{} & A_{23} & \multicolumn{4}{|c}{} \\ \cline{2-2} \end{array} \in \mathcal{O}_U(A).$$

Thus $\mathcal{O}_U(C) = \mathcal{O}_U(A)$ and $\mathbb{C}\mathcal{O}_U([A]) = \mathbb{C}\mathcal{O}_U([C])$. Since C only depends on A, the staircase core pattern is determined uniquely. Thus

$$\mathfrak{F}_6 = \{\mathbb{C}\mathcal{O}_U([D_{12}e_{12} + D_{23}e_{23} + D_{17}^*e_{17}]) \mid D_{12} \in \mathbb{F}_{q^3}, D_{23} \in \mathbb{F}_q, D_{17}^* \in \mathbb{F}_q^*\}.$$

Similarly, all of the statements are proved. $\square$

2.2.51 Remark. *Let $A \in V$. In 2.2.50, if $\mathbb{C}\mathcal{O}_U([A]) \subseteq \mathfrak{F}_3$ and $A_{2,3} \neq 0$, then $\mathbb{C}\mathcal{O}_U([A])$ is not a staircase orbit module. In all the other cases, the orbit modules $\mathbb{C}\mathcal{O}_U([A])$ are staircase orbit modules.*

2.2.52 Proposition (U-stabilizer)**.** *Let $A \in V$, then $\mathrm{Stab}_U(A)$ is established in Table 2.2.*

Table 2.2.: ${}^3D_4^{syl}(q^3)$-stabilizers

	$A \in V$	$\mathrm{Stab}_U(A)$
$\mathfrak{F}_6$	$\begin{array}{\|c\|c\|c\|c\|c\|c\|} \hline A_{12} & A_{13} & A_{15}^q & A_{15} & A_{16} & A_{17}^* \\ \hline & A_{23} \\ \cline{2-2} \end{array}$	$x(t_1, 0, -\frac{A_{16}t_1}{A_{17}^*}, \frac{A_{15}t_1 + A_{15}^{q^2}t_1^q + A_{16}t_1^{q+1}}{A_{17}^*}, t_5, t_6)$ $\forall\ t_1 \in \mathbb{F}_{q^3},\ t_5, t_6 \in \mathbb{F}_q$
$\mathfrak{F}_5$	$\begin{array}{\|c\|c\|c\|c\|c\|c\|} \hline A_{12} & A_{13} & A_{15}^q & A_{15} & A_{16}^* & \\ \hline & A_{23} \\ \cline{2-2} \end{array}$	$x(0, t_2, t_3, \frac{-A_{13}^{q^2}t_2 - A_{15}t_3 - A_{15}^{q^2}t_3^q}{A_{16}^*}, t_5, t_6)$ $\forall\ t_3 \in \mathbb{F}_{q^3},\ t_2, t_5, t_6 \in \mathbb{F}_q$
$\mathfrak{F}_4$	$\begin{array}{\|c\|c\|c\|c\|c\|c\|} \hline A_{12} & A_{13} & {A_{15}^*}^q & A_{15}^* & & \\ \hline & A_{23} \\ \cline{2-2} \end{array}$	$x(0, t_2, -\frac{A_{13}^{q^2}{A_{15}^*}^q + A_{13}^q A_{15}^* - A_{13}{A_{15}^*}^{q^2}}{2{A_{15}^*}^{q+1}}t_2, t_4, t_5, t_6)$ $\forall\ t_4 \in \mathbb{F}_{q^3},\ t_2, t_5, t_6 \in \mathbb{F}_q$
$\mathfrak{F}_3$	$\begin{array}{\|c\|c\|c\|c\|c\|c\|} \hline A_{12} & A_{13}^* & & & & \\ \hline & A_{23} \\ \cline{2-2} \end{array}$	$x(t_1, 0, t_3, t_4, t_5, t_6)$ $\forall\ t_1, t_3, t_4 \in \mathbb{F}_{q^3},\ t_5, t_6 \in \mathbb{F}_q$
$\mathfrak{F}_{1,2}$	$\begin{array}{\|c\|c\|c\|c\|c\|c\|} \hline A_{12} & & & & & \\ \hline & A_{23} \\ \cline{2-2} \end{array}$	$x(t_1, t_2, t_3, t_4, t_5, t_6)$ $\forall\ t_1, t_3, t_4 \in \mathbb{F}_{q^3},\ t_2, t_5, t_6 \in \mathbb{F}_q$

where $A_{13}^*, A_{15}^* \in \mathbb{F}_{q^3}^*$ and $A_{16}^*, A_{17}^* \in \mathbb{F}_q^*$.

Proof. Let $A \in \mathfrak{F}_4$ (i.e. $A_{17} = A_{16} = 0$, $A_{15} \neq 0$) and $x := x(t_1, t_2, t_3, t_4, t_5, t_6) \in U$, then

$$A.x = \begin{array}{|c|c|c|c|c|c|} \hline \begin{array}{l} A_{12} \\ -A_{13}t_2 \\ -{A_{15}^*}^q t_3^q \\ -A_{15}^* t_3^{q^2} \end{array} & \begin{array}{l} A_{13} \\ -{A_{15}^*}^q t_1^q \\ -A_{15}^* t_1^{q^2} \end{array} & {A_{15}^*}^q & A_{15}^* & & \\ \hline & A_{23} \\ \cline{2-2} \end{array} = A$$

$$\iff \begin{cases} -{A_{15}^*}^q t_1^q - A_{15}^* t_1^{q^2} = 0 \\ -A_{13}t_2 - {A_{15}^*}^q t_3^q - A_{15}^* t_3^{q^2} = 0 \end{cases} \overset{2.2.39}{\iff} \begin{cases} t_1 = 0 \\ A_{13}t_2 + {A_{15}^*}^q t_3^q + A_{15}^* t_3^{q^2} = 0 \end{cases}$$

$$\iff \begin{cases} t_1 = 0 \\ A_{13}{A_{15}^*}^{q^2} t_2 + {A_{15}^*}^{q^2+q} t_3^q + {A_{15}^*}^{q^2+1} t_3^{q^2} = 0 \\ A_{13}^q A_{15}^* t_2 + {A_{15}^*}^{q^2+1} t_3^{q^2} + {A_{15}^*}^{q+1} t_3 = 0 \\ A_{13}^{q^2} {A_{15}^*}^q t_2 + {A_{15}^*}^{q+1} t_3 + {A_{15}^*}^{q^2+q} t_3^q = 0 \end{cases}$$

$$\iff \begin{cases} t_1 = 0 \\ t_3 = -\frac{A_{13}^{q^2} {A_{15}^*}^q + A_{13}^q A_{15}^* - A_{13} {A_{15}^*}^{q^2}}{2{A_{15}^*}^{q+1}} t_2 \end{cases}$$

Thus

$$\begin{aligned} &\mathrm{Stab}_U(A) \\ = &\left\{ x_2(t_2)x_3(t_3)x_4(t_4)x_5(t_5)x_6(t_6) \;\middle|\; \begin{cases} t_3, t_4 \in \mathbb{F}_{q^3} \\ t_2, t_5, t_6 \in \mathbb{F}_q \\ {A_{15}^*}^q t_3^q + A_{15}^* t_3^{q^2} + A_{13} t_2 = 0 \end{cases} \right\} \\ = &\left\{ x_2(t_2)x_3\Big(-\frac{A_{13}^{q^2} {A_{15}^*}^q + A_{13}^q A_{15}^* - A_{13} {A_{15}^*}^{q^2}}{2{A_{15}^*}^{q+1}} t_2\Big)x_4(t_4)x_5(t_5)x_6(t_6) \;\middle|\; \begin{cases} t_4 \in \mathbb{F}_{q^3} \\ t_2, t_5, t_6 \in \mathbb{F}_q \end{cases} \right\}. \end{aligned}$$

□

2.2.53 Comparison. *(1) (Classification of orbit modules). Every (staircase) $A_n(q)$-orbit module has precisely one (staircase) verge pattern (see [Yan10, Theorem 3.2]). Every (staircase) $D_n^{syl}(q)$-orbit module has one and only one (staircase) core pattern (see [Jed13, 3.2.29]). Neither (staircase) verge patterns nor core patterns can do the classification for (staircase) ${}^3D_4^{syl}(q^3)$-orbit modules (e.g. the Family $\mathfrak{F}_3$ of 2.2.50).*

(2) (Stabilizer). Every (staircase) $A_n(q)$-orbit module has a basis element whose stabilizer is a pattern subgroup (see [Yan10, §3.3]). This does not hold for $D_n^{syl}(q)$-orbit modules (see [Jed13, 3.2.25]), or for ${}^3D_4^{syl}(q^3)$-orbit modules (e.g. the Family $\mathfrak{F}_5$ of 2.2.52).

2.2.3. $G_8(q^3)$-orbit modules

Let $G := G_8(q^3)$, $A_{12}, A_{13}, A_{15} \in \mathbb{F}_{q^3}$, $A_{16}, A_{17}, A_{23} \in \mathbb{F}_q$ and $A_{ij}^* \in \mathbb{F}_-^*$. In this subsection, we obtain the stabilizers $\mathrm{Stab}_G(A)$ for all $A \in V$ (2.2.56) and a classification of G-orbit modules (2.2.57).

2.2.54 Notation. *Set*

$$\begin{aligned} g :=& \tilde{x}_{23}(t_{23}) \cdot \tilde{x}_{67}(t_{67}) \\ & \cdot \tilde{x}_{12}(t_{12}) \cdot \tilde{x}_{78}(t_{78}) \cdot \tilde{x}_{34}(t_{34})\tilde{x}_{35}(t_{34}^q) \cdot \tilde{x}_{56}(t_{56})\tilde{x}_{46}(t_{56}^q) \cdot \tilde{x}_{36}(t_{36}) \\ & \cdot \tilde{x}_{13}(t_{13}) \cdot \tilde{x}_{68}(t_{68}) \cdot \tilde{x}_{24}(t_{24})\tilde{x}_{25}(t_{24}^q) \cdot \tilde{x}_{57}(t_{57})\tilde{x}_{47}(t_{57}^q) \cdot \tilde{x}_{27}(t_{27}) \\ & \cdot \tilde{x}_{14}(t_{14})\tilde{x}_{58}(t_{58}) \cdot \tilde{x}_{26}(t_{26})\tilde{x}_{37}(t_{37}) \cdot \tilde{x}_{15}(t_{15})\tilde{x}_{48}(t_{48}) \cdot \tilde{x}_{18}(t_{18}) \\ & \cdot \tilde{x}_{16}(t_{16}) \cdot \tilde{x}_{38}(t_{38}) \cdot \tilde{x}_{17}(t_{17}) \cdot \tilde{x}_{28}(t_{28}) \\ & \in G. \end{aligned}$$

2.2.55 Lemma (G-orbit modules)**.** *Let $g \in G$ and $A = (A_{ij}) \in V$, then*

$A.g =$

A_{12} $-A_{13}t_{23}$ $-A_{15}^q t_{24}$ $+A_{16}t_{56}^q t_{24}$ $-A_{17}t_{67}t_{56}^q t_{24}$ $-A_{15}t_{24}^q$ $+A_{16}t_{56}t_{24}^q$ $-A_{17}t_{67}t_{56}t_{24}^q$ $-A_{17}t_{27}$ $-A_{16}t_{26}$ $+A_{17}t_{67}t_{26}$	A_{13} $-A_{15}^q t_{34}$ $-A_{15}t_{34}^q$ $-A_{16}t_{36}$ $+A_{17}t_{67}t_{36}$ $-A_{17}t_{37}$	A_{15}^q $-A_{16}t_{56}^q$ $+A_{17}t_{67}t_{56}^q$ $-A_{17}t_{57}^q$	A_{15} $-A_{16}t_{56}$ $+A_{17}t_{67}t_{56}$ $-A_{17}t_{57}$	A_{16} $-A_{17}t_{67}$	A_{17}
	A_{23}				

.

Proof. By calculation, we get the formula. □

2.2.56 Proposition (G-stabilizer)**.** *Let $A \in V$, then $\mathrm{Stab}_G(A)$ is determined as follows:*

$G_8\left(q^3\right)$-stabilizers

	$A \in V$	$\mathrm{Stab}_G(A)$
$\mathfrak{F}_6$	$\begin{array}{cccccc} \boxed{A_{12}} & \boxed{A_{13}} & \boxed{A_{15}^q} & \boxed{A_{15}} & \boxed{A_{16}} & \boxed{A_{17}^*} \\ & \boxed{A_{23}} & & & & \end{array}$	$\begin{cases} t_{67} = 0 \\ A_{16}t_{56} + A_{17}^*t_{57} = 0 \\ A_{15}^q t_{34} + A_{15}t_{34}^q + A_{16}t_{36} \\ +A_{17}^*t_{37} = 0 \\ -A_{13}t_{23} - A_{15}^q t_{24} + A_{16}t_{56}^q t_{24} \\ -A_{15}t_{24}^q + A_{16}t_{56}t_{24}^q - A_{17}^*t_{27} \\ -A_{16}t_{26} = 0 \end{cases}$
$\mathfrak{F}_5$	$\begin{array}{cccccc} \boxed{A_{12}} & \boxed{A_{13}} & \boxed{A_{15}^q} & \boxed{A_{15}} & \boxed{A_{16}^*} & \boxed{} \\ & \boxed{A_{23}} & & & & \end{array}$	$\begin{cases} t_{56} = 0 \\ A_{15}^q t_{34} + A_{15}t_{34}^q + A_{16}^*t_{36} = 0 \\ A_{13}t_{23} + A_{15}^q t_{24} + A_{15}t_{24}^q \\ +A_{16}^*t_{26} = 0 \end{cases}$
$\mathfrak{F}_4$	$\begin{array}{cccccc} \boxed{A_{12}} & \boxed{A_{13}} & \boxed{{A_{15}^*}^q} & \boxed{A_{15}^*} & \boxed{} & \boxed{} \\ & \boxed{A_{23}} & & & & \end{array}$	$\begin{cases} t_{34} = 0 \\ t_{24} = -\frac{A_{13}{A_{15}^*}^{q^2} + A_{13}^{q^2}{A_{15}^*}^q - A_{13}^q A_{15}^*}{2{A_{15}^*}^{q^2+q}} t_{23} \end{cases}$
$\mathfrak{F}_3$	$\begin{array}{cccccc} \boxed{A_{12}} & \boxed{A_{13}^*} & \boxed{} & \boxed{} & \boxed{} & \boxed{} \\ & \boxed{A_{23}} & & & & \end{array}$	$t_{23} = 0$
$\mathfrak{F}_{1,2}$	$\begin{array}{cccccc} \boxed{A_{12}} & \boxed{} & \boxed{} & \boxed{} & \boxed{} & \boxed{} \\ & \boxed{A_{23}} & & & & \end{array}$	G

where $A_{13}^*, A_{15}^* \in \mathbb{F}_{q^3}^*$ and $A_{16}^*, A_{17}^* \in \mathbb{F}_q^*$.

Proof. It is straightforward to calculate. $\square$

Now we get the following classification of $G_8\,(q^3)$-orbit modules.

2.2.57 Proposition (Classification of $G_8\,(q^3)$-orbit modules)**.** *Let $A \in V$, then every G-orbit module $\mathbb{C}\mathcal{O}_G([A])$ is of one of the forms in the following table which are all irreducible.*

$G_8\,(q^3)$-orbit modules

	$\mathbb{C}\mathcal{O}_G([A])\ (A \in V)$	$\dim_{\mathbb{C}}\mathbb{C}\mathcal{O}_G([A])$	#orbit modules
F_6	$\mathbb{C}\mathcal{O}_G\left(\begin{bmatrix} \cdot & & & & & & A_{17}^* & \cdot \\ & \cdot & A_{23} & & & & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & & \cdot\ \cdot & & & \end{bmatrix}\right)$	q^{10}	$(q-1)q$
F_5	$\mathbb{C}\mathcal{O}_G\left(\begin{bmatrix} \cdot & & & & & A_{16}^* & & \cdot \\ & \cdot & A_{23} & & & & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & & \cdot\ \cdot & & & \end{bmatrix}\right)$	q^9	$(q-1)q$
F_4	$\mathbb{C}\mathcal{O}_G\left(\begin{bmatrix} \cdot & & & A_{15}^{*\,q} & A_{15}^* & & & \cdot \\ & \cdot & A_{23} & & & & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & & \cdot\ \cdot & & & \end{bmatrix}\right)$	q^6	$(q^3-1)q$
F_3	$\mathbb{C}\mathcal{O}_G\left(\begin{bmatrix} \cdot & \bar{A}_{12}^{A_{13}^*} & A_{13}^* & & & & & \cdot \\ & \cdot & A_{23} & & & & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & & \cdot\ \cdot & & & \end{bmatrix}\right)$	q	$(q^3-1)q^2 \cdot q$
$F_{1,2}$	$\mathbb{C}\mathcal{O}_G\left(\begin{bmatrix} \cdot & A_{12} & & & & & & \cdot \\ & \cdot & A_{23} & & & & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & & \cdot\ \cdot & & & \end{bmatrix}\right)$	1	q^4

where $\bar{A}_{12}^{A_{13}^*} \in T^{A_{13}^*}$.

Proof. We obtain the classifications by 2.2.55 (c.f. the proof of 2.2.50). Let $A \in V$ and ψ_A^G be the character of the G-orbit module $\mathbb{C}\mathcal{O}_G([A])$. By 2.2.56, we calculate and get the inner product $\langle \psi_A^G, \psi_A^G \rangle_G = 1$ (c.f. the proof of 2.2.79). Thus the proposition is proved. $\square$

Compare 2.2.50 and 2.2.57 (c.f. 2.3.17), we obtain the following relations.

2.2.58 Corollary. *Let* $\bar{A}_{12}^{A_{13}^*} \in T^{A_{13}^*}$*, then*

$$\begin{aligned}
\mathcal{O}_G(A_{12}e_{12} + A_{23}e_{23}) =& \mathcal{O}_U(A_{12}e_{12} + A_{23}e_{23}),\\
\mathcal{O}_G(A_{13}^*e_{13} + \bar{A}_{12}^{A_{13}^*}e_{12}) =& \mathcal{O}_U(A_{13}^*e_{13} + \bar{A}_{12}^{A_{13}^*}e_{12}),\\
\mathcal{O}_G({A_{15}^*}^{q^2}e_{14} + A_{15}^*e_{15} + A_{23}e_{23}) =& \mathcal{O}_U({A_{15}^*}^{q^2}e_{14} + A_{15}^*e_{15} + A_{23}e_{23}),\\
\mathcal{O}_G(A_{16}^*e_{16} + A_{23}e_{23}) =& \bigcup_{A_{13}\in\mathbb{F}_{q^3}} \mathcal{O}_U(A_{16}^*e_{16} + A_{23}e_{23} + A_{13}e_{13}),\\
\mathcal{O}_G(A_{17}^*e_{17} + A_{23}e_{23}) =& \bigcup_{A_{12}\in\mathbb{F}_{q^3}} \mathcal{O}_U(A_{17}^*e_{17} + A_{23}e_{23} + A_{12}e_{12}).
\end{aligned}$$

2.2.4. Homomorphisms between orbit modules

Let $U = {}^3D_4^{syl}(q^3)$. In this subsection, we define a truncated row operation of U on V (2.2.64). Then we show that every U-orbit module is isomorphic to a staircase orbit module (2.2.75). After that, we prove that every U-orbit module is isomorphic to some hook-separated staircase module (2.2.79). Finally, some irreducible modules are determined, and any two orbit modules are shown to be orthogonal when the 1st verges are different (2.2.83). In this subsection, the properties are adaptions of the corresponding statements in [Jed13, §3.3]. They are proved similarly.

The following property is well known.

2.2.59 Notation/Lemma. *Every $\varphi \in \mathrm{End}_{\mathbb{C}U}(\mathbb{C}U)$ is of the form*

$$\lambda_a \colon \mathbb{C}U \to \mathbb{C}U : x \mapsto ax \quad \textit{for a unique } a \in \mathbb{C}U.$$

2.2.60 Lemma. *Let $g \in U$ and $A \in V$, then the map*

$$\lambda_g|_{\mathbb{C}\mathcal{O}_U([A])} \colon \mathbb{C}\mathcal{O}_U([A]) \to \mathrm{Im}(\lambda_g|_{\mathbb{C}\mathcal{O}_U([A])}) = g\mathbb{C}\mathcal{O}_U([A])$$

is a $\mathbb{C}U$-isomorphism.

2.2.61 Lemma. *Let $A \in V$ and $g \in U$, then*

$$\lambda_g([A]) \overset{Def.}{=} g[A] = \frac{1}{|U|}\sum_{y\in U} \overline{\vartheta\kappa_q(g^{-\top}A, y)}y.$$

Proof. Let $g \in U$ and $A \in V$, then

$$\begin{aligned}
\lambda_g([A]) &= g[A] = \frac{1}{|U|}\sum_{z\in U}\overline{\chi_A(z)}gz \\
&\overset{y:=gz}{=} \frac{1}{|U|}\sum_{y\in U}\overline{\chi_A(g^{-1}y)}y \\
&= \frac{1}{|U|}\sum_{y\in U}\overline{\vartheta\kappa_q(A, f(g^{-1}y))}y \\
&\overset{f(g^{-1}y)\equiv g^{-1}y \mod V^\perp}{=} \frac{1}{|U|}\sum_{y\in U}\overline{\vartheta\kappa_q(A, g^{-1}y)}y \\
&= \frac{1}{|U|}\sum_{y\in U}\overline{\vartheta\kappa_q(g^{-\top}A, y)}y.
\end{aligned}$$

□

2.2.62 Lemma. *Let $A \in V$ and $\pi_{\triangledown} \colon V_0 \to V_{\triangledown} : A \mapsto \sum_{(i,j)\in\triangledown} A_{ij}e_{ij}$. Then*

$$\pi_{\triangledown}(x^{-\top}A)$$
$$= \begin{pmatrix}
0 & A_{12} & A_{13} & A_{15}^q & A_{15} & A_{16} & A_{17} & 0 \\
 & -t_1A_{12} & \begin{array}{c}-t_1A_{13}\\+A_{23}\end{array} & -t_1A_{15}^q & -t_1A_{15} & -t_1A_{16} & -t_1A_{17} & 0 \\
 & & \begin{array}{c}t_1t_2A_{13}\\-t_2A_{23}\\+t_3A_{13}\end{array} & \begin{array}{c}t_1t_2A_{15}^q\\+t_3A_{15}^q\end{array} & \begin{array}{c}t_1t_2A_{15}\\+t_3A_{15}\end{array} & \begin{array}{c}t_1t_2A_{16}\\+t_3A_{16}\end{array} & \begin{array}{c}t_1t_2A_{17}\\+t_3A_{17}\end{array} & 0 \\
 & & & \begin{array}{c}-t_1^qt_3A_{15}^q\\-t_4A_{15}^q\end{array} & \begin{array}{c}-t_1^qt_3A_{15}\\-t_4A_{15}\end{array} & \begin{array}{c}-t_1^qt_3A_{16}\\-t_4A_{16}\end{array} & \begin{array}{c}-t_1^qt_3A_{17}\\-t_4A_{17}\end{array} & 0 \\
 & & & & \begin{array}{c}-t_1^{q^2}t_3A_{15}\\-t_4^{q^2}A_{15}\end{array} & \begin{array}{c}-t_1^{q^2}t_3A_{16}\\-t_4^{q^2}A_{16}\end{array} & \begin{array}{c}-t_1^{q^2}t_3A_{17}\\-t_4^{q^2}A_{17}\end{array} & 0 \\
 & & & & & \begin{array}{c}-t_1^{q^2+q}t_3A_{16}\\-t_1^qt_4^{q^2}A_{16}\\-t_1^{q^2}t_4A_{16}\\-t_5A_{16}\end{array} & \begin{array}{c}-t_1^{q^2+q}t_3A_{17}\\-t_1^qt_4^{q^2}A_{17}\\-t_1^{q^2}t_4A_{17}\\-t_5A_{17}\end{array} & 0 \\
 & & & & & & \begin{array}{c}-t_1^{q^2+q}t_2t_3A_{17}\\-t_1^qt_2t_4^{q^2}A_{17}\\-t_1^{q^2}t_2t_4A_{17}\\-t_2t_5A_{17}\\-t_3^qt_4^{q^2}A_{17}\\-t_3^{q^2}t_4A_{17}\\-t_6A_{17}\end{array} & 0 \\
 & & & & & & & 0
\end{pmatrix}.$$

2.2.63 Proposition. *Let $A \in V$ and $x := x(t_1, t_2, t_3, t_4, t_5, t_6) \in U$, then*

$$x.A = A - \pi_q(t_1A_{13})e_{23}.$$

Proof. Let $A \in V$ and $x := x(t_1, t_2, t_3, t_4, t_5, t_6) \in U$, then

$$x.A = \pi(x^{-\top}A) \overset{2.2.62}{=} A - \pi_q(t_1A_{13})e_{23}.$$

Thus we get the result. □

2.2.64 Definition/Lemma (${}^3D_4^{syl}(q^3)$-truncated row operation)**.** *The map*

$$U \times V \to V : (u, A) \mapsto u.A := \pi(u^{-\top}A)$$

defines a group operation, which is called the ***truncated row operation****.*

Note that the elements of U act as $\mathbb{F}_q$-automorphisms on V.

Proof. Since π is an $\mathbb{F}_q$-linear map by 2.2.16, it is enough to prove that $(gh).A = g.(h.A)$ for all $g := x(t_1, t_2, t_3, t_4, t_5, t_6) \in U$, $h := x(s_1, s_2, s_3, s_4, s_5, s_6) \in U$ and $A = (A_{ij}) \in V$. Let $gh := x(r_1, r_2, r_3, r_4, r_5, r_6)$, then $r_1 = t_1 + s_1$ by 2.1.12. By 2.2.63, we have $g.(h.A) = g.(A - \pi_q(s_1A_{13})e_{23}) = A - (\pi_q(t_1A_{13}) + \pi_q(s_1A_{13}))e_{23} = A - \pi_q(r_1A_{13})e_{23} = (gh).A$. $\square$

2.2.65 Corollary. *Let $A \in V$, $x_i(t_i) \in U$ $(i = 1, 2, 3, 4, 5, 6)$ and $t_1, t_3, t_4 \in \mathbb{F}_{q^3}$, $t_2, t_5, t_6 \in \mathbb{F}_q$, then*

(i) $x_1(t_1).A$ is obtained from A by replacing A_{23} with $A_{23} - \pi_q(t_1A_{13})$, i.e.

$$x_1(t_1).A = \left(\begin{array}{|c|c|c|c|c|c|} \hline A_{12} & A_{13} & A_{15}^q & A_{15} & A_{16} & A_{17} \\ \hline & \begin{array}{c} A_{23} \\ -\pi_q(t_1A_{13}) \end{array} & & & & \\ \cline{2-2} \end{array} \right).$$

(ii) $x_i(t_i).A = A$ for all $i \in \{2, 3, 4, 5, 6\}$.

2.2.66 Notation. *$\mathcal{O}_U^{\mathrm{row}}(A)$ and $\mathrm{Stab}_U^{\mathrm{row}}(A)$ are defined under the 'truncated row operation', and $\mathcal{O}_U(A)$ and $\mathrm{Stab}_U(A)$ are used for the 'truncated column operation'.*

2.2.67 Remark. *From Corollary 2.2.65, we obtain*

$$x_1(t_1).A = x_{12}(t_1).A = \tilde{x}_{12}(t_1).A$$

and

$$\mathcal{O}^{\mathrm{row}}_{{}^3D_4^{syl}(q^3)}(A) = \mathcal{O}^{\mathrm{row}}_{D_4^{syl}(q^3)}(A) = \mathcal{O}^{\mathrm{row}}_{A_8(q^3)}(A).$$

2.2.68 Remark. *In general, $g.(A.u) \neq (g.A).u$. For example: let $t_1, t_4 \in \mathbb{F}_{q^3}^*$ and $A_{17}^* \in \mathbb{F}_q^*$, then*

$$\big(x_1(t_1).(A17^*e_{17})\big).x_4(t_4) = \left(\begin{array}{|c|c|c|c|c|c|} \hline \; & t_4^qA_{17}^* & \; & \; & \; & A_{17}^* \\ \hline & & & & & \\ \cline{2-2} \end{array} \right)$$

but

$$x_1(t_1).\big((A17^*e_{17}).x_4(t_4)\big) = \left(\begin{array}{|c|c|c|c|c|c|} \hline \; & t_4^qA_{17}^* & \; & \; & \; & A_{17}^* \\ \hline & -t_1t_4^qA_{17}^* & & & & \\ \cline{2-2} \end{array} \right).$$

2.2.69 Lemma. *Let $B \in V$, $g := x(t_1, t_2, t_3, t_4, t_5, t_6) \in U$ and $y \in U$ such that*

$$\mathrm{supp}(g^{-\top}B) \cap \mathrm{supp}(y - 1) \subseteq J.$$

Then $\vartheta\kappa_q(g^{-\top}B, y - 1) = \chi_{g.B}(y)$.

Proof. Let $B \in V$, $g := x(t_1, t_2, t_3, t_4, t_5, t_6) \in U$ and $y \in U$ as supposed, then

$$\begin{aligned}
\vartheta\kappa_q(g^{-\top}B, y-1) &= \vartheta\kappa_q(B - t_1 B_{13} e_{23}, y-1) \\
&= \vartheta\big(\kappa_q(B, y-1) + \kappa_q(-t_1 B_{13} e_{23}, y-1)\big) \\
&= \vartheta\big(\kappa_q(B, y-1) + \pi_q((-t_1 B_{13}) y_{23})\big) \\
&\overset{y_{23} \in \mathbb{F}_q}{=} \vartheta\big(\kappa_q(B, y-1) + \pi_q(-t_1 B_{13}) \cdot y_{23}\big) \\
&= \vartheta\big(\kappa_q(B, y-1) + \pi_q(\pi_q(-t_1 B_{13}) y_{23})\big) \\
&= \vartheta\big(\kappa_q(B, y-1) + \kappa_q(\pi_q(-t_1 B_{13}) e_{23}, y-1)\big) \\
&= \vartheta\kappa_q(B - \pi_q(t_1 B_{13}) e_{23}, y-1) \\
&\overset{2.2.63}{=} \vartheta\kappa_q(g.B, f(y)) \\
&= \chi_{g.B}(y).
\end{aligned}$$

$\square$

2.2.70 Proposition. *Let* $g \in U$ *and* $A \in V$*, such that* $\operatorname{supp}(g^{-\top}A) \cap \triangledown \subseteq J \cup \{(4,5)\}$. *Then*

$$\lambda_g([B]) = \chi_{g.B}(g)[g.B] \quad \textit{for all } B \in \mathcal{O}_U(A).$$

Proof. Let $B \in \mathcal{O}_U(A)$, then $\operatorname{main}(B) = \operatorname{main}(A)$. Thus,

$$\operatorname{supp}(g^{-\top}B) \cap \triangledown \subseteq J \cup \{(4,5)\} \quad \text{for all } B \in \mathcal{O}_U(A).$$

Then for all $y \in U$, $\operatorname{supp}(g^{-\top}B) \cap \operatorname{supp}(y-1) \subseteq J$, since $\operatorname{supp}(y-1) \subseteq \triangledown \backslash \{(4,5)\}$.
Then

$$\begin{aligned}
g[B] &\overset{2.2.61}{=} \frac{1}{|U|} \sum_{y \in U} \overline{\vartheta\kappa_q(g^{-\top}B, y)} y = \frac{1}{|U|} \sum_{y \in U} \overline{\vartheta\kappa_q(g^{-\top}B, y-1) \vartheta\kappa_q(g^{-\top}B, 1)} y \\
&\overset{2.2.69}{=} \overline{\vartheta\kappa_q(B, f(g^{-1}))} \cdot \frac{1}{|U|} \sum_{y \in U} \overline{\chi_{g.B}(y)} y = \overline{\chi_B(g^{-1})}[g.B].
\end{aligned}$$

We have

$$\begin{aligned}
\chi_{g.B}(g) &\overset{2.2.69}{=} \vartheta\kappa_q(g^{-\top}B, g-1) = \vartheta\kappa_q(B, 1-g^{-1}) = \overline{\vartheta\kappa_q(B, g^{-1}-1)} \\
&= \overline{\vartheta\kappa_q(B, f(g^{-1}))} = \overline{\chi_B(g^{-1})}.
\end{aligned}$$

Hence $\lambda_g([B]) = \chi_{g.B}(g)[g.B]$ for all $B \in \mathcal{O}_U(A)$. $\square$

2.2.71 Corollary. *Let* $g \in U$ *and* $A \in V$*, such that* $\operatorname{supp}(g^{-\top}A) \cap \triangledown \subseteq J \cup \{(4,5)\}$. *Then*

$$\operatorname{im}(\lambda_g|_{\mathbb{C}\mathcal{O}_U([A])}) = \mathbb{C}\mathcal{O}_U([g.A]).$$

Proof. Let $u \in U$, then $\mathbb{C}g[A.u] = \mathbb{C}g([A]u) = \mathbb{C}(g[A])u \overset{2.2.70}{=} \mathbb{C}[g.A]u$. Hence $\operatorname{im}(\lambda_g|_{\mathbb{C}\mathcal{O}_U([A])}) = \mathbb{C}\mathcal{O}_U([g.A])$. $\square$

2.2.72 Corollary. *Let* $g \in U$ *and* $A \in V$*, such that* $\operatorname{supp}(g^{-\top}A) \cap \text{◹} \subseteq J \cup \{(4,5)\}$. *Then*

$$g.(B.u) = (g.B).u \quad \textit{for all } B \in \mathcal{O}_U(A),\ u \in U.$$

Proof. Let $g \in U$ and $A \in V$, such that $\operatorname{supp}(g^{-\top}A) \cap \text{◹} \subseteq J \cup \{(4,5)\}$. Then

$$\mathbb{C}[(g.B).u] = \mathbb{C}[g.B]u \overset{2.2.70}{=} \mathbb{C}(g[B])u = \mathbb{C}g([B]u) = \mathbb{C}g[B.u] = \mathbb{C}[g.(B.u)].$$

Hence $g.(B.u) = (g.B).u$. $\square$

2.2.73 Corollary (Simple row move). *Let* $A \in V$, $x_i(t_i) \in U$ $(i = 1,2,3,4,5,6)$ *and* $t_1, t_3, t_4 \in \mathbb{F}_{q^3}$, $t_2, t_5, t_6 \in \mathbb{F}_q$*, such that* $\operatorname{supp}(x_i(t_i)^{-\top}A) \cap \text{◹} \subseteq J \cup \{(4,5)\}$ *then*

$$\mathbb{C}\mathcal{O}_U([A]) \cong \mathbb{C}\mathcal{O}_U([x_i(t_i).A]).$$

2.2.74 Corollary. *Let* $A \in V$ $(A_{17} = A_{16} = A_{15} = 0,\ A_{13} = A_{13}^* \neq 0)$*,* $x_1(t_1) \in U$ *such that* $\pi_q(t_1 A_{13}) = A_{23}$*, then* $\mathbb{C}\mathcal{O}_U([A]) \cong \mathbb{C}\mathcal{O}_U([x_1(t_1).A]) \cong \mathbb{C}\mathcal{O}_U([A - A_{23}e_{23}])$*, i.e.*

$$\mathbb{C}\mathcal{O}_U(\left[\begin{array}{|c|c|c|c|c|c|}\hline A_{12} & A_{13}^* & & & & \\ \hline & A_{23} & \multicolumn{4}{c}{} \\ \cline{2-2}\end{array}\right]) \cong \mathbb{C}\mathcal{O}_U(\left[\begin{array}{|c|c|c|c|c|c|}\hline A_{12} & A_{13}^* & & & & \\ \hline & 0 & \multicolumn{4}{c}{} \\ \cline{2-2}\end{array}\right]).$$

2.2.75 Corollary. *Every* U*-orbit module is isomorphic to a (not necessarily unique) staircase module, and the isomorphism is given by the left multiplication by a group element.*

Proof. Let $A \in V$ and $(1,3),(2,3) \in \operatorname{main}(A)$, then $(2,3)$ is deleted by 2.2.74. By 2.2.51, the claim is proved. $\square$

2.2.76 Lemma. *Let* $A \in V$ *with* $A_{17} = A_{17}^* \in \mathbb{F}_q^*$*,* $x_5(s_5) \in U$ *and* $s_5 \in \mathbb{F}_q$*, then*

$$\lambda_{x_5(s_5)}([A]) = \vartheta(s_5 A_{16})[A + s_5 A_{17}^* e_{23}].$$

Proof. Let $A \in V$ with $A_{17} = A_{17}^* \in \mathbb{F}_q^*$ and $x_5(s_5) \in U$, then

$$\begin{aligned}
\lambda_{x_5(s_5)}([A]) =& x_5(s_5)[A] \\
\overset{2.2.61}{=}& \frac{1}{|U|}\sum_{y\in U} \overline{\vartheta\kappa_q(x_5(s_5)^{-\top}A, y)}y \\
\overset{2.2.62}{=}& \frac{1}{|U|}\sum_{y\in U} \overline{\vartheta\kappa_q(A - s_5A_{16}e_{66} - s_5A_{17}^*e_{67}, y)}y \\
=& \frac{1}{|U|}\sum_{y\in U} \left(\overline{\vartheta\kappa_q(A, y)} \cdot \overline{\vartheta\pi_q(-s_5A_{16})} \cdot \overline{\vartheta\pi_q(-s_5A_{17}^*y_{67})}\right)y \\
=& \vartheta(s_5A_{16}) \cdot \frac{1}{|U|}\sum_{y\in U} \left(\overline{\vartheta\kappa_q(A, y)} \cdot \overline{\vartheta\pi_q(-s_5A_{17}^*y_{67})}\right)y \\
\overset{2.1.12}{=}& \vartheta(s_5A_{16}) \cdot \frac{1}{|U|}\sum_{y\in U} \left(\overline{\vartheta\kappa_q(A, y)} \cdot \overline{\vartheta\pi_q(s_5A_{17}^*y_{23})}\right)y \\
=& \vartheta(s_5A_{16}) \cdot \frac{1}{|U|}\sum_{y\in U} \overline{\vartheta\kappa_q(A + s_5A_{17}^*e_{23}, y)}y \\
=& \vartheta(s_5A_{16}) \cdot \frac{1}{|U|}\sum_{y\in U} \overline{\vartheta\kappa_q(A + s_5A_{17}^*e_{23}, f(y))}y \\
=& \vartheta(s_5A_{16})[A + s_5A_{17}^*e_{23}].
\end{aligned}$$

□

2.2.77 Proposition. *Let* $A, B \in V$*,* $A_{17} = A_{17}^* \in \mathbb{F}_q^*$*, and*

$$A := \begin{array}{|c|c|c|c|c|c|} \hline A_{12} & A_{13} & A_{15}^q & A_{15} & A_{16} & A_{17}^* \\ \hline \multicolumn{1}{c|}{} & A_{23} & \multicolumn{4}{c}{} \\ \cline{2-2} \end{array} \quad ,$$

$$B := \begin{array}{|c|c|c|c|c|c|} \hline A_{12} & A_{13} & A_{15}^q & A_{15} & A_{16} & A_{17}^* \\ \hline \multicolumn{1}{c|}{} & 0 & \multicolumn{4}{c}{} \\ \cline{2-2} \end{array}$$

then $\mathbb{C}\mathcal{O}_U([A]) \cong \mathbb{C}\mathcal{O}_U([B])$.

Proof. Let $C \in \mathcal{O}_U(A)$ and $s_5 := -\frac{A_{23}}{A_{17}^*} \in \mathbb{F}_q$. By 2.2.76, we get $\lambda_{x_5(s_5)}([C]) = \vartheta(s_5C_{16})[C + s_5A_{17}^*e_{23}]$, where $C + s_5A_{17}^*e_{23} \in \mathcal{O}_U(B)$. Thus $\mathbb{C}\mathcal{O}_U([A]) \cong \mathbb{C}\mathcal{O}_U([B])$ by 2.2.60. □

2.2.78 Definition. *Let* $1 \leq i \leq N$*. The* ***i-th hook*** *of* J *is defined to be*

$$H_i := \{(a, b) \in J \mid b = i \text{ or } a = \bar{i}\}.$$

In particular, $H_7 = \{(1,7),(2,3)\}$. Let $A \in V$ be a pattern. Then A is called ***hook-separated****, if on every hook H_i of J lies at most one main condition of A. Note that hook-separated patterns are always staircase patterns. Let $A \in V$ be hook-separated, then $\mathbb{C}\mathcal{O}_U([A])$ is called a* ***hook-separated staircase module****.*

2.2.79 Corollary. *Every U-orbit module is isomorphic to a certain hook-separated staircase module.*

Proof. By 2.2.75, every U-orbit module is isomorphic to a staircase module. By 2.2.77, we get the desired conclusion. $\square$

2.2.80 Lemma. *Let $A, B \in V$. Then $\mathrm{Hom}_{\mathbb{C}U}(\mathbb{C}\mathcal{O}_U([A]), \mathbb{C}\mathcal{O}_U([B])) = \{0\}$ if and only if for all $C \in \mathcal{O}_U(A)$ and $D \in \mathcal{O}_U(B)$ holds $\mathrm{Hom}_{\mathrm{Stab}_U(C,D)}(\mathbb{C}[C], \mathbb{C}[D]) = \{0\}$.*

Proof. Let ψ_A be the character of $\mathbb{C}\mathcal{O}_U([A])$ and ψ_B denote the character of $\mathbb{C}\mathcal{O}_U([B])$. Then

$$\begin{aligned}
0 =&\dim_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}U}(\mathbb{C}\mathcal{O}_U([A]), \mathbb{C}\mathcal{O}_U([B])) = \langle \psi_A, \psi_B\rangle_U \\
=&\frac{1}{|U|}\sum_{u\in U}\psi_A(u)\overline{\psi_B(u)} = \frac{1}{|U|}\sum_{u\in U}\Bigg(\sum_{\substack{C\in\mathcal{O}_U(A)\\ C.u=C}}\chi_C(u)\Bigg)\Bigg(\sum_{\substack{D\in\mathcal{O}_U(B)\\ D.u=D}}\overline{\chi_D(u)}\Bigg) \\
=&\frac{1}{|U|}\sum_{\substack{C\in\mathcal{O}_U(A)\\ D\in\mathcal{O}_U(B)}}\sum_{\substack{u\in U\\ C.u=C\\ D.u=D}}\chi_C(u)\overline{\chi_D(u)} = \frac{1}{|U|}\sum_{\substack{C\in\mathcal{O}_U(A)\\ D\in\mathcal{O}_U(B)}}\sum_{u\in\mathrm{Stab}_U(C,D)}\chi_C(u)\overline{\chi_D(u)} \\
=&\sum_{\substack{C\in\mathcal{O}_U(A)\\ D\in\mathcal{O}_U(B)}}\frac{|\mathrm{Stab}_U(C,D)|}{|U|}\langle\chi_C,\chi_D\rangle_{\mathrm{Stab}_U(C,D)} \\
=&\sum_{\substack{C\in\mathcal{O}_U(A)\\ D\in\mathcal{O}_U(B)}}\frac{|\mathrm{Stab}_U(C,D)|}{|U|}\Big(\dim_{\mathbb{C}}\mathrm{Hom}_{\mathrm{Stab}_U(C,D)}(\mathbb{C}[C],\mathbb{C}[D])\Big),
\end{aligned}$$

where χ_C is the linear character of the $\mathbb{C}\mathrm{Stab}_U(C,D)$-module $\mathbb{C}[C]$, while χ_D is the linear character of the $\mathbb{C}\mathrm{Stab}_U(C,D)$-module $\mathbb{C}[D]$. Thus $\langle\chi_C,\chi_D\rangle_{\mathrm{Stab}_U(C,D)} = 0$ or 1. Hence

$$\begin{aligned}
&0 = \dim_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}U}(\mathbb{C}\mathcal{O}_U([A]), \mathbb{C}\mathcal{O}_U([B])) \\
\Longleftrightarrow\ &\langle\chi_C,\chi_D\rangle_{\mathrm{Stab}_U(C,D)} = 0 \quad \text{for all } C\in\mathcal{O}_U(A), D\in\mathcal{O}_U(B) \\
\Longleftrightarrow\ &\dim_{\mathbb{C}}\mathrm{Hom}_{\mathrm{Stab}_U(C,D)}(\mathbb{C}[C],\mathbb{C}[D]) = 0 \quad \text{for all } C\in\mathcal{O}_U(A), D\in\mathcal{O}_U(B) \\
\Longleftrightarrow\ &\mathrm{Hom}_{\mathrm{Stab}_U(C,D)}(\mathbb{C}[C],\mathbb{C}[D]) = \{0\} \quad \text{for all } C\in\mathcal{O}_U(A), D\in\mathcal{O}_U(B).
\end{aligned}$$

$\square$

2.2.81 Lemma. *Let $A, B \in V$ and $y \in U$. Then*

$$\mathrm{Hom}_{\mathrm{Stab}_U(A,B)}(\mathbb{C}[A],\mathbb{C}[B]) = \mathrm{Hom}_{\mathrm{Stab}_U(A.y,B.y)}(\mathbb{C}[A.y],\mathbb{C}[B.y])$$

as $\mathbb{C}$-vector spaces.

Proof. c.f. the proof of Proposition 3.3.21 of [Jed13]. $\square$

2.2.82 Corollary (Homomorphism criterion). *Let $A, B \in V$. Then*

$$\mathrm{Hom}_{\mathbb{C}U}(\mathbb{C}\mathcal{O}_U([A]), \mathbb{C}\mathcal{O}_U([B])) = \{0\}$$

if and only if $\mathrm{Hom}_{\mathrm{Stab}_U(A,D)}(\mathbb{C}[A], \mathbb{C}[D]) = \{0\}$ *for all* $D \in \mathcal{O}_U(B)$.

Proof. Let ψ_A denote the character of $\mathbb{C}\mathcal{O}_U([A])$ and ψ_B denote the character of $\mathbb{C}\mathcal{O}_U([B])$. By the proof of Lemma 2.2.80, we get

$$\begin{aligned}
0 =& \dim_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}U}(\mathbb{C}\mathcal{O}_U([A]), \mathbb{C}\mathcal{O}_U([B])) = \langle \psi_A, \psi_B \rangle_U \\
=& \sum_{\substack{C \in \mathcal{O}_U(A) \\ D \in \mathcal{O}_U(B)}} \frac{|\mathrm{Stab}_U(C,D)|}{|U|} \dim_{\mathbb{C}}\mathrm{Hom}_{\mathrm{Stab}_U(C,D)}(\mathbb{C}[C], \mathbb{C}[D]) \\
=& \sum_{u \in U} \sum_{v \in U} \frac{|\mathrm{Stab}_U(A.u, B.v)|}{|U||\mathrm{Stab}_U(A)||\mathrm{Stab}_U(B)|} \dim_{\mathbb{C}}\mathrm{Hom}_{\mathrm{Stab}_U(A.u,B.v)}(\mathbb{C}[A.u], \mathbb{C}[B.v]) \\
\overset{v':=vu^{-1}}{=}& \sum_{u \in U} \sum_{v' \in U} \frac{|\mathrm{Stab}_U(A.u, B.(v'u))|}{|U||\mathrm{Stab}_U(A)||\mathrm{Stab}_U(B)|} \dim_{\mathbb{C}}\mathrm{Hom}_{\mathrm{Stab}_U(A.u,B.(v'u))}(\mathbb{C}[A.u], \mathbb{C}[B.(v'u)]) \\
=& \sum_{u \in U} \sum_{v' \in U} \frac{|\mathrm{Stab}_U(A.u, (B.v').u))|}{|U||\mathrm{Stab}_U(A)||\mathrm{Stab}_U(B)|} \\
& \cdot \dim_{\mathbb{C}}\mathrm{Hom}_{\mathrm{Stab}_U(A.u,(B.v').u)}(\mathbb{C}[A.u], \mathbb{C}[(B.v').u)]) \\
\overset{2.2.81}{=}& \sum_{u \in U} \sum_{v \in U} \frac{|\mathrm{Stab}_U(A, B.v))|}{|U||\mathrm{Stab}_U(A)||\mathrm{Stab}_U(B)|} \dim_{\mathbb{C}}\mathrm{Hom}_{\mathrm{Stab}_U(A,B.v)}(\mathbb{C}[A], \mathbb{C}[B.v]) \\
=& \sum_{v \in U} \frac{|\mathrm{Stab}_U(A, B.v))|}{|\mathrm{Stab}_U(A)||\mathrm{Stab}_U(B)|} \dim_{\mathbb{C}}\mathrm{Hom}_{\mathrm{Stab}_U(A,B.v)}(\mathbb{C}[A], \mathbb{C}[B.v]) \\
=& \sum_{D \in \mathcal{O}_U(B)} \frac{|\mathrm{Stab}_U(A, D))|}{|\mathrm{Stab}_U(A)|} \dim_{\mathbb{C}}\mathrm{Hom}_{\mathrm{Stab}_U(A,D)}(\mathbb{C}[A], \mathbb{C}[D]) \\
\iff& \dim_{\mathbb{C}}\mathrm{Hom}_{\mathrm{Stab}_U(A,D)}(\mathbb{C}[A], \mathbb{C}[D]) = 0 \quad \text{for all } D \in \mathcal{O}_U(B) \\
\iff& \mathrm{Hom}_{\mathrm{Stab}_U(A,D)}(\mathbb{C}[A], \mathbb{C}[D]) = \{0\} \quad \text{for all } D \in \mathcal{O}_U(B).
\end{aligned}$$

$\square$

2.2.83 Proposition. *Every U-orbit module is isomorphic to a hook-separated staircase module in Table 2.3, and they satisfy the following properties:*

(1) Let $A, B \in V$. If $\mathrm{verge}_1(A) \neq \mathrm{verge}_1(B)$,

$$\mathrm{Hom}_{\mathbb{C}U}(\mathbb{C}\mathcal{O}_U([A]), \mathbb{C}\mathcal{O}_U([B])) = \{0\}.$$

In particular, let $\mathbb{C}\mathcal{O}_U([A]) \in \mathfrak{F}_i$, $\mathbb{C}\mathcal{O}_U([B]) \in \mathfrak{F}_j$ *and* $i \neq j$, *then*

$$\mathrm{Hom}_{\mathbb{C}U}(\mathbb{C}\mathcal{O}_U([A]), \mathbb{C}\mathcal{O}_U([B])) = \{0\}.$$

(2) *In the Family $\mathfrak{F}_{1,2}$, the q^4 hook-separated staircase modules are irreducible and pairwise orthogonal.*

(3) *In the Family $\mathfrak{F}_3$, the $(q^3-1)q^2$ hook-separated staircase modules are irreducible and pairwise orthogonal.*

(4) *In the Family $\mathfrak{F}_4$, $\mathfrak{F}_5$ and $\mathfrak{F}_6$, the hook-separated staircase modules are reducible.*

Table 2.3.: Hook-separated staircase ${}^3D_4^{syl}(q^3)$-orbit modules

Family	$\mathbb{C}\mathcal{O}_U([A])$ $(A \in V)$	$\dim_{\mathbb{C}}\mathbb{C}\mathcal{O}_U([A])$	Irreducible
$\mathfrak{F}_6$	$\mathbb{C}\mathcal{O}_U\left(\left[\begin{array}{cccccccc} \cdot & \boxed{A_{12}} & \boxed{} & \boxed{} & \boxed{} & \boxed{} & \boxed{A^*_{17}} & \cdot \\ & \cdot & \boxed{0} & & & & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & \cdot & \cdot & & & \end{array}\right]\right)$	q^7	NO
$\mathfrak{F}_5$	$\mathbb{C}\mathcal{O}_U\left(\left[\begin{array}{cccccccc} \cdot & \boxed{} & \boxed{A_{13}} & \boxed{} & \boxed{} & \boxed{A^*_{16}} & \boxed{} & \cdot \\ & \cdot & \boxed{A_{23}} & & & & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & \cdot & \cdot & & & \end{array}\right]\right)$	q^6	NO
$\mathfrak{F}_4$	$\mathbb{C}\mathcal{O}_U\left(\left[\begin{array}{cccccccc} \cdot & \boxed{} & \boxed{} & \boxed{{A^*_{15}}^q} & \boxed{A^*_{15}} & \boxed{} & \boxed{} & \cdot \\ & \cdot & \boxed{A_{23}} & & & & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & \cdot & \cdot & & & \end{array}\right]\right)$	q^6	NO
$\mathfrak{F}_3$	$\mathbb{C}\mathcal{O}_U\left(\left[\begin{array}{cccccccc} \cdot & \boxed{\bar{A}_{12}^{A^*_{13}}} & \boxed{A^*_{13}} & \boxed{} & \boxed{} & \boxed{} & \boxed{} & \cdot \\ & \cdot & \boxed{0} & & & & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & \cdot & \cdot & & & \end{array}\right]\right)$	q	YES
$\mathfrak{F}_{1,2}$	$\mathbb{C}\mathcal{O}_U\left(\left[\begin{array}{cccccccc} \cdot & \boxed{A_{12}} & \boxed{} & \boxed{} & \boxed{} & \boxed{} & \boxed{} & \cdot \\ & \cdot & \boxed{A_{23}} & & & & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & \cdot & \cdot & & & \end{array}\right]\right)$	1	YES

where $A^*_{12}, A^*_{13}, A^*_{15} \in \mathbb{F}^*_{q^3}$, $A^*_{16}, A^*_{17}, A^*_{23} \in \mathbb{F}^*_q$ and $\bar{A}_{12}^{A^*_{13}} \in T^{A^*_{13}}$.

Proof. By 2.2.75 and 2.2.79, every U-orbit module is isompiphic to a hook-separated staircase module in Table 2.3.

Let $A, B \in V$ be hook-separated staircase core patterns of the Family $\mathfrak{F}_4$ and $C \in \mathcal{O}_U(B)$, i.e.

$$A := \begin{array}{cccccc} \boxed{} & \boxed{} & \boxed{{A^*_{15}}^q} & \boxed{A^*_{15}} & \boxed{} & \boxed{} \\ & \boxed{A_{23}} & & & & \end{array} \quad \text{and } B := \begin{array}{cccccc} \boxed{} & \boxed{} & \boxed{{B^*_{15}}^q} & \boxed{B^*_{15}} & \boxed{} & \boxed{} \\ & \boxed{B_{23}} & & & & \end{array}.$$

By 2.2.52, we get

$$\begin{aligned}
&\mathrm{Stab}_U(A)\\
=&\left\{x(0,t_2,t_3,t_4,t_5,t_6)\;\middle|\; {A_{15}^*}^q t_3^q + A_{15}^* t_3^{q^2} + A_{13}t_2 = 0,\ t_3,t_4\in\mathbb{F}_{q^3},\ t_2,t_5,t_6\in\mathbb{F}_q\right\}\\
\overset{A_{13}=0}{=}&X_2X_4X_5X_6.
\end{aligned}$$

Then by 2.2.39, $\mathrm{Stab}_U(A,C) = \begin{cases} X_2X_4X_5X_6, & C_{13}=0\\ X_4X_5X_6, & C_{13}\neq 0\end{cases}$. We calculate the inner product

$$\begin{aligned}
\langle \chi_A,\chi_C\rangle_{\mathrm{Stab}_U(A,C)} =& \frac{1}{|\mathrm{Stab}_U(A,C)|}\sum_{y\in\mathrm{Stab}_U(A,C)} \vartheta\kappa_q(A-C,f(y))\\
=&\frac{1}{|\mathrm{Stab}_U(A,C)|}\\
&\cdot\sum_{y\in\mathrm{Stab}_U(A,C)}\vartheta\kappa_q\left(\begin{array}{|c|c|c|c|c|c|}\hline -C_{12} & -C_{13} & (A_{15}^*-B_{15}^*)^q & A_{15}^*-B_{15}^* & & \\ \hline \multicolumn{1}{c|}{} & A_{23}-B_{23} & \multicolumn{4}{l}{} \\ \cline{2-2}\end{array}, f(y)\right).
\end{aligned}$$

If $C_{13}=0$, we have

$$C\in\left\{\begin{array}{|c|c|c|c|c|c|}\hline \begin{matrix}-{B_{15}^*}^q s_3^q\\ -B_{15}^* s_3^{q^2}\end{matrix} & 0 & {B_{15}^*}^q & B_{15}^* & & \\ \hline \multicolumn{1}{c|}{} & B_{23} & \multicolumn{4}{l}{} \\ \cline{2-2}\end{array}\;\middle|\; s_3\in\mathbb{F}_{q^3}\right\}.$$

$$\begin{aligned}
0\neq&\dim_{\mathbb{C}}\mathrm{Hom}_{\mathrm{Stab}_U(A,C)}(\mathbb{C}[A],\mathbb{C}[C]) = \langle\chi_A,\chi_C\rangle_{\mathrm{Stab}_U(A,C)}\\
=&\frac{1}{|X_2X_4X_5X_6|}\sum_{\substack{t_4\in\mathbb{F}_{q^3}\\ t_2,t_5,t_6\in\mathbb{F}_q}}\Bigg(\vartheta\pi_q\Big((A_{23}-B_{23})t_2\\
&\qquad + (A_{15}^*-B_{15}^*)^q t_4 + (A_{15}^*-B_{15}^*)t_4^{q^2}\Big)\Bigg)\\
=&\frac{1}{q^6}\sum_{\substack{t_4\in\mathbb{F}_{q^3}\\ t_2,t_5,t_6\in\mathbb{F}_q}}\left(\vartheta\Big((A_{23}-B_{23})t_2\Big)\cdot\vartheta\phi_0\Big((A_{15}^*-B_{15}^*)t_4^{q^2}(\eta+\eta^q)\Big)\right)\\
=&\left(\frac{1}{q}\sum_{t_2\in\mathbb{F}_q}\vartheta((A_{23}-B_{23})t_2)\right)\left(\frac{1}{q^3}\sum_{t_4\in\mathbb{F}_{q^3}}\vartheta\phi_0\Big((A_{15}^*-B_{15}^*)t_4^{q^2}(\eta+\eta^q)\Big)\right)\\
\iff&\{B_{23}=A_{23}\}\wedge\{B_{15}^*=A_{15}^*\}.
\end{aligned}$$

If $C_{13}\neq 0$,

$$C\in\left\{\begin{array}{|c|c|c|c|c|c|}\hline \begin{matrix}-{B_{15}^*}^q s_3^q\\ -B_{15}^* s_3^{q^2}\end{matrix} & \begin{matrix}-{B_{15}^*}^q {s_1^*}^q\\ -B_{15}^* {s_1^*}^{q^2}\end{matrix} & {B_{15}^*}^q & B_{15}^* & & \\ \hline \multicolumn{1}{c|}{} & B_{23} & \multicolumn{4}{l}{} \\ \cline{2-2}\end{array}\;\middle|\; s_1^*\in\mathbb{F}_{q^3}^*, s_3\in\mathbb{F}_{q^3}\right\}.$$

Therefore,

$$\begin{aligned}
0 \neq &\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{Stab}_U(A,C)}(\mathbb{C}[A], \mathbb{C}[C]) = \langle \chi_A, \chi_C \rangle_{\mathrm{Stab}_U(A,C)} \\
= &\frac{1}{|X_4X_5X_6|} \sum_{\substack{t_4 \in \mathbb{F}_{q^3} \\ t_5, t_6 \in \mathbb{F}_q}} \Big(\vartheta\pi_q \Big((A_{15}^* - B_{15}^*)^q t_4 + (A_{15}^* - B_{15}^*) t_4^{q^2} \Big) \Big) \\
= &\frac{1}{q^5} \sum_{\substack{t_4 \in \mathbb{F}_{q^3} \\ t_5, t_6 \in \mathbb{F}_q}} \Big(\vartheta\phi_0 \Big((A_{15}^* - B_{15}^*) t_4^{q^2} (\eta + \eta^q) \Big) \Big) \\
= &\frac{1}{q^3} \sum_{t_4 \in \mathbb{F}_{q^3}} \vartheta\phi_0 \Big((A_{15}^* - B_{15}^*) t_4^{q^2} (\eta + \eta^q) \Big) \\
\iff & B_{15}^* = A_{15}^*.
\end{aligned}$$

We get

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{Stab}_U(A,C)}(\mathbb{C}[A], \mathbb{C}[C]) \neq \{0\} \iff \langle \chi_A, \chi_C \rangle_{\mathrm{Stab}_U(A,C)} \neq 0 \quad (i.e. = 1) \\
\iff &\big\{ \{B_{23} = A_{23}\} \wedge \{B_{15}^* = A_{15}^*\} \big\} \wedge \{B_{15}^* = A_{15}^*\} \iff B_{15}^* = A_{15}^*.
\end{aligned}$$

Thus $\mathrm{Hom}_{\mathbb{C}U}(\mathbb{C}\mathcal{O}_U([A]), \mathbb{C}\mathcal{O}_U([B])) = \{0\} \iff B_{15}^* \neq A_{15}^*$.

Let $A \in \mathfrak{F}_i$ and $B \in \mathfrak{F}_j$, ψ_A denote the character of $\mathbb{C}\mathcal{O}_U([A])$ and ψ_B denote the character of $\mathbb{C}\mathcal{O}_U([B])$. In a similar way, we calculate $\langle \psi_A, \psi_B \rangle_U$. Then the statements of (1) are proved.

The q^4 hook-separated staircase modules of $\mathfrak{F}_{1,2}$ are of dimension 1, so they are irreducible, and they are pairwise orthogonal by calculating inner product.

Let $A, B \in V$ be hook-separated staircase patterns of the Family $\mathfrak{F}_3$ and $A \neq B$, then $\langle \psi_A, \psi_A \rangle_U = 1$ and $\langle \psi_A, \psi_B \rangle_U = 0$, thus the statements of (3) are proved.

Let $A \in V$ be a hook-separated staircase core pattern of the Family $\mathfrak{F}_4$, then the orbit module $\mathbb{C}\mathcal{O}_U([A])$ is reducible. Suppose it is irreducible, then by (1) and (2) we get

$$\big(\dim_{\mathbb{C}} \mathbb{C}\mathcal{O}_U([A]) \big)^2 = q^{12} < |U| - q^4 = q^{12} - q^4.$$

This is a contradiction. Thus the orbit modules of the Family $\mathfrak{F}_4$ are reducible. Similarly, (5) and (6) are proved. □

2.2.84 Remark. *There exist two hook-separated staircase modules such that they are neither orthogonal nor isomorphic. For example: let $A, B \in V$ be hook-separated staircase core patterns of the Family $\mathfrak{F}_4$ with $A_{15}^* = B_{15}^* \in \mathbb{F}_{q^3}^*$ and $A_{23} \neq B_{23}$, i.e.*

$$A := \begin{array}{|c|c|c|c|c|c|}\hline \; & \; & {A_{15}^*}^q & A_{15}^* & \; & \; \\ \hline & A_{23} & & & & \\ \cline{2-2} \end{array} \quad \textit{and } B := \begin{array}{|c|c|c|c|c|c|}\hline \; & \; & {A_{15}^*}^q & A_{15}^* & \; & \; \\ \hline & B_{23} & & & & \\ \cline{2-2} \end{array},$$

then $\mathbb{C}\mathcal{O}_U([A])$ and $\mathbb{C}\mathcal{O}_U([B])$ are neither orthogonal nor isomorphic.

Proof. (1) Let $D \in \mathcal{O}_U(A)$ and ψ_A denote the character of $\mathbb{C}\mathcal{O}_U([A])$. Then the $\mathbb{C}\mathrm{Stab}_U(A,D)$-modules $\mathbb{C}[A]$ and $\mathbb{C}[D]$ satisfy

$$\begin{aligned}
\mathbb{C}\left[\begin{array}{cccccc} \boxed{0} & \boxed{0} & \boxed{{A^*_{15}}^q} & \boxed{A^*_{15}} & \boxed{} & \boxed{} \\ & \boxed{A_{23}} & & & & \end{array}\right] &\cong \mathbb{C}\left[\begin{array}{cccccc} \boxed{D_{12}} & \boxed{0} & \boxed{{A^*_{15}}^q} & \boxed{A^*_{15}} & \boxed{} & \boxed{} \\ & \boxed{A_{23}} & & & & \end{array}\right] \\
\mathbb{C}\left[\begin{array}{cccccc} \boxed{0} & \boxed{0} & \boxed{{A^*_{15}}^q} & \boxed{A^*_{15}} & \boxed{} & \boxed{} \\ & \boxed{A_{23}} & & & & \end{array}\right] &\cong \mathbb{C}\left[\begin{array}{cccccc} \boxed{D_{12}} & \boxed{D^*_{13}} & \boxed{{A^*_{15}}^q} & \boxed{A^*_{15}} & \boxed{} & \boxed{} \\ & \boxed{A_{23}} & & & & \end{array}\right].
\end{aligned}$$

By the proof of 2.2.82, we get

$$\begin{aligned}
&\dim_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}U}(\mathbb{C}\mathcal{O}_U([A]),\mathbb{C}\mathcal{O}_U([A])) = \langle\psi_A,\psi_A\rangle_U \\
=&\sum_{D\in\mathcal{O}_U(A)} \frac{|\mathrm{Stab}_U(A,D)|}{|\mathrm{Stab}_U(A)|}\dim_{\mathbb{C}}\mathrm{Hom}_{\mathrm{Stab}_U(A,D)}(\mathbb{C}[A],\mathbb{C}[D]) \\
=&\frac{q^6\cdot q^3}{q^6} + \frac{q^5\cdot(q^3-1)q^3}{q^6} = q^5+q^3-q^2 \neq 1.
\end{aligned}$$

Thus, $\mathbb{C}\mathcal{O}_U([A])$ is not irreducible.

(2) Let ψ_A denote the character of $\mathbb{C}\mathcal{O}_U([A])$ and ψ_B the character of $\mathbb{C}\mathcal{O}_U([B])$. By the proof of 2.2.83, $\mathbb{C}\mathcal{O}_U([A])$ and $\mathbb{C}\mathcal{O}_U([B])$ are not orthogonal and by (1)

$$\langle\psi_A,\psi_A\rangle_U = q^5+q^3-q^2 = \langle\psi_B,\psi_B\rangle_U.$$

Let $D \in \mathcal{O}_U(B)$. The $\mathbb{C}\mathrm{Stab}_U(A,D)$-modules $\mathbb{C}[A]$ and $\mathbb{C}[D]$ satisfy

$$\begin{aligned}
\mathbb{C}\left[\begin{array}{cccccc} \boxed{0} & \boxed{0} & \boxed{{A^*_{15}}^q} & \boxed{A^*_{15}} & \boxed{} & \boxed{} \\ & \boxed{A_{23}} & & & & \end{array}\right] &\perp \mathbb{C}\left[\begin{array}{cccccc} \boxed{D_{12}} & \boxed{0} & \boxed{{A^*_{15}}^q} & \boxed{A^*_{15}} & \boxed{} & \boxed{} \\ & \boxed{B_{23}} & & & & \end{array}\right] \\
\mathbb{C}\left[\begin{array}{cccccc} \boxed{0} & \boxed{0} & \boxed{{A^*_{15}}^q} & \boxed{A^*_{15}} & \boxed{} & \boxed{} \\ & \boxed{A_{23}} & & & & \end{array}\right] &\cong \mathbb{C}\left[\begin{array}{cccccc} \boxed{D_{12}} & \boxed{D^*_{13}} & \boxed{{A^*_{15}}^q} & \boxed{A^*_{15}} & \boxed{} & \boxed{} \\ & \boxed{B_{23}} & & & & \end{array}\right].
\end{aligned}$$

By the proof of 2.2.82,

$$\begin{aligned}
&\dim_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}U}(\mathbb{C}\mathcal{O}_U([A]),\mathbb{C}\mathcal{O}_U([B])) = \langle\psi_A,\psi_B\rangle_U \\
=&\sum_{D\in\mathcal{O}_U(B)} \frac{|\mathrm{Stab}_U(A,D)|}{|\mathrm{Stab}_U(A)|}\dim_{\mathbb{C}}\mathrm{Hom}_{\mathrm{Stab}_U(A,D)}(\mathbb{C}[A],\mathbb{C}[D]) \\
=&\frac{q^5\cdot(q^3-1)q^3}{q^6} = q^5-q^2 \neq \langle\psi_A,\psi_A\rangle_U.
\end{aligned}$$

Thus, $\mathbb{C}\mathcal{O}_U([A])$ and $\mathbb{C}\mathcal{O}_U([B])$ are not isomorphic.

□

2.2.85 Comparison. *(1) (Classification of staircase U-modules). Let U be $A_n(q)$, $D_n^{syl}(q)$ or ${}^3D_4^{syl}(q^3)$, then every U-orbit module is isomorphic to a staircase U-module (see [Yan10, Prop. 2.2 and Thm. 3.2], [Jed13, 3.3.15] and 2.2.75).*

(2) (Irreducible U-modules). Every irreducible $A_n(q)$-module is a constituent of precisely one staircase module (see [Yan10, Thm. 2.4 and Cor. 2.7]). Every irreducible $D_n^{syl}(q)$-module is a constituent of an unique hook-separated staircase module (see [Jed13, 3.3.19 and 3.3.43]). Every irreducible $D_n^{syl}(q)$-module is a constituent of a (not necessarily unique) hook-separated staircase module (see 2.2.79, 2.2.83 and 2.2.84).

2.3. Supercharacter theories for ${}^3D_4^{syl}(q^3)$

Let $G := G_8(q^3)$, $U := {}^3D_4^{syl}(q^3)$, $t_1^*, t_3^*, t_4^* \in \mathbb{F}_{q^3}^*$ and $t_2^*, t_5^*, t_6^* \in \mathbb{F}_q^*$.

In this section, we construct a supercharacter theory for ${}^3D_4^{syl}(q^3)$ (2.3.21), establish the supercharacter table of ${}^3D_4^{syl}(q^3)$ in Table 2.5, and determine the character table of ${}^3D_4^{syl}(q^3)$ in Table 2.9.

2.3.1. A partition of ${}^3D_4^{syl}(q^3)$

In this subsection, a partition of ${}^3D_4^{syl}(q^3)$ is determined (see 2.3.8) which is a set of superclasses proved in the next subsection 2.3.2.

2.3.1 Lemma. *Let 1 denote $I_8 \in G$, then*

$$V_G := G - 1 = \{g - 1 \mid g \in G\}$$

is a nilpotent associative $\mathbb{F}_q$-algebra (G is an algebra group).

Proof. Let $g, h \in G$ and $k \in \mathbb{F}_q$, then

$$\begin{aligned} (g-1)+(h-1) &\in G-1,\\ k(g-1) &\in G-1,\\ (g-1)(h-1) &\in G-1. \end{aligned}$$

Thus the lemma follows. □

2.3.2 Notation/Lemma. *Let 1 denote $I_8 \in G$, $g \in G$ and $u \in U$, then set*

$$\begin{aligned} G(g-1)G :=& \{x(g-1)y \mid x, y \in G\} \subseteq V_G,\\ C_g^G :=& \{1 + x(g-1)y \mid x, y \in G\} = 1 + G(g-1)G \subseteq G,\\ C_u^U :=& \{1 + x(u-1)y \mid x, y \in G\} \cap U \subseteq C_u^G. \end{aligned}$$

2.3.3 Lemma. *Let 1 denote $I_8 \in G$ and $g, h \in G$, then the following statements are equivalent:*

(1) There exist $x, y \in G$, such that $g - 1 = x(h-1)y$,

(2) $C_g^G = C_h^G$,

(3) $g \in C_h^G$.

2.3.4 Corollary. *(1) The set $\{C_g^G \mid g \in G\}$ forms a partition of G with respect to the equivalence relation of 2.3.3 (1).*

(2) Let $g \in G$, then C_g^G is a union of conjugacy classes of G.

2.3.5 Lemma. *Let $u, v \in U$, then the following statements are equivalent:*

(1) There exist $x, y \in G$, such that $u - 1 = x(v-1)y$,

(2) $C_u^U = C_v^U$,

(3) $u \in C_v^U$.

2.3.6 Corollary. *(1) The set $\{C_u^U \mid u \in U\}$ forms a partition of U with respect to the equivalence relation of 2.3.5 (1).*

(2) Let $u \in U$, then C_u^U is a union of conjugacy classes of U.

We obtain a partition of ${}^3D_4^{syl}(q^3)$ by straightforward calculation.

2.3.7 Proposition (A partition of ${}^3D_4^{syl}(q^3)$)**.** *Let $T^{t_1^*}$ ($t_1^* \in \mathbb{F}_{q^3}$) be the transversal for $t_1^*\mathbb{F}_q^+$ in $\mathbb{F}_{q^3}^+$. Then the C_u^U with $u \in U$ are given in Table 2.4:*

2.3.8 Notation/Lemma. *Set*

$$C_i(t_i^*) := C^U_{x_i(t_i^*)} \quad \textit{for all } i = 3,4,5,6,$$

$$C_2(t_2^*) := \Big(\dot{\bigcup_{t_4^* \in \mathbb{F}_{q^3}^*}} C^U_{x_2(t_2^*)x_4(t_4^*)}\Big)\dot{\bigcup}\Big(\dot{\bigcup_{t_5 \in \mathbb{F}_q}} C^U_{x_2(t_2^*)x_5(t_5)}\Big),$$

$$C_{1,3}(t_1^*, \bar{t}_3^{t_1^*}) := C^U_{x_1(t_1^*)x_3(\bar{t}_3^{t_1^*})}, \quad C_{1,2}(t_1^*, t_2^*) := C^U_{x_2(t_2^*)x_1(t_1^*)}, \quad C_0 := \{1_U\} = \{I_8\}.$$

Note that these sets form a partition of U, denoted by $\mathcal{K}$.

2.3.9 Notation/Lemma. *Let*

$$C_i := \dot{\bigcup_{t_i \in \mathbb{F}_{q^3}^*}} C_i(t_i^*) \quad \textit{for all } i = 3,4, \qquad C_i := \dot{\bigcup_{t_i \in \mathbb{F}_q^*}} C_i(t_i^*) \quad \textit{for all } i = 2,5,6,$$

$$C_{1,3} := \dot{\bigcup_{\substack{t_1^* \in \mathbb{F}_{q^3}^* \\ \bar{t}_3^{t_1^*} \in T^{t_1^*}}}} C_{1,3}(t_1^*, \bar{t}_3^{t_1^*}), \quad C_{1,2} := \dot{\bigcup_{\substack{t_1^* \in \mathbb{F}_{q^3}^* \\ t_2^* \in \mathbb{F}_q^*}}} C_{1,2}(t_1^*, t_2^*), \quad C_0 := \{1_U\} = \{I_8\}.$$

Then

$$\mathcal{K} = \Big(\dot{\bigcup_{i \in \{0,2,3,4,5,6\}}} C_i\Big) \dot{\bigcup} C_{1,2} \dot{\bigcup} C_{1,3}.$$

2.3.10 Comparison (Superclasses)**.** *(1) The superclasses of $A_n(q)$ are the sets*

$$C_g^{AY} := \{I_n + x(g - I_n)y \mid x, y \in A_n(q)\} \quad \textit{for all } g \in A_n(q),$$

*which are called **André-Yan superclasses** (see [Jed13, 2.2.30 and 3.5.3]).*

Table 2.4.: A partition of ${}^3D_4^{syl}(q^3)$

$u \in U$	$\#C_u^U$	C_u^U	$\lvert C_u^U \rvert$
1	1	$x(0,0,0,0,0,0)$	1
$x_6(t_6^*)$	$q-1$	$x(0,0,0,0,0,t_6^*)$	1
$x_5(t_5^*)$	$q-1$	$x(0,0,0,0,t_5^*,s_6)$ $s_6 \in \mathbb{F}_q$	q
$x_4(t_4^*)$	q^3-1	$x(0,0,0,t_4^*,s_5,s_6)$ $s_5,s_6 \in \mathbb{F}_q$	q^2
$x_3(t_3^*)$	q^3-1	$x(0,0,t_3^*,s_4,s_5,s_6)$ $s_4 \in \mathbb{F}_{q^3}, s_5,s_6 \in \mathbb{F}_q$	q^5
$x_2(t_2^*)x_4(t_4^*)$	$(q-1)(q^3-1)$	$x(0,t_2^*,s_3,t_4^*-\frac{s_3^{q+1}}{t_2^*},s_5,s_6)$ $s_3 \in \mathbb{F}_{q^3}, s_5,s_6 \in \mathbb{F}_q$	q^5
$x_2(t_2^*)x_5(t_5)$	$(q-1)q$	$x(0,t_2^*,s_3,-\frac{s_3^{q+1}}{t_2^*},t_5-\frac{s_3^{q^2+q+1}}{t_2^{*2}},s_6)$ $s_3 \in \mathbb{F}_{q^3}, s_6 \in \mathbb{F}_q$	q^4
$x_1(t_1^*)x_3(\bar{t}_3^{t_1^*})$	$(q^3-1)q^2$	$x(t_1^*,0,\bar{t}_3^{t_1^*}+t_1^*s,s_4,s_5,s_6)$ $s_4 \in \mathbb{F}_{q^3}, s,s_5,s_6 \in \mathbb{F}_q$	q^6
$x_2(t_2^*)x_1(t_1^*)$	$(q-1)(q^3-1)$	$x(t_1^*,t_2^*,s_3,s_4,s_5,s_6)$ $s_3,s_4 \in \mathbb{F}_{q^3}, s_5,s_6 \in \mathbb{F}_q$	q^8

where $\bar{t}_3^{t_1^*} \in T^{t_1^*}$.

(2) The superclasses of $D_n^{syl}(q)$ are the sets $C_u^{AN} := D_n^{syl}(q) \cap \{I_{2n} + x(u - I_{2n})y \mid x, y \in A_{2n}(q)\}$ for all $u \in D_n^{syl}(q)$ (see [AN09a, page 1279]), which are called ***André-Neto superclasses*** *(see [Jed13, 3.5.5]).*

(3) The superclasses of ${}^3D_4^{syl}(q^3)$ are the elements of $\mathcal{K}$ (see ***2.3.8*** *and* ***2.3.21****).*

2.3.2. A supercharacter theory for ${}^3D_4^{syl}(q^3)$

In [DI08], P. Diaconis and I. M. Isaacs introduced the supercharacter theory for a finite group. In this subsection, we determine a supercharacter theory for ${}^3D_4^{syl}(q^3)$ (2.3.21), establish the supercharacter table of ${}^3D_4^{syl}(q^3)$ in Table 2.5 and list supercharacters, supermodules and superclasses of ${}^3D_4^{syl}(q^3)$ in Table 2.6.

Let $U := {}^3D_4^{syl}(q^3)$, $A_{12}^*, A_{13}^*, A_{15}^* \in \mathbb{F}_{q^3}^*$ and $A_{16}^*, A_{17}^*, A_{23}^* \in \mathbb{F}_q^*$.

2.3.11 Definition. *Let G be a finite group. Suppose that $\mathcal{K}$ is a partition of G and that $\mathcal{X}$ is a set of (nonzero) complex characters of G, such that*

(a) $|\mathcal{X}| = |\mathcal{K}|$,

(b) every character $\chi \in \mathcal{X}$ is constant on each member of $\mathcal{K}$ and

(c) the elements of $\mathcal{X}$ are pairwise orthogonal.

*Then $(\mathcal{X}, \mathcal{K})$ is called a **pre-supercharacter theory** for G. The function $\varphi\colon G \to \mathbb{C}$ is called a **superclass function**, if it satisfies (b). In particular, the superclass functions form a $\mathbb{C}$-vector space.*

2.3.12 Notation. *Let G be a finite group. If χ is a complex character of G, let $\operatorname{Irr}(\chi)$ denote the set of all irreducible constituents of χ.*

2.3.13 Definition/Lemma (§2 of [DI08]/3.6.2 of [Jed13]). *Let G be a finite group and $(\mathcal{X}, \mathcal{K})$ be a pre-supercharacter theory for G. Set $\sigma_\chi := \sum_{\psi \in \operatorname{Irr}(\chi)} \psi(1)\psi$. Then the following statements are equivalent.*

(1) The set $\{1\}$ is a member of $\mathcal{K}$.

(2) $\cup_{\chi \in \mathcal{X}} \operatorname{Irr}(\chi) = \operatorname{Irr}(G)$ and every character $\chi \in \mathcal{X}$ is a constant multiple of σ_χ.

(3) Every irreducible character ψ of G is a constituent of one character $\chi \in \mathcal{X}$.

*The pair $(\mathcal{X}, \mathcal{K})$ is called a **supercharacter theory** for G, if one of the three statements holds. We refer to the elements of $\mathcal{X}$ as **supercharacters**, and to the elements of $\mathcal{K}$ as **superclasses** of G. Note that $\sum_{\chi \in \mathcal{X}} \sigma_\chi = \operatorname{reg}_G$, the regular character of G.*

2.3.14 Remark. *The definition of the supercharacter theory in 2.3.13 is equivalent to the one of [DI08]. Let $(\mathcal{X}, \mathcal{K})$ be a supercharacter theory for a finite group G in this thesis, then the partition $\{\operatorname{Irr}(\chi) \mid \chi \in \mathcal{X}\}$ of $\operatorname{Irr}(G)$, the partition of G, and the supercharacters $\mathcal{X}$ form a supercharacter theory for G in [DI08].*

2.3.15 Definition. *Let G be a finite group and $(\mathcal{X}, \mathcal{K})$ be a supercharacter theory for G. A $\mathbb{C}G$-module is called a $\mathbb{C}G$-**supermodule**, if it affords a supercharacter of G.*

2.3.16 Remark. *(1) Roughly, a supercharacter theory replaces irreducible characters by supercharacters, conjugacy classes by superclasses, irreducible modules by supermodules. In such a way, a* ***supercharacter table*** *is constructed as a replacement for a character table.*

(2) For a finite group G, there are two 'trivial' supercharacter theories as follows: $(\mathrm{Irr}(G), \mathrm{ConjCl}(G))$, $(\{\mathrm{triv}_G, \mathrm{reg}_G - \mathrm{triv}_G\}, \{\{1\}, G\backslash\{1\}\})$, where $\mathrm{ConjCl}(G)$ denotes the set of conjugacy classes of G, and triv_G is the trivial character of G.

(3) Superclasses are unions of conjugacy classes ([DI08], Theorem 2.2(c)).

(4) Different rows of supercharacter tables are orthogonal, but the different columns may not be orthogonal. For example: let G be a finite group with $|G| > 2$, then $(\{\mathrm{triv}_G, \mathrm{reg}_G - \mathrm{triv}_G\}, \{\{1\}, G\backslash\{1\}\})$ is a supercharacter theory, and the supercharacter table is determined

	$\{1\}$	$g \in G\backslash\{1\}$
triv_G	1	1
$\mathrm{reg}_G - \mathrm{triv}_G$	$\|G\| - 1$	-1

Then

$$\begin{aligned}&\mathrm{triv}_G(1) \cdot \overline{\mathrm{triv}_G(g)} + (\mathrm{reg}_G - \mathrm{triv}_G)(1) \cdot \overline{(\mathrm{reg}_G - \mathrm{triv}_G)(g)}\\ =&1 - (|G| - 1) = 2 - |G| \neq 0,\end{aligned}$$

so the 1st and 2nd columns are not orthogonal.

2.3.17 Notation/Lemma. *Let $A = (A_{ij}) \in V$, set*

$$M(A_{12}e_{12} + A_{23}e_{23}) := \mathbb{C}\mathcal{O}_U\left(\left[\begin{array}{|c|c|c|c|c|c|}\hline A_{12} & & & & & \\ \hline \multicolumn{1}{c|}{} & A_{23} & \multicolumn{4}{c}{} \\ \cline{2-2}\end{array}\right]\right) = \mathbb{C}[A_{12}e_{12} + A_{23}e_{23}],$$

$$\begin{aligned}M(A_{13}^* e_{13} + \bar{A}_{12}^{A_{13}^*} e_{12}) :=&\mathbb{C}\left\{\left[\begin{array}{|c|c|c|c|c|c|}\hline \bar{A}_{12}^{A_{13}^*} + s_2 A_{13}^* & A_{13}^* & & & & \\ \hline \multicolumn{1}{c|}{} & & \multicolumn{4}{c}{} \\ \cline{2-2}\end{array}\right] \middle| s_2 \in \mathbb{F}_q\right\}\\ =&\mathbb{C}\mathcal{O}_U\left(\left[\begin{array}{|c|c|c|c|c|c|}\hline \bar{A}_{12}^{A_{13}^*} & A_{13}^* & & & & \\ \hline \multicolumn{1}{c|}{} & & \multicolumn{4}{c}{} \\ \cline{2-2}\end{array}\right]\right),\end{aligned}$$

$$\begin{aligned}M({A_{15}^*}^q e_{14} + A_{15}^* e_{15}) :=&\mathbb{C}\left\{\left[\begin{array}{|c|c|c|c|c|c|}\hline A_{12} & A_{13} & {A_{15}^*}^q & A_{15}^* & & \\ \hline \multicolumn{1}{c|}{} & A_{23} & \multicolumn{4}{c}{} \\ \cline{2-2}\end{array}\right] \middle| \begin{cases} A_{12}, A_{13} \in \mathbb{F}_{q^3} \\ A_{23} \in \mathbb{F}_q \end{cases}\right\}\\ =&\bigoplus_{A_{23} \in \mathbb{F}_q} \mathbb{C}\mathcal{O}_U\left(\left[\begin{array}{|c|c|c|c|c|c|}\hline & & {A_{15}^*}^q & A_{15}^* & & \\ \hline \multicolumn{1}{c|}{} & A_{23} & \multicolumn{4}{c}{} \\ \cline{2-2}\end{array}\right]\right),\end{aligned}$$

$$M(A_{16}^*e_{16}) := \mathbb{C}\left\{\left[\begin{array}{|c|c|c|c|c|c|}\hline A_{12} & A_{13} & A_{15}^q & A_{15} & {\color{red}A_{16}^*} & \\ \hline \multicolumn{1}{c|}{} & A_{23} & \multicolumn{4}{c}{} \\ \cline{2-2}\end{array}\right] \;\middle|\; \begin{cases} A_{12}, A_{13}, A_{15} \in \mathbb{F}_{q^3} \\ A_{23} \in \mathbb{F}_q \end{cases}\right\}$$
$$= \bigoplus_{\substack{A_{13}\in\mathbb{F}_{q^3} \\ A_{23}\in\mathbb{F}_q}} \mathbb{C}\mathcal{O}_U\left(\left[\begin{array}{|c|c|c|c|c|c|}\hline & A_{13} & & & {\color{red}A_{16}^*} & \\ \hline \multicolumn{1}{c|}{} & A_{23} & \multicolumn{4}{c}{} \\ \cline{2-2}\end{array}\right]\right),$$

$$M(A_{17}^*e_{17}) := \mathbb{C}\left\{\left[\begin{array}{|c|c|c|c|c|c|}\hline A_{12} & A_{13} & A_{15}^q & A_{15} & A_{16} & {\color{red}A_{17}^*} \\ \hline \multicolumn{1}{c|}{} & & \multicolumn{4}{c}{} \\ \cline{2-2}\end{array}\right] \;\middle|\; \begin{cases} A_{12}, A_{13}, A_{15} \in \mathbb{F}_{q^3} \\ A_{16} \in \mathbb{F}_q \end{cases}\right\}$$
$$= \bigoplus_{A_{12}\in\mathbb{F}_{q^3}} \mathbb{C}\mathcal{O}_U\left(\left[\begin{array}{|c|c|c|c|c|c|}\hline A_{12} & & & & & {\color{red}A_{17}^*} \\ \hline \multicolumn{1}{c|}{} & & \multicolumn{4}{c}{} \\ \cline{2-2}\end{array}\right]\right).$$

2.3.18 Notation. *Denote by $\mathcal{M}$ the set of all of the $\mathbb{C}U$-modules in 2.3.17.*

2.3.19 Notation. *(1) Let $M \in \mathcal{M}$, then the complex character of the $\mathbb{C}U$-module M is denoted by Ψ_M.*

(2) Set $\mathcal{X} := \{\Psi_M \mid M \in \mathcal{M}\}$.

2.3.20 Corollary. *Let $A = (A_{ij}) \in V$, and ψ_A be the character of $\mathbb{C}\mathcal{O}_U([A])$. Then*

$$\Psi_{M(A_{12}e_{12}+A_{23}e_{23})} = \psi_{A_{12}e_{12}+A_{23}e_{23}}, \qquad \Psi_{M(A_{13}^*e_{13}+\bar{A}_{12}^{A_{13}^*}e_{12})} = \psi_{\bar{A}_{12}^{A_{13}^*}e_{12}+A_{13}^*e_{13}},$$
$$\Psi_{M({A_{15}^*}^q e_{14}+A_{15}^*e_{15})} = \sum_{A_{23}\in\mathbb{F}_q} \psi_{A_{23}e_{23}+{A_{15}^*}^q e_{14}+A_{15}^*e_{15}},$$
$$\Psi_{M(A_{16}^*e_{16})} = \sum_{\substack{A_{13}\in\mathbb{F}_{q^3} \\ A_{23}\in\mathbb{F}_q}} \psi_{A_{13}e_{13}+A_{23}e_{23}+A_{16}^*e_{16}}, \qquad \Psi_{M(A_{17}^*e_{17})} = \sum_{A_{12}\in\mathbb{F}_{q^3}} \psi_{A_{12}e_{12}+A_{17}^*e_{17}}.$$

2.3.21 Proposition (Supercharacter theory for ${}^3D_4^{syl}(q^3)$)**.** *$(\mathcal{X}, \mathcal{K})$ is a supercharacter theory for ${}^3D_4^{syl}(q^3)$, where $\mathcal{K}$ is defined in 2.3.8, and $\mathcal{X}$ is defined in 2.3.19.*

Proof. By 2.3.8, $\mathcal{K}$ is a partition of U. We know that $\mathcal{X}$ is a set of nonzero complex charaxcters of U.

(a) *Claim that $|\mathcal{X}| = |\mathcal{K}|$.*

By 2.3.8, 2.3.17 and 2.3.19, $|\{\Psi_{M(A_{17}^*e_{17})} \mid A_{17}^* \in \mathbb{F}_q^*\}| = |\{M(A_{17}^*e_{17}) \mid A_{17}^* \in \mathbb{F}_q^*\}| = |C_6|$. Similarly, we obtain the Table 2.6. Thus $|\mathcal{X}| = |\mathcal{K}|$.

(b) *Claim that the characters $\chi \in \mathcal{X}$ are constant on the members of $\mathcal{K}$.*

Let $A_{15}^* \in \mathbb{F}_{q^3}^*$ and

$$\mathcal{B}_{15}(A_{15}^*) := \left\{\begin{array}{|c|c|c|c|c|c|}\hline C_{12} & C_{13} & {A_{15}^*}^q & A_{15}^* & & \\ \hline \multicolumn{1}{c|}{} & C_{23} & \multicolumn{4}{c}{} \\ \cline{2-2}\end{array} \;\middle|\; \begin{cases} C_{12}, C_{13} \in \mathbb{F}_{q^3} \\ C_{23} \in \mathbb{F}_q \end{cases}\right\}.$$

Let $y \in U$, then

$$\Psi_{M({A_{15}^*}^q e_{14}+A_{15}^* e_{15})}(y) = \sum_{\substack{C\in\mathcal{B}_{15}(A_{15}^*)\\ C.y=C}} \chi_C(y) = \sum_{\substack{C\in\mathcal{B}_{15}(A_{15}^*)\\ y\in\mathrm{Stab}_U(C)}} \chi_C(y).$$

If $y = x(0,0,0,t_4,t_5,t_6) \in C_0 \cup C_4 \cup C_5 \cup C_6$, $y \in \mathrm{Stab}_U(C)$ for all $C \in \mathcal{B}_{15}(A_{15}^*)$ by 2.2.52. Thus

$$\begin{aligned}\Psi_{M({A_{15}^*}^q e_{14}+A_{15}^* e_{15})}(y) = \sum_{C\in\mathcal{B}_{15}(A_{15}^*)} \chi_C(y) &= \sum_{C\in\mathcal{B}_{15}(A_{15}^*)} \vartheta\pi_q({A_{15}^*}^q t_4 + A_{15}^* {t_4}^{q^2})\\ &= q^7 \cdot \vartheta\pi_q({A_{15}^*}^q t_4 + A_{15}^* t_4^{q^2}).\end{aligned}$$

If $y \in C_{1,2} \cup C_{1,3} \cup C_3$, $y \notin \mathrm{Stab}_U(C)$ for all $C \in \mathcal{B}_{15}(A_{15}^*)$ by 2.2.52. Thus

$$\Psi_{M({A_{15}^*}^q e_{14}+A_{15}^* e_{15})}(y) = 0.$$

If $y = x(0,t_2^*,s_3,s_4,s_5,s_6) \in C_2$, by 2.2.52

$$\begin{aligned}\Psi_{M({A_{15}^*}^q e_{14}+A_{15}^* e_{15})}(y) &= \sum_{\substack{C\in\mathcal{B}_{15}(A_{15}^*)\\ {A_{15}^*}^q s_3^q + A_{15}^* s_3^{q^2} + C_{13} t_2^* = 0}} \chi_C(y)\\
&= \sum_{\substack{C_{12}\in\mathbb{F}_{q^3}\\ C_{23}\in\mathbb{F}_q}} \vartheta\kappa_q\left(\begin{array}{|c|c|c|c|c|c|}\hline C_{12} & -\frac{{A_{15}^*}^q s_3^q + A_{15}^* s_3^{q^2}}{t_2^*} & {A_{15}^*}^q & A_{15}^* & & \\ \hline & C_{23} \\ \cline{2-2}\end{array}, \begin{array}{|c|c|c|c|c|c|}\hline 0 & -s_3 & s_4 & s_4^{q^2} & * & * \\ \hline & t_2^* \\ \cline{2-2}\end{array}\right)\\
&= \sum_{\substack{C_{12}\in\mathbb{F}_{q^3}\\ C_{23}\in\mathbb{F}_q}} \vartheta\pi_q(C_{23}t_2^* + \frac{{A_{15}^*}^q s_3^{q+1} + A_{15}^* s_3^{q^2+1}}{t_2^*} + {A_{15}^*}^q s_4 + A_{15}^* s_4^{q^2})\\
&= q^3 \cdot \vartheta\pi_q\Big(\frac{{A_{15}^*}^q s_3^{q+1} + A_{15}^* s_3^{q^2+1}}{t_2^*} + {A_{15}^*}^q s_4 + A_{15}^* s_4^{q^2}\Big) \cdot \sum_{C_{23}\in\mathbb{F}_q} \vartheta\pi_q(C_{23}t_2^*)\\
&= 0.\end{aligned}$$

Similarly, we calculate the other values of the Table 2.5. Thus the claim is proved.

(c) The elements of $\mathcal{X}$ are pairwise orthogonal by 2.2.83.

(d) The set $\{1\}$ is a member of $\mathcal{K}$.

By 2.3.11 and 2.3.13, $(\mathcal{X},\mathcal{K})$ is a supercharacter theory for ${}^3D_4^{syl}(q^3)$. □

Table 2.5.: Supercharacter table of ${}^3D_4^{syl}(q^3)$

	C_0	$C_1(t_1^*, \bar{t}_3^{t_1^*})$	$C_2(t_2^*)$	$C_{1,2}(t_1^*, t_2^*)$	$C_3(t_3^*)$	$C_4(t_4^*)$	$C_5(t_5^*)$	$C_6(t_6^*)$
$\Psi_{M(0)}$	1	1	1	1	1	1	1	1
$\Psi_{M(A_{12}^* e_{12})}$	1	$\vartheta\pi_q(A_{12}^* t_1^*)$	1	$\vartheta\pi_q(A_{12}^* t_1^*)$	1	1	1	1
$\Psi_{M(A_{23}^* e_{23})}$	1	1	$\vartheta(A_{23}^* t_2^*)$	$\vartheta(A_{23}^* t_2^*)$	1	1	1	1
$\Psi_{M(A_{12}^* e_{12}+A_{23}^* e_{23})}$	1	$\vartheta\pi_q(A_{12}^* t_1^*)$	$\vartheta(A_{23}^* t_2^*)$	$\vartheta\pi_q(A_{12}^* t_1^*)$ $\cdot\vartheta(A_{23}^* t_2^*)$	1	1	1	1
$\Psi_{M(A_{13}^* e_{13}+\bar{A}_{12}^{A_{13}^*} e_{12})}$	q	$\vartheta\pi_q(\bar{A}_{12}^{A_{13}^*} t_1^* - \bar{t}_3^{t_1^*} A_{13}^*)$ $\cdot\sum_{r_2\in\mathbb{F}_q}\vartheta\pi_q(-A_{13}^* t_1^* r_2)$	0	0	$\vartheta\pi_q(-A_{13}^* t_3^*)$ $\cdot q$	q	q	q
$\Psi_{M({A_{15}^*}^q e_{14}+A_{15}^* e_{15})}$	q^7	0	0	0	0	$\vartheta\pi_q({A_{15}^*}^q t_4^*)$ $\cdot\vartheta\pi_q(A_{15}^* {t_4^*}^{q^2})$ $\cdot q^7$	q^7	q^7
$\Psi_{M(A_{16}^* e_{16})}$	q^{10}	0	0	0	0	0	$\vartheta(A_{16}^* t_5^*)$ $\cdot q^{10}$	q^{10}
$\Psi_{M(A_{17}^* e_{17})}$	q^{10}	0	0	0	0	0	0	$\vartheta(A_{17}^* t_6^*)$ $\cdot q^{10}$

2.3.22 Corollary. *The supercharacters, supermodules and superclasses of ${}^3D_4^{syl}(q^3)$ are listed in Table 2.6.*

Table 2.6.: Supercharacters, supermodules and superclasses of ${}^3D_4^{syl}(q^3)$

Supercharacters $\Psi \in \mathcal{X}$	Supermodules $M \in \mathcal{M}$	Superclasses $C \in \mathcal{K}$	$\#\{\Psi\}=\#\{M\}=\#\{C\}$
$\Psi_{M(A_{17}^*e_{17})}$	$M(A_{17}^*e_{17})$	$C_6(t_6^*)$	$q-1$
$\Psi_{M(A_{16}^*e_{16})}$	$M(A_{16}^*e_{16})$	$C_5(t_5^*)$	$q-1$
$\Psi_{M({A_{15}^*}^q e_{14}+A_{15}^*e_{15})}$	$M({A_{15}^*}^q e_{14}+A_{15}^*e_{15})$	$C_4(t_4^*)$	q^3-1
$\Psi_{M(A_{13}^*e_{13}+\bar{A}_{12}^{A_{13}^*}e_{12})}$	$M(A_{13}^*e_{13}+\bar{A}_{12}^{A_{13}^*}e_{12})$	$C_{1,3}(t_1^*, \bar{t}_3^{t_1^*})$	$(q^3-1)q^2$
$\Psi_{M(A_{12}^*e_{12}+A_{23}^*e_{23})}$	$M(A_{12}^*e_{12}+A_{23}^*e_{23})$	$C_{1,2}(t_1^*, t_2^*)$	$(q^3-1)(q-1)$
$\Psi_{M(A_{23}^*e_{23})}$	$M(A_{23}^*e_{23})$	$C_2(t_2^*)$	$q-1$
$\Psi_{M(A_{12}^*e_{12})}$	$M(A_{12}^*e_{12})$	$C_3(t_3^*)$	q^3-1
$\Psi_{M(0)}$	$M(0)$	C_0	1

2.3.23 Corollary. *The number of the supercharacters of ${}^3D_4^{syl}(q^3)$ is*

$$\begin{aligned} &|\mathcal{X}| = |\mathcal{M}| = |\mathcal{K}| \\ =&q^5+q^4+q^3-q^2+2q-3 \\ =&(q-1)^5+6(q-1)^4+15(q-1)^3+18(q-1)^2+12(q-1)+1. \end{aligned}$$

2.3.24 Comparison (Supercharacters). *(1) Supercharacters of $A_n(q)$ are the characters of staircase $A_n(q)$-modules, which are called **André-Yan supercharacters** (see [Jed13, 2.2.27]).*

*(2) Supercharacters of $D_n^{syl}(q)$ are the characters of the sums of all hook-separated staircase $D_n^{syl}(q)$-modules with the same verge (see [Jed13, 3.6.13 and 3.6.16] and [AN09a, page 1278]), which are called **André-Neto supercharacters**.*

(3) Supercharacters of ${}^3D_4^{syl}(q^3)$ of families $\mathfrak{F}_4$, $\mathfrak{F}_5$ and $\mathfrak{F}_6$ are the characters of sums of all hook-separated staircase ${}^3D_4^{syl}(q^3)$-modules with the same 1st verge (see 2.3.17 and 2.3.21).

2.3.3. Character table of ${}^3D_4^{syl}(q^3)$

In [Le13], Tung Le constructs and counts all complex irreducible characters of the Sylow p-subgroup ${}^3D_4^{syl}(q^3)$. In this subsection, we give more specific constructions of the irreducible characters of ${}^3D_4^{syl}(q^3)$ (2.3.45). Then calculate the conjugacy classes of ${}^3D_4^{syl}(q^3)$ (2.3.51), establish the character table of ${}^3D_4^{syl}(q^3)$ (2.3.52) and determine the relations between the supercharacters and the irreducible characters of ${}^3D_4^{syl}(q^3)$ (2.3.53).

Let G be a finite group, N a normal subgroup of G, K a field, and $U := {}^3D_4^{syl}(q^3)$.

Now we define more notations, then recall Clifford's Theorem. We refer to [CR81], [Dip13] and [Kar89].

2.3.25 Reminder. *Let G be a finite group, then $\mathrm{Irr}(G)$ is the set of all irreducible characters of G, and triv_G is the trivial character of G.*

2.3.26 Notation. *Let G be a finite group, H a subgroup of G, $\chi \in \mathrm{Irr}(G)$ and $\lambda \in \mathrm{Irr}(H)$.*

(1) Denote by $\mathrm{Ind}_H^G\lambda$ the character induced from λ.

(2) Denote by $\mathrm{Res}_H^G\chi$ the restriction of χ to H.

(3) The center of G is denoted by $Z(G)$.

(4) The kernel of χ is denoted by $\ker\chi := \{g \in G \mid \chi(g) = \chi(1)\}$.

(5) The commutator subgroup of G is $G' = \langle [x, y] \mid x, y \in G\rangle$, where $[x, y] = x^{-1}y^{-1}xy$.

2.3.27 Definition. *Let L be a (right) KN-module and $g \in G$. The* ***conjugate*** *KN-module L^g of L is defined to be the same vector space L with a new N-action given by*

$$L \times N \to L : (l, n) \mapsto l \times n = lgng^{-1}, \qquad \forall\, l \in L, n \in N.$$

2.3.28 Definition. *Let L be a (right) KN-module. The* ***inertia group*** *$I_G(L)$ in G is defined to be*

$$I_G(L) = \{g \in G \mid L^g \cong L \text{ as } KN\text{-module}\}.$$

In particular, $N \trianglelefteq I_G(L) \leqslant G$.

2.3.29 Notation. *Let N be a normal subgroup of G and $\lambda \in \mathrm{Irr}(N)$, then the inertia group is defined to be*

$$I_G(\lambda) = \{g \in G \mid \lambda^g = \lambda\},$$

where $\lambda^g(n) = \lambda(gng^{-1})$ for all $n \in N$.

2.3.30 Notation. *Let H be a subgroup of a group G, then denote by $T_{H\backslash G}$ a complete set of right coset representatives of H in G.*

2.3.31 Theorem (Clifford 1937, [Cli37])**.** *Let G be a finite group, $N \trianglelefteq G$, K a field and let M be an irreducible (right) KG-module. Then the following statements hold:*

(1) The restriction $\mathrm{Res}_N^G M$ contains an irreducible KN-module L.

(2) Let $H = I_G(L)$ be the inertia group of L in G, then there exists $e \in \mathbb{N}^*$ such that the restriction $\mathrm{Res}_N^G M$ is isomorphic to a direct sum of e copies of the KN-modules

$$\bigoplus_{t \in T_{H\backslash G}} L^t,$$

and the set $\{L^t \mid t \in T_{H\backslash G}\}$ is precisely the set of non-isomorphic conjugate of L in G.

(3) Let W denote the sum of all submodules of $\mathrm{Res}_N^G M$ which are isomorphic to L, then W is an irreducible KH-modules such that

$$\mathrm{Res}_N^H W \cong eL(= \bigoplus e \text{ many copies of } L),$$

and $M = \mathrm{Ind}_H^G W$.

We list some important facts (see [Dip13]) of Clifford's Theorem.

2.3.32 Remark. *We use the notations of the above Theorem.*

(1) Let $H = I_G(L) = N$, then for $g_1, g_2 \in T_{N\backslash G}$, $g_1 \neq g_2$, and $L^{g_1} \ncong L^{g_2}$. Thus $L = W$ is the only irreducible submodule of $\mathrm{Res}_N^G M$ which is isomorphic to L, and $M = \mathrm{Ind}_N^G L$, $L = W$, $e = 1$.

(2) Let $I_G(L) = G$, i.e. $L^g \cong L$ for all $g \in G$. It can happen that $\mathrm{Res}_N^G M = L$ $(e = 1)$. $\mathrm{Res}_N^G \mathrm{Ind}_N^G L = \bigoplus_{g \in T_{N\backslash G}} L \otimes g \cong [G : N]L$. For $2 \leq e \leq [G : N]$, $\mathrm{Res}_N^G M = L$ can happen, too.

(3) $t = [G : I_G(L)]$, $e \mid [I_G(L) : N]$ and $e = \frac{\dim_K W}{\dim_K L}$.

2.3.33 Notation. *Let $x, u \in U$, then the conjugate of x by u is ${}^u x := uxu^{-1}$, and the conjugacy class of x is ${}^U x := \{vxv^{-1} \mid v \in U\}$.*

We calculate and obtain the following conjugate elements by 2.1.10.

2.3.34 Proposition. *Let $u := x(r_1,r_2,r_3,r_4,r_5,r_6) \in U$, $x_i(t_i) \in U$ and $\bar{t}_3 \in \mathbb{F}_{q^3}$, then*

$$\begin{aligned}
{}^u x_6(t_6) =& x_6(t_6),\\
{}^u x_5(t_5) =& x_5(t_5)\cdot x_6(r_2t_5),\\
{}^u x_4(t_4) =& x_4(t_4)\cdot x_5(\phi_0(r_1t_4^q))\cdot x_6(\phi_0(r_1r_2t_4^q)+\phi_0(r_3t_4^q)),\\
{}^u x_3(t_3) =& x_3(t_3)\cdot x_4(r_1t_3^q+r_1^qt_3)\cdot x_5(\phi_0(r_1^{q^2+q}t_3))\\
&\cdot x_6(\phi_0(r_1^{q^2+q}r_2t_3)+\phi_0(-r_1t_3^{q^2+q})+\phi_0(-t_3r_4^q)),\\
{}^u x_2(t_2) =& x_2(t_2)\cdot x_3(-r_1t_2)\cdot x_4(-t_2r_1^{q+1})\cdot x_5(-t_2r_1^{q^2+q+1})\\
&\cdot x_6(-t_2r_5-t_2^2r_1^{q^2+q+1}-t_2r_1^{q^2+q+1}r_2),\\
{}^u x_1(t_1) =& x_1(t_1)\cdot x_3(r_2t_1)\cdot x_4(-r_2t_1^{q+1}-t_1r_3^q-t_1^qr_3)\\
&\cdot x_5(r_2t_1^{q^2+q+1}+\phi_0(r_1^{q^2}(-r_3t_1^q-r_3^qt_1))+\phi_0(r_3^{q^2}t_1^{q+1})+\phi_0(-t_1r_4^q))\\
&\cdot x_6(2r_2^2t_1^{q^2+q+1}+\phi_0(r_1^{q^2}r_2(-r_3t_1^q-r_3^qt_1))+\phi_0(r_2r_3^{q^2}t_1^{q+1})\\
&\quad+\phi_0(-r_2r_4^qt_1)+\phi_0(-t_1r_3^{q^2+q})),
\end{aligned}$$

and

$$\begin{aligned}
{}^u\big(x_3(t_3)x_5(t_5)\big) =& x_3(t_3)\cdot x_4(r_1t_3^q+r_1^qt_3)\cdot x_5(t_5+\phi_0(r_1^{q^2+q}t_3))\\
&\cdot x_6(r_2t_5+\phi_0(r_1^{q^2+q}r_2t_3)+\phi_0(-r_1t_3^{q^2+q})+\phi_0(-t_3r_4^q)),
\end{aligned}$$

$$\begin{aligned}
{}^u\big(x_2(t_2)x_4(t_4)x_5(t_5)\big) =& x_2(t_2)\cdot x_3(-r_1t_2)\cdot x_4(t_4-t_2r_1^{q+1})\\
&\cdot x_5(t_5-t_2r_1^{q^2+q+1}+\phi_0(r_1t_4^q))\\
&\cdot x_6(-t_2r_5-t_2^2r_1^{q^2+q+1}-t_2r_1^{q^2+q+1}r_2+\phi_0(r_1r_2t_4^q)\\
&\quad+\phi_0(r_3t_4^q)+r_2t_5),
\end{aligned}$$

$$\begin{aligned}
{}^u\big(x_1(t_1)x_3(\bar{t}_3)\big) =& x_1(t_1)\cdot x_3(r_2t_1+\bar{t}_3)\\
&\cdot x_4(-r_2t_1^{q+1}-t_1r_3^q-t_1^qr_3+r_1\bar{t}_3^q+r_1^q\bar{t}_3)\\
&\cdot x_5(r_2t_1^{q^2+q+1}+\phi_0(r_1^{q^2}(-r_3t_1^q-r_3^qt_1))+\phi_0(r_3^{q^2}t_1^{q+1})\\
&\quad+\phi_0(-t_1r_4^q)+\phi_0(r_1^{q^2+q}\bar{t}_3))\\
&\cdot x_6(2r_2^2t_1^{q^2+q+1}+\phi_0(r_1^{q^2}r_2(-r_3t_1^q-r_3^qt_1))\\
&\quad+\phi_0(r_2r_3^{q^2}t_1^{q+1})+\phi_0(-r_2r_4^qt_1)+\phi_0(-t_1r_3^{q^2+q})\\
&\quad+\phi_0(r_1^{q^2+q}r_2\bar{t}_3)+\phi_0(-r_1\bar{t}_3^{q^2+q})+\phi_0(-\bar{t}_3r_4^q)\\
&\quad+\phi_0(\bar{t}_3^{q^2}(r_2t_1^{q+1}+t_1r_3^q+t_1^qr_3))),
\end{aligned}$$

$$
\begin{aligned}
{}^u\big(x_2(t_2)x_1(t_1)\big) =& x_2(t_2)x_1(t_1)\cdot x_3(r_2t_1-r_1t_2)\\
&\cdot x_4(-r_2t_1^{q+1}-t_1r_3^q-t_1^qr_3-t_2r_1^{q+1}+t_1t_2r_1^q+t_1^qt_2r_1)\\
&\cdot x_5(r_2t_1^{q^2+q+1}+\phi_0\big(r_1^{q^2}(-r_3t_1^q-r_3^qt_1)\big)\\
&\quad+\phi_0\big(r_3^{q^2}t_1^{q+1}\big)+\phi_0\big(-t_1r_4^q\big)-t_2r_1^{q^2+q+1}\\
&\quad+\phi_0\big(-r_1t_1^{q^2+q}t_2\big)+\phi_0\big(t_1^{q^2}t_2r_1^{q+1}\big))\\
&\cdot x_6(2r_2^2t_1^{q^2+q+1}+\phi_0\big(r_1^{q^2}r_2(-r_3t_1^q-r_3^qt_1)\big)\\
&\quad+\phi_0\big(r_2r_3^{q^2}t_1^{q+1}\big)+\phi_0\big(-r_2r_4^qt_1\big)+\phi_0\big(-t_1r_3^{q^2+q}\big)\\
&\quad-t_2r_5-t_2^2r_1^{q^2+q+1}-t_2r_1^{q^2+q+1}r_2\\
&\quad+\phi_0\big(-2r_1r_2t_1^{q^2+q}t_2\big)+\phi_0\big(t_1^{q^2}t_2r_1^{q+1}r_2\big)\\
&\quad+\phi_0\big(r_1^{q^2+q}t_1t_2^2\big)).
\end{aligned}
$$

2.3.35 Lemma. *Let $a\in\mathbb{F}_{q^3}$, $b\in\mathbb{F}_q$ and*

$$
\begin{aligned}
&\vartheta_a\colon \mathbb{F}_{q^3}^+\to\mathbb{C}^*: x\mapsto\vartheta\pi_q(ax),\\
&\vartheta_b\colon \mathbb{F}_{q}^+\to\mathbb{C}^*: y\mapsto\vartheta\pi_q(by)=\vartheta(by).
\end{aligned}
$$

then $\operatorname{Irr}(\mathbb{F}_{q^3}^+)=\{\vartheta_a\mid a\in\mathbb{F}_{q^3}\}$ and $\operatorname{Irr}(\mathbb{F}_q^+)=\{\vartheta_b\mid b\in\mathbb{F}_q\}$.

Proof. Let $a\in\mathbb{F}_{q^3}$. By 1.2.22 and 2.2.13, $\vartheta_a(x+y)=\vartheta_a(x)\cdot\vartheta_a(y)$ for all $x,y\in\mathbb{F}_{q^3}$, so $\operatorname{Irr}(\mathbb{F}_{q^3}^+)\supseteq\{\vartheta_a\mid a\in\mathbb{F}_{q^3}\}$. Let $a,c\in\mathbb{F}_{q^3}$ and $\vartheta_a=\vartheta_c$, then $\vartheta_a(x)=\vartheta_c(x)$ for all $x\in\mathbb{F}_{q^3}$, i.e. $\vartheta\pi_q(ax)=\vartheta\pi_q(cx)$ for all $x\in\mathbb{F}_{q^3}$. Thus $a=c$ by 2.2.13. Otherwise $\vartheta(y)=1$ for all $y\in\mathbb{F}_{q^3}$. this is a contradiction since ϑ is nontrivial by 1.2.22. Thus $\#\{\vartheta_a\mid a\in\mathbb{F}_{q^3}\}=q^3$. Therefore, $\operatorname{Irr}(\mathbb{F}_{q^3}^+)=\{\vartheta_a\mid a\in\mathbb{F}_{q^3}\}$. Similarly, we obtain the second formula. □

2.3.36 Corollary. *Let $u\in\mathbb{F}_{q^3}$, and $\tilde\vartheta_u\colon\mathbb{F}_{q^3}^+\to\mathbb{C}^*: t\mapsto\vartheta\pi_q(u^qt+ut^{q^2})$, then $\tilde\vartheta_u=\vartheta_{\eta^{-1}(\eta+\eta^q)u^q}$, and $\operatorname{Irr}(\mathbb{F}_{q^3}^+)=\{\tilde\vartheta_u\mid u\in\mathbb{F}_{q^3}\}$.*

Proof. For all $t\in\mathbb{F}_{q^3}^+$,

$$
\begin{aligned}
\tilde\vartheta_u(t)=&\vartheta\pi_q(u^qt+ut^{q^2})\overset{2.2.13}{=}\vartheta\phi_0(\eta u^qt+\eta ut^{q^2})\overset{2.2.10}{=}\vartheta\phi_0((\eta+\eta^q)u^qt)\\
=&\vartheta\pi_q(\eta^{-1}(\eta+\eta^q)u^qt)=\vartheta_{\eta^{-1}(\eta+\eta^q)u^q}(t).
\end{aligned}
$$

By 2.2.12, $\eta+\eta^q\neq0$. By 2.3.35, we get the second formula. □

2.3.37 Lemma. *Let G be a finite group, $Z(G)\subseteq N\trianglelefteq G$, and a complex character $\chi\in\operatorname{Irr}(G)$. Let $\lambda\in\operatorname{Irr}(N)$ such that $\langle\operatorname{Res}_N^G\chi,\lambda\rangle_N=e>0$. Then for all $g\in Z(G)$,*

$$\left(\operatorname{Res}_N^G\chi\right)(g)=e\frac{|G|}{|I_G(\lambda)|}\lambda(g)$$

and $g\notin\ker\chi\iff g\notin\ker\lambda$. In particular, let $X\leqslant Z(G)$, then $X\nsubseteq\ker\chi$ if and only if $X\nsubseteq\ker\lambda$.

Proof. By Clifford's Theorem 2.3.31, we have for all $g \in Z(G)$

$$\left(\mathrm{Res}_N^G\right)\chi(g) = e \sum_{h \in T_{I_G(\lambda)\backslash G}} \lambda^h(g) \overset{g\in Z(G)}{=} e\frac{|G|}{|I_G(\lambda)|}\lambda(g).$$

Let 1 be the identity element of G and $g_0 \in Z(G)$, then

$$\begin{aligned} \chi(g_0) \neq \chi(1) &\iff \left(\mathrm{Res}_N^G\chi\right)(g_0) \neq \left(\mathrm{Res}_N^G\chi\right)(1) \\ \iff e\frac{|G|}{|I_G(\lambda)|}\lambda(g_0) \neq e\frac{|G|}{|I_G(\lambda)|}\lambda(1) &\iff \lambda(g_0) \neq \lambda(1). \end{aligned}$$

Thus $g \notin \ker\chi \iff g \notin \ker\lambda$. □

2.3.38 Definition (2.1, [Le13]). *Let χ be an irreducible character of a group G. χ is said to be* ***almost faithful*** *if $Z(G) \nleq \ker\chi$.*

2.3.39 Lemma. *Let $T := X_2X_3X_4X_5X_6$, $N := X_4X_5X_6$, and $H := X_1X_4X_5X_6$.*

(1) The subgroup N is abelian, $N \trianglelefteq U$, $T \trianglelefteq U$ and $H \leqslant U$ as follows:

$$\begin{tikzcd}
 & U=TH=X_2X_1X_3X_4X_5X_6 \arrow[dl, dash, "\trianglelefteq"'] \arrow[dr, dash, "\geqslant"] & \\
T=X_2X_3X_4X_5X_6 \arrow[dr, dash, "\trianglerighteq"'] & & H=X_1X_4X_5X_6 \arrow[dl, dash, "\trianglelefteq"] \\
 & N=T\cap H=X_4X_5X_6 &
\end{tikzcd}$$

(2) Let $\lambda \in \mathrm{Irr}(N)$ and $\mathrm{Res}_{X_6}^N\lambda \neq \mathrm{triv}_{X_6}$. If λ satisfies that $\mathrm{Res}_{X_5}^N\lambda = \mathrm{triv}_{X_5}$, we have

$$I_U(\lambda) = \{u \in U \mid \lambda^u = \lambda\} = H.$$

(3) Let $\lambda \in \mathrm{Irr}(N)$, then the inertia group $I_T(\lambda)$ is

$$I_T(\lambda) = \begin{cases} T & \textit{if } \mathrm{Res}_{X_6}^N\lambda = \mathrm{triv}_{X_6} \\ N & \textit{if } \mathrm{Res}_{X_6}^N\lambda \neq \mathrm{triv}_{X_6} \end{cases}.$$

(4) Let $\lambda \in \mathrm{Irr}(N)$, then the inertia group $I_H(\lambda)$ is

$$I_H(\lambda) = \begin{cases} H & \textit{if } \mathrm{Res}_{X_5}^N\lambda = \mathrm{triv}_{X_5} \\ N & \textit{if } \mathrm{Res}_{X_5}^N\lambda \neq \mathrm{triv}_{X_5} \end{cases}.$$

(5) Let $\psi \in \mathrm{Irr}(T)$ and $X_6 = Z(T) \nsubseteq \ker\psi$, then the inertia group $I_U(\psi)$ is

$$I_U(\psi) = \{u \in U \mid \psi^u = \psi\} = U.$$

Proof. (1) By 2.1.10 and 2.3.34, N is abelian, N and T are normal subgroups of U, and H is a subgroup of U. Then N is a normal subgroup of N and H, since N is a normal subgroup of U.

(2) Let $\lambda \in \mathrm{Irr}(N)$, $u := x(r_1, r_2, r_3, r_4, r_5, r_6) \in U$ and $x := x_4(t_4)x_5(t_5)x_6(t_6) \in N$, then λ is a linear character since N is abelian. We have

$$\begin{aligned} \lambda^u(x) =&\lambda(u \cdot x \cdot u^{-1}) \\ \overset{2.3.34}{=}&\lambda\left(x_4(t_4)x_5(t_5 + \phi_0(r_1t_4^q))x_6(t_6 + r_2t_5 + \phi_0(r_1r_2t_4^q) + \phi_0(r_3t_4^q))\right) \\ \overset{\mathrm{Res}^N_{X_5}\lambda=\mathrm{triv}_{X_5}}{=}&\lambda(x_4(t_4)) \cdot \lambda(x_6(t_6 + r_2t_5 + \phi_0(r_1r_2t_4^q) + \phi_0(r_3t_4^q))). \end{aligned}$$

Since $X_6 \nsubseteq \ker\lambda$ and $X_5 \subseteq \ker\lambda$, we get

$$\begin{aligned} &I_U(\lambda) = \{u \in U \mid \lambda^u = \lambda\} = \{u \in U \mid \lambda(u \cdot x \cdot u^{-1}) = \lambda(x) \text{ for all } x \in N\} \\ \overset{2.2.10}{=}&\{u := x(r_1, r_2, r_3, r_4, r_5, r_6) \in U \mid r_2 = r_3 = 0\} = X_1X_4X_5X_6 = H. \end{aligned}$$

(3) Similar to the proof of (2).

(4) Similar to the proof of (2).

(5) Let $x_6(t_6) \in X_6$. Since $X_6 = Z(T) \trianglelefteq N \trianglelefteq T$, there exists $\lambda \in \mathrm{Irr}(N)$ such that $0 < e = \langle \mathrm{Res}^T_N\psi, \lambda\rangle_N$, then

$$\left(\mathrm{Res}^T_N\psi\right)(x_6(t_6)) = e\frac{|T|}{|I_T(\lambda)|}\lambda(x_6(t_6)).$$

By 2.3.37, $X_6 = Z(T) \nsubseteq \ker\lambda$ since $X_6 = Z(T) \nsubseteq \ker\psi$, then $\mathrm{Res}^N_{X_6}\lambda \neq \mathrm{triv}_{X_6}$. Then $I_T(\lambda) = N$ by (3). Thus $\mathrm{Ind}^T_N\lambda \in \mathrm{Irr}(T)$ by Clifford's Theorem 2.3.31. Thus $e = 1$ and $\psi = \mathrm{Ind}^T_N\lambda$.

Let $u := x(s_1, s_2, s_3, s_4, s_5, s_6) \in U$ and $h := x(0, t_2, t_3, t_4, t_5, t_6) \in T$, then

$$\psi(h) = \left(\mathrm{Ind}^T_N\lambda\right)(h) = \frac{1}{|N|}\sum_{\substack{y\in T\\ y\cdot h\cdot y^{-1}\in N}}\lambda(y \cdot h \cdot y^{-1}).$$

– Let $t_2 \neq 0$ or $t_3 \neq 0$, then $\psi(h) = \left(\mathrm{Ind}^T_N\lambda\right)(h) = 0$ and

$$(\psi^u)(h) = \left(\mathrm{Ind}^T_N\lambda\right)^u(h) = \left(\mathrm{Ind}^T_N\lambda\right)(u \cdot h \cdot u^{-1}) \overset{2.3.34}{=} 0.$$

– Let $t_2 = t_3 = 0$, by 2.3.34

$$\begin{aligned} \psi(h) =& \left(\mathrm{Ind}^T_N\lambda\right)(h) = \frac{1}{|N|}\sum_{\substack{y\in T\\ y\cdot h\cdot y^{-1}\in N}}(y \cdot h \cdot y^{-1}) = \frac{1}{|N|}\sum_{y\in T}\lambda(y \cdot h \cdot y^{-1}) \\ \overset{y:=x(0,r_2,r_3,r_4,r_5,r_6)}{=}&\frac{1}{|N|}\sum_{\substack{r_2,r_5,r_6\in\mathbb{F}_q\\ r_3,r_4\in\mathbb{F}_{q^3}}}(x_4(t_4)x_5(t_5)x_6(t_6 + r_2t_5 + \phi_0(r_3t_4^q))), \end{aligned}$$

and

$$
\begin{aligned}
&(\psi^u)(h) = \left(\operatorname{Ind}_N^T \lambda\right)^u (h) = \left(\operatorname{Ind}_N^T \lambda\right)(u \cdot h \cdot u^{-1})\\
=&\left(\operatorname{Ind}_N^T \lambda\right)(x_4(t_4)x_5(t_5 + \phi_0(s_1t_4^q))x_6(t_6 + s_2t_5 + \phi_0(s_1s_2t_4^q) + \phi_0(s_3t_4^q)))\\
=&\frac{1}{|N|}\sum_{\substack{r_2,r_5,r_6\in\mathbb{F}_q\\ r_3,r_4\in\mathbb{F}_{q^3}}} \lambda(x_4(t_4)x_5(t_5+\phi_0(s_1t_4^q))x_6(t_6+s_2t_5+\phi_0(s_1s_2t_4^q)+\phi_0(s_3t_4^q))\\
&\qquad\qquad\qquad \cdot x_6(r_2t_5 + r_2\phi_0(s_1t_4^q) + \phi_0(r_3t_4^q))).
\end{aligned}
$$

Since $X_6 \nsubseteq \ker\lambda$, we have for all $u \in U$

$$
(\psi^u)(h) = \psi(h) = \begin{cases} q^4\lambda(x_6(t_6)) & \text{if } t_2 = t_3 = t_4 = t_5 = 0\\ 0 & \text{otherwise}\end{cases}.
$$

Thus $I_U(\psi) = U$. $\square$

Now we determine the irreducible characters of the subgroup $H = X_1X_4X_5X_6$ of U.

2.3.40 Lemma. *Let $H := X_1X_4X_5X_6$ and $\tilde{\chi} \in \operatorname{Irr}(H)$, then $H' = X_5$.*

(1) If $X_5 \subseteq \ker\tilde{\chi}$, let $\bar{H}_{146} := X_5\backslash H \cong \bar{X}_1\bar{X}_4\bar{X}_6$, $\bar{\chi}^{A_{17},A_{15},A_{12}} \in \operatorname{Irr}(\bar{H}_{146})$,

$$
\bar{\chi}^{A_{17},A_{15},A_{12}}(\bar{x}_1(t_1)\bar{x}_4(t_4)\bar{x}_6(t_6)) := \vartheta(A_{17}t_6)\cdot\vartheta\pi_q(A_{15}^qt_4 + A_{15}t_4^{q^2})\cdot\vartheta\pi_q(A_{12}t_1),
$$

and $\tilde{\chi}^{A_{17},A_{15},A_{12}}$ be the lift of $\bar{\chi}^{A_{17},A_{15},A_{12}}$ to H. Then

$$
\begin{aligned}
\operatorname{Irr}(H)_1 :=&\{\tilde{\chi} \in \operatorname{Irr}(H) \mid X_5 \subseteq \ker\tilde{\chi}\}\\
=&\{\tilde{\chi}^{A_{17},A_{15},A_{12}} \in \operatorname{Irr}(H) \mid A_{17} \in \mathbb{F}_q, A_{15}, A_{12} \in \mathbb{F}_{q^3}\}.
\end{aligned}
$$

(2) If $X_5 \nsubseteq \ker\tilde{\chi}$, let $N := X_4X_5X_6$, $\lambda^{A_{17},A_{16},A_{15}} \in \operatorname{Irr}(N)$, and

$$
\lambda^{A_{17},A_{16},A_{15}}(x_4(t_4)x_5(t_5)x_6(t_6)) := \vartheta(A_{17}t_6)\cdot\vartheta(A_{16}t_5)\cdot\vartheta\pi_q(A_{15}^qt_4 + A_{15}t_4^{q^2}).
$$

Then

$$
\operatorname{Irr}(H)_2 :=\{\tilde{\chi} \in \operatorname{Irr}(H) \mid X_5 \nsubseteq \ker\tilde{\chi}\} = \{\operatorname{Ind}_N^H\lambda^{A_{17},A_{16}^*,0} \mid A_{17}\in\mathbb{F}_q, A_{16}^* \in \mathbb{F}_q^*\}.
$$

Thus, $\operatorname{Irr}(H) = \operatorname{Irr}(H)_1\dot\cup\operatorname{Irr}(H)_2$, i.e. H has q^7 linear characters and $(q-1)q$ irreducible characters of degree q^3.

Proof. Let $\tilde{\chi} \in \operatorname{Irr}(H)$. By 2.1.10, $H' = X_5$ and $Z(H) = X_5X_6$.

(1) If $X_5 \subseteq \ker\tilde{\chi}$, $\tilde{\chi}$ is linear, and all linear characters of H are precisely the lifts to H of the irreducible characters of the abelian quotient group $X_5\backslash H$.

(2) If $X_5 \nsubseteq \ker\tilde{\chi}$, let $N := X_4X_5X_6$, $\lambda \in \mathrm{Irr}(N)$ and $\langle \mathrm{Res}_N^H\tilde{\chi}, \lambda\rangle_N > 0$. Since $X_5 \trianglelefteq Z(H) \trianglelefteq N \trianglelefteq H$, by 2.3.37 $X_5 \nsubseteq \ker\lambda$ i.e. $\mathrm{Res}_{X_5}^N\lambda \neq \mathrm{triv}_{X_5}$. Then $I_H(\lambda) = N$ by (4) of 2.3.39. By Clifford's Theorem 2.3.31, $\tilde{\chi} = \mathrm{Ind}_N^H\lambda$ and $\deg(\tilde{\chi}) = q^3$. Let $\lambda^{A_{17},A_{16},A_{15}} \in \mathrm{Irr}(N)$ for all $A_{17}, A_{16} \in \mathbb{F}_q$ and $A_{15} \in \mathbb{F}_{q^3}$,

$$\lambda^{A_{17},A_{16},A_{15}}(x_4(t_4)x_5(t_5)x_6(t_6)) := \vartheta(A_{17}t_6)\cdot\vartheta(A_{16}t_5)\cdot\vartheta\pi_q(A_{15}^qt_4 + A_{15}t_4^{q^2}).$$

Then there exists $\lambda^{A_{17},A_{16}^*,A_{15}}$ such that $\lambda = \lambda^{A_{17},A_{16}^*,A_{15}}$. We know that X_1 is a transversal of N in H. Let $r_1 \in \mathbb{F}_{q^3}$, for all $x_4(t_4)x_5(t_5)x_6(t_6) \in N$,

$$\begin{aligned}
&\left(\lambda^{A_{17},A_{16}^*,A_{15}}\right)^{x_1(r_1)}(x_4(t_4)x_5(t_5)x_6(t_6))\\
=&\lambda^{A_{17},A_{16}^*,A_{15}}(x_1(r_1)\cdot x_4(t_4)x_5(t_5)x_6(t_6)\cdot x_1(r_1)^{-1})\\
=&\lambda^{A_{17},A_{16}^*,A_{15}}(x_4(t_4)x_5(t_5+\phi_0(r_1t_4^q))x_6(t_6))\\
=&\vartheta(A_{17}t_6)\cdot\vartheta(A_{16}^*(t_5+\phi_0(r_1t_4^q)))\cdot\vartheta\pi_q(A_{15}^qt_4 + A_{15}t_4^{q^2})\\
=&\vartheta(A_{17}t_6)\cdot\vartheta(A_{16}^*t_5)\cdot\vartheta\big(\pi_q(A_{15}^qt_4 + A_{15}t_4^{q^2}) + A_{16}^*\phi_0(r_1t_4^q)\big)\\
\overset{2.3.36}{=}&\vartheta(A_{17}t_6)\cdot\vartheta(A_{16}^*t_5)\cdot\vartheta\pi_q(\eta^{-1}(\eta+\eta^q)A_{15}^qt_4 + \eta^{-1}A_{16}^*r_1^{q^2}t_4)\\
=&\vartheta(A_{17}t_6)\cdot\vartheta(A_{16}^*t_5)\cdot\vartheta\pi_q(\eta^{-1}((\eta+\eta^q)A_{15}^q + A_{16}^*r_1^{q^2})t_4).
\end{aligned}$$

By Clifford's Theorem, $\tilde{\chi} = \mathrm{Ind}_N^H\lambda^{A_{17},A_{16}^*,0}$ and

$$\mathrm{Res}_N^H\mathrm{Ind}_N^H\lambda^{A_{17},A_{16}^*,0} = \sum_{r_1\in\mathbb{F}_{q^3}}\left(\lambda^{A_{17},A_{16}^*,A_{15}}\right)^{x_1(r_1)} = \sum_{B_{15}\in\mathbb{F}_{q^3}}\lambda^{A_{17},A_{16}^*,B_{15}}.$$

Thus H has q^7 linear characters and $(q-1)q$ irreducible characters of degree q^3. $\square$

By the commutator relations of 2.3.34, we determine the conjugacy classes of H, and obtain the character table of H.

2.3.41 Corollary. *Let $x := x(t_1, 0, 0, t_4, t_5, t_6) \in H$ be a representative of one conjugacy class of H, then the character table of H is following:*

Character table of H

$\lvert{}^H x\rvert$	1	q	q
x	$x_5(t_5)x_6(t_6)$	$x_4(t_4^*)x_6(t_6)$	$x_1(t_1^*)x_4(t_4)x_6(t_6)$
$\tilde{\chi}^{A_{17},A_{15},A_{12}}$	$\vartheta(A_{17}t_6)$	$\vartheta(A_{17}t_6)$ $\cdot\vartheta\pi_q(A_{15}^q t_4^* + A_{15}t_4^{*q^2})$	$\vartheta(A_{17}t_6)$ $\cdot\vartheta\pi_q(A_{15}^q t_4 + A_{15}t_4^{q^2})$ $\cdot\vartheta\pi_q(A_{12}t_1^*)$
$\mathrm{Ind}_N^H\lambda^{A_{17},A_{16}^*,0}$	$q^3\vartheta(A_{17}t_6)$ $\cdot\vartheta(A_{16^*}t_5)$	0	0

We obtain the irreducible characters of the normal subgroup $X_2X_3X_4X_5X_6$ of U.

2.3.42 Lemma. *Let $T := X_2X_3X_4X_5X_6$ and $\psi \in \mathrm{Irr}(T)$, then $T' = X_6$.*

(1) If $X_6 \subseteq \ker\psi$, let $\bar{H}_{2345} := X_6\backslash T \cong \bar{X}_2\bar{X}_3\bar{X}_4\bar{X}_5$, $\bar{\chi}^{A_{16},A_{15},A_{13},A_{23}} \in \mathrm{Irr}(\bar{H}_{2345})$,

$$\begin{aligned}&\bar{\chi}^{A_{16},A_{15},A_{13},A_{23}}(\bar{x}_2(t_2)\bar{x}_3(t_3)\bar{x}_4(t_4)\bar{x}_5(t_5))\\ &:= \vartheta(A_{16}t_5)\cdot\vartheta\pi_q(A_{15}^q t_4 + A_{15}t_4^{q^2})\cdot\vartheta\pi_q(-A_{13}t_3)\cdot\vartheta(A_{23}t_2),\end{aligned}$$

and $\psi^{A_{16},A_{15},A_{13},A_{23}}$ be the lift of $\bar{\chi}^{A_{16},A_{15},A_{13},A_{23}}$ to T. Then

$$\begin{aligned}\mathrm{Irr}(T)_1 :=&\{\psi \in \mathrm{Irr}(T) \mid X_6 \subseteq \ker\psi\}\\ =&\{\psi^{A_{16},A_{15},A_{13},A_{23}} \mid A_{16},A_{23} \in \mathbb{F}_q, A_{13},A_{15} \in \mathbb{F}_{q^3}\}.\end{aligned}$$

(2) If $X_6 \not\subseteq \ker\psi$, let $N := X_4X_5X_6$, $\lambda^{A_{17},A_{16},A_{15}} \in \mathrm{Irr}(N)$, and

$$\lambda^{A_{17},A_{16},A_{15}}(x_4(t_4)x_5(t_5)x_6(t_6)) := \vartheta(A_{17}t_6)\cdot\vartheta(A_{16}t_5)\cdot\vartheta\pi_q(A_{15}^q t_4 + A_{15}t_4^{q^2}).$$

We get $\mathrm{Irr}(T)_2 := \{\psi \in \mathrm{Irr}(T) \mid X_6 \not\subseteq \ker\psi\} = \{\mathrm{Ind}_N^T\lambda^{A_{17}^*,0,0} \mid A_{17} \in \mathbb{F}_q^*\}$.

Thus, $\mathrm{Irr}(T) = \mathrm{Irr}(T)_1\dot{\cup}\mathrm{Irr}(T)_2$, i.e. T has q^8 linear characters and $(q-1)$ irreducible characters of degree q^4.

Proof. Let $\psi \in \mathrm{Irr}(T)$. By 2.1.10, $T' = Z(T) = X_6$.

(1) If $X_6 \subseteq \ker\psi$, ψ is linear, and all linear characters of T are precisely the lifts to T of the irreducible characters of the abelian quotient group $X_6\backslash T$.

(2) If $X_6 \not\subseteq \ker\psi$, let $N := X_4X_5X_6$, $\lambda \in \mathrm{Irr}(N)$ and $\langle \mathrm{Res}_N^T\psi,\lambda\rangle_N > 0$. Since $X_6 = Z(T) \trianglelefteq N \trianglelefteq T$, by 2.3.37 $X_6 \not\subseteq \ker\lambda$ i.e. $\mathrm{Res}_{X_6}^N\lambda \neq \mathrm{triv}_{X_6}$. Then $I_T(\lambda) = N$ by (3) of 2.3.39. By Clifford's Theorem 2.3.31, $\psi = \mathrm{Ind}_N^T\lambda$ and $\deg(\psi) = q^4$. Let $\lambda^{A_{17},A_{16},A_{15}} \in \mathrm{Irr}(N)$ for all $A_{17},A_{16} \in \mathbb{F}_q, A_{15} \in \mathbb{F}_{q^3}$ and

$$\lambda^{A_{17},A_{16},A_{15}}(x_4(t_4)x_5(t_5)x_6(t_6)) := \vartheta(A_{17}t_6)\cdot\vartheta(A_{16}t_5)\cdot\vartheta\pi_q(A_{15}^q t_4 + A_{15}t_4^{q^2}),$$

Then there exists $\lambda^{A_{17}^*,A_{16},A_{15}}$ such that $\lambda = \lambda^{A_{17}^*,A_{16},A_{15}}$. We get that X_2X_3 is a transversal of N in T. Let $r_2 \in \mathbb{F}_q$, $r_3 \in \mathbb{F}_{q^3}$, then for all $x_4(t_4)x_5(t_5)x_6(t_6) \in N$,

$$\begin{aligned}
&\left(\lambda^{A_{17}^*,A_{16},A_{15}}\right)^{x_2(s_2)x_3(s_3)}(x_4(t_4)x_5(t_5)x_6(t_6))\\
=&\lambda^{A_{17}^*,A_{16},A_{15}}(x_2(s_2)x_3(s_3)\cdot x_4(t_4)x_5(t_5)x_6(t_6)\cdot(x_2(s_2)x_3(s_3))^{-1})\\
=&\lambda^{A_{17}^*,A_{16},A_{15}}(x_4(t_4)x_5(t_5)x_6(t_6+s_2t_5+\phi_0(s_3t_4^q)))\\
=&\vartheta(A_{17}^*(t_6+s_2t_5+\phi_0(s_3t_4^q)))\cdot\vartheta(A_{16}t_5)\cdot\vartheta\pi_q(A_{15}^qt_4+A_{15}t_4^{q^2})\\
=&\vartheta(A_{17}^*t_6)\cdot\vartheta((A_{16}+A_{17}^*s_2)t_5)\cdot\vartheta\pi_q(A_{15}^qt_4+A_{15}t_4^{q^2}+A_{17}^*\phi_0(s_3t_4^q))\\
=&\vartheta(A_{17}^*t_6)\cdot\vartheta((A_{16}+A_{17}^*s_2)t_5)\cdot\vartheta\phi_0((A_{15}^qt_4+A_{15}t_4^{q^2})\eta+A_{17}^*s_3t_4^q)\\
=&\vartheta(A_{17}^*t_6)\cdot\vartheta((A_{16}+A_{17}^*s_2)t_5)\cdot\vartheta\pi_q(\eta^{-1}((\eta+\eta^q)A_{15}^q+A_{17}^*s_3^{q^2})t_4).
\end{aligned}$$

By Clifford's Theorem, $\psi = \mathrm{Ind}_N^T\lambda^{A_{17}^*,0,0}$ and

$$\mathrm{Res}_N^T\mathrm{Ind}_N^T\lambda^{A_{17}^*,0,0} = \sum_{\substack{r_2\in\mathbb{F}_q\\ r_3\in\mathbb{F}_{q^3}}}\left(\lambda^{A_{17}^*,B_{16},A_{15}}\right)^{x_2(r_2)x_3(r_3)} = \sum_{\substack{B_{15}\in\mathbb{F}_{q^3}\\ B_{16}\in\mathbb{F}_q}}\lambda^{A_{17}^*,B_{16},B_{15}}.$$

Thus T has q^8 linear characters and $(q-1)$ irreducible characters of degree q^4. □

By the commutator relations, we also determine the conjugacy classes and the character table of T.

2.3.43 Corollary. *Let $x := x(0,t_2,t_3,t_4,t_5,t_6) \in T$ be a representative of one conjugacy class of T, then the character table of T is shown in the following table:*

Character table of T

$\lvert {}^Tx\rvert$	1	q	q	q	q
x	$x_6(t_6)$	$x_5(t_5^*)$	$x_4(t_4^*)x_5(t_5)$	$x_3(t_3^*)x_4(t_4)x_5(t_5)$	$x_2(t_2^*)x_3(t_3)$ $\cdot x_4(t_4)x_5(t_5)$
$\psi^{A_{16},A_{15},A_{13},A_{23}}$	1	$\vartheta(A_{16}t_5^*)$	$\vartheta(A_{16}t_5)$ $\cdot\vartheta\pi_q(A_{15}^qt_4^*)$ $\cdot\vartheta\pi_q(A_{15}t_4^{*q^2})$	$\vartheta(A_{16}t_5)$ $\cdot\vartheta\pi_q(A_{15}^qt_4)$ $\cdot\vartheta\pi_q(A_{15}t_4^{q^2})$ $\cdot\vartheta\pi_q(-A_{13}t_3^*)$	$\vartheta(A_{16}t_5)$ $\cdot\vartheta\pi_q(A_{15}^qt_4)$ $\cdot\vartheta\pi_q(A_{15}t_4^{q^2})$ $\cdot\vartheta\pi_q(-A_{13}t_3)$ $\cdot\vartheta(A_{23}t_2^*)$
$\psi^{A_{17}^*}$	$q^4\vartheta(A_{17}^*t_6)$	0	0	0	0

2.3.44 Remark. *Let $H := X_1X_4X_5X_6$, $T := X_2X_3X_4X_5X_6$.*
Let $J := \{(1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(2,3)\}$, V as in §2.2.1.

(a) Let

$$\begin{aligned}
J_H :=& \bigcup_{i\in\{1,4,5,6\}}(\overline{\mathrm{supp}}(X_i)\cap J) = \{(1,2),(1,4),(1,5),(1,6),(1,7)\} \subseteq J,\\
V_H :=& V_{J_H}\bigcap V = \left\{A=(A_{ij})\in V_{J_H} \mid A_{16},\ A_{17}\in\mathbb{F}_q,\ A_{14}=A_{15}^q\in\mathbb{F}_{q^3}\right\}.
\end{aligned}$$

Similar to ${}^3D_4^{syl}(q^3)$, we construct a bijective 1-cocycle of H and a monomial $\mathbb{C}H$-module $\mathbb{C}H$, and get a supercharacter theory for H. The superclasses are the conjugacy classes, and the supercharacters conclude q^7 linear characters and $(q-1)q$ reducible characters of degree q^6.

(b) We exhibit a monomial $\mathbb{C}T$-module $\mathbb{C}T$ and a supercharacter theory for T as well. The superclasses are also the conjugacy classes, and the supercharacters conclude q^8 linear characters and $(q-1)$ reducible characters of degree q^8.

Now we give the explicit constructions of all complex irreducible characters of U.

2.3.45 Proposition. *Let $A_{12}, A_{13}, A_{15} \in \mathbb{F}_{q^3}$, and $A_{23}, A_{16}, A_{17} \in \mathbb{F}_q$. Let $A_{12}^*, A_{13}^*, A_{15}^* \in \mathbb{F}_{q^3}^*$, $A_{23}^*, A_{16}^*, A_{17}^* \in \mathbb{F}_q^*$ and $T^{A_{13}^*}$ be a transversal of $A_{13}^*\mathbb{F}_q^+$ in $\mathbb{F}_{q^3}^+$.*

(1) Let $\bar{U} := X_3X_4X_5X_6\backslash U = \bar{X}_2\bar{X}_1$, $\bar{\chi}_{lin}^{A_{12},A_{23}} \in \operatorname{Irr}(\bar{U})$, and $\bar{\chi}_{lin}^{A_{12},A_{23}}(\bar{x}_2(t_2)\bar{x}_1(t_1)) := \vartheta\pi_q(A_{12}t_1)\cdot\vartheta(A_{23}t_2)$. Let $\chi_{lin}^{A_{12},A_{23}}$ denote the lift of $\bar{\chi}_{lin}^{A_{12},A_{23}}$ to U, then

$$\mathfrak{F}_{lin} :=\{\chi \in \operatorname{Irr}(U) \mid X_3X_4X_5X_6 \subseteq \ker\chi\} = \{\chi_{lin}^{A_{12},A_{23}} \mid A_{12}\in\mathbb{F}_{q^3}, A_{23}\in\mathbb{F}_q\}.$$

(2) Let $\bar{A}_{12}^{A_{13}^*} \in T^{A_{13}^*}$, $\bar{U} := X_4X_5X_6\backslash U = \bar{X}_2\bar{X}_1\bar{X}_3$, $\bar{H} := \bar{X}_1\bar{X}_3$, $\bar{\chi}_{3,q}^{A_{13},A_{12}} \in \operatorname{Irr}(\bar{H})$ with $\bar{\chi}_{3,q}^{A_{13},A_{12}}(\bar{x}_1(t_1)\bar{x}_3(t_3)) := \vartheta\pi_q(A_{12}t_1)\cdot\vartheta\pi_q(-A_{13}t_3)$. Let $\chi_{3,q}^{A_{13}^*,\bar{A}_{12}^{A_{13}^*}}$ denote the lift of $\operatorname{Ind}_{\bar{H}}^{\bar{U}}\bar{\chi}_{3,q}^{A_{13}^*,\bar{A}_{12}^{A_{13}^*}}$ to U, then

$$\begin{aligned}\mathfrak{F}_3 :=&\{\chi\in\operatorname{Irr}(U)\mid X_4X_5X_6\subseteq\ker\chi,\ X_3\nsubseteq\ker\chi\}\\ =&\{\chi_{3,q}^{A_{13}^*,\bar{A}_{12}^{A_{13}^*}} \mid A_{13}^*\in\mathbb{F}_{q^3}^*, \bar{A}_{12}^{A_{13}^*}\in T^{A_{13}^*}\}.\end{aligned}$$

(3) Let $\bar{U} := X_5X_6\backslash U = \bar{X}_2\bar{X}_1\bar{X}_3\bar{X}_4$, $\bar{H} := \bar{X}_2\bar{X}_3\bar{X}_4$, $\bar{\chi}_{4,q^3}^{A_{15},A_{23},A_{13}} \in \operatorname{Irr}(\bar{H})$, and

$$\bar{\chi}_{4,q^3}^{A_{15},A_{23},A_{13}}(\bar{x}_2(t_2)\bar{x}_3(t_3)\bar{x}_4(t_4)) := \vartheta(A_{23}t_2)\cdot\vartheta\pi_q(-A_{13}t_3)\cdot\vartheta\pi_q(A_{15}^qt_4 + A_{15}t_4^{q^2}).$$

Let $\chi_{4,q^3}^{A_{15}^*,A_{23}}$ denote the lift of $\operatorname{Ind}_{\bar{H}}^{\bar{U}}\bar{\chi}_{4,q^3}^{A_{15}^*,A_{23},0}$ to U, then

$$\begin{aligned}\mathfrak{F}_4 :=&\{\chi\in\operatorname{Irr}(U)\mid X_5X_6\subseteq\ker\chi,\ X_4\nsubseteq\ker\chi\}\\ =&\{\chi_{4,q^3}^{A_{15}^*,A_{23}} \mid A_{15}^*\in\mathbb{F}_{q^3}^*, A_{23}\in\mathbb{F}_q\}.\end{aligned}$$

(4) Let $\bar{U} := X_6\backslash U = \bar{X}_2\bar{X}_1\bar{X}_3\bar{X}_4\bar{X}_5$, $\bar{H} := \bar{X}_2\bar{X}_3\bar{X}_4\bar{X}_5$, $\bar{\chi}_{5,q^3}^{A_{16},A_{23},A_{13},A_{15}} \in \operatorname{Irr}(\bar{H})$, and

$$\begin{aligned}&\bar{\chi}_{5,q^3}^{A_{16},A_{23},A_{13},A_{15}}(\bar{x}_2(t_2)\bar{x}_3(t_3)\bar{x}_4(t_4)\bar{x}_5(t_5))\\ :=&\vartheta(A_{23}t_2)\cdot\vartheta\pi_q(-A_{13}t_3)\cdot\vartheta\pi_q(A_{15}^qt_4+A_{15}t_4^{q^2})\cdot\vartheta(A_{16}t_5).\end{aligned}$$

Let $\chi_{5,q^3}^{A_{16}^*,A_{23},A_{13}}$ *denote the lift of* $\mathrm{Ind}_{\bar{H}}^{\bar{U}}\bar{\chi}_{5,q^3}^{A_{16}^*,A_{23},A_{13},0}$ *to* U*, then*

$$\begin{aligned}\mathfrak{F}_5 :=&\{\chi \in \mathrm{Irr}(U) \mid X_6 \subseteq \ker\chi,\ X_5 \nsubseteq \ker\chi\}\\ =&\{\chi_{5,q^3}^{A_{16}^*,A_{23},A_{13}} \mid A_{16}^* \in \mathbb{F}_q^*, A_{23} \in \mathbb{F}_q, A_{13} \in \mathbb{F}_{q^3}\}.\end{aligned}$$

(5) *Let* $H := X_1X_4X_5X_6$*,* $\bar{H} := X_4X_5\backslash H \cong \bar{X}_1\bar{X}_6$*,* $\bar{\chi}_{6,q^4}^{A_{17},A_{12}} \in \mathrm{Irr}(\bar{H})$*, and*

$$\bar{\chi}_{6,q^4}^{A_{17},A_{12}}(\bar{x}_1(t_1)\bar{x}_6(t_6)) := \vartheta\pi_q(A_{12}t_1)\cdot\vartheta(A_{17}t_6).$$

Let $\tilde{\chi}_{6,q^4}^{A_{17},A_{12}}$ *denote the lift of* $\bar{\chi}_{6,q^4}^{A_{17},A_{12}}$ *from* $\bar{H}$ *to* H*, and* $\chi_{6,q^4}^{A_{17}^*,A_{12}} := \mathrm{Ind}_H^U\tilde{\chi}_{6,q^4}^{A_{17}^*,A_{12}}$*, then* $\mathfrak{F}_6 := \{\chi \in \mathrm{Irr}(U) \mid X_6 \nsubseteq \ker\chi\} = \{\chi_{6,q^4}^{A_{17}^*,A_{12}} \mid A_{17}^* \in \mathbb{F}_q^*, A_{12} \in \mathbb{F}_{q^3}\}$.

Hence $\mathrm{Irr}(U) = \mathfrak{F}_{lin}\dot{\cup}\mathfrak{F}_3\dot{\cup}\mathfrak{F}_4\dot{\cup}\mathfrak{F}_5\dot{\cup}\mathfrak{F}_6$.

Proof. Let $\chi \in \mathrm{Irr}(U)$.

(1) *Family* $\mathfrak{F}_{lin}$*, where* $X_3X_4X_5X_6 \subseteq \ker\chi$.

Since $U' = X_3X_4X_5X_6$, all linear characters of U are precisely the lifts to U of the irreducible characters of the abelian quotient group $X_3X_4X_5X_6\backslash U$. If $X_3X_4X_5X_6 \subseteq \ker\chi$, χ is linear. Let $\bar{U} := X_3X_4X_5X_6\backslash U = \bar{X}_2\bar{X}_1$, $\bar{\chi}_{lin}^{A_{12},A_{23}} \in \mathrm{Irr}(\bar{U})$, and $\bar{\chi}_{lin}^{A_{12},A_{23}}(\bar{x}_2(t_2)\bar{x}_1(t_1)) := \vartheta\pi_q(A_{12}t_1)\cdot\vartheta(A_{23}t_2)$. Since $\bar{U}$ is abelian, we have $\mathrm{Irr}(\bar{U}) = \{\bar{\chi}_{lin}^{A_{12},A_{23}} \mid A_{12} \in \mathbb{F}_{q^3}, A_{23} \in \mathbb{F}_q\}$. Let $\chi_{lin}^{A_{12},A_{23}}$ denote the lift of $\bar{\chi}_{lin}^{A_{12},A_{23}}$ to U, then

$$\mathfrak{F}_{lin} =\{\chi \in \mathrm{Irr}(U) \mid X_3X_4X_5X_6 \subseteq \ker\chi\} = \{\chi_{lin}^{A_{12},A_{23}} \mid A_{12} \in \mathbb{F}_{q^3}, A_{23} \in \mathbb{F}_q\}.$$

(2) *Family* $\mathfrak{F}_3$*, where* $X_4X_5X_6 \subseteq \ker\chi$ *and* $X_3 \nsubseteq \ker\chi$.

Let $\bar{U} := X_4X_5X_6\backslash U = \bar{X}_2\bar{X}_1\bar{X}_3$, then by 2.1.10, the commutator relation in $\bar{U}$ is $[\bar{x}_2(t_2), \bar{x}_1(t_1)] = \bar{x}_3(t_2t_1)$. Let $\bar{H} := \bar{X}_1\bar{X}_3$, $\bar{\chi}_{3,q}^{A_{13},A_{12}} \in \mathrm{Irr}(\bar{H})$ and

$$\bar{\chi}_{3,q}^{A_{13},A_{12}}(\bar{x}_1(t_1)\bar{x}_3(t_3)) := \vartheta\pi_q(A_{12}t_1)\cdot\vartheta\pi_q(-A_{13}t_3).$$

We note that $\bar{X}_2$ is a transversal of $\bar{H}$ in $\bar{U}$, and $Z(\bar{U}) = \bar{X}_3$. For all $s_2 \in \mathbb{F}_q$,

$$\begin{aligned}&\left(\bar{\chi}_{3,q}^{A_{13}^*,A_{12}}\right)^{\bar{x}_2(s_2)}(\bar{x}_1(t_1)\bar{x}_3(t_3)) = \bar{\chi}_{3,q}^{A_{13}^*,A_{12}}(\bar{x}_2(s_2)\cdot\bar{x}_1(t_1)\bar{x}_3(t_3)\cdot\bar{x}_2(s_2)^{-1})\\ =&\bar{\chi}_{3,q}^{A_{13}^*,A_{12}}(\bar{x}_1(t_1)\bar{x}_3(t_3+s_2t_1)) = \vartheta\pi_q(A_{12}t_1 - A_{13}^*(t_3+s_2t_1))\\ =&\vartheta\pi_q((A_{12}-s_2A_{13}^*)t_1 - A_{13}^*t_3) = \vartheta\pi_q((A_{12}-s_2A_{13}^*)t_1)\cdot\vartheta\pi_q(-A_{13}^*t_3),\end{aligned}$$

so $I_{\bar{U}}(\bar{\chi}_{3,q}^{A_{13}^*,A_{12}}) = \bar{H}$. By Clifford theory, $\tilde{\chi}_{3,q}^{A_{13}^*,A_{12}} := \mathrm{Ind}_{\bar{H}}^{\bar{U}}\bar{\chi}_{3,q}^{A_{13}^*,A_{12}} \in \mathrm{Irr}(\bar{U})$ and

$$\mathrm{Res}_{\bar{H}}^{\bar{U}}\tilde{\chi}_{3,q}^{A_{13}^*,A_{12}} = \sum_{s_2\in\mathbb{F}_q}\left(\bar{\chi}_{3,q}^{A_{13}^*,A_{12}}\right)^{\bar{x}_2(s_2)} = \sum_{s_2\in\mathbb{F}_q}\bar{\chi}_{3,q}^{A_{13}^*,A_{12}-s_2A_{13}^*}.$$

By Clifford theory, there are $q^2(q^3-1)$ almost faithful irreducible characters of $\bar{U}$, i.e. $\{\tilde{\chi}_{3,q}^{A_{13}^*,\bar{A}_{12}^{A_{13}^*}} \mid A_{13}^* \in \mathbb{F}_{q^3}^*, \bar{A}_{12}^{A_{13}^*} \in T^{A_{13}^*}\}$. Let $\chi_{3,q}^{A_{13}^*,\bar{A}_{12}^{A_{13}^*}}$ be the lift of $\tilde{\chi}_{3,q}^{A_{13}^*,\bar{A}_{12}^{A_{13}^*}}$ to U. Therefore,

$$\begin{aligned}\mathfrak{F}_3 =&\{\chi \in \operatorname{Irr}(U) \mid X_4X_5X_6 \subseteq \ker\chi,\ X_3 \nsubseteq \ker\chi\}\\ =&\{\chi_{3,q}^{A_{13}^*,\bar{A}_{12}^{A_{13}^*}} \mid A_{13}^* \in \mathbb{F}_{q^3}^*, \bar{A}_{12}^{A_{13}^*} \in T^{A_{13}^*}\}.\end{aligned}$$

(3) *Family $\mathfrak{F}_4$, where $X_5X_6 \subseteq \ker\chi$ and $X_4 \nsubseteq \ker\chi$.*

Let $\bar{U} := X_5X_6\backslash U = \bar{X}_2\bar{X}_1\bar{X}_3\bar{X}_4$. By 2.1.10, the commutator relations of $\bar{U}$ are given as follows:

$$[\bar{x}_1(t_1), \bar{x}_2(t_2)] = \bar{x}_3(-t_2t_1)\cdot\bar{x}_4(t_2t_1^{q+1}), \qquad [\bar{x}_1(t_1), \bar{x}_3(t_3)] = \bar{x}_4(t_1t_3^q + t_1^qt_3).$$

Let $\bar{H} := \bar{X}_2\bar{X}_3\bar{X}_4$, $\bar{\chi}_{4,q^3}^{A_{15},A_{23},A_{13}} \in \operatorname{Irr}(\bar{H})$, and

$$\bar{\chi}_{4,q^3}^{A_{15},A_{23},A_{13}}(\bar{x}_2(t_2)\bar{x}_3(t_3)\bar{x}_4(t_4)) := \vartheta(A_{23}t_2)\cdot\vartheta\pi_q(-A_{13}t_3)\cdot\vartheta\pi_q(A_{15}^qt_4 + A_{15}t_4^{q^2}).$$

Then $\bar{H}$ is abelian and $\bar{H} \trianglelefteq \bar{U}$. Note that $\bar{X}_1$ is a transversal of $\bar{H}$ in $\bar{U}$. For each $\bar{x}_1(s_1) \in \bar{X}_1$, we consider $\left(\bar{\chi}_{4,q^3}^{A_{15}^*,A_{23},A_{13}}\right)^{\bar{x}_1(s_1)}$. For all $\bar{x}_2(t_2)\bar{x}_3(t_3)\bar{x}_4(t_4) \in \bar{H}$,

$$\begin{aligned}&\left(\bar{\chi}_{4,q^3}^{A_{15}^*,A_{23},A_{13}}\right)^{x_1(s_1)}(\bar{x}_2(t_2)\bar{x}_3(t_3)\bar{x}_4(t_4))\\
=&\bar{\chi}_{4,q^3}^{A_{15}^*,A_{23},A_{13}}(\bar{x}_1(s_1)\cdot\bar{x}_2(t_2)\bar{x}_3(t_3)\bar{x}_4(t_4)\cdot\bar{x}_1(s_1)^{-1})\\
=&\bar{\chi}_{4,q^3}^{A_{15}^*,A_{23},A_{13}}(\bar{x}_2(t_2)\bar{x}_3(t_3 - s_1t_2)\bar{x}_4(t_4 + s_1t_3^q + s_1^qt_3 - t_2s_1^{q+1}))\\
=&\vartheta(A_{23}t_2)\cdot\vartheta\pi_q(-A_{13}(t_3 - s_1t_2))\\
&\cdot\vartheta\pi_q({A_{15}^*}^q(t_4 + s_1t_3^q + s_1^qt_3 - t_2s_1^{q+1}) + A_{15}^*(t_4 + s_1t_3^q + s_1^qt_3 - t_2s_1^{q+1})^{q^2})\\
=&\vartheta(\pi_q(A_{23} + A_{13}s_1 - ({A_{15}^*}^qs_1^{q+1} + A_{15}^*s_1^{1+q^2}))t_2)\\
&\cdot\vartheta\pi_q(-A_{13}t_3 + {A_{15}^*}^q(s_1t_3^q + s_1^qt_3) + A_{15}^*(s_1t_3^q + s_1^qt_3)^{q^2})\\
&\cdot\vartheta\pi_q({A_{15}^*}^qt_4 + A_{15}^*t_4^{q^2})\\
=&\vartheta(\pi_q(A_{23} + A_{13}s_1 - ({A_{15}^*}^qs_1^{q+1} + A_{15}^*s_1^{1+q^2}))t_2)\\
&\cdot\vartheta\pi_q(-A_{13}t_3 + ({A_{15}^*}^qs_1^q + A_{15}^*s_1^{q^2})t_3 + ({A_{15}^*}^qs_1t_3^q + A_{15}^*s_1t_3^{q^2}))\\
&\cdot\vartheta\pi_q({A_{15}^*}^qt_4 + A_{15}^*t_4^{q^2})\end{aligned}$$

$$
\begin{aligned}
&\overset{2.2.13}{=}\vartheta(\pi_q(A_{23}+A_{13}s_1-({A_{15}^*}^q s_1^{q+1}+A_{15}^* s_1^{1+q^2}))t_2)\\
&\quad\cdot\vartheta\pi_q(-A_{13}t_3+({A_{15}^*}^q s_1^q+A_{15}^* s_1^{q^2})t_3+\eta^{-1}(\eta^{q^2}A_{15}^* s_1^{q^2}+\eta^q {A_{15}^*}^q s_1^q)t_3)\\
&\quad\cdot\vartheta\pi_q({A_{15}^*}^q t_4+A_{15}^* t_4^{q^2})\\
&=\vartheta(\pi_q(A_{23}+A_{13}s_1-({A_{15}^*}^q s_1^{q+1}+A_{15}^* s_1^{1+q^2}))t_2)\\
&\quad\cdot\vartheta\pi_q(-A_{13}t_3+\eta^{-1}((\eta+\eta^{q^2})A_{15}^* s_1^{q^2}+(\eta+\eta^q){A_{15}^*}^q s_1^q)t_3)\\
&\quad\cdot\vartheta\pi_q({A_{15}^*}^q t_4+A_{15}^* t_4^{q^2}).
\end{aligned}
$$

By 2.2.12, $\eta+\eta^q\neq 0$. Thus $I_{\bar U}(\bar\chi_{4,q^3}^{A_{15}^*,A_{23},A_{13}})=\bar H$ by 2.2.39. By Clifford theory, we get $\mathrm{Ind}_{\bar H}^{\bar U}\bar\chi_{4,q^3}^{A_{15}^*,A_{23},A_{13}}\in\mathrm{Irr}(\bar U)$, $\deg(\bar\chi_{4,q^3}^{A_{15}^*,A_{23},A_{13}})=q^3$ and

$$
\mathrm{Res}_{\bar H}^{\bar U}\mathrm{Ind}_{\bar H}^{\bar U}\bar\chi_{4,q^3}^{A_{15}^*,A_{23},A_{13}}=\sum_{\bar x_1(s_1)\in\bar X_1}\left(\bar\chi_{4,q^3}^{A_{15}^*,A_{23},A_{13}}\right)^{x_1(s_1)}=\sum_{B_{13}\in\mathbb{F}_{q^3}}\bar\chi_{4,q^3}^{A_{15}^*,\tilde A_{23},B_{13}},
$$

where $\tilde A_{23}$ is determined by A_{23}, A_{15}^*, A_{13} and B_{13}. Let $\chi_{4,q^3}^{A_{15}^*,A_{23}}$ denote the lift of $\mathrm{Ind}_{\bar H}^{\bar U}\bar\chi_{4,q^3}^{A_{15}^*,A_{23},0}$ to U, then

$$
\begin{aligned}
\mathfrak{F}_4=&\{\chi\in\mathrm{Irr}(U)\mid X_5X_6\subseteq\ker\chi,\ X_4\nsubseteq\ker\chi\}\\
=&\{\chi_{4,q^3}^{A_{15}^*,A_{23}}\mid A_{15}^*\in\mathbb{F}_{q^3}^*,A_{23}\in\mathbb{F}_q\}.
\end{aligned}
$$

(4) *Family $\mathfrak{F}_5$, where $X_6\subseteq\ker\chi$ and $X_5\nsubseteq\ker\chi$.*
Let $\bar U:=X_6\backslash U=\bar X_2\bar X_1\bar X_3\bar X_4\bar X_5$. By 2.1.10, the commutator relations of $\bar U$ are given as follows:

$$
\begin{aligned}
[\bar x_1(t_1),\bar x_2(t_2)]=&\bar x_3(-t_2t_1)\cdot\bar x_4(t_2t_1^{q+1})\cdot\bar x_5(-t_2t_1^{q^2+q+1}),\\
[\bar x_1(t_1),\bar x_3(t_3)]=&\bar x_4(t_1t_3^q+t_1^qt_3)\cdot\bar x_5(\phi_0(-t_1^{q^2+q}t_3)),\\
[\bar x_1(t_1),\bar x_4(t_4)]=&\bar x_5(\phi_0(t_1t_4^q)).
\end{aligned}
$$

Let $\bar H:=\bar X_2\bar X_3\bar X_4\bar X_5$, $\bar\chi_{5,q^3}^{A_{16},A_{23},A_{13},A_{15}}\in\mathrm{Irr}(\bar H)$, and

$$
\begin{aligned}
&\bar\chi_{5,q^3}^{A_{16},A_{23},A_{13},A_{15}}(\bar x_2(t_2)\bar x_3(t_3)\bar x_4(t_4)\bar x_5(t_5))\\
:=&\vartheta(A_{23}t_2)\cdot\vartheta\pi_q(-A_{13}t_3)\cdot\vartheta\pi_q(A_{15}^qt_4+A_{15}t_4^{q^2})\cdot\vartheta(A_{16}t_5).
\end{aligned}
$$

Then $\bar H$ is abelian and $\bar H\trianglelefteq\bar U$. Note that $\bar X_1$ is a transversal of $\bar H$ in $\bar U$. For each $\bar x_1(s_1)\in\bar X_1$, we consider $\left(\bar\chi_{5,q^3}^{A_{16}^*,A_{23},A_{13},A_{15}}\right)^{\bar x_1(s_1)}$.
For all $\bar x_2(t_2)\bar x_3(t_3)\bar x_4(t_4)\bar x_5(t_5)\in\bar H$,

$$\begin{aligned}
&\left(\bar{\chi}_{5,q^3}^{A_{16}^*,A_{23},A_{13},A_{15}}\right)^{x_1(s_1)}(\bar{x}_2(t_2)\bar{x}_3(t_3)\bar{x}_4(t_4)\bar{x}_5(t_5))\\
=&\bar{\chi}_{5,q^3}^{A_{16}^*,A_{23},A_{13},A_{15}}(\bar{x}_1(s_1)\cdot\bar{x}_2(t_2)\bar{x}_3(t_3)\bar{x}_4(t_4)\bar{x}_5(t_5)\cdot\bar{x}_1(s_1)^{-1})\\
=&\bar{\chi}_{5,q^3}^{A_{16}^*,A_{23},A_{13},A_{15}}(\bar{x}_2(t_2)\bar{x}_3(t_3-s_1t_2)\bar{x}_4(t_4+s_1t_3^q+s_1^qt_3-t_2s_1^{q+1})\\
&\qquad\qquad\cdot\bar{x}_5(t_5+\phi_0(s_1t_4^q)+\phi_0(s_1^{q^2+q}t_3)-t_2s_1^{q^2+q+1}))\\
=&\vartheta(A_{23}t_2)\cdot\vartheta\pi_q(-A_{13}(t_3-s_1t_2))\\
&\cdot\vartheta\pi_q(A_{15}^q(t_4+s_1t_3^q+s_1^qt_3-t_2s_1^{q+1})+A_{15}(t_4+s_1t_3^q+s_1^qt_3-t_2s_1^{q+1})^{q^2})\\
&\cdot\vartheta(A_{16}^*(t_5+\phi_0(s_1t_4^q)+\phi_0(s_1^{q^2+q}t_3)-t_2s_1^{q^2+q+1}))\\
=&\vartheta(\pi_q(A_{23}+A_{13}s_1-(A_{15}^qs_1^{q+1}+A_{15}s_1^{1+q^2})-A_{16}^*s_1^{q^2+q+1})t_2)\\
&\cdot\vartheta\pi_q(-A_{13}t_3+A_{15}^q(s_1t_3^q+s_1^qt_3)+A_{15}(s_1t_3^q+s_1^qt_3)^{q^2}+\phi_0(A_{16}^*s_1^{q^2+q}t_3))\\
&\cdot\vartheta\pi_q(A_{15}^qt_4+A_{15}t_4^{q^2}+\phi_0(A_{16}^*s_1t_4^q))\cdot\vartheta(A_{16}^*t_5)\\
=&\vartheta(\pi_q(A_{23}+A_{13}s_1-(A_{15}^qs_1^{q+1}+A_{15}s_1^{1+q^2})-A_{16}^*s_1^{q^2+q+1})t_2)\\
&\cdot\vartheta\pi_q(-A_{13}t_3+(A_{15}^qs_1^q+A_{15}s_1^{q^2})t_3+(A_{15}^qs_1t_3^q+A_{15}s_1t_3^{q^2})+\phi_0(A_{16}^*s_1^{q^2+q}t_3))\\
&\cdot\vartheta\pi_q(A_{15}^qt_4+A_{15}t_4^{q^2}+\phi_0(A_{16}^*s_1t_4^q))\cdot\vartheta(A_{16}^*t_5)\\
\overset{2.2.13}{=}&\vartheta(\pi_q(A_{23}+A_{13}s_1-(A_{15}^qs_1^{q+1}+A_{15}s_1^{1+q^2})-A_{16}^*s_1^{q^2+q+1})t_2)\\
&\cdot\vartheta\pi_q(-A_{13}t_3+(A_{15}^qs_1^q+A_{15}s_1^{q^2})t_3+\eta^{-1}(\eta^{q^2}A_{15}s_1^{q^2}+\eta^qA_{15}^qs_1^q)t_3+\eta^{-1}A_{16}^*s_1^{q^2+q}t_3)\\
&\cdot\vartheta\pi_q(\eta^{-1}(\eta+\eta^q)A_{15}^qt_4+\eta^{-1}A_{16}^*s_1^{q^2}t_4)\cdot\vartheta(A_{16}^*t_5)\\
=&\vartheta(\pi_q(A_{23}+A_{13}s_1-(A_{15}^qs_1^{q+1}+A_{15}s_1^{1+q^2})-A_{16}^*s_1^{q^2+q+1})t_2)\\
&\cdot\vartheta\pi_q(-A_{13}t_3+\eta^{-1}((\eta+\eta^{q^2})A_{15}s_1^{q^2}+(\eta+\eta^q)A_{15}^qs_1^q)t_3+\eta^{-1}A_{16}^*s_1^{q^2+q}t_3)\\
&\cdot\vartheta\pi_q(\eta^{-1}((\eta+\eta^q)A_{15}^q+A_{16}^*s_1^{q^2})t_4)\cdot\vartheta(A_{16}^*t_5).
\end{aligned}$$

Thus $I_{\bar{U}}(\bar{\chi}_{5,q^3}^{A_{16}^*,A_{23},A_{13},A_{15}})=\bar{H}$. By Clifford theory, $\mathrm{Ind}_{\bar{H}}^{\bar{U}}\bar{\chi}_{5,q^3}^{A_{16}^*,A_{23},A_{13},A_{15}}\in\mathrm{Irr}(\bar{U})$, $\deg(\bar{\chi}_{5,q^3}^{A_{16}^*,A_{23},A_{13},A_{15}})=q^3$ and

$$\mathrm{Res}_{\bar{H}}^{\bar{U}}\mathrm{Ind}_{\bar{H}}^{\bar{U}}\bar{\chi}_{5,q^3}^{A_{16}^*,A_{23},A_{13},A_{15}}=\sum_{\bar{x}_1(s_1)\in\bar{X}_1}\left(\bar{\chi}_{5,q^3}^{A_{16}^*,A_{23},A_{13},A_{15}}\right)^{x_1(s_1)}=\sum_{B_{15}\in\mathbb{F}_{q^3}}\bar{\chi}_{5,q^3}^{A_{16}^*,\tilde{A}_{23},\tilde{A}_{13},B_{15}},$$

where $\tilde{A}_{13}$ is determined by $\{A_{13},A_{16}^*,A_{15},B_{15}\}$, and $\tilde{A}_{23}$ is determined by $\{A_{23},A_{16}^*,A_{13},A_{15},B_{15}\}$. Let $\chi_{5,q^3}^{A_{16}^*,A_{23},A_{13}}$ denote the lift of $\mathrm{Ind}_{\bar{H}}^{\bar{U}}\bar{\chi}_{5,q^3}^{A_{16}^*,A_{23},A_{13},0}$ to U, then

$$\begin{aligned}
\mathfrak{F}_5=&\{\chi\in\mathrm{Irr}(U)\mid X_6\subseteq\ker\chi,\ X_5\nsubseteq\ker\chi\}\\
=&\{\chi_{5,q^3}^{A_{16}^*,A_{23},A_{13},0}\mid A_{16}^*\in\mathbb{F}_q^*,A_{23}\in\mathbb{F}_q,A_{13}\in\mathbb{F}_{q^3}\}.
\end{aligned}$$

(5) *Family $\mathfrak{F}_6$, where $X_6 \nsubseteq \ker\chi$.*

Let $T := X_2X_3X_4X_5X_6$, $N := X_4X_5X_6$, and $\chi \in \mathrm{Irr}(U)$ such that $X_6 \nsubseteq \ker(\chi)$, then $Z(T) = Z(U) = X_6$. If $\psi \in \mathrm{Irr}(T)$ and $\psi \in \mathrm{Irr}\left(\mathrm{Res}_T^U\chi\right)$, $X_6 \nsubseteq \ker\psi$ by 2.3.37. Let $\lambda^{A_{17},A_{16},A_{15}} \in \mathrm{Irr}(N)$,

$$\lambda^{A_{17},A_{16},A_{15}}(x_4(t_4)x_5(t_5)x_6(t_6)) := \vartheta(A_{17}t_6)\cdot\vartheta(A_{16}t_5)\cdot\vartheta\pi_q(A_{15}^qt_4 + A_{15}t_4^{q^2}),$$

and $\psi^{A_{17}^*} := \mathrm{Ind}_N^T\lambda^{A_{17}^*,0,0}$, then by 2.3.42

$$\{\psi\in\mathrm{Irr}(T)\mid X_6\nsubseteq\ker\psi\} = \{\mathrm{Ind}_N^T\lambda^{A_{17}^*,0,0}\mid A_{17}^*\in\mathbb{F}_q^*\} = \{\psi^{A_{17}^*}\mid A_{17}^*\in\mathbb{F}_q^*\}.$$

By (5) of 2.3.39, $I_U(\psi^{A_{17}^*}) = U$, so $\mathrm{Res}_T^U\chi = z^*\psi^{A_{17}^*}$ for some $z^*\in\mathbb{N}^*$. Thus

$$\begin{aligned}\mathfrak{F}_6 =& \{\chi\in\mathrm{Irr}(U)\mid X_6\nsubseteq\ker\chi\} = \bigcup_{\substack{\psi\in\mathrm{Irr}(T)\\ X_6\nsubseteq\ker\psi}}\{\chi\in\mathrm{Irr}(U)\mid\chi\in\mathrm{Irr}\left(\mathrm{Ind}_T^U\psi\right)\}\\ =& \bigcup_{A_{17}^*\in\mathbb{F}_q^*}\{\chi\in\mathrm{Irr}(U)\mid\chi\in\mathrm{Irr}\left(\mathrm{Ind}_T^U\psi^{A_{17}^*}\right)\}\\ =& \bigcup_{A_{17}^*\in\mathbb{F}_q^*}\{\chi\in\mathrm{Irr}(U)\mid\chi\in\mathrm{Irr}\left(\mathrm{Ind}_N^U\lambda^{A_{17}^*,0,0}\right)\}.\end{aligned}$$

Let $H := X_1X_4X_5X_6$, then $H' = X_5$ and $Z(H) = X_4X_5 \trianglelefteq H$. Let $\tilde{\chi}^{A_{17},A_{15},A_{12}} \in \mathrm{Irr}(H)$ as in (1) of Lemma 2.3.40. For all $x_4(t_4)x_5(t_5)x_6(t_6)\in N$,

$$\begin{aligned}&\left(\mathrm{Res}_N^H\tilde{\chi}^{A_{17}^*,0,A_{12}}\right)(x_4(t_4)x_5(t_5)x_6(t_6)) = \tilde{\chi}^{A_{17}^*,0,A_{12}}(x_4(t_4)x_5(t_5)x_6(t_6))\\ =&\bar{\chi}^{A_{17}^*,0,A_{12}}(\bar{x}_4(t_4)\bar{x}_6(t_6)) = \vartheta(A_{17}^*t_6) = \lambda^{A_{17}^*,0,0}(x_4(t_4)x_5(t_5)x_6(t_6)).\end{aligned}$$

Thus $\mathrm{Res}_N^H\tilde{\chi}^{A_{17}^*,0,A_{12}} = \lambda^{A_{17}^*,0,0}$ for all $A_{12}\in\mathbb{F}_{q^3}$. By (4) of 2.3.39 $I_H(\lambda^{A_{17}^*,0,0}) = H$, thus $\mathrm{Ind}_N^H\lambda^{A_{17}^*,0,0} = \sum_{A_{12}\in\mathbb{F}_q}\tilde{\chi}^{A_{17}^*,0,A_{12}}$. By (2) of 2.3.39 $I_U(\lambda^{A_{17}^*,0,0}) = H$. Then by Clifford's Theorem, $\mathrm{Ind}_H^U\tilde{\chi}^{A_{17}^*,0,A_{12}}\in\mathrm{Irr}(U)$ for all $A_{17}^*\in\mathbb{F}_q^*$. Thus

$$\begin{aligned}\mathfrak{F}_6 =& \bigcup_{A_{17}^2.3.52*\in\mathbb{F}_q^*}\{\chi\in\mathrm{Irr}(U)\mid\chi\in\mathrm{Irr}\left(\mathrm{Ind}_H^U\mathrm{Ind}_N^H\lambda^{A_{17}^*,0,0}\right)\}\\ =& \bigcup_{\substack{A_{17}^*\in\mathbb{F}_q^*\\ A_{12}\in\mathbb{F}_{q^3}}}\{\chi\in\mathrm{Irr}(U)\mid\chi\in\mathrm{Irr}\left(\mathrm{Ind}_H^U\tilde{\chi}^{A_{17}^*,0,A_{12}}\right)\}\\ =&\{\mathrm{Ind}_H^U\tilde{\chi}^{A_{17}^*,0,A_{12}}\mid A_{17}^*\in\mathbb{F}_q^*, A_{12}\in\mathbb{F}_{q^3}\}.\end{aligned}$$

For $A_{17}^*\in\mathbb{F}_q^*$ and $A_{12}\in\mathbb{F}_{q^3}$, $X_4X_5\subseteq\ker(\tilde{\chi}^{A_{17}^*,0,A_{12}})$ and $X_4X_5\trianglelefteq H$. Thus $\tilde{\chi}^{A_{17}^*,0,A_{12}}$ is the lift to H of some irreducible character of $X_4X_5\backslash H\cong\bar{X}_1\bar{X}_6$. Let $\bar{H} := X_4X_5\backslash H\cong\bar{X}_1\bar{X}_6$, $\bar{\chi}_{6,q^4}^{A_{17}^*,A_{12}}\in\mathrm{Irr}(\bar{H})$, $\bar{\chi}_{6,q^4}^{A_{17}^*,A_{12}}(\bar{x}_1(t_1)\bar{x}_6(t_6)) := \vartheta\pi_q(A_{12}t_1)\cdot\vartheta(A_{17}^*t_6)$, and let $\tilde{\chi}_{6,q^4}^{A_{17}^*,A_{12}}$ denote the lift of $\bar{\chi}_{6,q^4}^{A_{17}^*,A_{12}}$ from $\bar{H}$ to H, then $\tilde{\chi}_{6,q^4}^{A_{17}^*,A_{12}} = \tilde{\chi}^{A_{17}^*,0,A_{12}}$. Let $\chi_{6,q^4}^{A_{17}^*,A_{12}} := \mathrm{Ind}_H^U\tilde{\chi}_{6,q^4}^{A_{17}^*,A_{12}}$, then

$$\mathfrak{F}_6=\{\mathrm{Ind}_H^U\tilde{\chi}^{A_{17}^*,0,A_{12}}\mid A_{17}^*\in\mathbb{F}_q^*, A_{12}\in\mathbb{F}_{q^3}\}=\{\chi_{6,q^4}^{A_{17}^*,A_{12}}\mid A_{17}^*\in\mathbb{F}_q^*, A_{12}\in\mathbb{F}_{q^3}\}.$$

□

2.3.46 Remark. *From the proof (5) of Propositon 2.3.45, we obtain that*

$$\begin{aligned}\operatorname{Ind}_T^U \psi^{A_{17}^*} &= \operatorname{Ind}_N^U \lambda^{A_{17}^*,0,0} = \operatorname{Ind}_H^U\Big(\sum_{A_{12}\in\mathbb{F}_q} \tilde{\chi}^{A_{17}^*,0,A_{12}}\Big)\\ =&\operatorname{Ind}_H^U\Big(\sum_{A_{12}\in\mathbb{F}_q} \tilde{\chi}_{6,q^4}^{A_{17}^*,A_{12}}\Big) = \sum_{A_{12}\in\mathbb{F}_q} \operatorname{Ind}_H^U \tilde{\chi}_{6,q^4}^{A_{17}^*,A_{12}} = \sum_{A_{12}\in\mathbb{F}_q} \chi_{6,q^4}^{A_{17}^*,A_{12}}\end{aligned}$$

By Clifford theory, Tung Le ([Le13]) determined the numbers and degrees of all irredubible characters of ${}^3D_4^{syl}(q^3)$. We also obtain the numbers and degrees by 2.3.45.

2.3.47 Corollary. *The irreducible characters of ${}^3D_4^{syl}(q^3)$ are classified into five families as listed in Table 2.7*

Table 2.7.: Irreducible characters of ${}^3D_4^{syl}(q^3)$

Family	Notation	Parameter Set	Number	Degree	Properties
$\mathfrak{F}_6$	$\chi_{6,q^4}^{A_{17}^*,A_{12}}$	$\begin{cases} A_{17}^* \in \mathbb{F}_q^* \\ A_{12} \in \mathbb{F}_{q^3} \end{cases}$	$(q-1)q^3$	q^4	$X_6 \nsubseteq \ker \chi_{6,q^4}^{A_{17}^*,A_{12}}$
$\mathfrak{F}_5$	$\chi_{5,q^3}^{A_{16}^*,A_{23},A_{13}}$	$\begin{cases} A_{16}^* \in \mathbb{F}_q^* \\ A_{23} \in \mathbb{F}_q \\ A_{13} \in \mathbb{F}_{q^3} \end{cases}$	$(q-1)q^4$	q^3	$\begin{cases} X_6 \subseteq \ker \chi_{5,q^3}^{A_{16}^*,A_{23},A_{13}} \\ X_5 \nsubseteq \ker \chi_{5,q^3}^{A_{16}^*,A_{23},A_{13}} \end{cases}$
$\mathfrak{F}_4$	$\chi_{4,q^3}^{A_{15}^*,A_{23}}$	$\begin{cases} A_{15}^* \in \mathbb{F}_{q^3}^* \\ A_{23} \in \mathbb{F}_q \end{cases}$	$(q^3-1)q$	q^3	$\begin{cases} X_5X_6 \subseteq \ker \chi_{4,q^3}^{A_{15}^*,A_{23}} \\ X_4 \nsubseteq \ker \chi_{4,q^3}^{A_{15}^*,A_{23}} \end{cases}$
$\mathfrak{F}_3$	$\chi_{3,q}^{A_{13}^*,\bar{A}_{12}^{A_{13}^*}}$	$\begin{cases} A_{13}^* \in \mathbb{F}_{q^3}^* \\ \bar{A}_{12}^{A_{13}^*} \in T^{A_{13}^*} \end{cases}$	$(q^3-1)q^2$	q	$\begin{cases} X_4X_5X_6 \subseteq \ker \chi_{3,q}^{A_{13}^*,\bar{A}_{12}^{A_{13}^*}} \\ X_3 \nsubseteq \ker \chi_{3,q}^{A_{13}^*,\bar{A}_{12}^{A_{13}^*}} \end{cases}$
$\mathfrak{F}_{lin}$	$\chi_{lin}^{A_{12},A_{23}}$	$\begin{cases} A_{12} \in \mathbb{F}_{q^3} \\ A_{23} \in \mathbb{F}_q \end{cases}$	q^4	1	$X_3X_4X_5X_6 \subseteq \ker \chi_{lin}^{A_{12},A_{23}}$

2.3.48 Remark. *Let $U := {}^3D_4^{syl}(q^3)$.*
Let $J := \{(1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(2,3)\}$, V as in §2.2.1.

(a) *For the abelian groups appearing in 2.3.45, we can obtain their irreducible characters by constructing monomial modules. For example: let* $\bar{U} := X_5X_6\backslash U = \bar{X}_2\bar{X}_1\bar{X}_3\bar{X}_4$ *and* $\bar{H} := \bar{X}_2\bar{X}_3\bar{X}_4$*. For the abelian group* $\bar{H}$*, let*

$$\begin{aligned} J_{\bar{H}} &:= \bigcup_{i\in\{2,3,4\}} (\overline{\operatorname{supp}}(\bar{X}_i) \cap J) = \{(1,3),(1,4),(1,5),(2,3)\} \subseteq J, \\ V_{\bar{H}} &:= V_{J_{\bar{H}}} \bigcap V = \left\{A = (A_{ij}) \in V_{J_{\bar{H}}} \mid A_{23} \in \mathbb{F}_q,\ A_{14} = A_{15}^q\right\} \\ &= \{A = (A_{ij}) \in V \mid \operatorname{supp}(A) \subseteq J_{\bar{H}}, A_{23} \in \mathbb{F}_q,\ A_{13}, A_{14} = A_{15}^q \in \mathbb{F}_{q^3}\}. \end{aligned}$$

Similar to ${}^3D_4^{syl}(q^3)$*, we construct a bijective 1-cocycle of* $\bar{H}$ *and a monomial* $\mathbb{C}\bar{H}$*-module. The irreducible characters of* $\bar{H}$ *are obtained, since* $\bar{H}$ *is abelian. Thus*

$$\operatorname{Irr}(\bar{H}) = \{\bar{\chi}^{A_{15},A_{23},A_{13}} \mid A_{13}, A_{15} \in \mathbb{F}_{q^3}, A_{23} \in \mathbb{F}_q\}$$

where

$$\bar{\chi}^{A_{15},A_{23},A_{13}}(\bar{x}_2(t_2)\bar{x}_3(t_3)\bar{x}_4(t_4)) := \vartheta(A_{23}t_2) \cdot \vartheta\pi_q(-A_{13}t_3) \cdot \vartheta\pi_q(A_{15}^q t_4 + A_{15}t_4^{q^2}).$$

(b) *Similar to* $U = {}^3D_4^{syl}(q^3)$*, we can construct linearisations for the quotient groups* $X_3X_4X_5X_6\backslash U$*,* $X_4X_5X_6\backslash U$*,* $X_5X_6\backslash U$ *and* $X_6\backslash U$*, and get their supermodules, superclasses and supercharacters. By 2.3.34, their conjugacy classes are determined. We also obtain their irreducible character by Clifford theory and calculate their character tables.*

Take $\bar{U} := X_5X_6\backslash U = \bar{X}_2\bar{X}_1\bar{X}_3\bar{X}_4$ *for example, let*

$$\begin{aligned} J_{\bar{U}} &:= \bigcup_{i\in\{1,2,3,4\}} (\overline{\operatorname{supp}}(\bar{X}_i) \cap J) = \{(1,2),(1,3),(1,4),(1,5),(2,3)\} \subseteq J, \\ V_{\bar{U}} &:= V_{J_{\bar{U}}} \bigcap V = \left\{A = (A_{ij}) \in V_{J_{\bar{U}}} \mid A_{23} \in \mathbb{F}_q,\ A_{14} = A_{15}^q\right\} \\ &= \{A = (A_{ij}) \in V \mid \operatorname{supp}(A) \subseteq J_{\bar{U}}, A_{23} \in \mathbb{F}_q,\ A_{12}, A_{13}, A_{14} = A_{15}^q \in \mathbb{F}_{q^3}\}. \end{aligned}$$

Similar to ${}^3D_4^{syl}(q^3)$*, we construct a bijective 1-cocycle of* $\bar{U}$ *and a monomial* $\mathbb{C}\bar{U}$*-module. With a similar analysis of the families* $\mathfrak{F}_{1,2}$*,* $\mathfrak{F}_3$*,* $\mathfrak{F}_4$ *of the orbit modules of* U*, we construct the supercharacters of* $X_5X_6\backslash U$*.*

With a similar analysis of the families $\mathfrak{F}_{lin}$*,* $\mathfrak{F}_3$*,* $\mathfrak{F}_4$ *of the irreducible characters of* U*, we obtain all the irreducible characters and the character table of* $X_5X_6\backslash U$*.*

Before we get the conjugacy classes of ${}^3D_4^{syl}(q^3)$, we need the following lemma.

2.3.49 Lemma. *Let* $u \in \mathbb{F}_{q^3}^+\backslash\mathbb{F}_q$*,* $\ker\phi_0$ *be the kernel of* ϕ_0*,* $S := u\ker\phi_0$ *and*

$$\phi_0|_S \colon u\ker\phi_0 \to \mathbb{F}_q : t \mapsto t^{q^2} + t^q + t.$$

Then $\phi_0|_S$ *is an* $\mathbb{F}_q$*-epimorphism.*

Proof. The kernel $\ker\phi_0$ is an $\mathbb{F}_q$-vector subspace of $\mathbb{F}_{q^3}$ and $u \in \mathbb{F}_{q^3}$, so $u\ker\phi_0$ is an $\mathbb{F}_q$-vector subspace of $\mathbb{F}_{q^3}$. Thus $\phi_0|_S$ is an $\mathbb{F}_q$-homomorphism by 2.2.10. Let $t_0 \in \ker\phi_0$ and $ut_0 \in \ker(\phi_0|_S)$, then $t_0^{q^2} + t_0^q + t_0 = 0$ and $u^{q^2}t_0^{q^2} + u^q t_0^q + ut_0 = 0$, so $u^{q^2}t_0^{q^2} + u^{q^2}t_0^q + u^{q^2}t_0 = 0$ since $u^{q^2} \neq 0$. Then $(u^q - u^{q^2})t_0^q + (u - u^{q^2})t_0 = 0$. Since $u \notin \mathbb{F}_q$, $u^q - u^{q^2} \neq 0$. Then $(u^q - u^{q^2})^{-1}t_0 = (u^q - u^{q^2})^{-q}t_0^q$, so $(u^q - u^{q^2})^{-1}t_0 \in \mathbb{F}_q$ and $t_0 \in (u^q - u^{q^2})\mathbb{F}_q$. Thus $|\ker(\phi_0|_S)| \leq |x\mathbb{F}_q| = q$, then $|\mathrm{im}(\phi_0|_S)| \geq \frac{|S|}{|\ker(\phi_0|_S)|} \overset{2.2.10}{=} \frac{q^2}{q} = q$. We have $|\mathrm{im}(\phi_0|_S)| = q$ since $|\mathrm{im}(\phi_0|_S)| \leq |\mathbb{F}_q| = q$. Therefore, $\phi_0|_S$ is an $\mathbb{F}_q$-epimorphism. $\square$

2.3.50 Reminder. *By 2.2.49, let $a^* \in \mathbb{F}_{q^3}^* = \mathbb{F}_{q^3}\backslash\{0\}$, then T^{a^*} denotes a complete set of coset representatives (i.e. a transversal) of $(a^*\mathbb{F}_q^+)$ in $\mathbb{F}_{q^3}^+$. If $\bar{t}_0 \in T^{a^*}$ and $\bar{t}_0 \in a^*\mathbb{F}_q^+$, we set $\bar{t}_0 = 0$.*

2.3.51 Proposition (Conjugacy classes of ${}^3D_4^{syl}(q^3)$). *Let $t_1^* \in \mathbb{F}_{q^3}$ and $T^{t_1^*}$ be the complete set of coset representatives for $t_1^*\mathbb{F}_q^+$ in $\mathbb{F}_{q^3}^+$. The conjugacy classes of ${}^3D_4^{syl}(q^3)$ are listed in Table 2.8.*

Proof. We prove the hard cases of the proposition. Let $u := x(r_1, r_2, r_3, r_4, r_5, r_6) \in U$, $0 \neq t_1 \in \mathbb{F}_{q^3}^*$, $\bar{t}_3 \in \mathbb{F}_{q^3}$, $t_6 \in \mathbb{F}_q$ and

$$x(a_1, a_2, a_3, a_4, a_5, a_6) := {}^u\big(x_1(t_1)x_3(\bar{t}_3)x_6(t_6)\big).$$

(1) Let $\bar{t}_3 \in t_1\mathbb{F}_q^+$, then there exists $s \in \mathbb{F}_q^+$ such that $\bar{t}_3 = st_1$. By 2.3.34,

$$\begin{aligned}
a_1 =& t_1, \quad a_2 = 0, \quad a_3 = r_2t_1 + \bar{t}_3 = (r_2 + s)t_1,\\
a_4 =& -r_2t_1^{q+1} - (r_3 - sr_1)t_1^q - (r_3 - sr_1)^q t_1,\\
a_5 =& r_2t_1^{q^2+q+1} + \phi_0\big(r_1^{q^2}(-r_3t_1^q - r_3^q t_1)\big) + \phi_0\big(r_3^{q^2}t_1^{q+1}\big)\\
&+ \phi_0\big(-t_1r_4^q\big) + \phi_0\big(sr_1^{q^2+q}t_1\big),\\
a_6 =& t_6 + r_2^2t_1^{q^2+q+1} + r_2a_5 + \phi_0\big(-t_1r_3^{q^2+q}\big) + \phi_0\big(-s^2r_1t_1^{q^2+q}\big)\\
&+ \phi_0\big(-st_1r_4^q\big) + \phi_0\big(st_1^{q^2}(r_2t_1^{q+1} + t_1r_3^q + t_1^q r_3)\big)\\
=& t_6 + r_2^2t_1^{q^2+q+1} + (r_2 + s)a_5 + 2sr_2t_1^{q^2+q+1}\\
&+ \phi_0\big(s(r_3 - sr_1)t_1^{q^2+q}\big) - \phi_0\big((r_3 - sr_1)^{q^2+q}t_1\big).
\end{aligned}$$

Let a_3, a_4 and a_5 be fixed, then a_6 is determined uniquely by 2.2.39 and 2.2.10. Hence we get the conjugacy class of $x_1(t_1)x_6(t_6)$.

$${}^U\big(x_1(t_1)x_6(t_6)\big) = \{x(t_1, 0, s_2t_1, s_4, s_5, \hat{s}_6) \mid s_4 \in \mathbb{F}_{q^3}, s_2, s_5 \in \mathbb{F}_q\}.$$

Table 2.8.: Conjugacy classes of ${}^3D_4^{syl}(q^3)$

Representatives	#Conjugacy Classes	Conjugacy Classes	|Conjugacy Class|
I_8	1	$x(0,0,0,0,0,0)$	1
$x_6(t_6^*)$	$q-1$	$x(0,0,0,0,0,t_6^*)$	1
$x_5(t_5^*)$	$q-1$	$x(0,0,0,0,t_5^*,s_6)$ $s_6 \in \mathbb{F}_q$	q
$x_4(t_4^*)$	q^3-1	$x(0,0,0,t_4^*,s_5,s_6)$ $s_5,s_6 \in \mathbb{F}_q$	q^2
$x(0,0,t_3^*,0,t_5,0)$	$(q^3-1)q$	$x(0,0,t_3^*,s_4,\hat{s}_5,s_6)$ $s_4 \in \mathbb{F}_{q^3}, s_6 \in \mathbb{F}_q$	q^4
$x(0,t_2^*,0,t_4,t_5,0)$	$(q-1)q^4$	$x(0,t_2^*,s_3,\hat{s}_4,\hat{s}_5,s_6)$ $s_3 \in \mathbb{F}_{q^3}, s_6 \in \mathbb{F}_q$	q^4
$x(t_1^*,0,0,0,0,t_6)$	$(q^3-1)q$	$x(t_1^*,0,t_1^*s_2,s_4,s_5,\hat{s}_6)$ $s_4 \in \mathbb{F}_{q^3}, s_2,s_5 \in \mathbb{F}_q$	q^5
$x(t_1^*,0,\bar{t}_3^*,0,0,0)$	$(q^3-1)(q^2-1)$	$x(t_1^*,0,\bar{t}_3^*+t_1^*s_2,s_4,s_5,s_6)$ $s_4 \in \mathbb{F}_{q^3}, s_2,s_5,s_6 \in \mathbb{F}_q$	q^6
$x(t_1^*,t_2^*,0,0,0,0)$	$(q-1)(q^3-1)$	$x(t_1^*,t_2^*,s_3,s_4,s_5,s_6)$ $s_3,s_4 \in \mathbb{F}_{q^3}, s_5,s_6 \in \mathbb{F}_q$	q^8

where $0 \neq \bar{t}_3^* \in T^{t_1^*}$, $\hat{s}_-$ is determined by some of t_-^*, t_- and s_-.

(2) Let $\bar{t}_3 \notin t_1\mathbb{F}_q^+$. By 2.3.34,

$$\begin{aligned} a_1 =& t_1, \quad a_2 = 0, \quad a_3 = \bar{t}_3 + r_2 t_1, \\ a_4 =& - r_2 t_1^{q+1} - t_1 r_3^q - t_1^q r_3 + r_1 \bar{t}_3^q + r_1^q \bar{t}_3, \\ a_5 =& r_2 t_1^{q^2+q+1} + \phi_0\big(r_1^{q^2}(-r_3 t_1^q - r_3^q t_1)\big) + \phi_0\big(r_3^{q^2} t_1^{q+1}\big) \\ &+ \phi_0\big(- t_1 r_4^q\big) + \phi_0\big(r_1^{q^2+q}\bar{t}_3\big), \\ a_6 =& t_6 + r_2^2 t_1^{q^2+q+1} + r_2 a_5 + \phi_0\big(- t_1 r_3^{q^2+q}\big) + \phi_0\big(- r_1 \bar{t}_3^{q^2+q}\big) \\ &+ \phi_0\big(- \bar{t}_3 r_4^q\big) + \phi_0\big(\bar{t}_3^{q^2}(r_2 t_1^{q+1} + t_1 r_3^q + t_1^q r_3)\big). \end{aligned}$$

For the fixed r_1, r_2, r_3 and a_5, let

$$T := \{t_1 r_4^q \mid r_4 \in \mathbb{F}_{q^3}, \text{and } r_1, r_2, r_3, a_5 \text{ are fixed}\}.$$

By 2.2.10, $|T| = q^2$. Let $x_0 \in T$, then $T = x_0 + \ker\phi_0$ and $\bar{t}_3 r_4^q \in \frac{\bar{t}_3 x_0}{t_1} + \frac{\bar{t}_3}{t_1}\ker\phi_0$. We know $\bar{t}_3 \notin t_1\mathbb{F}_q^+$, so $\frac{\bar{t}_3}{t_1} \in \mathbb{F}_{q^3}^+ \backslash \mathbb{F}_q^+$. Hence

$$\{\phi_0(\bar{t}_3 r_4^q) \mid t_1 r_4^q \in T\} = \left\{\phi_0(\frac{\bar{t}_3 x_0}{t_1}) + \phi_0(t) \mid t \in \frac{\bar{t}_3}{t_1}\ker\phi_0\right\} \overset{2.3.49}{=} \mathbb{F}_q^+.$$

Thus a_6 can be every element of $\mathbb{F}_q$ for the fixed r_1, r_2, r_3 and a_5. Therefore,

$$^U\big(x_1(t_1)x_3(\bar{t}_3)\big) = \{x(t_1, 0, \bar{t}_3 + s_2 t_1, s_4, s_5, s_6) \mid s_4 \in \mathbb{F}_{q^3}, s_2, s_5, s_6 \in \mathbb{F}_q\}.$$

By 2.3.34, the other conjugacy classes are determined. Then the proposition is obtained. □

2.3.52 Proposition (Character table of ${}^3D_4^{syl}(q^3)$)**.** *The character table of* ${}^3D_4^{syl}(q^3)$ *is the one in Table 2.9.*

Table 2.9.: Character table of ${}^3D_4^{syl}(q^3)$

	I_8	$x_1(t_1^*) \cdot x_6(t_6)$	$x_1(t_1^*) \cdot x_3(\bar{t}_3^*)$	$x_2(t_2^*) \cdot x_4(t_4) \cdot x_5(t_5)$	$x_2(t_2^*) \cdot x_1(t_1^*)$	$x_3(t_3^*) \cdot x_5(t_5)$	$x_4(t_4^*)$	$x_5(t_5^*)$	$x_6(t_6^*)$
$\chi_{lin}^{0,0}$	1	1	1	1	1	1	1	1	1
$\chi_{lin}^{A_{12}^*,0}$	1	$\vartheta\pi_q(A_{12}^*t_1^*)$	$\vartheta\pi_q(A_{12}^*t_1^*)$	1	$\vartheta\pi_q(A_{12}^*t_1^*)$	1	1	1	1
χ_{lin}^{0,A_{23}^*}	1	1	1	$\vartheta(A_{23}^*t_2^*)$	$\vartheta(A_{23}^*t_2^*)$	1	1	1	1
$\chi_{lin}^{A_{12}^*,A_{23}^*}$	1	$\vartheta\pi_q(A_{12}^*t_1^*)$	$\vartheta\pi_q(A_{12}^*t_1^*)$	$\vartheta(A_{23}^*t_2^*)$	$\vartheta\pi_q(A_{12}^*t_1^*) \cdot\vartheta(A_{23}^*t_2^*)$	1	1	1	1
$\chi_{3,q}^{A_{13}^*,\bar{A}_{12}^{A_{13}^*}}$	q	⋇	⋇	0	0	$\vartheta\pi_q(-A_{13}^*t_3^*) \cdot q$	q	q	q
$\chi_{4,q^3}^{A_{15}^*,A_{23}}$	q^3	0	0	⋇	0	0	$\vartheta\pi_q({A_{15}^*}^q t_4^*) \cdot\vartheta\pi_q(A_{15}^* {t_4^*}^{q^2}) \cdot q^3$	q^3	q^3
$\chi_{5,q^3}^{A_{16}^*,A_{23},A_{13}}$	q^3	0	0	⋇	0	⋇	0	$\vartheta(A_{16}^*t_5^*) \cdot q^3$	q^3
$\chi_{6,q^4}^{A_{17}^*,A_{12}}$	q^4	⋇	0	0	0	0	0	0	$\vartheta(A_{17}^*t_6^*) \cdot q^4$

where $\bar{A}_{12}^{A_{13}^*} \in T^{A_{13}^*}$, $0 \neq \bar{t}_3^* \in T^{t_1^*}$, and

$$\begin{aligned}
&\chi_{6,q^4}^{A_{17}^*,A_{12}}(x_1(t_1^*)x_6(t_6))\\
=&\sum_{r_3\in\mathbb{F}_{q^3}} \vartheta\pi_q\left(A_{12}t_1^* + A_{17}^*\phi_0(-t_1^*r_3^{q^2+q}) + A_{17}^*t_6\right)\\
=&\vartheta\pi_q(A_{12}t_1^* + A_{17}^*t_6)\cdot \sum_{r_3\in\mathbb{F}_{q^3}} \vartheta\left(A_{17}^*\phi_0(-t_1^*r_3^{q^2+q})\right),
\end{aligned}$$

$$\begin{aligned}
&\chi_{5,q^3}^{A_{16}^*,A_{23},A_{13}}(x_3(t_3^*)x_5(t_5))\\
=&\sum_{r_1\in\mathbb{F}_{q^3}} \vartheta\pi_q\left(-A_{13}t_3^* + A_{16}^*\phi_0(t_3^*r_1^{q^2+q}) + A_{16}^*t_5\right)\\
=&\vartheta\pi_q(-A_{13}t_3^* + A_{16}^*t_5)\cdot \sum_{r_1\in\mathbb{F}_{q^3}} \vartheta\left(A_{16}^*\phi_0(t_3^*r_1^{q^2+q})\right),
\end{aligned}$$

$$\begin{aligned}
&\chi_{5,q^3}^{A_{16}^*,A_{23},A_{13}}(x_2(t_2^*)x_4(t_4)x_5(t_5))\\
=&\sum_{r_1\in\mathbb{F}_{q^3}} \vartheta\pi_q(A_{23}t_2^* + A_{13}r_1t_2^* + A_{16}^*(\phi_0\left(r_1t_4^q\right) - t_2^*r_1^{q^2+q+1}) + A_{16}^*t_5)\\
=&\vartheta(A_{23}t_2^* + A_{16}^*t_5)\cdot \sum_{r_1\in\mathbb{F}_{q^3}} \vartheta\pi_q\left(A_{13}r_1t_2^* + A_{16}^*(\phi_0\left(r_1t_4^q\right) - t_2^*r_1^{q^2+q+1})\right),
\end{aligned}$$

$$\begin{aligned}
&\chi_{4,q^3}^{A_{15}^*,A_{23}}(x_2(t_2^*)x_4(t_4)x_5(t_5))\\
=&\sum_{r_1\in\mathbb{F}_{q^3}} \vartheta\pi_q(A_{23}t_2^* + {A_{15}^*}^q(t_4 - t_2^*r_1^{q+1}) + A_{15}^*(t_4^{q^2} - t_2^*r_1^{q^2+1}))\\
=&\vartheta(A_{23}t_2^*)\cdot \sum_{r_1\in\mathbb{F}_{q^3}} \vartheta\pi_q\left({A_{15}^*}^q(t_4 - t_2^*r_1^{q+1}) + A_{15}^*(t_4^{q^2} - t_2^*r_1^{q^2+1})\right),
\end{aligned}$$

$$\begin{aligned}
&\chi_{3,q}^{A_{13}^*,\bar{A}_{12}^{A_{13}^*}}(x_1(t_1^*)x_6(t_6)) = \sum_{r_2\in\mathbb{F}_q} \vartheta\pi_q\left((\bar{A}_{12}^{A_{13}^*} - A_{13}^*r_2)t_1^*\right),\\
&\chi_{3,q}^{A_{13}^*,\bar{A}_{12}^{A_{13}^*}}(x_1(t_1^*)x_3(\bar{t}_3^*)) = \sum_{r_2\in\mathbb{F}_q} \vartheta\pi_q\left((\bar{A}_{12}^{A_{13}^*} - A_{13}^*r_2)t_1^* - \bar{t}_3^*A_{13}^*\right).
\end{aligned}$$

Proof. Let $x := x(t_1,t_2,t_3,t_4,t_5,t_6) \in U = {}^3D_4^{syl}(q^3)$.

(1) *The values of* $\chi_{6,q^4}^{A_{17}^*,A_{12}}$ *for all* $A_{17}^* \in \mathbb{F}_q^*$ *and* $A_{12} \in \mathbb{F}_{q^3}$.

Let $H := X_1X_4X_5X_6$, $\bar{H} := X_4X_5\backslash H \cong \bar{X}_1\bar{X}_6$, $\bar{\chi}_{6,q^4}^{A_{17}^*,A_{12}} \in \mathrm{Irr}(\bar{H})$ with

$$\bar{\chi}_{6,q^4}^{A_{17}^*,A_{12}}(\bar{x}_1(t_1)\bar{x}_6(t_6)) = \vartheta\pi_q(A_{12}t_1)\cdot\vartheta(A_{17}^*t_6).$$

Let $\tilde{\chi}_{6,q^4}^{A_{17}^*,A_{12}}$ denote the lift of $\bar{\chi}_{6,q^4}^{A_{17}^*,A_{12}}$ from $\bar{H}$ to H, and $\chi_{6,q^4}^{A_{17}^*,A_{12}} := \mathrm{Ind}_H^U\tilde{\chi}_{6,q^4}^{A_{17}^*,A_{12}}$. Then

$$\begin{aligned}\chi_{6,q^4}^{A_{17}^*,A_{12}}(x) &= \left(\mathrm{Ind}_H^U\tilde{\chi}_{6,q^4}^{A_{17}^*,A_{12}}\right)(x) = \frac{1}{|H|}\sum_{\substack{g\in U\\ g\cdot x\cdot g^{-1}\in H}} \tilde{\chi}_{6,q^4}^{A_{17}^*,A_{12}}(g\cdot x\cdot g^{-1})\\ &=\frac{1}{|H|}\sum_{\substack{g\in U\\ g\cdot x\cdot g^{-1}\in H}} \bar{\chi}_{6,q^4}^{A_{17}^*,A_{12}}(X_4X_5\backslash(g\cdot x\cdot g^{-1})).\end{aligned}$$

Thus,

$$\begin{aligned}&\chi_{6,q^4}^{A_{17}^*,A_{12}}(x_1(t_1^*)x_3(\bar{t}_3^*)) = \chi_{6,q^4}^{A_{17}^*,A_{12}}(x_2(t_2^*)x_4(t_4)x_5(t_5))\\ =&\chi_{6,q^4}^{A_{17}^*,A_{12}}(x_2(t_2^*)x_1(t_1^*)) = \chi_{6,q^4}^{A_{17}^*,A_{12}}(x_3(t_3^*)x_5(t_5)) \overset{gxg^{-1}\notin H}{=} 0.\end{aligned}$$

We have

$$\begin{aligned}&\chi_{6,q^4}^{A_{17}^*,A_{12}}(x_4(t_4)x_5(t_5)x_6(t_6))\\ =&\frac{1}{|H|}\sum_{\substack{g\in U\\ g\cdot x_4(t_4)x_5(t_5)x_6(t_6)\cdot g^{-1}\in H}} \tilde{\chi}_{6,q^4}^{A_{17}^*,A_{12}}(g\cdot x_4(t_4)x_5(t_5)x_6(t_6)\cdot g^{-1})\\ \overset{g:=x(r_1,r_2,r_3,r_4,r_5,r_6)}{=}&\frac{1}{|H|}\sum_{\substack{r_1,r_3,r_4\in\mathbb{F}_{q^3}\\ r_2,r_5,r_6\in\mathbb{F}_q}} \tilde{\chi}_{6,q^4}^{A_{17}^*,A_{12}}(x_4(t_4)x_5(t_5+\phi_0(r_1t_4^q))\\ &\qquad\cdot x_6(t_6+r_2t_5+\phi_0(r_1r_2t_4^q)+\phi_0(r_3t_4^q)))\\ =&\frac{1}{|H|}\sum_{\substack{r_1,r_3,r_4\in\mathbb{F}_{q^3}\\ r_2,r_5,r_6\in\mathbb{F}_q}} \bar{\chi}_{6,q^4}^{A_{17}^*,A_{12}}(\bar{x}_6(t_6+r_2t_5+\phi_0(r_1r_2t_4^q)+\phi_0(r_3t_4^q)))\\ =&\frac{1}{q^3}\sum_{\substack{r_1,r_3\in\mathbb{F}_{q^3}\\ r_2\in\mathbb{F}_q}} \bar{\chi}_{6,q^4}^{A_{17}^*,A_{12}}(\bar{x}_6(t_6+r_2t_5+\phi_0(r_1r_2t_4^q)+\phi_0(r_3t_4^q))),\end{aligned}$$

so,

$$\begin{aligned}&\chi_{6,q^4}^{A_{17}^*,A_{12}}(I_8) = q^4, \qquad \chi_{6,q^4}^{A_{17}^*,A_{12}}(x_4(t_4^*)) = \chi_{6,q^4}^{A_{17}^*,A_{12}}(x_5(t_5^*)) = 0,\\ &\chi_{6,q^4}^{A_{17}^*,A_{12}}(x_6(t_6^*)) = q^4\cdot\vartheta(A_{17}^*t_6^*).\end{aligned}$$

Then

$$\begin{aligned}
\chi_{6,q^4}^{A_{17}^*,A_{12}}(x_1(t_1^*)x_6(t_6)) =& \frac{1}{|H|} \sum_{\substack{g\in U\\ g\cdot x_1(t_1^*)x_6(t_6)\cdot g^{-1}\in H}} \tilde{\chi}_{6,q^4}^{A_{17}^*,A_{12}}(g\cdot x_1(t_1^*)x_6(t_6)\cdot g^{-1})\\
\overset{g:=x(r_1,r_2,r_3,r_4,r_5,r_6)}{=}& \frac{1}{|H|} \sum_{\substack{r_2=0\\ r_1,r_3,r_4\in\mathbb{F}_{q^3}\\ r_5,r_6\in\mathbb{F}_q}} \bar{\chi}_{6,q^4}^{A_{17}^*,A_{12}}(\bar{x}_1(t_1^*)\bar{x}_6(t_6+\phi_0\left(-t_1^*r_3^{q^2+q}\right)))\\
=& \sum_{r_3\in\mathbb{F}_{q^3}} \bar{\chi}_{6,q^4}^{A_{17}^*,A_{12}}(\bar{x}_1(t_1^*)\bar{x}_6(t_6+\phi_0\left(-t_1^*r_3^{q^2+q}\right)))\\
=& \vartheta\pi_q(A_{12}t_1^*+A_{17}^*t_6)\cdot \sum_{r_3\in\mathbb{F}_{q^3}} \vartheta\left(A_{17}^*\phi_0(-t_1^*r_3^{q^2+q})\right).
\end{aligned}$$

Thus all the values of $\chi_{6,q^4}^{A_{17}^*,A_{12}}$ are obtained.

(2) *The values of* $\chi_{3,q}^{A_{13}^*,\bar{A}_{12}^{A_{13}^*}}$ *for all* $A_{13}^*\in\mathbb{F}_{q^3}^*$ *and* $\bar{A}_{12}^{A_{13}^*}\in T^{A_{13}^*}$.

Let $\bar{U}:=X_4X_5X_6\backslash U=\bar{X}_2\bar{X}_1\bar{X}_3$, $\bar{H}:=\bar{X}_1\bar{X}_3$, $\bar{\chi}_{3,q}^{A_{13},A_{12}}\in\operatorname{Irr}(\bar{H})$ with

$$\bar{\chi}_{3,q}^{A_{13},A_{12}}(\bar{x}_1(t_1)\bar{x}_3(t_3)):=\vartheta\pi_q(A_{12}t_1)\cdot\vartheta\pi_q(-A_{13}t_3),$$

and $\chi_{3,q}^{A_{13}^*,\bar{A}_{12}^{A_{13}^*}}$ denote the lift of $\operatorname{Ind}_{\bar{H}}^{\bar{U}}\bar{\chi}_{3,q}^{A_{13}^*,\bar{A}_{12}^{A_{13}^*}}$ to U. For $x=x(t_1,t_2^*,t_3,t_4,t_5,t_6)\in U$ with $t_2^*\in\mathbb{F}_q^*$, we have

$$\begin{aligned}
\chi_{3,q}^{A_{13}^*,\bar{A}_{12}^{A_{13}^*}}(x) &= \left(\operatorname{Ind}_{\bar{H}}^{\bar{U}}\bar{\chi}_{3,q}^{A_{13}^*,\bar{A}_{12}^{A_{13}^*}}\right)(\bar{x}) = \frac{1}{|\bar{H}|}\sum_{\substack{\bar{u}\in\bar{U}\\ \bar{u}\cdot\bar{x}\cdot\bar{u}^{-1}\in\bar{H}}} \bar{\chi}_{3,q}^{A_{13}^*,\bar{A}_{12}^{A_{13}^*}}(\bar{u}\cdot\bar{x}\cdot\bar{u}^{-1})\\
&\overset{\bar{u}\cdot\bar{x}\cdot\bar{u}^{-1}\notin\bar{H}}{=} 0.
\end{aligned}$$

If $x=x(t_1,t_2^*,t_3,t_4,t_5,t_6)\in U$, we have

$$\begin{aligned}
&\chi_{3,q}^{A_{13}^*,\bar{A}_{12}^{A_{13}^*}}(x(t_1,0,t_3,t_4,t_5,t_6)) = \left(\operatorname{Ind}_{\bar{H}}^{\bar{U}}\bar{\chi}_{3,q}^{A_{13}^*,\bar{A}_{12}^{A_{13}^*}}\right)(\bar{x}_1(t_1)\bar{x}_3(t_3))\\
=&\frac{1}{|\bar{H}|}\sum_{\substack{\bar{u}:=\bar{x}_2(r_2)\bar{x}_1(r_1)\bar{x}_3(r_3)\in\bar{U}\\ \bar{u}\cdot\bar{x}_1(t_1)\bar{x}_3(t_3)\cdot\bar{u}^{-1}\in\bar{H}}} \bar{\chi}_{3,q}^{A_{13}^*,\bar{A}_{12}^{A_{13}^*}}(\bar{u}\cdot\bar{x}_1(t_1)\bar{x}_3(t_3)\cdot\bar{u}^{-1})\\
=&\frac{1}{|\bar{H}|}\sum_{\substack{r_1,r_3\in\mathbb{F}_{q^3}\\ r_2\in\mathbb{F}_q}} \bar{\chi}_{3,q}^{A_{13}^*,\bar{A}_{12}^{A_{13}^*}}(\bar{x}_1(t_1)\bar{x}_3(t_3+r_2t_1))\\
=&\sum_{r_2\in\mathbb{F}_q}\bar{\chi}_{3,q}^{A_{13}^*,\bar{A}_{12}^{A_{13}^*}}(\bar{x}_1(t_1)\bar{x}_3(t_3+r_2t_1)) = \sum_{r_2\in\mathbb{F}_q}\vartheta\pi_q\left(\bar{A}_{12}^{A_{13}^*}t_1-A_{13}^*(t_3+r_2t_1)\right).
\end{aligned}$$

Thus we get all the values of $\chi_{3,q}^{A_{13}^*,\bar{A}_{12}^{A_{13}^*}}$.

Similarly, we calculate all of the other values of the character table. $\square$

2.3.53 Proposition (Supercharacters and irreducible characters)**.** *The relations between the supercharacters and the irreducible characters of ${}^3D_4^{syl}(q^3)$ are determined:*

$$\Psi_{M(A_{17}^*e_{17})} = q^3 \sum_{A_{12}\in\mathbb{F}_{q^3}} \chi_{6,q^4}^{A_{17}^*,A_{12}}, \qquad \Psi_{M(A_{16}^*e_{16})} = q^3 \sum_{\substack{A_{23}\in\mathbb{F}_q\\ A_{13}\in\mathbb{F}_{q^3}}} \chi_{5,q^3}^{A_{16}^*,A_{23},A_{13}},$$

$$\Psi_{M(A_{15}^{*}{}^q e_{14}+A_{15}^*e_{15})} = q^3 \sum_{A_{23}\in\mathbb{F}_q} \chi_{4,q^3}^{A_{15}^*,A_{23}}, \qquad \Psi_{M(A_{13}^*e_{13}+\bar{A}_{12}^{A_{13}^*}e_{12})} = \chi_{3,q}^{A_{13}^*,\bar{A}_{12}^{A_{13}^*}},$$

$$\Psi_{M(A_{12}^*e_{12}+A_{23}^*e_{23})} = \chi_{lin}^{A_{12}^*,A_{23}^*}, \qquad \Psi_{M(A_{23}^*e_{23})} = \chi_{lin}^{0,A_{23}^*},$$

$$\Psi_{M(A_{12}^*e_{12})} = \chi_{lin}^{A_{12}^*,0}, \qquad \Psi_{M(0)} = \chi_{lin}^{0,0} = \mathrm{triv}_U.$$

Proof. Take $\Psi_{M(A_{15}^{*}{}^q e_{14}+A_{15}^*e_{15})} = q^3 \sum_{A_{23}\in\mathbb{F}_q} \chi_{4,q^3}^{A_{15}^*,A_{23}}$ for example. We must show that $\Psi_{M(A_{15}^{*}{}^q e_{14}+A_{15}^*e_{15})}(u) = q^3 \sum_{A_{23}\in\mathbb{F}_q} \chi_{4,q^3}^{A_{15}^*,A_{23}}(u)$ for all $u\in U$.

By Table 2.5 and Table 2.9, it is sufficient to prove that

$$0 = \Psi_{M(A_{15}^{*}{}^q e_{14}+A_{15}^*e_{15})}(u) = q^3 \sum_{A_{23}\in\mathbb{F}_q} \chi_{4,q^3}^{A_{15}^*,A_{23}}(u) \qquad \text{for all } u\in C_2.$$

Let $u := x(0,t_2^*,t_3,t_4,t_5,t_6)$, then

$$\begin{aligned}
&q^3 \sum_{A_{23}\in\mathbb{F}_q} \chi_{4,q^3}^{A_{15}^*,A_{23}}(u)\\
=&q^3 \sum_{A_{23}\in\mathbb{F}_q} \sum_{r_1\in\mathbb{F}_{q^3}} \vartheta\pi_q(A_{23}t_2^* + A_{15}^{*}{}^q(t_4 - t_2^*r_1^{q+1} + r_1t_3^q + r_1^qt_3)\\
&\qquad\qquad + A_{15}^*(t_4^{q^2} - t_2^*r_1^{q^2+1} + r_1^{q^2}t_3 + r_1t_3^{q^2}))\\
=&q^3 \sum_{A_{23}\in\mathbb{F}_q} \vartheta(A_{23}t_2^*)\\
&\cdot \sum_{r_1\in\mathbb{F}_{q^3}} \vartheta\pi_q(A_{15}^{*}{}^q(t_4 - t_2^*r_1^{q+1} + r_1t_3^q + r_1^qt_3) + A_{15}^*(t_4^{q^2} - t_2^*r_1^{q^2+1} + r_1^{q^2}t_3 + r_1t_3^{q^2}))\\
=&0.
\end{aligned}$$

Thus $\Psi_{M(A_{15}^{*}{}^q e_{14}+A_{15}^*e_{15})} = q^3 \sum_{A_{23}\in\mathbb{F}_q} \chi_{4,q^3}^{A_{15}^*,A_{23}}$ is proved. Similarly, we get the other formulae. $\square$

In this section, p is an odd prime. By Propositions 2.3.51, 2.3.45 and 2.3.52, we obtain the number of the conjugacy classes of ${}^3D_4^{syl}(q^3)$, and determine the numbers of the complex irreducible characters of degree q^c with $c \in \mathbb{N}$. In [Le13], T. Le counted the numbers of irreducible characters for all primes p.

2.3.54 Corollary. *Let* $\#\mathrm{Irr}$ *be the number of irreducible characters of* ${}^3D_4^{syl}(q^3)$*, and* $\#\mathrm{Irr}_c$ *be the number of irreducible characters of* ${}^3D_4^{syl}(q^3)$ *of dimension* q^c *with* $c \in \mathbb{N}$*, then*

$$\begin{aligned}
\#\mathrm{Irr}_4 &= q^4 - q^3 &&= (q-1)^4 + 3(q-1)^3 + 3(q-1)^2 + (q-1),\\
\#\mathrm{Irr}_3 &= q^5 - q &&= (q-1)^5 + 5(q-1)^4 + 10(q-1)^3 + 10(q-1)^2 + 4(q-1),\\
\#\mathrm{Irr}_1 &= q^5 - q^2 &&= (q-1)^5 + 5(q-1)^4 + 10(q-1)^3 + 9(q-1)^2 + 3(q-1),\\
\#\mathrm{Irr}_0 &= q^4 &&= (q-1)^4 + 4(q-1)^3 + 6(q-1)^2 + 4(q-1) + 1,
\end{aligned}$$

and

$$\begin{aligned}
\#\mathrm{Irr} =& \#\{\textit{Irreducible Characters of } {}^3D_4^{syl}\left(q^3\right)\}\\
=& \#\{\textit{Conjugacy Classes of } {}^3D_4^{syl}\left(q^3\right)\}\\
=& 2q^5 + 2q^4 - q^3 - q^2 - q\\
=& 2(q-1)^5 + 12(q-1)^4 + 27(q-1)^3 + 28(q-1)^2 + 12(q-1) + 1.
\end{aligned}$$

2.3.55 Remark. *We consider the analogue of Higman's conjecture, Lehrer's conjecture and Isaacs' conjecture of* $U_n(q)$ *for* ${}^3D_4^{syl}(q^3)$*. By 2.3.54, the conjectures are true for* ${}^3D_4^{syl}(q^3)$*.*

3. Supercharacter theories for $G_2^{syl}(q)$

In this chapter, we firstly construct a Lie algebra $\mathcal{L}_{G_2}$ of type G_2 which is a subalgebra of $\mathcal{L}_{D_4}$ (3.1.5), and then determine the Sylow p-subgroup $G_2^{syl}(q)$ of the Chevalley group of type $\mathcal{L}_{G_2}$ over the field $\mathbb{F}_q$ (3.1.27).

After that, we construct a 1-cocycle f and determine a monomial linearisation $(f, \kappa|_{V\times V})$ for $G_8(q)$ (3.2.29), for which $f|_{G_2^{syl}(q)}$ is bijective (3.2.28). We obtain a monomial $G_8(q)$-module $\mathbb{C}G_2^{syl}(q)$ (3.2.31).

Finally, we determine the supercharacter theory for $G_2^{syl}(q)$ (3.3.13), establish the supercharacter table of $G_2^{syl}(q)$ in Table 3.6. When $p \notin \{2,3\}$, we determine the relations between the supercharacters and the irreducible characters of $G_2^{syl}(q)$ (3.3.27), and calculate the character table of $G_2^{syl}(q)$ (3.3.26).

In this chapter, we use many definitions analogously to previous chapters, for example: the main condition, the verge pattern, the core pattern the staircase pattern and the hook-separated module.

3.1. Sylow p-subgroup $G_2^{syl}(q)$

In this section, we construct a Lie algebra of type G_2 and its corresponding Chevalley basis (see 3.1.5), and then determine the Sylow p-subgroup $G_2^{syl}(q)$ of the Chevalley group of type $\mathcal{L}_{G_2}$ over the field $\mathbb{F}_q$ (see 3.1.27).

3.1.1. Chevalley basis of a Lie algebra of type G_2

In this subsection, we construct a Lie algebra $\mathcal{L}_{G_2}$ of type G_2, which is a subalgebra of $\mathcal{L}_{D_4}$. Then a Chevalley basis of $\mathcal{L}_{G_2}$ is determined.

3.1.1 Notation. *Define $h_i \in \mathcal{H}_{D_4}$ $(i = 1,2,3,4)$ as follows:*

$$h_1 := e_{11} - e_{88}, \quad h_2 := e_{22} - e_{77}, \quad h_3 := e_{33} - e_{66}, \quad h_4 := e_{44} - e_{55}.$$

3.1.2 Reminder. *By 2.1.1, $h_{r_i} \in \mathcal{H}_{D_4}$ $(i = 1,2,3,4)$ and*

$$h_{r_1} = h_1 - h_2, \quad h_{r_2} = h_2 - h_3, \quad h_{r_3} = h_3 - h_4, \quad h_{r_4} = h_3 + h_4.$$

3.1.3 Notation. *Define* $e_i, f_i \in \mathcal{L}_{D_4}$ $(i = 1, 2, \ldots, 6)$ *and* $\tilde{h}_k \in \mathcal{H}_{D_4}$ $(k = 1, 2)$ *as follows:*

	$\sum_{i,j} c_{ij} e_{i,j}$	$\sum_\gamma c_\gamma e_\gamma$ or $\sum_\gamma c_\gamma h_\gamma$
$\tilde{h}_1$	$(e_{1,1} - e_{8,8})$ $-(e_{2,2} - e_{7,7})$ $+2(e_{3,3} - e_{6,6})$	$h_{r_1} + h_{r_3} + h_{r_4}$
$\tilde{h}_2$	$(e_{2,2} - e_{7,7})$ $-(e_{3,3} - e_{6,6})$	h_{r_2}
e_1	$(e_{1,2} - e_{7,8})$ $+(e_{3,4} - e_{5,6})$ $+(e_{3,5} - e_{4,6})$	$e_{r_1} + e_{r_3} + e_{r_4}$
e_2	$e_{2,3} - e_{6,7}$	e_{r_2}
e_3	$-(e_{1,3} - e_{6,8})$ $+(e_{2,4} - e_{5,7})$ $+(e_{2,5} - e_{4,7})$	$e_{r_1+r_2} + e_{r_2+r_3} + e_{r_2+r_4}$
e_4	$(e_{1,4} - e_{5,8})$ $+(e_{2,6} - e_{3,7})$ $+(e_{1,5} - e_{4,8})$	$e_{r_1+r_2+r_3} + e_{r_2+r_3+r_4} + e_{r_1+r_2+r_4}$
e_5	$e_{1,6} - e_{3,8}$	$e_{r_1+r_2+r_3+r_4}$
e_6	$e_{1,7} - e_{2,8}$	$e_{r_1+2r_2+r_3+r_4}$
f_1	$(e_{2,1} - e_{8,7})$ $+(e_{4,3} - e_{6,5})$ $+(e_{5,3} - e_{6,4})$	$e_{-r_1} + e_{-r_3} + e_{-r_4}$
f_2	$e_{3,2} - e_{7,6}$	e_{-r_2}
f_3	$-(e_{3,1} - e_{8,6})$ $+(e_{4,2} - e_{7,5})$ $+(e_{5,2} - e_{7,4})$	$e_{-r_1-r_2} + e_{-r_2-r_3} + e_{-r_2-r_4}$
f_4	$(e_{4,1} - e_{8,5})$ $+(e_{6,2} - e_{7,3})$ $+(e_{5,1} - e_{8,4})$	$e_{-r_1-r_2-r_3} + e_{-r_2-r_3-r_4} + e_{-r_1-r_2-r_4}$
f_5	$e_{6,1} - e_{8,3}$	$e_{-r_1-r_2-r_3-r_4}$
f_6	$e_{7,1} - e_{8,2}$	$e_{-r_1-2r_2-r_3-r_4}$

3.1.4 Notation/Lemma. *Define two vector subspaces of $\mathcal{L}_{D_4}$*

$$\begin{aligned}\tilde{\mathcal{H}} :=& \mathbb{C}\text{-}span\{\tilde{h}_1, \tilde{h}_2\}\\ =& \left\{\sum_{i=1}^{3} \lambda_i h_i \,\middle|\, \lambda_1 - \lambda_2 - \lambda_3 = 0\right\}\\ =& \left\{\sum_{i=1}^{4} \lambda_i h_i \,\middle|\, \lambda_1 - \lambda_2 - \lambda_3 = 0,\ \lambda_4 = 0\right\} \subseteq \mathcal{H}_{D_4},\\ \tilde{\mathcal{L}} :=& \mathbb{C}\text{-}span\{\tilde{h}_1, \tilde{h}_2, e_i, f_j \mid i, j = 1,2,3,4,5,6\}\\ =& \tilde{\mathcal{H}} \oplus \sum_{i=1}^{6} \mathbb{C}e_i \oplus \sum_{j=1}^{6} \mathbb{C}f_j.\end{aligned}$$

Proof. Let $\tilde{h} = \lambda_1\tilde{h}_1 + \lambda_2\tilde{h}_2 \in \tilde{\mathcal{H}}$, then

$$\tilde{h} = \lambda_1(h_1 - h_2 + 2h_3) + \lambda_2(h_2 - h_3) = \lambda_1 h_1 + (-\lambda_1 + \lambda_2)h_2 + (2\lambda_1 - \lambda_2)h_3,$$

so $\tilde{h} \in \{\sum_{i=1}^{3} \lambda_i h_i \mid \lambda_1 - \lambda_2 - \lambda_3 = 0\}$.

Conversely, let $\tilde{h} = \sum_{i=1}^{3} \lambda_i h_i \in \{\sum_{i=1}^{3} \lambda_i h_i \mid \lambda_1 - \lambda_2 - \lambda_3 = 0\}$, then

$$\begin{aligned}\tilde{h} =& \sum_{i=1}^{3} \lambda_i h_i = \lambda_1 h_1 + \lambda_2 h_2 + (\lambda_1 - \lambda_2)h_3\\ =& \lambda_1(h_1 - h_2 + 2h_3) + \lambda_1 h_2 - 2\lambda_1 h_3 + \lambda_2 h_2 + (\lambda_1 - \lambda_2)h_3\\ =& \lambda_1\tilde{h}_1 + (\lambda_1 + \lambda_2)\tilde{h}_2 \in \tilde{\mathcal{H}}.\end{aligned}$$

Hence $\tilde{\mathcal{H}} = \{\sum_{i=1}^{3} \lambda_i h_i \mid \lambda_1 - \lambda_2 - \lambda_3 = 0\}$. □

Motivated by [HRT01, §3.4 and §3.6], we construct a Lie algebra of type G_2 which is a subalgebra of $\mathcal{L}_{D_4}$.

3.1.5 Proposition (Lie algebra of type G_2). *(1) $\tilde{\mathcal{L}}$ is a 14-dimensional subalgebra of the Lie algebra $\mathcal{L}_{D_4}$,*

(2) $\tilde{\mathcal{L}}$ is a Lie algebra of type G_2,

(3) $\{\tilde{h}_k \mid k = 1, 2\} \cup \{e_i, f_i \mid i = 1, 2, \ldots, 6\}$ is a Chevalley basis of $\tilde{\mathcal{L}}$.

Proof. (The proof is an adaption of [Car05, §8.2] and [Car72, §11.2].)

(1) *We claim that $\tilde{\mathcal{L}}$ is closed under the Lie bracket* $[\ ,\ ]$.

Thus $[h, h'] = 0$ for all $h, h' \in \tilde{\mathcal{H}}$. Let $\tilde{h} := \sum_{i=1}^{3} \lambda_i h_i \in \tilde{\mathcal{H}}$, then

$$\begin{aligned}
[\tilde{h}, e_1] =& (\lambda_1 - \lambda_2)e_{r_1} + (\lambda_3 - \lambda_4)e_{r_3} + (\lambda_3 + \lambda_4)e_{r_4} \\
\overset{\lambda_4=0}{=}& (\lambda_1 - \lambda_2)e_{r_1} + \lambda_3 e_{r_3} + \lambda_3 e_{r_4} \\
\overset{\lambda_1-\lambda_2-\lambda_3=0}{=}& (\lambda_1 - \lambda_2)(e_{r_1} + e_{r_3} + e_{r_4}) \\
=& \frac{\lambda_1 - \lambda_2 + 2\lambda_3}{3} e_1, \\
[\tilde{h}, e_2] =& [\tilde{h}, e_2] = (\lambda_2 - \lambda_3)e_{r_2} = (\lambda_2 - \lambda_3)e_2.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
[\tilde{h}, e_3] =& \frac{\lambda_1 + 2\lambda_2 - \lambda_3}{3} e_3, & [\tilde{h}, e_4] =& \frac{2\lambda_1 + \lambda_2 + \lambda_3}{3} e_4, \\
[\tilde{h}, e_5] =& (\lambda_1 + \lambda_3)e_5, & [\tilde{h}, e_6] =& (\lambda_1 + \lambda_2)e_6.
\end{aligned}$$

Then

$$\begin{aligned}
[\tilde{h}, f_1] =& -\frac{\lambda_1 - \lambda_2 + 2\lambda_3}{3} f_1, & [\tilde{h}, f_2] =& -(\lambda_2 - \lambda_3)f_2, & [\tilde{h}, f_3] =& -\frac{\lambda_1 + 2\lambda_2 - \lambda_3}{3} f_3, \\
[\tilde{h}, f_4] =& -\frac{2\lambda_1 + \lambda_2 + \lambda_3}{3} f_4 & [\tilde{h}, f_5] =& -(\lambda_1 + \lambda_3)f_5, & [\tilde{h}, f_6] =& -(\lambda_1 + \lambda_2)f_6.
\end{aligned}$$

For $1 \leq i \leq j \leq 6$, by A.5.6 and 2.1.1, the non-trivial Lie brackets $[e_i, e_j]$ and $[f_i, f_j]$ are

$$\begin{aligned}
[e_1, e_2] =& -e_3, & [e_1, e_3] =& 2e_4, & [e_1, e_4] =& 3e_5, & [e_2, e_5] =& e_6, & [e_3, e_4] =& e_6, \\
[f_1, f_2] =& f_3, & [f_1, f_3] =& -2f_4, & [f_1, f_4] =& -3f_5, & [f_2, f_5] =& -f_6, & [f_3, f_4] =& -f_6.
\end{aligned}$$

The non-trivial $[e_i, f_i]$ $(1 \leq i \leq 6)$ are determined as follows:

$$\begin{aligned}
[e_1, f_1] =& h_1 - h_2 + 2h_3 & =& \tilde{h}_1 & =& h_{r_1} + h_{r_3} + h_{r_4}, \\
[e_2, f_2] =& h_2 - h_3 & =& \tilde{h}_2 & =& h_{r_2}, \\
\tilde{h}_3 := [e_3, f_3] =& h_1 + 2h_2 - h_3 & =& \tilde{h}_1 + 3\tilde{h}_2 & =& h_{r_1+r_2} + h_{r_2+r_3} + h_{r_2+r_4}, \\
\tilde{h}_4 := [e_4, f_4] =& 2h_1 + h_2 + h_3 & =& 2\tilde{h}_1 + 3\tilde{h}_2 & =& h_{r_1+r_2+r_3} + h_{r_2+r_3+r_4} + h_{r_1+r_2+r_4}, \\
\tilde{h}_5 := [e_5, f_5] =& h_1 + h_3 & =& \tilde{h}_1 + \tilde{h}_2 & =& h_{r_1+r_2+r_3+r_4}, \\
\tilde{h}_6 := [e_6, f_6] =& h_1 + h_2 & =& \tilde{h}_1 + 2\tilde{h}_2 & =& h_{r_1+2r_2+r_3+r_4}.
\end{aligned}$$

The non-trivial $[e_i, f_j]$ $(1 \leq i \leq j \leq 6)$ are determined.

$$\begin{aligned}
[e_1, f_3] =& 3f_2, & [e_1, f_4] =& -2f_3, & [e_1, f_5] =& -f_4, & [e_2, f_3] =& -f_1, & [e_2, f_6] =& -f_5, \\
[e_3, f_4] =& 2f_1, & [e_3, f_6] =& -f_4, & [e_4, f_5] =& f_1, & [e_4, f_6] =& f_2, & [e_5, f_6] =& f_1.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&[f_1,e_3]=3e_2, \quad [f_1,e_4]=-2e_3, \quad [f_1,e_5]=-e_4, \quad [f_2,e_3]=-e_1, \quad [f_2,e_6]=-e_5,\\
&[f_3,e_4]=2e_1, \quad [f_3,e_6]=-e_4, \quad [f_4,e_5]=e_1, \quad [f_4,e_6]=e_2, \quad [f_5,e_6]=e_1.
\end{aligned}$$

Thus, $\tilde{\mathcal{L}}$ is a subalgebra of the Lie algebra $\mathcal{L}_{D_4}$ with $\dim\tilde{\mathcal{L}}=14$.

(2) Let $\tilde{h}:=\sum_{i=1}^{3}\lambda_i h_i=\lambda_1\tilde{h}_1+(\lambda_1+\lambda_2)\tilde{h}_2\in\tilde{\mathcal{H}}$. By (1) we have

$$\begin{aligned}
[\tilde{h},e_1]&=\frac{\lambda_1-\lambda_2+2\lambda_3}{3}e_1, & [\tilde{h},f_1]&=-\frac{\lambda_1-\lambda_2+2\lambda_3}{3}f_1,\\
[\tilde{h},e_2]&=(\lambda_2-\lambda_3)e_2, & [\tilde{h},f_2]&=-(\lambda_2-\lambda_3)f_2,\\
[\tilde{h},e_3]&=\frac{\lambda_1+2\lambda_2-\lambda_3}{3}e_3, & [\tilde{h},f_3]&=-\frac{\lambda_1+2\lambda_2-\lambda_3}{3}f_3,\\
[\tilde{h},e_4]&=\frac{2\lambda_1+\lambda_2+\lambda_3}{3}e_4, & [\tilde{h},f_4]&=-\frac{2\lambda_1+\lambda_2+\lambda_3}{3}f_4,\\
[\tilde{h},e_5]&=(\lambda_1+\lambda_3)e_5, & [\tilde{h},f_5]&=-(\lambda_1+\lambda_3)f_5,\\
[\tilde{h},e_6]&=(\lambda_1+\lambda_2)e_6, & [\tilde{h},f_6]&=-(\lambda_1+\lambda_2)f_6.
\end{aligned}$$

The functions $\alpha,\beta\colon\tilde{\mathcal{H}}\to\mathbb{C}$ are given by

$$\alpha(\tilde{h})=\frac{\lambda_1-\lambda_2+2\lambda_3}{3}, \qquad \beta(\tilde{h})=\lambda_2-\lambda_3,$$

then set $\tilde{\Phi}:=\pm\{\alpha,\ \beta,\ \alpha+\beta,\ 2\alpha+\beta,\ 3\alpha+\beta,\ 3\alpha+2\beta\}$. Thus we write $[\tilde{h},e_{\tilde{\alpha}}]=\tilde{\alpha}(\tilde{h})e_{\tilde{\alpha}}$ and $\tilde{\mathcal{L}}_{\tilde{\alpha}}=\mathbb{C}e_{\tilde{\alpha}}$ for all $\tilde{\alpha}\in\tilde{\Phi}$ and $e_{\tilde{\alpha}}\in\{e_i,\ f_i\mid i=1,2,\dots,6\}$.

We claim that $\tilde{\mathcal{H}}$ is a Cartan subalgebra. By (1) $\tilde{\mathcal{H}}$ is abelian. Now it is sufficient to show that

$$\tilde{\mathcal{H}}=N_{\tilde{\mathcal{L}}}(\tilde{\mathcal{H}})=\{x\in\tilde{\mathcal{L}}\mid [x,h]\in\tilde{\mathcal{H}},\ \forall\, h\in\tilde{\mathcal{H}}\}.$$

Let $x\in N_{\tilde{\mathcal{L}}}(\tilde{\mathcal{H}})$, then $x=h'+\sum_{i=1}^{6}(a_ie_i+b_if_i)$ for $h'\in\tilde{\mathcal{H}}$ and $a_i,b_i\in\mathbb{C}$. Let $h=4h_1+3h_2+h_3\in\tilde{\mathcal{H}}$, then

$$\begin{aligned}
[h,x]&=(a_1e_1-b_1f_1)+2(a_2e_2-b_2f_2)+3(a_3e_3-b_3f_3)\\
&\quad+4(a_4e_4-b_4f_4)+5(a_5e_5-b_5f_5)+7(a_6e_6-b_6f_6)\in\tilde{\mathcal{H}}\\
\Longrightarrow\ & a_i=b_i=0 \text{ for all } i=1,2,3,4,5,6.
\end{aligned}$$

Thus $x\in\tilde{\mathcal{H}}$. Thus $\tilde{\mathcal{H}}$ is a Cartan subalgebra of $\tilde{\mathcal{L}}$. Hence

$$\tilde{\mathcal{L}}=\tilde{\mathcal{H}}\oplus\sum_{i=1}^{6}\mathbb{C}e_i\oplus\sum_{j=1}^{6}\mathbb{C}f_j=\tilde{\mathcal{H}}\oplus\sum_{\tilde{\alpha}\in\tilde{\Phi}}\mathbb{C}e_{\tilde{\alpha}}.$$

is a Cartan decomposition with respect to $\tilde{\mathcal{H}}$.

We claim that $\tilde{\mathcal{L}}$ *is semisimple*. Suppose there exists a non-zero ideal $\tilde{I}$ of $\tilde{\mathcal{L}}$. Then $[\tilde{\mathcal{H}}, \tilde{I}] \subseteq \tilde{I}$. We may regard $\tilde{I}$ as a $\tilde{\mathcal{H}}$-module and decompose it into weight spaces as follows:

$$\tilde{I} = (\tilde{\mathcal{H}} \cap \tilde{I}) \oplus \sum_{\tilde{\alpha} \in \tilde{\Phi}} (\mathbb{C}e_{\tilde{\alpha}} \cap \tilde{I}).$$

Let $x \in \tilde{I}$, then $x = x_0 + \sum_{\tilde{\alpha} \in \tilde{\Phi}} x_{\tilde{\alpha}}$ where $x_0 \in \tilde{\mathcal{H}}$ and $x_{\tilde{\alpha}} \in \tilde{\mathcal{L}}_{\tilde{\alpha}}$. We verify that $x_0 \in \tilde{I}$ and $x_{\tilde{\alpha}} \in \tilde{I}$. Let $\tilde{\alpha}_0 \in \tilde{\Phi}$ and $h = 4\tilde{h}_1 + 7\tilde{h}_2 = 4h_1 + 3h_2 + h_3 \in \tilde{\mathcal{H}}$, then $\tilde{\alpha}_0(h) \neq 0$ and $\tilde{\beta}(h) \neq \tilde{\alpha}_0(h)$ for all $\tilde{\beta} \in \tilde{\Phi}$ with $\tilde{\beta} \neq \tilde{\alpha}$. Then

$$\operatorname{ad} h \prod_{\substack{\tilde{\beta} \in \tilde{\Phi} \\ \tilde{\beta} \neq \tilde{\alpha}_0}} (\operatorname{ad} h - \tilde{\beta}(h)\mathrm{id}_{\tilde{\mathcal{L}}})(x) = \tilde{\alpha}_0(h) \prod_{\substack{\tilde{\beta} \in \tilde{\Phi} \\ \tilde{\beta} \neq \tilde{\alpha}_0}} (\tilde{\alpha}_0(h) - \tilde{\beta}(h))x_{\tilde{\alpha}_0} \in \tilde{I},$$

so $x_{\tilde{\alpha}_0} \in \tilde{I}$. Thus $x_0 \in \tilde{I}$. Hence

$$\tilde{I} = (\tilde{\mathcal{H}} \cap \tilde{I}) \oplus \sum_{\tilde{\alpha} \in \tilde{\Phi}} (\mathbb{C}e_{\tilde{\alpha}} \cap \tilde{I}).$$

We claim that $\mathbb{C}e_{\tilde{\alpha}} \cap \tilde{I} = \{0\}$. Suppose that $\mathbb{C}e_{\tilde{\alpha}} \cap \tilde{I} \neq \{0\}$ for some $\tilde{\alpha} \in \tilde{\Phi}$, then $e_{\tilde{\alpha}} \in \tilde{I}$. Then $\tilde{h}_{\tilde{\alpha}} = [e_{\tilde{\alpha}}, e_{-\tilde{\alpha}}] \in \tilde{I}$ and $[\tilde{h}_{\tilde{\alpha}}, e_{\tilde{\alpha}}] = 2e_{\tilde{\alpha}}$. This is a contradiction to that $\tilde{I}$ is abelian. Thus $\mathbb{C}e_{\tilde{\alpha}} \cap \tilde{I} = \{0\}$ and $\tilde{I} \subseteq \tilde{\mathcal{H}}$. Let $x \in \tilde{I}$, then $[x, e_{\tilde{\alpha}}] = \tilde{\alpha}(x)e_{\tilde{\alpha}} \in \tilde{I}$ for all $\tilde{\alpha} \in \tilde{\Phi}$. Thus $\tilde{\alpha}(x) = 0$. Thus $x = 0$. Hence $\tilde{I} = \{0\}$, a contradiction. Therefore, $\tilde{\mathcal{L}}$ is semisimple.

The functions $\tilde{\alpha} \in \tilde{\Phi}$ are the roots of $\tilde{\mathcal{L}}$ with respect to $\tilde{\mathcal{H}}$. A system of fundamental roots is given by $\{\alpha,\ \beta\}$, since all the other roots are integeral combinations of these with coefficients all non-negative or non-positive. Thus the set of the roots is $\tilde{\Phi}$. We determine the Cartan matrix of $\tilde{\mathcal{L}}$. The α-chain of roots through β is

$$\beta,\ \beta + \alpha,\ \beta + 2\alpha,\ \beta + 3\alpha.$$

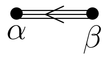
Then β-chain of roots through α is

$$\alpha,\ \alpha + \beta.$$

Then $A_{\alpha,\beta} = 0 - 3 = -3$ and $A_{\beta,\alpha} = -1$. Thus the Cartan matrix of $\tilde{\mathcal{L}}$ is

$$\begin{array}{c c} & \begin{array}{cc} \alpha & \beta \end{array} \\ \begin{array}{c} \alpha \\ \beta \end{array} & \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \end{array}$$

It is a Cartan matrix of type G_2. The Dynkin diagram of G_2 is



Then the Lie algebra $\tilde{\mathcal{L}}$ is simple since the Cartan matrix is indecomposable. Therefore, $\tilde{\mathcal{L}}$ is a simple Lie algebra of type G_2.

(3) The co-roots of $\tilde{\mathcal{L}}$ are $\tilde{h}_i = [e_i, f_i]$ $(i = 1, 2, \ldots, 6)$. For in this case $[\tilde{h}_i, e_i] = 2e_i$ with $i = 1, 2, \ldots, 6$. We know that $\tilde{\theta}(x) = -x^\top$ is an automorphism of $\tilde{\mathcal{L}}$ with $\tilde{\theta}(e_i) = -f_i$. Hence $[e_{\tilde{\alpha}}, e_{\tilde{\beta}}] = \pm(n_{\tilde{\alpha},\tilde{\beta}} + 1)e_{\tilde{\alpha}+\tilde{\beta}}$. Thus the fundamental co-roots $\tilde{h}_k$ $(k = 1, 2)$ together with e_i, f_i $(i = 1, 2, \ldots, 6)$ form a Chevalley basis of the Lie algebra $\tilde{\mathcal{L}}$.

□

3.1.6 Notation/Lemma. *Let $\mathcal{L}_{G_2} := \tilde{\mathcal{L}}$ and $\mathcal{H}_{G_2} := \tilde{\mathcal{H}}$. Then*

$$\mathcal{L}_{G_2} = \mathcal{H}_{G_2} \oplus \sum_{i=1}^{6} \mathbb{C}e_i \oplus \sum_{j=1}^{6} \mathbb{C}f_j.$$

Let $\mathcal{V}_{G_2} := \langle \mathcal{H}^*_{G_2} \rangle_{\mathbb{R}}$, then $\Delta_{G_2} = \{\alpha,\ \beta\}$ is a basis of $\mathcal{V}_{G_2}$ and $(\alpha, \alpha) = \frac{2}{24}$, $(\alpha, \beta) = -\frac{3}{24}$, $(\beta, \beta) = \frac{6}{24}$. The set of the root is

$$\Phi_{G_2} = \pm\{\alpha,\ \beta,\ \alpha + \beta,\ 2\alpha + \beta,\ 3\alpha + \beta,\ 3\alpha + 2\beta\}.$$

The set of positive roots is denoted by

$$\Phi^+_{G_2} = \{\alpha,\ \beta,\ \alpha + \beta,\ 2\alpha + \beta,\ 3\alpha + \beta,\ 3\alpha + 2\beta\}.$$

3.1.7 Notation. *Set*

$$\begin{aligned}
h_\alpha &:= \tilde{h}_1 & e_\alpha &:= e_1, & e_{-\alpha} &:= f_1,\\
h_\beta &:= \tilde{h}_2 & e_\beta &:= e_2, & e_{-\beta} &:= f_2,\\
h_{\alpha+\beta} &:= \tilde{h}_3 & e_{\alpha+\beta} &:= e_3, & e_{-(\alpha+\beta)} &:= f_3,\\
h_{2\alpha+\beta} &:= \tilde{h}_4 & e_{2\alpha+\beta} &:= e_4, & e_{-(2\alpha+\beta)} &:= f_4,\\
h_{3\alpha+\beta} &:= \tilde{h}_5 & e_{3\alpha+\beta} &:= e_5, & e_{-(3\alpha+\beta)} &:= f_5,\\
h_{3\alpha+2\beta} &:= \tilde{h}_6 & e_{3\alpha+2\beta} &:= e_6, & e_{-(3\alpha+2\beta)} &:= f_6.
\end{aligned}$$

3.1.8 Corollary. $\{h_\alpha, h_\beta\} \cup \{e_{\pm r} \mid r \in \Phi^+_{G_2}\}$ *is a Chevalley basis of* $\mathcal{L}_{G_2}$.

3.1.9 Definition. *Let $r := x_1\alpha + x_2\beta \in \mathcal{V}_{G_2}$, $s := y_1\alpha + y_2\beta \in \mathcal{V}_{G_2}$, then we write*

$$r \prec s,$$

if $\sum_{i=1}^2 x_i < \sum_{i=1}^2 y_i$, or if $\sum_{i=1}^2 x_i = \sum_{i=1}^2 y_i$ and the first non-zero coefficient $x_i - y_i$ is positive.

The total order on $\Phi^+_{G_2}$ is determined:

$$0 \prec \alpha \prec \beta \prec \alpha+\beta \prec 2\alpha+\beta \prec 3\alpha+\beta \prec 3\alpha+2\beta.$$

3.1.10 Corollary. *The Lie algebra* $\mathcal{L}_{G_2}$ *has the following structure constants:*

extraspecial pair (r,s)	$N_{r,s}$
(α,β)	-1
$(\alpha,\alpha+\beta)$	2
$(\alpha,2\alpha+\beta)$	3
$(\beta,3\alpha+\beta)$	1

3.1.2. Sylow p-subgroup $G_2^{syl}(q)$

In this subsection, we get the Sylow p-subgroup $G_2^{syl}(q)$ of the Chevalley group $G_2(q)$, and determine the commutators of $G_2^{syl}(q)$.

By calculation, we get the following formulae:

3.1.11 Lemma.

$$\begin{aligned} e_\alpha^2 &=-2e_{3,6}, & e_{\alpha+\beta}^2 &=-2e_{2,7}, & e_{2\alpha+\beta}^2 &=-2e_{1,8},\\ e_\alpha^3 &=0, & e_{\alpha+\beta}^3 &=0, & e_{2\alpha+\beta}^3 &=0,\\ e_\beta^2 &=0, & e_{3\alpha+\beta}^2 &=0, & e_{3\alpha+2\beta}^2 &=0. \end{aligned}$$

Similar to §11.3 of [Car72], the coefficients of $e_{i,j}$ in $\exp(te_r)=I_8+te_r+\frac{1}{2}t^2e_r^2$ for all $r\in\Phi_{G_2}$ are of the form ±1, $\pm t$ or $\pm t^2$ (i.e. the non-zero entries of $\exp(te_r)$ are from ±1, $\pm t$ and $\pm t^2$). This is because the coefficient of e_r^2 with $r\in\Phi_{G_2}$ are all divisible by 2. This fact enables us to transfer to an arbitrary field. For each matrix e_r in the above representation and each element t in an arbitrary field K, $\exp(te_r)$ is a well-defined non-singular matrix over K. We are interested in the Chevalley group of type $\mathcal{L}_{G_2}$ over the finite field $\mathbb{F}_q$ with $\operatorname{Char}\mathbb{F}_q\neq 2$.

3.1.12 Notation. *The Chevalley group of type $\mathcal{L}_{G_2}$ over the field $\mathbb{F}_q$ is denoted by*

$$G_2(q):=\langle\ \exp(t\operatorname{ad}e_r)\,|\,r\in\Phi_{G_2},\ t\in\mathbb{F}_q\,\rangle\,,$$

and its Sylow p-subgroup is $U_{G_2}(q):=\left\langle\ \exp(t\operatorname{ad}e_r)\,\middle|\,r\in\Phi_{G_2}^+,\ t\in\mathbb{F}_q\right\rangle$.

3.1.13 Notation. *For all $r\in\Phi_{G_2}$ and $t\in\mathbb{F}_q$, set $y_r(t):=\exp(te_r)=I_8+te_r+\frac{1}{2}t^2e_r^2$.*

3.1.14 Notation. *Write $\bar{U}_{G_2}:=\left\langle y_r(t)\,\middle|\,r\in\Phi_{G_2}^+,\ t\in\mathbb{F}_q\right\rangle$.*

3.1.15 Proposition. *The root subgroups of $\bar{U}_{G_2}(q)$ are given as*

$$Y_i:=\{y_i(t)\,|\,t\in\mathbb{F}_q\}\,,\quad i=1,2,3,4,5,6,$$

where

$$\begin{aligned} y_1(t):=&y_\alpha(t)=I_8+te_\alpha+\frac{1}{2}t^2e_\alpha^2\\ =&\begin{pmatrix} 1 & t & & & & & & \cdot\\ & 1 & & & & & \cdot & \\ & & 1 & t & t & -t^2 & & \\ & & & 1 & \cdot & -t & & \\ & & & & 1 & -t & & \\ & & & & & 1 & & \\ & & & & & & 1 & -t\\ & & & & & & & 1 \end{pmatrix}\\ =&x_{\varepsilon_1-\varepsilon_2}(t)x_{\varepsilon_3-\varepsilon_4}(t)x_{\varepsilon_3+\varepsilon_4}(t)\\ =&x_1(t), \end{aligned}$$

$$\begin{aligned}
y_2(t) :=& y_\beta(t) = I_8 + te_\beta \\
=& \begin{pmatrix}
1 & & & & & & & \cdot \\
 & 1 & t & & & & \cdot & \\
 & & 1 & & & \cdot & & \\
 & & & 1 & \cdot & & & \\
 & & & & 1 & & & \\
 & & & & & 1 & -t & \\
 & & & & & & 1 & \\
 & & & & & & & 1
\end{pmatrix} \\
=& x_{\varepsilon_2-\varepsilon_3}(t) \\
=& x_2(t),
\end{aligned}$$

$$\begin{aligned}
y_3(t) :=& y_{\alpha+\beta}(t) = I_8 + te_{\alpha+\beta} + \frac{1}{2}t^2 e_{\alpha+\beta}^2 \\
=& \begin{pmatrix}
1 & & -t & & & & & \cdot \\
 & 1 & & t & t & & -t^2 & \\
 & & 1 & & & \cdot & & \\
 & & & 1 & \cdot & & -t & \\
 & & & & 1 & & -t & \\
 & & & & & 1 & & t \\
 & & & & & & 1 & \\
 & & & & & & & 1
\end{pmatrix} \\
=& x_{\varepsilon_1-\varepsilon_3}(t) x_{\varepsilon_2-\varepsilon_4}(t) x_{\varepsilon_2+\varepsilon_4}(t) \\
=& x_3(t),
\end{aligned}$$

$$\begin{aligned}
y_4(t) :=& y_{2\alpha+\beta}(t) = I_8 + te_{2\alpha+\beta} + \frac{1}{2}t^2 e_{2\alpha+\beta}^2 \\
=& \begin{pmatrix}
1 & & & t & t & & & -t^2 \\
 & 1 & & & & t & \cdot & \\
 & & 1 & & & \cdot & -t & \\
 & & & 1 & \cdot & & & -t \\
 & & & & 1 & & & -t \\
 & & & & & 1 & & \\
 & & & & & & 1 & \\
 & & & & & & & 1
\end{pmatrix} \\
=& x_{\varepsilon_1-\varepsilon_4}(t) x_{\varepsilon_2+\varepsilon_3}(t) x_{\varepsilon_1+\varepsilon_4}(t) \\
=& x_4(t),
\end{aligned}$$

$$\begin{aligned} y_5(t) :=& y_{3\alpha+\beta}(t) = I_8 + te_{3\alpha+\beta} \\ =& \begin{pmatrix} 1 & & & & & t & & \cdot \\ & 1 & & & & & \cdot & \\ & & 1 & & & \cdot & & -t \\ & & & 1 & \cdot & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{pmatrix} \\ =& x_{\varepsilon_1+\varepsilon_3}(t) \\ =& x_5(t), \end{aligned}$$

$$\begin{aligned} y_6(t) :=& y_{3\alpha+2\beta}(t) = I_8 + te_{3\alpha+2\beta} \\ =& \begin{pmatrix} 1 & & & & & & t & \cdot \\ & 1 & & & & & \cdot & -t \\ & & 1 & & & \cdot & & \\ & & & 1 & \cdot & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{pmatrix} \\ =& x_{\varepsilon_1+\varepsilon_2}(t) \\ =& x_6(t). \end{aligned}$$

3.1.16 Corollary. *$Y_i \leqslant X_i$ for all $i = 1, 3, 4$, $Y_i = X_i$ for all $i = 2, 5, 6$, and $Y_i \leqslant {}^3D_4^{syl}(q^3)$.*

3.1.17 Reminder. *We note that $y_i(t) = x_i(t)$ for all $t \in \mathbb{F}_q$ and $i = 1, 2, \ldots, 6$. Thus we also write $y_i(t) = x_i(t)$ for all $t \in \mathbb{F}_q$ and $i = 1, 2, \ldots, 6$ in the rest of the section.*

Then we get the following lemma.

3.1.18 Lemma. *$\bar{U}_{G_2} \leqslant {}^3D_4^{syl}(q^3) \leqslant D_4^{syl}(q^3)$ and $\bar{U}_{G_2} \leqslant D_4^{syl}(q)$.*

We get the following proposition.

3.1.19 Proposition.

$$\bar{U}_{G_2} = \left\{ \prod_{r \in \Phi^+_{G_2}} y_r(t_r) \,\middle|\, t_r \in \mathbb{F}_q, r \in \Phi^+_{G_2} \right\} = \left\{ \prod_{i \in \{1,2,\ldots,6\}} x_i(t_i) \,\middle|\, t_i \in \mathbb{F}_q \right\}$$

where the product can be taken in an arbitrary, but fixed, order.

3.1.20 Corollary. $|\bar{U}_{G_2}| = q^6$.

3.1.21 Reminder. *Let $p > 2$, then*

$$\begin{aligned}
|G_2(q)| =& q^6(q^2-1)(q^6-1),\\
|U_{G_2}| =& q^6,\\
|D_4(q)| =& \frac{1}{4}q^{12}(q^2-1)(q^4-1)(q^6-1)(q^4-1),\\
|D_4\left(q^3\right)| =& \frac{1}{4}q^{36}(q^6-1)(q^{12}-1)(q^{18}-1)(q^{12}-1),\\
|D_4^{syl}(q)| =& q^{12},\\
|{}^3D_4\left(q^3\right)| =& q^{12}(q^2-1)(q^6-1)(q^8+q^4+1),\\
|{}^3D_4^{syl}\left(q^3\right)| =& q^{12}.
\end{aligned}$$

3.1.22 Proposition. *Let*

$$\sigma_{\bar{U}_{G_2}} : \bar{U}_{G_2} \to U_{G_2} : \exp(te_r) \mapsto \exp(t\,\mathrm{ad}\, e_r),$$

where $r \in \Phi^+_{G_2}$ and $t \in K$. Then $\sigma_{\bar{U}_{n,K}}$ is a group isomorphism map.

Proof. We know $\sigma_{\bar{U}_{G_2}}$ is a group epimorphism. Since $|\bar{U}_{G_2}| = |U_{G_2}| = q^6$, $\sigma_{\bar{U}_{G_2}}$ is an isomorphism. □

3.1.23 Notation. *Set $G_2^{syl}(q) := \bar{U}_{G_2}$.*

3.1.24 Definition. *A subgroup $P \leqslant G_2^{syl}(q)$ is a **pattern subgroup**, if it is generated by some root subgroups, i.e. $P := \langle Y_i \,|\, i \in I \subseteq \{1,2,\dots,6\} \rangle \leqslant G_2^{syl}(q)$.*

We get the commutators of $G_2^{syl}(q)$ by calculation.

3.1.25 Proposition. *Let $t_1, t_2, t_3, t_4, t_5, t_6 \in \mathbb{F}_q$ and define the commutators*

$$[x_i(t_i), x_j(t_j)] := x_i(t_i)^{-1}x_j(t_j)^{-1}x_i(t_i)x_j(t_j)$$

Then the non-trivial commutators of $G_2^{syl}(q)$ are determined as follows:

$$\begin{aligned}
[x_1(t_1), x_2(t_2)] =& x_3(-t_2t_1)\cdot x_4(t_2t_1^2)\cdot x_5(-t_2t_1^3)\cdot x_6(2t_2^2t_1^3),\\
[x_2(t_2), x_1(t_1)] =& x_3(t_2t_1)\cdot x_4(-t_2t_1^2)\cdot x_5(t_2t_1^3)\cdot x_6(t_2^2t_1^3),\\
[x_1(t_1), x_3(t_3)] =& x_4(2t_1t_3)\cdot x_5(-3t_1^2t_3)\cdot x_6(-3t_1t_3^2),\\
[x_1(t_1), x_4(t_4)] =& x_5(3t_1t_4),\\
[x_3(t_3), x_4(t_4)] =& x_6(3t_3t_4),\\
[x_2(t_2), x_5(t_5)] =& x_6(t_2t_5).
\end{aligned}$$

In particular, let $\mathrm{Char}\mathbb{F}_q = 3$*, then the commutators are*

$$\begin{aligned}
[x_1(t_1), x_2(t_2)] =& x_3(-t_2t_1) \cdot x_4(t_2t_1^2) \cdot x_5(-t_2t_1^3) \cdot x_6(2t_2^2t_1^3),\\
[x_2(t_2), x_1(t_1)] =& x_3(t_2t_1) \cdot x_4(-t_2t_1^2) \cdot x_5(t_2t_1^3) \cdot x_6(t_2^2t_1^3),\\
[x_1(t_1), x_3(t_3)] =& x_4(2t_1t_3),\\
[x_2(t_2), x_5(t_5)] =& x_6(t_2t_5).
\end{aligned}$$

3.1.26 Notation. *Let* $t_i \in \mathbb{F}_q$ *with* $i \in \{1, 2, \dots, 6\}$*, write*

$$\begin{aligned}
y(t_1, t_2, t_3, t_4, t_5, t_6) :=& y_2(t_2)y_1(t_1)y_3(t_3)y_4(t_4)y_5(t_5)y_6(t_6)\\
=& x(t_1, t_2, t_3, t_4, t_5, t_6) \in G_2^{syl}(q).
\end{aligned}$$

3.1.27 Proposition (Sylow p-subgroup $G_2^{syl}(q)$)**.** *A Sylow* p*-subgroup* $G_2^{syl}(q)$ *of the Chevalley group* $G_2(q)$ *is written as follows:*

$$\begin{aligned}
G_2^{syl}(q) = \bar{U}_{G_2} &= \{y(t_1, t_2, t_3, t_4, t_5, t_6) \mid t_1, t_2, t_3, t_4, t_5, t_6 \in \mathbb{F}_q\}\\
&= \{x(t_1, t_2, t_3, t_4, t_5, t_6) \mid t_1, t_2, t_3, t_4, t_5, t_6 \in \mathbb{F}_q\},
\end{aligned}$$

where

$$y(t_1, t_2, t_3, t_4, t_5, t_6)$$
$$= \begin{pmatrix}
1 & t_1 & -t_3 & t_1t_3+t_4 & t_1t_3+t_4 & \begin{matrix} t_1t_4 \\ +t_5 \end{matrix} & \begin{matrix} -t_1t_3^2 \\ +t_3t_4+t_6 \end{matrix} & \begin{matrix} -2t_1t_3t_4-t_1t_6 \\ +t_3t_5-t_4^2 \end{matrix} \\
 & 1 & t_2 & t_1t_2+t_3 & t_1t_2+t_3 & \begin{matrix} -t_1^2t_2 \\ +t_4 \end{matrix} & \begin{matrix} -2t_1t_2t_3 \\ -t_2t_4-t_3^2 \end{matrix} & \begin{matrix} -t_1^2t_2t_3-2t_1t_2t_4 \\ -t_2t_5-2t_3t_4-t_6 \end{matrix} \\
 & & 1 & t_1 & t_1 & -t_1^2 & -2t_1t_3-t_4 & -t_1^2t_3-2t_1t_4-t_5 \\
 & & & 1 & 0 & -t_1 & -t_3 & -t_1t_3-t_4 \\
 & & & & 1 & -t_1 & -t_3 & -t_1t_3-t_4 \\
 & & & & & 1 & -t_2 & t_1t_2+t_3 \\
 & & & & & & 1 & -t_1 \\
 & & & & & & & 1
\end{pmatrix}.$$

Proof. By 3.1.19 and 3.1.22, $G_2^{syl}(q)$ is a Sylow p-subgroup of $G_2(q)$. By calculation, we get the matrix form as claimed. $\square$

3.1.28 Corollary. $G_2^{syl}(q) \leqslant {}^3D_4^{syl}(q^3) \leqslant D_4^{syl}(q^3) \leqslant A_8(q^3)$ *and* $G_2^{syl}(q) \leqslant D_4^{syl}(q) \leqslant D_4^{syl}(q^3) \leqslant A_8(q^3)$.

3.2. Monomial $G_2^{syl}(q)$-module

Let $N = 8$, $n = 4$, $G := G_8(q)$ and $U := G_2^{syl}(q)$.

In this section, we construct a bigger group $G_8(q)$ such that $G_2^{syl}(q) \leqslant G_8(q) \leqslant A_8(q)$, and determine a monomial linearisation $(f, \kappa|_{V\times V})$ for $G_8(q)$, where f is a surjective 1-cocycle and $\kappa|_{V\times V}$ is a non-degenerate bilinear form (3.2.29). In particular, $f|_U$ is bijective (3.2.28). Then we give classifications of U-orbit modules (3.2.37). By the homomorphisms between orbit modules, a new classification of U-orbit modules is established in which every U-orbit module is isomorphic to some hook-separated staircase module (3.2.57). Finally, some irreducible modules are determined, and any two orbit modules from different families are orthogonal (3.2.61).

3.2.1. The group $G_8(q)$

In this subsection, we construct a group $G_8(q)$ such that $G_2^{syl}(q) \leqslant G_8(q) \leqslant A_8(q)$.

For $t \in \mathbb{F}_q$, we have $x_{i,j}(t) = 1 + te_{i,j} - te_{\bar{j},\bar{i}} \in G_2^{syl}(q)$ for all $(i, j) \in \triangledown$.

3.2.1 Definition/Lemma (A bigger group $G_8(q)$). *We set*

$$G_8(q) := \left\{ u = (u_{i,j}) \in A_8(q) \;\middle|\; \begin{cases} u_{i,j} = 0 & \text{if } (i,j) = (4,5) \\ u_{i,j+1} = u_{i,j} & \text{if } (i,j) \in \{(2,4),(3,4)\} \\ u_{i-1,j} = u_{i,j} & \text{if } (i,j) \in \{(5,6),(5,7)\} \end{cases} \right\}$$

$$= \left\{ \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & u_{12} & u_{13} & u_{14} & u_{15} & u_{16} & u_{17} & u_{18} \\ \hline & 1 & u_{23} & u_{24} & u_{24} & u_{26} & u_{27} & u_{28} \\ \hline & & 1 & u_{34} & u_{34} & u_{36} & u_{37} & u_{38} \\ \hline & & & 1 & 0 & u_{56} & u_{57} & u_{48} \\ \hline & & & & 1 & u_{56} & u_{57} & u_{58} \\ \hline & & & & & 1 & u_{67} & u_{68} \\ \hline & & & & & & 1 & u_{78} \\ \hline & & & & & & & 1 \\ \hline \end{array} \right) \;\middle|\; u_{ij} \in \mathbb{F}_q \right\}.$$

Then $G_8(q)$ is a subgroup of $A_8(q)$ and $|G_8(q)| = q^{23}$.

3.2.2 Definition. *For $(i, j) \in \ddot{J}$, define the subgroups of $G_8(q)$ as follows:*

$$\dot{Y}_{i,j} := \{\dot{x}_{ij}(t) \mid t \in \mathbb{F}_q\}.$$

3.2.3 Proposition.

$$G_8(q) = \left\{ \prod_{(i,j)\in\ddot{J}} \dot{x}_{ij}(t_{ij}) \;\middle|\; t_{ij} \in \mathbb{F}_q \right\},$$

where the product can be taken in an arbitrary, but fixed, order.

3.2.4 Reminder.

$$\mathrm{G}_2^{syl}(q) := \{x_2(t_2)x_1(t_1)x_3(t_3)x_4(t_4)x_5(t_5)x_6(t_6) \,|\, t_1,t_2,t_3,t_4,t_5,t_6 \in \mathbb{F}_q\}$$

$$= \left\{ \begin{array}{l|l} \tilde{x}_{23}(t_2)\tilde{x}_{67}(-t_2) & \\ \cdot\tilde{x}_{12}(t_1)\tilde{x}_{78}(-t_1)\cdot\tilde{x}_{34}(t_1)\tilde{x}_{35}(t_1) & \\ \cdot\tilde{x}_{56}(-t_1)\tilde{x}_{46}(-t_1)\cdot\tilde{x}_{36}(t_1^2) & \\ \cdot\tilde{x}_{13}(-t_3)\tilde{x}_{68}(t_3)\cdot\tilde{x}_{24}(t_3)\tilde{x}_{25}(t_3) & \\ \cdot\tilde{x}_{57}(-t_3)\tilde{x}_{47}(-t_3)\cdot\tilde{x}_{27}(t_3^2) & t_1,t_2,t_3,t_4,t_5,t_6 \in \mathbb{F}_q \\ \cdot\tilde{x}_{14}(t_4)\tilde{x}_{58}(-t_4)\cdot\tilde{x}_{26}(t_4)\tilde{x}_{37}(-t_4) & \\ \cdot\tilde{x}_{15}(t_4)\tilde{x}_{48}(-t_4)\cdot\tilde{x}_{16}(t_5)\tilde{x}_{38}(-t_5) & \\ \cdot\tilde{x}_{17}(t_6)\tilde{x}_{28}(-t_6) & \end{array} \right\}.$$

3.2.5 Corollary.

$$G_8(q) = \left\{ \begin{array}{l|l} \tilde{x}_{23}(t_{23})\cdot\tilde{x}_{67}(t_{67}) & \\ \cdot\tilde{x}_{12}(t_{12})\cdot\tilde{x}_{78}(t_{78})\cdot\tilde{x}_{34}(t_{34})\tilde{x}_{35}(t_{34}) & \\ \cdot\tilde{x}_{56}(t_{56})\tilde{x}_{46}(t_{56})\cdot\tilde{x}_{36}(t_{36}) & \\ \cdot\tilde{x}_{13}(t_{13})\cdot\tilde{x}_{68}(t_{68})\cdot\tilde{x}_{24}(t_{24})\tilde{x}_{25}(t_{24}) & \\ \cdot\tilde{x}_{57}(t_{57})\tilde{x}_{47}(t_{57})\cdot\tilde{x}_{27}(t_{27}) & t_{ij} \in \mathbb{F}_q \\ \cdot\tilde{x}_{14}(t_{14})\tilde{x}_{58}(t_{58})\cdot\tilde{x}_{26}(t_{26})\tilde{x}_{37}(t_{37}) & \\ \cdot\tilde{x}_{15}(t_{15})\tilde{x}_{48}(t_{48})\cdot\tilde{x}_{18}(t_{18}) & \\ \cdot\tilde{x}_{16}(t_{16})\cdot\tilde{x}_{38}(t_{38}) & \\ \cdot\tilde{x}_{17}(t_{17})\cdot\tilde{x}_{28}(t_{28}) & \end{array} \right\}.$$

$$
=\left\{
\begin{array}{|c|c|c|c||c|c|c|c|}
\hline
1 & t_{12} & t_{13} & \begin{array}{c} t_{12}t_{24} \\ +t_{14} \end{array} & \begin{array}{c} t_{12}t_{24} \\ +t_{15} \end{array} & t_{12}t_{26}+t_{16} & \begin{array}{r} 2t_{12}t_{24}t_{57} \\ +t_{12}t_{27} \\ +t_{13}t_{37} \\ +t_{17} \end{array} & \begin{array}{r} t_{12}t_{24}t_{58} \\ +t_{12}t_{24}t_{48} \\ +t_{14}t_{48} \\ +t_{13}t_{38} \\ +t_{12}t_{28} \\ +t_{18} \end{array} \\
\hline
 & 1 & t_{23} & \begin{array}{c} t_{23}t_{34} \\ +t_{24} \end{array} & \begin{array}{c} t_{23}t_{34} \\ +t_{24} \end{array} & \begin{array}{r} 2t_{23}t_{34}t_{56} \\ +t_{23}t_{36} \\ +t_{26} \end{array} & \begin{array}{r} 2t_{24}t_{57} \\ +2t_{23}t_{34}t_{57} \\ +t_{23}t_{37} \\ +t_{27} \end{array} & \begin{array}{r} 2t_{23}t_{34}t_{56}t_{68} \\ +t_{23}t_{36}t_{68} \\ +t_{24}t_{58} \\ +t_{24}t_{48} \\ +t_{23}t_{34}t_{58} \\ +t_{23}t_{34}t_{48} \\ +t_{23}t_{38} \\ +t_{28} \end{array} \\
\hline
 & & 1 & t_{34} & t_{34} & \begin{array}{r} 2t_{34}t_{56} \\ +t_{36} \end{array} & \begin{array}{r} 2t_{34}t_{57} \\ +t_{37} \end{array} & \begin{array}{r} 2t_{34}t_{56}t_{68} \\ +t_{36}t_{68} \\ +t_{34}t_{58} \\ +t_{34}t_{48} \\ +t_{38} \end{array} \\
\hline
 & & & 1 & 0 & t_{56} & t_{57} & t_{56}t_{68}+t_{48} \\
\hline\hline
 & & & & 1 & t_{56} & t_{57} & t_{56}t_{68}+t_{58} \\
\hline
 & & & & & 1 & t_{67} & t_{67}t_{78}+t_{68} \\
\hline
 & & & & & & 1 & t_{78} \\
\hline
 & & & & & & & 1 \\
\hline
\end{array}
\;\middle|\; t_{ij} \in \mathbb{F}_q \right\}.
$$

3.2.6 Corollary. $|G_2^{syl}(q)| = q^6$, $|G_8(q)| = q^{23}$, $|A_8(q)| = q^{28}$ *and*

$$G_2^{syl}(q) \leqslant G_8(q) \leqslant A_8(q).$$

3.2.2. Monomial $G_2^{syl}(q)$-module

Let $p \neq 2$, $G := G_8(q)$ and $U := G_2^{syl}(q)$. In this subsection, we get a monomial linearisation $(f, \kappa|_{V\times V})$ for $G_8(q)$, where f is a surjective 1-cocycle and $\kappa|_{V\times V}$ is a non-degenerate bilinear form (3.2.29). The restriction $f|_U$ is bijective (3.2.28). Then we make $\mathbb{C}U$ into a monomial $G_8(q)$-module (3.2.31).

It is essential to choose a suitable vector space V for the 1-cocycle and a suitable non-degenerate bilinear form κ on V_0.

We define the following notations for $G_2^{syl}(q)$.

3.2.7 Notation/Lemma. *Let $V_0 := \mathrm{Mat}_{8\times 8}(q)$, then V_0 is an $\mathbb{F}_q$-vector space.*

3.2.8 Notation/Lemma. *The map*

$$\kappa\colon V_0 \times V_0 \to \mathbb{F}_q : (A, B) \mapsto \mathrm{tr}(A^\top B)$$

*is called the **trace form**, and κ is a non-degenerate symmetric bilinear form on V_0.*

3.2.9 Notation/Lemma. *Set $J := \{(1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (2,3)\}$ and*

$$\begin{aligned} V_J &:= \bigoplus_{(i,j)\in J} \mathbb{F}_q e_{ij} \\ &= \left\{ \left(\begin{array}{cccccccc} \cdot & \boxed{A_{12}} & \boxed{A_{13}} & \boxed{A_{14}} & \boxed{A_{15}} & \boxed{A_{16}} & \boxed{A_{17}} & \cdot \\ & \cdot & \boxed{A_{23}} & & & & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & \cdot & \cdot & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{array} \right) \in V_0 \;\middle|\; A_{i,j} \in \mathbb{F}_q \right\}. \end{aligned}$$

Then V_J is a 7-dimensional subspace of V_0 over $\mathbb{F}_q$.

3.2.10 Lemma. *Let $V_J^\perp$ denote the orthogonal complement of V_J in V_0 with respect to the trace form κ, i.e.*

$$V_J^\perp := \{B \in V_0 \mid \kappa(A, B) = 0,\ \forall\ A \in V_J\}.$$

Then $V_J^\perp = V_{\square\setminus J}$ and $V_0 = V_J \oplus V_J^\perp$.

3.2.11 Corollary. *$\kappa|_{V_J\times V_J} : V_J \times V_J \to \mathbb{F}_q$ is a non-degenerate bilinear form.*

3.2.12 Notation/Lemma. *Let*

$$\pi_J\colon V_0 = V_J \oplus V_J^\perp \to V_J : A \mapsto \sum_{(i,j)\in J} A_{i,j} e_{i,j}.$$

Then π_J is a projection of V_0 to the first component and is an $\mathbb{F}_q$-linear map.

3.2.13 Lemma. *Suppose* $A, B \in V_0$*, such that* $\operatorname{supp}(A) \cap \operatorname{supp}(B) \subseteq J$*. Then*

$$\begin{aligned}\kappa(A,B) &= \kappa(\pi_J(A),B) = \kappa(A,\pi_J(B))\\ &=\kappa(\pi_J(A),\pi_J(B)) = \kappa|_{V_J\times V_J}(\pi_J(A),\pi_J(B)).\end{aligned}$$

We define an important vector space V as follows.

3.2.14 Notation/Lemma. *Let*

$$\begin{aligned}V :=& \{A = (A_{ij}) \in V_0 \mid \operatorname{supp}(A) \in J,\ A_{14} = A_{15}\}\\ =& \left\{ \begin{pmatrix} \cdot & A_{12} & A_{13} & A_{15} & A_{15} & A_{16} & A_{17} & \cdot \\ & \cdot & A_{23} & & & & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & \cdot & \cdot & & & \\ & & & & & & & \\ & & & & & & & \end{pmatrix} \in V_0 \;\middle|\; \operatorname{supp}(A) \in J,\ A_{i,j} \in \mathbb{F}_q \right\},\end{aligned}$$

then V *is a 6-dimensional subspace of* V_J *over* $\mathbb{F}_q$ *and* $\operatorname{supp}(V) = J$.

We define the following map π, which will play a crucial role in our later statement.

3.2.15 Notation/Lemma. *Let*

$$\pi\colon V_0 \to V : A \mapsto \begin{array}{l} A_{12}e_{12} + A_{13}e_{13} + \frac{A_{14}+A_{15}}{2}e_{14} + \frac{A_{14}+A_{15}}{2}e_{15} \\ +A_{16}e_{16} + A_{17}e_{17} + A_{23}e_{23} \end{array},$$

i.e.

$$\pi(A) = \begin{pmatrix} & A_{12} & A_{13} & \frac{A_{14}+A_{15}}{2} & \frac{A_{14}+A_{15}}{2} & A_{16} & A_{17} & \\ & & A_{23} & & & & & \\ & & & & & & & \end{pmatrix}.$$

Then π *is* $\mathbb{F}_q$*-epimorphism. In particular,* $\pi|_V = \mathrm{id}_V$*,* $\pi^2 = \pi$ *and* $\pi(I_8) = O_8$.

3.2.16 Proposition. *Let* $V^\perp$ *denote the orthogonal complement of* V *in* V_0 *with respect to the trace form* κ*, i.e.*

$$V^\perp := \{B \in V_0 \mid \kappa(A,B) = 0 \ \textit{for all}\ A \in V\},$$

and

$$\begin{aligned}W :=& \bigoplus_{(i,j)\notin J} \mathbb{F}_q e_{ij} + \{x(e_{15}-e_{14}) \mid x \in \mathbb{F}_q\}\\ =&\{A=(A_{ij}) \in V_0 \mid A_{12}=A_{13}=A_{16}=A_{17}=A_{23}=0,\ A_{14}=-A_{15}\}\\ =&\left\{ \begin{array}{|c|c|c|c|c|c|c|c|} \hline A_{11} & 0 & 0 & -A_{15} & A_{15} & 0 & 0 & A_{18} \\ \hline A_{21} & A_{22} & 0 & A_{24} & A_{25} & A_{26} & A_{27} & A_{28} \\ \hline A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & A_{36} & A_{37} & A_{38} \\ \hline A_{41} & A_{42} & A_{43} & A_{44} & A_{45} & A_{46} & A_{47} & A_{48} \\ \hline A_{51} & A_{52} & A_{53} & A_{54} & A_{55} & A_{56} & A_{57} & A_{58} \\ \hline A_{61} & A_{62} & A_{63} & A_{64} & A_{65} & A_{66} & A_{67} & A_{68} \\ \hline A_{71} & A_{72} & A_{73} & A_{74} & A_{75} & A_{76} & A_{77} & A_{78} \\ \hline A_{81} & A_{82} & A_{83} & A_{84} & A_{85} & A_{86} & A_{87} & A_{88} \\ \hline \end{array} \in V_0 \;\middle|\; A_{ij} \in \mathbb{F}_q \right\}.\end{aligned}$$

Then $W = V^\perp$.

Proof. (1) $V^\perp \supseteq W$.

For all $B=(B_{ij}) \in W$ and for all $A \in V$, we get $\kappa(A,B) = A_{15}B_{15} - A_{15}B_{15} = 0$, so $B \in V^\perp$ for all $B \in W$. Thus $W \subseteq V^\perp$.

(2) $V^\perp \subseteq W$.

For all $B=(B_{ij}) \in V^\perp \subseteq V_0$ and for all $A=(A_{ij}) \in V$,

$$\begin{aligned}0 =&\kappa(A,B)\\ =&A_{12}B_{12}+A_{13}B_{13}+A_{15}B_{14}+A_{15}B_{15}+A_{16}B_{16}+A_{17}B_{17}+A_{23}B_{23}.\end{aligned}$$

(2.1) We claim $B_{12}=B_{13}=B_{16}=B_{17}=B_{23}=0$.

Let $A=e_{12}$, then $0=\kappa(A,B)=B_{12}$. Thus $B_{12}=0$. Similarly, $B_{13}=B_{16}=B_{17}=B_{23}=0$.

(2.2) We claim $B_{14}=-B_{15}$.

Let $A=e_{14}+e_{15}$, then $0=\kappa(A,B)=B_{14}+B_{15}$. Thus $B_{14}=-B_{15}$.

Hence $B \in W$, and $V^\perp \subseteq W$.

Finally, we get $V^\perp = W$. □

3.2.17 Lemma. *$\kappa|_{V\times V}$ is a non-degenerate bilinear form.*

Proof. It is sufficient to prove that let $A \in V$ and $\kappa_q(A,B)=0$ $(\forall\ B \in V)$ then $A=0$.

$$0=\kappa(A,B)=A_{12}B_{12}+A_{13}B_{13}+2A_{15}B_{15}+A_{16}B_{16}+A_{17}B_{17}+A_{23}B_{23}.$$

(1) Let $B := e_{12}$, then $0=\kappa(A,B)=A_{12}$. Similarly, $A_{12}=A_{13}=A_{16}=A_{17}=A_{23}=0$.

(2) Assume $A_{15} \neq 0$. Let $B := e_{14} + e_{15}$, then $0 = \kappa(A, B) = 2A_{15} \overset{\text{Char } \mathbb{F}_q \neq 2}{\Longrightarrow} A_{15} = 0$.

Hence $A = 0$. □

3.2.18 Corollary. *$V_0 = V \oplus V^{\perp}$, and $\pi\colon V_0 \to V$ is the projective map to the first component.*

3.2.19 Corollary. *Let $A \in V_0$ and $\pi_J(A) \in V$, then $\pi(A) = \pi_J(A)$.*

3.2.20 Corollary. *Suppose $A, B \in V_0$, such that $\operatorname{supp}(A) \cap \operatorname{supp}(B) \subseteq J$ and $\pi_J(A) \in V$. Then*

$$\begin{aligned} \kappa(A, B) &= \kappa(\pi(A), B) = \kappa(A, \pi(B)) \\ &= \kappa(\pi(A), \pi(B)) = \kappa|_{V \times V}(\pi(A), \pi(B)). \end{aligned}$$

Proof. We have

$$\begin{aligned} \kappa(A, B) &\overset{3.2.13}{=} \kappa(\pi_J(A), B) \overset{3.2.19}{=} \kappa(\pi(A), B) \\ &\overset{3.2.17}{=} \kappa(\pi(A), \pi(B)) = \kappa|_{V \times V}(\pi(A), \pi(B)) \overset{\pi(B) \in V}{=} \kappa(A, \pi(B)). \end{aligned}$$

Then the result is proved. □

3.2.21 Lemma. *Let $A \in V$ and $g \in G$, then*

$$\pi_J(Ag^{\top}) \in V.$$

In particular, $\pi_J(Ag^{\top}) = \pi(Ag^{\top})$.

Proof. Let $A \in V$ and $g \in G$. It is sufficient to prove

$$(Ag^{\top})_{15} = (Ag^{\top})_{14}.$$

We know that

$$\begin{aligned} (Ag^{\top})_{15} &= \sum_{j=1}^{8} A_{1j} g^{\top}_{j5} = \sum_{j=1}^{8} A_{1j} g_{5j} = \sum_{j=2}^{7} A_{1j} g_{5j} = \sum_{j=5}^{7} A_{1j} g_{5j} \\ &= A_{15} + A_{16} g_{56} + A_{17} g_{57} \\ &\overset{\left\{\substack{A_{14} = A_{15} \\ g_{45} = 0}\right.}{=} A_{14} + A_{15} g_{45} + A_{16} g_{46} + A_{17} g_{47} \\ &= (Ag^{\top})_{14}. \end{aligned}$$

Thus $\pi_J(Ag^{\top}) \in V$.

□

3.2.22 Lemma. *Let $A, B \in V$ and $g, h \in G$, then*

$$\operatorname{supp}(Bh^\top) \cap \operatorname{supp}(Ag) \subseteq J.$$

3.2.23 Proposition (Group action of G on V)**.** *The map*

$$- \circ -\colon V \times G \to V : (A, g) \mapsto A \circ g := \pi(Ag)$$

is a group action, and the elements of the group G act as $\mathbb{F}_q$-automorphisms.

Proof. It is enough to prove that $A \circ (gh) = (A \circ g) \circ h$ for all $A \in V$ and $g, h \in G$. We have

$$\begin{aligned}\kappa(B, A \circ (gh)) =& \kappa(B, \pi(A(gh))) \overset{3.2.20}{=} \kappa(B, A(gh))\\ =& \kappa(Bh^\top, Ag) \overset{3.2.20}{=} \kappa(\pi(Bh^\top), Ag) \overset{3.2.20}{=} \kappa(\pi(Bh^\top), \pi(Ag))\\ =& \kappa(\pi(Bh^\top), A \circ g) \overset{3.2.20}{=} \kappa(Bh^\top, A \circ g) = \kappa(B, (A \circ g)h)\\ =& \kappa(B, (A \circ g) \circ h).\end{aligned}$$

Thus $A \circ (gh) = (A \circ g) \circ h$ by 3.2.17. □

By 1.2.21, we get a new action:

3.2.24 Corollary. *There exists an unique linear action $-.-$ of G on V:*

$$-.-\colon V \times G \to V : (A, g) \mapsto A.g := A \circ g^{-\top}$$

such that $\kappa|_{V\times V}(A.g, B) = \kappa|_{V\times V}(A, B \circ g^{-1})$ for all $B \in V$. In particular, for all $A, B \in V$ and $g \in G$,

$$\kappa|_{V\times V}(A, B \circ g) = \kappa|_{V\times V}(A.g^{-1}, B) = \kappa|_{V\times V}(A \circ g^\top, B).$$

3.2.25 Notation. *Set $f := \pi|_G \colon G \to V$.*

3.2.26 Lemma. *Let $x, g \in G$, then*

$$f(x)g \equiv (x-1)g \mod V^\perp.$$

In particular, $f(x) \equiv x - 1 \mod V^\perp$.

Proof. We have

(a) $f(x) = \pi(x) = \pi(x) - \pi(1) = \pi(x-1)$, so $f(x) \equiv x - 1 \mod V^\perp$.

(b) For all $A \in V$,

$$\begin{aligned}\kappa(A, f(x)g) = \kappa(Ag^\top, f(x)) &\overset{3.2.20}{=} \kappa(\pi(Ag^\top), f(x))\\ \overset{(a)}{=} \kappa(\pi(Ag^\top), x-1) &\overset{3.2.20}{=} \kappa(Ag^\top, x-1) = \kappa(A, (x-1)g),\end{aligned}$$

so $f(x)g \equiv (x-1)g \mod V^\perp$ by 3.2.17.

□

3.2.27 Proposition. *Let $x, g \in G$, then*

$$f(xg) = f(x) \circ g + f(g).$$

Proof. For all $x, g \in G$,

$$f(xg) \overset{3.2.26}{\equiv} xg - 1 = (x-1)g + (g-1) \overset{3.2.26}{\equiv} f(x)g + f(g) \mod V^{\perp}$$

$$\begin{aligned} &\overset{3.2.18}{\implies} \quad \pi(f(xg)) = \pi(f(x)g + f(g)) \\ &\implies \quad f(xg) = \pi(f(x)g) + \pi(f(g)) = f(x) \circ g + f(g). \end{aligned}$$

Hence $f(xg) = f(x) \circ g + f(g)$. □

3.2.28 Proposition (Bijective 1-cocycle of $G_2^{syl}(q)$)**.** *Let $U = G_2^{syl}(q)$, then $f|_U := \pi|_U \colon U \to V$ is a bijection. In particular, $f|_U$ is a bijective 1-cocycle of U.*

Proof. Let $x := x(t_1, t_2, t_3, t_4, t_5, t_6) \in U$, then

$$f|_U(x) = \pi(x(t_1, t_2, t_3, t_4, t_5, t_6))$$

$$= \left(\begin{array}{|c|c|c|c|c|c|} \hline t_1 & -t_3 & t_1t_3 + t_4 & t_1t_3 + t_4 & t_1t_4 + t_5 & -t_1t_3^2 + t_3t_4 + t_6 \\ \hline \multicolumn{1}{c|}{} & t_2 & \multicolumn{4}{c}{} \\ \cline{2-2} \end{array} \right).$$

(1) Since $f \colon G \to V$ is well defined, $f|_U$ is well defined.

(2) $f|_U$ is surjective.

For all $A \in V$, let $A := A_{12}e_{12} + A_{13}e_{13} + A_{15}e_{14} + A_{15}e_{15} + A_{16}e_{16} + A_{17}e_{17} + A_{23}e_{23}$. We find a $x := x(t_1, t_2, t_3, t_4, t_5, t_6) \in U$ such that $f|_U(x) = A$, where

$$\begin{aligned} t_1 &= A_{12}, \qquad t_2 = A_{23}, \qquad t_3 = -A_{13}, \qquad t_4 = A_{15} - t_1t_3 = A_{15} + A_{12}A_{13}, \\ t_5 &= A_{16} - t_1t_4 = A_{16} - A_{12}(A_{15} + A_{12}A_{13}) = A_{16} - A_{12}^2A_{13} - A_{12}A_{15}, \\ t_6 &= A_{17} + t_1t_3^2 - t_3t_4 = A_{17} + A_{12}^2A_{13} + A_{13}(A_{15} + A_{12}A_{13}) \\ &= A_{17} + 2A_{12}A_{13}^2 + A_{13}A_{15}. \end{aligned}$$

Thus, $f|_U$ is surjective.

(3) $f|_U$ is bijective, since $|U| = q^6 = |V|$.

□

3.2.29 Corollary (Monomial linearisation for $G_8(q)$)**.** *$f = \pi|_G : G \to V$ is a surjective 1-cocycle of G in V, and $(f, \kappa|_{V \times V})$ is a monomial linearisation for $G = G_8(q)$.*

3.2.30 Corollary (Monomial linearisation for $G_2^{syl}(q)$). *$(f|_{G_2^{syl}(q)}, \kappa|_{V\times V})$ is a monomial linearisation for $G_2^{syl}(q)$.*

Now we obtain the monomial G-module $\mathbb{C}G_2^{syl}(q)$, which is essential for the construction of the supercharacter theory for $G_2^{syl}(q)$.

3.2.31 Theorem (Fundamental theorem for $G_2^{syl}(q)$). *Let $G = G_8(q)$, $U = G_2^{syl}(q)$ and*

$$[A] = \frac{1}{|U|}\sum_{u\in U}\overline{\chi_A(u)}u \qquad \textit{for all } A \in V.$$

Then the set $\{[A] \mid A \in V\}$ forms a $\mathbb{C}$-basis for the complex group algebra $\mathbb{C}U$. Let

$$[A] * g := \chi_{A.g}(g)[A.g] = \vartheta\kappa(A.g, f(g))[A.g] \qquad \textit{for all } g \in G, A \in V,$$

then $\mathbb{C}U$ is a monomial $\mathbb{C}G$-module. The restriction of the $*$-operation to U is given by the usual right multiplication of U on $\mathbb{C}U$, i.e.

$$[A] * u = [A]u = \frac{1}{|U|}\sum_{y\in U}\overline{\chi_A(y)}yu \qquad \textit{for all } u \in U, A \in V.$$

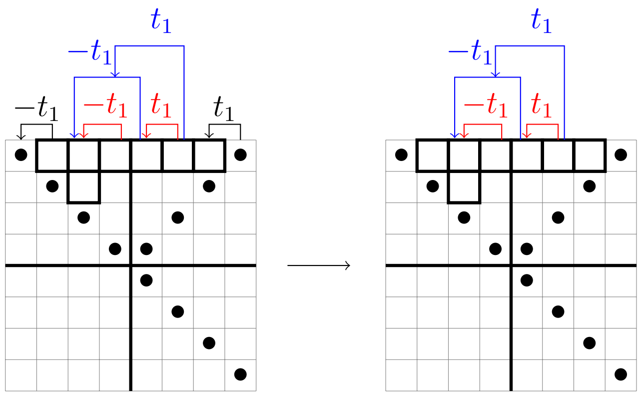
Proof. By 3.2.29, $(f, \kappa|_{V\times V})$ is a monomial linearisation for G, satisfying that $f|_U$ is a bijective map. By 2.2.27, $A.u := \pi(Au^{-\top})$. Hence the whole theorem is proved by 1.2.24. $\square$

3.2.3. $G_2^{syl}(q)$-orbit modules

Let $U := G_2^{syl}(q)$, $A \in V$, $y_i(t_i) = x_i(t_i) \in U$ and $t_i \in \mathbb{F}_q$ ($i = 1, 2, \ldots, 6$). In this subsection, we get a classification of U-orbit modules (3.2.37) and obtain the stabilizers $\mathrm{Stab}_U(A)$ for all $A \in V$ (3.2.39).

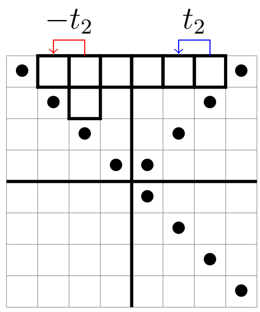
3.2.32 Lemma. *Let $A \in V$, $x_i(t_i) \in U$ and $t_i \in \mathbb{F}_q$ with $i \in \{1, 2, \ldots, 6\}$. Then $A.x_i(t_i)$ and the corresponding figures of moves are obtained as follows:*

(1) $A.x_1(t_1) = A.\left(x_{12}(t_1)x_{34}(t_1)x_{35}(t_1)\right) = A.\left(x_{34}(t_1)x_{35}(t_1)\right)$.



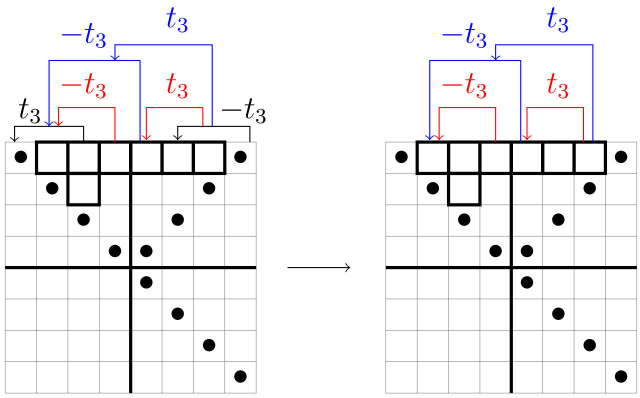
$A.x_1(t_1)$ $\longrightarrow$ $A.(x_{34}(t_1)x_{35}(t_1))$

(2) $A.x_2(t_2) = A.x_{23}(t_2)$.



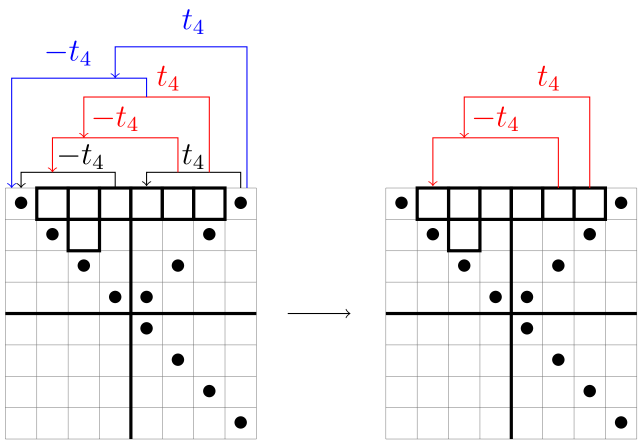
$A.x_2(t_2)$

(3) $A.x_3(t_3) = A.\left(x_{13}(-t_3)x_{24}(t_3)x_{25}(t_3)\right) = A.\left(x_{24}(t_3)x_{25}(t_3)\right)$.



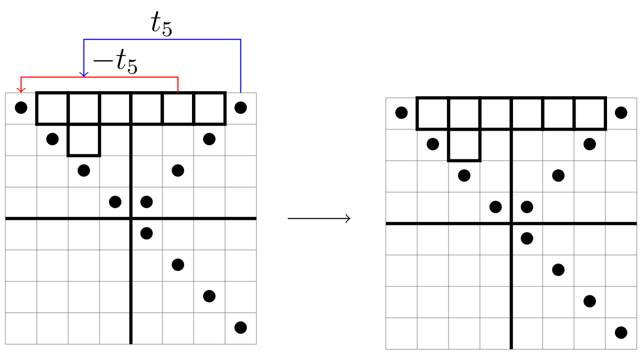
$A.x_3(t_3)$ $\qquad$ $A.(x_{24}(t_3)x_{25}(t_3))$

(4) $A.x_4(t_4) = A.\left(x_{14}(t_4)x_{26}(t_4)x_{15}(t_4)\right) = A.x_{26}(t_4)$.



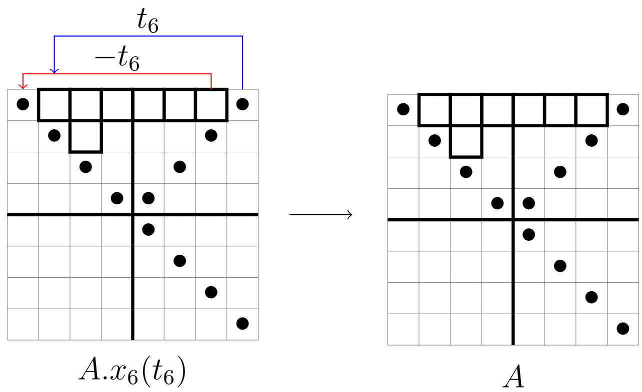
$A.x_4(t_4)$ $\qquad$ $A.x_{26}(t_4)$

(5) $A.x_5(t_5) = A.x_{16}(t_5) = A$.



$A.x_5(t_5)$ $\qquad$ A

(6) $A.x_6(t_6) = A.x_{17}(t_6) = A.$



For $A \in V$, we often only draw the entries A_{ij} where $(i,j) \in J$, since $\mathrm{supp}(A) \subseteq J$. By 3.2.32, we calculate $[A].x_i(t_i)$ for all $A \in V$ and $x_i(t_i) \in U$.

3.2.33 Lemma. *Let* $A \in V$, $x_i(t_i) \in U$ *and* $t_i \in \mathbb{F}_q$ *with* $i \in \{1,2,3,4,5,6\}$. *Then*

(1)

$$\begin{aligned}&[A].x_1(t_1) = \vartheta(A_{12}t_1)[A.x_1(t_1)]\\ =&\vartheta(A_{12}t_1)\left[\begin{array}{|c|c|c|c|c|c|}\hline A_{12} & \begin{matrix}A_{13}-2A_{15}t_1\\ -A_{16}t_1^2\end{matrix} & \begin{matrix}A_{16}t_1\\ +A_{15}\end{matrix} & \begin{matrix}A_{16}t_1\\ +A_{15}\end{matrix} & A_{16} & A_{17} \\ \hline & A_{23} & & & & \\ \hline\end{array}\right],\end{aligned}$$

(2)

$$\begin{aligned}&[A].x_2(t_2) = \vartheta(A_{23}t_2)[A.x_2(t_2)]\\ =&\vartheta(A_{23}t_2)\left[\begin{array}{|c|c|c|c|c|c|}\hline A_{12}-A_{13}t_2 & A_{13} & A_{15} & A_{15} & A_{16}+A_{17}t_2 & A_{17} \\ \hline & A_{23} & & & & \\ \hline\end{array}\right],\end{aligned}$$

(3)

$$\begin{aligned}&[A].x_3(t_3) = \vartheta(-A_{13}t_3)[A.x_3(t_3)]\\ =&\vartheta(-A_{13}t_3)\left[\begin{array}{|c|c|c|c|c|c|}\hline \begin{matrix}A_{12}-2A_{15}t_3\\ -A_{17}t_3^2\end{matrix} & A_{13} & \begin{matrix}A_{17}t_3\\ +A_{15}\end{matrix} & \begin{matrix}A_{17}t_3\\ +A_{15}\end{matrix} & A_{16} & A_{17} \\ \hline & A_{23} & & & & \\ \hline\end{array}\right],\end{aligned}$$

(4)

$$\begin{aligned}&[A].x_4(t_4) = \vartheta(2A_{15}t_4)[A.x_4(t_4)]\\ =&\vartheta(2A_{15}t_4)\left[\begin{array}{|c|c|c|c|c|c|}\hline A_{12}-A_{16}t_4 & A_{13}+A_{17}t_4 & A_{15} & A_{15} & A_{16} & A_{17} \\ \hline & A_{23} & & & & \\ \hline\end{array}\right],\end{aligned}$$

(5) $[A].x_5(t_5) = \vartheta(A_{16}t_5)[A]$,

(6) $[A].x_6(t_6) = \vartheta(A_{17}t_6)[A]$.

3.2.34 Lemma ($G_2^{syl}(q)$-orbit modules)**.** *Let $A = (A_{ij}) \in V$, the U-orbit module $\mathbb{C}\mathcal{O}_U([A])$ $(A \in V)$ is determined.*

$$
\begin{aligned}
&\mathbb{C}\mathcal{O}_U([A])\\
=&\mathbb{C}\left\{\left[\begin{array}{|c|c|c|c|c|c|}
\hline
\begin{matrix} A_{12} \\ -A_{13}t_2 \\ -2A_{15}t_3 \\ -2A_{16}t_1t_3 \\ -2A_{17}t_2t_1t_3 \\ -A_{17}t_3^2 \\ -A_{16}t_4 \\ -A_{17}t_2t_4 \end{matrix} &
\begin{matrix} A_{13} \\ -2A_{15}t_1 \\ -A_{16}t_1^2 \\ -A_{17}t_2t_1^2 \\ +A_{17}t_4 \end{matrix} &
\begin{matrix} A_{15} \\ +A_{16}t_1 \\ +A_{17}t_2t_1 \\ +A_{17}t_3 \end{matrix} &
\begin{matrix} A_{15} \\ +A_{16}t_1 \\ +A_{17}t_2t_1 \\ +A_{17}t_3 \end{matrix} &
\begin{matrix} A_{16} \\ +A_{17}t_2 \end{matrix} &
A_{17} \\
\hline
\multicolumn{1}{c|}{} & A_{23} & \multicolumn{4}{c}{} \\
\cline{2-2}
\end{array}\right]\right.\\
&\left.\middle|\, t_1, t_2, t_3, t_4 \in \mathbb{F}_q\right\}.
\end{aligned}
$$

Proof. By 3.2.33, we calculate the general orbit modules directly. □

3.2.35 Reminder. *By 2.2.47, a pattern $A \in V$ is called a core pattern if*

$$\operatorname{supp}(A) \subseteq \operatorname{core}(A) = \operatorname{main}(A) \cup \operatorname{minor}(A).$$

3.2.36 Notation. *Define the families of U-orbit modules as follows:*

$$
\begin{aligned}
\mathfrak{F}_6 :=&\{\mathbb{C}\mathcal{O}_U(A) \mid A \in V,\ A_{17} \neq 0\},\\
\mathfrak{F}_5 :=&\{\mathbb{C}\mathcal{O}_U(A) \mid A \in V,\ A_{16} \neq 0,\ A_{17} = 0\},\\
\mathfrak{F}_4 :=&\{\mathbb{C}\mathcal{O}_U(A) \mid A \in V,\ A_{15} \neq 0,\ A_{16} = A_{17} = 0\},\\
\mathfrak{F}_3 :=&\{\mathbb{C}\mathcal{O}_U(A) \mid A \in V,\ A_{13} \neq 0,\ A_{15} = A_{16} = A_{17} = 0\},\\
\mathfrak{F}_{1,2} :=&\{\mathbb{C}\mathcal{O}_U(A) \mid A \in V,\ A_{13} = A_{15} = A_{16} = A_{17} = 0\}.
\end{aligned}
$$

Let $A \in V$, we also say $A \in \mathfrak{F}_i$, if $\mathbb{C}\mathcal{O}_U([A]) \in \mathfrak{F}_i$.

3.2.37 Proposition (Classification of $G_2^{syl}(q)$-orbit modules)**.** *Every U-orbit module is one of the following forms in Table 3.1. In particular, in every U-orbit module there exists precisely a staircase core pattern.*

Table 3.1.: $G_2^{syl}(q)$-orbit modules

Family	$\mathbb{C}\mathcal{O}_U([A])$ $(A \in V)$	$\dim_{\mathbb{C}} \mathbb{C}\mathcal{O}_U([A])$	$\#\mathbb{C}\mathcal{O}_U([A])$
$\mathfrak{F}_6$	$\mathbb{C}\mathcal{O}_U\left(\begin{bmatrix} \cdot & \boxed{A_{12}} & \boxed{} & \boxed{} & \boxed{} & \boxed{} & \boxed{A^*_{17}} & \cdot \\ & \cdot & \boxed{A_{23}} & & & & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & \cdot & \cdot & & & \end{bmatrix}\right)$	q^3	$(q-1)q^2$
$\mathfrak{F}_5$	$\mathbb{C}\mathcal{O}_U\left(\begin{bmatrix} \cdot & \boxed{} & \boxed{A_{13}} & \boxed{} & \boxed{} & \boxed{A^*_{16}} & \boxed{} & \cdot \\ & \cdot & \boxed{A_{23}} & & & & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & \cdot & \cdot & & & \end{bmatrix}\right)$	q^2	$(q-1)q^2$
$\mathfrak{F}_4$	$\mathbb{C}\mathcal{O}_U\left(\begin{bmatrix} \cdot & \boxed{} & \boxed{} & \boxed{A^*_{15}} & \boxed{A^*_{15}} & \boxed{} & \boxed{} & \cdot \\ & \cdot & \boxed{A_{23}} & & & & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & \cdot & \cdot & & & \end{bmatrix}\right)$	q^2	$(q-1)q$
$\mathfrak{F}_3$	$\mathbb{C}\mathcal{O}_U\left(\begin{bmatrix} \cdot & \boxed{} & \boxed{A^*_{13}} & \boxed{} & \boxed{} & \boxed{} & \boxed{} & \cdot \\ & \cdot & \boxed{A_{23}} & & & & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & \cdot & \cdot & & & \end{bmatrix}\right)$	q	$(q-1)q$
$\mathfrak{F}_{1,2}$	$\mathbb{C}\mathcal{O}_U\left(\begin{bmatrix} \cdot & \boxed{A_{12}} & \boxed{} & \boxed{} & \boxed{} & \boxed{} & \boxed{} & \cdot \\ & \cdot & \boxed{A_{23}} & & & & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & \cdot & \cdot & & & \end{bmatrix}\right)$	1	q^2

where $A^*_{13}, A^*_{15}, A^*_{16}, A^*_{17} \in \mathbb{F}_q^*$ and $\#\mathbb{C}\mathcal{O}_U([A]) := \#\{\mathbb{C}\mathcal{O}_U([A]) \in \mathfrak{F}_i\}$ for $i \in \{\{1,2\},3,4,5,6\}$.

Proof. Let $A = (A_{ij}) \in V$ with $A_{17} = A^*_{17} \in \mathbb{F}_q^*$, then

$$\begin{aligned}
&\mathbb{C}\mathcal{O}_U([A]) \\
=&\mathbb{C}\left\{ \left[\begin{bmatrix} \boxed{\begin{matrix} A_{12} \\ +\frac{A_{13}A_{16}+A_{15}^2}{A^*_{17}} \\ -\frac{B_{13}B_{16}+B_{15}^2}{A^*_{17}} \end{matrix}} & \boxed{B_{13}} & \boxed{B_{15}} & \boxed{B_{15}} & \boxed{B_{16}} & \boxed{A^*_{17}} \\ & \boxed{A_{23}} & & & & \end{bmatrix}\right] \middle| \; B_{13}, B_{15}, B_{16} \in \mathbb{F}_q \right\}.
\end{aligned}$$

Thus $\dim_{\mathbb{C}}\mathbb{C}\mathcal{O}_U([A]) = q^3$. Let $u := x(t_1, -\frac{A_{16}}{A_{17}^*}, -\frac{A_{15}}{A_{17}^*}, -\frac{A_{13}-2A_{15}t_1}{A_{17}^*}, t_5, t_6)$, then there is a staircase core pattern

$$C := A.u = \begin{array}{|c|c|c|c|c|c|}\hline A_{12} + \frac{A_{13}A_{16}+A_{15}^2}{A_{17}^*} & & & & & A_{17}^* \\ \hline \multicolumn{1}{c|}{} & A_{23} & \multicolumn{4}{c}{} \\ \cline{2-2} \end{array} \in \mathcal{O}_U(A).$$

Thus $\mathcal{O}_U(C) = \mathcal{O}_U(A)$ and $\mathbb{C}\mathcal{O}_U([A]) = \mathbb{C}\mathcal{O}_U([C])$. We shall show that the staircase core pattern is determined uniquely. Suppose that there is another staircase core pattern $D = D_{12}e_{12} + D_{23}e_{23} + D_{17}e_{17} \in \mathcal{O}_U(A)$, then there exists an element $v \in U$ such that $D = C.v$. We obtain that $v = 1$, i.e. $D = C$. Thus

$$\begin{aligned} &\#\mathbb{C}\mathcal{O}_U([A]) = \#\{\mathbb{C}\mathcal{O}_U([A]) \in \mathfrak{F}_i\} \\ =&\# \left\{D = D_{12}e_{12} + D_{23}e_{23} + D_{17}^*e_{17} \in V \,\middle|\, D_{12}, D_{23} \in \mathbb{F}_q, D_{17}^* \in \mathbb{F}_q^*\right\} \\ =&(q-1)q^2. \end{aligned}$$

Similarly, all of the statements are proved. □

3.2.38 Remark. *Let $A \in V$. In 3.2.37, the orbit modules $\mathbb{C}\mathcal{O}_U([A])$ are staircase modules except that $\mathbb{C}\mathcal{O}_U([A]) \subseteq \mathfrak{F}_3$ and $A_{2,3} \neq 0$.*

3.2.39 Proposition ($G_2^{syl}(q)$-stabilizer)**.** *Let $A \in V$, then $\mathrm{Stab}_U(A)$ is established in Table 3.2.*

Proof. We know that $\mathrm{Stab}_U(A) = \{u \in U \mid A.u = A\}$, so the results are obtained by calculation. □

Table 3.2.: $G_2^{syl}(q)$-stabilizers

	$A \in V$	$\mathrm{Stab}_U(A)$
$\mathfrak{F}_6$	$\begin{array}{cccccccc} \cdot & A_{12} & A_{13} & A_{15} & A_{15} & A_{16} & A_{17}^* & \cdot \\ & \cdot & A_{23} & & & & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & \cdot & \cdot & & & \end{array}$	$x(t_1, 0, -\frac{A_{16}t_1}{A_{17}^*}, \frac{2A_{15}t_1+A_{16}t_1^2}{A_{17}^*}, t_5, t_6)$ $\forall\ t_1, t_5, t_6 \in \mathbb{F}_q$
$\mathfrak{F}_5$	$\begin{array}{cccccccc} \cdot & A_{12} & A_{13} & A_{15} & A_{15} & A_{16}^* & & \cdot \\ & \cdot & A_{23} & & & & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & \cdot & \cdot & & & \end{array}$	$x(0, t_2, t_3, \frac{-A_{13}t_2-2A_{15}t_3}{A_{16}^*}, t_5, t_6)$ $\forall\ t_2, t_3, t_5, t_6 \in \mathbb{F}_q$
$\mathfrak{F}_4$	$\begin{array}{cccccccc} \cdot & A_{12} & A_{13} & A_{15}^* & A_{15}^* & & & \cdot \\ & \cdot & A_{23} & & & & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & \cdot & \cdot & & & \end{array}$	$x(0, t_2, \frac{-A_{13}t_2}{2A_{15}^*}, t_4, t_5, t_6)$ $\forall\ t_2, t_4, t_5, t_6 \in \mathbb{F}_q$
$\mathfrak{F}_3$	$\begin{array}{cccccccc} \cdot & A_{12} & A_{13}^* & & & & & \cdot \\ & \cdot & A_{23} & & & & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & \cdot & \cdot & & & \end{array}$	$x(t_1, 0, t_3, t_4, t_5, t_6)$ $\forall\ t_1, t_3, t_4, t_5, t_6 \in \mathbb{F}_q$
$\mathfrak{F}_{1,2}$	$\begin{array}{cccccccc} \cdot & A_{12} & & & & & & \cdot \\ & \cdot & A_{23} & & & & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & \cdot & \cdot & & & \end{array}$	$x(t_1, t_2, t_3, t_4, t_5, t_6)$ $\forall\ t_1, t_2, t_3, t_4, t_5, t_6 \in \mathbb{F}_q$

where $A_{13}^*, A_{15}^*, A_{16}^*, A_{17}^* \in \mathbb{F}_q^*$.

3.2.4. Homomorphisms between orbit modules

Let $U := G_2^{syl}(q)$. In this subsection, we define a truncated row operation of U on V. Then we classify the U-orbit modules: every U-orbit module is isomorphic to some hook-separated staircase module (3.2.57). Finally, some irreducible modules are determined, and some irreducible modules are determined, and any two orbit modules are shown to be orthogonal when the 1st verges are different (3.2.61).

3.2.40 Lemma. *Let $A \in V$ and $g \in U$, then*

$$\lambda_g([A]) \overset{Def.}{=} g[A] = \frac{1}{|U|}\sum_{y \in U} \overline{\vartheta\kappa(g^{-\top}A, y)}y.$$

3.2.41 Lemma. *Let $A \in V$ and $\pi_{\triangledown}\colon V_0 \to V_{\triangledown} : A \mapsto \sum_{(i,j)\in\triangledown} A_{ij}e_{ij}$. Then*

$$\pi_{\triangledown}(x^{-\top}A)$$

$$= \begin{pmatrix}
0 & A_{12} & A_{13} & A_{15} & A_{15} & A_{16} & A_{17} & 0 \\
 & -t_1A_{12} & \substack{-t_1A_{13}\\+A_{23}} & -t_1A_{15} & -t_1A_{15} & -t_1A_{16} & -t_1A_{17} & 0 \\
 & & \substack{t_1t_2A_{13}\\-t_2A_{23}\\+t_3A_{13}} & \substack{t_1t_2A_{15}\\+t_3A_{15}} & \substack{t_1t_2A_{15}\\+t_3A_{15}} & \substack{t_1t_2A_{16}\\+t_3A_{16}} & \substack{t_1t_2A_{17}\\+t_3A_{17}} & 0 \\
 & & & \substack{-t_1t_3A_{15}\\-t_4A_{15}} & \substack{-t_1t_3A_{15}\\-t_4A_{15}} & \substack{-t_1t_3A_{16}\\-t_4A_{16}} & \substack{-t_1t_3A_{17}\\-t_4A_{17}} & 0 \\
 & & & & \substack{-t_1t_3A_{15}\\-t_4A_{15}} & \substack{-t_1t_3A_{16}\\-t_4A_{16}} & \substack{-t_1t_3A_{17}\\-t_4A_{17}} & 0 \\
 & & & & & \substack{-t_1^2t_3A_{16}\\-2t_1t_4A_{16}\\-t_5A_{16}} & \substack{-t_1^2t_3A_{17}\\-2t_1t_4A_{17}\\-t_5A_{17}} & 0 \\
 & & & & & & \substack{-t_1^2t_2t_3A_{17}\\-2t_1t_2t_4A_{17}\\-t_2t_5A_{17}\\-2t_3t_4A_{17}\\-t_6A_{17}} & 0 \\
 & & & & & & & 0
\end{pmatrix}.$$

3.2.42 Proposition. *Let $A \in V$ and $x := x(t_1, t_2, t_3, t_4, t_5, t_6) \in U$, then*

$$\pi(x^{-\top}A) = A - t_1A_{13}e_{23}.$$

Proof. Let $A \in V$ and $x := x(t_1, t_2, t_3, t_4, t_5, t_6) \in U$, then $\pi(x^{-\top}A) \overset{3.2.41}{=} A - t_1A_{13}e_{23}$. Thus we get the result. $\square$

3.2.43 Definition/Lemma ($G_2^{syl}(q)$-truncated row operation)**.** *The map*

$$U \times V \to V : (u, A) \mapsto g.A := \pi(g^{-\top}A)$$

defines a group operation, which is called the ***truncated row operation****.*
Note that the elements of U act as $\mathbb{F}_q$-automorphisms on V.

3.2.44 Corollary. *Let $A \in V$, $t_i \in \mathbb{F}_q$ and $x_i(t_i) \in U$ $(i = 1, 2, 3, 4, 5, 6)$, then*
(i) $x_1(t_1).A$ is obtained from A by replacing A_{23} with $A_{23} - t_1A_{13}$, i.e.

$$x_1(t_1).A = \left(\begin{array}{|c|c|c|c|c|c|} \hline A_{12} & A_{13} & A_{15} & A_{15} & A_{16} & A_{17} \\ \hline & A_{23} - t_1A_{13} & & & & \\ \cline{2-2} \end{array} \right).$$

(ii) $x_i(t_i).A = A$ for all $i \in \{2, 3, 4, 5, 6\}$.

3.2.45 Notation. *$\mathcal{O}_U^{\mathrm{row}}(A)$ and $\mathrm{Stab}_U^{\mathrm{row}}(A)$ are defined under the 'truncated row operation', and $\mathcal{O}_U(A)$ and $\mathrm{Stab}_U(A)$ are used for the 'truncated column operation'.*

3.2.46 Remark. *From Corollary 3.2.44 we get*

$$x_1(t_1).A = x_{12}(t_1).A = \tilde{x}_{12}(t_1).A$$

and

$$\mathcal{O}^{\mathrm{row}}_{G_2^{syl}(q)}(A) = \mathcal{O}^{\mathrm{row}}_{D_4^{syl}(q)}(A) = \mathcal{O}^{\mathrm{row}}_{A_8(q)}(A).$$

3.2.47 Remark. *In general, $g.(A.u) \neq (g.A).u$. For example: let $t_1, t_4 \in \mathbb{F}_q^*$ and $A_{17}^* \in \mathbb{F}_q^*$, then*

$$\big(x_1(t_1).(A^*_{17}e_{17})\big).x_4(t_4) = \left(\begin{array}{|c|c|c|c|c|c|} \hline & t_4A^*_{17} & & & & A^*_{17} \\ \hline & & & & & \\ \cline{2-2} \end{array} \right)$$

but

$$x_1(t_1).\big((A^*_{17}e_{17}).x_4(t_4)\big) = \left(\begin{array}{|c|c|c|c|c|c|} \hline & t_4A^*_{17} & & & & A^*_{17} \\ \hline & -t_1t_4A^*_{17} & & & & \\ \cline{2-2} \end{array} \right).$$

3.2.48 Lemma. *Let $B \in V$, $g := x(t_1, t_2, t_3, t_4, t_5, t_6) \in U$ and $y \in U$ such that*

$$\mathrm{supp}(g^{-\top}B) \cap \mathrm{supp}(y - 1) \subseteq J.$$

Then $\vartheta\kappa(g^{-\top}B, y - 1) = \chi_{g.B}(y)$.

3.2.49 Proposition. *Let $g \in U$ and $A \in V$, such that $\mathrm{supp}(g^{-\top}A) \cap \text{◹} \subseteq J \cup \{(4, 5)\}$. Then*

$$\lambda_g([B]) = \chi_{g.B}(g)[g.B] \quad \textit{for all } B \in \mathcal{O}_U(A).$$

Proof. Let $B \in \mathcal{O}_U(A)$, then $\mathrm{main}(B) = \mathrm{main}(A)$. Thus,

$$\mathrm{supp}(g^{-\top}B) \cap \triangledown \subseteq J \cup \{(4,5)\} \quad \text{for all } B \in \mathcal{O}_U(A).$$

Then for all $y \in U$, $\mathrm{supp}(g^{-\top}B) \cap \mathrm{supp}(y-1) \subseteq J$, since $\mathrm{supp}(y-1) \subseteq \triangledown \backslash \{(4,5)\}$.
Then

$$\begin{aligned} g[B] &\overset{3.2.40}{=} \frac{1}{|U|} \sum_{y \in U} \overline{\vartheta\kappa(g^{-\top}B, y)} y \\ &= \frac{1}{|U|} \sum_{y \in U} \overline{\vartheta\kappa(g^{-\top}B, y-1)\vartheta\kappa(g^{-\top}B, 1)} y \\ &\overset{3.2.48}{=} \overline{\vartheta\kappa(B, g^{-1})} \cdot \frac{1}{|U|} \sum_{y \in U} \overline{\chi_{g.B}(y)} y \\ &= \overline{\chi_B(g^{-1})}[g.B], \end{aligned}$$

and

$$\begin{aligned} \chi_{g.B}(g) &= \vartheta\kappa(g.B, f(g)) = \vartheta\kappa(\pi(g^{-\top}B), f(g)) = \vartheta\kappa(g^{-\top}B, f(g)) = \vartheta\kappa(B, f(g)g^{-1}) \\ &= \vartheta\kappa(B, f(gg^{-1}) - f(g^{-1})) = \vartheta\kappa(B, f(1) - f(g^{-1})) = \vartheta\kappa(B, -f(g^{-1})) = \overline{\chi_B(g^{-1})}. \end{aligned}$$

Hence $\lambda_g([B]) = \chi_{g.B}(g)[g.B]$ for all $B \in \mathcal{O}_U(A)$. □

3.2.50 Corollary. *Let $g \in U$ and $A \in V$, such that* $\mathrm{supp}(g^{-\top}A) \cap \triangledown \subseteq J \cup \{(4,5)\}$. *Then* $\mathrm{im}(\lambda_g|_{\mathbb{C}\mathcal{O}_U([A])}) = \mathbb{C}\mathcal{O}_U([g.A])$.

Proof. Let $u \in U$, then $\mathbb{C}g[A.u] = \mathbb{C}g([A]u) = \mathbb{C}(g[A])u \overset{3.2.49}{=} \mathbb{C}[g.A]u$. Hence $\mathrm{im}(\lambda_g|_{\mathbb{C}\mathcal{O}_U([A])}) = \mathbb{C}\mathcal{O}_U([g.A])$. □

3.2.51 Corollary. *Let $g \in U$ and $A \in V$, such that* $\mathrm{supp}(g^{-\top}A) \cap \triangledown \subseteq J \cup \{(4,5)\}$. *Then*

$$g.(B.u) = (g.B).u \quad \textit{for all } B \in \mathcal{O}_U(A),\ u \in U.$$

Proof. Let $g \in U$ and $A \in V$, we have

$$\mathbb{C}[(g.B).u] = \mathbb{C}[g.B]u \overset{3.2.49}{=} \mathbb{C}(g[B])u = \mathbb{C}g([B]u) = \mathbb{C}g[B.u] = \mathbb{C}[g.(B.u)].$$

Hence $g.(B.u) = (g.B).u$. □

3.2.52 Corollary (Simple row move). *Let $A \in V$, $t_i \in \mathbb{F}_q$ and $x_i(t_i) \in U$ ($i = 1,2,3,4,5,6$), such that* $\mathrm{supp}(x_i(t_i)^{-\top}A) \cap \triangledown \subseteq J \cup \{(4,5)\}$ *then*

$$\mathbb{C}\mathcal{O}_U([A]) \cong \mathbb{C}\mathcal{O}_U([x_i(t_i).A]).$$

3.2.53 Corollary. *Let $A \in V$ ($A_{17} = A_{16} = A_{15} = 0$, $A_{13} = A_{13}^* \neq 0$), and $x_1(t_1) \in U$ such that $t_1 A_{13}^* = A_{23}$, then $\mathbb{C}\mathcal{O}_U([A]) \cong \mathbb{C}\mathcal{O}_U([x_1(t_1).A]) \cong \mathbb{C}\mathcal{O}_U([A - A_{23}e_{23}])$, i.e.*

$$\mathbb{C}\mathcal{O}_U(\left[\begin{array}{|c|c|c|c|c|c|}\hline A_{12} & A_{13}^* & & & & \\ \hline & A_{23} & \multicolumn{4}{c}{} \\ \cline{2-2}\end{array}\right]) \cong \mathbb{C}\mathcal{O}_U(\left[\begin{array}{|c|c|c|c|c|c|}\hline A_{12} & A_{13}^* & & & & \\ \hline & 0 & \multicolumn{4}{c}{} \\ \cline{2-2}\end{array}\right]).$$

3.2.54 Corollary. *Every U-orbit module is isomorphic to a (not necessarily unique) staircase module, and the isomorphism is given by the left multiplication by a group element.*

Proof. Let $A \in V$ and $(1,3),(2,3) \in \mathrm{main}(A)$, then $(2,3)$ is deleted by 3.2.53. By 3.2.38, the claim is proved. $\square$

3.2.55 Lemma. *Let $A \in V$ with $A_{17} = A_{17}^* \in \mathbb{F}_q^*$, $x_5(s_5) \in U$ and $s_5 \in \mathbb{F}_q$, then*

$$\lambda_{x_5(s_5)}([A]) = \vartheta(s_5 A_{16})[A + s_5 A_{17}^* e_{23}].$$

Proof. Let $A \in V$ with $A_{17} = A_{17}^* \in \mathbb{F}_q^*$ and $x_5(s_5) \in U$, then

$$\begin{aligned}
\lambda_{x_5(s_5)}([A]) &= x_5(s_5)[A] \\
&\overset{3.2.40}{=} \frac{1}{|U|} \sum_{y \in U} \overline{\vartheta\kappa(x_5(s_5)^{-\top}A, y)}y \\
&\overset{3.2.41}{=} \frac{1}{|U|} \sum_{y \in U} \overline{\vartheta\kappa(A - s_5 A_{16} e_{66} - s_5 A_{17}^* e_{67}, y)}y \\
&= \frac{1}{|U|} \sum_{y \in U} \left(\overline{\vartheta\kappa(A, y)} \cdot \overline{\vartheta(-s_5 A_{16})} \cdot \overline{\vartheta(-s_5 A_{17}^* y_{67})}\right)y \\
&= \vartheta(s_5 A_{16}) \cdot \frac{1}{|U|} \sum_{y \in U} \left(\overline{\vartheta\kappa(A, y)} \cdot \overline{\vartheta(-s_5 A_{17}^* y_{67})}\right)y \\
&\overset{3.1.27}{=} \vartheta(s_5 A_{16}) \cdot \frac{1}{|U|} \sum_{y \in U} \left(\overline{\vartheta\kappa(A, y)} \cdot \overline{\vartheta(s_5 A_{17}^* y_{23})}\right)y \\
&= \vartheta(s_5 A_{16}) \cdot \frac{1}{|U|} \sum_{y \in U} \overline{\vartheta\kappa(A + s_5 A_{17}^* e_{23}, y)}y \\
&= \vartheta(s_5 A_{16}) \cdot \frac{1}{|U|} \sum_{y \in U} \overline{\vartheta\kappa(A + s_5 A_{17}^* e_{23}, f(y))}y \\
&= \vartheta(s_5 A_{16})[A + s_5 A_{17}^* e_{23}].
\end{aligned}$$

$\square$

3.2.56 Proposition. *Let $A, B \in V$, $A_{17} = A_{17}^* \in \mathbb{F}_q^*$, and*

$$A := \begin{array}{|c|c|c|c|c|c|}\hline A_{12} & A_{13} & A_{15} & A_{15} & A_{16} & A_{17}^* \\ \hline \multicolumn{1}{c|}{} & A_{23} & \multicolumn{4}{c}{} \\ \cline{2-2}\end{array} \quad ,$$

$$B := \begin{array}{|c|c|c|c|c|c|}\hline A_{12} & A_{13} & A_{15} & A_{15} & A_{16} & A_{17}^* \\ \hline \multicolumn{1}{c|}{} & 0 & \multicolumn{4}{c}{} \\ \cline{2-2}\end{array}$$

then $\mathbb{C}\mathcal{O}_U([A]) \cong \mathbb{C}\mathcal{O}_U([B])$.

Proof. Let $C \in \mathcal{O}_U(A)$ and $s_5 := -\frac{A_{23}}{A_{17}^*} \in \mathbb{F}_q$. By 3.2.55, we get $\lambda_{x_5(s_5)}([C]) = \vartheta(s_5 C_{16})[C + s_5 A_{17}^* e_{23}]$, where $C + s_5 A_{17}^* e_{23} \in \mathcal{O}_U(B)$. Thus $\mathbb{C}\mathcal{O}_U([A]) \cong \mathbb{C}\mathcal{O}_U([B])$. $\square$

3.2.57 Proposition. *Every U-orbit module is isomorphic to a certain hook-separated staircase module.*

Proof. By 3.2.54, every U-orbit module is isomorphic to a staircase module. By 3.2.56, we get the desired conclusion. $\square$

As for the other groups, we get the following three properties for the $G_2^{syl}(q)$-orbit modules.

3.2.58 Lemma. *Let $A, B \in V$. Then $\operatorname{Hom}_{\mathbb{C}U}(\mathbb{C}\mathcal{O}_U([A]), \mathbb{C}\mathcal{O}_U([B])) = \{0\}$ if and only if for all $C \in \mathcal{O}_U(A)$ and $D \in \mathcal{O}_U(B)$ holds $\operatorname{Hom}_{\operatorname{Stab}_U(C,D)}(\mathbb{C}[C], \mathbb{C}[D]) = \{0\}$.*

3.2.59 Lemma. *Let $A, B \in V$ and $y \in U$. Then*

$$\operatorname{Hom}_{\operatorname{Stab}_U(A,B)}(\mathbb{C}[A], \mathbb{C}[B]) = \operatorname{Hom}_{\operatorname{Stab}_U(A.y,B.y)}(\mathbb{C}[A.y], \mathbb{C}[B.y])$$

as $\mathbb{C}$-vector spaces.

3.2.60 Corollary (Homomorphism criterion)**.** *Let $A, B \in V$. Then*

$$\operatorname{Hom}_{\mathbb{C}U}(\mathbb{C}\mathcal{O}_U([A]), \mathbb{C}\mathcal{O}_U([B])) = \{0\}$$

if and only if $\operatorname{Hom}_{\operatorname{Stab}_U(A,D)}(\mathbb{C}[A], \mathbb{C}[D]) = \{0\}$ for all $D \in \mathcal{O}_U(B)$.

3.2.61 Proposition. *All of the hook-separated staircase $G_2^{syl}(q)$-modules are listed in Table 3.3, and they satisfy the following properties.*

(1) Let $A, B \in V$. If $\operatorname{verge}_1(A) \neq \operatorname{verge}_1(B)$, $\operatorname{Hom}_{\mathbb{C}U}(\mathbb{C}\mathcal{O}_U([A]), \mathbb{C}\mathcal{O}_U([B])) = \{0\}$. In particular, let $\mathbb{C}\mathcal{O}_U([A]) \in \mathfrak{F}_i$, $\mathbb{C}\mathcal{O}_U([B]) \in \mathfrak{F}_j$ and $i \neq j$, then

$$\operatorname{Hom}_{\mathbb{C}U}(\mathbb{C}\mathcal{O}_U([A]), \mathbb{C}\mathcal{O}_U([B])) = \{0\}.$$

(2) In family $\mathfrak{F}_{1,2}$, the q^2 hook-separated staircase modules are irreducible and pairwise orthogonal.

(3) In family $\mathfrak{F}_3$, the $(q-1)$ hook-separated staircase modules are irreducible and pairwise orthogonal.

(4) In family $\mathfrak{F}_4$, $\mathfrak{F}_5$ and $\mathfrak{F}_6$, the hook-separated staircase modules are reducible.

Table 3.3.: Hook-separated staircase $G_2^{syl}(q)$-modules

Family	$\mathbb{C}\mathcal{O}_U([A])$ $(A \in V)$	$\dim_{\mathbb{C}}\mathbb{C}\mathcal{O}_U([A])$	Irreducible
$\mathfrak{F}_6$	$\mathbb{C}\mathcal{O}_U\left(\begin{bmatrix} \cdot & \boxed{A_{12}} & \square & \square & \square & \square & \boxed{A^*_{17}} & \cdot \\ & \cdot & \boxed{0} & & & & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & \cdot & \cdot & & & \end{bmatrix}\right)$	q^3	NO
$\mathfrak{F}_5$	$\mathbb{C}\mathcal{O}_U\left(\begin{bmatrix} \cdot & \square & \boxed{A_{13}} & \square & \square & \boxed{A^*_{16}} & \square & \cdot \\ & \cdot & \boxed{A_{23}} & & & & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & \cdot & \cdot & & & \end{bmatrix}\right)$	q^2	NO
$\mathfrak{F}_4$	$\mathbb{C}\mathcal{O}_U\left(\begin{bmatrix} \cdot & \square & \square & \boxed{A^*_{15}} & \boxed{A^*_{15}} & \square & \square & \cdot \\ & \cdot & \boxed{A_{23}} & & & & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & \cdot & \cdot & & & \end{bmatrix}\right)$	q^2	NO
$\mathfrak{F}_3$	$\mathbb{C}\mathcal{O}_U\left(\begin{bmatrix} \cdot & \square & \boxed{A^*_{13}} & \square & \square & \square & \square & \cdot \\ & \cdot & \boxed{0} & & & & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & \cdot & \cdot & & & \end{bmatrix}\right)$	q	YES
$\mathfrak{F}_{1,2}$	$\mathbb{C}\mathcal{O}_U\left(\begin{bmatrix} \cdot & \boxed{A_{12}} & \square & \square & \square & \square & \square & \cdot \\ & \cdot & \boxed{A_{23}} & & & & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & \cdot & \cdot & & & \end{bmatrix}\right)$	1	YES

where $A^*_{12}, A^*_{13}, A^*_{15}, A^*_{16}, A^*_{17}, A^*_{23} \in \mathbb{F}_q^*$.

Proof. By the definition of the hook-separated staircase module, we get all of the hook-separated staircase $G_2^{syl}(q)$-modules in Table 3.3.

(a) Let $A, B \in V$ be hook-separated staircase core patterns of family $\mathfrak{F}_4$ and $C \in \mathcal{O}_U(B)$, i.e.

$$A := \begin{array}{cccccc} \square & \square & \boxed{A^*_{15}} & \boxed{A^*_{15}} & \square & \square \\ & \boxed{A_{23}} & & & & \end{array}, \qquad B := \begin{array}{cccccc} \square & \square & \boxed{B^*_{15}} & \boxed{B^*_{15}} & \square & \square \\ & \boxed{B_{23}} & & & & \end{array}.$$

By 3.2.39, we get $\mathrm{Stab}_U(A) \overset{A_{13}=0}{=} Y_2Y_4Y_5Y_6$. Then,

$$\mathrm{Stab}_U(A,C) = \begin{cases} Y_2Y_4Y_5Y_6, & C_{13}=0 \\ Y_4Y_5Y_6, & C_{13}\neq 0 \end{cases}.$$

We calculate the inner product

$$\langle \chi_A, \chi_C \rangle_{\mathrm{Stab}_U(A,C)} = \frac{1}{|\mathrm{Stab}_U(A,C)|} \sum_{y\in \mathrm{Stab}_U(A,C)} \vartheta\kappa(A-C, f(y))$$

$$=\frac{1}{|\mathrm{Stab}_U(A,C)|}$$

$$\cdot \sum_{y\in \mathrm{Stab}_U(A,C)} \vartheta\kappa\left(\begin{array}{|c|c|c|c|c|c|}\hline -C_{12} & -C_{13} & A^*_{15}-B^*_{15} & A^*_{15}-B^*_{15} & & \\ \hline & A_{23}-B_{23} & \multicolumn{4}{c}{} \\ \cline{2-2} \end{array}, f(y) \right).$$

If $C_{13}=0$, we have

$$C \in \left\{ \begin{array}{|c|c|c|c|c|c|}\hline -2B^*_{15}s_3 & 0 & B^*_{15} & B^*_{15} & & \\ \hline & B_{23} & \multicolumn{4}{c}{} \\ \cline{2-2} \end{array} \;\middle|\; s_3\in\mathbb{F}_q \right\}.$$

Hence

$$\begin{aligned} 0 \neq & \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{Stab}_U(A,C)}(\mathbb{C}[A], \mathbb{C}[C]) = \langle \chi_A, \chi_C \rangle_{\mathrm{Stab}_U(A,C)} \\ =& \frac{1}{|Y_2Y_4Y_5Y_6|} \sum_{t_2,t_4,t_5,t_6\in\mathbb{F}_q} \vartheta\Big((A_{23}-B_{23})t_2 + 2(A^*_{15}-B^*_{15})t_4 \Big) \\ =& \left(\frac{1}{q}\sum_{t_2\in\mathbb{F}_q} \vartheta\Big((A_{23}-B_{23})t_2\Big) \right)\left(\frac{1}{q}\sum_{t_4\in\mathbb{F}_q} \vartheta\Big(2(A^*_{15}-B^*_{15})t_4\Big) \right) \\ \iff & \{B_{23}=A_{23}\} \wedge \{B^*_{15}=A^*_{15}\}. \end{aligned}$$

If $C_{13}\neq 0$,

$$C \in \left\{ \begin{array}{|c|c|c|c|c|c|}\hline -2B^*_{15}s_3 & -2B^*_{15}s^*_1 & {B^*_{15}}^q & B^*_{15} & & \\ \hline & B_{23} & \multicolumn{4}{c}{} \\ \cline{2-2} \end{array} \;\middle|\; s^*_1\in\mathbb{F}^*_q,\, s_3\in\mathbb{F}_q \right\}.$$

Therefore,

$$\begin{aligned} 0 \neq & \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{Stab}_U(A,C)}(\mathbb{C}[A], \mathbb{C}[C]) = \langle \chi_A, \chi_C \rangle_{\mathrm{Stab}_U(A,C)} \\ =& \frac{1}{|Y_4Y_5Y_6|} \sum_{t_4,t_5,t_6\in\mathbb{F}_q} \vartheta\Big(2(A^*_{15}-B^*_{15})t_4\Big) = \frac{1}{q}\sum_{t_4\in\mathbb{F}_q} \vartheta\Big(2(A^*_{15}-B^*_{15})t_4\Big) \\ \iff & B^*_{15}=A^*_{15}. \end{aligned}$$

We get

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{Stab}_U(A,C)}(\mathbb{C}[A], \mathbb{C}[C]) \neq \{0\} \iff \langle \chi_A, \chi_C \rangle_{\mathrm{Stab}_U(A,C)} \neq 0 \quad (i.e. =1) \\ \iff & \Big\{\{B_{23}=A_{23}\} \wedge \{B^*_{15}=A^*_{15}\}\Big\} \wedge \{B^*_{15}=A^*_{15}\} \iff B^*_{15}=A^*_{15}. \end{aligned}$$

Thus $\mathrm{Hom}_{\mathbb{C}U}(\mathbb{C}\mathcal{O}_U([A]), \mathbb{C}\mathcal{O}_U([B])) = \{0\} \iff B^*_{15}\neq A^*_{15}$.

(b) Let $A \in \mathfrak{F}_i$ and $B \in \mathfrak{F}_j$, ψ_A denote the character of $\mathbb{C}\mathcal{O}_U([A])$ and ψ_B the character of $\mathbb{C}\mathcal{O}_U([B])$. In the similar way to (a), we calculate $\langle \psi_A, \psi_B \rangle_U$. Then the statements of (1) are proved by 2.2.82.

(c) Let $A, B \in V$ be hook-separated staircase core patterns of family $\mathfrak{F}_4$. Let $D \in \mathcal{O}_U(A)$ and ψ_A denote the character of $\mathbb{C}\mathcal{O}_U([A])$. By (a), the $\mathbb{C}\mathrm{Stab}_U(A, D)$-modules $\mathbb{C}[A]$ and $\mathbb{C}[D]$ satisfy

$$\mathbb{C}\left[\begin{array}{|c|c|c|c|c|c|}\hline 0 & 0 & A_{15}^* & A_{15}^* & & \\ \hline \multicolumn{1}{c|}{} & A_{23} & \multicolumn{4}{c}{} \\ \cline{2-2}\end{array}\right] \cong \mathbb{C}\left[\begin{array}{|c|c|c|c|c|c|}\hline D_{12} & 0 & A_{15}^* & A_{15}^* & & \\ \hline \multicolumn{1}{c|}{} & A_{23} & \multicolumn{4}{c}{} \\ \cline{2-2}\end{array}\right]$$

$$\mathbb{C}\left[\begin{array}{|c|c|c|c|c|c|}\hline 0 & 0 & A_{15}^* & A_{15}^* & & \\ \hline \multicolumn{1}{c|}{} & A_{23} & \multicolumn{4}{c}{} \\ \cline{2-2}\end{array}\right] \cong \mathbb{C}\left[\begin{array}{|c|c|c|c|c|c|}\hline D_{12} & D_{13}^* & A_{15}^* & A_{15}^* & & \\ \hline \multicolumn{1}{c|}{} & A_{23} & \multicolumn{4}{c}{} \\ \cline{2-2}\end{array}\right]$$

where $D_{13}^* \in \mathbb{F}_q^*$. By the proof of 2.2.82, we get

$$\begin{aligned}
&\dim_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}U}(\mathbb{C}\mathcal{O}_U([A]), \mathbb{C}\mathcal{O}_U([A])) = \langle \psi_A, \psi_A \rangle_U \\
=&\sum_{D \in \mathcal{O}_U(A)} \frac{|\mathrm{Stab}_U(A, D)|}{|\mathrm{Stab}_U(A)|} \dim_{\mathbb{C}}\mathrm{Hom}_{\mathrm{Stab}_U(A,D)}(\mathbb{C}[A], \mathbb{C}[D]) \\
=&\frac{q^4 \cdot q}{q^4} + \frac{q^3 \cdot (q-1)q}{q^4} = 2q - 1 > 1.
\end{aligned}$$

Thus, $\mathbb{C}\mathcal{O}_U([A])$ is not irreducible.

(d) Let $A \in V$ be a hook-separated staircase core pattern of family $\mathfrak{F}_5$. In the similar way to (c), $\mathbb{C}\mathcal{O}_U([A])$ is not irreducible.

(e) Let $A, B \in V$ be hook-separated staircase core patterns of family $\mathfrak{F}_3$ and $A \neq B$. We have $\langle \psi_A, \psi_A \rangle_U = 1$ and $\langle \psi_A, \psi_B \rangle_U = 0$. Thus the statements of (3) are proved.

(f) The q^2 hook-separated staircase modules of $\mathfrak{F}_{1,2}$ are of dimension 1, so they are irreducible. They are pairwise orthogonal by calculating $\langle \psi_A, \psi_B \rangle_U$ (c.f. (a)).

(g) Let $A \in V$ be a hook-separated staircase core pattern of family $\mathfrak{F}_6$, then the orbit module $\mathbb{C}\mathcal{O}_U([A])$ is reducible. Suppose it is irreducible, then by (1) and (2) we get

$$\left(\dim_{\mathbb{C}} \mathbb{C}\mathcal{O}_U([A])\right)^2 = q^6 < |U| - q^2 = q^6 - q^2.$$

This is a contradiction. Thus the orbit modules of family $\mathfrak{F}_6$ are reducible.

$\square$

3.2.62 Remark. *There exist two hook-separated staircase modules such that they are neither orthogonal nor isomorphic. For example: let $A, B \in V$ be hook-separated staircase core patterns of family $\mathfrak{F}_4$ with $A^*_{15} = B^*_{15} \in \mathbb{F}_q^*$ and $A_{23} \neq B_{23}$, i.e.*

$$A := \begin{array}{|c|c|c|c|c|c|}\hline & & A^*_{15} & A^*_{15} & & \\ \hline \multicolumn{1}{c|}{} & A_{23} & \multicolumn{4}{c}{} \\ \cline{2-2} \end{array}\ , \qquad B := \begin{array}{|c|c|c|c|c|c|}\hline & & A^*_{15} & A^*_{15} & & \\ \hline \multicolumn{1}{c|}{} & B_{23} & \multicolumn{4}{c}{} \\ \cline{2-2} \end{array}\ ,$$

then $\mathbb{C}\mathcal{O}_U([A])$ and $\mathbb{C}\mathcal{O}_U([B])$ are neither orthogonal nor isomorphic.

Proof. Let ψ_A denote the character of $\mathbb{C}\mathcal{O}_U([A])$ and ψ_B the character of $\mathbb{C}\mathcal{O}_U([B])$.
By the proof of 3.2.61, $\mathbb{C}\mathcal{O}_U([A])$ and $\mathbb{C}\mathcal{O}_U([B])$ are not orthogonal and

$$\langle \psi_A, \psi_A \rangle_U = 2q - 1 = \langle \psi_B, \psi_B \rangle_U.$$

Let $D \in \mathcal{O}_U(B)$. The $\mathbb{C}\mathrm{Stab}_U(A, D)$-modules $\mathbb{C}[A]$ and $\mathbb{C}[D]$ satisfy

$$\mathbb{C}\left[\begin{array}{|c|c|c|c|c|c|}\hline 0 & 0 & A^*_{15} & A^*_{15} & & \\ \hline \multicolumn{1}{c|}{} & {\color{red}A_{23}} & \multicolumn{4}{c}{} \\ \cline{2-2} \end{array}\right] \perp \mathbb{C}\left[\begin{array}{|c|c|c|c|c|c|}\hline D_{12} & {\color{green}0} & A^*_{15} & A^*_{15} & & \\ \hline \multicolumn{1}{c|}{} & {\color{blue}B_{23}} & \multicolumn{4}{c}{} \\ \cline{2-2} \end{array}\right]$$

$$\mathbb{C}\left[\begin{array}{|c|c|c|c|c|c|}\hline 0 & 0 & A^*_{15} & A^*_{15} & & \\ \hline \multicolumn{1}{c|}{} & {\color{red}A_{23}} & \multicolumn{4}{c}{} \\ \cline{2-2} \end{array}\right] \cong \mathbb{C}\left[\begin{array}{|c|c|c|c|c|c|}\hline D_{12} & {\color{green}D^*_{13}} & A^*_{15} & A^*_{15} & & \\ \hline \multicolumn{1}{c|}{} & {\color{blue}B_{23}} & \multicolumn{4}{c}{} \\ \cline{2-2} \end{array}\right].$$

By the proof of 3.2.60, we get

$$\begin{aligned} &\dim_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}U}(\mathbb{C}\mathcal{O}_U([A]), \mathbb{C}\mathcal{O}_U([B])) = \langle \psi_A, \psi_B \rangle_U \\ =& \sum_{D \in \mathcal{O}_U(B)} \frac{|\mathrm{Stab}_U(A, D)|}{|\mathrm{Stab}_U(A)|} \dim_{\mathbb{C}}\mathrm{Hom}_{\mathrm{Stab}_U(A,D)}(\mathbb{C}[A], \mathbb{C}[D]) \\ =& \frac{q^3 \cdot (q-1)q}{q^4} = q - 1 \neq \langle \psi_A, \psi_A \rangle_U. \end{aligned}$$

Thus, $\mathbb{C}\mathcal{O}_U([A])$ and $\mathbb{C}\mathcal{O}_U([B])$ are not isomorphic.

$\square$

3.3. Supercharacter theories for $G_2^{syl}(q)$

Let $G := G_8(q)$, $U := G_2^{syl}(q)$, and $t_i^* \in \mathbb{F}_q^*$ for all $i = 1, 2, \cdots, 6$.

In this section, we obtain the conjugacy classes of $G_2^{syl}(q)$ (3.3.4).

Then we determine the supercharacter theory for $G_2^{syl}(q)$ (3.3.13), establish the supercharacter table of $G_2^{syl}(q)$ in Table 3.6 and list supercharacters, supermodules and superclasses of $G_2^{syl}(q)$ in Table 3.7.

After that, we construct the complex irreducible characters of $G_2^{syl}(q)$ $(p > 3)$ (3.3.23), and calculate the character table of $G_2^{syl}(q)$ $(p > 3)$ (3.3.26), determine the relations between the supercharacters and the irreducible characters of $G_2^{syl}(q)$ $(p > 3)$ (3.3.27).

3.3.1. Conjugacy classes of $G_2^{syl}(q)$

In this subsection, we determine the conjugacy classes of $G_2^{syl}(q)$ (3.3.4), and obtain one partition of $G_2^{syl}(q)$ (3.3.6) which is a set of the superclasses proved in Subsection 3.3.2.

3.3.1 Notation. *Let $x, u \in U$, then the conjugate of x by u is ${}^u x := uxu^{-1}$, and the conjugacy class of u is ${}^U x := \{vxv^{-1} \mid v \in U\}$.*

By 3.1.25, we obtain the following conjugate elements when $p \neq 3$.

3.3.2 Lemma. *Let $\operatorname{char} \mathbb{F}_q = p \notin \{2, 3\}$, $u := x(r_1, r_2, r_3, r_4, r_5, r_6) \in U$ and $x_i(t_i) \in U$, then*

$$
\begin{aligned}
{}^u x_6(t_6) =& x_6(t_6),\\
{}^u x_5(t_5) =& x_5(t_5) \cdot x_6(r_2 t_5),\\
{}^u x_4(t_4) =& x_4(t_4) \cdot x_5(3r_1 t_4) \cdot x_6(3r_1 r_2 t_4 + 3r_3 t_4),\\
{}^u x_3(t_3) =& x_3(t_3) \cdot x_4(2r_1 t_3) \cdot x_5(3r_1^2 t_3) \cdot x_6(3r_1^2 r_2 t_3 - 3r_1 t_3^2 - 3t_3 r_4),\\
{}^u x_2(t_2) =& x_2(t_2) \cdot x_3(-r_1 t_2) \cdot x_4(-t_2 r_1^2) \cdot x_5(-t_2 r_1^3)\\
& \cdot x_6(-t_2 r_5 - t_2^2 r_1^3 - t_2 r_1^3 r_2),\\
{}^u x_1(t_1) =& x_1(t_1) \cdot x_3(r_2 t_1) \cdot x_4(-r_2 t_1^2 - 2t_1 r_3)\\
& \cdot x_5(r_2 t_1^3 - 6r_1 r_3 t_1 + 3r_3 t_1^2 - 3t_1 r_4)\\
& \cdot x_6(2r_2^2 t_1^3 - 6r_1 r_2 r_3 t_1 + 3r_2 r_3 t_1^2 - 3r_2 r_4 t_1 - 3t_1 r_3^2),
\end{aligned}
$$

$$
{}^u\big(x_4(t_4) x_5(t_5) x_6(t_6)\big) = x_4(t_4) \cdot x_5(t_5 + 3r_1 t_4) \cdot x_6(t_6 + r_2 t_5 + 3r_1 r_2 t_4 + 3r_3 t_4),
$$

and

$$\begin{aligned}
{}^{u}\big(x_3(t_3)x_5(t_5)\big) =& x_3(t_3)\cdot x_4(2r_1t_3)\cdot x_5(t_5+3r_1^2t_3)\\
&\cdot x_6(r_2t_5+3r_1^2r_2t_3-3r_1t_3^2-3t_3r_4),\\
{}^{u}\big(x_2(t_2)x_4(t_4)x_5(t_5)\big) =& x_2(t_2)\cdot x_3(-r_1t_2)\cdot x_4(t_4-t_2r_1^2)\\
&\cdot x_5(t_5-t_2r_1^3+3r_1t_4)\\
&\cdot x_6(-t_2r_5-t_2^2r_1^3-t_2r_1^3r_2+3r_1r_2t_4+3r_3t_4+r_2t_5),\\
{}^{u}\big(x_2(t_2)x_1(t_1)\big) =& x_2(t_2)x_1(t_1)\cdot x_3(r_2t_1-r_1t_2)\\
&\cdot x_4(-r_2t_1^2-2t_1r_3-t_2r_1^2+2t_1t_2r_1)\\
&\cdot x_5(r_2t_1^3-6r_1r_3t_1+3r_3t_1^2-3t_1r_4\\
&\quad -t_2r_1^3-3r_1t_1^2t_2+3t_1t_2r_1^2)\\
&\cdot x_6(2r_2^2t_1^3-6r_1r_2r_3t_1+3r_2r_3t_1^2-3r_2r_4t_1-3t_1r_3^2\\
&\quad -t_2r_5-t_2^2r_1^3-t_2r_1^3r_2-6r_1r_2t_1^2t_2+3t_1t_2r_1^2r_2\\
&\quad +3r_1^2t_1t_2^2).
\end{aligned}$$

By 3.1.25, we also get the conjugate elements when $p=3$.

3.3.3 Lemma. *Let* $\operatorname{char}\mathbb{F}_q=p=3$*,* $u:=x(r_1,r_2,r_3,r_4,r_5,r_6)\in U$ *and* $x_i(t_i)\in U$*, then*

$$\begin{aligned}
{}^{u}x_6(t_6) =& x_6(t_6),\\
{}^{u}x_5(t_5) =& x_5(t_5)\cdot x_6(r_2t_5),\\
{}^{u}x_4(t_4) =& x_4(t_4),\\
{}^{u}x_3(t_3) =& x_3(t_3)\cdot x_4(2r_1t_3),\\
{}^{u}x_2(t_2) =& x_2(t_2)\cdot x_3(-r_1t_2)\cdot x_4(-t_2r_1^2)\cdot x_5(-t_2r_1^3)\cdot x_6(-t_2r_5-t_2^2r_1^3-t_2r_1^3r_2),\\
{}^{u}x_1(t_1) =& x_1(t_1)\cdot x_3(r_2t_1)\cdot x_4(-r_2t_1^2-2t_1r_3)\cdot x_5(r_2t_1^3)\cdot x_6(2r_2^2t_1^3),
\end{aligned}$$

and

$$\begin{aligned}
{}^{u}(x_4(t_4)x_5(t_5)) =& x_4(t_4)x_5(t_5)\cdot x_6(r_2t_5),\\
{}^{u}\big(x_3(t_3)x_5(t_5)\big) =& x_3(t_3)\cdot x_4(2r_1t_3)\cdot x_5(t_5)\cdot x_6(r_2t_5),\\
{}^{u}\big(x_2(t_2)x_4(t_4)x_5(t_5)\big) =& x_2(t_2)\cdot x_3(-r_1t_2)\cdot x_4(t_4-t_2r_1^2)\cdot x_5(t_5-t_2r_1^3)\\
&\cdot x_6(-t_2r_5-t_2^2r_1^3-t_2r_1^3r_2+r_2t_5),\\
{}^{u}(x_1(t_1)x_5(t_5)x_6(t_6)) =& x_1(t_1)\cdot x_3(r_2t_1)\cdot x_4(-r_2t_1^2-2t_1r_3)\cdot x_5(t_5+r_2t_1^3)\\
&\cdot x_6(t_6+2r_2^2t_1^3+r_2t_5),\\
{}^{u}\big(x_2(t_2)x_1(t_1)\big) =& x_2(t_2)x_1(t_1)\cdot x_3(r_2t_1-r_1t_2)\\
&\cdot x_4(-r_2t_1^2-2t_1r_3-t_2r_1^2+2t_1t_2r_1)\cdot x_5(r_2t_1^3-t_2r_1^3)\\
&\cdot x_6(2r_2^2t_1^3-t_2r_5-t_2^2r_1^3-t_2r_1^3r_2).
\end{aligned}$$

3.3.4 Proposition (Conjugacy classes of $G_2^{syl}(q)$)**.** *Let* $\operatorname{char}\mathbb{F}_q = p \notin \{2,3\}$. *The conjugacy classes of* $G_2^{syl}(q)$ *are listed in Table 3.4.*

Table 3.4.: Conjugacy classes of $G_2^{syl}(q)$ with $p \notin \{2,3\}$

Representatives	#Conjugacy Classes	Conjugacy Classes	|Conjugacy Class|
I_8	1	$x(0,0,0,0,0,0)$	1
$x_6(t_6^*)$	$q-1$	$x(0,0,0,0,0,t_6^*)$	1
$x_5(t_5^*)$	$q-1$	$x(0,0,0,0,t_5^*,s_6)$ $s_6 \in \mathbb{F}_q$	q
$x_4(t_4^*)$	$q-1$	$x(0,0,0,t_4^*,s_5,s_6)$ $s_5,s_6 \in \mathbb{F}_q$	q^2
$x(0,0,t_3^*,0,t_5,0)$	$(q-1)q$	$x(0,0,t_3^*,s_4,\hat{s}_5,s_6)$ $s_4,s_6 \in \mathbb{F}_q$	q^2
$x(0,t_2^*,0,t_4,t_5,0)$	$(q-1)q^2$	$x(0,t_2^*,s_3,\hat{s}_4,\hat{s}_5,s_6)$ $s_3,s_6 \in \mathbb{F}_q$	q^2
$x(t_1^*,0,0,0,0,t_6)$	$(q-1)q$	$x(t_1^*,0,s_3,s_4,s_5,\hat{s}_6)$ $s_3,s_4,s_5 \in \mathbb{F}_q$	q^3
$x(t_1^*,t_2^*,0,0,0,0)$	$(q-1)^2$	$x(t_1^*,t_2^*,s_3,s_4,s_5,s_6)$ $s_3,s_4,s_5,s_6 \in \mathbb{F}_q$	q^4

where $\hat{s}_-$ is determined by some of t_-^*, t_- and s_-.

Proof. Let $u := x(r_1,r_2,r_3,r_4,r_5,r_6) \in U$, $0 \neq t_1 \in \mathbb{F}_q^*$, $t_6 \in \mathbb{F}_q$, and

$$x(a_1,a_2,a_3,a_4,a_5,a_6) := {}^{u}\big(x_1(t_1)x_6(t_6)\big),$$

then by 3.3.2,

$$\begin{aligned} a_1 &= t_1, \qquad a_2 = 0, \qquad a_3 = r_2t_1, \qquad a_4 = -r_2t_1^2 - 2r_3t_1, \\ a_5 &= r_2t_1^3 - 6r_1r_3t_1 + 3r_3t_1^2 - 3t_1r_4, \qquad a_6 = t_6 + r_2^2t_1^3 + r_2a_5 - 3r_3^2t_1. \end{aligned}$$

Let a_3, a_4 and a_5 be fixed, then a_6 is determined uniquely. Hence we get the conjugacy class of $x_1(t_1)x_6(t_6)$.

$${}^{U}\big(x_1(t_1)x_6(t_6)\big) = \{x(t_1,0,s_3,s_4,s_5,\hat{s}_6) \mid s_3,s_4,s_5 \in \mathbb{F}_q\}.$$

By 3.3.2, other cases are determined analogously. Then all of the conjugacy classes are obtained. □

3.3.5 Notation/Lemma. *Set*

$$\begin{aligned}
&C_6(t_6^*) :={}^U x_6(t_6^*), \quad C_5(t_5^*) := {}^U x_5(t_5^*), \quad C_4(t_4^*) := {}^U x_4(t_4^*),\\
&C_3(t_3^*) := \bigcup_{t_5\in\mathbb{F}_q} {}^U\big(x_3(t_3^*)x_5(t_5)\big), \quad C_2(t_2^*) := \bigcup_{t_4,t_5\in\mathbb{F}_q} {}^U\big(x_2(t_2^*)x_4(t_4)x_5(t_5)\big),\\
&C_1(t_1^*) :={}^U x_1(t_1^*), \quad C_{1,2}(t_1^*,t_2^*) := {}^U\big(x_2(t_2^*)x_1(t_1^*)\big), \quad C_0 := \{1_U\} = \{1\}.
\end{aligned}$$

Note that they can form a partition of U as well, denoted by $\mathcal{K}$.

We get a partition of $G_2^{syl}(q)$ by 3.3.4.

3.3.6 Corollary (A partition of $G_2^{syl}(q)$)**.** *Let $u \in U$, then the partition $\mathcal{K}$ is given in Table 3.5:*

Table 3.5.: A partition $\mathcal{K}$ of $G_2^{syl}(q)$ with $p \notin \{2,3\}$

$u \in U$	$\#C_{1,2}(t_1^*,t_2^*)$ or $\#C_i(t_i^*)$	$C_{1,2}(t_1^*,t_2^*)$ or $C_i(t_i^*)$	$\lvert C_{1,2}(t_1^*,t_2^*)\rvert$ or $\lvert C_i(t_i^*)\rvert$
1	1	$x(0,0,0,0,0,0)$	1
$x_6(t_6^*)$	$q-1$	$x(0,0,0,0,0,t_6^*)$	1
$x_5(t_5^*)$	$q-1$	$x(0,0,0,0,t_5^*,s_6)$ $s_6 \in \mathbb{F}_q$	q
$x_4(t_4^*)$	$q-1$	$x(0,0,0,t_4^*,s_5,s_6)$ $s_5,s_6 \in \mathbb{F}_q$	q^2
$x_3(t_3^*)$	$q-1$	$x(0,0,t_3^*,s_4,s_5,s_6)$ $s_4,s_5,s_6 \in \mathbb{F}_q$	q^3
$x_2(t_2^*)$	$q-1$	$x(0,t_2^*,s_3,s_4,s_5,s_6)$ $s_3,s_4,s_5,s_6 \in \mathbb{F}_q$	q^4
$x_1(t_1^*)$	$q-1$	$x(t_1^*,0,s_3,s_4,s_5,s_6)$ $s_3,s_4,s_5,s_6 \in \mathbb{F}_q$	q^4
$x_2(t_2^*)x_1(t_1^*)$	$(q-1)^2$	$x(t_1^*,t_2^*,s_3,s_4,s_5,s_6)$ $s_3,s_4,s_5,s_6 \in \mathbb{F}_q$	q^4

3.3.7 Notation/Lemma. *Set*

$$C_i := \dot{\bigcup_{t_i\in\mathbb{F}_q^*}} C_i(t_i^*) \quad \textit{for all } i = 1,2,3,4,5,6,$$

$$C_{1,2} := \dot{\bigcup_{t_1^*,t_2^*\in\mathbb{F}_q^*}} C_{1,2}(t_1^*,t_2^*), \quad C_0 := \{1_U\} = \{1\}.$$

Then

$$\mathcal{K} = (\dot{\bigcup_{i\in\{0,1,\ldots,6\}}} C_i) \dot{\bigcup} C_{1,2}.$$

3.3.8 Remark. *We can also determine the partition for $G_2^{syl}(q)$ $(p > 2)$ in 3.3.6 by calculating $\{I_8 + x(u-1)y \mid x, y \in G\} \cap U$ for all $u \in U$ (c.f. 2.3.1).*

3.3.2. A supercharacter theory for $G_2^{syl}(q)$

In this subsection, let $U := G_2^{syl}(q)$, $A_{ij}^* \in \mathbb{F}_q^*$ and $A_{ij} \in \mathbb{F}_q$.

We determine the supercharacter theory for $G_2^{syl}(q)$ (3.3.13), establish the supercharacter table of $G_2^{syl}(q)$ in Table 3.6 and list supercharacters, supermodules and superclasses of $G_2^{syl}(q)$ in Table 3.7.

3.3.9 Notation/Lemma. *Let $A = (A_{ij}) \in V$, set*

$$M(A_{12}e_{12}+A_{23}e_{23}) := \mathbb{C}\mathcal{O}_U\left(\left[\begin{array}{|c|c|c|c|c|c|}\hline A_{12} & & & & & \\ \hline \multicolumn{1}{c|}{} & A_{23} & \multicolumn{4}{c}{} \\ \cline{2-2}\end{array}\right]\right) = \mathbb{C}[A_{12}e_{12}+A_{23}e_{23}],$$

$$\begin{aligned} M(A_{13}^*e_{13}) :=& \mathbb{C}\left\{\left[\begin{array}{|c|c|c|c|c|c|}\hline A_{12} & A_{13}^* & & & & \\ \hline \multicolumn{1}{c|}{} & & \multicolumn{4}{c}{} \\ \cline{2-2}\end{array}\right] \,\middle|\, A_{12} \in \mathbb{F}_q\right\} \\ =& \mathbb{C}\mathcal{O}_U\left(\left[\begin{array}{|c|c|c|c|c|c|}\hline & A_{13}^* & & & & \\ \hline \multicolumn{1}{c|}{} & & \multicolumn{4}{c}{} \\ \cline{2-2}\end{array}\right]\right), \end{aligned}$$

$$\begin{aligned} M(A_{15}^*(e_{14}+e_{15})) :=& \mathbb{C}\left\{\left[\begin{array}{|c|c|c|c|c|c|}\hline A_{12} & A_{13} & A_{15}^* & A_{15}^* & & \\ \hline \multicolumn{1}{c|}{} & A_{23} & \multicolumn{4}{c}{} \\ \cline{2-2}\end{array}\right] \,\middle|\, A_{12}, A_{13}, A_{23} \in \mathbb{F}_q\right\} \\ =& \bigoplus_{A_{23}\in\mathbb{F}_q} \mathbb{C}\mathcal{O}_U\left(\left[\begin{array}{|c|c|c|c|c|c|}\hline & & A_{15}^* & A_{15}^* & & \\ \hline \multicolumn{1}{c|}{} & A_{23} & \multicolumn{4}{c}{} \\ \cline{2-2}\end{array}\right]\right), \end{aligned}$$

$$\begin{aligned} M(A_{16}^*e_{16}) :=& \mathbb{C}\left\{\left[\begin{array}{|c|c|c|c|c|c|}\hline A_{12} & A_{13} & A_{15} & A_{15} & A_{16}^* & \\ \hline \multicolumn{1}{c|}{} & A_{23} & \multicolumn{4}{c}{} \\ \cline{2-2}\end{array}\right] \,\middle|\, A_{12}, A_{13}, A_{15}, A_{23} \in \mathbb{F}_q\right\} \\ =& \bigoplus_{A_{13},A_{23}\in\mathbb{F}_q} \mathbb{C}\mathcal{O}_U\left(\left[\begin{array}{|c|c|c|c|c|c|}\hline & A_{13} & & & A_{16}^* & \\ \hline \multicolumn{1}{c|}{} & A_{23} & \multicolumn{4}{c}{} \\ \cline{2-2}\end{array}\right]\right), \end{aligned}$$

$$\begin{aligned} M(A_{17}^*e_{17}) :=& \mathbb{C}\left\{\left[\begin{array}{|c|c|c|c|c|c|}\hline A_{12} & A_{13} & A_{15} & A_{15} & A_{16} & A_{17}^* \\ \hline \multicolumn{1}{c|}{} & & \multicolumn{4}{c}{} \\ \cline{2-2}\end{array}\right] \,\middle|\, A_{12}, A_{13}, A_{15}, A_{16} \in \mathbb{F}_q\right\} \\ =& \bigoplus_{A_{12}\in\mathbb{F}_q} \mathbb{C}\mathcal{O}_U\left(\left[\begin{array}{|c|c|c|c|c|c|}\hline A_{12} & & & & & A_{17}^* \\ \hline \multicolumn{1}{c|}{} & & \multicolumn{4}{c}{} \\ \cline{2-2}\end{array}\right]\right). \end{aligned}$$

3.3.10 Notation. *Denote by $\mathcal{M}$ the set of all of the $\mathbb{C}U$-modules in 3.3.9.*

3.3.11 Notation. *(1) Let $M \in \mathcal{M}$, then the complex character of the $\mathbb{C}U$-module M is denoted by Ψ_M.*

(2) Set $\mathcal{X} := \{\Psi_M \mid M \in \mathcal{M}\}$.

3.3.12 Corollary. *Let $A = (A_{ij}) \in V$, and ψ_A be the character of $\mathbb{C}\mathcal{O}_U([A])$. Then*

$$\Psi_{M(A_{12}e_{12}+A_{23}e_{23})} = \psi_{A_{12}e_{12}+A_{23}e_{23}}, \qquad \Psi_{M(A^*_{13}e_{13})} = \psi_{A^*_{13}e_{13}},$$
$$\Psi_{M(A^*_{15}(e_{14}+e_{15}))} = \sum_{A_{23}\in\mathbb{F}_q} \psi_{A_{23}e_{23}+A^*_{15}(e_{14}+e_{15})},$$
$$\Psi_{M(A^*_{16}e_{16})} = \sum_{A_{13},A_{23}\in\mathbb{F}_q} \psi_{A_{13}e_{13}+A_{23}e_{23}+A^*_{16}e_{16}}, \qquad \Psi_{M(A^*_{17}e_{17})} = \sum_{A_{12}\in\mathbb{F}_q} \psi_{A_{12}e_{12}+A^*_{17}e_{17}}.$$

3.3.13 Proposition (Supercharacter theory for $G_2^{syl}(q)$)**.** *$(\mathcal{X}, \mathcal{K})$ is a supercharacter theory for $G_2^{syl}(q)$, where $\mathcal{K}$ is defined in 3.3.5, and $\mathcal{X}$ is defined in 3.3.11.*

Proof. By 3.3.5, $\mathcal{K}$ is a partition of U. We know that $\mathcal{X}$ is a set of nonzero complex characters of U.

(a) *Claim that $|\mathcal{X}| = |\mathcal{K}|$.*

By 3.3.5, 3.3.9 and 3.3.11, $|\{\Psi_{M(A^*_{17}e_{17})} \mid A^*_{17} \in \mathbb{F}_q^*\}| = |\{M(A^*_{17}e_{17}) \mid A^*_{17} \in \mathbb{F}_q^*\}| = |C_6|$. Similarly, we obtain the Table 3.7. Thus $|\mathcal{X}| = |\mathcal{K}|$.

(b) *Claim that the characters $\chi \in \mathcal{X}$ are constant on the members of $\mathcal{K}$.*

Let $A^*_{15} \in \mathbb{F}_q^*$ and

$$\mathcal{B}_{15}(A^*_{15}) := \left\{ \begin{array}{|c|c|c|c|c|c|} \hline C_{12} & C_{13} & A^*_{15} & A^*_{15} & & \\ \hline \multicolumn{1}{c|}{} & C_{23} & \multicolumn{4}{c}{} \\ \cline{2-2} \end{array} \;\middle|\; C_{12}, C_{13}, C_{23} \in \mathbb{F}_q \right\}.$$

Let $y \in U$, then

$$\Psi_{M(A^*_{15}(e_{14}+e_{15}))}(y) = \sum_{\substack{C\in\mathcal{B}_{15}(A^*_{15})\\ C.y=C}} \chi_C(y) = \sum_{\substack{C\in\mathcal{B}_{15}(A^*_{15})\\ y\in\mathrm{Stab}_U(C)}} \chi_C(y).$$

If $y = x(0,0,0,t_4,t_5,t_6) \in C_0 \cup C_4 \cup C_5 \cup C_6$, $y \in \mathrm{Stab}_U(C)$ for all $C \in \mathcal{B}_{15}(A^*_{15})$ by 3.2.39. Thus

$$\Psi_{M(A^*_{15}(e_{14}+e_{15}))}(y) = \sum_{C\in\mathcal{B}_{15}(A^*_{15})} \chi_C(y) = \sum_{C\in\mathcal{B}_{15}(A^*_{15})} \vartheta(2A^*_{15}t_4) = q^3 \cdot \vartheta(2A^*_{15}t_4).$$

If $y \in C_1 \cup C_{1,2} \cup C_3$, $y \notin \mathrm{Stab}_U(C)$ for all $C \in \mathcal{B}_{15}(A^*_{15})$ by 3.2.39. Thus

$$\Psi_{M(A^*_{15}(e_{14}+e_{15}))}(y) = 0.$$

If $y = x(0, t_2^*, s_3, s_4, s_5, s_6) \in C_2$, by 3.2.39

$$
\begin{aligned}
\Psi_{M(A_{15}^*(e_{14}+e_{15}))}(y) &= \sum_{\substack{C \in \mathcal{B}_{15}(A_{15}^*) \\ C_{13} = -\frac{2A_{15}^* s_3}{t_2^*}}} \chi_C(y) \\
&= \sum_{C_{12}, C_{23} \in \mathbb{F}_q} \vartheta_K \left(\begin{array}{|c|c|c|c|c|c|} \hline C_{12} & -\frac{2A_{15}^* s_3}{t_2^*} & A_{15}^* & A_{15}^* & & \\ \hline \multicolumn{1}{c|}{} & C_{23} & \multicolumn{4}{|c}{} \\ \cline{2-2} \end{array}, \begin{array}{|c|c|c|c|c|c|} \hline 0 & -s_3 & s_4 & s_4 & * & * \\ \hline \multicolumn{1}{c|}{} & t_2^* & \multicolumn{4}{|c}{} \\ \cline{2-2} \end{array} \right) \\
&= \sum_{C_{12}, C_{23} \in \mathbb{F}_q} \vartheta(C_{23} t_2^* + \frac{2A_{15}^* s_3^2}{t_2^*} + 2A_{15}^* s_4) \\
&= q \cdot \vartheta(\frac{2A_{15}^* s_3^2}{t_2^*} + 2A_{15}^* s_4) \cdot \sum_{C_{23} \in \mathbb{F}_q} \vartheta(C_{23} t_2^*) \\
&= 0.
\end{aligned}
$$

Similarly, we calculate the other values of the Table 3.6. Thus the claim is proved.

(c) The elements of $\mathcal{X}$ are pairwise orthogonal by 3.2.61.

(d) The set $\{1\}$ is a member of $\mathcal{K}$.

By 2.3.11 and 2.3.13, $(\mathcal{X}, \mathcal{K})$ is a supercharacter theory for $G_2^{syl}(q)$. $\square$

Table 3.6.: Supercharacter table of $G_2^{syl}(q)$

	C_0	$C_1(t_1^*)$	$C_2(t_2^*)$	$C_{1,2}(t_1^*, t_2^*)$	$C_3(t_3^*)$	$C_4(t_4^*)$	$C_5(t_5^*)$	$C_6(t_6^*)$
$\Psi_{M(0)}$	1	1	1	1	1	1	1	1
$\Psi_{M(A_{12}^* e_{12})}$	1	$\vartheta(A_{12}^* t_1^*)$	1	$\vartheta(A_{12}^* t_1^*)$	1	1	1	1
$\Psi_{M(A_{23}^* e_{23})}$	1	1	$\vartheta(A_{23}^* t_2^*)$	$\vartheta(A_{23}^* t_2^*)$	1	1	1	1
$\Psi_{M(A_{12}^* e_{12}+A_{23}^* e_{23})}$	1	$\vartheta(A_{12}^* t_1^*)$	$\vartheta(A_{23}^* t_2^*)$	$\vartheta(A_{12}^* t_1^* + A_{23}^* t_2^*)$	1	1	1	1
$\Psi_{M(A_{13}^* e_{13})}$	q	0	0	0	$\vartheta(-A_{13}^* t_3^*)$ $\cdot q$	q	q	q
$\Psi_{M(A_{15}^* (e_{14}+e_{15}))}$	q^3	0	0	0	0	$\vartheta(2A_{15}^* t_4^*)$ $\cdot q^3$	q^3	q^3
$\Psi_{M(A_{16}^* e_{16})}$	q^4	0	0	0	0	0	$\vartheta(A_{16}^* t_5^*)$ $\cdot q^4$	q^4
$\Psi_{M(A_{17}^* e_{17})}$	q^4	0	0	0	0	0	0	$\vartheta(A_{17}^* t_6^*)$ $\cdot q^4$

3.3.14 Corollary. *The supercharacters, supermodules and superclasses of $G_2^{syl}(q)$ are listed in Table 3.7.*

Table 3.7.: Supercharacters, supermodules and superclasses of $G_2^{syl}(q)$

Supercharacters $\Psi \in \mathcal{X}$	Supermodules $M \in \mathcal{M}$	Superclasses $C \in \mathcal{K}$	$\#\{\Psi\}=\#\{M\}=\#\{C\}$
$\Psi_{M(A_{17}^*e_{17})}$	$M(A_{17}^*e_{17})$	$C_6(t_6^*)$	$q-1$
$\Psi_{M(A_{16}^*e_{16})}$	$M(A_{16}^*e_{16})$	$C_5(t_5^*)$	$q-1$
$\Psi_{M(A_{15}^*(e_{14}+e_{15}))}$	$M(A_{15}^*(e_{14}+e_{15}))$	$C_4(t_4^*)$	$q-1$
$\Psi_{M(A_{13}^*e_{13})}$	$M(A_{13}^*e_{13})$	$C_3(t_3^*)$	$q-1$
$\Psi_{M(A_{12}^*e_{12}+A_{23}^*e_{23})}$	$M(A_{12}^*e_{12}+A_{23}^*e_{23})$	$C_{1,2}(t_1^*,t_2^*)$	$(q-1)^2$
$\Psi_{M(A_{23}^*e_{23})}$	$M(A_{23}^*e_{23})$	$C_2(t_2^*)$	$q-1$
$\Psi_{M(A_{12}^*e_{12})}$	$M(A_{12}^*e_{12})$	$C_1(t_1^*)$	$q-1$
$\Psi_{M(0)}$	$M(0)$	C_0	1

3.3.15 Corollary. *The number of the supercharacters $G_2^{syl}(q)$ is*

$$|\mathcal{X}| = |\mathcal{M}| = |\mathcal{K}| = q^2+4q-4 = (q-1)^2+6(q-1)+1.$$

3.3.3. Character tables of $G_2^{syl}(q)$

Let $\operatorname{char}\mathbb{F}_q = p \notin \{2,3\}$ and $U := G_2^{syl}(q)$.

In this subsection, we construct the complex irreducible characters of $G_2^{syl}(q)$ (3.3.23), dtermine the character table of $G_2^{syl}(q)$ (3.3.26) and obtain the relations between the supercharacters and the irreducible characters of $G_2^{syl}(q)$ (3.3.27).

Let G be a finite group, then $\operatorname{Irr}(G)$ denotes the set of all complex irreducible characters of G. Let $\chi \in \operatorname{Irr}(U)$.

3.3.16 Lemma. *Let $Y_i \leqslant U$, then*

$$Z(U) = Y_6,\ Z(Y_6\backslash U) = \bar{Y}_5,\ Z(Y_5Y_6\backslash U) = \bar{Y}_4,\ Z(Y_4Y_5Y_6\backslash U) = \bar{Y}_3,$$

and $Y_4Y_4Y_5Y_6\backslash U$ is abelian.

Proof. From the commutator relations in 3.1.25, we get the centers of the groups. □

3.3.17 Lemma. *Let $T := Y_2Y_3Y_4Y_5Y_6$, $N := Y_4Y_5Y_6$, and $H := Y_1Y_4Y_5Y_6$.*

(1) The subgroup N is abelian, $N \trianglelefteq U$, $T \trianglelefteq U$ and $H \leqslant U$ as follows:

$$\begin{tikzcd}[column sep=small]
& \substack{U=TH=\\ Y_2Y_1Y_3Y_4Y_5Y_6} \arrow[dl, dash, "\trianglelefteq"'] \arrow[dr, dash, "\geqslant"] & \\
\substack{T=\\ Y_2Y_3Y_4Y_5Y_6} \arrow[dr, dash, "\trianglerighteq"'] & & \substack{H=\\ Y_1Y_4Y_5Y_6} \arrow[dl, dash, "\trianglelefteq"] \\
& \substack{N=T\cap H=\\ Y_4Y_5Y_6} &
\end{tikzcd}$$

(2) Let $\lambda \in \operatorname{Irr}(N)$ and $\operatorname{Res}^N_{Y_6}\lambda \neq \operatorname{triv}_{Y_6}$. If λ satisfies that $\operatorname{Res}^N_{Y_5}\lambda = \operatorname{triv}_{Y_5}$, we have

$$I_U(\lambda) = \{u \in U \mid \lambda^u = \lambda\} = H.$$

(3) Let $\lambda \in \operatorname{Irr}(N)$, then the inertia group $I_T(\lambda)$ is

$$I_T(\lambda) = \begin{cases} T & \textit{if } \operatorname{Res}^N_{Y_6}\lambda = \operatorname{triv}_{Y_6} \\ N & \textit{if } \operatorname{Res}^N_{Y_6}\lambda \neq \operatorname{triv}_{Y_6} \end{cases}.$$

(4) Let $\lambda \in \operatorname{Irr}(N)$, then the inertia group $I_H(\lambda)$ is

$$I_H(\lambda) = \begin{cases} H & \textit{if } \operatorname{Res}^N_{Y_5}\lambda = \operatorname{triv}_{Y_5} \\ N & \textit{if } \operatorname{Res}^N_{Y_5}\lambda \neq \operatorname{triv}_{Y_5} \end{cases}.$$

(5) *Let $\psi \in \mathrm{Irr}(T)$ and $Y_6 = Z(T) \nsubseteq \ker\psi$, then the inertia group $I_U(\psi)$ is*

$$I_U(\psi) = \{u \in U \mid \psi^u = \psi\} = U.$$

Proof. (1) By 3.1.25 and 3.3.2, N is abelian, N and T are normal subgroups of U, and H is a subgroup of U. Then N is a normal subgroup of N and H, since N is a normal subgroup of U.

(2) Let $\lambda \in \mathrm{Irr}(N)$, $u := x(r_1, r_2, r_3, r_4, r_5, r_6) \in U$ and $x := x_4(t_4)x_5(t_5)x_6(t_6) \in N$, then λ is a linear character since N is abelian. We have

$$\begin{aligned}\lambda^u(x) =&\lambda(u \cdot x \cdot u^{-1}) \overset{3.3.2}{=} \lambda\left(x_4(t_4)x_5(t_5 + 3r_1t_4)x_6(t_6 + r_2t_5 + 3r_1r_2t_4 + 3r_3t_4)\right)\\ \overset{\mathrm{Res}^N_{Y_5}\lambda=\mathrm{triv}_{Y_5}}{=}&\lambda(x_4(t_4)) \cdot \lambda(x_6(t_6 + r_2t_5 + 3r_1r_2t_4 + 3r_3t_4)).\end{aligned}$$

Since $Y_6 \nsubseteq \ker\lambda$ and $Y_5 \subseteq \ker\lambda$, we get

$$\begin{aligned}I_U(\lambda) &= \{u \in U \mid \lambda^u = \lambda\} = \{u \in U \mid \lambda(u \cdot x \cdot u^{-1}) = \lambda(x) \text{ for all } x \in N\}\\ &=\{u := x(r_1, r_2, r_3, r_4, r_5, r_6) \in U \mid r_2 = r_3 = 0\} = Y_1Y_4Y_5Y_6 = H.\end{aligned}$$

(3) Similar to the proof of (2).

(4) Similar to the proof of (2).

(5) Let $x_6(t_6) \in Y_6$. Since $Y_6 = Z(T) \trianglelefteq N \trianglelefteq T$, there exists $\lambda \in \mathrm{Irr}(N)$ such that $0 < e = \langle \mathrm{Res}^T_N\psi, \lambda\rangle_N$, then

$$\left(\mathrm{Res}^T_N\psi\right)(x_6(t_6)) = e\frac{|T|}{|I_T(\lambda)|}\lambda(x_6(t_6)).$$

By 2.3.37, $Y_6 = Z(T) \nsubseteq \ker\lambda$ since $Y_6 = Z(T) \nsubseteq \ker\psi$. Thus $\mathrm{Res}^N_{Y_6}\lambda \neq \mathrm{triv}_{Y_6}$. Then $I_T(\lambda) = N$ by (3). Thus $\mathrm{Ind}^T_N\lambda \in \mathrm{Irr}(T)$ by Clifford's Theorem 2.3.31. Thus $e = 1$ and $\psi = \mathrm{Ind}^T_N\lambda$.

Let $u := x(s_1, s_2, s_3, s_4, s_5, s_6) \in U$ and $h := x(0, t_2, t_3, t_4, t_5, t_6) \in T$, then

$$\psi(h) = \left(\mathrm{Ind}^T_N\lambda\right)(h) = \frac{1}{|N|}\sum_{\substack{y\in T\\ y\cdot h\cdot y^{-1}\in N}} \lambda(y \cdot h \cdot y^{-1}).$$

– Let $t_2 \neq 0$ or $t_3 \neq 0$, then $\psi(h) = \left(\mathrm{Ind}^T_N\lambda\right)(h) = 0$ and

$$(\psi^u)(h) = \left(\mathrm{Ind}^T_N\lambda\right)^u(h) = \left(\mathrm{Ind}^T_N\lambda\right)(u \cdot h \cdot u^{-1}) \overset{3.3.2}{=} 0.$$

– Let $t_2 = t_3 = 0$, by 3.3.2

$$\psi(h) = \left(\mathrm{Ind}_N^T\lambda\right)(h) = \frac{1}{|N|}\sum_{\substack{y\in T\\ y\cdot h\cdot y^{-1}\in N}}(y\cdot h\cdot y^{-1}) = \frac{1}{|N|}\sum_{y\in T}\lambda(y\cdot h\cdot y^{-1})$$

$$\overset{y:=x(0,r_2,r_3,r_4,r_5,r_6)}{=}\frac{1}{|N|}\sum_{r_2,r_3,r_4,r_5,r_6\in\mathbb{F}_q}(x_4(t_4)x_5(t_5)x_6(t_6+r_2t_5+3r_3t_4)),$$

and

$$\begin{aligned}(\psi^u)(h) &= \left(\mathrm{Ind}_N^T\lambda\right)^u(h) = \left(\mathrm{Ind}_N^T\lambda\right)(u\cdot h\cdot u^{-1})\\
&= \left(\mathrm{Ind}_N^T\lambda\right)(x_4(t_4)x_5(t_5+3s_1t_4)x_6(t_6+s_2t_5+3s_1s_2t_4+3s_3t_4))\\
&= \frac{1}{|N|}\sum_{r_2,r_3,r_4,r_5,r_6\in\mathbb{F}_q}\lambda(x_4(t_4)x_5(t_5+3s_1t_4)x_6(t_6+s_2t_5+3s_1s_2t_4+3s_3t_4)\\
&\qquad\cdot x_6(r_2t_5+3r_2s_1t_4+3r_3t_4))\\
&= \sum_{r_2,r_3\in\mathbb{F}_q}\lambda(x_4(t_4)x_5(t_5+3s_1t_4)x_6(t_6+s_2t_5+3s_1s_2t_4+3s_3t_4)\\
&\qquad\cdot x_6(r_2t_5+3r_2s_1t_4+3r_3t_4)).\end{aligned}$$

Since $Y_6\nsubseteq\ker\lambda$, then for all $u\in U$ we have

$$(\psi^u)(h) = \psi(h) = \begin{cases} q^2\lambda(x_6(t_6)) & \text{if } t_2=t_3=t_4=t_5=0\\ 0 & \text{otherwise}\end{cases}.$$

Thus $I_U(\psi) = U$. □

Now we determine the irreducible characters of the subgroup $H = Y_1Y_4Y_5Y_6$ of U.

3.3.18 Lemma. *Let $H := Y_1Y_4Y_5Y_6$ and $\tilde\chi\in\mathrm{Irr}(H)$.*

(1) If $Y_5\subseteq\ker\tilde\chi$, let $\bar H_{146} := Y_5\backslash H\cong\bar Y_1\bar Y_4\bar Y_6$, $\bar\chi^{A_{17},A_{15},A_{12}}\in\mathrm{Irr}(\bar H_{146})$,

$$\bar\chi^{A_{17},A_{15},A_{12}}(\bar x_1(t_1)\bar x_4(t_4)\bar x_6(t_6)) := \vartheta(A_{17}t_6)\cdot\vartheta(2A_{15}t_4)\cdot\vartheta(A_{12}t_1),$$

and $\tilde\chi^{A_{17},A_{15},A_{12}}$ be the lift of $\bar\chi^{A_{17},A_{15},A_{12}}$ to H. Then

$$\mathrm{Irr}(H)_1 := \{\tilde\chi\in\mathrm{Irr}(H)\mid Y_5\subseteq\ker\tilde\chi\} = \{\tilde\chi^{A_{17},A_{15},A_{12}}\in\mathrm{Irr}(H)\mid A_{17},A_{15},A_{12}\in\mathbb{F}_q\}.$$

(2) If $Y_5\nsubseteq\ker\tilde\chi$, let $N := Y_4Y_5Y_6$, $\lambda^{A_{17},A_{16},A_{15}}\in\mathrm{Irr}(N)$, and

$$\lambda^{A_{17},A_{16},A_{15}}(x_4(t_4)x_5(t_5)x_6(t_6)) := \vartheta(A_{17}t_6)\cdot\vartheta(A_{16}t_5)\cdot\vartheta(2A_{15}t_4).$$

Then

$$\mathrm{Irr}(H)_2 := \{\tilde\chi\in\mathrm{Irr}(H)\mid Y_5\nsubseteq\ker\tilde\chi\} = \{\mathrm{Ind}_N^H\lambda^{A_{17},A_{16}^*,0}\mid A_{17}\in\mathbb{F}_q, A_{16}^*\in\mathbb{F}_q^*\}.$$

Thus, $\mathrm{Irr}(H) = \mathrm{Irr}(H)_1 \dot{\cup} \mathrm{Irr}(H)_2$, *i.e.* H *has* q^3 *linear characters and* $(q-1)q$ *irreducible characters of degree* q.

Proof. Let $\tilde{\chi} \in \mathrm{Irr}(H)$. From the commutator relations, $H' = Y_5$ and $Z(H) = Y_5Y_6$.

(1) If $Y_5 \subseteq \ker \tilde{\chi}$, $\tilde{\chi}$ is linear. Since $H' = Y_5$, all linear characters of H are precisely the lifts to H of the irreducible characters of the abelian quotient group $Y_5 \backslash H$.

(2) If $Y_5 \nsubseteq \ker \tilde{\chi}$, let $N := Y_4Y_5Y_6$, $\lambda \in \mathrm{Irr}(N)$ and $\langle \mathrm{Res}_N^H \tilde{\chi}, \lambda \rangle_N > 0$. Since $Y_5 \trianglelefteq Z(H) \trianglelefteq N \trianglelefteq H$, by 2.3.37 $Y_5 \nsubseteq \ker \lambda$ i.e. $\mathrm{Res}_{Y_5}^N \lambda \neq \mathrm{triv}_{Y_5}$. Then $I_H(\lambda) = N$ by (4) of 3.3.17. By Clifford's Theorem 2.3.31, $\tilde{\chi} = \mathrm{Ind}_N^H \lambda$ and $\deg(\tilde{\chi}) = q$. Let $\lambda^{A_{17},A_{16},A_{15}} \in \mathrm{Irr}(N)$, for all $A_{17}, A_{16}, A_{15} \in \mathbb{F}_q$, and

$$\lambda^{A_{17},A_{16},A_{15}}(x_4(t_4)x_5(t_5)x_6(t_6)) := \vartheta(A_{17}t_6) \cdot \vartheta(A_{16}t_5) \cdot \vartheta(2A_{15}t_4).$$

Then there exists $\lambda^{A_{17},A_{16}^*,A_{15}}$ such that $\lambda = \lambda^{A_{17},A_{16}^*,A_{15}}$. Note that Y_1 is a transversal of N in H. Let $r_1 \in \mathbb{F}_q$, for all $x_4(t_4)x_5(t_5)x_6(t_6) \in N$,

$$\begin{aligned}
&\left(\lambda^{A_{17},A_{16}^*,A_{15}}\right)^{x_1(r_1)} (x_4(t_4)x_5(t_5)x_6(t_6)) \\
=&\lambda^{A_{17},A_{16}^*,A_{15}}(x_1(r_1) \cdot x_4(t_4)x_5(t_5)x_6(t_6) \cdot x_1(r_1)^{-1}) \\
=&\lambda^{A_{17},A_{16}^*,A_{15}}(x_4(t_4)x_5(t_5 + 3r_1t_4)x_6(t_6)) \\
=&\vartheta(A_{17}t_6) \cdot \vartheta(A_{16}^*(t_5 + 3r_1t_4)) \cdot \vartheta(2A_{15}t_4) \\
=&\vartheta(A_{17}t_6) \cdot \vartheta(A_{16}^*t_5) \cdot \vartheta((2A_{15} + 3A_{16}^*r_1)t_4).
\end{aligned}$$

By Clifford's Theorem, $\tilde{\chi} = \mathrm{Ind}_N^H \lambda^{A_{17},A_{16}^*,0}$ and

$$\mathrm{Res}_N^H \mathrm{Ind}_N^H \lambda^{A_{17},A_{16}^*,0} = \sum_{r_1 \in \mathbb{F}_q} \left(\lambda^{A_{17},A_{16}^*,A_{15}}\right)^{x_1(r_1)} = \sum_{B_{15} \in \mathbb{F}_q} \lambda^{A_{17},A_{16}^*,B_{15}}.$$

Thus H has q^3 linear characters and $(q-1)q$ irreducible characters of degree q. □

By the commutator relations, we determine the conjugacy classes of H, and obtain the character table of $H = Y_1Y_4Y_5Y_6$.

3.3.19 Corollary. *Let* $y := x(t_1, 0, 0, t_4, t_5, t_6) \in H = Y_1Y_4Y_5Y_6$ *be a representative of one conjugacy class of* H*, then the character table of* H *is shown in Table 3.8.*

We obtain the irreducible characters of the normal subgroup $T = Y_2Y_3Y_4Y_5Y_6$ of U.

3.3.20 Lemma. *Let* $T := Y_2Y_3Y_4Y_5Y_6$ *and* $\psi \in \mathrm{Irr}(T)$*, then* $T' = Y_6$.

Table 3.8.: Character table of $H = Y_1Y_4Y_5Y_6$

$\lvert {}^H y \rvert$	1	q	q
y	$x_5(t_5)x_6(t_6)$	$x_4(t_4^*)x_6(t_6)$	$x_1(t_1^*)x_4(t_4)x_6(t_6)$
$\tilde{\chi}^{A_{17},A_{15},A_{12}}$	$\vartheta(A_{17}t_6)$	$\vartheta(A_{17}t_6 + 2A_{15}t_4^*)$	$\vartheta(A_{17}t_6 + 2A_{15}t_4 + A_{12}t_1^*)$
$\mathrm{Ind}_N^H \lambda^{A_{17},A_{16}^*,0}$	$q\vartheta(A_{17}t_6 + A_{16}^*t_5)$	0	0

(1) If $Y_6 \subseteq \ker\psi$, let $\bar{H}_{2345} := Y_6\backslash T \cong \bar{Y}_2\bar{Y}_3\bar{Y}_4\bar{Y}_5$, $\bar{\chi}^{A_{16},A_{15},A_{13},A_{23}} \in \mathrm{Irr}(\bar{H}_{2345})$,

$$\bar{\chi}^{A_{16},A_{15},A_{13},A_{23}}(\bar{x}_2(t_2)\bar{x}_3(t_3)\bar{x}_4(t_4)\bar{x}_5(t_5)) := \vartheta(A_{16}t_5)\cdot\vartheta(2A_{15}t_4)\cdot\vartheta(-A_{13}t_3)\cdot\vartheta(A_{23}t_2),$$

and $\psi^{A_{16},A_{15},A_{13},A_{23}}$ be the lift of $\bar{\chi}^{A_{16},A_{15},A_{13},A_{23}}$ to T. Then

$$\mathrm{Irr}(T)_1 :=\{\psi \in \mathrm{Irr}(T) \mid Y_6 \subseteq \ker\psi\} = \{\psi^{A_{16},A_{15},A_{13},A_{23}} \mid A_{16}, A_{15}, A_{13}, A_{23} \in \mathbb{F}_q\}.$$

(2) If $Y_6 \nsubseteq \ker\psi$, let $N := Y_4Y_5Y_6$, $\lambda^{A_{17},A_{16},A_{15}} \in \mathrm{Irr}(N)$, and

$$\lambda^{A_{17},A_{16},A_{15}}(x_4(t_4)x_5(t_5)x_6(t_6)) := \vartheta(A_{17}t_6)\cdot\vartheta(A_{16}t_5)\cdot\vartheta(2A_{15}t_4).$$

Then $\mathrm{Irr}(T)_2 := \{\psi \in \mathrm{Irr}(T) \mid Y_6 \nsubseteq \ker\psi\} = \{\mathrm{Ind}_N^T \lambda^{A_{17}^*,0,0} \mid A_{17} \in \mathbb{F}_q^*\}$.

Thus, $\mathrm{Irr}(T) = \mathrm{Irr}(T)_1 \dot\cup \mathrm{Irr}(T)_2$, i.e. T has q^4 linear characters and $(q-1)$ irreducible characters of degree q^2.

By the commutator relations, we also determine the conjugacy classes of $T = Y_2Y_3Y_4Y_5Y_6$ and the character table of T.

3.3.21 Corollary. *Let $y := x(0, t_2, t_3, t_4, t_5, t_6) \in T = Y_2Y_3Y_4Y_5Y_6$ be a representative of one conjugacy class of T, then the character table of T is the one in Table 3.9.*

Table 3.9.: Character table of $T = Y_2Y_3Y_4Y_5Y_6$

$\lvert {}^T y \rvert$	1	q	q	q	q
y	$x_6(t_6)$	$x_5(t_5^*)$	$x_4(t_4^*)x_5(t_5)$	$x_3(t_3^*)x_4(t_4)x_5(t_5)$	$x_2(t_2^*)x_3(t_3)$ $\cdot x_4(t_4)x_5(t_5)$
$\psi^{A_{16},A_{15},A_{13},A_{23}}$	1	$\vartheta(A_{16}t_5^*)$	$\vartheta(A_{16}t_5)$ $\cdot\vartheta(2A_{15}t_4^*)$	$\vartheta(A_{16}t_5)$ $\cdot\vartheta(2A_{15}t_4)$ $\cdot\vartheta(-A_{13}t_3^*)$	$\vartheta(A_{16}t_5)$ $\cdot\vartheta(2A_{15}t_4)$ $\cdot\vartheta(-A_{13}t_3)$ $\cdot\vartheta(A_{23}t_2^*)$
$\psi^{A_{17}^*}$	$q^2\cdot\vartheta(A_{17}^*t_6)$	0	0	0	0

3.3.22 Remark. *Let $H := Y_1Y_4Y_5Y_6 \leqslant G_2^{syl}(q)$, $T := Y_2Y_3Y_4Y_5Y_6 \trianglelefteq G_2^{syl}(q)$. Let $J := \{(1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(2,3)\}$, V as in §3.2.2.*

(a) Let

$$J_H := \bigcup_{i\in\{1,4,5,6\}} (\overline{\operatorname{supp}}(Y_i) \cap J) = \{(1,2),(1,4),(1,5),(1,6),(1,7)\} \subseteq J,$$

$$V_H := V_{J_H} \bigcap V = \{A = (A_{ij}) \in V_{J_H} \mid A_{i,j}, A_{14} = A_{15} \in \mathbb{F}_q\}.$$

Similar to $G_2^{syl}(q)$, we construct a bijective 1-cocycle of H and a monomial $\mathbb{C}H$-module $\mathbb{C}H$, and get a supercharacter theory for H. The superclasses are the conjugacy classes, and the supercharacters conclude q^3 linear characters and $(q-1)q$ reducible characters of degree q^2.

(b) We establish a monomial $\mathbb{C}T$-module $\mathbb{C}T$ and a supercharacter theory for T as well. The superclasses are also the conjugacy classes, and the supercharacters conclude q^4 linear characters and $(q-1)$ reducible characters of degree q^4.

Now we give the constructions of the irreducible characters of $G_2^{syl}(q)$.

3.3.23 Proposition. *Let $A_{ij} \in \mathbb{F}_q$, and $A_{ij}^* \in \mathbb{F}_q^*$.*

(1) Let $\bar{U} := Y_3Y_4Y_5Y_6\backslash U = \bar{Y}_2\bar{Y}_1$, $\bar{\chi}_{lin}^{A_{12},A_{23}} \in \operatorname{Irr}(\bar{U})$ with $\bar{\chi}_{lin}^{A_{12},A_{23}}(\bar{x}_2(t_2)\bar{x}_1(t_1)) := \vartheta(A_{12}t_1)\cdot\vartheta(A_{23}t_2)$, and $\chi_{lin}^{A_{12},A_{23}}$ be the lift of $\bar{\chi}_{lin}^{A_{12},A_{23}}$ to U. Then

$$\mathfrak{F}_{lin} := \{\chi \in \operatorname{Irr}(U) \mid Y_3Y_4Y_5Y_6 \subseteq \ker\chi\} = \{\chi_{lin}^{A_{12},A_{23}} \mid A_{12}, A_{23} \in \mathbb{F}_q\}.$$

(2) Let $\bar{U} := Y_4Y_5Y_6\backslash U = \bar{Y}_2\bar{Y}_1\bar{Y}_3$, $\bar{H} := \bar{Y}_1\bar{Y}_3$, $\bar{\chi}_{3,q}^{A_{13},A_{12}} \in \operatorname{Irr}(\bar{H})$ with

$$\bar{\chi}_{3,q}^{A_{13},A_{12}}(\bar{x}_1(t_1)\bar{x}_3(t_3)) := \vartheta(A_{12}t_1 - A_{13}t_3),$$

and $\chi_{3,q}^{A_{13}^*}$ be the lift of $\operatorname{Ind}_{\bar{H}}^{\bar{U}}\bar{\chi}_{3,q}^{A_{13}^*,0}$ to U. Then

$$\mathfrak{F}_3 := \{\chi \in \operatorname{Irr}(U) \mid Y_4Y_5Y_6 \subseteq \ker\chi,\ Y_3 \nsubseteq \ker\chi\} = \{\chi_{3,q}^{A_{13}^*} \mid A_{13}^* \in \mathbb{F}_q^*\}.$$

(3) Let $\bar{U} := Y_5Y_6\backslash U = \bar{Y}_2\bar{Y}_1\bar{Y}_3\bar{Y}_4$, $\bar{H} := \bar{Y}_2\bar{Y}_3\bar{Y}_4$, $\bar{\chi}_{4,q}^{A_{15},A_{23},A_{13}} \in \operatorname{Irr}(\bar{H})$,

$$\bar{\chi}_{4,q}^{A_{15},A_{23},A_{13}}(\bar{x}_2(t_2)\bar{x}_3(t_3)\bar{x}_4(t_4)) := \vartheta(A_{23}t_2)\cdot\vartheta(-A_{13}t_3)\cdot\vartheta(2A_{15}t_4),$$

and $\chi_{4,q}^{A_{15}^*,A_{23}}$ be the lift of $\operatorname{Ind}_{\bar{H}}^{\bar{U}}\bar{\chi}_{4,q}^{A_{15}^*,A_{23},0}$ to U. Then

$$\mathfrak{F}_4 := \{\chi \in \operatorname{Irr}(U) \mid Y_5Y_6 \subseteq \ker\chi,\ Y_4 \nsubseteq \ker\chi\} = \{\chi_{4,q}^{A_{15}^*,A_{23}} \mid A_{15}^* \in \mathbb{F}_q^*, A_{23} \in \mathbb{F}_q\}.$$

(4) *Let* $\bar{U} := Y_6\backslash U = \bar{Y}_2\bar{Y}_1\bar{Y}_3\bar{Y}_4\bar{Y}_5$, $\bar{H} := \bar{Y}_2\bar{Y}_3\bar{Y}_4\bar{Y}_5$, $\bar{\chi}_{5,q}^{A_{16},A_{23},A_{13},A_{15}} \in \mathrm{Irr}(\bar{H})$,

$\bar{\chi}_{5,q}^{A_{16},A_{23},A_{13},A_{15}}(\bar{x}_2(t_2)\bar{x}_3(t_3)\bar{x}_4(t_4)\bar{x}_5(t_5)) := \vartheta(A_{23}t_2)\cdot\vartheta(-A_{13}t_3)\cdot\vartheta(2A_{15}t_4)\cdot\vartheta(A_{16}t_5)$,

and $\chi_{5,q}^{A_{16}^*,A_{23},A_{13}}$ *be the lift of* $\mathrm{Ind}_{\bar{H}}^{\bar{U}}\bar{\chi}_{5,q}^{A_{16}^*,A_{23},A_{13},0}$ *to* U*. Then*

$$\begin{aligned}\mathfrak{F}_5 :=&\{\chi\in\mathrm{Irr}(U)\mid Y_6\subseteq\ker\chi,\ Y_5\nsubseteq\ker\chi\}\\ =&\{\chi_{5,q}^{A_{16}^*,A_{23},A_{13}}\mid A_{16}^*\in\mathbb{F}_q^*, A_{23},A_{13}\in\mathbb{F}_q\}.\end{aligned}$$

(5) *Let* $H := Y_1Y_4Y_5Y_6$, $\bar{H} := Y_4Y_5\backslash H \cong \bar{Y}_1\bar{Y}_6$, $\bar{\chi}_{6,q^2}^{A_{17},A_{12}} \in \mathrm{Irr}(\bar{H})$*, and*

$$\bar{\chi}_{6,q^2}^{A_{17},A_{12}}(\bar{x}_1(t_1)\bar{x}_6(t_6)) := \vartheta(A_{12}t_1)\cdot\vartheta(A_{17}t_6).$$

Let $\tilde{\chi}_{6,q^2}^{A_{17},A_{12}}$ *denote the lift of* $\bar{\chi}_{6,q^2}^{A_{17},A_{12}}$ *from* $\bar{H}$ *to* H*, and* $\chi_{6,q^2}^{A_{17}^*,A_{12}} := \mathrm{Ind}_H^U\tilde{\chi}_{6,q^2}^{A_{17}^*,A_{12}}$. *Then* $\mathfrak{F}_6 := \{\chi\in\mathrm{Irr}(U)\mid Y_6\nsubseteq\ker\chi\} = \{\chi_{6,q^2}^{A_{17}^*,A_{12}}\mid A_{17}^*\in\mathbb{F}_q^*, A_{12}\in\mathbb{F}_q\}$.

Hence $\mathrm{Irr}(U) = \mathfrak{F}_{lin}\dot\cup\mathfrak{F}_3\dot\cup\mathfrak{F}_4\dot\cup\mathfrak{F}_5\dot\cup\mathfrak{F}_6$.

Proof. Let $\chi\in\mathrm{Irr}(U)$.

(1) *Family* $\mathfrak{F}_{lin}$*, where* $Y_3Y_4Y_5Y_6\subseteq\ker\chi$.

Since $U' = Y_3Y_4Y_5Y_6$, all linear characters of U are precisely the lifts to U of the irreducible characters of the abelian quotient group $Y_3Y_4Y_5Y_6\backslash U$. If $Y_3Y_4Y_5Y_6\subseteq\ker\chi$, χ is linear. Let $\bar{U} := Y_3Y_4Y_5Y_6\backslash U = \bar{Y}_2\bar{Y}_1$, $\bar{\chi}_{lin}^{A_{12},A_{23}}\in\mathrm{Irr}(\bar{U})$, with $\bar{\chi}_{lin}^{A_{12},A_{23}}(\bar{x}_2(t_2)\bar{x}_1(t_1)) := \vartheta(A_{12}t_1)\cdot\vartheta(A_{23}t_2)$. Since $\bar{U}$ is abelian, $\mathrm{Irr}(\bar{U}) = \{\bar{\chi}_{lin}^{A_{12},A_{23}}\mid A_{12},A_{23}\in\mathbb{F}_q\}$. Let $\chi_{lin}^{A_{12},A_{23}}$ denote the lift of $\bar{\chi}_{lin}^{A_{12},A_{23}}$ to U, then

$$\mathfrak{F}_{lin} =\{\chi\in\mathrm{Irr}(U)\mid Y_3Y_4Y_5Y_6\subseteq\ker\chi\} = \{\chi_{lin}^{A_{12},A_{23}}\mid A_{12},A_{23}\in\mathbb{F}_q\}.$$

(2) *Family* $\mathfrak{F}_3$*, where* $Y_4Y_5Y_6\subseteq\ker\chi$ *and* $Y_3\nsubseteq\ker\chi$.

Let $\bar{U} := Y_4Y_5Y_6\backslash U = \bar{Y}_2\bar{Y}_1\bar{Y}_3$, then by 3.1.25, the commutator relation in $\bar{U}$ is that $[\bar{x}_2(t_2),\bar{x}_1(t_1)] = \bar{x}_3(t_2t_1)$. Let $\bar{H} := \bar{Y}_1\bar{Y}_3$, $\bar{\chi}_{3,q}^{A_{13},A_{12}}\in\mathrm{Irr}(\bar{H})$ with $\bar{\chi}_{3,q}^{A_{13},A_{12}}(\bar{x}_1(t_1)\bar{x}_3(t_3)) := \vartheta(A_{12}t_1)\cdot\vartheta(-A_{13}t_3)$, and $\tilde{\chi}_{3,q}^{A_{13},A_{12}} := \mathrm{Ind}_{\bar{H}}^{\bar{U}}\bar{\chi}_{3,q}^{A_{13},A_{12}}$. We note that $\bar{Y}_2$ is a transversal of $\bar{H}$ in $\bar{U}$, and $Z(\bar{U}) = \bar{Y}_3$. For all $s_2\in\mathbb{F}_q$,

$$\begin{aligned}&\left(\bar{\chi}_{3,q}^{A_{13}^*,A_{12}}\right)^{\bar{x}_2(s_2)}(\bar{x}_1(t_1)\bar{x}_3(t_3)) = \bar{\chi}_{3,q}^{A_{13}^*,A_{12}}(\bar{x}_2(s_2)\cdot\bar{x}_1(t_1)\bar{x}_3(t_3)\cdot\bar{x}_2(s_2)^{-1})\\ =&\bar{\chi}_{3,q}^{A_{13}^*,A_{12}}(\bar{x}_1(t_1)\bar{x}_3(t_3+s_2t_1)) = \vartheta(A_{12}t_1 - A_{13}^*(t_3+s_2t_1))\\ =&\vartheta((A_{12}-s_2A_{13}^*)t_1 - A_{13}^*t_3) = \vartheta((A_{12}-s_2A_{13}^*)t_1)\cdot\vartheta(-A_{13}^*t_3).\end{aligned}$$

Thus $I_{\bar{U}}(\bar{\chi}_{3,q}^{A_{13}^*,A_{12}}) = \bar{H}$. By Clifford's Theorem, $\tilde{\chi}_{3,q}^{A_{13}^*,A_{12}} = \mathrm{Ind}_{\bar{H}}^{\bar{U}}\bar{\chi}_{3,q}^{A_{13}^*,A_{12}} \in \mathrm{Irr}(\bar{U})$ and

$$\mathrm{Res}_{\bar{H}}^{\bar{U}}\tilde{\chi}_{3,q}^{A_{13}^*,A_{12}} = \sum_{s_2\in\mathbb{F}_q}\left(\bar{\chi}_{3,q}^{A_{13}^*,A_{12}}\right)^{\bar{x}_2(s_2)} = \sum_{s_2\in\mathbb{F}_q}\bar{\chi}_{3,q}^{A_{13}^*,A_{12}-s_2A_{13}^*} = \sum_{B_{12}\in\mathbb{F}_q}\bar{\chi}_{3,q}^{A_{13}^*,B_{12}}.$$

By Clifford's Theorem, there are $q-1$ almost faithful irreducible characters of $\bar{U}$, i.e. $\{\tilde{\chi}_{3,q}^{A_{13}^*,0} \mid A_{13}^* \in \mathbb{F}_q^*\}$. Let $\chi_{3,q}^{A_{13}^*}$ be the lift of $\tilde{\chi}_{3,q}^{A_{13}^*,0}$ to U. Therefore,

$$\mathfrak{F}_3 = \{\chi \in \mathrm{Irr}(U) \mid Y_4Y_5Y_6 \subseteq \ker\chi,\ Y_3 \nsubseteq \ker\chi\} = \{\chi_{3,q}^{A_{13}^*} \mid A_{13}^* \in \mathbb{F}_q^*\}.$$

(3) *Family $\mathfrak{F}_4$, where $Y_5Y_6 \subseteq \ker\chi$ and $Y_4 \nsubseteq \ker\chi$.*
Let $\bar{U} := Y_5Y_6\backslash U = \bar{Y}_2\bar{Y}_1\bar{Y}_3\bar{Y}_4$. By 3.1.25, the commutator relations of $\bar{U}$ are given as follows:

$$[\bar{x}_1(t_1),\bar{x}_2(t_2)] = \bar{x}_3(-t_2t_1)\cdot\bar{x}_4(t_2t_1^2), \qquad [\bar{x}_1(t_1),\bar{x}_3(t_3)] = \bar{x}_4(2t_1t_3).$$

Let $\bar{H} := \bar{Y}_2\bar{Y}_3\bar{Y}_4$, $\bar{\chi}_{4,q}^{A_{15},A_{23},A_{13}} \in \mathrm{Irr}(\bar{H})$, and

$$\bar{\chi}_{4,q}^{A_{15},A_{23},A_{13}}(\bar{x}_2(t_2)\bar{x}_3(t_3)\bar{x}_4(t_4)) := \vartheta(A_{23}t_2)\cdot\vartheta(-A_{13}t_3)\cdot\vartheta(2A_{15}t_4).$$

Then $\bar{H}$ is abelian and $\bar{H} \trianglelefteq \bar{U}$. Note that $\bar{Y}_1$ is a transversal of $\bar{H}$ in $\bar{U}$. For each $\bar{x}_1(s_1) \in \bar{Y}_1$, we consider $\left(\bar{\chi}_{4,q}^{A_{15}^*,A_{23},A_{13}}\right)^{\bar{x}_1(s_1)}$. For all $\bar{x}_2(t_2)\bar{x}_3(t_3)\bar{x}_4(t_4) \in \bar{H}$,

$$\begin{aligned}
&\left(\bar{\chi}_{4,q}^{A_{15}^*,A_{23},A_{13}}\right)^{x_1(s_1)}(\bar{x}_2(t_2)\bar{x}_3(t_3)\bar{x}_4(t_4))\\
=&\bar{\chi}_{4,q}^{A_{15}^*,A_{23},A_{13}}(\bar{x}_1(s_1)\cdot\bar{x}_2(t_2)\bar{x}_3(t_3)\bar{x}_4(t_4)\cdot\bar{x}_1(s_1)^{-1})\\
=&\bar{\chi}_{4,q}^{A_{15}^*,A_{23},A_{13}}(\bar{x}_2(t_2)\bar{x}_3(t_3-s_1t_2)\bar{x}_4(t_4+2s_1t_3-t_2s_1^2))\\
=&\vartheta(A_{23}t_2)\cdot\vartheta(-A_{13}(t_3-s_1t_2))\cdot\vartheta(2A_{15}^*(t_4+2s_1t_3-t_2s_1^2))\\
=&\vartheta((A_{23}+A_{13}s_1-2A_{15}^*s_1^2)t_2)\cdot\vartheta((-A_{13}+4A_{15}^*s_1)t_3)\cdot\vartheta(2A_{15}^*t_4).
\end{aligned}$$

Thus $I_{\bar{U}}(\bar{\chi}_{4,q}^{A_{15}^*,A_{23},A_{13}}) = \bar{H}$. By Clifford theory, we get $\mathrm{Ind}_{\bar{H}}^{\bar{U}}\bar{\chi}_{4,q}^{A_{15}^*,A_{23},A_{13}} \in \mathrm{Irr}(\bar{U})$, $\deg(\bar{\chi}_{4,q}^{A_{15}^*,A_{23},A_{13}}) = q$ and

$$\mathrm{Res}_{\bar{H}}^{\bar{U}}\mathrm{Ind}_{\bar{H}}^{\bar{U}}\bar{\chi}_{4,q}^{A_{15}^*,A_{23},A_{13}} = \sum_{\bar{x}_1(s_1)\in\bar{Y}_1}\left(\bar{\chi}_{4,q}^{A_{15}^*,A_{23},A_{13}}\right)^{x_1(s_1)} = \sum_{B_{13}\in\mathbb{F}_q}\bar{\chi}_{4,q}^{A_{15}^*,\tilde{A}_{23},B_{13}},$$

where $\tilde{A}_{23}$ is determined by A_{23}, A_{15}^*, A_{13} and B_{13}. Let $\chi_{4,q}^{A_{15}^*,A_{23}}$ denote the lift of $\mathrm{Ind}_{\bar{H}}^{\bar{U}}\bar{\chi}_{4,q}^{A_{15}^*,A_{23},0}$ to U, then

$$\mathfrak{F}_4 = \{\chi \in \mathrm{Irr}(U) \mid Y_5Y_6 \subseteq \ker\chi,\ Y_4 \nsubseteq \ker\chi\} = \{\chi_{4,q}^{A_{15}^*,A_{23}} \mid A_{15}^* \in \mathbb{F}_q^*, A_{23} \in \mathbb{F}_q\}.$$

(4) *Family $\mathfrak{F}_5$, where $Y_6 \subseteq \ker\chi$ and $Y_5 \nsubseteq \ker\chi$.*
Let $\bar{U} := Y_6\backslash U = \bar{Y}_2\bar{Y}_1\bar{Y}_3\bar{Y}_4\bar{Y}_5$. By 3.1.25, the commutator relations of $\bar{U}$ are given as follows:

$$\begin{aligned}
[\bar{x}_1(t_1), \bar{x}_2(t_2)] =& \bar{x}_3(-t_2t_1) \cdot \bar{x}_4(t_2t_1^2) \cdot \bar{x}_5(-t_2t_1^3),\\
[\bar{x}_1(t_1), \bar{x}_3(t_3)] =& \bar{x}_4(2t_1t_3) \cdot \bar{x}_5(-3t_1^2t_3),\\
[\bar{x}_1(t_1), \bar{x}_4(t_4)] =& \bar{x}_5(3t_1t_4).
\end{aligned}$$

Let $\bar{H} := \bar{Y}_2\bar{Y}_3\bar{Y}_4\bar{Y}_5$, $\bar{\chi}_{5,q}^{A_{16},A_{23},A_{13},A_{15}} \in \mathrm{Irr}(\bar{H})$, and

$$\bar{\chi}_{5,q}^{A_{16},A_{23},A_{13},A_{15}}(\bar{x}_2(t_2)\bar{x}_3(t_3)\bar{x}_4(t_4)\bar{x}_5(t_5)) := \vartheta(A_{23}t_2) \cdot \vartheta(-A_{13}t_3) \cdot \vartheta(2A_{15}t_4) \cdot \vartheta(A_{16}t_5).$$

Then $\bar{H}$ is abelian and $\bar{H} \trianglelefteq \bar{U}$. Note that $\bar{Y}_1$ is a transversal of $\bar{H}$ in $\bar{U}$. For each $\bar{x}_1(s_1) \in \bar{Y}_1$, we consider $\left(\bar{\chi}_{5,q}^{A_{16}^*,A_{23},A_{13},A_{15}}\right)^{\bar{x}_1(s_1)}$. For all $\bar{x}_2(t_2)\bar{x}_3(t)\bar{x}_4(t_4)\bar{x}_5(t_5) \in \bar{H}$,

$$\begin{aligned}
&\left(\bar{\chi}_{5,q}^{A_{16}^*,A_{23},A_{13},A_{15}}\right)^{x_1(s_1)}(\bar{x}_2(t_2)\bar{x}_3(t_3)\bar{x}_4(t_4)\bar{x}_5(t_5))\\
=&\bar{\chi}_{5,q}^{A_{16}^*,A_{23},A_{13},A_{15}}(\bar{x}_1(s_1) \cdot \bar{x}_2(t_2)\bar{x}_3(t_3)\bar{x}_4(t_4)\bar{x}_5(t_5) \cdot \bar{x}_1(s_1)^{-1})\\
=&\bar{\chi}_{5,q}^{A_{16}^*,A_{23},A_{13},A_{15}}(\bar{x}_2(t_2)\bar{x}_3(t_3 - s_1t_2)\bar{x}_4(t_4 + 2s_1t_3 - t_2s_1^2)\\
&\qquad \cdot \bar{x}_5(t_5 + 3s_1t_4 + 3s_1^2t_3 - t_2s_1^3))\\
=&\vartheta(A_{23}t_2) \cdot \vartheta(-A_{13}(t_3 - s_1t_2)) \cdot \vartheta(2A_{15}(t_4 + 2s_1t_3 - t_2s_1^2))\\
&\cdot \vartheta(A_{16}^*(t_5 + 3s_1t_4 + 3s_1^2t_3 - t_2s_1^3))\\
=&\vartheta((A_{23} + A_{13}s_1 - 2A_{15}s_1^2 - A_{16}^*s_1^3)t_2) \cdot \vartheta((-A_{13} + 4A_{15}s_1 + 3A_{16}^*s_1^2)t_3))\\
&\cdot \vartheta((2A_{15} + 3A_{16}^*s_1)t_4) \cdot \vartheta(A_{16}^*t_5).
\end{aligned}$$

Thus $I_{\bar{U}}(\bar{\chi}_{5,q}^{A_{16}^*,A_{23},A_{13},A_{15}}) = \bar{H}$. By Clifford theory, $\mathrm{Ind}_{\bar{H}}^{\bar{U}}\bar{\chi}_{5,q}^{A_{16}^*,A_{23},A_{13},A_{15}} \in \mathrm{Irr}(\bar{U})$, $\deg(\bar{\chi}_{5,q}^{A_{16}^*,A_{23},A_{13},A_{15}}) = q$ and

$$\mathrm{Res}_{\bar{H}}^{\bar{U}}\mathrm{Ind}_{\bar{H}}^{\bar{U}}\bar{\chi}_{5,q}^{A_{16}^*,A_{23},A_{13},A_{15}} = \sum_{\bar{x}_1(s_1)\in\bar{Y}_1}\left(\bar{\chi}_{5,q}^{A_{16}^*,A_{23},A_{13},A_{15}}\right)^{x_1(s_1)} = \sum_{B_{15}\in\mathbb{F}_q}\bar{\chi}_{5,q}^{A_{16}^*,\tilde{A}_{23},\tilde{A}_{13},B_{15}},$$

where $\tilde{A}_{13}$ is determined by $\{A_{13}, A_{16}^*, A_{15}, B_{15}\}$, and $\tilde{A}_{23}$ is determined by $\{A_{23}, A_{16}^*, A_{13}, A_{15}, B_{15}\}$. Let $\chi_{5,q}^{A_{16}^*,A_{23},A_{13}}$ denote the lift of $\mathrm{Ind}_{\bar{H}}^{\bar{U}}\bar{\chi}_{5,q}^{A_{16}^*,A_{23},A_{13},0}$ to U, then

$$\begin{aligned}
\mathfrak{F}_5 =&\{\chi \in \mathrm{Irr}(U) \mid Y_6 \subseteq \ker\chi,\ Y_5 \nsubseteq \ker\chi\}\\
=&\{\chi_{5,q}^{A_{16}^*,A_{23},A_{13},0} \mid A_{16}^* \in \mathbb{F}_q^*,\ A_{23}, A_{13} \in \mathbb{F}_q\}.
\end{aligned}$$

(5) *Family $\mathfrak{F}_6$, where $Y_6 \nsubseteq \ker\chi$.*

Let $T := Y_2Y_3Y_4Y_5Y_6$, $N := Y_4Y_5Y_6$, and $\chi \in \mathrm{Irr}(U)$ such that $Y_6 \nsubseteq \ker(\chi)$, then $Z(T) = Z(U) = Y_6$. If $\psi \in \mathrm{Irr}(T)$ and $\psi \in \mathrm{Irr}\left(\mathrm{Res}_T^U\chi\right)$, $Y_6 \nsubseteq \ker\psi$ by 2.3.37. Let $\lambda^{A_{17},A_{16},A_{15}} \in \mathrm{Irr}(N)$,

$$\lambda^{A_{17},A_{16},A_{15}}(x_4(t_4)x_5(t_5)x_6(t_6)) := \vartheta(A_{17}t_6)\cdot\vartheta(A_{16}t_5)\cdot\vartheta(2A_{15}t_4),$$

and $\psi^{A_{17}^*} := \mathrm{Ind}_N^T\lambda^{A_{17}^*,0,0}$. Then by 3.3.20

$$\{\psi \in \mathrm{Irr}(T) \mid Y_6 \nsubseteq \ker\psi\} = \{\mathrm{Ind}_N^T\lambda^{A_{17}^*,0,0} \mid A_{17}^* \in \mathbb{F}_q^*\} = \{\psi^{A_{17}^*} \mid A_{17}^* \in \mathbb{F}_q^*\}.$$

By (5) of 3.3.17, $I_U(\psi^{A_{17}^*}) = U$, so $\mathrm{Res}_T^U\chi = z^*\psi^{A_{17}^*}$ for some $z^* \in \mathbb{N}^*$. Thus

$$\begin{aligned}\mathfrak{F}_6 &= \{\chi \in \mathrm{Irr}(U) \mid Y_6 \nsubseteq \ker\chi\} = \bigcup_{\substack{\psi\in\mathrm{Irr}(T)\\ Y_6\nsubseteq\ker\psi}} \{\chi \in \mathrm{Irr}(U) \mid \chi \in \mathrm{Irr}\left(\mathrm{Ind}_T^U\psi\right)\}\\ &= \bigcup_{A_{17}^*\in\mathbb{F}_q^*} \{\chi \in \mathrm{Irr}(U) \mid \chi \in \mathrm{Irr}\left(\mathrm{Ind}_T^U\psi^{A_{17}^*}\right)\}\\ &= \bigcup_{A_{17}^*\in\mathbb{F}_q^*} \{\chi \in \mathrm{Irr}(U) \mid \chi \in \mathrm{Irr}\left(\mathrm{Ind}_N^U\lambda^{A_{17}^*,0,0}\right)\}.\end{aligned}$$

Let $H := Y_1Y_4Y_5Y_6$, then $H' = Y_5$ and $Z(H) = Y_4Y_5 \trianglelefteq H$. Let $\tilde{\chi}^{A_{17},A_{15},A_{12}} \in \mathrm{Irr}(H)$ as in (1) of Lemma 3.3.18. For all $x_4(t_4)x_5(t_5)x_6(t_6) \in N$,

$$\begin{aligned}&\left(\mathrm{Res}_N^H\tilde{\chi}^{A_{17}^*,0,A_{12}}\right)(x_4(t_4)x_5(t_5)x_6(t_6)) = \tilde{\chi}^{A_{17}^*,0,A_{12}}(x_4(t_4)x_5(t_5)x_6(t_6))\\ =&\bar{\chi}^{A_{17}^*,0,A_{12}}(\bar{x}_4(t_4)\bar{x}_6(t_6)) = \vartheta(A_{17}^*t_6) = \lambda^{A_{17}^*,0,0}(x_4(t_4)x_5(t_5)x_6(t_6)).\end{aligned}$$

Thus $\mathrm{Res}_N^H\tilde{\chi}^{A_{17}^*,0,A_{12}} = \lambda^{A_{17}^*,0,0}$ for all $A_{12} \in \mathbb{F}_q$. By (4) of 3.3.17 $I_H(\lambda^{A_{17}^*,0,0}) = H$, thus $\mathrm{Ind}_N^H\lambda^{A_{17}^*,0,0} = \sum_{A_{12}\in\mathbb{F}_q}\tilde{\chi}^{A_{17}^*,0,A_{12}}$. By (2) of 3.3.17 $I_U(\lambda^{A_{17}^*,0,0}) = H$. Then by Clifford's Theorem, $\mathrm{Ind}_H^U\tilde{\chi}^{A_{17}^*,0,A_{12}} \in \mathrm{Irr}(U)$ for all $A_{17}^* \in \mathbb{F}_q^*$. Thus

$$\begin{aligned}\mathfrak{F}_6 &= \bigcup_{A_{17}^*\in\mathbb{F}_q^*} \{\chi \in \mathrm{Irr}(U) \mid \chi \in \mathrm{Irr}\left(\mathrm{Ind}_H^U\mathrm{Ind}_N^H\lambda^{A_{17}^*,0,0}\right)\}\\ &= \bigcup_{\substack{A_{17}^*\in\mathbb{F}_q^*\\ A_{12}\in\mathbb{F}_q}} \{\chi \in \mathrm{Irr}(U) \mid \chi \in \mathrm{Irr}\left(\mathrm{Ind}_H^U\tilde{\chi}^{A_{17}^*,0,A_{12}}\right)\}\\ &= \{\mathrm{Ind}_H^U\tilde{\chi}^{A_{17}^*,0,A_{12}} \mid A_{17}^* \in \mathbb{F}_q^*, A_{12} \in \mathbb{F}_q\}.\end{aligned}$$

For $A_{17}^* \in \mathbb{F}_q^*$ and $A_{12} \in \mathbb{F}_q$, $Y_4Y_5 \subseteq \ker(\tilde{\chi}^{A_{17}^*,0,A_{12}})$ and $Y_4Y_5 \trianglelefteq H$. Thus $\tilde{\chi}^{A_{17}^*,0,A_{12}}$ is the lift to H of some irreducible character of $Y_4Y_5\backslash H \cong \bar{Y}_1\bar{Y}_6$. Let $\bar{H} := Y_4Y_5\backslash H \cong \bar{Y}_1\bar{Y}_6$, $\bar{\chi}_{6,q^2}^{A_{17}^*,A_{12}} \in \mathrm{Irr}(\bar{H})$, $\bar{\chi}_{6,q^2}^{A_{17}^*,A_{12}}(\bar{x}_1(t_1)\bar{x}_6(t_6)) := \vartheta(A_{12}t_1)\cdot\vartheta(A_{17}^*t_6)$, and $\tilde{\chi}_{6,q^2}^{A_{17}^*,A_{12}}$ denote the lift of $\bar{\chi}_{6,q^2}^{A_{17}^*,A_{12}}$ from $\bar{H}$ to H, then $\tilde{\chi}_{6,q^2}^{A_{17}^*,A_{12}} = \tilde{\chi}^{A_{17}^*,0,A_{12}}$. Let $\chi_{6,q^2}^{A_{17}^*,A_{12}} := \mathrm{Ind}_H^U\tilde{\chi}_{6,q^2}^{A_{17}^*,A_{12}}$, then

$$\mathfrak{F}_6 = \{\mathrm{Ind}_H^U\tilde{\chi}^{A_{17}^*,0,A_{12}} \mid A_{17}^* \in \mathbb{F}_q^*, A_{12} \in \mathbb{F}_q\} = \{\chi_{6,q^2}^{A_{17}^*,A_{12}} \mid A_{17}^* \in \mathbb{F}_q^*, A_{12} \in \mathbb{F}_q\}.$$

$\square$

By the above argument, we construct all irredubible characters of $G_2^{syl}(q)$.

3.3.24 Corollary. *The irreducible characters of $G_2^{syl}(q)$ are classified into five families as listed in Table 3.10*

Table 3.10.: Irreducible characters of $G_2^{syl}(q)$

Family	Notation	Parameter Set	Number	Degree	Properties
$\mathfrak{F}_6$	$\chi_{6,q^2}^{A_{17}^*,A_{12}}$	$\begin{cases} A_{17}^* \in \mathbb{F}_q^* \\ A_{12} \in \mathbb{F}_q \end{cases}$	$(q-1)q$	q^2	$Y_6 \nsubseteq \ker\chi_{6,q^2}^{A_{17}^*,A_{12}}$
$\mathfrak{F}_5$	$\chi_{5,q}^{A_{16}^*,A_{23},A_{13}}$	$\begin{cases} A_{16}^* \in \mathbb{F}_q^* \\ A_{23} \in \mathbb{F}_q \\ A_{13} \in \mathbb{F}_q \end{cases}$	$(q-1)q^2$	q	$\begin{cases} Y_6 \subseteq \ker\chi_{5,q}^{A_{16}^*,A_{23},A_{13}} \\ Y_5 \nsubseteq \ker\chi_{5,q}^{A_{16}^*,A_{23},A_{13}} \end{cases}$
$\mathfrak{F}_4$	$\chi_{4,q}^{A_{15}^*,A_{23}}$	$\begin{cases} A_{15}^* \in \mathbb{F}_q^* \\ A_{23} \in \mathbb{F}_q \end{cases}$	$(q-1)q$	q	$\begin{cases} Y_5Y_6 \subseteq \ker\chi_{4,q}^{A_{15}^*,A_{23}} \\ Y_4 \nsubseteq \ker\chi_{4,q}^{A_{15}^*,A_{23}} \end{cases}$
$\mathfrak{F}_3$	$\chi_{3,q}^{A_{13}^*}$	$A_{13}^* \in \mathbb{F}_q^*$	$q-1$	q	$\begin{cases} Y_4Y_5Y_6 \subseteq \ker\chi_{3,q}^{A_{13}^*} \\ Y_3 \nsubseteq \ker\chi_{3,q}^{A_{13}^*} \end{cases}$
$\mathfrak{F}_{lin}$	$\chi_{lin}^{A_{12},A_{23}}$	$A_{12}, A_{23} \in \mathbb{F}_q$	q^2	1	$Y_3Y_4Y_5Y_6 \subseteq \ker\chi_{lin}^{A_{12},A_{23}}$

3.3.25 Remark. *Let $U := G_2^{syl}(q)$.*

Let $J := \{(1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(2,3)\}$, V as in §3.2.2.

(a) *For the abelian groups appearing in 3.3.23, we can obtain their irreducible characters by constructing monomial modules. For example: let $\bar{U} := Y_5Y_6\backslash U = \bar{Y}_2\bar{Y}_1\bar{Y}_3\bar{Y}_4$ and $\bar{H} := \bar{Y}_2\bar{Y}_3\bar{Y}_4$. For the abelian group $\bar{H}$, let*

$$J_{\bar{H}} := \bigcup_{i\in\{2,3,4\}} (\overline{\mathrm{supp}}(\bar{Y}_i) \cap J) = \{(1,3),(1,4),(1,5),(2,3)\} \subseteq J,$$

$$V_{\bar{H}} := V_{J_{\bar{H}}} \bigcap V = \left\{A = (A_{ij}) \in V_{J_{\bar{H}}} \mid A_{i,j} \in \mathbb{F}_q,\ A_{14} = A_{15}\right\}.$$

Analogous to $G_2^{syl}(q)$, we construct a bijective 1-cocycle of $\bar{H}$ and a monomial $\mathbb{C}\bar{H}$-module. The irreducible characters of $\bar{H}$ are obtained, since $\bar{H}$ is abelian. Thus we get

$$\mathrm{Irr}(\bar{H}) = \{\bar{\chi}^{A_{15},A_{23},A_{13}} \mid A_{13}, A_{15}, A_{23} \in \mathbb{F}_q\}$$

where

$$\bar{\chi}^{A_{15},A_{23},A_{13}}(\bar{x}_2(t_2)\bar{x}_3(t_3)\bar{x}_4(t_4)) := \vartheta(A_{23}t_2)\cdot\vartheta(-A_{13}t_3)\cdot\vartheta(2A_{15}t_4).$$

(b) Similar arguments to $U = G_2^{syl}(q)$, we can construct linearisations for the quotient groups $Y_3Y_4Y_5Y_6\backslash U$, $Y_4Y_5Y_6\backslash U$, $Y_5Y_6\backslash U$ and $Y_6\backslash U$, and get their supermodules, superclasses and supercharacters. By 3.3.2, their conjugacy classes are determined. We also obtain their irreducible character by Clifford theory and calculate their character tables.

Take $\bar{U} := Y_5Y_6\backslash U = \bar{Y}_2\bar{Y}_1\bar{Y}_3\bar{Y}_4$ for example, let

$$J_{\bar{U}} := \bigcup_{i\in\{1,2,3,4\}} (\overline{\mathrm{supp}}(\bar{Y}_i)\cap J) = \{(1,2),(1,3),(1,4),(1,5),(2,3)\} \subseteq J,$$
$$V_{\bar{U}} := V_{J_{\bar{U}}} \bigcap V = \left\{A = (A_{ij}) \in V_{J_{\bar{U}}} \mid A_{i,j} \in \mathbb{F}_q,\ A_{14} = A_{15}\right\}.$$

Similar to $G_2^{syl}(q)$, we construct a bijective 1-cocycle of $\bar{U}$ and a monomial $\mathbb{C}\bar{U}$-module. With the similar analysis of the families $\mathfrak{F}_{1,2}$, $\mathfrak{F}_3$, $\mathfrak{F}_4$ of the orbit modules of U, we construct the supercharacters of $Y_5Y_6\backslash U$.

With the similar analysis of the families $\mathfrak{F}_{lin}$, $\mathfrak{F}_3$, $\mathfrak{F}_4$ of the irreducible characters of U, we obtain all the irreducible characters and the character table of $Y_5Y_6\backslash U$.

3.3.26 Proposition (Character table of $G_2^{syl}(q)$)**.** *The character table of $G_2^{syl}(q)$ is shown in Table 3.11.*

Table 3.11.: Character table of $G_2^{syl}(q)$ with $p \notin \{2,3\}$

	I_8	$x_1(t_1^*) \cdot x_6(t_6)$	$x_2(t_2^*) \cdot x_4(t_4) \cdot x_5(t_5)$	$x_2(t_2^*) \cdot x_1(t_1^*)$	$x_3(t_3^*) \cdot x_5(t_5)$	$x_4(t_4^*)$	$x_5(t_5^*)$	$x_6(t_6^*)$
$\chi_{lin}^{0,0}$	1	1	1	1	1	1	1	1
$\chi_{lin}^{A_{12}^*,0}$	1	$\vartheta(A_{12}^*t_1^*)$	1	$\vartheta(A_{12}^*t_1^*)$	1	1	1	1
χ_{lin}^{0,A_{23}^*}	1	1	$\vartheta(A_{23}^*t_2^*)$	$\vartheta(A_{23}^*t_2^*)$	1	1	1	1
$\chi_{lin}^{A_{12}^*,A_{23}^*}$	1	$\vartheta(A_{12}^*t_1^*)$	$\vartheta(A_{23}^*t_2^*)$	$\vartheta(A_{12}^*t_1^*) \cdot\vartheta(A_{23}^*t_2^*)$	1	1	1	1
$\chi_{3,q}^{A_{13}^*}$	q	0	0	0	$\vartheta(-A_{13}^*t_3^*) \cdot q$	q	q	q
$\chi_{4,q}^{A_{15}^*,A_{23}}$	q	0	$\sum_{r_1\in\mathbb{F}_q} \vartheta(-2A_{15}^*t_2^*r_1^2) \cdot\vartheta(2A_{15}^*t_4)\cdot\vartheta(A_{23}t_2^*)$	0	0	$\vartheta(2A_{15}^*t_4^*) \cdot q$	q	q
$\chi_{5,q}^{A_{16}^*,A_{23},A_{13}}$	q	0	$\sum_{r_1\in\mathbb{F}_q} \vartheta(A_{13}t_2^*r_1 - A_{16}^*t_2^*r_1^3+3A_{16}^*t_4r_1) \cdot\vartheta(A_{16}^*t_5)\cdot\vartheta(A_{23}t_2^*)$	0	$\sum_{r_1\in\mathbb{F}_q} \vartheta(3A_{16}^*t_3^*r_1^2) \cdot\vartheta(A_{16}^*t_5)\cdot\vartheta(-A_{13}t_3^*)$	0	$\vartheta(A_{16}^*t_5^*) \cdot q$	q
$\chi_{6,q^2}^{A_{17}^*,A_{12}}$	q^2	$\vartheta(A_{17}^*t_6)\cdot\vartheta(A_{12}t_1^*) \sum_{r_3\in\mathbb{F}_q} \vartheta(-3A_{17}t_1^*r_3^2)$	0	0	0	0	0	$\vartheta(A_{17}^*t_6^*) \cdot q^2$

Proof. Let $y := y(t_1, t_2, t_3, t_4, t_5, t_6) = x(t_1, t_2, t_3, t_4, t_5, t_6) \in U = G_2^{syl}(q)$.

(1) *The values of* $\chi_{6,q^2}^{A_{17}^*,A_{12}}$ *for all* $A_{17}^* \in \mathbb{F}_q^*$ *and* $A_{12} \in \mathbb{F}_{q^3}$.

Let $H := Y_1Y_4Y_5Y_6$, $\bar{H} := Y_4Y_5\backslash H \cong \bar{Y}_1\bar{Y}_6$, $\bar{\chi}_{6,q^2}^{A_{17},A_{12}} \in \mathrm{Irr}(\bar{H})$ with

$$\bar{\chi}_{6,q^2}^{A_{17},A_{12}}(\bar{x}_1(t_1)\bar{x}_6(t_6)) := \vartheta(A_{12}t_1) \cdot \vartheta(A_{17}t_6).$$

Let $\tilde{\chi}_{6,q^2}^{A_{17},A_{12}}$ denote the lift of $\bar{\chi}_{6,q^2}^{A_{17},A_{12}}$ from $\bar{H}$ to H, and $\chi_{6,q^2}^{A_{17}^*,A_{12}} := \mathrm{Ind}_H^U \tilde{\chi}_{6,q^2}^{A_{17}^*,A_{12}}$. Then

$$\begin{aligned}\chi_{6,q^2}^{A_{17}^*,A_{12}}(y) &= \left(\mathrm{Ind}_H^U \tilde{\chi}_{6,q^2}^{A_{17}^*,A_{12}}\right)(y) = \frac{1}{|H|}\sum_{\substack{g\in U\\ g\cdot y\cdot g^{-1}\in H}} \tilde{\chi}_{6,q^2}^{A_{17}^*,A_{12}}(g\cdot y\cdot g^{-1})\\ &=\frac{1}{|H|}\sum_{\substack{g\in U\\ g\cdot y\cdot g^{-1}\in H}} \bar{\chi}_{6,q^2}^{A_{17}^*,A_{12}}(Y_4Y_5\backslash(g\cdot y\cdot g^{-1})).\end{aligned}$$

Thus

$$\begin{aligned}&\chi_{6,q^2}^{A_{17}^*,A_{12}}(x_2(t_2^*)x_4(t_4)x_5(t_5)) = \chi_{6,q^2}^{A_{17}^*,A_{12}}(x_2(t_2^*)x_1(t_1^*))\\ =&\chi_{6,q^2}^{A_{17}^*,A_{12}}(x_3(t_3^*)x_5(t_5)) \overset{gxg^{-1}\notin H}{=} 0,\end{aligned}$$

and

$$\begin{aligned}&\chi_{6,q^2}^{A_{17}^*,A_{12}}(x_4(t_4)x_5(t_5)x_6(t_6))\\ =&\frac{1}{|H|}\sum_{\substack{g:=x(r_1,r_2,r_3,r_4,r_5,r_6)\in U\\ g\cdot x_4(t_4)x_5(t_5)x_6(t_6)\cdot g^{-1}\in H}} \tilde{\chi}_{6,q^2}^{A_{17}^*,A_{12}}(g\cdot x_4(t_4)x_5(t_5)x_6(t_6)\cdot g^{-1})\\ =&\frac{1}{|H|}\sum_{r_1,r_2,r_3,r_4,r_5,r_6\in\mathbb{F}_q} \tilde{\chi}_{6,q^2}^{A_{17}^*,A_{12}}(x_4(t_4)x_5(t_5+3r_1t_4)x_6(t_6+r_2t_5+3r_1r_2t_4+3r_3t_4))\\ =&\frac{1}{|H|}\sum_{r_1,r_2,r_3,r_4,r_5,r_6\in\mathbb{F}_q} \bar{\chi}_{6,q^2}^{A_{17}^*,A_{12}}(\bar{x}_6\left(t_6+r_2t_5+3r_1r_2t_4+3r_3t_4\right))\\ =&\frac{1}{q}\sum_{r_1,r_2,r_3\in\mathbb{F}_q} \bar{\chi}_{6,q^2}^{A_{17}^*,A_{12}}(\bar{x}_6\left(t_6+r_2t_5+3r_1r_2t_4+3r_3t_4\right)).\end{aligned}$$

Then $\chi_{6,q^2}^{A_{17}^*,A_{12}}(I_8) = q^2$, $\chi_{6,q^2}^{A_{17}^*,A_{12}}(x_4(t_4^*)) = \chi_{6,q^2}^{A_{17}^*,A_{12}}(x_5(t_5^*)) = 0$, $\chi_{6,q^2}^{A_{17}^*,A_{12}}(x_6(t_6^*)) =$

$q^2 \cdot \vartheta(A_{17}^* t_6^*)$, and

$$\begin{aligned}\chi_{6,q^2}^{A_{17}^*,A_{12}}(x_1(t_1^*)x_6(t_6)) &= \frac{1}{|H|} \sum_{\substack{g\in U\\ g\cdot x_1(t_1^*)x_6(t_6)\cdot g^{-1}\in H}} \tilde{\chi}_{6,q^2}^{A_{17}^*,A_{12}}(g\cdot x_1(t_1^*)x_6(t_6)\cdot g^{-1})\\
&\overset{g:=x(r_1,r_2,r_3,r_4,r_5,r_6)}{=} \frac{1}{|H|} \sum_{\substack{r_2=0\\ r_1,r_3,r_4,r_5,r_6\in\mathbb{F}_q}} \bar{\chi}_{6,q^2}^{A_{17}^*,A_{12}}(\bar{x}_1(t_1^*)\bar{x}_6(t_6-3t_1^*r_3^2))\\
&= \sum_{r_3\in\mathbb{F}_q} \bar{\chi}_{6,q^2}^{A_{17}^*,A_{12}}(\bar{x}_1(t_1^*)\bar{x}_6(t_6-3t_1^*r_3^2)) = \vartheta(A_{12}t_1^*+A_{17}^*t_6)\cdot \sum_{r_3\in\mathbb{F}_q}\vartheta(-3A_{17}^*t_1^*r_3^2).\end{aligned}$$

Thus we get all of the values of $\chi_{6,q^2}^{A_{17}^*,A_{12}}$.

(2) *The values of $\chi_{4,q}^{A_{15}^*,A_{23}}$ for all $A_{15}^*\in\mathbb{F}_q^*$ and $A_{23}\in\mathbb{F}_q$.*

Let $\bar{U} := Y_5Y_6\backslash U = \bar{Y}_2\bar{Y}_1\bar{Y}_3\bar{Y}_4$, $\bar{H} := \bar{Y}_2\bar{Y}_3\bar{Y}_4$, $\bar{\chi}_{4,q}^{A_{15},A_{23},A_{13}} \in \mathrm{Irr}(\bar{H})$ with

$$\bar{\chi}_{4,q}^{A_{15},A_{23},A_{13}}(\bar{x}_2(t_2)\bar{x}_3(t_3)\bar{x}_4(t_4)) := \vartheta(A_{23}t_2)\cdot\vartheta(-A_{13}t_3)\cdot\vartheta(2A_{15}t_4),$$

and $\chi_{4,q}^{A_{15}^*,A_{23}}$ denote the lift of $\mathrm{Ind}_{\bar{H}}^{\bar{U}}\bar{\chi}_{4,q}^{A_{15}^*,A_{23},0}$ to U. We have

$$\begin{aligned}\chi_{4,q}^{A_{15}^*,A_{23}}(y) &= \left(\mathrm{Ind}_{\bar{H}}^{\bar{U}}\bar{\chi}_{4,q}^{A_{15}^*,A_{23},0}\right)\left(\bar{x}_2(t_2)\bar{x}_1(t_1)\bar{x}_3(t_3)\bar{x}_4(t_4)\right)\\
&=\frac{1}{|\bar{H}|}\sum_{\substack{\bar{u}:=\bar{x}_2(r_2)\bar{x}_1(r_1)\bar{x}_3(r_3)\bar{x}_4(r_4)\in\bar{U}\\ \bar{u}\cdot\bar{x}_2(t_2)\bar{x}_1(t_1)\bar{x}_3(t_3)\bar{x}_4(t_4)\cdot\bar{u}^{-1}\in\bar{H}}} \bar{\chi}_{4,q}^{A_{15}^*,A_{23},0}\left(\bar{u}\cdot\bar{x}_2(t_2)\bar{x}_1(t_1)\bar{x}_3(t_3)\bar{x}_4(t_4)\cdot\bar{u}^{-1}\right),\end{aligned}$$

then

$$\begin{aligned}\chi_{4,q}^{A_{15}^*,A_{23}}(x_5(t_5)x_6(t_6)) &= \bar{\chi}_{4,q}^{A_{15}^*,A_{23},0}(\bar{I}_8) = q,\\
\chi_{4,q}^{A_{15}^*,A_{23}}(x_1(t_1^*)x_6(t_6)) &= \chi_{4,q}^{A_{15}^*,A_{23}}(x_1(t_1^*)x_2(t_2^*)) = 0,\\
\chi_{4,q}^{A_{15}^*,A_{23}}(x_4(t_4^*)) &= q\cdot\bar{\chi}_{4,q}^{A_{15}^*,A_{23},0}(x_4(t_4^*)) = q\cdot\vartheta(2A_{15}^*t_4^*),\end{aligned}$$

$$
\begin{aligned}
\chi_{4,q}^{A_{15}^*,A_{23}}(x_3(t_3^*)x_5(t_5)) =&\mathrm{Ind}_{\bar{H}}^{\bar{U}}\bar{\chi}_{4,q}^{A_{15}^*,A_{23},0}\big(\bar{x}_3(t_3^*)\big)\\
=&\frac{1}{|\bar{H}|}\sum_{\substack{\bar{u}:=\bar{x}_2(r_2)\bar{x}_1(r_1)\bar{x}_3(r_3)\bar{x}_4(r_4)\in\bar{U}\\ \bar{u}\cdot\bar{x}_3(t_3^*)\cdot\bar{u}^{-1}\in\bar{H}}}\bar{\chi}_{4,q}^{A_{15}^*,A_{23},0}\big(\bar{u}\cdot\bar{x}_3(t_3^*)\cdot\bar{u}^{-1}\big)\\
=&\sum_{r_1\in\mathbb{F}_q}\bar{\chi}_{4,q}^{A_{15}^*,A_{23},0}\big(\bar{x}_3(t_3^*)\bar{x}_4(2r_1t_3^*)\big)\\
=&\sum_{r_1\in\mathbb{F}_q}\vartheta\big(4A_{15}^*r_1t_3^*\big)\\
=&0,\\
\chi_{4,q}^{A_{15}^*,A_{23}}(x_2(t_2^*)x_4(t_4)x_5(t_5)) =&\mathrm{Ind}_{\bar{H}}^{\bar{U}}\bar{\chi}_{4,q}^{A_{15}^*,A_{23},0}\big(\bar{x}_2(t_2^*)\bar{x}_4(t_4)\big)\\
=&\frac{1}{|\bar{H}|}\sum_{\substack{\bar{u}\in\bar{U}\\ \bar{u}\cdot\bar{x}_2(t_2)\bar{x}_4(t_4)\cdot\bar{u}^{-1}\in\bar{H}}}\bar{\chi}_{4,q}^{A_{15}^*,A_{23},0}\big(\bar{u}\cdot\bar{x}_2(t_2^*)\bar{x}_4(t_4)\cdot\bar{u}^{-1}\big)\\
\overset{\bar{u}:=\bar{x}_2(r_2)\bar{x}_1(r_1)\bar{x}_3(r_3)\bar{x}_4(r_4)}{=}&\sum_{r_1\in\mathbb{F}_q}\bar{\chi}_{4,q}^{A_{15}^*,A_{23},0}\big(\bar{x}_2(t_2^*)\bar{x}_3(-t_2^*r_1)\bar{x}_4(t_4-t_2^*r_1^2)\big)\\
=&\vartheta(A_{23}t_2^*)\cdot\sum_{r_1\in\mathbb{F}_q}\vartheta\big(2A_{15}^*(t_4-t_2^*r_1^2)\big).
\end{aligned}
$$

Thus all the values of $\chi_{4,q}^{A_{15}^*,A_{23}}$ are obtained.

All the other values are determined by similar calculation. $\square$

3.3.27 Proposition (Supercharacters and irreducible characters)**.** *The following relations between the supercharacters and the irreducible characters of $G_2^{syl}(q)$ are obtained:*

$$
\begin{aligned}
&\Psi_{M(A_{17}^*e_{17})}=q\sum_{A_{12}\in\mathbb{F}_q}\chi_{6,q^2}^{A_{17}^*,A_{12}}, && \Psi_{M(A_{16}^*e_{16})}=q\sum_{A_{13},A_{23}\in\mathbb{F}_q}\chi_{5,q}^{A_{16}^*,A_{23},A_{13}},\\
&\Psi_{M(A_{15}^*(e_{14}+e_{15}))}=q\sum_{A_{23}\in\mathbb{F}_q}\chi_{4,q}^{A_{15}^*,A_{23}}, && \Psi_{M(A_{13}^*e_{13})}=\chi_{3,q}^{A_{13}^*},\\
&\Psi_{M(A_{12}^*e_{12}+A_{23}^*e_{23})}=\chi_{lin}^{A_{12}^*,A_{23}^*}, && \Psi_{M(A_{23}^*e_{23})}=\chi_{lin}^{0,A_{23}^*},\\
&\Psi_{M(A_{12}^*e_{12})}=\chi_{lin}^{A_{12}^*,0}, && \Psi_{M(0)}=\chi_{lin}^{0,0}=\mathrm{triv}_U.
\end{aligned}
$$

Proof. Take $\Psi_{M(A_{15}^*(e_{14}+e_{15}))}=q\sum_{A_{23}\in\mathbb{F}_q}\chi_{4,q^3}^{A_{15}^*,A_{23}}$ for example. We must show that $\Psi_{M(A_{15}^*(e_{14}+e_{15}))}(u)=q\sum_{A_{23}\in\mathbb{F}_q}\chi_{4,q}^{A_{15}^*,A_{23}}(u)$ for all $u\in U$.

By Table 3.6 and Table 3.11, it is sufficient to to prove that

$$0=\Psi_{M(A_{15}^*(e_{14}+e_{15}))}(u)=q\sum_{A_{23}\in\mathbb{F}_q}\chi_{4,q}^{A_{15}^*,A_{23}}(u)\qquad\text{for all } u\in C_2.$$

Let $u := y(0, t_2^*, t_3, t_4, t_5, t_6)$, then

$$\begin{aligned} q \sum_{A_{23}\in\mathbb{F}_q} \chi_{4,q}^{A_{15}^*,A_{23}}(u) &= q \sum_{A_{23}\in\mathbb{F}_q} \sum_{r_1\in\mathbb{F}_q} \vartheta(A_{23}t_2^* + 2A_{15}^*(t_4 - t_2^* r_1^2 + 2r_1t_3)) \\ =& q \sum_{A_{23}\in\mathbb{F}_q} \vartheta(A_{23}t_2^*) \sum_{r_1\in\mathbb{F}_q} \vartheta(2A_{15}^*(t_4 - t_2^* r_1^2 + 2r_1t_3)) = 0. \end{aligned}$$

Thus $\Psi_{M(A_{15}^*(e_{14}+e_{15})} = q\sum_{A_{23}\in\mathbb{F}_q} \chi_{4,q}^{A_{15}^*,A_{23}}$ is proved.

Similarly, we get the other formulae. □

S. M. Goodwin and G. Röhrle [GR09] obtained an algorithm and calculated the number $k(G_2^{syl}(q))$ of conjugacy classes of $G_2^{syl}(q)$, but did not get the conjugacy classes. F. Himstedt, T. Le and K. Magaard [HLM16] determined most irreducible characters of $G_2^{syl}(q)$ with $p > 3$ by parameterizing *midafis* (minimal degree almost faithful irreducible characters). By Propositions 3.3.4, 3.3.23 and 3.3.26, we obtain the number of the conjugacy classes of $G_2^{syl}(q)$ and determine the numbers of the complex irreducible characters of degree q^c with $c \in \mathbb{N}$ (also see [GR09, Table 1] and [GMR16, Table 3]).

3.3.28 Corollary. *Let* $\#\mathrm{Irr}$ *be the number of irreducible characters of* $G_2^{syl}(q)$*, and* $\#\mathrm{Irr}_c$ *be the number of irreducible characters of* $G_2^{syl}(q)$ *of dimension* q^c *with* $c \in \mathbb{N}$*, then*

$$\begin{aligned} \#\mathrm{Irr}_2 &= q^2 - q &&= (q-1)^2 + (q-1), \\ \#\mathrm{Irr}_1 &= q^3 - 1 &&= (q-1)^3 + 3(q-1)^2 + 3(q-1), \\ \#\mathrm{Irr}_0 &= q^2 &&= (q-1)^2 + 2(q-1) + 1, \end{aligned}$$

and

$$\begin{aligned} \#\mathrm{Irr} =& \#\{\textit{Irreducible Characters of } G_2^{syl}(q)\} \\ =& \#\{\textit{Conjugacy Classes of } G_2^{syl}(q)\} \\ =& q^3 + 2q^2 - q - 1 \\ =& (q-1)^3 + 5(q-1)^2 + 6(q-1) + 1. \end{aligned}$$

3.3.29 Remark. *We consider the analogue of Higman's conjecture, Lehrer's conjecture and Isaacs' conjecture of* $U_n(q)$ *for* $G_2^{syl}(q)$*, and the conjectures are true for* $G_2^{syl}(q)$ *(see 3.3.28 and [GR09, GMR16]).*

4. Supercharacter theories for ${}^{2}G_{2}^{syl}\left(3^{2m+1}\right)$

In this chapter, we firstly choose suitable structure constants of extraspecial pairs, with those we can construct a Sylow 3-subgroup ${}^{2}G_{2}^{syl}\left(3^{2m+1}\right)$ of the Ree group ${}^{2}G_{2}\left(3^{2m+1}\right)$.

Then we determine a monomial linearisation $(f, \kappa|_{V\times V})$ for $A_8(q)$ (4.2.16), and a monomial linearisation $(f|_{{}^{2}G_{2}^{syl}(3^{2m+1})}, \kappa|_{V\times V})$ for ${}^{2}G_{2}^{syl}\left(3^{2m+1}\right)$ (4.2.17). We get a monomial $A_8(q)$-module $\mathbb{C}U$ (4.2.18), and obtain a classification of U-orbit modules (4.2.26).

Finally, we establish a supercharacter theory for ${}^{2}G_{2}^{syl}\left(3^{2m+1}\right)$ (4.3.14), and calculate the character table of ${}^{2}G_{2}^{syl}(3)$ (4.3.27).

In this chapter, we use many definitions analogously to previous chapters, for example: the main condition and the truncated column operation.

4.1. Sylow p-subgroup ${}^{2}G_{2}^{syl}\left(3^{2m+1}\right)$

In this section, we get a new Chevalley basis of the Lie algebra of type G_2, and construct a Sylow 3-subgroup ${}^{2}G_{2}^{syl}\left(3^{2m+1}\right)$ of the Ree group ${}^{2}G_{2}\left(3^{2m+1}\right)$.

4.1.1. New Chevalley basis of the Lie algebra of type G_2

In this subsection, we choose different signs of the structure constants of the Lie algebra $\mathcal{L}_{G_2}$ from those in Section 3.1. Then we get a new Chevalley basis of the Lie algebra $\mathcal{L}_{G_2}$. But we get exactly the same Sylow p-subgroup $G_2^{syl}(q)$ and root subgroups Y_i $(i = 1, 2, \ldots, 6)$ of the Chevalley group $G_2(q)$ as those in subsection 3.1.2.

4.1.1 Example (New structure constants of Lie algebra $\mathcal{L}_{G_2}$). *We choose the following* *signs of the structure constants of Lie algebra $\mathcal{L}_{G_2}$:*

extraspecial pair (r, s)	$N_{r,s}$
(α, β)	-1
$(\alpha, \alpha+\beta)$	-2
$(\alpha, 2\alpha+\beta)$	3
$(\beta, 3\alpha+\beta)$	1

i.e. the sign of $N_{\alpha,\alpha+\beta}$ is different from that in 3.1.10. The signs satisfy that $N_{\alpha,\alpha+\beta} = -2N_{\beta,3\alpha+\beta}$ and $N_{\alpha,2\alpha+\beta} = 3N_{\beta,3\alpha+\beta}$. Then all of the structure constants of $\mathcal{L}_{G_2}$ are determined:

Table 4.1.: New structure constants of the Lie algebra $\mathcal{L}_{G_2}$

$N_{r,s}$	α	β	α $+\beta$	2α $+\beta$	3α $+\beta$	3α $+2\beta$	$-\alpha$	$-\beta$	$-\alpha$ $-\beta$	-2α $-\beta$	-3α $-\beta$	-3α -2β
α	$\cdot$	-1	-2	3			$*$		3	2	-1	
β	1	$\cdot$			1			$*$	-1			-1
$\alpha+\beta$	2		$\cdot$	3			3	-1	$*$	-2		-1
$2\alpha+\beta$	-3		-3	$\cdot$			2		-2	$*$	1	1
$3\alpha+\beta$		-1			$\cdot$		-1			1	$*$	1
$3\alpha+2\beta$						$\cdot$		-1	-1	1	1	$*$
$-\alpha$	$*$		-3	-2	1		$\cdot$	1	2	-3		
$-\beta$		$*$	1			1	-1	$\cdot$			-1	
$-\alpha-\beta$	-3	1	$*$	2		1	-2		$\cdot$	-3		
$-2\alpha-\beta$	-2		2	$*$	-1	-1	3		3	$\cdot$		
$-3\alpha-\beta$	1			-1	$*$	-1		1			$\cdot$	
$-3\alpha-2\beta$		1	1	-1	-1	$*$						$\cdot$

According to the signs of the structure constants of the Lie algebra $\mathcal{L}_{G_2}$ in Example 4.1.1, we get a Chevalley basis for the Lie algebra $\mathcal{L}_{G_2}$.

4.1.2 Proposition (New Chevalley basis of the Lie algebra $\mathcal{L}_{G_2}$)**.** *A Chevalley basis of the Lie algebra $\mathcal{L}_{G_2}$ is*

	$\sum_{i,j} c_{ij} e_{i,j}$	$\sum_\gamma c_\gamma e_\gamma$ or $\sum_\gamma c_\gamma h_\gamma$
h_α	$(e_{1,1}-e_{8,8})$ $-(e_{2,2}-e_{7,7})$ $+2(e_{3,3}-e_{6,6})$	$h_{r_1}+h_{r_3}+h_{r_4}$
h_β	$(e_{2,2}-e_{7,7})$ $-(e_{3,3}-e_{6,6})$	h_{r_2}
e_α	$(e_{1,2}-e_{7,8})$ $+(e_{3,4}-e_{5,6})$ $+(e_{3,5}-e_{4,6})$	$e_{r_1}+e_{r_3}+e_{r_4}$
e_β	$e_{2,3}-e_{6,7}$	e_{r_2}
$e_{\alpha+\beta}$	$-(e_{1,3}-e_{6,8})$ $+(e_{2,4}-e_{5,7})$ $+(e_{2,5}-e_{4,7})$	$e_{r_1+r_2}+e_{r_2+r_3}+e_{r_2+r_4}$
$e_{2\alpha+\beta}$	$-(e_{1,4}-e_{5,8})$ $-(e_{2,6}-e_{3,7})$ $-(e_{1,5}-e_{4,8})$	$-e_{r_1+r_2+r_3}-e_{r_2+r_3+r_4}-e_{r_1+r_2+r_4}$
$e_{3\alpha+\beta}$	$-(e_{1,6}-e_{3,8})$	$-e_{r_1+r_2+r_3+r_4}$
$e_{3\alpha+2\beta}$	$-(e_{1,7}-e_{2,8})$	$-e_{r_1+2r_2+r_3+r_4}$
$e_{-\alpha}$	$(e_{2,1}-e_{8,7})$ $+(e_{4,3}-e_{6,5})$ $+(e_{5,3}-e_{6,4})$	$e_{-r_1}+e_{-r_3}+e_{-r_4}$
$e_{-\beta}$	$e_{3,2}-e_{7,6}$	e_{-r_2}
$e_{-\alpha-\beta}$	$-(e_{3,1}-e_{8,6})$ $+(e_{4,2}-e_{7,5})$ $+(e_{5,2}-e_{7,4})$	$e_{-r_1-r_2}+e_{-r_2-r_3}+e_{-r_2-r_4}$
$e_{-2\alpha-\beta}$	$-(e_{4,1}-e_{8,5})$ $-(e_{6,2}-e_{7,3})$ $-(e_{5,1}-e_{8,4})$	$-e_{-r_1-r_2-r_3}-e_{-r_2-r_3-r_4}-e_{-r_1-r_2-r_4}$
$e_{-3\alpha-\beta}$	$-(e_{6,1}-e_{8,3})$	$-e_{-r_1-r_2-r_3-r_4}$
$e_{-3\alpha-2\beta}$	$-(e_{7,1}-e_{8,2})$	$-e_{-r_1-2r_2-r_3-r_4}$

According to the new Chevalley basis of the Lie algebra $\mathcal{L}_{G_2}$, we define the following notations.

4.1.3 Notation. *Define a matrix group*

$$\bar{G}_2(q) := \langle\, \exp(te_r)) \,|\, r \in \Phi_{G_2},\ t \in \mathbb{F}_q \,\rangle\,,$$

and the Chevalley group of type $\mathcal{L}_{G_2}$ over the field $\mathbb{F}_q$

$$G_2(q) := \langle\, \exp(t \operatorname{ad} e_r)) \mid r \in \Phi_{G_2},\ t \in \mathbb{F}_q \,\rangle\,.$$

For $t \in \mathbb{F}_q$, we have $y_i(t) = x_i(t)$.

4.1.4 Notation. *We set*

$$y_r(t) := \exp(te_r) = I_8 + te_r + \frac{1}{2}t^2e_r^2 \quad \textit{for all } r \in \Phi_{G_2},\ t \in \mathbb{F}_q.$$

In particular,

$$\begin{aligned} y_1(t) :=& y_\alpha(t) = x_1(t), & y_2(t) :=& y_\beta(t) = x_2(t), & y_3(t) :=& y_{\alpha+\beta}(t) = x_3(t),\\ y_4(t) :=& y_{2\alpha+\beta}(t) = x_4(-t), & y_5(t) :=& y_{3\alpha+\beta}(t) = x_5(-t), & y_6(t) :=& y_{3\alpha+2\beta}(t) = x_6(-t). \end{aligned}$$

4.1.5 Notation/Lemma. *The positive root subgroups of $G_2(q)$ (up to isomorphisms) are given as*

$$Y_i := \{y_i(t) \mid t \in \mathbb{F}_q\}\,,\ i = 1, 2, 3, 4, 5, 6.$$

By calculation, we get the following commutators.

4.1.6 Proposition. *Let $t_1, t_2, t_3, t_4, t_5, t_6 \in \mathbb{F}_q$ and define the commutator*

$$[y_i(t_i), y_j(t_j)] := y_i(t_i)^{-1}y_j(t_j)^{-1}y_i(t_i)y_j(t_j)$$

then the non-trivial commutators are determined

$$\begin{aligned} [y_1(t_1), y_2(t_2)] =& y_3(-t_2t_1) \cdot y_4(-t_2t_1^2) \cdot y_5(t_2t_1^3) \cdot y_6(-2t_2^2t_1^3),\\ [y_2(t_2), y_1(t_1)] =& y_3(t_2t_1) \cdot y_4(t_2t_1^2) \cdot y_5(-t_2t_1^3) \cdot y_6(-t_2^2t_1^3),\\ [y_1(t_1), y_3(t_3)] =& y_4(-2t_1t_3) \cdot y_5(3t_1^2t_3) \cdot y_6(3t_1t_3^2),\\ [y_1(t_1), y_4(t_4)] =& y_5(3t_1t_4),\\ [y_3(t_3), y_4(t_4)] =& y_6(3t_3t_4),\\ [y_2(t_2), y_5(t_5)] =& y_6(t_2t_5). \end{aligned}$$

In particular, let $\operatorname{Char}\mathbb{F}_q = 3$, then

$$\begin{aligned} [y_1(t_1), y_2(t_2)] =& y_3(-t_2t_1) \cdot y_4(-t_2t_1^2) \cdot y_5(t_2t_1^3) \cdot y_6(-2t_2^2t_1^3)\\ =& y_3(-t_2t_1) \cdot y_4(-t_2t_1^2) \cdot y_5(t_2t_1^3) \cdot y_6(t_2^2t_1^3),\\ [y_2(t_2), y_1(t_1)] =& y_3(t_2t_1) \cdot y_4(t_2t_1^2) \cdot y_5(-t_2t_1^3) \cdot y_6(-t_2^2t_1^3),\\ [y_1(t_1), y_3(t_3)] =& y_4(-2t_1t_3) = y_4(t_1t_3),\\ [y_2(t_2), y_5(t_5)] =& y_6(t_2t_5). \end{aligned}$$

4.1.7 Notation. *Let $t_i \in \mathbb{F}_q$ with $i \in \{1, 2, \ldots, 6\}$, write*

$$\begin{aligned} y(t_1, t_2, t_3, t_4, t_5, t_6) :=& y_2(t_2)y_1(t_1)y_3(t_3)y_4(t_4)y_5(t_5)y_6(t_6)\\ =& x(t_1, t_2, t_3, -t_4, -t_5, -t_6) \in G_2^{syl}(q). \end{aligned}$$

4.1.8 Proposition (Sylow p-subgroup $G_2^{syl}(q)$)**.** *A matrix Sylow p-subgroup $G_2^{syl}(q)$ of $G_2(q)$ (up to isomorphism) is*

$$\begin{aligned} G_2^{syl}(q) &:= \{y_2(t_2)y_1(t_1)y_3(t_3)y_4(t_4)y_5(t_5)y_6(t_6) \mid t_1,t_2,t_3,t_4,t_5,t_6 \in \mathbb{F}_q\} \\ &= \{x(t_1,t_2,t_3,-t_4,-t_5,-t_6) \mid t_1,t_2,t_3,t_4,t_5,t_6 \in \mathbb{F}_q\} \end{aligned}$$

where

$$y_2(t_2)y_1(t_1)y_3(t_3)y_4(t_4)y_5(t_5)y_6(t_6)$$
$$= \begin{pmatrix} 1 & t_1 & -t_3 & t_1t_3-t_4 & t_1t_3-t_4 & \begin{smallmatrix}-t_1t_4\\ -t_5\end{smallmatrix} & \begin{smallmatrix}-t_1t_3^2\\ -t_3t_4-t_6\end{smallmatrix} & \begin{smallmatrix}2t_1t_3t_4+t_1t_6\\ -t_3t_5-t_4^2\end{smallmatrix} \\ & 1 & t_2 & t_1t_2+t_3 & t_1t_2+t_3 & \begin{smallmatrix}-t_1^2t_2\\ -t_4\end{smallmatrix} & \begin{smallmatrix}-2t_1t_2t_3\\ +t_2t_4-t_3^2\end{smallmatrix} & \begin{smallmatrix}-t_1^2t_2t_3+2t_1t_2t_4\\ +t_2t_5+2t_3t_4+t_6\end{smallmatrix} \\ & & 1 & t_1 & t_1 & -t_1^2 & -2t_1t_3+t_4 & -t_1^2t_3+2t_1t_4+t_5 \\ & & & 1 & 0 & -t_1 & -t_3 & -t_1t_3+t_4 \\ & & & & 1 & -t_1 & -t_3 & -t_1t_3+t_4 \\ & & & & & 1 & -t_2 & t_1t_2+t_3 \\ & & & & & & 1 & -t_1 \\ & & & & & & & 1 \end{pmatrix}.$$

Proof. Similar to the proof of 3.1.27. $\square$

4.1.9 Reminder ([Car72], Theorem 9.4.10)**.**

$$|G_2(q)| = q^6(q^2-1)(q^6-1).$$

4.1.2. Sylow p-subgroup ${}^2G_2^{syl}\,(3^{2m+1})$

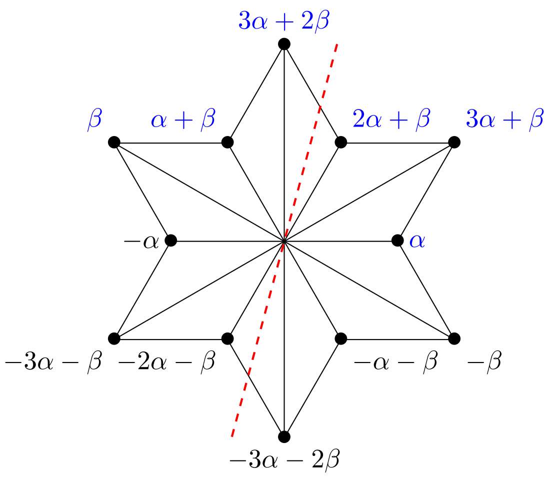
In this subsection, we use the notations of the subsection 3.1.1, and construct a Sylow 3-subgroup ${}^2G_2^{syl}\,(3^{2m+1})$ of the Ree group ${}^2G_2\,(3^{2m+1})$.

Let $p := 3$, $q := 3^{2m+1}$ $(m \in \mathbb{N})$ and $\theta := 3^m$. Define two field automorphisms of $G_2(q)$ as follows: for all $r \in \Phi_{G_2}$,

$$F_\theta \colon y_r(t) \mapsto y_r(t^\theta) = y_r(t^{3^m}), \qquad F_3 \colon y_r(t) \mapsto y_r(t^3).$$

Then $F_3 F_\theta^2 = \mathrm{id}_{G_2(q)}$.

Let $\rho \colon r \mapsto \bar{r}$ be a non-trivial symmetry of the Dynkin diagram of type G_2 (interchanging α and β). For every $r \in \Phi_{G_2}$, $\bar{r}$ is obtained by reflecting r in the line bisecting α and β.



Let $\epsilon_i = \pm 1$ $(i = 1,2,3,4)$ satisfy

$$N_{\alpha,\beta} = \epsilon_1, \quad N_{\alpha,\alpha+\beta} = 2\epsilon_2, \quad N_{\alpha,2\alpha+\beta} = 3\epsilon_3, \quad N_{\beta,3\alpha+\beta} = \epsilon_4.$$

Since the structure constants of $\mathcal{L}_{G_2}$ in Example 4.1.1 satisfy $-\epsilon_2 = \epsilon_3 = \epsilon_4 = 1$, by [Car72, Proposition 12.4.1] the map

$$y_r(t) \mapsto \begin{cases} y_{\bar{r}}(t), & \bar{r} \text{ is short} \\ y_{\bar{r}}(t^3), & \bar{r} \text{ is long} \end{cases}, \qquad r \in \Phi_{G_2},\ t \in \mathbb{F}_q,$$

can be extended to a graph automorphism $\tilde{\rho}$ of $G_2(q)$.

Let $F := \tilde{\rho} F_\theta = F_\theta \tilde{\rho}$, then

$$F \colon G_2(q) \to G_2(q) : y_r(t) \mapsto \begin{cases} y_{\bar{r}}(t^\theta), & \bar{r} \text{ is short} \\ y_{\bar{r}}(t^{3\theta}), & \bar{r} \text{ is long} \end{cases}, \qquad r \in \Phi_{G_2},\ t \in \mathbb{F}_q.$$

For a subgroup X of $G_2(q)$, we set $X^F := \{x \in X \mid F(x) = x\}$. By §13.4 of [Car72], $G_2(q)^F = {}^2G_2(q)$ and $G_2^{syl}(q)^F$ is a subgroup of ${}^2G_2(q)$. Then $G_2^{syl}(q)^F$ is also a Sylow p-subgroup of ${}^2G_2(q)$ (14.1.2 of [Car72]).

4.1.10 Lemma ([Car72], Theorem 14.3.2).

$$\begin{aligned}|{}^2G_2(q)| =& q^3(q-1)(q^3+1) = 3^{3(2m+1)}(3^{2m+1}-1)(3^{3(2m+1)}+1)\\ =& 3^{6m+3}(3^{2m+1}-1)(3^{6m+3}+1).\end{aligned}$$

4.1.11 Proposition (Sylow p-subgroup ${}^2G_2^{syl}(3^{2m+1})$). *A matrix Sylow 3-subgroup ${}^2G_2^{syl}(q)$ of ${}^2G_2(q)$ is*

$${}^2G_2^{syl}(q) := \left\{ y_2(t_1^{3\theta}) \cdot y_1(t_1)y_3(t_3)y_4(t_4) \cdot y_5(t_3^{3\theta} + t_1^{3\theta+3})y_6(t_4^{3\theta} + t_1^{6\theta+3}) \,\middle|\, t_1, t_3, t_4 \in \mathbb{F}_q \right\},$$

where

$$\begin{aligned}&y_2(t_1^{3\theta}) \cdot y_1(t_1)y_3(t_3)y_4(t_4) \cdot y_5(t_3^{3\theta} + t_1^{3\theta+3})y_6(t_4^{3\theta} + t_1^{6\theta+3})\\ =&\begin{pmatrix} 1 & t_1 & -t_3 & \begin{array}{c} t_1t_3\\ -t_4\end{array} & \begin{array}{c} t_1t_3\\ -t_4\end{array} & \begin{array}{c} -t_1t_4\\ -t_1^{3\theta+3}\\ -t_3^{3\theta}\end{array} & \begin{array}{c} -t_1t_3^2\\ -t_3t_4\\ -t_1^{6\theta+3}\\ -t_4^{3\theta}\end{array} & \begin{array}{c} 2t_1t_3t_4 + t_1^{6\theta+4}\\ +t_1t_4^{3\theta} - t_3t_1^{3\theta+3}\\ -t_3^{3\theta+1} - t_4^2\end{array}\\ & 1 & t_1^{3\theta} & \begin{array}{c} t_1^{3\theta+1}\\ +t_3\end{array} & \begin{array}{c} t_1^{3\theta+1}\\ +t_3\end{array} & \begin{array}{c} -t_1^{3\theta+2}\\ -t_4\end{array} & \begin{array}{c} -2t_1^{3\theta+1}t_3\\ +t_1^{3\theta}t_4\\ -t_3^2\end{array} & \begin{array}{c} -t_1^{3\theta+2}t_3\\ +2t_1^{3\theta+1}t_4\\ +t_1^{3\theta}t_3^{3\theta} + 2t_3t_4\\ +t_4^{3\theta} + 2t_1^{6\theta+3}\end{array}\\ & & 1 & t_1 & t_1 & -t_1^2 & \begin{array}{c} -2t_1t_3\\ +t_4\end{array} & \begin{array}{c} -t_1^2t_3 + 2t_1t_4\\ +t_3^{3\theta} + t_1^{3\theta+3}\end{array}\\ & & & 1 & 0 & -t_1 & -t_3 & -t_1t_3 + t_4\\ & & & & 1 & -t_1 & -t_3 & -t_1t_3 + t_4\\ & & & & & 1 & -t_1^{3\theta} & t_1^{3\theta+1} + t_3\\ & & & & & & 1 & -t_1\\ & & & & & & & 1\end{pmatrix}.\end{aligned}$$

Proof. We know $\operatorname{Char}\mathbb{F}_q = 3$ and $t^{3\theta^2} = t$ for all $t \in \mathbb{F}_q$. We choose the structure constants in the same way as in 4.1.1. Let $t_i \in \mathbb{F}_q$ and $y(t_1, t_2, t_3, t_4, t_5, t_6) \in G_2^{syl}(q)^F$. Then

$$\begin{aligned}&y(t_1, t_2, t_3, t_4, t_5, t_6) = F\big(y(t_1, t_2, t_3, t_4, t_5, t_6)\big)\\ =&F\big(y_2(t_2)y_1(t_1)y_3(t_3)y_4(t_4)y_5(t_5)y_6(t_6)\big) = y_1(t_2^\theta)y_2(t_1^{3\theta})y_5(t_3^{3\theta})y_6(t_4^{3\theta})y_3(t_5^\theta)y_4(t_6^\theta)\\ \overset{4.1.6}{=}&y_2(t_1^{3\theta})y_1(t_2^\theta)y_3(-t_1^{3\theta}t_2^\theta)y_4(-t_1^{3\theta}t_2^{2\theta})y_5(t_1^{3\theta}t_2^{3\theta})y_6(t_1^{6\theta}t_2^{3\theta}) \cdot y_3(t_5^\theta)y_4(t_6^\theta)y_5(t_3^{3\theta})y_6(t_4^{3\theta})\\ =&y\big(t_2^\theta,\ t_1^{3\theta},\ t_5^\theta - t_1^{3\theta}t_2^\theta,\ t_6^\theta - t_1^{3\theta}t_2^{2\theta},\ t_3^{3\theta} + t_1^{3\theta}t_2^{3\theta},\ t_4^{3\theta} + t_1^{6\theta}t_2^{3\theta}\big).\end{aligned}$$

Thus

$$t_1 = t_2^{\theta}, \quad t_3 = t_5^{\theta} - t_1^{3\theta} t_2^{\theta} = t_5^{\theta} - t_1^{3\theta+1}, \quad t_5 = t_3^{3\theta} + t_1^{3\theta} t_2^{3\theta} = t_3^{3\theta} + t_1^{3\theta+3},$$
$$t_2 = t_1^{3\theta}, \quad t_4 = t_6^{\theta} - t_1^{3\theta} t_2^{2\theta} = t_6^{\theta} - t_1^{3\theta+2}, \quad t_6 = t_4^{3\theta} + t_1^{6\theta} t_2^{3\theta} = t_4^{3\theta} + t_1^{6\theta+3}.$$

Hence $y(t_1, t_2, t_3, t_4, t_5, t_6) = y\big(t_1,\ t_1^{3\theta},\ t_3,\ t_4,\ t_3^{3\theta} + t_1^{3\theta+3},\ t_4^{3\theta} + t_1^{6\theta+3}\big)$, and $\left|G_2^{syl}(q)^F\right| = q^3$.

Therefore ${}^2G_2^{syl}(q) := G_2^{syl}(q)^F$ is a Sylow p-subgroup of ${}^2G_2(q)$. We get the matrix form by calculation. □

4.1.12 Corollary. ${}^2G_2^{syl}\,(3^{2m+1}) \leqslant G_2^{syl}\,(3^{2m+1})$.

4.1.13 Notation/Lemma. *Let $i \in \{1,3,4\}$ and $t_i \in \mathbb{F}_q$, then set*

$$a(t_1) := y_2(t_1^{3\theta}) y_1(t_1) y_5(t_1^{3\theta+3}) y_6(t_1^{6\theta+3})$$
$$= \begin{pmatrix} 1 & t_1 & & & & -t_1^{3\theta+3} & -t_1^{6\theta+3} & t_1^{6\theta+4} \\ & 1 & t_1^{3\theta} & t_1^{3\theta+1} & t_1^{3\theta+1} & -t_1^{3\theta+2} & 0 & 2t_1^{6\theta+3} \\ & & 1 & t_1 & t_1 & -t_1^{2} & 0 & t_1^{3\theta+3} \\ & & & 1 & 0 & -t_1 & 0 & 0 \\ & & & & 1 & -t_1 & 0 & 0 \\ & & & & & 1 & -t_1^{3\theta} & t_1^{3\theta+1} \\ & & & & & & 1 & -t_1 \\ & & & & & & & 1 \end{pmatrix},$$

$$b(t_3) := y_3(t_3) y_5(t_3^{3\theta})$$
$$= \begin{pmatrix} 1 & & -t_3 & & & -t_3^{3\theta} & & -t_3^{3\theta+1} \\ & 1 & & t_3 & t_3 & & -t_3^{2} & \\ & & 1 & & & \cdot & & t_3^{3\theta} \\ & & & 1 & \cdot & & -t_3 & \\ & & & & 1 & & -t_3 & \\ & & & & & 1 & & t_3 \\ & & & & & & 1 & \\ & & & & & & & 1 \end{pmatrix},$$

$$
\begin{aligned}
c(t_4) :=& y_4(t_4)y_6(t_4^{3\theta}) \\
=& \begin{pmatrix}
1 & & & -t_4 & -t_4 & & -t_4^{3\theta} & -t_4^2 \\
 & 1 & & & & -t_4 & \cdot & t_4^{3\theta} \\
 & & 1 & & & \cdot & t_4 & \\
 & & & 1 & \cdot & & & t_4 \\
 & & & & 1 & & & t_4 \\
 & & & & & 1 & & \\
 & & & & & & 1 & \\
 & & & & & & & 1
\end{pmatrix}.
\end{aligned}
$$

Let $Y(t_1, t_3, t_4) := a(t_1)b(t_3)c(t_4)$, then

$$
\begin{aligned}
& y_2(t_1^{3\theta}) \cdot y_1(t_1)y_3(t_3)y_4(t_4) \cdot y_5(t_3^{3\theta} + t_1^{3\theta+3})y_6(t_4^{3\theta} + t_1^{6\theta+3}) \\
=& a(t_1)b(t_3)c(t_4) = Y(t_1, t_3, t_4).
\end{aligned}
$$

By calculation, we get the following properties.

4.1.14 Proposition. *Let $i \in \{1, 3, 4\}$ and $t_i, s_i \in \mathbb{F}_q$, then*

$$
\begin{aligned}
& Y(t_1, t_3, t_4) \cdot Y(s_1, s_3, s_4) \\
=& Y(t_1 + s_1,\ t_3 + s_3 - t_1 s_1^{3\theta},\ t_4 + s_4 + t_1 s_1^{3\theta+1} - t_1^2 s_1^{3\theta} - t_3 s_1), \\
& Y(t_1, t_3, t_4)^{-1} = Y(-t_1,\ -t_3 - t_1^{3\theta+1},\ -t_4 + t_1^{3\theta+2} - t_1 t_3).
\end{aligned}
$$

In particular,

$$
\begin{aligned}
a(t_1) \cdot a(s_1) =& Y(t_1 + s_1,\ -t_1 s_1^{3\theta},\ t_1 s_1^{3\theta+1} - t_1^2 s_1^{3\theta}), \\
a(t_1)^{-1} =& Y(-t_1,\ -t_1^{3\theta+1},\ t_1^{3\theta+2}), \\
b(t_3) \cdot b(s_3) =& b(t_3 + s_3), \\
c(t_4) \cdot c(s_4) =& c(t_4 + s_4).
\end{aligned}
$$

4.1.15 Proposition. *Let $i \in \{1, 3, 4\}$ and $t_i, s_i \in \mathbb{F}_q$, then the commutators of ${}^2G_2^{syl}(q)$ are*

$$
\begin{aligned}
& [Y(t_1, t_3, t_4), Y(s_1, s_3, s_4)] \\
=& Y\big(0,\ t_1^{3\theta} s_1 - t_1 s_1^{3\theta},\ (t_1 s_1^{3\theta+1} - t_1^{3\theta+1} s_1) + (t_1^{3\theta} s_1^2 - t_1^2 s_1^{3\theta}) + (t_1 s_3 - t_3 s_1)\big), \\
& [Y(t_1, t_3, t_4)^{-1}, Y(s_1, s_3, s_4)^{-1}] \\
=& Y\big(0,\ t_1^{3\theta} s_1 - t_1 s_1^{3\theta},\ (t_1^2 s_1^{3\theta} - t_1^{3\theta} s_1^2) + (t_1 s_3 - t_3 s_1)\big).
\end{aligned}
$$

In particular,

$$
\begin{aligned}
& [a(t_1), a(s_1)] = b(t_1^{3\theta} s_1 - t_1 s_1^{3\theta}) \cdot c(t_1^{3\theta} s_1^2 - t_1^2 s_1^{3\theta} + t_1 s_1^{3\theta+1} - t_1^{3\theta+1} s_1), \\
& [a(t_1)^{-1}, a(s_1)^{-1}] = b(t_1^{3\theta} s_1 - t_1 s_1^{3\theta}) \cdot c(t_1^2 s_1^{3\theta} - t_1^{3\theta} s_1^2), \\
& [a(t_1), b(s_3)] = c(t_1 s_3), \\
& [a(t_1)^{-1}, b(s_3)^{-1}] = c(t_1 s_3).
\end{aligned}
$$

4.1.16 Proposition. *Let $t_i, s_i \in \mathbb{F}_q$ with $i \in \{1,3,4\}$, then the conjugate of $Y(t_1,t_3,t_4)$ is*

$$\begin{aligned} & Y(s_1,s_3,s_4)\cdot Y(t_1,t_3,t_4)\cdot Y(s_1,s_3,s_4)^{-1} \\ =& Y\big(t_1,\ t_3+t_1s_1^{3\theta}-t_1^{3\theta}s_1,\ t_4+(t_1^2s_1^{3\theta}+t_1^{3\theta}s_1^2)+t_1^{3\theta+1}s_1+(t_3s_1-t_1s_3)\big). \end{aligned}$$

In particular,

$$\begin{aligned} & Y(s_1,s_3,s_4)\cdot a(t_1)\cdot Y(s_1,s_3,s_4)^{-1} \\ =& Y\big(t_1,\ t_1s_1^{3\theta}-t_1^{3\theta}s_1,\ (t_1^2s_1^{3\theta}+t_1^{3\theta}s_1^2)+t_1^{3\theta+1}s_1-t_1s_3\big), \\ & Y(s_1,s_3,s_4)\cdot b(t_3)\cdot Y(s_1,s_3,s_4)^{-1}=Y(0,\ t_3,\ t_3s_1), \\ & Y(s_1,s_3,s_4)\cdot c(t_4)\cdot Y(s_1,s_3,s_4)^{-1}=c(t_4). \end{aligned}$$

4.2. Monomial ${}^2G_2^{syl}\,(3^{2m+1})$-module

Let $N = 8$, $n = 4$, $q = 3^{2m+1}$, $G := A_8(q)$, and $U := {}^2G_2^{syl}\,(3^{2m+1})$.

In this section, we determine a monomial linearisation $(f, \kappa|_{V\times V})$ for $A_8(q)$, where f is a surjective 1-cocycle and $\kappa|_{V\times V}$ is a non-degenerate bilinear form (4.2.16). In particular, $f|_U$ is bijective (4.2.15). We get a monomial $A_8(q)$-module $\mathbb{C}U$ (4.2.18). Then we obtain a classification of U-orbit modules (4.2.26).

4.2.1. Monomial ${}^2G_2^{syl}\,(3^{2m+1})$-module

Let $q = 3^{2m+1}$, $G := A_8(q)$, and $U := {}^2G_2^{syl}\,(3^{2m+1})$. In this subsection, we determine a monomial linearisation $(f, \kappa|_{V\times V})$ for $A_8(q)$, where f is a surjective 1-cocycle and $\kappa|_{V\times V}$ is a non-degenerate bilinear form (4.2.16). The restriction $f|_U$ is bijective (4.2.15). Then we make $\mathbb{C}U$ into a monomial $A_8(q)$-module (4.2.18).

It is essential to choose a suitable vector space V for the 1-cocycle and a suitable non-degenerate bilinear form κ on V_0. We define the following notations for ${}^2G_2^{syl}\,(3^{2m+1})$.

4.2.1 Notation/Lemma. *Let $V_0 := \mathrm{Mat}_{8\times 8}(q)$, then V_0 is an $\mathbb{F}_q$-vector space.*

4.2.2 Notation/Lemma. *The map $\kappa\colon V_0 \times V_0 \to \mathbb{F}_q : (A, B) \mapsto \mathrm{tr}(A^\top B)$ is called the* ***trace form****, and κ is a non-degenerate symmetric bilinear form on V_0.*

4.2.3 Notation/Lemma. *Set $J := \{(1,2), (1,3), (1,4)\}$ and*

$$V := V_J = \left\{ \left(\begin{array}{cccccccc} \cdot & \boxed{A_{12}} & \boxed{A_{13}} & \boxed{A_{14}} & & & & \cdot \\ & \cdot & & & & & \cdot & \\ & & \cdot & & & \cdot & & \\ & & & \cdot & \cdot & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & \cdot & \\ & & & & & & & \cdot \end{array} \right) \;\middle|\; A_{12}, A_{13}, A_{14} \in \mathbb{F}_q \right\},$$

Then V is a 3-dimensional subspace of V_0 over $\mathbb{F}_q$.

4.2.4 Lemma. *Let $V^\perp$ denote the orthogonal complement of V in V_0 with respect to the trace form κ, i.e. $V^\perp := \{B \in V_0 \mid \kappa(A, B) = 0,\ \forall\, A \in V\}$. then $V^\perp = V_{\Box\setminus J}$ and $V_0 = V \oplus V^\perp$.*

4.2.5 Corollary. *$\kappa|_{V\times V}\colon V \times V \to \mathbb{F}_q$ is a non-degenerate bilinear form.*

4.2.6 Notation/Lemma. *Set $\pi := \pi_J$ i.e.*

$$\pi\colon V_0 = V \oplus V^{\perp} \to V : A \mapsto \sum_{(i,j)\in J} A_{i,j}e_{i,j} = A_{12}e_{12} + A_{13}e_{13} + A_{14}e_{14}.$$

Then π is a projection to the first component and is an $\mathbb{F}_q$-linear map.

4.2.7 Lemma. *Suppose $A, B \in V_0$, such that $\operatorname{supp}(A) \cap \operatorname{supp}(B) \subseteq J$. Then*

$$\kappa(A,B) = \kappa(\pi(A),B) = \kappa(A,\pi(B)) = \kappa(\pi(A),\pi(B)) = \kappa|_{V\times V}(\pi(A),\pi(B)).$$

4.2.8 Lemma. *Let $A \in V$ and $g \in G$, then $\pi_{\triangledown}(Ag^{\top}) \in V$.*

4.2.9 Lemma. *Let $A, B \in V$ and $g, h \in G$, then $\operatorname{supp}(Bh^{\top}) \cap \operatorname{supp}(Ag) \subseteq J$.*

4.2.10 Proposition (Group action of G on V)**.** *The map*

$$- \circ - \colon V \times G \to V : (A,g) \mapsto A \circ g := \pi(Ag)$$

is a group action, and the elements of the group G act as $\mathbb{F}_q$-automorphisms.

Proof. With straightforward calculation or c.f. the proof of 3.2.23. □

By 1.2.21, we get a new action:

4.2.11 Corollary. *There exists an unique linear action $-.-$ of G on V:*

$$-.-\colon V \times G \to V : (A,g) \mapsto A.g := A \circ g^{-\top}$$

such that $\kappa|_{V\times V}(A.g, B) = \kappa|_{V\times V}(A, B \circ g^{-1})$ for all $B \in V$. In particular, for all $A, B \in V$ and $g \in G$,

$$\kappa|_{V\times V}(A, B\circ g) = \kappa|_{V\times V}(A.g^{-1}, B) = \kappa|_{V\times V}(A \circ g^{\top}, B).$$

4.2.12 Notation. *Set $f := \pi|_G \colon G \to V$.*

The following lemma is proved by straightforward calculation or c.f. the proof of 3.2.26.

4.2.13 Lemma. *Let $x, g \in G$, then $f(x)g \equiv (x-1)g \mod V^{\perp}$. In particular, $f(x) \equiv x - 1 \mod V^{\perp}$.*

4.2.14 Proposition. *Let $x, g \in G$, then $f(xg) = f(x) \circ g + f(g)$.*

Proof. With straightforward calculation or c.f. the proof of 3.2.27. □

4.2.15 Proposition (Bijective 1-cocycle of ${}^2G_2^{syl}(3^{2m+1})$)**.** *Let $U = {}^2G_2^{syl}(3^{2m+1})$, then $f|_U := \pi|_U \colon U \to V$ is a bijection. In particular, $f|_U$ is a bijective 1-cocycle of U.*

Proof. By 4.1.11 and 4.2.6, $f|_U$ is a bijection. □

4.2.16 Corollary (Monomial linearisation for $A_8(q)$)**.** *$f := \pi|_G : G \to V$ is a surjective 1-cocycle of G in V, and $(f, \kappa|_{V\times V})$ is a monomial linearisation for $G = A_8(q)$.*

4.2.17 Corollary (Monomial linearisation for ${}^2G_2^{syl}\,(3^{2m+1})$)**.** *$(f|_U, \kappa|_{V\times V})$ is a monomial linearisation for U.*

Now we establish the monomial G-module $\mathbb{C}\left({}^2G_2^{syl}\,(3^{2m+1})\right)$, which is essential for the construction of the supercharacter theory for ${}^2G_2^{syl}\,(3^{2m+1})$.

4.2.18 Theorem (Fundamental theorem for ${}^2G_2^{syl}\,(3^{2m+1})$)**.** *Let $G = A_8(q)$, $U = {}^2G_2^{syl}\,(3^{2m+1})$ and*

$$[A] = \frac{1}{|U|}\sum_{u\in U}\overline{\chi_A(u)}u \qquad \textit{for all } A \in V.$$

Then the set $\{[A] \mid A \in V\}$ forms a $\mathbb{C}$-basis for the complex group algebra $\mathbb{C}U$. Let

$$[A] * g := \chi_{A.g}(g)[A.g] = \vartheta\kappa(A.g, f(g))[A.g] \qquad \textit{for all } g \in G, A \in V,$$

then $\mathbb{C}U$ is a monomial $\mathbb{C}G$-module. The restriction of the $*$-operation to U is given by the usual right multiplication of U on $\mathbb{C}U$, i.e.

$$[A] * u = [A]u = \frac{1}{|U|}\sum_{y\in U}\overline{\chi_A(y)}yu \qquad \textit{for all } u \in U, A \in V.$$

Proof. By 4.2.16, $(f, \kappa|_{V\times V})$ is a monomial linearisation for G, satisfying that $f|_U$ is a bijective map (see 4.2.15). By 4.2.11, $A.u := \pi(Au^{-\top})$. Thus the whole theorem is proved in view of 1.2.24. □

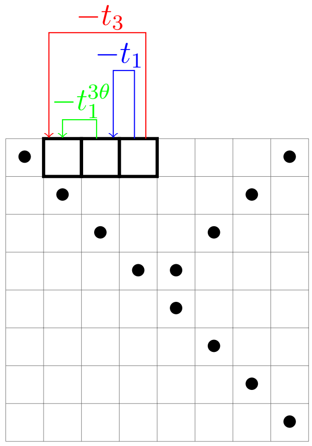
4.2.2. ${}^2G_2^{syl}(3^{2m+1})$-orbit modules

Let $U := {}^2G_2^{syl}(q) = {}^2G_2^{syl}(3^{2m+1})$, $A \in V$, $y_i(t_i) \in U$ and $t_i \in \mathbb{F}_q$ $(i = t_1, t_3, t_4)$.

In this subsection, we get the stabilizers $\mathrm{Stab}_U(A)$ for all $A \in V$ (4.2.22) and obtain a classification of U-orbit modules (4.2.26).

4.2.19 Lemma. *Let $A \in V$, $Y(t_1, t_3, t_4) \in U$ and $t_i \in \mathbb{F}_q$ with $i \in \{1, 3, 4\}$. Then $A.Y(t_1, t_3, t_4)$ and the corresponding figures of moves are given as follows:*

$$\begin{aligned} A.Y(t_1,t_3,t_4) &= A.\big(y_2(t_1^{3\theta})y_1(t_1)y_3(t_3)\big) \\ =& A.\big(x_2(t_1^{3\theta})x_1(t_1)x_3(t_3)\big) = A.\big(\tilde{x}_{23}(t_1^{3\theta})\tilde{x}_{34}(t_1)\tilde{x}_{24}(t_3)\big). \end{aligned}$$



For $A \in V$, we often only draw the entries A_{ij} where $(i, j) \in J$.

4.2.20 Notation. *Set*

$$\begin{aligned} \mathfrak{F}_4 :=& \{\mathbb{C}\mathcal{O}_U(A) \mid A \in V,\ A_{14} \neq 0\}, \\ \mathfrak{F}_3 :=& \{\mathbb{C}\mathcal{O}_U(A) \mid A \in V,\ A_{13} \neq 0,\ A_{14} = 0\}, \\ \mathfrak{F}_1 :=& \{\mathbb{C}\mathcal{O}_U(A) \mid A \in V,\ A_{12} \neq 0,\ A_{13} = A_{14} = 0\}. \end{aligned}$$

Let $A \in V$, we also say $A \in \mathfrak{F}_i$, if $\mathbb{C}\mathcal{O}_U([A]) \in \mathfrak{F}_i$.

4.2.21 Proposition (${}^2G_2^{syl}(q)$-orbit modules)**.** *Let $A = (A_{ij}) \in V$, then the U-orbit module $\mathbb{C}\mathcal{O}_U([A])$ $(A \in V)$ is determined as follows:*

$$\begin{aligned} &\mathbb{C}\mathcal{O}_U([A_{12}e_{12} + A_{13}e_{13} + A_{14}e_{14}]) \\ =&\mathbb{C}\left\{ \left[\begin{array}{c|c|c|c|c} \cdot & \begin{matrix} A_{12} \\ -A_{13}t_1^{3\theta} \\ -A_{14}t_3 \end{matrix} & \begin{matrix} A_{13} \\ -A_{14}t_1 \end{matrix} & A_{14} & \cdot \\ & \cdot & & & \cdot \\ & & \cdot & \cdot & \\ & & \cdot \quad \cdot & & \end{array} \right] \,\middle|\, t_1, t_3 \in \mathbb{F}_q \right\}. \end{aligned}$$

Proof. By 4.2.19, we calculate the orbit modules directly. □

4.2.22 Proposition (${}^2G_2^{syl}(q)$-stabilizer)**.** *Let $A = (A_{ij}) \in V$, then $\mathrm{Stab}_U(A)$ is established in Table 4.2.*

Table 4.2.: ${}^2G_2^{syl}(q)$-stabilizers

	$A \in V$	$\mathrm{Stab}_U(A)$	$\lvert\mathrm{Stab}_U(A)\rvert$	$\dim_{\mathbb{C}} \mathbb{C}\mathcal{O}_U([A])$
$\mathfrak{F}_4$	$\cdot\ \boxed{A_{12}}\boxed{A_{13}}\boxed{A_{14}^*}\ \cdot$ $\cdot\quad\cdot$ $\cdot\quad\cdot$ $\cdot\ \cdot$	$Y(0,0,t_4)$	q	q^2
$\mathfrak{F}_3$	$\cdot\ \boxed{A_{12}}\boxed{A_{13}^*}\boxed{}\ \cdot$ $\cdot\quad\cdot$ $\cdot\quad\cdot$ $\cdot\ \cdot$	$Y(0,t_3,t_4)$	q^2	q
$\mathfrak{F}_1$	$\cdot\ \boxed{A_{12}}\boxed{}\boxed{}\ \cdot$ $\cdot\quad\cdot$ $\cdot\quad\cdot$ $\cdot\ \cdot$	U	q^3	1

Proof. By 4.2.21, we calculate the stabilizers. □

As for the other groups, we get the following three statements for the ${}^2G_2^{syl}(3^{2m+1})$-orbit modules.

4.2.23 Lemma. *Let $A, B \in V$. Then $\mathrm{Hom}_{\mathbb{C}U}(\mathbb{C}\mathcal{O}_U([A]), \mathbb{C}\mathcal{O}_U([B])) = \{0\}$ if and only if for all $C \in \mathcal{O}_U(A)$ and $D \in \mathcal{O}_U(B)$ holds $\mathrm{Hom}_{\mathrm{Stab}_U(C,D)}(\mathbb{C}[C], \mathbb{C}[D]) = \{0\}$.*

4.2.24 Lemma. *Let $A, B \in V$ and $y \in U$. Then*

$$\mathrm{Hom}_{\mathrm{Stab}_U(A,B)}(\mathbb{C}[A], \mathbb{C}[B]) = \mathrm{Hom}_{\mathrm{Stab}_U(A.y,B.y)}(\mathbb{C}[A.y], \mathbb{C}[B.y])$$

as $\mathbb{C}$-vector spaces.

4.2.25 Corollary (Homomorphism criterion)**.** *Let $A, B \in V$. Then*

$$\mathrm{Hom}_{\mathbb{C}U}(\mathbb{C}\mathcal{O}_U([A]), \mathbb{C}\mathcal{O}_U([B])) = \{0\}$$

if and only if $\mathrm{Hom}_{\mathrm{Stab}_U(A,D)}(\mathbb{C}[A], \mathbb{C}[C]) = \{0\}$ for all $D \in \mathcal{O}_U(B)$.

4.2.26 Proposition (Classification of ${}^2G_2^{syl}(q)$-orbit modules)**.** *Every U-orbit module is one of the following forms in Table 4.3, and they satisfy the following properties.*

(1) *Let* $A, B \in V$. *If* $\operatorname{verge}(A) \neq \operatorname{verge}(B)$, *then* $\operatorname{Hom}_{\mathbb{C}U}(\mathbb{C}\mathcal{O}_U([A]), \mathbb{C}\mathcal{O}_U([B])) = \{0\}$. *In particular, if* $\mathbb{C}\mathcal{O}_U([A]) \in \mathfrak{F}_i$, $\mathbb{C}\mathcal{O}_U([B]) \in \mathfrak{F}_j$ *and* $i \neq j$, *then*

$$\operatorname{Hom}_{\mathbb{C}U}(\mathbb{C}\mathcal{O}_U([A]), \mathbb{C}\mathcal{O}_U([B])) = \{0\}.$$

(2) *In the Family* $\mathfrak{F}_1$, *the* U*-orbit modules are irreducible and pairwise orthogonal.*

(3) *In the Family* $\mathfrak{F}_3$ *and* $\mathfrak{F}_4$, *the* U*-orbit modules are reducible.*

Table 4.3.: ${}^2G_2^{syl}(q)$-orbit modules

Family	$\mathbb{C}\mathcal{O}_U([A])$ $(A \in V)$	$\dim_{\mathbb{C}}\mathbb{C}\mathcal{O}_U([A])$	$\#\mathbb{C}\mathcal{O}_U([A])$	Irr.
$\mathfrak{F}_4$	$\mathbb{C}\mathcal{O}_U\left(\begin{bmatrix} \cdot & \boxed{}\boxed{}\boxed{A^*_{14}} & & \cdot \\ & \cdot & \cdot & \\ & \cdot & \cdot & \\ & \cdot & \cdot & \end{bmatrix}\right)$	q^2	$q-1$	NO
$\mathfrak{F}_3$	$\mathbb{C}\mathcal{O}_U\left(\begin{bmatrix} \cdot & \boxed{}\boxed{A^*_{13}}\boxed{} & & \cdot \\ & \cdot & \cdot & \\ & \cdot & \cdot & \\ & \cdot & \cdot & \end{bmatrix}\right)$	q	$q-1$	NO
$\mathfrak{F}_1$	$\mathbb{C}\mathcal{O}_U\left(\begin{bmatrix} \cdot & \boxed{A_{12}}\boxed{}\boxed{} & & \cdot \\ & \cdot & \cdot & \\ & \cdot & \cdot & \\ & \cdot & \cdot & \end{bmatrix}\right)$	1	q	YES

Proof. (a) By 4.2.21, we classify ${}^2G_2^{syl}(q)$-orbit modules.

(b) Using 4.2.25, (1) is proved.

(c) The orbit modules in the Family $\mathfrak{F}_1$ are 1-dimensional, so (2) is obtained.

(d) Let $A = A_{12}e_{12} + A^*_{13}e_{13} \in \mathfrak{F}_3$ and $C \in \mathcal{O}_U(A)$. By 4.2.22, $\operatorname{Stab}_U(A, C) = \operatorname{Stab}_U(A) = \{Y(0, t_3, t_4) \mid t_3, t_4 \in \mathbb{F}_q\}$. The inner product is $\langle \chi_A, \chi_C \rangle_{\operatorname{Stab}_U(A,C)} = 1$. Let ψ_A denote the character of $\mathbb{C}\mathcal{O}_U([A])$, then

$$\begin{aligned} &\dim_{\mathbb{C}}\operatorname{Hom}_{\mathbb{C}U}(\mathbb{C}\mathcal{O}_U([A]), \mathbb{C}\mathcal{O}_U([A])) = \langle \psi_A, \psi_A \rangle_U \\ &= \sum_{C \in \mathcal{O}_U(A)} \frac{|\operatorname{Stab}_U(A, C)|}{|\operatorname{Stab}_U(A)|} \dim_{\mathbb{C}}\operatorname{Hom}_{\operatorname{Stab}_U(A,C)}(\mathbb{C}[A], \mathbb{C}[C]) = q > 1. \end{aligned}$$

Thus, $\mathbb{C}\mathcal{O}_U([A])$ is not irreducible.

(e) If $A \in V$ is a pattern of the Family $\mathfrak{F}_4$, then the orbit module $\mathbb{C}\mathcal{O}_U([A])$ is reducible. Suppose it is irreducible, then $\left(\dim_C \mathbb{C}\mathcal{O}_U([A])\right)^2 = q^4 < |U| = q^3$. This is a contradiction. Thus the orbit modules of the Family $\mathfrak{F}_4$ are reducible. □

4.2.27 Corollary. *Let $A, B \in V$, then two orbit modules $\mathbb{C}\mathcal{O}_U([A])$ and $\mathbb{C}\mathcal{O}_U([B])$ are either $\mathbb{C}\mathcal{O}_U([A]) = \mathbb{C}\mathcal{O}_U([B])$ or $\mathrm{Hom}_{\mathbb{C}U}(\mathbb{C}\mathcal{O}_U([A]), \mathbb{C}\mathcal{O}_U([B])) = \{0\}$.*

4.3. Supercharacter theories for ${}^2G_2^{syl}\left(3^{2m+1}\right)$

In this section, let $q = 3^{2m+1}$, $G := A_8(q)$, $U := {}^2G_2^{syl}\left(3^{2m+1}\right)$, and $t^*, t_1^*, t_3^*, t_4^* \in \mathbb{F}_q^*$.

We determine a supercharacter theory for ${}^2G_2^{syl}\left(3^{2m+1}\right)$ (4.3.14), establish the supercharacter table of ${}^2G_2^{syl}\left(3^{2m+1}\right)$ in Table 4.6. Then we obtain the character table of ${}^2G_2^{syl}(3)$ (4.3.27) and give the relations between supercharacters and irreducible characters of ${}^2G_2^{syl}(3)$ (4.3.28).

4.3.1. Conjugacy classes of ${}^2G_2^{syl}\left(3^{2m+1}\right)$

In this subsection, we determine the conjugacy classes of ${}^2G_2^{syl}\left(3^{2m+1}\right)$ (4.3.4), and obtain one partition of ${}^2G_2^{syl}\left(3^{2m+1}\right)$ (4.3.8) which is a set of the superclasses proved in the next subsetion.

4.3.1 Notation. *Let $x, u \in U$, then the conjugate of x by u is ${}^u x := uxu^{-1}$, and the conjugacy class of u is ${}^U x := \{vxv^{-1} \mid v \in U\}$.*

4.3.2 Lemma. *Let $t \in \mathbb{F}_q^*$, $\theta := 3^m$, $\mathbb{F}_q^+$ be the additive group of $\mathbb{F}_q$ and*

$$\varsigma_t \colon \mathbb{F}_q^+ \to \mathbb{F}_q^+ : s \mapsto ts^{3\theta} - t^{3\theta}s,$$

then ς_t is a homomorphism and $|\operatorname{im}\varsigma_t| = \theta^2 = 3^{2m}$.

Proof. We know that ς_t is a homomorphism. *We claim that* $\ker\varsigma_t = \{0, t, -t\}$. Note that $\ker\varsigma_t \supseteq \{0, t, -t\}$. Let $s \in \ker\varsigma_t$, then $ts^{3\theta} - t^{3\theta}s = 0$. We know $0 \in \ker\varsigma_t \cap \{0, t, -t\}$. Let $s \neq 0$, then

$$(t^{-1}s)^{3\theta-1} = 1 \implies |t^{-1}s| \Big| (3\theta - 1).$$

Since $(t^{-1}s)^{q-1} = (t^{-1}s)^{3\theta^2-1} = 1$, the order $|t^{-1}s|$ divides the greatest common divisor $(3\theta - 1, 3\theta^2 - 1) = (3^{m+1} - 1, 3^{2m+1} - 1)$. If $m = 0$, $(3^{m+1} - 1, 3^{2m+1} - 1) = (2, 2) = 2$. If $m > 0$, $(3^{m+1} - 1, 3^{2m+1} - 1) = 2$ by the Euclidean algorithm. Thus,

$$|t^{-1}s| \Big| 2 \implies (t^{-1}s)^2 = 1 \implies t^{-1}s = \pm 1 \implies s = \pm t.$$

Thus $\ker\varsigma_t \subseteq \{0, t, -t\}$. The claim is proved. Therefore,

$$|\operatorname{im}\varsigma_t| = \frac{|\mathbb{F}_q^+|}{|\ker\varsigma_t|} = \frac{3^{2m+1}}{3} = 3^{2m}.$$

□

4.3.3 Notation. *For $t^* \in \mathbb{F}_q^*$ denote by ${}^{t^*}T$ a complete set of coset representatives (i.e. a transversal) of $\operatorname{im}\varsigma_{t^*}$ in $\mathbb{F}_q^+$. Thus $|{}^{t^*}T| = 3$.*

Table 4.4.: Conjugacy classes of ${}^2G_2^{syl}\left(3^{2m+1}\right)$

Representatives	#Conjugacy Classes	Conjugacy Classes	|Conjugacy Class|
I_8	1	$Y(0,0,0)$	1
$Y(0,0,t_4^*)$	$q-1$	$Y(0,0,t_4^*)$	1
$Y(0,t_3^*,0)$	$q-1$	$Y(0,t_3^*,s_4)$ $s_4 \in \mathbb{F}_q$	q
$Y(t_1^*, {}^{t_1^*}t_3, 0)$	$(q-1)\cdot 3$	$Y(t_1^*, \tilde{s}_3, s_4)$ $\tilde{s}_3 \in {}^{t_1^*}t_3 + \operatorname{im}\varsigma_{t_1^*},\ s_4 \in \mathbb{F}_q$	$q \cdot 3^{2m}$

where ${}^{t_1^*}t_3 \in {}^{t_1^*}T$.

4.3.4 Proposition (Conjugacy classes of ${}^2G_2^{syl}\left(3^{2m+1}\right)$). *Let $U = {}^2G_2^{syl}\left(3^{2m+1}\right)$, then the conjugacy classes of U are listed in Table 4.4.*

Proof. By 4.1.16 and 4.3.2, we get the conjugacy classes of U. □

4.3.5 Corollary.

$$\begin{aligned}&\#\{\textit{Irreducible Characters of } {}^2G_2^{syl}\left(3^{2m+1}\right)\}\\ =&\#\{\textit{Conjugacy Classes of } {}^2G_2^{syl}\left(3^{2m+1}\right)\}\\ =&5q-4 = 5(q-1)+1.\end{aligned}$$

4.3.6 Remark. *We also consider the analogue of Higman's conjecture for ${}^2G_2^{syl}\left(3^{2m+1}\right)$. By 4.3.5, the conjecture is true for ${}^2G_2^{syl}\left(3^{2m+1}\right)$.*

4.3.7 Notation/Lemma. *Set*

$$\begin{aligned} C_4(t_4^*) &:= {}^U Y(0,0,t_4^*), & C_3(t_3^*) &:= {}^U Y(0,t_3^*,0),\\ C_1(t_1^*) &:= \bigcup_{{}^{t_1^*}t_3 \in {}^{t_1^*}T} {}^U Y(t_1^*, {}^{t_1^*}t_3, 0), & C_0 &:= \{I_8\}. \end{aligned}$$

Note that they can form a partition of U, denoted by $\mathcal{K}$, i.e.

$$\mathcal{K} := \{C_0, C_1(t_1^*), C_3(t_3^*), C_4(t_4^*) \mid t_1^*, t_3^*, t_4^* \in \mathbb{F}_q^*\}.$$

We get a partition of ${}^2G_2^{syl}\left(3^{2m+1}\right)$ by 4.3.4.

Table 4.5.: A partition $\mathcal{K}$ of ${}^2G_2^{syl}\left(3^{2m+1}\right)$

$u \in U$	$\#C_i(t_i^*)$or $\#C_0$	$C_i(t_i^*)$ or C_0	$\lvert C_i(t_i^*)\rvert$ or $\lvert C_0)\rvert$
I_8	1	$Y(0,0,0)$	1
$Y(0,0,t_4^*)$	$q-1$	$Y(0,0,t_4^*)$	1
$Y(0,t_3^*,0)$	$q-1$	$Y(0,t_3^*,s_4)$ $s_4 \in \mathbb{F}_q$	q
$Y(t_1^*,0,0)$	$q-1$	$Y(t_1^*,s_3,s_4)$ $s_3,s_4 \in \mathbb{F}_q$	q^2

4.3.8 Proposition (A partition of ${}^2G_2^{syl}\left(3^{2m+1}\right)$)**.** *Let $u \in U$, then the partition $\mathcal{K}$ is shown in Table 4.5:*

4.3.9 Notation/Lemma. *Set $C_i := \dot\bigcup_{t_i \in \mathbb{F}_q^*} C_i(t_i^*)$ for all $i = 1, 3, 4$, and $C_0 := \{I_8\}$. Then*

$$\mathcal{K} = \dot{\bigcup_{i \in \{0,1,3,4\}}} C_i.$$

Note that we can also determine the partition for ${}^2G_2^{syl}(q)$ in 4.3.8 by calculating $\{I_8 + x(u-1)y \mid x, y \in G\} \cap U = \{I_8 + (u-1)y \mid y \in G\} \cap U$ for all $u \in U$ (c.f. §2.3.1).

4.3.2. A supercharacter theory for ${}^2G_2^{syl}\left(3^{2m+1}\right)$

In this subsection, let $A_{12}^*, A_{13}^*, A_{14}^* \in \mathbb{F}_q^*$. We determine the supercharacter theory for ${}^2G_2^{syl}\left(3^{2m+1}\right)$ (4.3.14), establish the supercharacter table of ${}^2G_2^{syl}\left(3^{2m+1}\right)$ in Table 4.6 and list supercharacters, supermodules and superclasses of ${}^2G_2^{syl}\left(3^{2m+1}\right)$ in Table 4.7.

4.3.10 Notation. *Let $A = (A_{ij}) \in V$, set*

$$\begin{aligned} M(0) &:=\{0\}, & M(A_{12}^*e_{12}) &:=\mathbb{C}\mathcal{O}_U([A_{12}^*e_{12}]), \\ M(A_{13}^*e_{13}) &:=\mathbb{C}\mathcal{O}_U([A_{13}^*e_{13}]), & M(A_{14}^*e_{14}) &:=\mathbb{C}\mathcal{O}_U([A_{14}^*e_{14}]). \end{aligned}$$

4.3.11 Notation. *Denote by $\mathcal{M}$ the set of all of the $\mathbb{C}U$-modules in 4.3.10, i.e.*

$$\begin{aligned} \mathcal{M} :=&\{M(0), M(A_{12}^*e_{12}), M(A_{13}^*e_{13}), M(A_{14}^*e_{14}) \mid A_{12}^*, A_{13}^*, A_{14}^* \in \mathbb{F}_q^*\} \\ =&\{\{0\}, \mathbb{C}\mathcal{O}_U([A_{12}^*e_{12}]), \mathbb{C}\mathcal{O}_U([A_{13}^*e_{13}]), \mathbb{C}\mathcal{O}_U([A_{14}^*e_{14}]) \mid A_{12}^*, A_{13}^*, A_{14}^* \in \mathbb{F}_q^*\}. \end{aligned}$$

4.3.12 Notation. *(1) Let $M \in \mathcal{M}$, then the complex character of the $\mathbb{C}U$-module M is denoted by Ψ_M.*

(2) Set $\mathcal{X} := \{\Psi_M \mid M \in \mathcal{M}\}$.

4.3.13 Corollary. *Let $A \in V$ and ψ_A be the character of $\mathbb{C}\mathcal{O}_U([A])$. Thus*

$$\Psi_{M(0)} =\psi_0, \quad \Psi_{M(A_{12}^*e_{12})} = \psi_{A_{12}^*e_{12}}, \quad \Psi_{M(A_{13}^*e_{13})} = \psi_{A_{13}^*e_{13}}, \quad \Psi_{M(A_{14}^*e_{14})} = \psi_{A_{14}^*e_{14}}.$$

4.3.14 Proposition (Supercharacter theory for ${}^2G_2^{syl}\left(3^{2m+1}\right)$)**.** *$(\mathcal{X}, \mathcal{K})$ is a supercharacter theory for ${}^2G_2^{syl}\left(3^{2m+1}\right)$, where $\mathcal{K}$ is defined in 4.3.7, and $\mathcal{X}$ is defined in 4.3.12.*

Proof. By 4.3.7, $\mathcal{K}$ is a partition of U. We know that $\mathcal{X}$ is a set of nonzero complex charaxcters of U.

(a) *Claim that $|\mathcal{X}| = |\mathcal{K}|$.*

By 4.3.7, 4.3.10 and 4.3.12, $|\{\Psi_{M(A_{14}^*e_{14})} \mid A_{14}^* \in \mathbb{F}_q^*\}| = |\{M(A_{14}^*e_{14}) \mid A_{14}^* \in \mathbb{F}_q^*\}| = |C_4|$, Similarly, we obtain the Table 4.7. Thus $|\mathcal{X}| = |\mathcal{K}|$.

(b) *Claim that the characters $\chi \in \mathcal{X}$ are constant on the members of $\mathcal{K}$.*

Let $A_{13}^* \in \mathbb{F}_q^*$ and $y \in U$, then

$$\Psi_{M(A_{13^*}e_{13})}(y) = \sum_{\substack{C \in \mathcal{O}_U(A_{13}^*e_{13}) \\ C.y=C}} \chi_C(y) = \sum_{\substack{C \in \mathcal{O}_U(A_{13}^*e_{13}) \\ y \in \mathrm{Stab}_U(C)}} \chi_C(y).$$

If $y = Y(0, t_3, t_4) \in C_0 \cup C_3 \cup C_4$, $y \in \mathrm{Stab}_U(C)$ for all $C \in \mathcal{O}_U(A_{13}^* e_{13})$ by 4.2.22. Thus

$$\Psi_{M(A_{13}^* e_{13})}(y) = \sum_{C \in \mathcal{O}_U(A_{13}^* e_{13})} \chi_C(y) = \sum_{C \in \mathcal{O}_U(A_{13}^* e_{13})} \vartheta(-A_{13}^* t_3) = q \cdot \vartheta(-A_{13}^* t_3).$$

If $y \in C_1$, $y \notin \mathrm{Stab}_U(C)$ for all $C \in \mathcal{O}_U(A_{13}^* e_{13})$ by 4.2.22. Thus

$$\Psi_{M(A_{13}^* e_{13})}(y) = 0.$$

Similarly, we calculate the other values of the Table 4.6. Thus the claim is proved.

(c) The elements of $\mathcal{X}$ are pairwise orthogonal by 4.2.26.

(d) The set $\{I_8\}$ is a member of $\mathcal{K}$.

By 2.3.11 and 2.3.13, $(\mathcal{X}, \mathcal{K})$ is a supercharacter theory for ${}^2G_2^{syl}\,(3^{2m+1})$. □

Table 4.6.: Supercharacter table of ${}^2G_2^{syl}\,(3^{2m+1})$

	C_0	$C_1(t_1^*)$	$C_3(t_3^*)$	$C_4(t_4^*)$
$\Psi_{M(0)}$	1	1	1	1
$\Psi_{M(A_{12}^* e_{12})}$	1	$\vartheta(A_{12}^* t_1^*)$	1	1
$\Psi_{M(A_{13}^* e_{13})}$	q	0	$q \cdot \vartheta(-A_{13}^* t_3^*)$	q
$\Psi_{M(A_{14}^* e_{14})}$	q^2	0	0	$q^2 \cdot \vartheta(-A_{14}^* t_4^*)$

4.3.15 Corollary. *The supercharacters, supermodules and superclasses of* ${}^2G_2^{syl}\,(3^{2m+1})$ *are listed in Table 4.7.*

4.3.16 Corollary. *The number of the supercharacters* ${}^2G_2^{syl}(q)$ *is*

$$|\mathcal{X}| = |\mathcal{M}| = |\mathcal{K}| = 3q - 2 = 3(q-1) + 1.$$

Table 4.7.: Supercharacters, supermodules and superclasses of ${}^2G_2^{syl}\,(3^{2m+1})$

Supercharacters $\Psi \in \mathcal{X}$	Supermodules $M \in \mathcal{M}$	Superclasses $C \in \mathcal{K}$	$\#\{\Psi\}{=}\#\{M\}{=}\#\{C\}$
$\Psi_{M(A_{14}^* e_{14})}$	$M(A_{14}^* e_{14})$	$C_4(t_4^*)$	$q-1$
$\Psi_{M(A_{13}^* e_{13})}$	$M(A_{13}^* e_{13})$	$C_3(t_3^*)$	$q-1$
$\Psi_{M(A_{12}^* e_{12})}$	$M(A_{12}^* e_{12})$	$C_1(t_1^*)$	$q-1$
$\Psi_{M(0)}$	$M(0)$	C_0	1

4.3.3. Character table of ${}^2G_2^{syl}(3)$

In this subsection, let $q = 3$ and $U := {}^2G_2^{syl}(3)$. We obtain the character table of ${}^2G_2^{syl}(3)$ (4.3.27) and give the relations between supercharacters and irreducible characters of ${}^2G_2^{syl}(3)$ (4.3.28).

4.3.17 Notation. *Set*

$$Y_a := \{a(t_1) \mid t_1 \in \mathbb{F}_q\}, \quad Y_b := \{b(t_3) \mid t_3 \in \mathbb{F}_q\}, \quad Y_c := \{c(t_4) \mid t_4 \in \mathbb{F}_q\}.$$

In this subsection, $m = 0$ and $\theta = 3^m = 1$ since $p = 3$. We rewrite the properties in §4.1.2.

4.3.18 Proposition (Multiplication)**.** *Let $i \in \{1,3,4\}$ and $t_i, s_i \in \mathbb{F}_3$, then*

$$\begin{aligned} &Y(t_1,t_3,t_4)\cdot Y(s_1,s_3,s_4)\\ =&Y(t_1+s_1,\ t_3+s_3-t_1s_1,\ t_4+s_4+t_1s_1^2-t_1^2s_1-t_3s_1),\\ &Y(t_1,t_3,t_4)^{-1} = Y(-t_1,\ -t_3-t_1^2,\ -t_4+t_1-t_1t_3).\end{aligned}$$

In particular,

$$\begin{aligned} a(t_1)\cdot a(s_1) =& Y(t_1+s_1, -t_1s_1, t_1s_1^2-t_1^2s_1),\\ a(t_1)^{-1} =& Y(-t_1, -t_1^2, t_1),\\ b(t_3)\cdot b(s_3) =& b(t_3+s_3),\\ c(t_4)\cdot c(s_4) =& c(t_4+s_4).\end{aligned}$$

4.3.19 Corollary. *Y_b and Y_c are subgroups of ${}^2G_2^{syl}(3)$, but Y_a is not a subgroup of ${}^2G_2^{syl}(3)$.*

4.3.20 Proposition (Commutator relations of ${}^2G_2^{syl}(3)$)**.** *Let $i \in \{1,3,4\}$ and $t_i, s_i \in \mathbb{F}_3$, then the commutators of ${}^2G_2^{syl}(3)$ are*

$$[Y(t_1,t_3,t_4), Y(s_1,s_3,s_4)] = Y\big(0,\ 0,\ (t_1^2 s_1 - t_1 s_1^2) + (t_1 s_3 - t_3 s_1)\big),$$
$$[Y(t_1,t_3,t_4)^{-1}, Y(s_1,s_3,s_4)^{-1}] = Y\big(0,\ 0,\ (t_1^2 s_1 - t_1 s_1^2) + (t_1 s_3 - t_3 s_1)\big).$$

In particular,

$$[a(t_1), a(s_1)] = c(t_1^2 s_1 - t_1 s_1^2), \qquad [a(t_1)^{-1}, a(s_1)^{-1}] = c(t_1^2 s_1 - t_1 s_1^2),$$
$$[a(t_1), b(s_3)] = c(t_1 s_3), \qquad [a(t_1)^{-1}, b(s_3)^{-1}] = c(t_1 s_3).$$

4.3.21 Corollary. *Y_c and Y_bY_c are normal subgroups of ${}^2G_2^{syl}(3)$.*

4.3.22 Corollary. *The center $Z(U) = Y_c$ and $Y_c \backslash U$ is abelian.*

4.3.23 Proposition (Conjugacy classes)**.** *Let $t_i, s_i \in \mathbb{F}_q$ with $i \in \{1,3,4\}$, then the conjugate of $Y(t_1,t_3,t_4)$ is*

$$\begin{aligned} &Y(s_1,s_3,s_4)\cdot Y(t_1,t_3,t_4)\cdot Y(s_1,s_3,s_4)^{-1} \\ =&Y\big(t_1,\ t_3,\ t_4 + (t_1 s_1^2 - t_1^2 s_1) + (t_3 s_1 - t_1 s_3)\big). \end{aligned}$$

In particular,

$$Y(s_1,s_3,s_4)\cdot a(t_1)\cdot Y(s_1,s_3,s_4)^{-1} = Y\big(t_1,\ 0,\ t_1 s_1^2 - t_1^2 s_1 - t_1 s_3\big),$$
$$Y(s_1,s_3,s_4)\cdot b(t_3)\cdot Y(s_1,s_3,s_4)^{-1} = Y(0,\ t_3,\ t_3 s_1),$$
$$Y(s_1,s_3,s_4)\cdot c(t_4)\cdot Y(s_1,s_3,s_4)^{-1} = c(t_4).$$

Let G be a finite group, and $\mathrm{Irr}(G)$ be the set of all irreducible characters of G. Let $\chi \in \mathrm{Irr}(U)$.

4.3.24 Proposition. *Let $A_{ij} \in \mathbb{F}_q$ and $A_{ij}^* \in \mathbb{F}_q^*$.*

(1) Let $\bar{U} := Y_c\backslash U = \bar{Y}_a\bar{Y}_b$, $\bar{\chi}_{lin}^{A_{12},A_{13}} \in \mathrm{Irr}(\bar{U})$, $\bar{\chi}_{lin}^{A_{12},A_{13}}(\bar{a}(t_1)\bar{b}(t_3)) := \vartheta(A_{12}t_1)\cdot\vartheta(-A_{13}t_3)$, and $\chi_{lin}^{A_{12},A_{13}}$ be the lift of $\bar{\chi}_{lin}^{A_{12},A_{13}}$ to U. Then

$$\mathfrak{F}_{lin} := \{\chi \in \mathrm{Irr}(U) \mid Y_c \subseteq \ker\chi\} = \{\chi_{lin}^{A_{12},A_{13}} \mid A_{12}, A_{13} \in \mathbb{F}_q\}.$$

(2) Let $H := Y_bY_c$, $\lambda^{A_{14},A_{13}} \in \mathrm{Irr}(H)$, $\lambda^{A_{14},A_{13}}(b(t_3)c(t_4)) := \vartheta(-A_{14}t_4 - A_{13}t_3)$, $\chi_{2,q}^{A_{14}} := \mathrm{Ind}_H^U\lambda^{A_{14},0}$. Then

$$\mathfrak{F}_2 := \{\chi \in \mathrm{Irr}(U) \mid Y_c \nsubseteq \ker\chi\} = \{\chi_{2,q}^{A_{14}^*} \mid A_{14}^* \in \mathbb{F}_{1q}^*\}.$$

Proof. Let $\chi \in \mathrm{Irr}(U)$.

(1) *Family $\mathfrak{F}_{lin}$, where $Y_c \subseteq \ker\chi$.*

Since the commutator subgroup is $U' = Y_c$, all linear characters of U are precisely the lifts to U of the irreducible characters of the abelian quotient group $Y_c\backslash U$. If $Y_c \subseteq \ker\chi$, χ is linear. Let $\bar{U} := Y_aY_b\backslash U = \bar{Y}_a\bar{Y}_b$, $\bar{\chi}_{lin}^{A_{12},A_{13}} \in \mathrm{Irr}(\bar{U})$ with $\bar{\chi}_{lin}^{A_{12},A_{13}}(\bar{a}(t_1)\bar{b}(t_3)) := \vartheta(A_{12}t_1)\cdot\vartheta(-A_{13}t_3)$. Since $\bar{U}$ is abelian, $\mathrm{Irr}(\bar{U}) = \{\bar{\chi}_{lin}^{A_{12},A_{13}} \mid A_{12}, A_{13} \in \mathbb{F}_q\}$. Let $\chi_{lin}^{A_{12},A_{13}}$ be the lift of $\bar{\chi}_{lin}^{A_{12},A_{13}}$ to U, then

$$\mathfrak{F}_{lin} = \{\chi \in \mathrm{Irr}(U) \mid Y_c \subseteq \ker\chi\} = \{\chi_{lin}^{A_{12},A_{13}} \mid A_{12}, A_{13} \in \mathbb{F}_q\}.$$

(2) *Family $\mathfrak{F}_2$, where $Y_c \nsubseteq \ker\chi$.*

Let $H := Y_bY_c$, $\lambda^{A_{14},A_{13}} \in \mathrm{Irr}(H)$ with $\lambda^{A_{14},A_{13}}(b(t_3)c(t_4)) := \vartheta(-A_{14}t_4 - A_{13}t_3)$. We note that Y_a is a transversal of H in U, and $Z(U) = Y_c$. For all $s_1 \in \mathbb{F}_q$,

$$\begin{aligned}&\big(\lambda^{A_{14}^*,A_{13}}\big)^{a(s_1)}(b(t_3)c(t_4)) = \lambda^{A_{14}^*,A_{13}}\big(a(s_1)\cdot b(t_3)c(t_4)\cdot a(s_1)^{-1}\big)\\ =&\lambda^{A_{14}^*,A_{13}}\big(b(t_3)c(t_4+s_1t_3)\big) = \vartheta\big(-A_{14}^*(t_4+s_1t_3) - A_{13}t_3\big)\\ =&\vartheta\big(-A_{14}^*t_4 - (A_{13}+s_1A_{14}^*)t_3\big).\end{aligned}$$

Thus $I_U(\lambda^{A_{14}^*,A_{13}}) = H$. By Clifford's Theorem, $\mathrm{Ind}_H^U\lambda^{A_{14}^*,A_{13}} \in \mathrm{Irr}(U)$ and

$$\mathrm{Res}_H^U\mathrm{Ind}_H^U\lambda^{A_{14}^*,A_{13}} = \sum_{s_1\in\mathbb{F}_q}\big(\lambda^{A_{14}^*,A_{13}}\big)^{a(s_1)} = \sum_{s_1\in\mathbb{F}_q}\lambda^{A_{14}^*,(A_{13}-s_1A_{14}^*)} = \sum_{B_{13}\in\mathbb{F}_q}\lambda^{A_{14}^*,B_{13}}.$$

By Clifford's Theorem, there are $q-1$ almost faithful irreducible characters of $\bar{U}$, i.e.

$$\mathfrak{F}_3 = \{\chi \in \mathrm{Irr}(U) \mid Y_c \nsubseteq \ker\chi\} = \{\chi_{2,q}^{A_{13}^*} \mid A_{13}^* \in \mathbb{F}_q^*\}.$$

$\square$

By the above argument, we obtain all irredubible characters of ${}^2G_2^{syl}(3)$.

4.3.25 Theorem (Irreducible characters of ${}^2G_2^{syl}(3)$)**.** *The irreducible characters of $G_2^{syl}(3)$ are classified into two families as listed in Table 4.8*

By 4.3.23, we obtain the conjugacy classes of ${}^2G_2^{syl}(3)$.

4.3.26 Proposition (Conjugacy classes of ${}^2G_2^{syl}(3)$)**.** *The conjugacy classes of ${}^2G_2^{syl}(3)$ are listed in Table 4.9.*

4.3.27 Proposition. *The character table of ${}^2G_2^{syl}(3)$ is the one in Table 4.10.*

Table 4.8.: Irreducible characters of ${}^2G_2^{syl}(3)$

Family	Notation	Parameter Set	Number	Degree	Property
$\mathfrak{F}_2$	$\chi_{2,q}^{A_{14}^*}$	$\mathbb{F}_q^*$	$q-1$	q	$Y_c \nsubseteq \ker \chi_{2,q}^{A_{14}^*}$
$\mathfrak{F}_{lin}$	$\chi_{lin}^{A_{12},A_{13}}$	$\mathbb{F}_q \times \mathbb{F}_q$	q^2	1	$Y_c \subseteq \ker \chi_{lin}^{A_{12},A_{13}}$

Table 4.9.: Conjugacy classes of ${}^2G_2^{syl}(3)$

Representatives	#Conjugacy Classes	Conjugacy Classes	|Conjugacy Class|
I_8	1	$Y(0,0,0)$	1
$Y(0,0,t_4^*)$	$q-1$	$Y(0,0,t_4^*)$	1
$Y(0,t_3^*,0)$	$q-1$	$Y(0,t_3^*,s_4)$ $s_4 \in \mathbb{F}_q$	q
$Y(t_1^*,t_3,0)$	$(q-1)\cdot q$	$Y(t_1^*,t_3,s_4)$ $s_4 \in \mathbb{F}_q$	q

Table 4.10.: Character table of ${}^2G_2^{syl}(3)$

	I_8	$Y(t_1^*,t_3,0)$	$Y(0,t_3^*,0)$	$Y(0,0,t_4^*)$
$\chi_{lin}^{0,0}$	1	1	1	1
$\chi_{lin}^{A_{12}^*,0}$	1	$\vartheta(A_{12}^*t_1^*)$	1	1
$\chi_{lin}^{A_{12},A_{13}^*}$	1	$\vartheta(A_{12}t_1^*)\cdot\vartheta(-A_{13}^*t_3)$	$\vartheta(-A_{13}^*t_3^*)$	1
$\chi_{2,q}^{A_{14}^*}$	q	0	0	$q\cdot\vartheta(-A_{14}^*t_4^*)$

Proof. Let $u = Y(t_1, t_3, t_4) \in U$, $H := Y_bY_c$, $\lambda^{A_{14},A_{13}} \in \mathrm{Irr}(H)$ with

$$\lambda^{A_{14},A_{13}}(b(t_3)c(t_4)) := \vartheta(-A_{14}t_4 - A_{13}t_3).$$

We have

$$\chi_{2,q}^{A_{14}^*}(u) = \mathrm{Ind}_H^U \lambda^{A_{14}^*,0}(u) = \frac{1}{|H|} \sum_{\substack{g\in U \\ g\cdot Y(t_1,t_3,t_4)\cdot g^{-1}\in H}} \lambda^{A_{14}^*,0}\big(g\cdot Y(t_1,t_3,t_4)\cdot g^{-1}\big),$$

then

$$\begin{aligned}
\chi_{2,q}^{A_{14}^*}(c(t_4)) =& q\cdot \lambda^{A_{14}^*,0}(c(t_4)) = q\cdot\vartheta(-A_{14}^*t_4),\\
\chi_{2,q}^{A_{14}^*}(b(t_3^*)) =& \frac{1}{|H|} \sum_{\substack{g:=Y(s_1,s_3,s_4)\in U \\ g\cdot b(t_3^*)\cdot g^{-1}\in H}} \lambda^{A_{14}^*,0}\big(g\cdot b(t_3^*)\cdot g^{-1}\big) = \sum_{s_1\in\mathbb{F}_q} \lambda^{A_{14}^*,0}\big(Y(0,t_3^*,t_3^*s_1)\big)\\
=& \sum_{s_1\in\mathbb{F}_q} \vartheta(-A_{1,4}^*t_3^*s_1) = 0,\\
\chi_{2,q}^{A_{14}^*}(Y(t_1^*,t_3)) =& 0.
\end{aligned}$$

We obtain other values similarly. □

4.3.28 Proposition (Supercharacters and irreducible characters)**.** *The following relations between supercharacters and irreducible characters of* ${}^2G_2^{syl}(3)$ *are obtained.*

$$\begin{aligned}
\Psi_{M(A_{14}^*e_{14})} =& q\cdot\chi_{2,q}^{A_{14}^*}, \qquad & \Psi_{M(A_{13}^*e_{13})} =& \sum_{A_{23}\in\mathbb{F}_q} \chi_{lin}^{A_{23},A_{13}^*},\\
\Psi_{M(A_{12}^*e_{12})} =& \chi_{lin}^{A_{12}^*,0}, & \Psi_{M(0)} =& \chi_{lin}^{0,0} = \mathrm{triv}_U.
\end{aligned}$$

Proof. Compare Table 4.6 and Table 4.10, the formulae are obtained. □

4.3.29 Corollary. *Let* $\#\mathrm{Irr}$ *be the number of irreducible characters of* ${}^2G_2^{syl}(3)$*,* $\#\mathrm{Irr}_c$ *be the number of irreducible characters of* ${}^2G_2^{syl}(3)$ *of dimension* q^c *with* $c\in\mathbb{N}$ *and* $q = 3$*, then*

$$\begin{aligned}
\#\mathrm{Irr}_1 &= q-1 &&= (q-1),\\
\#\mathrm{Irr}_0 &= q^2 &&= (q-1)^2 + 2(q-1) + 1,
\end{aligned}$$

and

$$\begin{aligned}
\#\mathrm{Irr} =& \#\{\textit{Irreducible Characters of } {}^2G_2^{syl}(3)\}\\
=& \#\{\textit{Conjugacy Classes of } {}^2G_2^{syl}(3)\}\\
=& q^2 + q - 1\\
=& (q-1)^2 + 3(q-1) + 1.
\end{aligned}$$

We consider the analogue of Higman's conjecture, Lehrer's conjecture and Isaacs' conjecture of $U_n(q)$ *for* ${}^2G_2^{syl}(3)$*, and the conjectures hold for* ${}^2G_2^{syl}(3)$*.*

5. Monomial $E_6^{syl}(q)$-, $F_4^{syl}(q)$-, ${}^2E_6^{syl}(q^2)$- and $E_7^{syl}(q)$-modules

In this chapter, we determine monomial linearisations for the Sylow p-subgroups $E_6^{syl}(q)$, $F_4^{syl}(q)$, ${}^2E_6^{syl}(q^2)$ and $E_7^{syl}(q)$, and then obtain monomial $E_6^{syl}(q)$-, $F_4^{syl}(q)$-, ${}^2E_6^{syl}(q^2)$- and $E_7^{syl}(q)$-modules.

5.1. Monomial $E_6^{syl}(q)$-module

In this section, we construct a Chevalley basis for the Lie algebra $\mathcal{L}_{E_6}$ of type E_6 (see 5.1.3) and determine the Sylow p-subgroup $E_6^{syl}(q)$ for the Chevalley group $E_6(q)$ (see 5.1.25). Then we get a monomial linearisation $(f, \kappa|_{V\times V})$ for $E_6^{syl}(q)$, where f is a bijective 1-cocycle and $\kappa|_{V\times V}$ is a non-degenerate bilinear form (5.1.45). Finally, we make $\mathbb{C}E_6^{syl}(q)$ into a monomial $E_6^{syl}(q)$-module (5.1.46).

5.1.1. Chevalley basis of the Lie algebra of type E_6

In this subsection, we obtain a Chevalley basis for the Lie algebra $\mathcal{L}_{E_6}$ of type E_6 (5.1.3). By [HRT01, §3.3], there is a Lie algebra $\mathcal{L}_{E_6}$ of type E_6 generated by $\{e_r \mid \pm r \in \Delta_{E_6}\}$, denoted by

$$\mathcal{L}_{E_6} := \langle\, e_r \in \mathrm{Mat}_{27\times 27}(\mathbb{C}) \mid \pm r \in \Delta_{E_6} \,\rangle_{\text{Lie}}\,,$$

where $\Delta_{E_6} = \{a, b, c, d, e, f\}$, $e_{-r} = e_r^\top$ for all $r \in \Delta_{E_6}$ and

$$\begin{aligned}
e_a &:= e_{1,2} + e_{11,13} + e_{14,16} + e_{17,18} + e_{19,20} + e_{21,22},\\
e_b &:= e_{4,5} + e_{6,7} + e_{8,10} + e_{19,21} + e_{20,22} + e_{23,24},\\
e_c &:= e_{2,3} + e_{9,11} + e_{12,14} + e_{15,17} + e_{20,23} + e_{22,24},\\
e_d &:= e_{3,4} + e_{7,9} + e_{10,12} + e_{17,19} + e_{18,20} + e_{24,25},\\
e_e &:= e_{4,6} + e_{5,7} + e_{12,15} + e_{14,17} + e_{16,18} + e_{25,26},\\
e_f &:= e_{6,8} + e_{7,10} + e_{9,12} + e_{11,14} + e_{13,16} + e_{26,27}.
\end{aligned}$$

Let $\mathcal{H}_{E_6}$ be the Cartan subalgebra, then

$$\mathcal{H}_{E_6} = \langle\, [e_r, e_{-r}] \in \mathrm{Mat}_{27\times 27}(\mathbb{C}) \mid r \in \Delta_{E_6} \,\rangle_{\mathbb{C}}\,.$$

Let $\mathcal{H}_{E_6}^*$ be the dual space of $\mathcal{H}_{E_6}$, we obtain a basis $\{\varepsilon_1+\varepsilon_2+\varepsilon_8, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7\}$ of $\mathcal{H}_{E_6}^*$ such that

$$a = -\frac{1}{2}\sum_{i=1}^{8}\varepsilon_i,\ b = \varepsilon_6-\varepsilon_7,\ c = \varepsilon_6+\varepsilon_7,\ d = \varepsilon_5-\varepsilon_6,\ e = \varepsilon_4-\varepsilon_5,\ f = \varepsilon_3-\varepsilon_4.$$

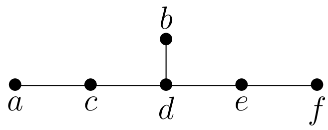
Let $\mathcal{V}_{E_6} := \mathbb{R}\text{-span}\{\varepsilon_1+\varepsilon_2+\varepsilon_8, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7\}$ be a subspace of $\mathcal{H}_{E_6}^*$, then $\mathcal{V}_{E_6}$ becomes a Euclidean space (see Appendix A.3). The root system of type E_6 is

$$\begin{aligned}\Phi_{E_6} =& \{\pm\varepsilon_i \pm \varepsilon_j \mid i<j,\ i,j=3,4,5,6,7\}\\ & \bigcup \left\{ \pm\frac{1}{2}\sum_{i=1}^{8}\epsilon_1\varepsilon_i \,\middle|\, \epsilon_i = \pm 1,\ \epsilon_1=\epsilon_2=\epsilon_8=1,\ \prod_{i=1}^{8}\epsilon_i = 1\right\}.\end{aligned}$$

The set of positive roots of Φ_{E_6} is

$$\begin{aligned}\Phi_{E_6}^+ =& \{\varepsilon_i \pm \varepsilon_j \mid i<j,\ i,j=3,4,5,6,7\}\\ & \bigcup \left\{ -\frac{1}{2}\sum_{i=1}^{8}\epsilon_1\varepsilon_i \,\middle|\, \epsilon_i = \pm 1,\ \epsilon_1=\epsilon_2=\epsilon_8=1,\ \prod_{i=1}^{8}\epsilon_i = 1\right\}.\end{aligned}$$

The Dynkin diagram of E_6 is



Let $r_1 := a,\ r_2 := b,\ r_3 := c,\ r_4 := d,\ r_5 := e,\ r_6 := f$, then the Cartan matrix M of E_6 is

$$\begin{array}{c c}
 & \begin{array}{cccccc} a & b & c & d & e & f \end{array}\\
\begin{array}{c} a\\ b\\ c\\ d\\ e\\ f \end{array} &
\begin{pmatrix}
2 & 0 & -1 & 0 & 0 & 0\\
0 & 2 & 0 & -1 & 0 & 0\\
-1 & 0 & 2 & -1 & 0 & 0\\
0 & -1 & -1 & 2 & -1 & 0\\
0 & 0 & 0 & -1 & 2 & -1\\
0 & 0 & 0 & 0 & -1 & 2
\end{pmatrix}_{6\times 6}
\end{array}$$

where $M_{r_i,r_j} = M_{i,j} = \langle r_i, r_j\rangle = \frac{2\kappa(r_i,r_j)}{\kappa(r_i,r_i)}$ $(r_i, r_j \in \Delta_{E_6})$.

Let Φ_{E_6} be the root system of type E_6. Each root $r \in \Phi_{E_6}$ can be written as $r = \sum_{\alpha\in\Delta_{E_6}} k_\alpha \alpha$, the height of r is $\mathrm{ht}(r) = \sum_{\alpha\in\Delta_{E_6}} k_\alpha$.

5.1.1 Definition. *For $r, s \in \Phi_{E_6}^+$, we write $r \prec s$, if $\mathrm{ht}(r) < \mathrm{ht}(s)$ (e.g. $a \prec a+c$) or if $\mathrm{ht}(r) = \mathrm{ht}(s)$, this is the lexicographical order (e.g. $a \prec b \prec c \prec d \prec e \prec f$).*

We take the structure constants $N_{r,s}$ to be positive, if (r,s) is an extraspecial pair according to the total order in Definition 5.1.1. We give all the extraspecial pairs and the corresponding structure constants as follows.

5.1.2 Notation/Lemma. *We have the following extraspecial pairs of roots in $\Phi_{E_6}^+ \backslash \Delta_{E_6}$ and the corresponding structure constants $N_{r,s}$ $(r, s \in \Phi_{E_6}^+)$. According to column 2 and column 3 of the following table, we define r_i $(i = 7, 8, \ldots, 36)$ to be the root in column 3 which is according to i in column 2 (e.g. $r_7 = a + c$).*

Height	i	$r_i \in \Phi_{E_6}^+ \backslash \Delta_{E_6}$	*Extraspecial pairs* (r,s)	$N_{r,s}$
2	7	$a+c$	(a,c)	1
	8	$b+d$	(b,d)	1
	9	$c+d$	(c,d)	1
	10	$d+e$	(d,e)	1
	11	$e+f$	(e,f)	1
3	12	$a+c+d$	$(a,c+d)$	1
	13	$b+c+d$	$(b,c+d)$	1
	14	$b+d+e$	$(b,d+e)$	1
	15	$c+d+e$	$(c,d+e)$	1
	16	$d+e+f$	$(d,e+f)$	1
4	17	$a+b+c+d$	$(a,b+c+d)$	1
	18	$a+c+d+e$	$(a,c+d+e)$	1
	19	$b+c+d+e$	$(b,c+d+e)$	1
	20	$b+d+e+f$	$(b,d+e+f)$	1
	21	$c+d+e+f$	$(c,d+e+f)$	1
5	22	$a+b+c+d+e$	$(a,b+c+d+e)$	1
	23	$a+c+d+e+f$	$(a,c+d+e+f)$	1
	24	$b+c+2d+e$	$(d,b+c+d+e)$	1
	25	$b+c+d+e+f$	$(b,c+d+e+f)$	1
6	26	$a+b+c+2d+e$	$(a,b+c+2d+e)$	1
	27	$a+b+c+d+e+f$	$(a,b+c+d+e+f)$	1
	28	$b+c+2d+e+f$	$(d,b+c+d+e+f)$	1
7	29	$a+b+2c+2d+e$	$(c,a+b+c+2d+e)$	1
	30	$a+b+c+2d+e+f$	$(a,b+c+2d+e+f)$	1
	31	$b+c+2d+2e+f$	$(e,b+c+2d+e+f)$	1
8	32	$a+b+2c+2d+e+f$	$(c,a+b+c+2d+e+f)$	1
	33	$a+b+c+2d+2e+f$	$(a,b+c+2d+2e+f)$	1
9	34	$a+b+2c+2d+2e+f$	$(c,a+b+c+2d+2e+f)$	1
10	35	$a+b+2c+3d+2e+f$	$(d,a+b+2c+2d+2e+f)$	1
11	36	$a+2b+2c+3d+2e+f$	$(b,a+b+2c+3d+2e+f)$	1

Then the structure constants N_{r_i,r_j} of $\mathcal{L}_{E_6}$ are obtained:

The structure constants $N(r_i, r_j)$ of the Lie algebra $\mathcal{L}_{E_6}$

$N(r_i,r_j)$	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	r_9	r_{10}	r_{11}	r_{12}	r_{13}	r_{14}	r_{15}	r_{16}
r_1	·		1						1				1		1	
r_2		·		1					1	1		1			1	1
r_3	−1		·	1				1		1				1		1
r_4		−1	−1	·	1		−1				1					
r_5				−1	·	1		−1	−1			−1	−1			
r_6					−1	·				−1				−1	−1	
r_7				1			·	1		1				1		1
r_8			−1		1		−1	·			1				−1	
r_9	−1	−1			1				·		1			−1		
r_{10}		−1	−1			1	−1			·			−1			
r_{11}				−1				−1	−1		·	−1	−1			
r_{12}		−1			1						1	·		−1		
r_{13}	−1				1					1	1		·			1
r_{14}			−1			1	−1		1			1		·		
r_{15}	−1	−1				1		1							·	
r_{16}		−1	−1				−1						−1			·
r_{17}					1					1	1				1	1
r_{18}		−1				1		1					1			
r_{19}	−1			−1		1						1				−1
r_{20}			−1				−1		1			1			−1	
r_{21}	−1	−1						1						−1		
r_{22}				−1		1			−1							−1
r_{23}		−1						1					1	−1		
r_{24}	−1					1	1				1					
r_{25}	−1			−1						1		1				
r_{26}			−1			1					1					
r_{27}				−1					−1	1					1	
r_{28}	−1				−1		1									
r_{29}						1					1					1
r_{30}			−1		−1										1	
r_{31}	−1						1					−1				
r_{32}					−1					−1				−1		
r_{33}			−1						1				1			
r_{34}				−1				−1								
r_{35}		−1														
r_{36}																

$N(r_i,r_j)$	r_{17}	r_{18}	r_{19}	r_{20}	r_{21}	r_{22}	r_{23}	r_{24}	r_{25}	r_{26}	r_{27}	r_{28}	r_{29}	r_{30}	r_{31}	r_{32}	r_{33}	r_{34}	r_{35}	r_{36}
r_1			1		1			1	1			1			1					
r_2		1			1		1												1	
r_3				1						1				1			1			
r_4			1			1			1		1							1		
r_5	−1											1		1		1				
r_6		−1	−1			−1		−1		−1			−1							
r_7				1				−1				−1			−1					
r_8		−1			−1		−1											1		
r_9				−1		1					1						−1			
r_{10}	−1								−1		−1					1				
r_{11}	−1							−1		−1			−1							
r_{12}			−1	−1					−1						1					
r_{13}		−1					−1										−1			
r_{14}					1		1									1				
r_{15}	−1			1							−1			−1						
r_{16}	−1		1			1							−1							
r_{17}	·				1										1					
r_{18}		·		1					1			1								
r_{19}			·				1							−1						
r_{20}		−1		·									−1							
r_{21}	−1				·	1				1										
r_{22}					−1	·						1								
r_{23}			−1				·	−1												
r_{24}							1	·			1									
r_{25}		−1							·	1										
r_{26}					−1				−1	·										
r_{27}								−1			·									
r_{28}		−1				−1						·								
r_{29}				1									·							
r_{30}			1											·						
r_{31}	−1														·					
r_{32}																·				
r_{33}																	·			
r_{34}																		·		
r_{35}																			·	
r_{36}																				·

where the dots and empty entries are zeros.

By 5.1.2, we determine a Chevalley basis of the Lie algebra $\mathcal{L}_{E_6}$.

5.1.3 Lemma (Chevalley basis of $\mathcal{L}_{E_6}$). *For $r \in \Delta_{E_6}$, set $h_r := [e_r, e_{-r}]$. Let $r \in \Phi_{E_6}^+ \backslash \Delta_{E_6}$, and (s_1, s_2) the extraspecial pair of r, set*

$$e_r := \frac{1}{N_{s_1,s_2}}[e_{s_1}, e_{s_2}], \quad e_{-r} := \frac{1}{N_{-s_1,-s_2}}[e_{-s_1}, e_{-s_2}].$$

Then the set

$$\{h_r \in \mathrm{Mat}_{27\times 27}(\mathbb{C}) \mid r \in \Delta_{E_6}\} \cup \{e_{\pm r} \in \mathrm{Mat}_{27\times 27}(\mathbb{C}) \mid r \in \Phi_{E_6}^+\}$$

is a Chevalley basis of the Lie algebra $\mathcal{L}_{E_6}$.

Proof. We know that the Lie algebra $\mathcal{L}_{E_6}$ of type E_6 is generated by

$$\{e_{\pm r} \mid r \in \Delta_{E_6}\}.$$

The fundamental co-roots h_r for all $r \in \Delta_{E_6}$ are determined

$$h_r = [e_r, e_{-r}] = [e_r, e_r^\top] \quad \text{for all } r \in \Delta_{E_6},$$

since $[[e_r, e_{-r}], e_r] = 2e_r$.

We claim that $e_{-r} = e_r^\top$ for all $r \in \Phi_{E_6}^+$. Let $r \in \Phi_{E_6}^+$, we use the induction on $\mathrm{ht}(r)$. If $\mathrm{ht}(r) = 1$, $r \in \Delta_{E_6}$. Thus $e_{-r} = e_r^\top$. Suppose that $e_{-r} = e_r^\top$ if $\mathrm{ht}(r) \leq k$. If $\mathrm{ht}(r) = k+1$, there exist $s_1, s_2 \in \Phi_{E_6}^+$ such that $r = s_1 + s_2$ and $\mathrm{ht}(s_1) < \mathrm{ht}(r)$, $\mathrm{ht}(s_2) < \mathrm{ht}(r)$. Thus $[e_{-s_1}, e_{-s_2}] = N_{-s_1,-s_2}e_{-r} = -N_{s_1,s_2}e_{-r}$ and

$$[e_{-s_1}, e_{-s_2}] = [e_{s_1}^\top, e_{s_2}^\top] = -[e_{s_1}, e_{s_2}]^\top = -N_{s_1,s_2}e_r^\top.$$

Then $e_{-r} = e_r^\top$ when $\mathrm{ht}(r) = k+1$. Therefore $e_{-r} = e_r^\top$ $(r \in \Phi_{E_6}^+)$.

We obtain that $[h_r, e_r] = 2e_r$ and $h_r = [e_r, e_{-r}]$ for all $r \in \Phi_{E_6}^+$.

Let θ be the map of $\mathcal{L}_{E_6}$ into itself: $\theta(x) = -x^\top$, where $x^\top$ is the transpose of x. Since

$$\theta([x,y]) = -[x,y]^\top = -(xy-yx)^\top = -(y^\top x^\top - x^\top y^\top) = [x^\top, y^\top] = [\theta(x), \theta(y)],$$

θ is an automorphism of $\mathcal{L}_{E_6}$. We also have $\theta(e_r) = -e_r^\top = -e_{-r}$ for all $r \in \Phi_{E_6}$. Then $N_{r,s} = \pm(n_{r,s}+1)$ $(r, s \in \Phi_{E_6})$ are obained. Thus the fundamental co-roots h_r together with e_r, e_{-r} (r positive) form a Chevalley basis. $\square$

5.1.4 Corollary. *All e_r $(r \in \Phi^+_{E_6})$ are given as follows:*

Height	i	e_{r_i} $(r_i \in \Phi^+_{E_6})$
1	1	$e_{1,2} + e_{11,13} + e_{14,16} + e_{17,18} + e_{19,20} + e_{21,22}$
	2	$e_{4,5} + e_{6,7} + e_{8,10} + e_{19,21} + e_{20,22} + e_{23,24}$
	3	$e_{2,3} + e_{9,11} + e_{12,14} + e_{15,17} + e_{20,23} + e_{22,24}$
	4	$e_{3,4} + e_{7,9} + e_{10,12} + e_{17,19} + e_{18,20} + e_{24,25}$
	5	$e_{4,6} + e_{5,7} + e_{12,15} + e_{14,17} + e_{16,18} + e_{25,26}$
	6	$e_{6,8} + e_{7,10} + e_{9,12} + e_{11,14} + e_{13,16} + e_{26,27}$
2	7	$e_{1,3} - e_{9,13} - e_{12,16} - e_{15,18} + e_{19,23} + e_{21,24}$
	8	$-e_{3,5} + e_{6,9} + e_{8,12} - e_{17,21} - e_{18,22} + e_{23,25}$
	9	$e_{2,4} - e_{7,11} - e_{10,14} + e_{15,19} - e_{18,23} + e_{22,25}$
	10	$e_{3,6} - e_{5,9} + e_{10,15} - e_{14,19} - e_{16,20} + e_{24,26}$
	11	$e_{4,8} + e_{5,10} - e_{9,15} - e_{11,17} - e_{13,18} + e_{25,27}$
3	12	$e_{1,4} + e_{7,13} + e_{10,16} - e_{15,20} - e_{17,23} + e_{21,25}$
	13	$-e_{2,5} - e_{6,11} - e_{8,14} - e_{15,21} + e_{18,24} + e_{20,25}$
	14	$-e_{3,7} - e_{4,9} + e_{8,15} + e_{14,21} + e_{16,22} + e_{23,26}$
	15	$e_{2,6} + e_{5,11} - e_{10,17} - e_{12,19} + e_{16,23} + e_{22,26}$
	16	$e_{3,8} - e_{5,12} - e_{7,15} + e_{11,19} + e_{13,20} + e_{24,27}$
4	17	$-e_{1,5} + e_{6,13} + e_{8,16} + e_{15,22} + e_{17,24} + e_{19,25}$
	18	$e_{1,6} - e_{5,13} + e_{10,18} + e_{12,20} + e_{14,23} + e_{21,26}$
	19	$-e_{2,7} + e_{4,11} - e_{8,17} + e_{12,21} - e_{16,24} + e_{20,26}$
	20	$-e_{3,10} - e_{4,12} - e_{6,15} - e_{11,21} - e_{13,22} + e_{23,27}$
	21	$e_{2,8} + e_{5,14} + e_{7,17} + e_{9,19} - e_{13,23} + e_{22,27}$
5	22	$-e_{1,7} - e_{4,13} + e_{8,18} - e_{12,22} - e_{14,24} + e_{19,26}$
	23	$e_{1,8} - e_{5,16} - e_{7,18} - e_{9,20} - e_{11,23} + e_{21,27}$
	24	$e_{2,9} + e_{3,11} + e_{8,19} + e_{10,21} + e_{16,25} + e_{18,26}$
	25	$-e_{2,10} + e_{4,14} + e_{6,17} - e_{9,21} + e_{13,24} + e_{20,27}$
6	26	$e_{1,9} - e_{3,13} - e_{8,20} - e_{10,22} + e_{14,25} + e_{17,26}$
	27	$-e_{1,10} - e_{4,16} - e_{6,18} + e_{9,22} + e_{11,24} + e_{19,27}$
	28	$e_{2,12} + e_{3,14} - e_{6,19} - e_{7,21} - e_{13,25} + e_{18,27}$
7	29	$-e_{1,11} - e_{2,13} + e_{8,23} + e_{10,24} + e_{12,25} + e_{15,26}$
	30	$e_{1,12} - e_{3,16} + e_{6,20} + e_{7,22} - e_{11,25} + e_{17,27}$
	31	$-e_{2,15} - e_{3,17} - e_{4,19} - e_{5,21} + e_{13,26} + e_{16,27}$
8	32	$-e_{1,14} - e_{2,16} - e_{6,23} - e_{7,24} - e_{9,25} + e_{15,27}$
	33	$-e_{1,15} + e_{3,18} + e_{4,20} + e_{5,22} + e_{11,26} + e_{14,27}$
9	34	$e_{1,17} + e_{2,18} - e_{4,23} - e_{5,24} + e_{9,26} + e_{12,27}$
10	35	$-e_{1,19} - e_{2,20} - e_{3,23} + e_{5,25} + e_{7,26} + e_{10,27}$
11	36	$e_{1,21} + e_{2,22} + e_{3,24} + e_{4,25} + e_{6,26} + e_{8,27}$

5.1.2. Sylow p-subgroup $E_6^{syl}(q)$

Let K be a field. In this subsection, we obtain the Sylow p-subgroup $E_6^{syl}(q)$ for the Chevalley group $E_6(q)$ (see 5.1.25).

5.1.5 Notation. *Let* $\square := \{(i,j) \mid 1 \le i, j \le 27\}$ *and* $\nabla := \{(i,j) \mid 1 \le i < j \le 27\}$.

5.1.6 Reminder. $\mathcal{L}_{E_6}$ *is the Lie algebra of type* E_6 *over the field* $\mathbb{C}$.

5.1.7 Corollary. *For all* $r \in \Phi_{E_6}$, $e_r^2 = 0$.

Proof. By 5.1.3 and calculation, we get the result. $\square$

Similar to §11.3 of [Car72], the coefficients of $\exp(te_r) = I_{27} + te_r$ are all of the form ± 1 or $\pm t$ (i.e. the non-zero entries of $\exp(te_r)$ are of the form ± 1 and $\pm t$). This fact enables us to transfer to an arbitrary field. For each matrix e_r in the above representation and each element t in an arbitrary field K, $\exp(te_r)$ is a well-defined non-singular matrix over K.

5.1.8 Notation. *Set*

$$\begin{aligned} \bar{x}_{r,K}(t) &:= \exp(te_r) = I_{27} + t \cdot e_r, \quad r \in \Phi_{E_6},\ t \in K, \\ x_{r,K}(t) &:= \exp(t \operatorname{ad} e_r), \quad r \in \Phi_{E_6},\ t \in K. \end{aligned}$$

5.1.9 Notation. *Set a matrix group* $\bar{E}_6(K) := \langle\, \bar{x}_{r,K}(t) \mid r \in \Phi_{E_6},\ t \in K \,\rangle$*, and the Chevalley group of type* $\mathcal{L}_{E_6}$ *over the field* K *is*

$$E_6(K) := \langle\, x_{r,K}(t) \mid r \in \Phi_{E_6},\ t \in K \,\rangle\,.$$

5.1.10 Notation. *Set a subgroup of* $\bar{E}_6(K)$

$$\bar{X}_{r,K} := \langle\, \bar{x}_{r,K}(t) \mid t \in K \,\rangle \qquad \textit{for all } r \in \Phi_{E_6},$$

and root subgroups of the Chevalley group $E_6(K)$ *are*

$$X_{r,K} := \langle\, x_{r,K}(t) \mid t \in K \,\rangle \qquad \textit{for all } r \in \Phi_{E_6}.$$

By [Car72, 4.5.1], $(\exp(t \operatorname{ad} e_r))(x) = \exp(te_r) \cdot x \cdot \exp(te_r)^{-1}$ for all $x \in \mathcal{L}_{E_6}$. Then we get a group epimorphism between the matrix group $\bar{E}_6(K)$ and the Chevalley group $E_6(K)$ of type $\mathcal{L}_{E_6}$ over the field K as follows.

5.1.11 Proposition. *Let* K *be a field, and*

$$\sigma\colon \bar{E}_6(K) \to E_6(K) : \exp(te_r) \mapsto \exp(t \operatorname{ad} e_r),$$

then σ *is a group epimorphism, and the kernel* $\ker\sigma$ *is the centre* $\bar{Z}$ *of* $\bar{E}_6(K)$.

5.1.12 Notation/Lemma. *Set*

$$\begin{aligned}\bar{U}_K &:= \left\langle \bar{x}_{r,K}(t) \,\middle|\, r \in \Phi^+_{E_6},\ t \in K \right\rangle \leqslant \bar{E}_6(K),\\ U_K &:= \left\langle x_{r,K}(t) \,\middle|\, r \in \Phi^+_{E_6},\ t \in K \right\rangle \leqslant E_6(K).\end{aligned}$$

5.1.13 Corollary. *Let K be a field, and*

$$\sigma_{\bar{U}_K} : \bar{U}_K \to U_K : \exp(te_r) \mapsto \exp(t \operatorname{ad} e_r),$$

then σ is a group epimorphism.

By calculation, we get the following commutators.

5.1.14 Proposition (Commutators of $\bar{U}_K$)**.** *Let $r_i, r_j \in \Phi^+_{E_6}$, then*

$$[\bar{y}_{r_j,K}(t_j), \bar{y}_{r_i,K}(t_i)] = \prod_{\substack{k,l>0\\ kr_i+lr_j \in \Phi^+_{E_6}}} \bar{x}_{kr_i+lr_j,K}(C_{klr_ir_j}(-t_i)^k t_j^l).$$

Actually, if $kr_i + lr_j \in \Phi^+_{E_6}$ with $k, l > 0$, we have $kr_i + lr_j = r_i + r_j$ and

$$[\bar{y}_{r_j,K}(t_j), \bar{y}_{r_i,K}(t_i)] = \bar{x}_{r_i+r_j,K}(-C_{11r_ir_j} t_i t_j) = \bar{x}_{r_i+r_j,K}(N_{r_j,r_i} t_i t_j).$$

5.1.15 Lemma. *Let $r, s \in \Phi^+_{E_6}$ and $r \neq s$, then*

$$\overline{\operatorname{supp}}(\bar{X}_{r,K}) \cap \overline{\operatorname{supp}}(\bar{X}_{s,K}) = \overline{\operatorname{supp}}(e_r) \cap \overline{\operatorname{supp}}(e_s) = \emptyset.$$

Proof. By 1.1.8, we have $\overline{\operatorname{supp}}(\bar{X}_{r,K}) \cap \overline{\operatorname{supp}}(\bar{X}_{s,K}) = \overline{\operatorname{supp}}(e_r) \cap \overline{\operatorname{supp}}(e_s)$. By 5.1.3, we get $\overline{\operatorname{supp}}(e_r) \cap \overline{\operatorname{supp}}(e_s) = \emptyset$. □

In the remainder of this section, we consider $K = \mathbb{F}_q$. The following property is well known.

5.1.16 Lemma. *For $(i, j) \in \triangledown$ and $t \in \mathbb{F}_q$, let $\tilde{x}_{i,j}(t) := I_{27} + te_{i,j}$. Then*

$$A_{27}(\mathbb{F}_q) = \Big\{ \prod_{(i,j)\in\triangledown} \tilde{x}_{i,j}(t_{i,j}) \,\Big|\, t_{i,j} \in \mathbb{F}_q \Big\},$$

where the product can be taken in an arbitrary, but fixed, order. In particular,

$$\prod_{(i,j)\in\triangledown} \tilde{x}_{i,j}(t_{i,j}) = \prod_{(i,j)\in\triangledown} \tilde{x}_{i,j}(s_{i,j}) \Longleftrightarrow t_{i,j} = s_{i,j} \text{ for all } (i,j) \in \triangledown.$$

5.1.17 Proposition. *Each element of $\bar{U}_{\mathbb{F}_q}$ is uniquely expressible in the form*

$$\prod_{r_i \in \Phi^+_{E_6}} \bar{x}_{r_i,\mathbb{F}_q}(t_i),$$

where the product is taken over all positive roots in the increasing order.

Proof. Each element of $\bar{U}_{\mathbb{F}_q}$ is a product of elements $\bar{x}_{r,\mathbb{F}_q}(t)$ with $r \in \Phi^+_{E_6}, t \in \mathbb{F}_q$. If there is a pair of consecutive terms $\bar{x}_{r_j,\mathbb{F}_q}(t_j)\bar{x}_{r_i,\mathbb{F}_q}(t_i)$ with $r_i \prec r_j$, by 5.1.14 we have

$$\bar{x}_{r_j,\mathbb{F}_q}(t_j)\bar{x}_{r_i,\mathbb{F}_q}(t_i) = \bar{x}_{r_i,\mathbb{F}_q}(t_i)\bar{x}_{r_j,\mathbb{F}_q}(t_j) \prod_{\substack{k,l>0 \\ kr_i+lr_j\in\Phi^+_{E_6}}} \bar{x}_{kr_i+lr_j,\mathbb{F}_q}(C_{klr_ir_j}(-t_i)^k t_j^l).$$

Since $\mathrm{ht}(kr_i + lr_j) \geq \mathrm{ht}(r_i) + \mathrm{ht}(r_j)$, then after a finite number of steps we get the desired product.

For the uniqueness, suppose $\prod_{r_i\in\Phi^+_{E_6}} \bar{x}_{r_i,\mathbb{F}_q}(t_{r_i}) = \prod_{r_i\in\Phi^+_{E_6}} \bar{x}_{r_i,\mathbb{F}_q}(s_{r_i})$. Then

$$\bar{x}_{r_i,\mathbb{F}_q}(t_{r_i}) = \prod_{k=1}^{6}(I_{27} + \epsilon_{r_i,k}t_{r_i}e_{i_k,j_k}),$$

where $\epsilon_{r_i,k} \in \{1, -1\}$. The Lemmas 5.1.15 and 5.1.16 imply $t_{r_i} = s_{r_i}$ $(r_i \in \Phi^+_{E_6})$. $\square$

5.1.18 Corollary. $|\bar{U}_{\mathbb{F}_q}| = q^{|\Phi^+_{E_6}|} = q^{36}$.

5.1.19 Corollary. $\bar{U}_{\mathbb{F}_q} \cong U_{\mathbb{F}_q}$.

Proof. By 5.1.13, there is an epimorphism $\sigma_{\bar{U}_{\mathbb{F}_q}} : \bar{U}_{\mathbb{F}_q} \to U_{\mathbb{F}_q} : \bar{x}_{r,\mathbb{F}_q}(t_i) \mapsto x_{r,\mathbb{F}_q}(t_i)$. We know that $|\bar{U}_{\mathbb{F}_q}| \overset{5.1.18}{=} q^{36} \overset{\text{[Car72, §8.6]}}{=} |\bar{U}_{\mathbb{F}_q}|$, so $\sigma_{\bar{U}_{\mathbb{F}_q}}$ is an isomorphism. Thus $\bar{U}_{\mathbb{F}_q} \cong U_{\mathbb{F}_q}$. $\square$

5.1.20 Notation/Lemma (Root subgroups of $\bar{E}_6(q)$). *Let*

$$\begin{aligned} x_r(t) &:= \exp(te_r) = I_{27} + t \cdot e_r = \bar{x}_{r,\mathbb{F}_q}(t), \quad r \in \Phi_{E_6},\ t \in \mathbb{F}_q, \\ x_i(t) &:= \exp(te_{r_i}) = I_{27} + t \cdot e_{r_i} = x_{r_i}(t), \quad i = 1, 2, \ldots, 36,\ t \in \mathbb{F}_q. \end{aligned}$$

Then the root subgroups of $\bar{E}_6(q)$ are given as

$$\begin{aligned} X_r &:= \{x_r(t) \mid t \in \mathbb{F}_q\} = \bar{X}_{r,\mathbb{F}_q}, \quad r \in \Phi_{E_6}, \\ X_i &:= \{x_{r_i}(t) \mid t \in \mathbb{F}_q\} = X_{r_i}, \quad i = 1, 2, \ldots, 36. \end{aligned}$$

5.1.21 Notation. *Set* $U := \left\langle x_r(t) \,\middle|\, r \in \Phi^+_{E_6},\ t \in \mathbb{F}_q \right\rangle = \bar{U}_{\mathbb{F}_q}$.

5.1.22 Lemma ([Car65]). *The order of $E_6(q)$ is*

$$|E_6(q)| = \frac{1}{(3, q-1)} q^{36}(q^2 - 1)(q^5 - 1)(q^6 - 1)(q^8 - 1)(q^9 - 1)(q^{12} - 1),$$

where $(3, q-1)$ is the greatest common divisor of 3 and $q-1$.

5.1.23 Corollary. $|U| = q^{36}$ *and U is a Sylow p-subgroup of the Chevalley group $E_6(q)$ under isomorphism.*

Proof. By 5.1.18, $\bar{U}_{\mathbb{F}_q}$ is a Sylow p-subgroup of $E_6(q)$. By 5.1.19, $|U| = |\bar{U}_{\mathbb{F}_q}| = q^{36}$ and U is a Sylow p-subgroup of $E_6(q)$ under isomorphism. □

5.1.24 Proposition.

$$U = \Big\{ \prod_{r\in\Phi^+_{E_6}} x_r(t_r) \,\Big|\, t_r \in \mathbb{F}_q \Big\},$$

where the product can be taken in an arbitrary, but fixed, order.

Proof. We know $U \overset{\text{definition}}{\supseteq} \big\{ \prod_{r\in\Phi^+_{E_6}} x_r(t_r) \,\big|\, t_r \in \mathbb{F}_q \big\}$, where the product is taken in an arbitrary, but fixed, order. Suppose $\prod_{r\in\Phi^+_{E_6}} x_r(t_r) = \prod_{r\in\Phi^+_{E_6}} x_r(s_r)$. We have

$$x_r(t_r) = \prod_{k=1}^{6} (I_{27} + \epsilon_{r,k} t_r e_{i_k,j_k}),$$

where $\epsilon_{r,k} = \pm 1$. By 5.1.15 and 5.1.16, we get $t_r = s_r$ $(r \in \Phi^+_{E_6})$. Thus

$$\Big| \Big\{ \prod_{r\in\Phi^+_{E_6}} x_r(t_r) \,\Big|\, t_r \in \mathbb{F}_q \Big\} \Big| = q^{36} = |U|.$$

Therefore, the desired formula is obtained. □

By 5.1.23 and 5.1.24, we get the following Sylow p-subgroup of $E_6(q)$.

5.1.25 Proposition (Sylow p-subgroup $E_6^{syl}(q)$)**.** *Let $t_i \in \mathbb{F}_q$ $(i = 1, 2, \ldots, 36)$ and*

$$E_6^{syl}(q) := U.$$

Then $|E_6^{syl}(q)| = q^{36}$, $E_6^{syl}(q)$ is a Sylow p-subgroup of the Chevalley group $E_6(q)$ under isomorphism and the matrix elements of the Sylow p-subgroup $E_6^{syl}(q)$ are

$$\begin{aligned}
&x_6(t_6)x_{11}(t_{11})x_5(t_5)x_2(t_2) \cdot x_{20}(t_{20})x_{16}(t_{16})x_{14}(t_{14})x_{10}(t_{10})x_8(t_8)x_4(t_4)\\
&\cdot x_{31}(t_{31})x_{28}(t_{28})x_{25}(t_{25})x_{24}(t_{24})x_{21}(t_{21})x_{19}(t_{19})x_{15}(t_{15})x_{13}(t_{13})x_9(t_9)x_3(t_3)\\
&\cdot x_{36}(t_{36})x_{35}(t_{35})x_{34}(t_{34})x_{33}(t_{33})x_{32}(t_{32})x_{30}(t_{30})x_{29}(t_{29})x_{27}(t_{27})x_{26}(t_{26})x_{23}(t_{23})\\
&\cdot x_{22}(t_{22})x_{18}(t_{18})x_{17}(t_{17})x_{12}(t_{12})x_7(t_7)x_1(t_1)
\end{aligned}$$

$$= \left(\begin{array}{*{27}{c}}
1 & t_1 & t_7 & t_{12} & -t_{17} & t_{18} & -t_{22} & t_{23} & t_{26} & -t_{27} & -t_{29} & t_{30} & \begin{array}{c}-t_1t_{29}\\-t_7t_{26}\\-t_{12}t_{22}\\+t_{17}t_{18}\end{array} & -t_{32} & -t_{33} & \begin{array}{c}-t_1t_{32}\\-t_7t_{30}\\-t_{12}t_{27}\\+t_{17}t_{23}\end{array} & t_{34} & \begin{array}{c}t_1t_{34}\\+t_7t_{33}\\-t_{18}t_{27}\\+t_{22}t_{23}\end{array} & -t_{35} & \begin{array}{c}-t_1t_{35}\\+t_{12}t_{33}\\+t_{18}t_{30}\\-t_{23}t_{26}\end{array} & t_{36} & * & * & * & * & * & *\\
0 & 1 & t_3 & t_9 & -t_{13} & t_{15} & -t_{19} & t_{21} & t_{24} & -t_{25} & \begin{array}{c}t_3t_{24}\\+t_9t_{19}\\-t_{13}t_{15}\end{array} & t_{28} & \begin{array}{c}t_1t_3t_{24}\\+t_1t_9t_{19}\\-t_1t_{13}t_{15}\\-t_3t_{26}\\-t_7t_{24}\\-t_9t_{22}\\-t_{12}t_{19}\\+t_{13}t_{18}\\+t_{15}t_{17}\\-t_{29}\end{array} & \begin{array}{c}t_3t_{28}\\+t_9t_{25}\\-t_{13}t_{21}\end{array} & -t_{31} & * & * & * & * & * & * & * & * & * & * & * & *\\
0 & 0 & 1 & t_4 & -t_8 & t_{10} & -t_{14} & t_{16} & \begin{array}{c}-t_4t_{14}\\+t_8t_{10}\end{array} & -t_{20} & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & *\\
0 & 0 & 0 & 1 & t_2 & t_5 & t_2t_5 & t_{11} & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & *\\
0 & 0 & 0 & 0 & 1 & 0 & t_5 & 0 & \begin{array}{c}t_4t_5\\-t_{10}\end{array} & t_{11} & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & *\\
0 & 0 & 0 & 0 & 0 & 1 & t_2 & t_6 & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & *\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & *\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & *\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & *\\
\vdots & & & & & & & & & & & \ddots & \ddots & \ddots & & & & & & & & & & & & & \vdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & *\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & *\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)_{27\times 27}$$

5.1.26 Corollary (36 variables)**.** *The matrix elements of Sylow* p*-subgroup* E_6^{syl} *are written as follows:*

$$\begin{array}{cc}
 & \begin{array}{*{27}{c}} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ \vdots \\ 25 \\ 26 \\ 27 \end{array} &
\left(\begin{array}{*{27}{c}}
1 & t_1 & t_7 & t_{12} & -t_{17} & t_{18} & -t_{22} & t_{23} & t_{26} & -t_{27} & -t_{29} & t_{30} & * & -t_{32} & -t_{33} & * & t_{34} & * & -t_{35} & * & t_{36} & * & * & * & * & * & * \\
0 & 1 & t_3 & t_9 & -t_{13} & t_{15} & -t_{19} & t_{21} & t_{24} & -t_{25} & * & t_{28} & * & * & -t_{31} & * & * & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 1 & t_4 & -t_8 & t_{10} & -t_{14} & t_{16} & * & -t_{20} & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 1 & t_2 & t_5 & * & t_{11} & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 1 & 0 & * & 0 & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 1 & * & t_6 & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
\vdots & & & & & & & & & & & \ddots & \ddots & \ddots & & & & & & & & & & & & & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & t_6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)
\end{array}$$

where $t_i \in \mathbb{F}_q$ $(i = 1, 2, \ldots, 36)$.

5.1.27 Corollary. *Let $u = (u_{i,j}) \in E_6^{syl}(q)$, then*

$$\begin{cases} u_{13,j_{13}} = 0 & \textit{if } j_{13} \in \{14, 15, 17, 19, 21\} \\ u_{16,j_{16}} = 0 & \textit{if } j_{16} \in \{17, 19, 21\} \\ u_{18,j_{18}} = 0 & \textit{if } j_{18} \in \{19, 21\} \\ u_{20,21} = 0 & \\ u_{11,j_{11}} = 0 & \textit{if } j_{11} \in \{12, 15\} \\ u_{14,15} = 0 & \\ u_{9,10} = 0 & \\ u_{7,8} = 0 & \end{cases} .$$

5.1.3. Monomial $E_6^{syl}(q)$-module

Let $U := E_6^{syl}(q)$ and $V_0 := \mathrm{Mat}_{27\times 27}(\mathbb{F}_q)$. In this subsection, we get a monomial linearisation $(f, \kappa|_{V\times V})$ for $E_6^{syl}(q)$, where f is a bijective 1-cocycle and $\kappa|_{V\times V}$ is a non-degenerate bilinear form (5.1.45). Then we make $\mathbb{C}E_6^{syl}(q)$ into a monomial $E_6^{syl}(q)$-module (5.1.46).

It is essential to choose a suitable vector space V for the 1-cocycle and a suitable non-degenerate bilinear form κ on V_0.

5.1.28 Notation/Lemma. *The map*

$$\kappa\colon V_0 \times V_0 \to \mathbb{F}_q : (X, Y) \mapsto \mathrm{tr}(X^\top Y)$$

is called the ***trace form****, and κ is a non-degenerate symmetric bilinear form on V_0.*

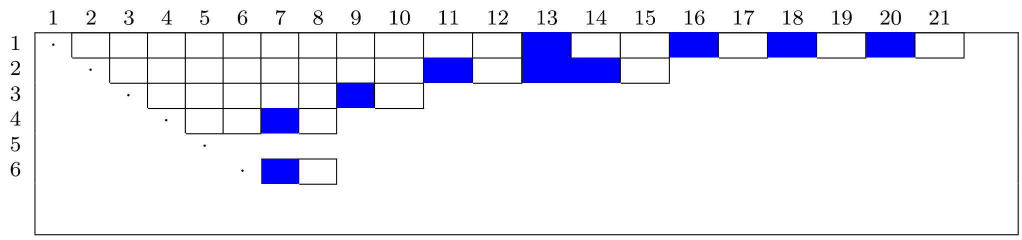
5.1.29 Notation. *Set*

$$J := \left\{ (i,j) \in \triangledown \;\middle|\; \begin{cases} j \in \{2,3,\dots,11,12,14,15,17,19,21\} & \text{if } i = 1 \\ j \in \{3,4,5,6,7,8,9,10,12,15\} & \text{if } i = 2 \\ j \in \{4,5,6,7,8,10\} & \text{if } i = 3 \\ j \in \{5,6,8\} & \text{if } i = 4 \\ j = 8 & \text{if } i = 6 \end{cases} \right\},$$

and

$$J' := \left\{ (i,j) \in \triangledown \;\middle|\; \begin{cases} j \in \{13,16,18,20\} & \text{if } i = 1 \\ j \in \{11,13,14\} & \text{if } i = 2 \\ j = 9 & \text{if } i = 3 \\ j = 7 & \text{if } i = 4 \\ j = 7 & \text{if } i = 6 \end{cases} \right\}.$$

Then J is the set of the coordinates of the empty blocks of the following 27×27-matrix, while J' is the set of the coordinates of the blue blocks of the following 27×27-matrix.



5.1.30 Reminder. *Let $J \subseteq \triangledown$, then by 1.1.9 the subset J^c of $\square$ is defined as follows:*

$$\begin{aligned} J^c := & J \dot\cup \{(i,i+1) \in \triangledown \backslash J \mid (i,i+2) \in J\} \\ & \dot\cup \{(i,j) \in \triangledown \backslash J \mid \exists\, j_1 < j < j_2 \text{ and } (i,j_1),(i,j_2) \in J\}. \end{aligned}$$

5.1.31 Remark. *(1)* $J^c = J \dot\cup J'$,

(2) $\#J = 36$, $\#J' = 10$ *and* $\#J^c = 46$.

5.1.32 Notation/Lemma. *Set*

$$V := V_J = \Big\{ \sum_{(i,j)\in J} A_{i,j}e_{i,j} \in V_0 \,\Big|\, A_{i,j} \in \mathbb{F}_q \Big\},$$

where

$$\sum_{(i,j)\in J} A_{i,j}e_{i,j} =$$

$$\left(\begin{array}{ccccccccccccccccccccc}
\cdot & A_{1,2} & A_{1,3} & A_{1,4} & A_{1,5} & A_{1,6} & A_{1,7} & A_{1,8} & A_{1,9} & A_{1,10} & A_{1,11} & A_{1,12} & 0 & A_{1,14} & A_{1,15} & 0 & A_{1,17} & 0 & A_{1,19} & 0 & A_{1,21} \\ \cline{2-21}
 & \cdot & A_{2,3} & A_{2,4} & A_{2,5} & A_{2,6} & A_{2,7} & A_{2,8} & A_{2,9} & A_{2,10} & 0 & A_{2,12} & 0 & 0 & A_{2,15} & & & & & & \\ \cline{3-15}
 & & \cdot & A_{3,4} & A_{3,5} & A_{3,6} & A_{3,7} & A_{3,8} & 0 & A_{3,10} & & & & & & & & & & & \\ \cline{4-10}
 & & & \cdot & A_{4,5} & A_{4,6} & 0 & A_{4,8} & & & & & & & & & & & & & \\ \cline{5-8}
 & & & & \cdot & 0 & 0 & 0 & & & & & & & & & & & & & \\ \cline{7-8}
 & & & & & \cdot & 0 & A_{6,8} & & & & & & & & & & & & & \\ \cline{7-8}
 & & & & & & & & & & & & & & & & & & & &
\end{array}\right)$$

Then V is a 36-dimensional subspace of V_0 over $\mathbb{F}_q$, and $\mathrm{supp}(V) = J$.

5.1.33 Lemma. *Let $V^\perp$ denote the orthogonal complement of the subspace V of V_0 with respect to the trace form κ, i.e. $V^\perp := \{B \in V_0 \mid \kappa(A,B) = 0,\ \forall\ A \in V\}$. Then $V^\perp = V_{\square\setminus J}$ and $V_0 = V \oplus V^\perp$.*

5.1.34 Notation/Lemma. *Let*

$$\pi\colon V_0 = V \oplus V^\perp \to V : A \mapsto \sum_{(i,j)\in J} A_{i,j}e_{i,j}.$$

Then π is a projection to the first component and is an $\mathbb{F}_q$-linear map.

5.1.35 Corollary. *$\kappa|_{V\times V}\colon V \times V \to \mathbb{F}_q$ is a non-degenerate bilinear form.*

5.1.36 Lemma. *Suppose $A, B \in V_0$, such that* $\mathrm{supp}(A) \cap \mathrm{supp}(B) \subseteq J$*. Then*

$$\kappa(A,B) = \kappa(\pi(A),B) = \kappa(A,\pi(B)) = \kappa(\pi(A),\pi(B)) = \kappa|_{V\times V}(\pi(A),\pi(B)).$$

5.1.37 Lemma. *Let $A \in V$ and $g \in U$, then* $\mathrm{supp}(Ag^\top) \cap \triangledown \subseteq J$*. In particular $\pi_{J^c}(Ag^\top) \in V$.*

Proof. Let $A \in V$, $g \in U$. It is sufficient to prove $(Ag^\top)_{i,j} = 0$, and $(i,j) \in J^c \backslash J = J'$. We have

$$(Ag^\top)_{1,13} = \sum_{j=1}^{27} A_{1,j} g^\top_{j,13} = \sum_{j=2}^{21} A_{1,j} g^\top_{j,13} = \sum_{j=2}^{21} A_{1,j} g_{13,j} = \sum_{j=13}^{21} A_{1,j} g_{13,j}$$
$$\overset{\substack{A_{1,j}=0\\ j\in\{13,16,18,20\}}}{=} \sum_{j\in\{14,15,17,19,21\}} A_{1,j} g_{13,j} \overset{5.1.27}{=} 0,$$

Similarly, we get $(Ag^\top)_{i,j} = 0$ for all $(i,j) \in J'$. Thus $\operatorname{supp}(Ag^\top) \cap \triangledown \subseteq J$ and $\pi_{J^c}(Ag^\top) = \pi_J(Ag^\top) \in V$. □

5.1.38 Corollary. *Let $A, B \in V$ and $g, h \in U$, then* $\operatorname{supp}(Bh^\top) \cap \operatorname{supp}(Ag) \subseteq J$.

5.1.39 Proposition (Group action of $E_6^{syl}(q)$ on V)**.** *The map*

$$- \circ -\colon V \times U \to V : (A, g) \mapsto A \circ g := \pi(Ag)$$

is a group action, and the elements of the group U act as $\mathbb{F}_q$-automorphisms.

Proof. c.f. the proof of 2.2.25. □

By 1.2.21, we get a new action:

5.1.40 Corollary. *There exists an unique linear action $-.-$ of G on V:*

$$-.-\colon V \times G \to V : (A, g) \mapsto A.g := A \circ g^{-\top}$$

such that $\kappa|_{V\times V}(A.g, B) = \kappa|_{V\times V}(A, B \circ g^{-1})$ for all $B \in V$. In particular, for all $A, B \in V$ and $g \in G$,

$$\kappa|_{V\times V}(A, B \circ g) = \kappa|_{V\times V}(A.g^{-1}, B) = \kappa|_{V\times V}(A \circ g^\top, B).$$

5.1.41 Notation. $f := \pi|_U\colon U \to V$.

5.1.42 Proposition. *$f\colon U \to V$ is bijective.*

Proof. By 5.1.25 and 5.1.32, we obtain that $f\colon U \to V$ is bijective. □

5.1.43 Lemma. *Let $x, g \in U$, then $f(x)g \equiv (x-1)g \mod V^\perp$. In particular, $f(x) \equiv x - 1 \mod V^\perp$.*

Proof. c.f. the proof of 2.2.29. □

5.1.44 Proposition. *Let $x, g \in U$, then $f(xg) = f(x) \circ g + f(g)$.*

Proof. c.f. the proof of 2.2.30 or by 5.1.27 as follows: for all $j_0 = 1, 2, \ldots, 12$,

$$\begin{aligned}
&\big(f(x) \circ g + f(g)\big)_{1,j_0} = \big(f(x) \circ g\big)_{1,j_0} + f(g)_{1,j_0} = \big(f(x)g\big)_{1,j_0} + f(g)_{1,j_0} \\
= &\sum_{k=2}^{j_0} f(x)_{1,k} g_{k,j_0} + g_{1,j_0} = \sum_{k=2}^{j_0} x_{1,k} g_{k,j_0} + g_{1,j_0} = \sum_{k=1}^{j_0} x_{1,k} g_{k,j_0} = (xg)_{1,j_0} = f(xg)_{1,j_0},
\end{aligned}$$

and

$$\begin{aligned}
&\big(f(x) \circ g + f(g)\big)_{1,14} = \big(f(x) \circ g\big)_{1,14} + f(g)_{1,14} = \big(f(x)g\big)_{1,14} + f(g)_{1,14} \\
= &\sum_{k=2}^{14} f(x)_{1,k} g_{k,14} + g_{1,14} = \sum_{k=2}^{12} f(x)_{1,k} g_{k,14} + f(x)_{1,13} g_{13,14} + f(x)_{1,14} g_{14,14} + g_{1,14} \\
= &\sum_{k=2}^{12} x_{1,k} g_{k,14} + 0 \cdot g_{13,14} + x_{1,14} g_{14,14} + g_{1,14} \\
= &\sum_{k=1}^{14} x_{1,k} g_{k,14} - x_{1,13} g_{13,14} \overset{5.1.27}{=} \sum_{k=1}^{14} x_{1,k} g_{k,14} = (xg)_{1,14} = f(xg)_{1,14}.
\end{aligned}$$

Similarly, $f(xg)_{i,j} = \big(f(x) \circ g + f(g)\big)_{i,j}$ for all $(i,j) \in J$. Hence $f(xg) = f(x) \circ g + f(g)$. $\square$

By 5.1.42 and 5.1.44, we obtain a bijective 1-cocycle of $E_6^{syl}(q)$.

5.1.45 Corollary (Monomial linearisation for $E_6^{syl}(q)$)**.** *f is a bijective 1-cocycle of $E_6^{syl}(q)$, and $(f, \kappa|_{V \times V})$ is a monomial linearisation for $U = E_6^{syl}(q)$.*

Now we obtain the monomial U-module $\mathbb{C}E_6^{syl}(q)$, which is essential for the construction of the supercharacter theory for $E_6^{syl}(q)$.

5.1.46 Theorem (Fundamental theorem for $E_6^{syl}(q)$)**.** *Let $U = E_6^{syl}(q)$ and*

$$[A] = \frac{1}{|U|} \sum_{u \in U} \overline{\chi_A(u)} u \qquad \textit{for all } A \in V.$$

Then the set $\{[A] \mid A \in V\}$ forms a $\mathbb{C}$-basis for the complex group algebra $\mathbb{C}U$. The operation of U on $\mathbb{C}U$ is given by the usual right multiplication, i.e.

$$[A]u = \frac{1}{|U|} \sum_{y \in U} \overline{\chi_A(y)} yu = \chi_{A.u}(g)[A.u] = \vartheta\kappa(A.u, f(u))[A.u] \qquad \textit{for all } u \in U, A \in V,$$

then $\mathbb{C}U$ is a monomial $\mathbb{C}U$-module.

Proof. By 5.1.45, $(f, \kappa|_{V \times V})$ is a monomial linearisation for U, satisfying that f is a bijective map. By 5.1.40, $A.u := \pi(Au^{-\top})$. Thus the whole theorem is proved in view of 1.2.24. $\square$

5.2. Monomial $F_4^{syl}(q)$-module

In this section, we construct a Chevalley basis for the Lie algebra $\mathcal{L}_{F_4}$ of type F_4 (see 5.2.3) and determine the Sylow p-subgroup $F_4^{syl}(q)$ for the Chevalley group $F_4(q)$ (see 5.2.26). Then we get a monomial linearisation $(f, \kappa|_{V\times V})$ for $E_6^{syl}(q)$, where f is a bijective 1-cocycle and $\kappa|_{V\times V}$ is a non-degenerate bilinear form (5.2.49). Finally, we make $\mathbb{C}F_4^{syl}(q)$ into a monomial $F_4^{syl}(q)$-module (5.2.50).

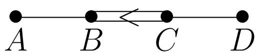
5.2.1. Chevalley basis of the Lie algebra of type F_4

In this subsection, we determine a Chevalley basis for the Lie algebra $\mathcal{L}_{F_4}$ of type F_4 (see 5.2.3)

Let ρ be a linear transformation of $\mathcal{V}_{E_6}$ into itself arising from a non-trivial symmetry of the Dynkin diagram of $\mathcal{L}_{E_6}$ sending a to f, c to e, and fixing b and d. Then $\rho(\Phi_{E_6}) = \Phi_{E_6}$ (see [Car72, 12.2.2]). There is an automorphism of $\mathcal{L}_{E_6}$ arising from ρ, also denoted by ρ, such that $e_r^\rho := \rho(e_r) = \pm e_{\rho(r)}$ and $h_r^\rho := \rho(h_r) = h_{\rho(r)}$ for all $r \in \Phi_{E_6}$ (see [Car72, 12.2.3]).

By §3.4 and §3.5 of [HRT01], let $\mathcal{L}_{E_6}^\rho := \{x \in \mathcal{L}_{E_6} \mid \rho(x) = x\}$, then the subalgebra $\mathcal{L}_{E_6}^\rho$ of $\mathcal{L}_{E_6}$ is a simple Lie algebra of type F_4, denoted by $\mathcal{L}_{F_4}$. The Cartan subalgebra is $\mathcal{H}_{F_4} = \mathcal{H}_{E_6}^\rho = \mathcal{H}_{E_6} \cap \mathcal{L}_{E_6}^\rho$. The restriction of a, c, b and d to $\mathcal{H}_{F_4}$ are the fundamental roots A, B, C and D. Thus the base of F_4 is $\Delta_{F_4} = \{A, B, C, D\}$. Let Φ_{F_4} denote the root system of type F_4, and $\Phi_{F_4}^+$ denote the positive roots of Φ_{F_4}. We have $e_A = e_a + e_f$, $e_B = e_c + e_e$, $e_C = e_d$, $e_D = e_b$, and $e_{-r} = e_r^\top$ for all $r \in \Delta_{F_4}$. Thus $\mathcal{L}_{F_4}$ is generated by $\{e_r \mid \pm r \in \Delta_{F_4}\}$.

The Dynkin diagram of F_4 is



With respect to the ordering A, B, C and D, the Cartan matrix M of F_4 is

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \end{array}$$

where $M_{i,j} = \langle R_i, R_j \rangle = \frac{2\kappa(R_i,R_j)}{\kappa(R_i,R_i)}$ $(R_i, R_j \in \Delta_{F_4})$.

Let Φ_{F_4} be the root system of type F_4. Each root $r \in \Phi_{F_4}$ can be written as $r = \sum_{\alpha\in\Delta_{F_4}} k_\alpha \alpha$, the height of r is $\mathrm{ht}(r) = \sum_{\alpha\in\Delta_{F_4}} k_\alpha$.

5.2.1 Definition. *For $r, s \in \Phi_{F_4}^+$, we write $r \prec s$, if $\mathrm{ht}(r) < \mathrm{ht}(s)$ (e.g. $A \prec A + B$) or if $\mathrm{ht}(r) = \mathrm{ht}(s)$, this is the lexicographical order (e.g. $A \prec B \prec C \prec D$).*

We fix the structure constants $N_{r,s}$ to be positive, if (r, s) is an extraspecial pair according to the total order in Definition 5.2.1. We give all the extraspecial pairs and the corresponding structure constants as follows.

5.2.2 Notation/Lemma. *We have the following extraspecial pairs of roots in $\Phi^+_{F_4} \backslash \Delta_{F_4}$ and the corresponding structure constants $N_{r,s}$ ($r, s \in \Phi^+_{F_4}$). According to column 2 and column 3 of the following table, we define R_i ($i = 1, 2, \dots, 24$) to be the root in column 3 which is according to i in column 2 (e.g. $R_5 = A + B$).*

Height	i	*Positive Roots* $R_i \in \Phi^+_{F_4}$	*Extraspecial Pair* (r, s)	$N_{r,s}$
1	1	A		
	2	B		
	3	C		
	4	D		
2	5	$A+B$	(A, B)	1
	6	$B+C$	(B, C)	1
	7	$C+D$	(C, D)	1
3	8	$A+B+C$	$(A, B+C)$	1
	9	$2B+C$	$(B, B+C)$	2
	10	$B+C+D$	$(B, C+D)$	1
4	11	$A+2B+C$	$(A, 2B+C)$	1
	12	$A+B+C+D$	$(A, B+C+D)$	1
	13	$2B+C+D$	$(B, B+C+D)$	2
5	14	$2A+2B+C$	$(A, A+2B+C)$	2
	15	$A+2B+C+D$	$(A, 2B+C+D)$	1
	16	$2B+2C+D$	$(C, 2B+C+D)$	1
6	17	$2A+2B+C+D$	$(A, A+2B+C+D)$	2
	18	$A+2B+2C+D$	$(A, 2B+2C+D)$	1
7	19	$2A+2B+2C+D$	$(A, A+2B+2C+D)$	2
	20	$A+3B+2C+D$	$(B, A+2B+2C+D)$	1
8	21	$2A+3B+2C+D$	$(A, A+3B+2C+D)$	1
9	22	$2A+4B+2C+D$	$(B, 2A+3B+2C+D)$	2
10	23	$2A+4B+3C+D$	$(C, 2A+4B+2C+D)$	1
11	24	$2A+4B+3C+2D$	$(D, 2A+4B+3C+D)$	1

Then the structure constants N_{R_i,R_j} of $\mathcal{L}_{F_4}$ are determined:

The structure constants $N(e_{R_i}, e_{R_j})$ of the Lie algebra $\mathcal{L}_{F_4}$

$N(e_{R_i}, e_{R_j})$	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}	R_{11}	R_{12}	R_{13}	R_{14}	R_{15}	R_{16}	R_{17}	R_{18}	R_{19}	R_{20}	R_{21}	R_{22}	R_{23}	R_{24}
R_1	·	1				1			1	1	2		1		2	1		2		1				
R_2	−1	·	1			2	1	1		2		1						1	1		2			
R_3		−1	·	1	−1								1		1		1					1		
R_4			−1	·		−1		−1	−1		−1			−1									1	
R_5			1		·	1	1	2		1		2				−1		−1		−2				
R_6	−1	−2		1	−1	·				−2		−1			1		1				−2			
R_7		−1			−1		·		−1		−1			−1								−1		
R_8		−1		1	−2			·		−1		−2	−1		−1					2				
R_9	−1			1			1		·			−1					1		1					
R_{10}	−1	−2			−1	2		1		·	−1			−1							2			
R_{11}	−2			1			1			1	·	−1			−2			−2						
R_{12}		−1			−2	1		2	1		1	·								−2				
R_{13}	−1		−1					1					·	−1					−1					
R_{14}				1			1			1			1	·		1								
R_{15}	−2		−1			−1		1			2				·			2						
R_{16}	−1				1									−1		·	1							
R_{17}			−1			−1			−1							−1	·							
R_{18}	−2	−1			1						2				−2			·						
R_{19}		−1							−1				1						·					
R_{20}	−1				2			−2				2								·				
R_{21}		−2				2				−2											·			
R_{22}			−1				1															·		
R_{23}				−1																			·	
R_{24}																								·

where the dots and empty entries are zeros.

By 5.2.2, we determine a Chevalley basis of the Lie algebra $\mathcal{L}_{F_4}$.

5.2.3 Lemma (Chevalley basis of $\mathcal{L}_{F_4}$). *For $R \in \Delta_{F_4}$, set $h_R := [e_R, e_{-R}]$. Let $R \in \Phi^+_{E_6} \backslash \Delta_{F_4}$, and (R_1, R_2) the extraspecial pair of R, set*

$$e_R := \frac{1}{N_{R_1,R_2}}[e_{R_1}, e_{R_2}], \quad e_{-R} := \frac{1}{N_{-R_1,-R_2}}[e_{-R_1}, e_{-R_2}].$$

Then the set

$$\{h_r \in \mathrm{Mat}_{27\times 27}(\mathbb{C}) \mid r \in \Delta_{F_4}\} \cup \{e_{\pm r} \in \mathrm{Mat}_{27\times 27}(\mathbb{C}) \mid r \in \Phi^+_{F_4}\}$$

is a Chevalley basis of the Lie algebra $\mathcal{L}_{F_4}$.

Proof. c.f. the proof of 5.1.3. □

5.2.4 Corollary. *All e_R $(R \in \Phi^+_{F_4})$ are given as follows:*

Height	i	*Positive roots* $R_i \in \Phi^+_{F_4}$	e_{R_i}	e_{R_i}
1	1	A	$e_{r_1} + e_{r_6}$	$e_a + e_f$
	2	B	$e_{r_3} + e_{r_5}$	$e_c + e_e$
	3	C	e_{r_4}	e_d
	4	D	e_{r_2}	e_b
2	5	$A+B$	$e_{r_7} - e_{r_{11}}$	$e_{a+c} - e_{e+f}$
	6	$B+C$	$e_{r_9} - e_{r_{10}}$	$e_{c+d} - e_{d+e}$
	7	$C+D$	$-e_{r_8}$	$-e_{b+d}$
3	8	$A+B+C$	$e_{r_{12}} + e_{r_{16}}$	$e_{a+c+d} + e_{d+e+f}$
	9	$2B+C$	$-e_{r_{15}}$	$-e_{c+d+e}$
	10	$B+C+D$	$-e_{r_{13}} + e_{r_{14}}$	$-e_{b+c+d} + e_{b+d+e}$
4	11	$A+2B+C$	$-e_{r_{18}} + e_{r_{21}}$	$-e_{a+c+d+e} + e_{c+d+e+f}$
	12	$A+B+C+D$	$-e_{r_{17}} - e_{r_{20}}$	$-e_{a+b+c+d} - e_{b+d+e+f}$
	13	$2B+C+D$	$e_{r_{19}}$	$e_{b+c+d+e}$
5	14	$2A+2B+C$	$e_{r_{23}}$	$e_{a+c+d+e+f}$
	15	$A+2B+C+D$	$e_{r_{22}} - e_{r_{25}}$	$e_{a+b+c+d+e} - e_{b+c+d+e+f}$
	16	$2B+2C+D$	$e_{r_{24}}$	$e_{b+c+2d+e}$
6	17	$2A+2B+C+D$	$-e_{r_{27}}$	$-e_{a+b+c+d+e+f}$
	18	$A+2B+2C+D$	$e_{r_{26}} - e_{r_{28}}$	$e_{a+b+c+2d+e} - e_{b+c+2d+e+f}$
7	19	$2A+2B+2C+D$	$-e_{r_{30}}$	$-e_{a+b+c+2d+e+f}$
	20	$A+3B+2C+D$	$e_{r_{29}} - e_{r_{31}}$	$e_{a+b+2c+2d+e} - e_{b+c+2d+2e+f}$
8	21	$2A+3B+2C+D$	$-e_{r_{32}} - e_{r_{33}}$	$-e_{a+b+c+2d+2e+f} - e_{a+b+2c+2d+e+f}$
9	22	$2A+4B+2C+D$	$-e_{r_{34}}$	$-e_{a+b+2c+2d+2e+f}$
10	23	$2A+4B+3C+D$	$-e_{r_{35}}$	$-e_{a+b+2c+3d+2e+f}$
11	24	$2A+4B+3C+2D$	$-e_{r_{36}}$	$-e_{a+2b+2c+3d+2e+f}$

5.2.2. Sylow p-subgroup $F_4^{syl}(q)$

Let K be a field. In this subsection, we obtain the Sylow p-subgroup $F_4^{syl}(q)$ for the Chevalley group $F_4(q)$ (see 5.2.26).

5.2.5 Notation. *Let* $\square := \{(i,j) \mid 1 \leq i,j \leq 27\}$ *and* $\triangledown := \{(i,j) \mid 1 \leq i < j \leq 27\}$.

5.2.6 Reminder. $\mathcal{L}_{F_4}$ *is the Lie algebra of type* F_4 *over the field* $\mathbb{C}$.

5.2.7 Corollary. *Let* $R_i \in \Phi_{F_4}^+$ *for all* $i \in \{1, 2, \ldots, 24\}$*, then*

$$\begin{aligned} e_{R_1}^2 &= 2e_{11,16}, & e_{R_2}^2 &= 2e_{12,17}, & e_{R_5}^2 &= -2e_{9,18}, & e_{R_6}^2 &= -2e_{10,19}, \\ e_{R_8}^2 &= 2e_{7,20}, & e_{R_{10}}^2 &= 2e_{8,21}, & e_{R_{11}}^2 &= -2e_{5,23}, & e_{R_{12}}^2 &= -2e_{6,22}, \\ e_{R_{15}}^2 &= 2e_{4,24}, & e_{R_{18}}^2 &= -2e_{3,25}, & e_{R_{20}}^2 &= 2e_{2,26}, & e_{R_{21}}^2 &= -2e_{1,27}, \end{aligned}$$

and

$$\begin{aligned} e_{R_i}^2 &= 0, & i &= 3, 4, 7, 9, 13, 14, 16, 17, 19, 22, 23, 24, \\ e_{R_i}^3 &= 0, & i &= 1, 2, \ldots, 24. \end{aligned}$$

Proof. By 5.2.3 and calculation, we get the result. $\square$

Similar to §11.3 of [Car72], the coefficients of $\exp(te_r) = I_{27} + te_r + \frac{1}{2}t^2 e_r^2$ ($r \in \Phi_{F_4}$) are all of the form ± 1, $\pm t$ or $\pm t^2$. Because the coefficients of e_r^2 (when this is not zero) are divisible by 2 (see 5.2.7). This fact enables us to transfer to an arbitrary field. For each matrix e_r in the above representation and each element t in an arbitrary field K, $\exp(te_r)$ is a well-defined non-singular matrix over K.

5.2.8 Notation. *Set*

$$\begin{aligned} \bar{y}_{r,K}(t) &:= \exp(te_r) = I_{27} + te_r + \frac{1}{2}t^2 e_r^2, \quad r \in \Phi_{F_4},\ t \in K, \\ y_{r,K}(t) &:= \exp(t \operatorname{ad} e_r), \quad r \in \Phi_{F_4},\ t \in K. \end{aligned}$$

5.2.9 Lemma. *For* $R \in \Phi_{F_4}$*, there are two cases as follows:*

(1) If there exists $r \in \Phi_{e_6}$ *such that* $e_R = e_r$*, we have*

$$\bar{y}_{R,K}(t) = \bar{x}_{r,K}(t) \quad \textit{for all } t \in K.$$

(2) If there exists $r_1, r_2 \in \Phi_{e_6}$ *such that* $e_R = \epsilon_1 e_{r_1} + \epsilon_2 e_{r_2}$ *and* $\epsilon_1, \epsilon_2 \in \{1, -1\}$*,*

$$\bar{y}_{R,K}(t) = \bar{x}_{r_1,K}(\epsilon_1 t) \cdot \bar{x}_{r_2,K}(\epsilon_1 t) \quad \textit{for all } t \in K.$$

Proof. By 5.2.7, the case (1) is obtained.

Let $r_1, r_2 \in \Phi_{E_6}$ such that $e_R = \epsilon_1 e_{r_1} + \epsilon_2 e_{r_2}$ and $\epsilon_1, \epsilon_2 \in \{1, -1\}$, then

$$\begin{aligned}\bar{x}_{r_1,K}(\epsilon_1 t) \cdot \bar{x}_{r_2,K}(\epsilon_1 t) =&(I_{27} + \epsilon_1 t \cdot e_{r_1}) \cdot (I_{27} + \epsilon_1 t \cdot e_{r_2})\\ =&I_{27} + t\epsilon_1 e_{r_1} + t\epsilon_2 e_{r_2} + t^2 \epsilon_1 e_{r_1} \cdot \epsilon_2 e_{r_2},\end{aligned}$$

and

$$\begin{aligned}\bar{y}_{R,K}(t) =&I_{27} + te_R + \frac{1}{2}t^2 e_R^2 = I_{27} + t(\epsilon_1 e_{r_1} + \epsilon_2 e_{r_2}) + \frac{1}{2}t^2(\epsilon_1 e_{r_1} + \epsilon_2 e_{r_2})^2\\ \overset{5.2.7}{=}&I_{27} + t(\epsilon_1 e_{r_1} + \epsilon_2 e_{r_2}) + t^2 \epsilon_1 e_{r_1} \cdot \epsilon_2 e_{r_2} = \bar{x}_{r_1,K}(\epsilon_1 t) \cdot \bar{x}_{r_2,K}(\epsilon_1 t).\end{aligned}$$

□

5.2.10 Notation. *Set a matrix group $\bar{F}_4(K)$*

$$\bar{F}_4(K) := \langle\, \bar{y}_{r,K}(t) \mid r \in \Phi_{F_4},\ t \in K \,\rangle\,,$$

and the Chevalley group of type $\mathcal{L}_{F_4}$ over the field K is

$$F_4(K) := \langle\, y_{r,K}(t) \mid r \in \Phi_{F_4},\ t \in K \,\rangle\,.$$

By 5.2.9, we get the following property.

5.2.11 Corollary. $\bar{F}_4(K) \leqslant \bar{E}_6(K)$.

5.2.12 Notation. *Set subgroups of $\bar{F}_4(K)$*

$$\bar{Y}_{r,K} := \langle\, \bar{y}_{r,K}(t) \mid t \in K \,\rangle \qquad \textit{for all } r \in \Phi_{F_4},$$

and root subgroups of the Chevalley group $F_4(K)$ are

$$Y_{r,K} := \langle\, y_{r,K}(t) \mid t \in K \,\rangle \qquad \textit{for all } r \in \Phi_{F_4}.$$

By [Car72, 4.5.1], $(\exp(t \operatorname{ad} e_r))\,(x) = \exp(te_r) \cdot x \cdot \exp(te_r)^{-1}$ for all $x \in \mathcal{L}_{F_4}$. Then we get a group epimorphism between the matrix group $\bar{F}_4(K)$ and the Chevalley group $F_4(K)$ of type $\mathcal{L}_{F_4}$ over the field K as follows.

5.2.13 Proposition. *Let K be a field, and*

$$\sigma\colon \bar{F}_4(K) \to F_4(K) : \exp(te_r) \mapsto \exp(t \operatorname{ad} e_r),$$

then σ is a group epimorphism, and the kernel $\ker\sigma$ is the centre $\bar{Z}$ of $\bar{F}_4(K)$.

5.2.14 Notation/Lemma. *Set*

$$\begin{aligned}\bar{U}_K &:= \left\langle \bar{y}_{r,K}(t) \,\middle|\, r \in \Phi_{F_4}^+,\ t \in K \right\rangle \leqslant \bar{F}_4(K),\\ U_K &:= \left\langle y_{r,K}(t) \,\middle|\, r \in \Phi_{F_4}^+,\ t \in K \right\rangle \leqslant F_4(K).\end{aligned}$$

5.2.15 Corollary. *Let K be a field, and*

$$\sigma_{\bar{U}_K} : \bar{U}_K \to U_K : \exp(te_r) \mapsto \exp(t \operatorname{ad} e_r),$$

then σ is a group epimorphism.

By calculation, we get the following commutators.

5.2.16 Proposition (Commutators of $\bar{U}_K$)**.** *Let $r_i, r_j \in \Phi^+_{F_4}$, then*

$$[\bar{y}_{r_j,K}(t_j), \bar{y}_{r_i,K}(t_i)] = \prod_{\substack{k,l>0 \\ kr_i+lr_j \in \Phi^+_{F_4}}} \bar{y}_{kr_i+lr_j,K}(C_{klr_ir_j}(-t_i)^k t_j^l).$$

In particular, if $r_i + 2r_j \in \Phi^+_{F_4}$, we have

$$[\bar{y}_{r_j,K}(t_j), \bar{y}_{r_i,K}(t_i)] = \bar{y}_{r_i+r_j,K}(N_{r_j,r_i} t_i t_j) \cdot \bar{y}_{r_i+2r_j,K}(-\frac{1}{2} N_{r_j,r_i} N_{r_j,r_j+r_i} t_i t_j^2).$$

If $r_i + r_j \in \Phi^+_{F_4}$, $2r_i + r_j \notin \Phi^+_{F_4}$ and $r_i + 2r_j \notin \Phi^+_{F_4}$,

$$[\bar{y}_{r_j,K}(t_j), \bar{y}_{r_i,K}(t_i)] = \bar{y}_{r_i+r_j,K}(-C_{11r_ir_j} t_i t_j) = \bar{y}_{r_i+r_j,K}(N_{r_j r_i} t_i t_j).$$

5.2.17 Lemma. *Let $r, s \in \Phi^+_{F_4}$ and $r \neq s$, then*

$$\overline{\operatorname{supp}}(\bar{Y}_{r,K}) \cap \overline{\operatorname{supp}}(\bar{Y}_{s,K}) = \emptyset.$$

Proof. Let $r, s \in \Phi^+_{F_4}$ and $r \neq s$, by 1.1.8 we have

$$\overline{\operatorname{supp}}(\bar{Y}_{r,K}) \cap \overline{\operatorname{supp}}(\bar{Y}_{s,K}) = \{\operatorname{supp}(e_r) \cap \operatorname{supp}(e_r^2)\} \bigcap \{\operatorname{supp}(e_s) \cap \operatorname{supp}(e_s^2)\}.$$

By 5.2.3, $\{\operatorname{supp}(e_r) \cap \operatorname{supp}(e_r^2)\} \bigcap \{\operatorname{supp}(e_s) \cap \operatorname{supp}(e_s^2)\} = \emptyset$. □

5.2.18 Proposition. *Each element of $\bar{U}_{\mathbb{F}_q}$ is uniquely expressible in the form*

$$\prod_{r_i \in \Phi^+_{F_4}} \bar{y}_{r_i,\mathbb{F}_q}(t_i),$$

where the product is taken over all positive roots in the increasing order.

Proof. Each element of $\bar{U}_{\mathbb{F}_q}$ is a product of elements $\bar{y}_{r,\mathbb{F}_q}(t)$ with $r \in \Phi^+_{F_4}, t \in \mathbb{F}_q$. If there is a pair of consecutive terms $\bar{y}_{r_j,\mathbb{F}_q}(t_j)\bar{y}_{r_i,\mathbb{F}_q}(t_i)$ with $r_i \prec r_j$, by 5.2.16 we have

$$\bar{x}_{r_j,\mathbb{F}_q}(t_j)\bar{x}_{r_i,\mathbb{F}_q}(t_i) = \bar{x}_{r_i,\mathbb{F}_q}(t_i)\bar{x}_{r_j,\mathbb{F}_q}(t_j) \prod_{\substack{k,l>0 \\ kr_i+lr_j \in \Phi^+_{E_6}}} \bar{x}_{kr_i+lr_j,\mathbb{F}_q}(C_{klr_ir_j}(-t_i)^k t_j^l).$$

Since $\mathrm{ht}(kr_i + lr_j) \geq \mathrm{ht}(r_i) + \mathrm{ht}(r_j)$, then after a finite number of steps we get the desired product.

For the uniqueness, suppose $\prod_{r_i\in\Phi^+_{F_4}} \bar{y}_{r_i,\mathbb{F}_q}(t_{r_i}) = \prod_{r_i\in\Phi^+_{F_4}} \bar{y}_{r_i,\mathbb{F}_q}(s_{r_i})$. Then

$$\bar{y}_{r_i,\mathbb{F}_q}(t_{r_i}) = \prod_k (I_{27} + \epsilon_{r_i,k} t_{r_i} e_{i_k,j_k}),$$

where $\epsilon_{r_i,k} \in \{1,-1\}$.

The Lemma 5.1.16 implies $t_{r_i} = s_{r_i}$ $(r_i \in \Phi^+_{F_4})$. □

5.2.19 Corollary. $|\bar{U}_{\mathbb{F}_q}| = q^{|\Phi^+_{F_4}|} = q^{24}$.

5.2.20 Corollary. $\bar{U}_{\mathbb{F}_q} \cong U_{\mathbb{F}_q}$.

Proof. c.f. the proof of 5.1.19. □

In the remainder of this section, we consider $K = \mathbb{F}_q$.

5.2.21 Notation/Lemma (Root subgroups of $\bar{F}_4(q)$)**.** *Let*

$$y_r(t) := \exp(te_r) = I_{27} + t\cdot e_r + \frac{1}{2}t^2e_r^2 = \bar{y}_{r,\mathbb{F}_q}(t), \quad r \in \Phi_{F_4},\ t \in \mathbb{F}_q,$$
$$y_i(t) := \exp(te_{r_i}) = I_{27} + t\cdot e_{r_i} + \frac{1}{2}t^2e_{r_i}^2 = y_{r_i}(t), \quad i = 1,2,\ldots,24,\ t \in \mathbb{F}_q.$$

Then the root subgroups of $\bar{F}_4(q)$,

$$Y_r := \{y_r(t)\,|\,t \in \mathbb{F}_q\} = \bar{Y}_{r,\mathbb{F}_q}, \quad r \in \Phi_{F_4},$$
$$Y_i := \{y_{r_i}(t)\,|\,t \in \mathbb{F}_q\} = Y_{r_i}, \quad i = 1,2,\ldots,24.$$

5.2.22 Notation. *Set*

$$U := \left\langle y_r(t) \,\middle|\, r \in \Phi^+_{F_4},\ t \in \mathbb{F}_q \right\rangle = \bar{U}_{\mathbb{F}_q}.$$

5.2.23 Lemma ([Car65])**.** *The order of $F_4(q)$ is*

$$|F_4(q)| = q^{24}(q^2-1)(q^6-1)(q^8-1)(q^{12}-1).$$

5.2.24 Corollary. $|U| = q^{24}$ *and U is a Sylow p-subgroup of the Chevalley group $F_4(q)$ under isomorphism.*

Proof. c.f. the proof of the 5.1.23. □

5.2.25 Proposition.

$$U = \left\{ \prod_{r\in\Phi^+_{F_4}} y_r(t_r) \,\middle|\, t_r \in \mathbb{F}_q \right\},$$

where the product can be taken in an arbitrary, but fixed, order.

Proof. c.f. the proof of the 5.1.24. □

By 5.2.24 and 5.2.25, we get the following Sylow p-subgroup of $F_4(q)$.

5.2.26 Proposition (Sylow p-subgroup $F_4^{syl}(q)$)**.** *Let $t_i \in \mathbb{F}_q$ $(i = 1, 2, \ldots, 24)$ and*

$$F_4^{syl}(q) := U.$$

Then $|F_4^{syl}(q)| = q^{24}$, $F_4^{syl}(q)$ is a Sylow p-subgroup of the Chevalley group $F_4(q)$ under isomorphism and the matrix elements of the Sylow p-subgroup $F_4^{syl}(q)$ are given as

$$\begin{aligned}
&y_4(t_4)y_7(t_7)y_3(t_3)y_{16}(t_{16}) \cdot y_{13}(t_{13})y_9(t_9)y_{10}(t_{10})y_6(t_6)x_2(t_2)\\
&\cdot y_{24}(t_{24})y_{23}(t_{23})y_{22}(t_{22})y_{21}(t_{21})y_{19}(t_{19})y_{20}(t_{20})\\
&\cdot y_{17}(t_{17})y_{18}(t_{18})y_{14}(t_{14})y_{15}(t_{15})\\
&\cdot y_{11}(t_{11})y_{12}(t_{12})y_8(t_8)y_5(t_5)y_1(t_1)
\end{aligned}$$

$$= \left(\begin{array}{ccccccccccccccccccccccccccc}
1 & t_1 & t_5 & t_8 & t_{12} & -t_{11} & -t_{15} & \begin{array}{c}-t_1t_{11}\\-t_5t_8\\+t_{14}\end{array} & t_{18} & \begin{array}{c}-t_1t_{15}\\-t_5t_{12}\\+t_{17}\end{array} & -t_{20} & \begin{array}{c}t_1t_{18}\\-t_8t_{12}\\-t_{19}\end{array} & * & \begin{array}{c}-t_1t_{20}\\+t_{21}\end{array} & \begin{array}{c}t_5t_{18}\\+t_8t_{15}\\-t_{11}t_{12}\\+t_{21}\end{array} & * & \begin{array}{c}-t_5t_{20}\\-t_{11}t_{15}\\-t_{22}\end{array} & * & \begin{array}{c}-t_8t_{20}\\+t_{11}t_{18}\\+t_{23}\end{array} & * & \begin{array}{c}-t_{12}t_{20}\\+t_{15}t_{18}\\-t_{24}\end{array} & * & * & * & * & * & * \\
0 & 1 & t_2 & t_6 & t_{10} & \begin{array}{c}t_2t_6\\-t_9\end{array} & \begin{array}{c}t_2t_{10}\\-t_{13}\end{array} & \begin{array}{c}t_1t_2t_6\\-t_1t_9\\+t_2t_8\\-t_5t_6\\+t_{11}\end{array} & \begin{array}{c}t_6t_{10}\\+t_{16}\end{array} & \begin{array}{c}t_1t_2t_{10}\\-t_1t_{13}\\+t_2t_{12}\\-t_5t_{10}\\+t_{15}\end{array} & * & \begin{array}{c}t_1t_6t_{10}\\+t_1t_{16}\\+t_6t_{12}\\-t_8t_{10}\\-t_{18}\end{array} & * & * & \begin{array}{c}t_2t_6t_{12}\\-t_2t_8t_{10}\\+t_5t_6t_{10}\\+t_5t_{16}\\+t_8t_{13}\\-t_9t_{12}\\+t_{20}\end{array} & * & * & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 1 & t_3 & t_7 & \begin{array}{c}t_2t_3\\-t_6\end{array} & \begin{array}{c}t_2t_7\\-t_{10}\end{array} & \begin{array}{c}t_1t_2t_3\\-t_1t_6\\-t_3t_5\\+t_8\end{array} & * & \begin{array}{c}t_1t_2t_7\\-t_1t_{10}\\-t_5t_7\\+t_{12}\end{array} & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 1 & t_4 & t_2 & * & \begin{array}{c}t_1t_2\\-t_5\end{array} & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 1 & 0 & * & 0 & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 1 & * & t_1 & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
\vdots & & & & & & & & & & & \ddots & \ddots & \ddots & & & & & & & & & & & & & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)_{27\times 27}$$

5.2.27 Corollary (36 variables)**.** *The matrix elements of Sylow p-subgroup F_4^{syl} are written as follows:*

$$
\begin{array}{cc}
 & \begin{array}{ccccccccccccccccccccccccccc} 1&2&3&4&5&6&7&8&9&10&11&12&13&14&15&16&17&18&19&20&21&22&23&24&25&26&27 \end{array} \\
\begin{array}{c} 1\\2\\3\\4\\5\\6\\7\\8\\9\\\vdots\\25\\26\\27 \end{array} &
\begin{pmatrix}
1 & t_1 & t_5 & t_8 & t_{12} & -t_{11} & -t_{15} & \substack{-t_1t_{11}\\-t_5t_8\\+t_{14}} & t_{18} & \substack{-t_1t_{15}\\-t_5t_{12}\\+t_{17}} & -t_{20} & \substack{t_1t_{18}\\-t_8t_{12}\\-t_{19}} & * & \substack{-t_1t_{20}\\+t_{21}} & \substack{t_5t_{18}\\+t_8t_{15}\\-t_{11}t_{12}\\+t_{21}} & * & \substack{-t_5t_{20}\\-t_{11}t_{15}\\-t_{22}} & * & \substack{-t_8t_{20}\\+t_{11}t_{18}\\+t_{23}} & * & \substack{-t_{12}t_{20}\\+t_{15}t_{18}\\-t_{24}} & * & * & * & * & * & * \\
0 & 1 & t_2 & t_6 & t_{10} & \substack{t_2t_6\\-t_9} & \substack{t_2t_{10}\\-t_{13}} & \substack{t_1t_2t_6\\-t_1t_9\\+t_2t_8\\-t_5t_6\\+t_{11}} & \substack{t_6t_{10}\\+t_{16}} & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 1 & t_3 & t_7 & * \\
0 & 0 & 0 & 1 & t_4 & * \\
0 & 0 & 0 & 0 & 1 & 0 & * & 0 & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 1 & * \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
\vdots & & & & & & & & & & & \ddots & \ddots & \ddots & & & & & & & & & & & & & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & t_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array}
$$

where $t_i \in \mathbb{F}_q$ $(i = 1, 2, \ldots, 24)$.

5.2.28 Corollary. $F_4^{syl}(q) \leqslant E_6^{syl}(q)$.

Proof. The corollary follows immediately by 5.2.9, 5.2.25 and 5.1.24. □

5.2.29 Corollary. *Let* $g = (g_{i,j}) \in F_4^{syl}(q)$*, then*

$$\begin{cases} g_{13,j_{13}} = 0 & \textit{if } j_{13} \in \{14, 15, 17, 19, 21\} \\ g_{16,j_{16}} = 0 & \textit{if } j_{16} \in \{17, 19, 21\} \\ g_{18,j_{18}} = 0 & \textit{if } j_{18} \in \{19, 21\} \\ g_{20,21} = 0 \\ g_{14,15} = 0 \\ g_{8,9} = 0 \end{cases}$$

and

$$\begin{cases} g_{14,17} = g_{15,17} \\ g_{14,19} = g_{15,19} \\ g_{14,21} = g_{15,21} \end{cases}.$$

5.2.3. Monomial $F_4^{syl}(q)$-module

Let $U := F_4^{syl}(q)$ and $V_0 := \mathrm{Mat}_{27\times 27}(\mathbb{F}_q)$. In this subsection, we get a monomial linearisation $(f, \kappa|_{V\times V})$ for $F_4^{syl}(q)$, where f is a bijective 1-cocycle and $\kappa|_{V\times V}$ is a non-degenerate bilinear form (5.2.49). Then we make $\mathbb{C}F_4^{syl}(q)$ into a monomial $F_4^{syl}(q)$-module (5.2.50).

5.2.30 Notation/Lemma. *The map*

$$\kappa\colon V_0 \times V_0 \to \mathbb{F}_q : (X, Y) \mapsto \mathrm{tr}(X^\top Y)$$

*is called the **trace form**, and κ is a non-degenerate symmetric bilinear form on V_0.*

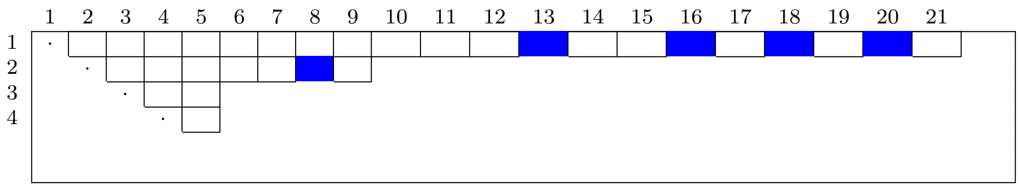
5.2.31 Notation. *Set*

$$J := \left\{ (i,j) \,\middle|\, \begin{cases} j \in \{2,3,\dots,11,12,14,15,17,19,21\} & \text{if } i=1 \\ j \in \{3,4,5,6,7,9\} & \text{if } i=2 \\ j \in \{4,5\} & \text{if } i=3 \\ j \in \{5\} & \text{if } i=4 \end{cases} \right\},$$

and

$$J' := \left\{ (i,j) \in \triangledown \,\middle|\, \begin{cases} j \in \{13,16,18,20\} & \text{if } i=1 \\ j = 8 & \text{if } i=2 \end{cases} \right\}.$$

Then J is the set of the coordinates of the empty blocks of the following 27×27-matrix, while J' is the set of the coordinates of the blue blocks of the following 27×27-matrix.



5.2.32 Remark. *(1) $J^c = J \dot\cup J'$,*

(2) $\#J = 25$, $\#J' = 5$ and $\#J^c = 30$.

5.2.33 Notation/Lemma. *Set*

$$V_J = \bigoplus_{(i,j)\in J} \mathbb{F}_q e_{i,j} = \Big\{ \sum_{(i,j)\in J} A_{i,j} e_{i,j} \in V_0 \,\Big|\, A_{i,j} \in \mathbb{F}_q \Big\},$$

where

$$\sum_{(i,j)\in J} A_{i,j} e_{i,j} =$$

$$\left(\begin{array}{ccccccccccccccccccccc} \cdot & A_{1,2} & A_{1,3} & A_{1,4} & A_{1,5} & A_{1,6} & A_{1,7} & A_{1,8} & A_{1,9} & A_{1,10} & A_{1,11} & A_{1,12} & 0 & A_{1,14} & A_{1,15} & 0 & A_{1,17} & 0 & A_{1,19} & 0 & A_{1,21} \\ & \cdot & A_{2,3} & A_{2,4} & A_{2,5} & A_{2,6} & A_{2,7} & 0 & A_{2,9} & & & & & & & & & & & & \\ & & \cdot & A_{3,4} & A_{3,5} & & & & & & & & & & & & & & & & \\ & & & \cdot & A_{4,5} & & & & & & & & & & & & & & & & \end{array}\right)$$

Then V_J is a 25-dimensional subspace of V_0 over $\mathbb{F}_q$, and $\mathrm{supp}(V) = J$.

5.2.34 Notation/Lemma. *Set*

$$V := \{A = (A_{i,j}) \in V_J \mid A_{i,j} \in \mathbb{F}_q,\ A_{1,14} = A_{1,15}\},$$

where

$A =$

$$\begin{pmatrix} \cdot & A_{1,2} & A_{1,3} & A_{1,4} & A_{1,5} & A_{1,6} & A_{1,7} & A_{1,8} & A_{1,9} & A_{1,10} & A_{1,11} & A_{1,12} & 0 & {\color{red}A_{1,15}} & {\color{red}A_{1,15}} & 0 & A_{1,17} & 0 & A_{1,19} & 0 & A_{1,21} \\ & \cdot & A_{2,3} & A_{2,4} & A_{2,5} & A_{2,6} & A_{2,7} & 0 & A_{2,9} & & & & & & & & & & & & \\ & & \cdot & A_{3,4} & A_{3,5} & & & & & & & & & & & & & & & & \\ & & & \cdot & A_{4,5} & & & & & & & & & & & & & & & & \\ & & & & & & & & & & & & & & & & & & & & \end{pmatrix}$$

Then V is a 24-dimensional subspace of V_J over $\mathbb{F}_q$, and $\mathrm{supp}(V) = J$.

5.2.35 Proposition. *Let $V^{\perp}$ denote the orthogonal complement of the subspace V of V_0 with respect to the trace form κ, i.e. $V^{\perp} := \{B \in V_0 \mid \kappa(A, B) = 0,\ \forall\ A \in V\}$. Then*

$$\begin{aligned} V^{\perp} &= \bigoplus_{(i,j)\notin J} \mathbb{F}_q e_{ij} + \{x e_{1,15} - x e_{1,14} \mid x \in \mathbb{F}_q\} \\ &= \{A = (A_{i,j}) \in V_0 \mid A_{1,14} = -A_{1,15},\ A_{i,j} = 0 \textit{ if } (i,j) \in J \backslash \{(1,14), (1,15)\}\}, \end{aligned}$$

where $A =$

$$\begin{pmatrix} A_{1,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{1,13} & {\color{red}-A_{1,15}} & {\color{red}A_{1,15}} & A_{1,16} & 0 & A_{18} & 0 & A_{1,20} & 0 & A_{1,22} & \cdots & A_{1,27} \\ A_{2,1} & A_{2,2} & 0 & 0 & 0 & 0 & 0 & A_{2,8} & 0 & A_{2,10} & & & & \cdots & & & & & & & & & & A_{2,27} \\ A_{3,1} & A_{3,2} & A_{3,3} & 0 & 0 & A_{3,6} & & & & & & & & \cdots & & & & & & & & & & A_{3,27} \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} & 0 & A_{4,6} & & & & & & & & \cdots & & & & & & & & & & A_{4,27} \\ A_{5,1} & A_{5,2} & A_{5,3} & A_{5,4} & A_{5,5} & A_{5,6} & & & & & & & & \cdots & & & & & & & & & & A_{5,27} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & & & & & & & & & & & & & & & \vdots \\ A_{27,1} & A_{27,2} & A_{27,3} & A_{27,4} & A_{27,5} & A_{27,6} & & & & & & & & \cdots & & & & & & & & & & A_{27,27} \end{pmatrix}.$$

Proof. Let

$$W := \{A = (A_{i,j}) \in V_0 \mid A_{1,14} = -A_{1,15},\ A_{i,j} = 0 \text{ if } (i,j) \in J \backslash \{(1,14), (1,15)\}\}.$$

Claim $V^{\perp} \subseteq W$. Let $B = (B_{i,j}) \in V^{\perp}$ and $A = (A_{i,j}) \in V$, then

$$\kappa(A, B) = \sum_{(i,j)\in J\backslash\{(1,14),(1,15)\}} A_{i,j} B_{i,j} + 2A_{1,15}(B_{1,14} + B_{1,15}).$$

Let $A := e_{i,j}$ $(i,j) \in J\backslash\{(1,14),(1,15)\}$, then $0 = \kappa(A,B) = B_{i,j}$. Let $A := e_{1,14} + e_{1,15}$, then $0 = \kappa(A,B) = B_{1,14} + B_{1,15}$, so $B_{1,14} = -B_{1,15}$. Thus $B \in W$. Therefore, the statement is proved since $V^{\perp} \supseteq W$. $\square$

5.2.36 Corollary. $V_0 = V \oplus V^{\perp}$.

5.2.37 Corollary. $\kappa|_{V\times V} : V \times V \to \mathbb{F}_q$ *is a non-degenerate bilinear form.*

Proof. It is sufficient to prove that $\kappa|_{V\times V}$ is non-degenerate. Let $A \in V$ such that $\kappa(A,B) = 0$ for all $B \in V$. Then $A \in V^\top \cap V = \{0\}$. $\square$

5.2.38 Notation/Lemma. *Let*

$$\pi\colon V_0 = V \oplus V^\perp \to V : A \mapsto \frac{A_{1,14}+A_{1,15}}{2}(e_{1,14}+e_{1,15}) + \sum_{(i,j)\in J\setminus\{(1,14),(1,15)\}} A_{i,j}e_{i,j}$$

i.e. $\pi(A) =$

$$\left(\begin{array}{l}
\cdot\ A_{1,2}\ A_{1,3}\ A_{1,4}\ A_{1,5}\ A_{1,6}\ A_{1,7}\ A_{1,8}\ A_{1,9}\ A_{1,10}\ A_{1,11}\ A_{1,12}\ 0\ \frac{A_{1,14}+A_{1,15}}{2}\ \frac{A_{1,14}+A_{1,15}}{2}\ 0\ A_{1,17}\ 0\ A_{1,19}\ 0\ A_{1,21} \\
\quad \cdot\ A_{2,3}\ A_{2,4}\ A_{2,5}\ A_{2,6}\ A_{2,7}\ 0\ A_{2,9} \\
\qquad \cdot\ A_{3,4}\ A_{3,5} \\
\qquad\quad \cdot\ A_{4,5}
\end{array}\right)$$

then π is a projection to the first component and is a $\mathbb{F}_q$-linear map.

Proof. It suffices to prove that π is a $\mathbb{F}_q$-linear map. Let $A, B \in V_0$ and $c \in \mathbb{F}_q$, then

$$\begin{aligned}
&\pi(A+B) = \pi(\sum_{i,j}(A_{i,j}+B_{i,j})) \\
=&\frac{A_{1,14}+B_{1,14}+A_{1,15}+B_{1,15}}{2}(e_{1,14}+e_{1,15}) + \sum_{(i,j)\in J\setminus\{(1,14),(1,15)\}}(A_{i,j}+B_{i,j})e_{i,j} \\
=&\pi(A)+\pi(B),
\end{aligned}$$

and

$$\begin{aligned}
\pi(cA) =&\pi(\sum_{i,j}(cA_{i,j})) = \frac{cA_{1,14}+cA_{1,15}}{2}(e_{1,14}+e_{1,15}) + \sum_{(i,j)\in J\setminus\{(1,14),(1,15)\}} cA_{i,j}e_{i,j} \\
=&c\pi(A).
\end{aligned}$$

$\square$

5.2.39 Lemma. $\pi = \pi \circ \pi_J$. *In particular, let $A \in V_0$ and $\pi_J(A) \in V$, then $\pi(A) = \pi_J(A)$.*

5.2.40 Corollary. *Suppose $A, B \in V_0$, such that* $\operatorname{supp}(A) \cap \operatorname{supp}(B) \subseteq J$ *and $\pi_J(A) \in V$. Then*

$$\kappa(A,B) = \kappa(\pi(A),B) = \kappa(A,\pi(B)) = \kappa(\pi(A),\pi(B)) = \kappa|_{V\times V}(\pi(A),\pi(B)).$$

Proof. c.f. the proof of 2.2.21. $\square$

5.2.41 Lemma. *Let $\pi_J\colon V_0 \to V_J : A \mapsto \sum_{(i,j)\in J} A_{ij}e_{ij}$, then*

$$\operatorname{supp}(Ag^\top) \cap \triangledown \subseteq J \qquad \textit{for all } A \in V,\ g \in U,$$

and $\pi_{J^c}(Ag^\top) \in V$. In particular, $\pi_J(Ag^\top) = \pi(Ag^\top)$.

Proof. Let $A \in V$, $g \in U$. It is sufficient to prove $(Ag^\top)_{i,j} = 0$ for all $(i,j) \in J^c \backslash J = J'$, and $(Ag^\top)_{1,14} = (Ag^\top)_{1,15}$. By 5.2.29,

$$(Ag^\top)_{1,13} = \sum_{j=1}^{27} A_{1,j} g^\top_{j,13} = \sum_{j=2}^{21} A_{1,j} g^\top_{j,13} = \sum_{j=2}^{21} A_{1,j} g_{13,j} = \sum_{j=13}^{21} A_{1,j} g_{13,j}$$
$$\overset{\substack{A_{1,j}=0\\ j\in\{13,16,18,20\}}}{=} \sum_{j\in\{14,15,17,19,21\}} A_{1,j} g_{13,j} \overset{\substack{g_{13,j}=0\\ j\in\{14,15,19,21\}}}{=} 0.$$

Similarly, $(Ag^\top)_{i,j} = 0$ for all $(i,j) \in J' = \{(1,16),(1,18),(1,20),(2,8)\}$. We have

$$\begin{aligned}(Ag^\top)_{1,14} &= \sum_{j=1}^{27} A_{1,j} g^\top_{j,14} = \sum_{j=2}^{21} A_{1,j} g^\top_{j,14} = \sum_{j=2}^{21} A_{1,j} g_{14,j} = \sum_{j=14}^{21} A_{1,j} g_{14,j}\\
&\overset{\substack{A_{1,j}=0\\ j\in\{16,18,20\}}}{=} \sum_{j\in\{14,15,17,19,21\}} A_{1,j} g_{14,j}\\
&= A_{1,14} + A_{1,15} g_{14,15} + A_{1,17} g_{14,17} + A_{1,19} g_{14,19} + A_{1,21} g_{14,21}\\
&\overset{g_{14,15}=0}{=} A_{1,14} + A_{1,17} g_{14,17} + A_{1,19} g_{14,19} + A_{1,21} g_{14,21}\\
&\overset{\substack{g_{14,j}=g_{15,j}\\ j\in\{17,19,21\}}}{=} A_{1,15} + A_{1,17} g_{15,17} + A_{1,19} g_{15,19} + A_{1,21} g_{15,21} = (Ag^\top)_{1,15}.\end{aligned}$$

Thus, $(Ag^\top)_{i,j} = 0$ for all $(i,j) \in J^c \backslash J = J'$, and $(Ag^\top)_{1,14} = (Ag^\top)_{1,15}$. Hence $\mathrm{supp}(Ag^\top) \cap \triangledown \subseteq J$ and $\pi_{J^c}(Ag^\top) = \pi_J(Ag^\top) \in V$. □

5.2.42 Corollary. *Let $A, B \in V$ and $g, h \in U$, then $\mathrm{supp}(Bh^\top) \cap \mathrm{supp}(Ag) \subseteq J$.*

5.2.43 Proposition (Group action of $F_4^{syl}(q)$ on V). *The map*

$$- \circ -: V \times U \to V : (A, g) \mapsto A \circ g := \pi(Ag)$$

is a group action, and the elements of the group U act as $\mathbb{F}_q$-automorphisms.

Proof. c.f. the proof of 2.2.25. □

By 1.2.21, we get a new action:

5.2.44 Corollary. *There exists an unique linear action $-.-$ of G on V:*

$$-.-: V \times G \to V : (A, g) \mapsto A.g := A \circ g^{-\top}$$

such that $\kappa|_{V\times V}(A.g, B) = \kappa|_{V\times V}(A, B \circ g^{-1})$ for all $B \in V$. In particular, for all $A, B \in V$ and $g \in G$, $\kappa|_{V\times V}(A, B \circ g) = \kappa|_{V\times V}(A.g^{-1}, B) = \kappa|_{V\times V}(A \circ g^\top, B)$.

5.2.45 Notation. $f := \pi|_U : U \to V$.

5.2.46 Proposition. *$f|_U : U \to V$ is bijective.*

Proof. By 5.2.26 and 5.2.34, we obtain that $f : U \to V$ is bijective. $\square$

5.2.47 Lemma. *Let $x, g \in U$, then $f(x)g \equiv (x-1)g \mod V^\perp$. In particular, $f(x) \equiv x - 1 \mod V^\perp$.*

Proof. c.f. the proof of 2.2.29. $\square$

5.2.48 Proposition. *Let $x, g \in U$, then $f(xg) = f(x) \circ g + f(g)$.*

Proof. c.f. the proof of 2.2.30 or by 5.2.29 as follows: for all $j_0 = 1, 2, \ldots, 12$, we have

$$\begin{aligned}
&\big(f(x)\circ g + f(g)\big)_{1,j_0} = \big(f(x)\circ g\big)_{1,j_0} + f(g)_{1,j_0} = \big(f(x)g\big)_{1,j_0} + f(g)_{1,j_0}\\
=&\sum_{k=2}^{j_0} f(x)_{1,k}g_{k,j_0} + g_{1,j_0} = \sum_{k=2}^{j_0} x_{1,k}g_{k,j_0} + g_{1,j_0} = \sum_{k=1}^{j_0} x_{1,k}g_{k,j_0} = (xg)_{1,j_0} = f(xg)_{1,j_0},
\end{aligned}$$

and by 5.2.29,

$$\begin{aligned}
&\big(f(x)\circ g + f(g)\big)_{1,14} = \frac{\big(f(x)g\big)_{1,14} + \big(f(x)g\big)_{1,15}}{2} + \frac{g_{1,14}+g_{1,15}}{2}\\
=&\frac{1}{2}\Big(\sum_{k=2}^{14} f(x)_{1,k}g_{k,14} + \sum_{k=2}^{15} f(x)_{1,k}g_{k,15}\Big) + \frac{g_{1,14}+g_{1,15}}{2}\\
=&\frac{g_{1,14}+g_{1,15}}{2} + \frac{1}{2}\Big(\sum_{k=2}^{12} x_{1,k}g_{k,14} + f(x)_{1,13}g_{13,14} + f(x)_{1,14}\Big)\\
&+\frac{1}{2}\Big(\sum_{k=2}^{12} x_{1,k}g_{k,15} + f(x)_{1,13}g_{13,15} + f(x)_{1,14}g_{14,15} + f(x)_{1,15}\Big)\\
\overset{\substack{g_{13,14}=0\\ g_{13,15}=0\\ g_{14,15}=0}}{=}&\frac{g_{1,14}+g_{1,15}}{2} + \frac{1}{2}\Big(\sum_{k=2}^{12} x_{1,k}g_{k,14} + x_{1,13}g_{13,14} + f(x)_{1,14}\Big)\\
&+\frac{1}{2}\Big(\sum_{k=2}^{12} x_{1,k}g_{k,15} + x_{1,13}g_{13,15} + x_{1,14}g_{14,15} + f(x)_{1,15}\Big)\\
=&\frac{g_{1,14}+g_{1,15}}{2} + \frac{1}{2}\Big(\sum_{k=2}^{13} x_{1,k}g_{k,14} + f(x)_{1,14}\Big) + \frac{1}{2}\Big(\sum_{k=2}^{14} x_{1,k}g_{k,15} + f(x)_{1,15}\Big)\\
=&\frac{g_{1,14}+g_{1,15}}{2} + \frac{1}{2}\Big(\sum_{k=2}^{13} x_{1,k}g_{k,14} + \frac{x_{1,14}+x_{1,15}}{2}\Big) + \frac{1}{2}\Big(\sum_{k=2}^{14} x_{1,k}g_{k,15} + \frac{x_{1,14}+x_{1,15}}{2}\Big)\\
=&\frac{1}{2}\Big(\sum_{k=2}^{13} x_{1,k}g_{k,14} + g_{1,14} + x_{1,14}\Big) + \frac{1}{2}\Big(\sum_{k=2}^{14} x_{1,k}g_{k,15} + g_{1,15} + x_{1,15}\Big)\\
=&\frac{(xg)_{1,14} + (xg)_{1,15}}{2} = f(xg)_{1,14},
\end{aligned}$$

and

$$
\begin{aligned}
\big(f(x)\circ g+f(g)\big)_{1,17} &= \big(f(x)g\big)_{1,17}+f(g)_{1,17}=\sum_{k=2}^{17} f(x)_{1,k}g_{k,17}+g_{1,17}\\
&=\sum_{k=2}^{12} x_{1,k}g_{k,17}+f(x)_{1,13}g_{13,17}+f(x)_{1,14}g_{14,17}+f(x)_{1,15}g_{15,17}\\
&\quad+f(x)_{1,16}g_{16,17}+f(x)_{1,17}+g_{1,17}\\
&\overset{\substack{g_{13,17}=0\\ g_{16,17}=0}}{=}\sum_{k=2}^{12} x_{1,k}g_{k,17}+x_{1,13}g_{13,17}+f(x)_{1,14}g_{14,17}+f(x)_{1,15}g_{15,17}\\
&\quad+x_{1,16}g_{16,17}+f(x)_{1,17}+g_{1,17}\\
&=\sum_{k=2}^{13} x_{1,k}g_{k,17}+f(x)_{1,14}g_{14,17}+f(x)_{1,15}g_{15,17}+x_{1,16}g_{16,17}+f(x)_{1,17}+g_{1,17}\\
&=\sum_{k=2}^{13} x_{1,k}g_{k,17}+\frac{x_{1,14}+x_{1,15}}{2}(g_{14,17}+g_{15,17})+x_{1,16}g_{16,17}+x_{1,17}+g_{1,17}\\
&\overset{x_{14,17}=x_{15,17}}{=}\left(\sum_{k=2}^{13} x_{1,k}g_{k,17}\right)+x_{1,14}g_{14,17}+x_{1,15}g_{15,17}+x_{1,16}g_{16,17}+x_{1,17}+g_{1,17}\\
&=(xg)_{1,17}=f(xg)_{1,17}.
\end{aligned}
$$

Similarly, $f(xg)_{i,j}=\big(f(x)\circ g+f(g)\big)_{i,j}$ for all $(i,j)\in J$. Hence, $f(xg)=f(x)\circ g+f(g)$. $\square$

By 5.2.46 and 5.2.48, we obtain a bijective 1-cocycle of $F_4^{syl}(q)$.

5.2.49 Corollary (Monomial linearisation for $F_4^{syl}(q)$)**.** *f is a bijective 1-cocycle of $F_4^{syl}(q)$, and $(f,\kappa|_{V\times V})$ is a monomial linearisation for $U=F_4^{syl}(q)$.*

Now we obtain the monomial U-module $\mathbb{C}F_4^{syl}(q)$, which is essential for the construction of the supercharacter theory for $F_4^{syl}(q)$.

5.2.50 Theorem (Fundamental theorem for $F_4^{syl}(q)$)**.** *Let $U=F_4^{syl}(q)$ and*

$$[A]=\frac{1}{|U|}\sum_{u\in U}\overline{\chi_A(u)}u \qquad \text{for all } A\in V.$$

Then the set $\{[A]\mid A\in V\}$ forms a $\mathbb{C}$-basis for the complex group algebra $\mathbb{C}U$. The operation of U on $\mathbb{C}U$ is given by the usual right multiplication, i.e.

$$[A]u=\frac{1}{|U|}\sum_{y\in U}\overline{\chi_A(y)}yu=\chi_{A.u}(u)[A.u]=\vartheta\kappa(A.u,f(u))[A.u] \qquad \text{for all } u\in U, A\in V,$$

then $\mathbb{C}U$ is a monomial $\mathbb{C}U$-module.

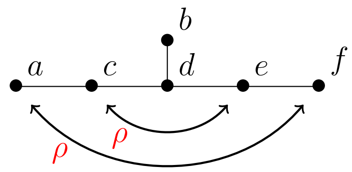
Proof. By 5.2.49, $(f,\kappa|_{V\times V})$ is a monomial linearisation for U, satisfing f is bijective. By 5.2.44, $A.u:=\pi(Au^{-\top})$. Thus the whole theorem is proved in view of 1.2.24. $\square$

5.3. Monomial ${}^2E_6^{syl}(q^2)$-module

In this section, we determine the matrix Sylow p-subgroup ${}^2E_6^{syl}(q^2)$ for the Steinberg group ${}^2E_6(q)$ (see 5.3.3). Then we get a monomial linearisation $(f, \kappa_q|_{V\times V})$ for $E_6^{syl}(q)$ (see 5.3.31). Finally, we obtain that $\mathbb{C}\left({}^2E_6^{syl}(q^2)\right)$ is a monomial ${}^2E_6^{syl}(q^2)$-module (5.3.32).

5.3.1. Sylow p-subgroup ${}^2E_6^{syl}(q^2)$

Let ρ be a linear transformation of $\mathcal{V}_{E_6}$ into itself arising from a non-trivial symmetry of the Dynkin diagram of $\mathcal{L}_{E_6}$ sending a to f, c to e, and fixing b and d.



Then $\rho(\Phi_{E_6}) = \Phi_{E_6}$ (see [Car72, 12.2.2]). There is an automorphism of $\mathcal{L}_{E_6}$ arising from ρ, also denoted by ρ, such that $e_r^\rho := \rho(e_r) = \pm e_{\rho(r)}$ and $h_r^\rho := \rho(h_r) = h_{\rho(r)}$ for all $r \in \Phi_{E_6}$ (see [Car72, 12.2.3]).

By §3.4 and §3.5 of [HRT01], let $\mathcal{L}_{E_6}^\rho := \{x \in \mathcal{L}_{E_6} \mid \rho(x) = x\}$, then the subalgebra $\mathcal{L}_{E_6}^\rho$ of $\mathcal{L}_{E_6}$ is a simple Lie algebra of type F_4, denoted by $\mathcal{L}_{F_4}$. The Cartan subalgebra is $\mathcal{H}_{F_4} = \mathcal{H}_{E_6}^\rho = \mathcal{H}_{E_6} \cap \mathcal{L}_{E_6}^\rho$. The restriction of a, c, b and d to $\mathcal{H}_{F_4}$ are the fundamental roots A, B, C and D. Thus the base of F_4 is $\Delta_{F_4} = \{A, B, C, D\}$. Let Φ_{F_4} denote the root system of type F_4, and $\Phi_{F_4}^+$ denote the positive roots of Φ_{F_4}. We have $e_A = e_a + e_f$, $e_B = e_c + e_e$, $e_C = e_d$, $e_D = e_b$, and $e_{-r} = e_r^\top$ for all $r \in \Delta_{F_4}$. Thus $\mathcal{L}_{F_4}$ is generated by $\{e_r \mid \pm r \in \Delta_{F_4}\}$.

Let an automorphism of the Lie algebra $\mathcal{L}_{E_6}$ be determined by

$$h_r \mapsto h_{\rho(r)}, \quad e_r \mapsto e_{\rho(r)}, \quad e_{-r} \mapsto e_{-\rho(r)}, \qquad (r \in \Delta_{E_6})$$

and for every $r \in \Phi_{E_6}$ satisfying $e_r \mapsto \gamma_r e_{\rho(r)}$. We choose the suitable structure constants such that $\gamma_r = 1$ for all $r \in \Delta_{E_6}$. Let $r \in \Phi_{E_6}^+ \setminus \Delta_{E_6}$, and $r = a + b$ where $a, b \in \Phi_{E_6}^+$, $\operatorname{ht}(a) < \operatorname{ht}(r)$ and $\operatorname{ht}(b) < \operatorname{ht}(r)$. Then $\gamma_r = \frac{\gamma_a \gamma_b N_{\rho(a),\rho(b)}}{N_{a,b}}$ (c.f. A.5.11). Thus we obtain γ_r for all $r \in \Phi_{E_6}^+$ as follows:

$r \in \Phi_{E_6}^+$	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	r_9	r_{10}	r_{11}	r_{12}
γ_r	1	1	1	1	1	1	-1	1	-1	-1	-1	1

$r \in \Phi_{E_6}^+$	r_{13}	r_{14}	r_{15}	r_{16}	r_{17}	r_{18}	r_{19}	r_{20}	r_{21}	r_{22}	r_{23}	r_{24}
γ_r	-1	-1	1	1	1	-1	1	1	-1	-1	1	1

$r \in \Phi_{E_6}^+$	r_{25}	r_{26}	r_{27}	r_{28}	r_{29}	r_{30}	r_{31}	r_{32}	r_{33}	r_{34}	r_{35}	r_{36}
γ_r	-1	-1	1	-1	-1	1	-1	1	1	1	1	1

The Chevalley group $E_6(q^2)$ has a field automorphism F_q sending $x_r(t)$ to $x_r(\gamma_r t^q)$, and a graph automorphism ρ sending $x_r(t)$ to $x_{\rho(r)}(t)$ $(r \in \Phi_{E_6})$ (see [Car72, 12.2.3]). Let $F := \rho F_q = F_q\rho$. For a subgroup X of $E_6(q^2)$, we write $X^F := \{x \in X | F(x) = x\}$. Then $E_6(q^2)^F = {}^2E_6(q^2)$. Let $r \in \Phi^+_{E_6}$ and $t \in \mathbb{F}_{q^2}$, we set

$$ {}^2x_{r^1}(t) := \begin{cases} x_r(t) & \text{if } \rho(r) = r,\ t^q = t \\ x_r(t)\cdot x_{\rho(r)}(\gamma_r t^q) & \text{if } \rho(r) \neq r,\ t^{q^2} = t \end{cases}. $$

5.3.1 Definition/Lemma. *Let* $t \in \mathbb{F}_{q^2}$*, we set*

$$
\begin{aligned}
{}^2x_1(t) &:= {}^2x_{r_1^1}(t) = {}^2x_{r_6^1}(t^q) = x_{r_1}(t)x_{r_6}(t^q), & {}^2x_3(t) &:= {}^2x_{r_4^1}(t) = x_{r_4}(t),\\
{}^2x_2(t) &:= {}^2x_{r_3^1}(t) = {}^2x_{r_5^1}(t^q) = x_{r_3}(t)x_{r_5}(t^q), & {}^2x_4(t) &:= {}^2x_{r_2^1}(t) = x_{r_2}(t),\\
{}^2x_5(t) &:= {}^2x_{r_7^1}(t) = {}^2x_{r_{11}^1}(-t^q) = x_{r_7}(t)x_{r_{11}}(-t^q), & {}^2x_7(t) &:= {}^2x_{r_8^1}(t) = x_{r_8}(t),\\
{}^2x_6(t) &:= {}^2x_{r_9^1}(t) = {}^2x_{r_{10}^1}(-t^q) = x_{r_9}(t)x_{r_{10}}(-t^q), & {}^2x_9(t) &:= {}^2x_{r_{15}^1}(t) = x_{r_{15}}(t),\\
{}^2x_8(t) &:= {}^2x_{r_{12}^1}(t) = {}^2x_{r_{16}^1}(t^q) = x_{r_{12}}(t)x_{r_{16}}(t^q), & {}^2x_{13}(t) &:= {}^2x_{r_{19}^1}(t) = x_{r_{19}}(t),\\
{}^2x_{10}(t) &:= {}^2x_{r_{13}^1}(t) = {}^2x_{r_{14}^1}(-t^q) = x_{r_{13}}(t)x_{r_{14}}(-t^q), & {}^2x_{14}(t) &:= {}^2x_{r_{23}^1}(t) = x_{r_{23}}(t),\\
{}^2x_{11}(t) &:= {}^2x_{r_{18}^1}(t) = {}^2x_{r_{21}^1}(-t^q) = x_{r_{18}}(t)x_{r_{21}}(-t^q), & {}^2x_{16}(t) &:= {}^2x_{r_{24}^1}(t) = x_{r_{24}}(t),\\
{}^2x_{12}(t) &:= {}^2x_{r_{17}^1}(t) = {}^2x_{r_{20}^1}(t^q) = x_{r_{17}}(t)x_{r_{20}}(t^q), & {}^2x_{17}(t) &:= {}^2x_{r_{27}^1}(t) = x_{r_{27}}(t),\\
{}^2x_{15}(t) &:= {}^2x_{r_{22}^1}(t) = {}^2x_{r_{25}^1}(-t^q) = x_{r_{22}}(t)x_{r_{25}}(-t^q), & {}^2x_{19}(t) &:= {}^2x_{r_{30}^1}(t) = x_{r_{30}}(t),\\
{}^2x_{18}(t) &:= {}^2x_{r_{26}^1}(t) = {}^2x_{r_{28}^1}(-t^q) = x_{r_{26}}(t)x_{r_{28}}(-t^q), & {}^2x_{22}(t) &:= {}^2x_{r_{34}^1}(t) = x_{r_{34}}(t),\\
{}^2x_{20}(t) &:= {}^2x_{r_{29}^1}(t) = {}^2x_{r_{31}^1}(-t^q) = x_{r_{29}}(t)x_{r_{31}}(-t^q), & {}^2x_{23}(t) &:= {}^2x_{r_{35}^1}(t) = x_{r_{35}}(t),\\
{}^2x_{21}(t) &:= {}^2x_{r_{32}^1}(t) = {}^2x_{r_{33}^1}(t^q) = x_{r_{32}}(t)x_{r_{33}}(t^q), & {}^2x_{24}(t) &:= {}^2x_{r_{36}^1}(t) = x_{r_{36}}(t).
\end{aligned}
$$

The root subgroups of ${}^2E_6(q^2)$ *are determined:*

$$
\begin{aligned}
{}^2X_i &:= \{{}^2x_i(t) \mid t \in \mathbb{F}_{q^2}\} \quad \textit{for all } i \in \{1,2,5,6,8,10,11,12,15,18,20,21\},\\
{}^2X_i &:= \{{}^2x_i(t) \mid t \in \mathbb{F}_{q}\} \quad \textit{for all } i \in \{3,4,7,9,13,14,16,17,19,22,23,24\}.
\end{aligned}
$$

A Sylow p-subgroup of ${}^2E_6(q^2)$ is determined

$$ {}^2E_6^{syl}(q^2) := \left\{ \prod_{i=1}^{24} {}^2x_i(t_i) \,\middle|\, \begin{cases} t_i \in \mathbb{F}_{q^2} & \text{if } i \in \{1,2,5,6,8,10,11,12,15,18,20,21\} \\ t_i \in \mathbb{F}_q & \text{if } i \in \{3,4,7,9,13,14,16,17,19,22,23,24\} \end{cases} \right\}, $$

where the product can be taken in an arbitrary, but fixed, order (c.f. [Car72, 13.6.1]). In particular, $\left|{}^2E_6^{syl}(q^2)\right| = q^{36}$. By [Car72, 14.3.2],

$$ \left|{}^2E_6(q^2)\right| = \frac{1}{(3,q+1)}q^{36}(q^2-1)(q^5+1)(q^6-1)(q^8-1)(q^9+1)(q^{12}-1). $$

5.3.2 Remark. $Y_i \subseteq {}^2X_i$ *for all* $i = 1,2,\ldots,24$*, and* $F_4^{syl}(q) \subseteq {}^2E_6^{syl}(q^2) \subseteq E_6^{syl}(q^2)$.

5.3.3 Corollary (Sylow p-subgroup ${}^2E_6^{syl}\,(q^2)$)**.** *A matrix Sylow p-subgroup ${}^2E_6^{syl}\,(q^2)$ of the Steinberg group ${}^2E_6\,(q^2)$ is determined, and every element of ${}^2E_6^{syl}\,(q^2)$ is expressed as follows:*

$$
\begin{aligned}
&{}^2x_4(t_4)\cdot{}^2x_7(-t_7)\cdot{}^2x_3(t_3)\cdot{}^2x_{16}(t_{16})\cdot{}^2x_{13}(t_{13})\cdot{}^2x_9(-t_9)\cdot{}^2x_{10}(-t_{10})\cdot{}^2x_6(t_6)\cdot x_2(t_2)\\
&\cdot{}^2x_{24}(-t_{24})\cdot{}^2x_{23}(-t_{23})\cdot{}^2x_{22}(-t_{22})\cdot{}^2x_{21}(-t_{21})\cdot{}^2x_{19}(-t_{19})\cdot{}^2x_{20}(t_{20})\\
&\cdot{}^2x_{17}(-t_{17})\cdot{}^2x_{18}(-t_{18})\cdot{}^2x_{14}(t_{14})\cdot{}^2x_{15}(t_{15})\cdot{}^2x_{11}(-t_{11})\cdot{}^2x_{12}(-t_{12})\cdot{}^2x_8(t_8)\cdot{}^2x_5(t_5)\cdot{}^2x_1(t_1)
\end{aligned}
$$

$$
= \begin{pmatrix}
1 & t_1 & t_5 & t_8 & t_{12} & -t_{11} & -t_{15} & \begin{matrix}-t_1^qt_{11}\\-t_5^qt_8\\+t_{14}\end{matrix} & t_{18} & \begin{matrix}-t_1^qt_{15}\\-t_5^qt_{12}\\+t_{17}\end{matrix} & -t_{20} & \begin{matrix}t_1^qt_{18}\\-t_8^qt_{12}\\-t_{19}\end{matrix} & * & \begin{matrix}-t_1^qt_{20}\\+t_{21}\end{matrix} & \begin{matrix}t_5^qt_{18}\\+t_8^qt_{15}\\-t_{11}t_{12}^q\\+t_{21}^q\end{matrix} & * & \begin{matrix}-t_5^qt_{20}\\-t_{11}^qt_{15}\\-t_{22}\end{matrix} & * & \begin{matrix}-t_8^qt_{20}\\+t_{11}^qt_{18}\\+t_{23}\end{matrix} & * & \begin{matrix}-t_{12}^qt_{20}\\+t_{15}^qt_{18}\\-t_{24}\end{matrix} & * & * & * & * & * & *\\
0 & 1 & t_2 & t_6 & t_{10} & \begin{matrix}t_2^qt_6\\-t_9\end{matrix} & \begin{matrix}t_2^qt_{10}\\-t_{13}\end{matrix} & \begin{matrix}t_1^qt_2^qt_6\\-t_1^qt_9\\+t_2t_8^q\\-t_5^qt_6\\+t_{11}^q\end{matrix} & \begin{matrix}t_6^qt_{10}\\+t_{16}\end{matrix} & \begin{matrix}t_1^qt_2^qt_{10}\\-t_1^qt_{13}\\+t_2t_{12}^q\\-t_5^qt_{10}\\+t_{15}^q\end{matrix} & * & \begin{matrix}t_1^qt_6^qt_{10}\\+t_1^qt_{16}\\+t_6t_{12}^q\\-t_8^qt_{10}\\-t_{18}^q\end{matrix} & * & * & \begin{matrix}t_2^qt_6t_{12}^q\\-t_2^qt_8^qt_{10}\\+t_5^qt_6^qt_{10}\\+t_5^qt_{16}\\+t_8^qt_{13}\\-t_9t_{12}^q\\+t_{20}^q\end{matrix} & * & * & * & * & * & * & * & * & * & * & * & *\\
0 & 0 & 1 & t_3 & t_7 & \begin{matrix}t_2^qt_3\\-t_6^q\end{matrix} & \begin{matrix}t_2^qt_7\\-t_{10}^q\end{matrix} & \begin{matrix}t_1^qt_2^qt_3\\-t_1^qt_6^q\\-t_3t_5^q\\+t_8^q\end{matrix} & * & \begin{matrix}t_1^qt_2^qt_7\\-t_1^qt_{10}^q\\-t_5^qt_7\\+t_{12}^q\end{matrix} & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & *\\
0 & 0 & 0 & 1 & t_4 & t_2^q & * & \begin{matrix}t_1^qt_2^q\\-t_5^q\end{matrix} & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & *\\
0 & 0 & 0 & 0 & 1 & 0 & * & 0 & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & *\\
0 & 0 & 0 & 0 & 0 & 1 & * & t_1^q & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & *\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & *\\
\vdots & & & & & & & & & & & \ddots & \ddots & \ddots & & & & & & & & & & & & & \vdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}_{27\times 27}
$$

5.3.4 Corollary (36 variables). *The matrix elements of Sylow p-subgroup ${}^2E_6^{syl}(q^2)$ are written as follows:*

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
1	1	t_1	t_5	t_8	t_{12}	$-t_{11}$	$-t_{15}$	$-t_1^q t_{11} -t_5^q t_8 +t_{14}$	t_{18}	$-t_1^q t_{15} -t_5^q t_{12} +t_{17}$	$-t_{20}$	$t_1^q t_{18} -t_8^q t_{12} -t_{19}$	$*$	$-t_1^q t_{20} +t_{21}$	$t_5^q t_{18} +t_8^q t_{15} -t_{11} t_{12}^q +t_{21}^q$	$*$	$-t_5^q t_{20} -t_{11}^q t_{15} -t_{22}$	$*$	$-t_8^q t_{20} +t_{11}^q t_{18} +t_{23}$	$*$	$-t_{12}^q t_{20} +t_{15}^q t_{18} -t_{24}$	$*$	$*$	$*$	$*$	$*$	$*$
2	0	1	t_2	t_6	t_{10}	$t_2^q t_6 -t_9$	$t_2^q t_{10} -t_{13}$	$*$	$t_6^q t_{10} +t_{16}$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
3	0	0	1	t_3	t_7	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
4	0	0	0	1	t_4	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
5	0	0	0	0	1	0	$*$	0	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
6	0	0	0	0	0	1	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
7	0	0	0	0	0	0	1	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
8	0	0	0	0	0	0	0	1	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
9	0	0	0	0	0	0	0	0	1	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
$\vdots$	$\vdots$											$\ddots$	$\ddots$	$\ddots$													$\vdots$
25	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	$*$	$*$
26	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	t_1
27	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

where $t_i \in \mathbb{F}_{q^2}$ for all $i \in \{1,2,5,6,8,10,11,12,15,18,20,21\}$ and $t_i \in \mathbb{F}_q$ for all $i \in \{3,4,7,9,13,14,16,17,19,22,23,24\}$.

5.3.5 Corollary. *Let $g = (g_{i,j}) \in {}^2E_6^{syl}(q^2)$, then*

$$\begin{cases} g_{13,j_{13}} = 0, & j_{13} \in \{14, 15, 17, 19, 21\} \\ g_{16,j_{16}} = 0, & j_{16} \in \{17, 19, 21\} \\ g_{18,j_{18}} = 0, & j_{18} \in \{19, 21\} \\ g_{20,21} = 0 \\ g_{14,15} = 0 \end{cases}, \qquad \begin{cases} g_{14,17}^q = g_{15,17} = t_2 \\ g_{14,19}^q = g_{15,19} = t_6 \\ g_{14,21}^q = g_{15,21} = t_{10} \end{cases},$$

$$\begin{cases} g_{8,j_8} = 0, & j_8 \in \{9, 11, 13\} \\ g_{10,j_{10}} = 0, & j_{10} \in \{11, 13\} \\ g_{12,13} = 0 \end{cases}, \qquad \begin{cases} g_{8,j_8} \in \mathbb{F}_q, & j_8 \in \{10, 12, 17, 19, 21\} \\ g_{10,j_{10}} \in \mathbb{F}_q, & j_{10} \in \{12, 17, 19, 21\} \\ g_{12,j_{12}} \in \mathbb{F}_q, & j_{12} \in \{17, 19, 21\} \\ g_{17,j_{17}} \in \mathbb{F}_q, & j_{17} \in \{19, 21\} \\ g_{19,21} \in \mathbb{F}_q \\ g_{6,j_6} \in \mathbb{F}_q, & j_6 \in \{7, 9\} \\ g_{7,9} \in \mathbb{F}_q, \\ g_{4,5} \in \mathbb{F}_q \end{cases},$$

and $g_{i_{14},14}^q = g_{i_{14},15}$ for all $i_{14} \in \{8, 10, 12\}$

5.3.2. Monomial ${}^2E_6^{syl}(q^2)$-module

Let $U := {}^2E_6^{syl}(q^2)$ and $V_0 := \mathrm{Mat}_{27\times 27}(\mathbb{F}_{q^2})$. In this subsection, we get a monomial linearisation $(f, \kappa_q|_{V\times V})$ for ${}^2E_6^{syl}(q^2)$, where f is a bijective 1-cocycle and $\kappa_q|_{V\times V}$ is a non-degenerate bilinear form (5.3.31). Then we make $\mathbb{C}\left({}^2E_6^{syl}(q^2)\right)$ into a monomial ${}^2E_6^{syl}(q^2)$-module (5.3.32).

5.3.6 Reminder. *Let* $\square := \{(i,j) \mid 1 \le i, j \le 27\}$ *and* $\nabla := \{(i,j) \mid 1 \le i < j \le 27\}$.

5.3.7 Notation/Lemma. *The map*

$$\kappa\colon V_0 \times V_0 \to \mathbb{F}_{q^2} : (X, Y) \mapsto \mathrm{tr}(X^\top Y)$$

is called the ***trace form****, and κ is a non-degenerate symmetric bilinear form on V_0.*

5.3.8 Notation. *Set*

$$J := \left\{ (i,j) \in \nabla \;\middle|\; \begin{cases} j \in \{2,3,\ldots,11,12,14,15,17,19,21\} & \text{if } i = 1 \\ j \in \{3,4,5,6,7,9\} & \text{if } i = 2 \\ j \in \{4,5\} & \text{if } i = 3 \\ j \in \{5\} & \text{if } i = 4 \end{cases} \right\},$$

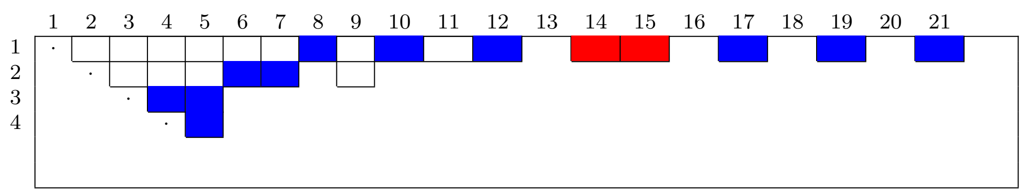
and

$$J' := \left\{ (i,j) \in \nabla \;\middle|\; \begin{cases} j \in \{13,16,18,20\} & \text{if } i = 1 \\ j = 8 & \text{if } i = 2 \end{cases} \right\}.$$

Set

$$J_1 := \left\{ (i,j) \in J \;\middle|\; \begin{cases} j \in \{2,3,4,5,6,7,9,11,14\} & \text{if } i = 1 \\ j \in \{3,4,5\} & \text{if } i = 2 \end{cases} \right\}.$$

and $J_2 := J \backslash (J_1 \dot\cup \{(1,15)\})$, then $|J_1| = |J_2| = 12$ and $J = J_1 \dot\cup J_2 \dot\cup \{(1,15)\}$. J_2 is the set of the coordinates of the blue blocks of the following 27×27-matrix.



5.3.9 Remark. *(1)* $J^c = J \dot\cup J'$,

(2) $\#J = 25$, $\#J' = 5$ *and* $\#J^c = 30$.

5.3.10 Notation/Lemma. *Set*

$$V_J = \bigoplus_{(i,j)\in J} \mathbb{F}_q e_{i,j} = \Big\{ \sum_{(i,j)\in J} A_{i,j}e_{i,j} \in V_0 \,\Big|\, A_{i,j} \in \mathbb{F}_{q^2} \Big\},$$

where

$$\sum_{(i,j)\in J} A_{i,j}e_{i,j} =$$

$$\left(\begin{array}{lllllllllllllllllllll}
\cdot & A_{1,2} & A_{1,3} & A_{1,4} & A_{1,5} & A_{1,6} & A_{1,7} & A_{1,8} & A_{1,9} & A_{1,10} & A_{1,11} & A_{1,12} & 0 & A_{1,14} & A_{1,15} & 0 & A_{1,17} & 0 & A_{1,19} & 0 & A_{1,21} \\
 & \cdot & A_{2,3} & A_{2,4} & A_{2,5} & A_{2,6} & A_{2,7} & 0 & A_{2,9} & & & & & & & & & & & & \\
 & & \cdot & A_{3,4} & A_{3,5} & & & & & & & & & & & & & & & & \\
 & & & \cdot & A_{4,5} & & & & & & & & & & & & & & & & \\
 & & & & & & & & & & & & & & & & & & & &
\end{array}\right)$$

Then V_J is an $\mathbb{F}_{q^2}$-subspace and also an $\mathbb{F}_q$-subspace of V_0 with $\dim_{\mathbb{F}_q} V_J = 50$. In particular, $V_{\square} = V_0$.

5.3.11 Lemma. *Let $\phi_0 \colon \mathbb{F}_{q^2} \to \mathbb{F}_q : t \mapsto t + t^q$, then ϕ_0 is an $\mathbb{F}_q$-epimorphism and $|\mathrm{ker}\phi_0| = q$.*

Proof. We know that ϕ_0 is an $\mathbb{F}_q$-homomorphism. Observe that $\ker\phi_0 = \{t \in \mathbb{F}_{q^2} \mid t + t^q = 0\}$, and that the degree of $t + t^q$ is q, so $|\ker\phi_0| \leq q$. On the other hand, $\operatorname{im}\phi_0 \subseteq \mathbb{F}_q \implies |\operatorname{im}\phi_0| \leq q$. We have $\mathbb{F}_{q^2}/\ker\phi_0 \cong \operatorname{im}\phi_0 \Longrightarrow |\ker\phi_0| = \frac{|\mathbb{F}_{q^2}|}{|\operatorname{im}\phi_0|} \geq q$. Thus, $|\mathrm{ker}\phi_0| = q$ and $\operatorname{im}\phi_0 = \mathbb{F}_q$. Therefore, ϕ_0 is an $\mathbb{F}_q$-epimorphism. $\square$

5.3.12 Proposition. *There exists an element $\eta \in \mathbb{F}_{q^2}\backslash\mathbb{F}_q$ such that $\eta^q + \eta = 1$.*

Proof. By 5.3.11, $\#\{t \in \mathbb{F}_{q^2} \mid t^q + t = 1\} = q > 2$. But $\#\{t \in \mathbb{F}_q \mid t^q + t = 2t = 1\} = 1$, so there exists at least one element of $\mathbb{F}_{q^2}$ such that $t^q + t = 1$. Then the claim is proved. $\square$

From now on in this chapter, we fix an element $\eta \in \mathbb{F}_{q^2}\backslash\mathbb{F}_q$ such that $\eta^q + \eta = 1$.

5.3.13 Notation/Lemma. *Let*

$$\pi_q \colon \mathbb{F}_{q^2} \to \mathbb{F}_q : x \mapsto \phi_0(\eta x) = (\eta x)^q + \eta x,$$

then π_q is an $\mathbb{F}_q$-epimorphism and $\mathbb{F}_{q^2} = \ker\pi_q \oplus \mathbb{F}_q$. In particular, $\pi_q|_{\mathbb{F}_q} = \mathrm{id}_{\mathbb{F}_q}$ and $\pi_q^2 = \pi_q$.

Proof. Since $\eta \neq 0$ and ϕ_0 is an $\mathbb{F}_q$-epimorphism, π_q is an $\mathbb{F}_q$-epimorphism.

Let $x \in \mathbb{F}_q$, then $\pi_q(x) = x\pi_q(1) = x\phi_0(\eta) = x(\eta^q + \eta) = x$, so $\pi_q|_{\mathbb{F}_q} = \mathrm{id}_{\mathbb{F}_q}$. Then $\pi_q^2 = \pi_q$ since the image $\operatorname{im}\pi_q = \mathbb{F}_q$.

The image $\operatorname{im}\pi_q = \mathbb{F}_q$ and the kernel $\ker\pi_q$ are $\mathbb{F}_q$-subspaces of $\mathbb{F}_{q^2}$, so $(\ker\pi_q + \mathbb{F}_q) \subseteq \mathbb{F}_{q^2}$. Let $x \in \ker\pi_q \cap \mathbb{F}_q$, then $x \overset{x\in\mathbb{F}_q}{=} \pi_q(x) \overset{x\in\ker\pi_q}{=} 0$, so $\ker\pi_q \cap \mathbb{F}_q = \{0\}$. Let $x \in \mathbb{F}_{q^2}$, then $\pi_q(x - \pi_q(x)) = (\pi_q - \pi_q^2)(x) = 0$, so $x - \pi_q(x) \in \ker\pi_q$. Hence $x = (x - \pi_q(x)) + \pi_q(x)$. Therefore, $\mathbb{F}_{q^2} = \ker\pi_q \oplus \mathbb{F}_q$. $\square$

5.3.14 Corollary. *Let $x \in \mathbb{F}_{q^2}$, then $\pi_q(xy) = 0$ for all $y \in \mathbb{F}_{q^2}$ if and only if $x = 0$.*

5.3.15 Proposition. *Define a map by*

$$\kappa_q := \pi_q \circ \kappa \colon V_0 \times V_0 \to \mathbb{F}_q,$$

where $\kappa_q(A, B) = \pi_q \circ \kappa(A, B) = \pi_q\big(\mathrm{tr}(A^\top B)\big)$. Then κ_q is a symmetric $\mathbb{F}_q$-bilinear form on V_0.

Proof. We know that κ is a symmetric bilinear map (5.3.7), and that π_q is an $\mathbb{F}_q$-epimorphism (5.3.13), then the claim is proved. □

5.3.16 Notation/Lemma. *Set*

$$V := \{A = (A_{i,j}) \in V_J \mid A_{1,15} = A_{1,14}^q,\ A_{i,j} \in \mathbb{F}_q \text{ for all } (i,j) \in J_2\},$$

where

$A =$

$$\left(\begin{array}{cccccccccccccccccccc}
\cdot & A_{1,2} & A_{1,3} & A_{1,4} & A_{1,5} & A_{1,6} & A_{1,7} & A_{1,8} & A_{1,9} & A_{1,10} & A_{1,11} & A_{1,12} & 0 & A_{1,14} & A_{1,14}^q & 0 & A_{1,17} & 0 & A_{1,19} & 0 & A_{1,21} \\
 & \cdot & A_{2,3} & A_{2,4} & A_{2,5} & A_{2,6} & A_{2,7} & 0 & A_{2,9} & & & & & & & & & & & & \\
 & & \cdot & A_{3,4} & A_{3,5} & & & & & & & & & & & & & & & & \\
 & & & \cdot & A_{4,5} & & & & & & & & & & & & & & & &
\end{array}\right)$$

Then V is a 36-dimensional subspace of V_J over $\mathbb{F}_q$, and $\mathrm{supp}(V) = J$.

5.3.17 Notation/Lemma. *The map*

$$\pi \colon V_0 \to V : A \mapsto \begin{array}{l} \sum_{(i,j)\in J_1\setminus\{(1,14)\}} A_{i,j} e_{i,j} \\ \frac{A_{1,14}\eta^{1-q} + A_{1,15}^q}{1+\eta^{1-q}} e_{1,14} + \left(\frac{A_{1,14}\eta^{1-q} + A_{1,15}^q}{1+\eta^{1-q}}\right)^q e_{1,15} \\ + \sum_{(i,j)\in J_2} \pi_q(A_{i,j}) e_{i,j} \end{array}$$

is an $\mathbb{F}_q$-epimorphism. In particular, $\pi|_V = \mathrm{id}_V$, $\pi^2 = \pi$ and $\pi(I_{27}) = O_{27}$.

5.3.18 Proposition. *Let $V^\perp$ denote the orthogonal complement of the subspace V of V_0 with respect to the trace form κ_q, i.e. $V^\perp := \{B \in V_0 \mid \kappa_q(A, B) = 0,\ \forall\, A \in V\}$, and*

$$\begin{aligned} W &:= \bigoplus_{(i,j)\notin J} \mathbb{F}_{q^2} e_{i,j} + \bigoplus_{(i,j)\in J_2} \ker \pi_q e_{i,j} + \{x e_{1,14} - x^q \eta^{q-1} e_{1,15} \mid x \in \mathbb{F}_{q^2}\} \\ &= \left\{ A = (A_{i,j}) \in V_0 \;\middle|\; \left\{ \begin{array}{ll} A_{i,j} = 0 & \text{if } (i,j) \in J_1\setminus\{(1,14)\} \\ A_{i,j} \in \ker \pi_q & \text{if } (i,j) \in J_2 \\ A_{1,15} = -A_{1,14}^q \eta^{q-1} & \end{array} \right. \right\}. \end{aligned}$$

Then $V^\perp = W$.

Proof. (1) *Claim that $V^\perp \supseteq W$.*

For all $B = (B_{i,j}) \in W$ and for all $A = (A_{i,j}) \in V$, we get

$$\begin{aligned}\kappa_q(A,B) = \pi_q\kappa(A,B) = \pi_q(&\sum_{(i,j)\in J_2}(A_{i,j}B_{i,j}) + A_{1,14}B_{1,14} - A_{14}^qB_{1,14}^q\eta^{q-1})\\ =&\pi_q(A_{1,14}B_{1,14} - A_{1,14}^qB_{1,14}^q\eta^{q-1})\\ =&(A_{1,14}B_{1,14}\eta)^q + A_{1,14}B_{1,14}\eta - \big((A_{1,14}^qB_{1,14}^q\eta^q)^q + A_{1,14}^qB_{1,14}^q\eta^q\big)\\ =&0.\end{aligned}$$

Then $B \in V^\perp$ for all $B \in W$, so $W \subseteq V^\perp$.

(2) *Claim that $V^\perp \subseteq W$.*

For all $B = (B_{i,j}) \in V^\perp \subseteq V_0$ and for all $A = (A_{i,j}) \in V$, then

$$0 = \kappa_q(A,B) = \pi_q\kappa(A,B) = \pi_q(\sum_{(i,j)\in J_1\cup J_2} A_{i,j}B_{i,j} + A_{1,14}^qB_{1,15}).$$

(2.1) *We claim that $B_{i,j} = 0$ for all $(i,j) \in J_1\backslash\{(1,14)\}$.*

Assume that $B_{1,2} \neq 0$. Let $A = A_{1,2}e_{12} = B_{1,2}^{-1}e_{1,2}$, then $0 = \kappa_q(A,B) = \pi_q(A_{12}B_{1,2}) = \pi_q(1) = 1$. This is a contradiction. Thus $B_{1,2} = 0$.

(2.2) *We claim $B_{i,j} \in \ker\pi_q$ for all $(i,j) \in J_2$.*

Let $A = A_{1,8}e_{1,8} = e_{1,8}$, then $0 = \kappa_q(A,B) = \pi_q(A_{1,8}B_{1,8}) = \pi_q(B_{1,8})$. Thus $B_{1,8} \in \ker\pi_q$.

(2.3) *We claim that $B_{1,15} = -B_{1,14}^q\eta^{q-1}$.*

Assume $B_{1,15} \neq -B_{1,14}^q\eta^{q-1}$, i.e. $B_{1,14}^q\eta^q + B_{1,15}\eta \neq 0$. Let $A = A_{1,14}e_{1,14} + A_{1,14}^qe_{1,15}$ $(A_{1,14} \in \mathbb{F}_{q^2})$, then

$$\begin{aligned}0 =&\kappa_q(A,B) = \pi_q(A_{1,14}B_{1,14} + A_{1,14}^qB_{1,15})\\ =&\big((A_{1,14}B_{1,14}\eta)^q + (A_{1,14}B_{1,14}\eta)\big) + \big((A_{1,14}^qB_{1,15}\eta)^q + (A_{1,14}^qB_{1,15}\eta)\big)\\ =&A_{1,14}(B_{1,14}\eta + B_{1,15}^q\eta^q) + A_{1,14}^q(B_{1,14}\eta + B_{1,15}^q\eta^q)^q\\ =&\phi_0\big(A_{1,14}(B_{1,14}\eta + B_{1,15}^q\eta^q)\big).\end{aligned}$$

Thus $\ker\phi_0 = \mathbb{F}_{q^2}$ since $B_{14}^{q^2}\eta^{q^2} + B_{15}\eta \neq 0$. So $|\ker\phi_0| = q^2 > q$. This is a contradiction since $|\ker\phi_0| = q$. Thus $B_{1,15} = -B_{1,14}^q\eta^{q-1}$.

Hence $B \in W$ and $V^\perp \subseteq W$.

Therefore, $V^\perp = W$. $\square$

5.3.19 Corollary. *$\kappa_q|_{V\times V}$ is a non-degenerate $\mathbb{F}_q$-bilinear form.*

5.3.20 Corollary. *$V_0 = V \oplus V^\perp$, and $\pi\colon V_0 = V \oplus V^\perp \to V$ is the projection to the first component V.*

Proof. We know that $V + V^\perp \subseteq V_0$ since V and $V^\perp$ are $\mathbb{F}_q$-subspaces of V_0. Let $A \in V \cap V^\perp$, then $\kappa_q(A, B) \overset{A\in V^\perp}{=} 0 \overset{A\in V}{=} \kappa_q|_{V\times V}(A, B)$ for all $B \in V$. We get $A = 0$ since $\kappa_q|_{V\times V}$ is non-degenerate by 5.3.19. Thus $V \cap V^\perp = \{0\}$. Let $A \in V_0$, then $A = \pi(A) + (A - \pi(A))$ and $\pi(A) \in V$. It is enough to show that $A - \pi(A) \in V^\perp$. Let $B = (B_{i,j}) = \pi(A) \in V$ and $C = (C_{i,j}) = A - \pi(A)$, by 5.3.13 $\mathbb{F}_{q^2} = \ker \pi_q \oplus \mathbb{F}_q$, thus it is sufficient to prove that $C_{1,15} = -C_{1,14}^q \eta^{q-1}$. We have

$$\begin{aligned} C_{1,14} =& A_{1,14} - \frac{A_{1,14}\eta^{1-q} + A_{1,15}^q}{1+\eta^{1-q}} = \frac{A_{1,14} - A_{1,15}^q}{1+\eta^{1-q}}, \\ C_{1,15} =& A_{1,15} - \left(\frac{A_{1,14}\eta^{1-q} + A_{1,15}^q}{1+\eta^{1-q}}\right)^q = \frac{A_{1,15} - A_{1,14}^q}{1+\eta^{q-1}}\eta^{q-1} = -C_{1,14}^q\eta^{q-1}. \end{aligned}$$

Hence $V_0 = V \oplus V^\perp$. $\square$

5.3.21 Lemma. *$\pi = \pi \circ \pi_J$. In particular, let $A \in V_0$ and $\pi_J(A) \in V$, then $\pi(A) = \pi_J(A)$.*

5.3.22 Corollary. *Suppose $A, B \in V_0$, such that $\operatorname{supp}(A) \cap \operatorname{supp}(B) \subseteq J$ and $\pi_J(A) \in V$. Then*

$$\kappa_q(A, B) = \kappa_q(\pi(A), B) = \kappa_q(A, \pi(B)) = \kappa_q(\pi(A), \pi(B)) = \kappa_q|_{V\times V}(\pi(A), \pi(B)).$$

Proof. c.f. the proof of 2.2.21. $\square$

5.3.23 Lemma. *Let $\pi_J\colon V_0 \to V_J : A \mapsto \sum_{(i,j)\in J} A_{ij}e_{ij}$, then*

$$\operatorname{supp}(Ag^\top) \cap \nabla \subseteq J \qquad \textit{for all } A \in V,\ g \in U,$$

and $\pi_{J^c}(Ag^\top) \in V$. In particular, $\pi_J(Ag^\top) = \pi(Ag^\top)$.

Proof. Let $A \in V$, $g \in U$. It is sufficient to prove that

$$\begin{cases} (Ag^\top)_{i,j} = 0, & (i,j) \in J^c\backslash J = J', \\ (Ag^\top)_{1,15} = (Ag^\top)_{1,14}^q, & \\ (Ag^\top)_{ij} \in \mathbb{F}_q, & (i,j) \in J_2. \end{cases}$$

(1) By 5.3.5, we have

$$\begin{aligned} (Ag^\top)_{1,13} =& \sum_{j=1}^{27} A_{1,j}g_{j,13}^\top = \sum_{j=2}^{21} A_{1,j}g_{j,13}^\top = \sum_{j=2}^{21} A_{1,j}g_{13,j} = \sum_{j=13}^{21} A_{1,j}g_{13,j} \\ \overset{\substack{A_{1,j}=0 \\ j\in\{13,16,18,20\}}}{=}& \sum_{j\in\{14,15,17,19,21\}} A_{1,j}g_{13,j} \overset{\substack{g_{13,j}=0 \\ j\in\{14,15,19,21\}}}{=} 0. \end{aligned}$$

Similarly, $(Ag^\top)_{i,j} = 0$ for all $(i,j) \in J' = \{(1,16), (1,18), (1,20), (2,8)\}$.

(2) By 5.3.5,

$$
\begin{aligned}
(Ag^\top)^q_{1,14} =& \Big(\sum_{j=1}^{27} A_{1,j} g^\top_{j,14}\Big)^q = \Big(\sum_{j=2}^{21} A_{1,j} g^\top_{j,14}\Big)^q = \Big(\sum_{j=2}^{21} A_{1,j} g_{14,j}\Big)^q \\
=& \Big(\sum_{j=14}^{21} A_{1,j} g_{14,j}\Big)^q \overset{\substack{A_{1,j}=0 \\ j\in\{13,16,18,20\}}}{=} \Big(\sum_{j\in\{14,15,17,19,21\}} A_{1,j} g_{14,j}\Big)^q \\
=& (A_{1,14} + A_{1,15} g_{14,15} + A_{1,17} g_{14,17} + A_{1,19} g_{14,19} + A_{1,21} g_{14,21})^q \\
\overset{g_{14,15}=0}{=}& A^q_{1,14} + A^q_{1,17} g^q_{14,17} + A^q_{1,19} g^q_{14,19} + A^q_{1,21} g^q_{14,21} \\
\overset{\substack{g^q_{14,j}=g_{15,j} \\ j\in\{17,19,21\}}}{=}& A_{1,15} + A_{1,17} g_{15,17} + A_{1,19} g_{15,19} + A_{1,21} g_{15,21} = (Ag^\top)_{1,15}.
\end{aligned}
$$

Thus, $(Ag^\top)^q_{1,14} = (Ag^\top)_{1,15}$.

(3) By 5.3.5,

$$
\begin{aligned}
(Ag^\top)_{1,10} =& \sum_{j=1}^{27} A_{1,j} g^\top_{j,10} = \sum_{j=2}^{21} A_{1,j} g^\top_{j,10} = \sum_{j=2}^{21} A_{1,j} g_{10,j} = \sum_{j=10}^{21} A_{1,j} g_{10,j} \\
\overset{A\in V}{=}& A_{1,10} + A_{1,11} g_{10,11} + A_{1,12} g_{10,12} + (A_{1,14} g_{10,14} + A^q_{1,14} g_{10,15}) \\
&+ A_{1,17} g_{10,17} + A_{1,19} g_{10,19} + A_{1,21} g_{10,21} \\
\overset{5.3.5}{\in}& \mathbb{F}_q,
\end{aligned}
$$

Similarly, $(Ag^\top)_{ij} \in \mathbb{F}_q$ for all $(i,j) \in J_2$.

Hence $\mathrm{supp}(Ag^\top) \cap \,\diagdown\!\!\!\rceil \subseteq J$ and $\pi_{J^c}(Ag^\top) = \pi_J(Ag^\top) \in V$. □

5.3.24 Corollary. *Let* $A, B \in V$ *and* $g, h \in U$*, then* $\mathrm{supp}(Bh^\top) \cap \mathrm{supp}(Ag) \subseteq J$.

5.3.25 Proposition (Group action of ${}^2E_6^{syl}(q^2)$ on V)**.** *The map*

$$- \circ - : V \times U \to V : (A, g) \mapsto A \circ g := \pi(Ag)$$

is a group action, and the elements of the group U *act as* $\mathbb{F}_q$*-automorphisms.*

Proof. c.f. the proof of 2.2.25. □

By 1.2.21, we get a new action:

5.3.26 Corollary. *There exists an unique linear action* $-.-$ *of* G *on* V*:*

$$-.- : V \times G \to V : (A, g) \mapsto A.g := A \circ g^{-\top}$$

such that $\kappa_q|_{V\times V}(A.g, B) = \kappa_q|_{V\times V}(A, B \circ g^{-1})$ *for all* $B \in V$*. In particular, for all* $A, B \in V$ *and* $g \in G$*,* $\kappa_q|_{V\times V}(A, B \circ g) = \kappa_q|_{V\times V}(A.g^{-1}, B) = \kappa_q|_{V\times V}(A \circ g^\top, B)$*.*

5.3.27 Notation. $f := \pi|_U : U \to V$.

5.3.28 Proposition. *$f|_U : U \to V$ is bijective.*

Proof. By 5.3.3 and 5.3.16, we obtain that $f : U \to V$ is bijective. □

5.3.29 Lemma. *Let $x, g \in U$, then $f(x)g \equiv (x-1)g \mod V^\perp$. In particular, $f(x) \equiv x - 1 \mod V^\perp$.*

Proof. c.f. the proof of 2.2.29. □

5.3.30 Proposition. *Let $x, g \in U$, then $f(xg) = f(x) \circ g + f(g)$.*

Proof. c.f. the proof of 2.2.30. □

By 5.3.28 and 5.3.30, we obtain a bijective 1-cocycle of ${}^2E_6^{syl}(q^2)$.

5.3.31 Corollary (Monomial linearisation for ${}^2E_6^{syl}(q^2)$)**.** *The map f is a bijective 1-cocycle of ${}^2E_6^{syl}(q^2)$, and $(f, \kappa_q|_{V\times V})$ is a monomial linearisation for $U = {}^2E_6^{syl}(q^2)$.*

Now we obtain the monomial U-module $\mathbb{C}\left({}^2E_6^{syl}(q^2)\right)$, which is essential for the construction of the supercharacter theory for ${}^2E_6^{syl}(q^2)$.

5.3.32 Theorem (Fundamental theorem for ${}^2E_6^{syl}(q^2)$)**.** *Let $U = {}^2E_6^{syl}(q^2)$ and*

$$[A] = \frac{1}{|U|} \sum_{u \in U} \overline{\chi_A(u)} u \qquad \textit{for all } A \in V.$$

Then the set $\{[A] \mid A \in V\}$ forms a $\mathbb{C}$-basis for the complex group algebra $\mathbb{C}U$. The operation of U on $\mathbb{C}U$ is given by the usual right multiplication, i.e.

$$[A]u = \frac{1}{|U|} \sum_{y \in U} \overline{\chi_A(y)} yu = \chi_{A.u}(u)[A.u] = \vartheta\kappa_q(A.u, f(u))[A.u] \quad \textit{for all } u \in U, A \in V,$$

then $\mathbb{C}U$ is a monomial $\mathbb{C}U$-module.

Proof. By 5.3.31, $(f, \kappa_q|_{V\times V})$ is a monomial linearisation for U, satisfing f is bijective. By 5.3.26, $A.u := \pi(Au^{-\top})$. Thus the whole theorem is proved in view of 1.2.24. □

5.4. Monomial $E_7^{syl}(q)$-module

In this section, we construct a Chevalley basis for the Lie algebra $\mathcal{L}_{E_7}$ of type E_7 (see 5.4.3) and determine the Sylow p-subgroup $E_7^{syl}(q)$ for the Chevalley group $E_7(q)$ (see 5.4.25). Then we get a monomial linearisation $(f, \kappa|_{V\times V})$ for $E_7^{syl}(q)$, where f is a bijective 1-cocycle and $\kappa|_{V\times V}$ is a non-degenerate bilinear form (5.4.44). Finally, we make $\mathbb{C}E_7^{syl}(q)$ into a monomial $E_7^{syl}(q)$-module (5.4.45).

5.4.1. Chevalley basis of the Lie algebra of type E_7

In this subsection, we determine a Chevalley basis for the Lie algebra $\mathcal{L}_{E_7}$ of type E_7 (see 5.4.3)

By [HRT01, §3.2], there is a Lie algebra $\mathcal{L}_{E_7}$ of type E_7 generated by $\{e_r \mid \pm r \in \Delta_{E_7}\}$, denoted by

$$\mathcal{L}_{E_7} := \langle\, e_r \in \mathrm{Mat}_{56\times 56}(\mathbb{C}) \mid \pm r \in \Delta_{E_7} \,\rangle_{\text{Lie}}\,,$$

where $\Delta_{E_7} = \{a, b, c, d, e, f, g\}$, $e_{-r} = e_r^{\top}$ for all $r \in \Delta_{E_7}$ and

$$\begin{aligned}
e_a :=&e_{7,8} + e_{9,10} + e_{11,12} + e_{13,15} + e_{16,18} + e_{19,22}\\
&+ e_{35,38} + e_{39,41} + e_{42,44} + e_{45,46} + e_{47,48} + e_{49,50},\\
e_b :=&e_{5,6} + e_{7,9} + e_{8,10} + e_{20,23} + e_{24,26} + e_{27,29}\\
&+ e_{28,30} + e_{31,33} + e_{34,37} + e_{47,49} + e_{48,50} + e_{51,52},\\
e_c :=&e_{5,7} + e_{6,9} + e_{12,14} + e_{15,17} + e_{18,21} + e_{22,25}\\
&+ e_{32,35} + e_{36,39} + e_{40,42} + e_{43,45} + e_{48,51} + e_{50,52},\\
e_d :=&e_{4,5} + e_{9,11} + e_{10,12} + e_{17,20} + e_{21,24} + e_{25,28}\\
&+ e_{29,32} + e_{33,36} + e_{37,40} + e_{45,47} + e_{46,48} + e_{52,53},,\\
e_e :=&e_{3,4} + e_{11,13} + e_{12,15} + e_{14,17} + e_{24,27} + e_{26,29}\\
&+ e_{28,31} + e_{30,33} + e_{40,43} + e_{42,45} + e_{44,46} + e_{53,54},\\
e_f :=&e_{2,3} + e_{13,16} + e_{15,18} + e_{17,21} + e_{20,24} + e_{23,26}\\
&+ e_{31,34} + e_{33,37} + e_{36,40} + e_{39,42} + e_{41,44} + e_{54,55},\\
e_g :=&e_{1,2} + e_{16,19} + e_{18,22} + e_{21,25} + e_{24,28} + e_{26,30}\\
&+ e_{27,31} + e_{29,33} + e_{32,36} + e_{35,39} + e_{38,41} + e_{55,56}.
\end{aligned}$$

Let $\mathcal{H}_{E_7}$ be the Cartan subalgebra, then

$$\mathcal{H}_{E_7} = \langle\, [e_r, e_{-r}] \in \mathrm{Mat}_{56\times 56}(\mathbb{C}) \,|\, r \in \Delta_{E_7} \,\rangle_{\mathbb{C}}\,.$$

Let $\mathcal{H}_{E_7}^*$ be the dual space of $\mathcal{H}_{E_7}$, we obtain a basis $\{\varepsilon_1 + \varepsilon_8, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7\}$ of

$\mathcal{H}_{E_7}^*$ such that

$$a = -\frac{1}{2}\sum_{i=1}^{8}\varepsilon_i,\ b = \varepsilon_6 - \varepsilon_7,\ c = \varepsilon_6 + \varepsilon_7,\ d = \varepsilon_5 - \varepsilon_6,$$
$$e = \varepsilon_4 - \varepsilon_5,\quad f = \varepsilon_3 - \varepsilon_4,\ g = \varepsilon_2 - \varepsilon_3.$$

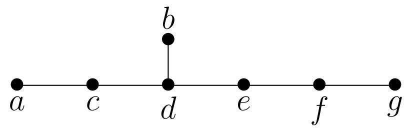
Let $\mathcal{V}_{E_7} := \mathbb{R}\text{-span}\{\varepsilon_1+\varepsilon_8, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7\}$ be a subspace of $\mathcal{H}_{E_7}^*$, then $\mathcal{V}_{E_7}$ becomes a Euclidean space (see Appendix A.3). The root system of type E_7 is

$$\begin{aligned}\Phi_{E_7} =& \{\pm\varepsilon_i \pm \varepsilon_j \mid i<j,\ i,j = 2,3,4,5,6,7\} \bigcup \{\pm(\varepsilon_1+\varepsilon_8)\} \\ & \bigcup \left\{ \pm\frac{1}{2}\sum_{i=1}^{8}\epsilon_1\varepsilon_i \,\middle|\, \epsilon_i = \pm 1,\ \epsilon_1 = \epsilon_8 = 1,\ \prod_{i=1}^{8}\epsilon_i = 1 \right\}.\end{aligned}$$

The set of positive roots of Φ_{E_7} is

$$\begin{aligned}\Phi_{E_7}^+ =& \{\varepsilon_i \pm \varepsilon_j \mid i<j,\ i,j = 2,3,4,5,6,7\} \bigcup \{-(\varepsilon_1+\varepsilon_8)\} \\ & \bigcup \left\{ -\frac{1}{2}\sum_{i=1}^{8}\epsilon_1\varepsilon_i \,\middle|\, \epsilon_i = \pm 1,\ \epsilon_1 = \epsilon_8 = 1,\ \prod_{i=1}^{8}\epsilon_i = 1 \right\}.\end{aligned}$$

Then $\Delta_{E_6} = \{a,b,c,d,e,f\}$ is a base of type E_6. Denote Φ_{E_6} by the root system of type E_6, and denote $\Phi_{E_6}^+$ by the positive roots of Φ_{E_6}. The Dynkin diagram of E_7 is



Let $r_1 := a,\ r_2 := b,\ r_3 := c,\ r_4 := d,\ r_5 := e,\ r_6 := f$ and $r_{37} = g$, then the Cartan matrix M of E_7 is

$$\begin{array}{c c} & \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ g \end{array} & \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}_{7\times 7} \end{array}$$

where $M_{i,j} = \langle r_i, r_j\rangle = \frac{2\kappa(r_i,r_j)}{\kappa(r_i,r_i)}$ $(r_i, r_j \in \Delta_{E_7})$.

Let Φ_{E_7} be the root system of type E_7. Each root $r \in \Phi_{E_7}$ can be written as $r = \sum_{\alpha\in\Delta_{E_7}} k_\alpha \alpha$, the height of r is $\mathrm{ht}(r) = \sum_{\alpha\in\Delta_{E_7}} k_\alpha$.

5.4.1 Definition. *For $r, s \in \Phi_{E_7}^+$, we write $r \prec s$, if* $\mathrm{ht}(r) < \mathrm{ht}(s)$ *(*e.g. $a \prec a+c$*) or if* $\mathrm{ht}(r) = \mathrm{ht}(s)$*, this is the lexicographical order (*e.g. $a \prec b \prec c \prec d \prec e \prec f \prec g$*).*

We take the structure constants $N_{r,s}$ to be positive, if (r, s) is an extraspecial pair according to the total order in Definition 5.4.1. We give all the extraspecial pairs and the corresponding structure constants as follows.

5.4.2 Notation/Lemma. *We have the following extraspecial pairs of roots in $\Phi_{E_7}^+ \backslash \Delta_{E_7}$ and the corresponding structure constants $N_{r,s}$ $(r, s \in \Phi_{E_7}^+)$. If $r \in \Phi_{E_6}^+$, see 5.1.2. If $r \in \Phi_{E_7}^+ \backslash \Phi_{E_6}^+$, see the following table. According to column 2 and column 3 of the following two tables, we define r_i $(i = 1, 2, \ldots, 63)$ to be the root in column 3 which is according to i in column 2 (e.g. $r_7 = a + c$).*

Height	i	*Positive Roots* $r_i \in \Phi_{E_7}^+ \backslash \Phi_{E_6}^+$	*Extraspecial Pair* (r, s)	$N_{r,s}$
1	37	g		
2	38	$f+g$	(f, g)	1
3	39	$e+f+g$	$(e, f+g)$	1
4	40	$d+e+f+g$	$(d, e+f+g)$	1
5	41	$b+d+e+f+g$	$(b, d+e+f+g)$	1
5	42	$c+d+e+f+g$	$(c, d+e+f+g)$	1
6	43	$a+c+d+e+f+g$	$(a, c+d+e+f+g)$	1
6	44	$b+c+d+e+f+g$	$(b, c+d+e+f+g)$	1
7	45	$a+b+c+d+e+f+g$	$(a, b+c+d+e+f+g)$	1
7	46	$b+c+2d+e+f+g$	$(d, b+c+d+e+f+g)$	1
8	47	$a+b+c+2d+e+f+g$	$(a, b+c+2d+e+f+g)$	1
8	48	$b+c+2d+2e+f+g$	$(e, b+c+2d+e+f+g)$	1
9	49	$a+b+2c+2d+e+f+g$	$(c, a+b+c+2d+e+f+g)$	1
9	50	$a+b+c+2d+2e+f+g$	$(a, b+c+2d+2e+f+g)$	1
9	51	$b+c+2d+2e+2f+g$	$(f, b+c+2d+2e+f+g)$	1
10	52	$a+b+2c+2d+2e+f+g$	$(c, a+b+c+2d+2e+f+g)$	1
10	53	$a+b+c+2d+2e+2f+g$	$(a, b+c+2d+2e+2f+g)$	1
11	54	$a+b+2c+3d+2e+f+g$	$(d, a+b+2c+2d+2e+f+g)$	1
11	55	$a+b+2c+2d+2e+2f+g$	$(c, a+b+c+2d+2e+2f+g)$	1
12	56	$a+2b+2c+3d+2e+f+g$	$(b, a+b+2c+3d+2e+f+g)$	1
12	57	$a+b+2c+3d+2e+2f+g$	$(d, a+b+2c+2d+2e+2f+g)$	1
13	58	$a+2b+2c+3d+2e+2f+g$	$(b, a+b+2c+3d+2e+2f+g)$	1
13	59	$a+b+2c+3d+3e+2f+g$	$(e, a+b+2c+3d+2e+2f+g)$	1
14	60	$a+2b+2c+3d+3e+2f+g$	$(b, a+b+2c+3d+3e+2f+g)$	1
15	61	$a+2b+2c+4d+3e+2f+g$	$(d, a+2b+2c+3d+3e+2f+g)$	1
16	62	$a+2b+3c+4d+3e+2f+g$	$(c, a+2b+2c+4d+3e+2f+g)$	1
17	63	$2a+2b+3c+4d+3e+2f+g$	$(a, a+2b+3c+4d+3e+2f+g)$	1

By 5.4.2, we determine a Chevalley basis of the Lie algebra $\mathcal{L}_{E_7}$.

5.4.3 Lemma (Chevalley basis of $\mathcal{L}_{E_7}$)**.** *For* $r \in \Delta_{E_7}$*, set* $h_r := [e_r, e_{-r}]$*. Let* $r \in \Phi_{E_7}^+ \backslash \Delta_{E_7}$*, and* (s_1, s_2) *the extraspecial pair of* r*, set*

$$e_r := \frac{1}{N_{s_1,s_2}}[e_{s_1}, e_{s_2}], \quad e_{-r} := \frac{1}{N_{-s_1,-s_2}}[e_{-s_1}, e_{-s_2}].$$

Then $\{h_r \in \mathrm{Mat}_{56\times 56}(\mathbb{C}) \mid r \in \Delta_{E_7}\} \cup \{e_{\pm r} \in \mathrm{Mat}_{56\times 56}(\mathbb{C}) \mid r \in \Phi_{E_7}^+\}$ *is a Chevalley basis of the Lie algebra* $\mathcal{L}_{E_7}$*.*

Proof. c.f. the proof of 5.1.3. □

5.4.4 Example.

$$\begin{aligned} e_{a+c} =& \frac{1}{N_{a,c}}[e_a, e_c] = [e_a, e_c] \\ =& - e_{5,8} - e_{6,10} + e_{11,14} + e_{13,17} + e_{16,21} + e_{19,25} \\ & - e_{32,38} - e_{36,41} - e_{40,44} - e_{43,46} + e_{47,51} + e_{49,52}. \end{aligned}$$

5.4.2. Sylow p-subgroup $E_7^{syl}(q)$

Let K be a field. In this subsection, we obtain the Sylow p-subgroup $E_7^{syl}(q)$ for the Chevalley group $E_7(q)$ (see 5.4.25).

5.4.5 Notation. *Let* $\square := \{(i,j) \mid 1 \le i, j \le 56\}$ *and* $\triangledown := \{(i,j) \mid 1 \le i < j \le 56\}$.

5.4.6 Reminder. $\mathcal{L}_{E_7}$ *is the Lie algebra of type* E_7 *over the field* $\mathbb{C}$.

5.4.7 Corollary. *For all* $r \in \Phi_{E_7}$, $e_r^2 = 0$.

Proof. By 5.4.3 and calculation, we get the result. $\square$

Similar to §11.3 of [Car72], the coefficients of $\exp(te_r) = I_{56} + te_r$ are all of the form ± 1 or $\pm t$ (i.e. the non-zero entries of $\exp(te_r)$ are of the form ± 1 and $\pm t$). This fact enables us to transfer to an arbitrary field. For each matrix e_r in the above representation and each element t in an arbitrary field K, $\exp(te_r)$ is a well-defined non-singular matrix over K.

5.4.8 Notation. *Set*

$$\begin{aligned} \bar{x}_{r,K}(t) &:= \exp(te_r) = I_{56} + t \cdot e_r, \quad r \in \Phi_{E_7},\ t \in K, \\ x_{r,K}(t) &:= \exp(t \operatorname{ad} e_r), \quad r \in \Phi_{E_7},\ t \in K. \end{aligned}$$

5.4.9 Notation. *Set a matrix group*

$$\bar{E}_7(K) := \langle\, \bar{x}_{r,K}(t) \mid r \in \Phi_{E_7},\ t \in K \,\rangle\,,$$

and the Chevalley group of type $\mathcal{L}_{E_7}$ *over the field* K *is*

$$E_7(K) := \langle\, x_{r,K}(t) \mid r \in \Phi_{E_7},\ t \in K \,\rangle\,.$$

5.4.10 Notation. *Set subgroups of* $\bar{E}_7(K)$

$$\bar{X}_{r,K} := \langle\, \bar{x}_{r,K}(t) \mid t \in K \,\rangle \qquad \textit{for all } r \in \Phi_{E_7},$$

and root subgroups of the Chevalley group $E_7(K)$ *are*

$$X_{r,K} := \langle\, x_{r,K}(t) \mid t \in K \,\rangle \qquad \textit{for all } r \in \Phi_{E_7}.$$

By [Car72, 4.5.1], $(\exp(t \operatorname{ad} e_r))\,(x) = \exp(te_r) \cdot x \cdot \exp(te_r)^{-1}$ for all $x \in \mathcal{L}_{E_7}$. Similar, we get a group epimorphism between the matrix group $\bar{E}_7(K)$ and the Chevalley group $E_7(K)$ of type $\mathcal{L}_{E_7}$ over the field K as follows.

5.4.11 Proposition. *Let* K *be a field, and*

$$\sigma\colon \bar{E}_7(K) \to E_7(K) : \exp(te_r) \mapsto \exp(t \operatorname{ad} e_r),$$

then σ *is a group epimorphism, and the kernel* $\ker \sigma$ *is the centre* $\bar{Z}$ *of* $\bar{E}_7(K)$.

5.4.12 Notation/Lemma. *Set*

$$\bar{U}_K := \left\langle \bar{x}_{r,K}(t) \,\middle|\, r \in \Phi_{E_7}^+,\ t \in K \right\rangle \leqslant \bar{E}_7(K),$$
$$U_K := \left\langle x_{r,K}(t) \,\middle|\, r \in \Phi_{E_7}^+,\ t \in K \right\rangle \leqslant E_7(K).$$

5.4.13 Corollary. *Let K be a field, and*

$$\sigma_{\bar{U}_K} : \bar{U}_K \to U_K : \exp(te_r) \mapsto \exp(t \operatorname{ad} e_r),$$

then σ is a group epimorphism.

By calculation, we get the following commutators.

5.4.14 Proposition (Commutators of $\bar{U}_K$)**.** *Let $r_i, r_j \in \Phi_{E_7}^+$, then*

$$[\bar{y}_{r_j,K}(t_j), \bar{y}_{r_i,K}(t_i)] = \prod_{\substack{k,l>0 \\ kr_i+lr_j \in \Phi_{E_7}^+}} \bar{x}_{kr_i+lr_j,K}(C_{klr_ir_j}(-t_i)^k t_j^l).$$

Actually, let $kr_i + lr_j \in \Phi_{E_7}^+$ with $k, l > 0$, then $kr_i + lr_j = r_i + r_j$ and

$$[\bar{y}_{r_j,K}(t_j), \bar{y}_{r_i,K}(t_i)] = \bar{x}_{r_i+r_j,K}(-C_{11r_ir_j} t_i t_j) = \bar{x}_{r_i+r_j,K}(N_{r_jr_i} t_i t_j).$$

5.4.15 Lemma. *Let $r, s \in \Phi_{E_7}^+$ and $r \neq s$, then*

$$\overline{\operatorname{supp}}(\bar{X}_{r,K}) \cap \overline{\operatorname{supp}}(\bar{X}_{s,K}) = \overline{\operatorname{supp}}(e_r) \cap \overline{\operatorname{supp}}(e_s) = \emptyset.$$

Proof. By 1.1.8, we have $\overline{\operatorname{supp}}(\bar{X}_{r,K}) \cap \overline{\operatorname{supp}}(\bar{X}_{s,K}) = \overline{\operatorname{supp}}(e_r) \cap \overline{\operatorname{supp}}(e_s)$. By 5.4.3, $\overline{\operatorname{supp}}(e_r) \cap \overline{\operatorname{supp}}(e_s) = \emptyset$. □

The following property is well known.

5.4.16 Lemma. *For $(i, j) \in \triangledown$ and $t \in \mathbb{F}_q$, let $\tilde{x}_{i,j}(t) := I_{56} + te_{i,j}$. Then*

$$A_{56}(\mathbb{F}_q) = \left\{ \prod_{(i,j) \in \triangledown} \tilde{x}_{i,j}(t_{i,j}) \,\middle|\, t_{i,j} \in \mathbb{F}_q \right\},$$

where the product can be taken in an arbitrary, but fixed, order. In particular,

$$\prod_{(i,j) \in \triangledown} \tilde{x}_{i,j}(t_{i,j}) = \prod_{(i,j) \in \triangledown} \tilde{x}_{i,j}(s_{i,j}) \Longleftrightarrow t_{i,j} = s_{i,j} \text{ for all } (i,j) \in \triangledown.$$

5.4.17 Proposition. *Each element of $\bar{U}_{\mathbb{F}_q}$ is uniquely expressible in the form*

$$\prod_{r_i \in \Phi_{E_7}^+} \bar{x}_{r_i,\mathbb{F}_q}(t_i),$$

where the product is taken over all positive roots in the increasing order.

Proof. c.f. the proof of 5.4.17. □

5.4.18 Corollary. $|\bar{U}_{\mathbb{F}_q}| = q^{|\Phi^+_{E_7}|} = q^{63}$.

5.4.19 Corollary. $\bar{U}_{\mathbb{F}_q} \cong U_{\mathbb{F}_q}$.

Proof. c.f. the proof of 5.1.19. □

In the remainder of this section, we consider $K = \mathbb{F}_q$.

5.4.20 Notation/Lemma (Root subgroups of $\bar{E}_7(q)$)**.** *Let*

$$\begin{aligned} x_r(t) &:= \exp(te_r) = I_{56} + t \cdot e_r = \bar{x}_{r,\mathbb{F}_q}(t), \quad r \in \Phi_{E_7},\ t \in \mathbb{F}_q, \\ x_i(t) &:= \exp(te_{r_i}) = I_{56} + t \cdot e_{r_i} = x_{r_i}(t), \quad i = 1, 2, \dots, 63,\ t \in \mathbb{F}_q. \end{aligned}$$

Then the root subgroups of $\bar{E}_7(q)$ are

$$\begin{aligned} X_r &:= \{x_r(t) \,|\, t \in \mathbb{F}_q\} = \bar{X}_{r,\mathbb{F}_q}, \quad r \in \Phi_{E_7}, \\ X_i &:= \{x_{r_i}(t) \,|\, t \in \mathbb{F}_q\} = X_{r_i}, \quad i = 1, 2, \dots, 63. \end{aligned}$$

5.4.21 Notation. *Set*

$$U := \left\langle x_r(t) \,\middle|\, r \in \Phi^+_{E_7},\ t \in \mathbb{F}_q \right\rangle = \bar{U}_{\mathbb{F}_q}.$$

5.4.22 Lemma ([Car65])**.** *The order of $E_7(q)$ is*

$$|E_7(q)| = \frac{1}{(2, q-1)} q^{63}(q^2-1)(q^6-1)(q^8-1)(q^{10}-1)(q^{12}-1)(q^{14}-1)(q^{18}-1),$$

where $(2, q-1)$ is the greatest common divisor of 2 and $q-1$.

5.4.23 Corollary. $|U| = q^{63}$ *and U is a Sylow p-subgroup of the Chevalley group $E_7(q)$ under isomorphism.*

Proof. c.f. the proof of 5.1.23. □

5.4.24 Proposition.

$$U = \left\{ \prod_{r \in \Phi^+_{E_7}} x_r(t_r) \,\middle|\, t_r \in \mathbb{F}_q \right\},$$

where the product can be taken in an arbitrary, but fixed, order.

Proof. c.f. the proof of 5.1.24. □

By 5.4.23 and 5.4.24, we get the following Sylow p-subgroup of $E_7(q)$.

5.4.25 Proposition (Sylow p-subgroup $E_7^{syl}(q)$)**.** *Let $t_i \in \mathbb{F}_q$ $(i = 1, 2, \ldots, 63)$ and*

$$E_7^{syl}(q) := U.$$

Then $|E_7^{syl}(q)| = q^{63}$, $E_7^{syl}(q)$ is a Sylow p-subgroup of the Chevalley group $E_7(q)$ under isomorphism and the matrix elements of the Sylow p-subgroup $E_7^{syl}(q)$ are given as

$$\begin{aligned}
&x_1(t_1)x_7(t_7)x_3(t_3)x_2(t_2)\\
&\cdot x_{17}(t_{17})x_{13}(t_{13})x_{12}(t_{12})x_9(t_9)x_8(t_8)x_4(t_4)\\
&\cdot x_{29}(t_{29})x_{26}(t_{26})x_{24}(t_{24})x_{22}(t_{22})x_{19}(t_{19})x_{18}(t_{18})x_{15}(t_{15})x_{14}(t_{14})x_{10}(t_{10})x_5(t_5)\\
&\cdot x_{36}(t_{36})x_{35}(t_{35})x_{34}(t_{34})x_{33}(t_{33})x_{32}(t_{32})x_{31}(t_{31})x_{30}(t_{30})x_{28}(t_{28})x_{27}(t_{27})x_{25}(t_{25})\\
&\cdot x_{23}(t_{23})x_{21}(t_{21})x_{20}(t_{20})x_{16}(t_{16})x_{11}(t_{11})x_6(t_6)\\
&\cdot x_{63}(t_{63})x_{62}(t_{62})x_{61}(t_{61})x_{60}(t_{60})x_{59}(t_{59})x_{58}(t_{58})x_{57}(t_{57})x_{56}(t_{56})x_{55}(t_{55})x_{54}(t_{54})\\
&\cdot x_{53}(t_{53})x_{52}(t_{52})x_{51}(t_{51})x_{50}(t_{50})x_{49}(t_{49})x_{48}(t_{48})x_{47}(t_{47})x_{46}(t_{46})x_{45}(t_{45})x_{44}(t_{44})\\
&\cdot x_{43}(t_{43})x_{42}(t_{42})x_{41}(t_{41})x_{40}(t_{40})x_{39}(t_{39})x_{38}(t_{38})x_{37}(t_{37})
\end{aligned}$$

$$= \left(\begin{array}{ccccccccccccccccccccccccccccccccccccccccccc}
1 & t_{37} & -t_{38} & t_{39} & -t_{40} & t_{41} & t_{42} & -t_{43} & -t_{44} & t_{45} & t_{46} & -t_{47} & -t_{48} & t_{49} & t_{50} & t_{51} & -t_{52} & -t_{53} & * & t_{54} & t_{55} & * & -t_{56} & -t_{57} & * & t_{58} & t_{59} & * & -t_{60} & * & * & t_{61} & * & * & -t_{62} & * & * & t_{63} & * & \cdots & * & * & * \\
0 & 1 & t_6 & -t_{11} & t_{16} & -t_{20} & -t_{21} & t_{23} & t_{25} & -t_{27} & -t_{28} & t_{30} & t_{31} & -t_{32} & -t_{33} & * & t_{34} & * & * & -t_{35} & * & * & t_{36} & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & \cdots & * & * & * \\
0 & 0 & 1 & t_5 & -t_{10} & t_{14} & t_{15} & -t_{18} & -t_{19} & t_{22} & t_{24} & -t_{26} & * & t_{29} & * & \cdots & * & * & * \\
0 & 0 & 0 & 1 & t_4 & -t_8 & -t_9 & t_{12} & t_{13} & -t_{17} & * & \cdots & * & * & * \\
0 & 0 & 0 & 0 & 1 & t_2 & t_3 & -t_7 & * & \cdots & * & * & * \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & t_3 & * & \cdots & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & t_1 & * & \cdots & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & \cdots & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & \cdots & * & * & * \\
\vdots & & & & & & & & & & & & & & & & & & & & & & \ddots & \ddots & \ddots & & & & & & & & & & & & & & & & & & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1
\end{array}\right)_{56\times 56}$$

where $t_i \in \mathbb{F}_q$ $(i = 1, 2, \ldots, 63)$.

5.4.26 Corollary. *Let $u = (u_{i,j}) \in E_7^{syl}(q)$, then*

$$\begin{cases} u_{19,j_{19}} = 0, & j_{19} \in \{20,21,23,24,26,27,29,32,35,38\} \\ u_{22,j_{22}} = 0, & j_{22} \in \{21,23,24,26,27,29,32,35,38\} \\ u_{25,j_{25}} = 0, & j_{25} \in \{26,27,29,32,35,38\} \\ u_{28,j_{28}} = 0, & j_{28} \in \{29,32,35,38\} \\ u_{30,j_{30}} = 0, & j_{30} \in \{32,35,38\} \\ u_{31,j_{31}} = 0, & j_{31} \in \{32,35,38\} \\ u_{33,j_{33}} = 0, & j_{33} \in \{35,38\} \\ u_{34,j_{34}} = 0, & j_{34} \in \{35,38\} \\ u_{36,38} = 0 & \\ u_{37,38} = 0 & \\ u_{16,j_{16}} = 0, & j_{16} \in \{17,20,23\} \\ u_{18,j_{18}} = 0, & j_{18} \in \{20,23\} \\ u_{19,j_{19}} = 0, & j_{19} \in \{20,23\} \\ u_{21,23} = 0 & \\ u_{22,23} = 0 & \\ u_{13,14} = 0 & \end{cases}.$$

5.4.3. Monomial $E_7^{syl}(q)$-module

Let $U := E_7^{syl}(q)$ and $V_0 := \mathrm{Mat}_{56\times 56}(\mathbb{F}_q)$. In this subsection, we get a monomial linearisation $(f, \kappa|_{V\times V})$ for $E_7^{syl}(q)$, where f is a bijective 1-cocycle and $\kappa|_{V\times V}$ is a non-degenerate bilinear form (5.4.44). Finally, we make $\mathbb{C}E_7^{syl}(q)$ into a monomial $E_7^{syl}(q)$-module (5.4.45).

It is essential to choose a suitable vector space V for the 1-cocycle and a suitable non-degenerate bilinear form κ on V_0.

5.4.27 Notation/Lemma. *The map*

$$\kappa\colon V_0 \times V_0 \to \mathbb{F}_q : (X, Y) \mapsto \mathrm{tr}(X^\top Y)$$

*is called the **trace form**, and κ is a non-degenerate symmetric bilinear form on V_0.*

5.4.28 Notation. *Set*

$$J := \left\{ (i,j) \in \triangledown \;\middle|\; \begin{cases} j \in \{2,3,\ldots,18,20,21,23,24,26,27,29,32,35,38\} & \textit{if } i = 1 \\ j \in \{3,4,\ldots,15,17,20,23\} & \textit{if } i = 2 \\ j \in \{4,5,\ldots,12,14\} & \textit{if } i = 3 \\ j \in \{5,6,\ldots,10\} & \textit{if } i = 4 \\ j \in \{6,7,8\} & \textit{if } i = 5 \\ j = 8 & \textit{if } i = 7 \end{cases} \right\},$$

and

$$J' := \left\{ (i,j) \in \triangledown \;\middle|\; \begin{cases} j \in \{19,22,25,28,30,31,33,34,36,37\} & \textit{if } i = 1 \\ j \in \{16,18,19,21,22\} & \textit{if } i = 2 \\ j = 13 & \textit{if } i = 3 \end{cases} \right\}.$$

5.4.29 Reminder. *Let $J \subseteq \triangledown$, then by 1.1.9 the subset J^c of $\square$ is defined as follows:*

$$\begin{aligned} J^c :=& J \bigcup \{(i,i+1) \in \triangledown \backslash J \mid (i,i+2) \in J\} \\ & \bigcup \{(i,j) \in \triangledown \backslash J \mid \exists\, j_1 < j < j_2 \textit{ and } (i,j_1),(i,j_2) \in J\}. \end{aligned}$$

5.4.30 Remark. *(1) $J^c = J \dot\cup J'$,*

(2) $\#J = 63$, $\#J' = 16$ and $\#J^c = 79$.

5.4.31 Notation/Lemma. *Set*

$$V := V_J = \left\{ \sum_{(i,j)\in J} A_{i,j} e_{i,j} \in V_0 \;\middle|\; A_{i,j} \in \mathbb{F}_q \right\}$$

where

$$\sum_{(i,j)\in J} A_{i,j}e_{i,j} =$$

$$\left(\begin{array}{l}
A_{1,2}\ A_{1,3}\ A_{1,4}\ A_{1,5}\ A_{1,6}\ A_{1,7}\ A_{1,8}\ A_{1,9}\ A_{1,10}\ A_{1,11}\ A_{1,12}\ A_{1,13}\ A_{1,14}\ A_{1,15}\ A_{1,16}\ A_{1,17}\ A_{1,18}\ 0\ A_{1,20}\ A_{1,21}\ 0\ A_{1,23}\ A_{1,24}\ 0\ A_{1,26}\ A_{1,27}\ 0\ A_{1,29}\ 0\ 0\ A_{1,32}\ 0\ 0\ A_{1,35}\ 0\ 0\ A_{1,38} \\
\quad A_{2,3}\ A_{2,4}\ A_{2,5}\ A_{2,6}\ A_{2,7}\ A_{2,8}\ A_{2,9}\ A_{2,10}\ A_{2,11}\ A_{2,12}\ A_{2,13}\ A_{2,14}\ A_{2,15}\ 0\ A_{2,17}\ 0\ 0\ A_{2,20}\ 0\ 0\ A_{2,23} \\
\qquad A_{3,4}\ A_{3,5}\ A_{3,6}\ A_{3,7}\ A_{3,8}\ A_{3,9}\ A_{3,10}\ A_{3,11}\ A_{3,12}\ 0\ A_{3,14} \\
\qquad\quad A_{4,5}\ A_{4,6}\ A_{4,7}\ A_{4,8}\ A_{4,9}\ A_{4,10} \\
\qquad\qquad A_{5,6}\ A_{5,7}\ A_{5,8} \\
\qquad\qquad\quad 0\ 0 \\
\qquad\qquad\qquad A_{7,8}
\end{array}\right)$$

Then V is a 63-dimensional subspace of V_0 over $\mathbb{F}_q$, $\mathrm{supp}(V) = J$.

5.4.32 Lemma. *Let $V^\perp$ denote the orthogonal complement of V in V_0 with respect to the trace form κ, i.e. $V^\perp := \{B \in V_0 \mid \kappa(A,B) = 0,\ \forall\, A \in V\}$. Then $V^\perp = V_{\square\setminus J}$ and $V_0 = V \oplus V^\perp$.*

5.4.33 Notation/Lemma. *Let*

$$\pi\colon V_0 = V \oplus V^\perp \to V : A \mapsto \sum_{(i,j)\in J} A_{i,j} e_{i,j}.$$

Then π is a projection to the first component and is an $\mathbb{F}_q$-linear map.

5.4.34 Corollary. $\kappa|_{V\times V}\colon V \times V \to \mathbb{F}_q$ *is a non-degenerate bilinear form.*

5.4.35 Lemma. *Suppose $A, B \in V_0$, such that* $\operatorname{supp}(A) \cap \operatorname{supp}(B) \subseteq J$. *Then*

$$\kappa(A,B) = \kappa(\pi(A),B) = \kappa(A,\pi(B)) = \kappa(\pi(A),\pi(B)) = \kappa|_{V\times V}(\pi(A),\pi(B)).$$

5.4.36 Lemma. *Let $A \in V$ and $g \in U$, then* $\operatorname{supp}(Ag^\top) \cap \boxslash \subseteq J$. *In particular $\pi_{J^c}(Ag^\top) \in V$.*

Proof. Let $A \in V$, $g \in G$. It is sufficient to prove $(Ag^\top)_{i,j} = 0$ for all $(i,j) \in J^c \setminus J = J'$. We have

$$\begin{aligned}(Ag^\top)_{1,19} &= \sum_{j=1}^{56} A_{1,j} g^\top_{j,19} = \sum_{j=2}^{38} A_{1,j} g^\top_{j,19} = \sum_{j=2}^{38} A_{1,j} g_{19,j} = \sum_{j=19}^{38} A_{1,j} g_{19,j} \\ &\overset{\substack{A_{1,j}=0\\(1,j)\notin J}}{=} \sum_{j\in\{20,21,23,24,26,27,29,32,35,38\}} A_{1,j} g_{19,j} \overset{5.4.26}{=} 0,\end{aligned}$$

Similarly, $(Ag^\top)_{i,j} = 0$ for all $(i,j) \in J'$. Thus, $\operatorname{supp}(Ag^\top) \cap \boxslash \subseteq J$, and $\pi_{J^c}(Ag^\top) = \pi_J(Ag^\top) \in V$. $\square$

5.4.37 Corollary. *Let $A, B \in V$ and $g, h \in U$, then* $\operatorname{supp}(Bh^\top) \cap \operatorname{supp}(Ag) \subseteq J$.

5.4.38 Proposition (Group action of $E_7^{syl}(q)$ on V). *The map*

$$-\circ-\colon V \times U \to V : (A,g) \mapsto A \circ g := \pi(Ag)$$

is a group action, and the elements of the group U act as $\mathbb{F}_q$-automorphisms.

Proof. c.f. the proof of 2.2.25. $\square$

By 1.2.21, we get a new action:

5.4.39 Corollary. *There exists an unique linear action $-.-$ of G on V:*

$$-.-\colon V\times G\to V : (A,g)\mapsto A.g := A\circ g^{-\top}$$

such that $\kappa|_{V\times V}(A.g,B) = \kappa|_{V\times V}(A, B\circ g^{-1})$ for all $B\in V$. In particular, for all $A,B\in V$ and $g\in G$,

$$\kappa|_{V\times V}(A, B\circ g) = \kappa|_{V\times V}(A.g^{-1}, B) = \kappa|_{V\times V}(A\circ g^{\top}, B).$$

5.4.40 Notation. $f := \pi|_U\colon U\to V$.

5.4.41 Proposition. *$f\colon U\to V$ is bijective.*

Proof. By 5.4.25 and 5.4.31, we obtain that $f\colon U\to V$ is bijective. □

5.4.42 Lemma. *Let $x,g\in U$, then $f(x)g\equiv (x-1)g \mod V^{\perp}$. In particular, $f(x)\equiv x-1 \mod V^{\perp}$.*

Proof. c.f. the proof of 2.2.29. □

5.4.43 Proposition. *Let $x,g\in U$, then $f(xg) = f(x)\circ g + f(g)$.*

Proof. c.f. the proof of 2.2.30 or by 5.4.26 as follows:
for all $j_0 = 1,2,\ldots,18$,

$$\big(f(x)\circ g + f(g)\big)_{1,j_0} = \big(f(x)g\big)_{1,j_0} + f(g)_{1,j_0} = \sum_{k=2}^{j_0} f(x)_{1,k}g_{k,j_0} + g_{1,j_0}$$
$$= \sum_{k=2}^{j_0} x_{1,k}g_{k,j_0} + g_{1,j_0} = \sum_{k=1}^{j_0} x_{1,k}g_{k,j_0} = (xg)_{1,j_0} = f(xg)_{1,j_0},$$

and

$$\big(f(x)\circ g + f(g)\big)_{1,20} = \big(f(x)g\big)_{1,20} + f(g)_{1,20} = \sum_{k=2}^{20} f(x)_{1,k}g_{k,20} + g_{1,20}$$
$$= \sum_{k=2}^{18} f(x)_{1,k}g_{k,20} + f(x)_{1,19}g_{19,20} + f(x)_{1,20}g_{20,20} + g_{1,20}$$
$$= \sum_{k=2}^{18} x_{1,k}g_{k,20} + 0\cdot g_{19,20} + x_{1,20}g_{20,20} + g_{1,20}$$
$$= \sum_{k=1}^{20} x_{1,k}g_{k,20} - x_{1,19}g_{19,20} \overset{5.4.26}{=} \sum_{k=1}^{20} x_{1,k}g_{k,20} = \big(xg\big)_{1,20} = f(xg)_{1,20}.$$

Similarly, $f(xg)_{i,j} = \big(f(x)\circ g + f(g)\big)_{i,j}$ for all $(i,j)\in J$. Thus $f(xg) = f(x)\circ g + f(g)$. □

By 5.4.41 and 5.4.43, we obtain a bijective 1-cocycle of $E_7^{syl}(q)$.

5.4.44 Corollary (Monomial linearisation for $E_7^{syl}(q)$)**.** *f is a bijective 1-cocycle of $E_7^{syl}(q)$, and $(f, \kappa|_{V\times V})$ is a monomial linearisation for $U = E_7^{syl}(q)$.*

Now we obtain the monomial U-module $\mathbb{C}E_7^{syl}(q)$, which is essential for the construction of the supercharacter theory for $E_7^{syl}(q)$.

5.4.45 Theorem (Fundamental theorem for $E_7^{syl}(q)$)**.** *Let $U = E_7^{syl}(q)$ and*

$$[A] = \frac{1}{|U|}\sum_{u\in U}\overline{\chi_A(u)}u \qquad \text{for all } A \in V.$$

Then the set $\{[A] \mid A \in V\}$ forms a $\mathbb{C}$-basis for the complex group algebra $\mathbb{C}U$. The operation of U on $\mathbb{C}U$ is given by the usual right multiplication, i.e.

$$[A]u = \frac{1}{|U|}\sum_{y\in U}\overline{\chi_A(y)}yu = \chi_{A.u}(g)[A.u] = \vartheta\kappa(A.u, f(u))[A.u] \qquad \text{for all } u \in U, A \in V,$$

then $\mathbb{C}U$ is a monomial $\mathbb{C}U$-module.

Proof. By 5.4.44, $(f, \kappa|_{V\times V})$ is a monomial linearisation for U, satisfying that f is bijective. By 5.4.39, $A.u := \pi(Au^{-\top})$. Thus the whole theorem is proved in view of 1.2.24. □

A. Appendix

A.1. The construction of monomial modules

In this section, we introduce some properties of the monomial $\mathbb{C}G$-modules. We mainly refer to [DG16].

Let G be a finite multiplictive group, U a subgroup of G, V an $\mathbb{F}_q$-vector space with finite dimension, and $f\colon G \to V$ a surjective 1-cocycle such that $f|_U$ is bijective.

A.1.1 Lemma. *Let $H := \{g \in G \mid f(g) = 0\}$, then $H \cap U = \{1\}$ and $G = HU$.*

Proof. Since $H \leqslant G$ and $U \leqslant G$, we have $G \supseteq HU$.

Let X be a complete set of right coset representatives of H in G. Let $g \in G$, then there exist $h \in H$ and $x \in X$ such that $g = hx$. We have $f(g) = f(x) \in V$, then there exists $u \in U$ such that $f(u) = f(x)$ since $f|_U$ is bijective. We know $f(xu^{-1}) = f(x) \circ u^{-1} + f(u^{-1}) = f(u) \circ u^{-1} + f(u^{-1}) = f(1) = 0$, so $x = h_x u$ for some $h_x \in H$. Thus $g = hh_x u \in HU$, i.e. $G \subseteq HU$. Therefore $G = HU$.

Let $g \in H \cap U$, then $f(g) = 0 = f|_U(g)$. So $g = 1_G$ since $f|_U$ is bijective. $\square$

A.1.2 Proposition. *Let*

$$e := \sum_{h\in H} h, \quad \tau_e := \sum_{h\in H} \tau_h \quad \textit{and} \quad [\chi] := [\hat{\chi} f] := \sum_{g\in G} \chi(g) g \quad \textit{for all } \hat{\chi} \in \hat{V},$$

and $\mathbb{C}_H = \mathbb{C}\{e\}$ be a trivial H-module. Then

$$\begin{aligned}
\operatorname{Ind}_H^G \mathbb{C}_H =& e\mathbb{C}G = \mathbb{C}\{eu \mid u \in U\} = e\mathbb{C}U = \mathbb{C}\{e[\chi|_U] \mid \hat{\chi} \in \hat{V}\} \\
=& \mathbb{C}\{[\chi] \mid \hat{\chi} \in \hat{V}\} \quad (\textit{as } \mathbb{C}\textit{-vector space}), \\
\operatorname{im} f^* =& \mathbb{C}\{\chi \mid \hat{\chi} \in \hat{V}\} = \mathbb{C}\{\tau_e * [\chi|_U] \mid \hat{\chi} \in \hat{V}\} = \tau_e * \mathbb{C}U \\
=& \mathbb{C}\{\tau_e * u \mid u \in U\} = \tau_e * \mathbb{C}G = \tau_e \mathbb{C}^G \quad (\textit{as } \mathbb{C}\textit{-vector space}).
\end{aligned}$$

In particular, $(\operatorname{im} f^*, *)_{\mathbb{C}G} \cong \operatorname{Ind}_H^G \mathbb{C}_H$ and $\chi = \tau_e * [\chi|_U]$.

Proof. It is enough to prove $\chi = \tau_e * [\chi|_U]$,

$$\begin{aligned}
\chi =& \sum_{g\in G} \chi(g)\tau_g = \sum_{h\in H}\sum_{u\in U} \chi(hu)\tau_{hu} = \sum_{h\in H}\sum_{u\in U} \chi(u)\tau_h * u \\
=& (\sum_{h\in H} \tau_h) * (\sum_{u\in U} \chi(u)u) = \tau_e * [\chi|_U].
\end{aligned}$$

$\square$

Now we give a summary of the isomporphic $\mathbb{C}G$-modules and the $\mathbb{C}$-basis:

$$\begin{tikzcd}
\mathbb{C}\{\chi|_U|\hat{\chi}\in\hat{V}\}= & (\mathbb{C}^U,*)_{\mathbb{C}G} \arrow[r, "\varPhi"] & (\mathbb{C}U,*)_{\mathbb{C}G} \arrow[d, "f_{\mathbb{C}U}"] & =\mathbb{C}\{[\chi|_U]|\hat{\chi}\in\hat{V}\} \\
\mathbb{C}\hat{V}= & (\mathbb{C}^V,*)_{\mathbb{C}G} \arrow[u, "f|_U^*"] \arrow[r, "\varPsi"] \arrow[d, "f^*"'] & (\mathbb{C}V,*)_{\mathbb{C}G} \arrow[d] & =\mathbb{C}\{[\hat{\chi}]|\hat{\chi}\in\hat{V}\} \\
\begin{array}{c}\mathbb{C}\{\tau_e*[\chi|_U]|\hat{\chi}\in\hat{V}\}=\\ \mathbb{C}\{\chi|\hat{\chi}\in\hat{V}\}=\end{array} & (\operatorname{im} f^*,*)_{\mathbb{C}G} \arrow[r, "\varUpsilon_{\operatorname{im} f^*}"] & \operatorname{Ind}_H^G\mathbb{C}_H & \begin{array}{c}=\mathbb{C}\{e[\chi|_U]|\hat{\chi}\in\hat{V}\}\\ =\mathbb{C}\{[\chi]|\hat{\chi}\in\hat{V}\}\end{array}
\end{tikzcd}$$

where $\varUpsilon\colon \mathbb{C}^G \to \mathbb{C}G : \tau \mapsto \sum_{g\in G}\tau(g)g$.

A.2. Chevalley basis

In this section, we recall some definitions and notations of Lie algebras, e.g. simple Lie algebras, semisimple Lie algebras, Cartan subalgebras, roots, Cartan decompositions, Chevalley bases and Chevalley groups. We refer to [Car72], [Hum78], [Car05] and [Gec15].

A.2.1 Definition. *Let K be a field. A* ***Lie algebra*** *over K is a K-vector space $\mathcal{L}$, with an operation*

$$[\ ,\ ]\colon \mathcal{L}\times\mathcal{L}\to\mathcal{L} : (x,y)\mapsto[x,y],$$

is defined satisfying the following axioms:

(i) $(x,y)\mapsto[x,y]$ *is bilinear,*

(ii) $[x,x]=0$ *for all* $x\in\mathcal{L}$,

(iii) $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$ *for all* $x,y,z\in\mathcal{L}$.

The operation $[\ ,\ ]$ of a Lie algebra $\mathcal{L}$ is called the ***Lie bracket****.*

Let $\mathcal{L}$ be a Lie algebra over K and $\mathcal{M}$, $\mathcal{N}$ be subspaces of $\mathcal{L}$. Then

$$[\mathcal{M},\mathcal{N}] := K\text{-span}\{[x,y]\mid x\in\mathcal{M},\ y\in\mathcal{N}\}.$$

A **subalgebra** $\mathcal{H}$ of $\mathcal{L}$ is a subspace of $\mathcal{L}$ such that $[\mathcal{H},\mathcal{H}]\subseteq\mathcal{H}$. An **ideal** $\mathcal{I}$ of $\mathcal{L}$ is a subspace of $\mathcal{L}$ such that $[\mathcal{H},\mathcal{L}]\subseteq\mathcal{H}$.

Let $\mathcal{L}_1$, $\mathcal{L}_2$ be Lie algebras over K, a linear map $\phi\colon\mathcal{L}_1\to\mathcal{L}_2$ is called a **homomorphism of Lie algebras** if $\phi([x,y])=[\phi(x),\phi(y)]$ for all $x,y\in\mathcal{L}_1$. A homomorphism $\phi\colon\mathcal{L}_1\to\mathcal{L}_2$ of Lie algebras is called an **isomorphism of Lie algebras** if ϕ is bijective.

Let $M_n(K)$ be the associative algebra of all $n\times n$-matrices over the field K with the usual matrix product. Let $gl_n(K)$ be the corresponding Lie algebra with the Lie bracket:

$$[x,y]:=xy-yx\qquad\text{for all } x,y\in M_n(K).$$

Similarly, if V is a K-vector space, then $\mathrm{End}(V):=\{\varphi\colon V\to V\mid\varphi\text{ linear}\}$ is the associative algebra with the composition of maps as product. The corresponding Lie algebra is denoted by $gl(V)$, and the Lie bracket is:

$$[\varphi,\psi]:=\varphi\circ\psi-\psi\circ\varphi\qquad\text{for all }\varphi,\psi\in\mathrm{End}(V).$$

A.2.2 Definition. *Let V be a K-vector space, a* ***representation*** *ρ of a Lie algebra $\mathcal{L}$ over K is a homomorphism of Lie algebras*

$$\rho\colon\mathcal{L}\to gl(V).$$

A.2.3 Definition. *Let $\mathcal{L}$ be a Lie algebra over K. A* ***(left) $\mathcal{L}$-module*** *V is a vector space over K together with a multiplication*

$$\mathcal{L} \times V \to V : (x, v) \mapsto x.v$$

satisfying the axioms:

(i) $(x, v) \mapsto x.v$ *is bilinear,*

(ii) $[x, y].v = x.(y.v) - y.(x.v)$ *for all* $x, y \in \mathcal{L}$ *and* $v \in V$.

In particular, $\mathcal{L}$ is itself a left $\mathcal{L}$-module, called ***adjoint module****.*

Let V be a $\mathcal{L}$-module and $n = \dim V < \infty$, $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ be a K-basis of V, and $M_{\mathcal{B}}(\rho(x))$ be the matrix of $\rho(x)$ with respect to $\mathcal{B}$ i.e.

$$x.v_j = \sum_{i=1}^{n} M_{\mathcal{B}}(\rho(x))_{i,j} e_i,$$

then a **matrix representation** $\rho_{\mathcal{B}}$ is determined

$$\rho_{\mathcal{B}} \colon \mathcal{L} \to gl_n(K) : x \mapsto M_{\mathcal{B}}(\rho(x)).$$

Let $\mathcal{L}$ be a Lie algebra and $x \in \mathcal{L}$, define a linear map $\operatorname{ad} x$ of $\mathcal{L}$ by

$$\operatorname{ad} x \colon \mathcal{L} \to \mathcal{L} : y \mapsto [x, y].$$

A Lie algebra $\mathcal{L}$ is **abelian**, if $[\mathcal{L}, \mathcal{L}] = \{0\}$. For $n \in \mathbb{N}^*$, set $\mathcal{L}^1 := \mathcal{L}$ and

$$\mathcal{L}^{n+1} := [\mathcal{L}^n, \mathcal{L}] \qquad \text{for all } n \geq 1.$$

A Lie algebra $\mathcal{L}$ is **nilpotent**, if $\mathcal{L}^n = \{0\}$ for some $n \geq 1$.

A.2.4 Definition. *A Lie algebra $\mathcal{L}$ is called* ***simple*** *if $\mathcal{L}$ is not abelian, and $\mathcal{L}$, $\{0\}$ are the only ideals of $\mathcal{L}$.*

Let $\mathbb{C}$ be the field of complex numbers, $\mathcal{L}$ a nilpotent Lie algebra over $\mathbb{C}$, V a $\mathcal{L}$-module with $\dim V < \infty$, $\rho \colon \mathcal{L} \to gl(V)$ the corresponding representation. If V is irreducible, then $\dim V = 1$ (see [Car05, Theorem 2.2]). Let $\lambda \colon \mathcal{L} \to \mathbb{C}$ be a 1-dimensional representation of $\mathcal{L}$, set

$$V_\lambda := \left\{ v \in V \mid \forall\ x \in \mathcal{L}, \exists\, l(x) \geq 1 \text{ such that } (\rho - \lambda(x)\mathrm{id}_V)^{l(x)}(x) = 0 \right\},$$

then

$$V = \bigoplus_{\lambda} V_\lambda$$

If $V_\lambda \neq \{0\}$, the 1-dimensional representation λ is called a **weight**, and V_λ is called the **weight space** of λ.

Let $\mathcal{H}$ be a subalgebra of a complex Lie algebra $\mathcal{L}$, set

$$N_{\mathcal{L}}(\mathcal{H}) := \{x \in \mathcal{L} \mid [h, x] \in \mathcal{H} \text{ for all } h \in \mathcal{H}\}.$$

$N_{\mathcal{L}}(\mathcal{H})$ is called the **normaliser** of $\mathcal{H}$ in $\mathcal{H}$.

A.2.5 Definition. *A subalgebra $\mathcal{H}$ of a complex Lie algebra $\mathcal{L}$ is called a **Cartan subalgebra**, if $\mathcal{H}$ is nilpotent and $\mathcal{H} = N_{\mathcal{L}}(\mathcal{H})$.*

The dimension of the cartan subalgebra of $\mathcal{L}$ is called the **rank** of $\mathcal{L}$.

Let $\mathcal{L}$ be a complex Lie algebra $\mathcal{L}$, $\mathcal{H}$ be a Cartan subalgebra of $\mathcal{L}$, and regard $\mathcal{L}$ as an $\mathcal{H}$-module. Since H is nilpotent, we have

$$L = \bigoplus_{\lambda} L_\lambda$$

where

$$\mathcal{L}_\lambda := \left\{x \in \mathcal{L} \mid \forall\ h \in \mathcal{H}, \exists\, n \in \mathbb{N}^* \text{ such that } (\operatorname{ad} h - \lambda(h)\mathrm{id}_{\mathcal{L}})^n(x) = 0\right\}.$$

Then $L_0 = H$ (see [Car05, Proposition 4.1]).

A.2.6 Definition. *Let $\mathcal{L}$ be a complex Lie algebra $\mathcal{L}$, $\mathcal{H}$ a Cartan subalgebra of $\mathcal{L}$. The 1-dimensional representation $\lambda\colon \mathcal{H} \to \mathbb{C}$ of $\mathcal{H}$ such that $\lambda \neq 0$ and $L_\lambda \neq \{0\}$ called the **root** of $\mathcal{L}$ with respect to $\mathcal{H}$. The set of roots of $\mathcal{L}$ with respect to $\mathcal{H}$ will be denoted by Φ.*

A.2.7 Definition. *Let $\mathcal{L}$ be a complex Lie algebra $\mathcal{L}$, $\mathcal{H}$ a Cartan subalgebra of $\mathcal{L}$, and Φ the set of roots of $\mathcal{L}$ with respect to $\mathcal{H}$, then the decomposition*

$$\mathcal{L} = \mathcal{H} \oplus \left(\bigoplus_{r \in \Phi} \mathcal{L}_r\right)$$

*is callled the **Cartan decompositon** of $\mathcal{L}$ with respect to $\mathcal{H}$. $\mathcal{L}_r$ is called the **root space** of r.*

A.2.8 Definition. *A complex Lie algebra $\mathcal{L}$ is called **semisimple**, if $\{0\}$ is the only abelian ideal of $\mathcal{L}$.*

A simple complex Lie algebra is semisimple.

We define the **Killing form** as follows:

$$\kappa\colon \mathcal{L} \times \mathcal{L} \to \mathbb{C} : (x, y) \mapsto \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y).$$

Let a complex Lie algebra $\mathcal{L}$ be semisimple, then the Killing form κ is non-degenerate (see [Car05, Theorem 4.10]), and the restriction of κ to $\mathcal{H}$ is also non-degenerate. There is a linear isomorphism $\mathcal{H} \to \mathcal{H}^* : h \mapsto h^*$ where $h^* \in \mathcal{H}^*$ is defined by

$$h^*(x) = \kappa(h, x) \qquad \text{for all } x \in \mathcal{H}.$$

For all $r \in \Phi$, there exists a unique element $t_r \in \mathcal{H}$ satisfying

$$r(h) = \kappa(t_r, h) = t_r^*(h) \qquad \text{for all } h \in \mathcal{H}$$

and $r = t_r^*$. We know that $\mathcal{H} = \langle t_r \mid r \in \Phi \rangle_{\mathbb{C}}$ (span as $\mathbb{C}$-vector space). Since $r(t_r) = \kappa(t_r, t_r) \neq 0$ for all $r \in \Phi$, we set $h_r := \frac{2t_r}{\kappa(t_r,t_r)}$, then $h_r \in [\mathcal{L}_r, \mathcal{L}_{-r}]$ and $r(h_r) = 2$. Let $r, s \in \Phi$ and $s \neq \pm r$, and

$$\begin{aligned}\tilde{n}_{r,s} :=& \max\{i \in \mathbb{N} \mid s + ir \in \Phi\},\\ n_{r,s} :=& \max\{i \in \mathbb{N} \mid s - ir \in \Phi\},\end{aligned}$$

then the set of roots

$$s - n_{r,s}r, \ldots, s - r, s, s + r, \ldots, s + \tilde{n}_{r,s}r$$

is called the r**-chain** of roots through s. Then $s(h_r) \overset{\text{Def.}}{=} 2\frac{\kappa(t_r,t_s)}{\kappa(t_r,t_r)} = n_{r,s} - \tilde{n}_{r,s} \in \mathbb{Z}$ (see [Car05, Proposition 4.22]). We set $\langle \mathcal{H} \rangle_{\mathbb{R}} := \langle t_r \mid r \in \Phi \rangle_{\mathbb{R}} \subseteq \mathcal{H}$ as $\mathbb{R}$-vector space. Since $\kappa(t_r, t_s) \in \mathbb{Q}$, the Killing form κ restricted to $\langle \mathcal{H} \rangle_{\mathbb{R}}$ is a map $\langle \mathcal{H} \rangle_{\mathbb{R}} \times \langle \mathcal{H} \rangle_{\mathbb{R}} \to \mathbb{R}$ which is a symmetic positive definite bilinear form. We denote by $\langle \mathcal{H}^* \rangle_{\mathbb{R}}$ the image of $\langle \mathcal{H} \rangle_{\mathbb{R}}$ under the isomorphism $\mathcal{H} \to \mathcal{H}^* : h \mapsto h^*$. $\langle \mathcal{H}^* \rangle_{\mathbb{R}}$ is the $\mathbb{R}$-subspace of $\mathcal{H}^*$ spanned by Φ. We define a symmetic positive definite bilinear form on $\langle \mathcal{H}^* \rangle_{\mathbb{R}}$ by

$$(\ ,\ ) \colon \langle \mathcal{H}^* \rangle_{\mathbb{R}} \times \langle \mathcal{H}^* \rangle_{\mathbb{R}} \to \mathbb{R} : (h_1^*, h_2^*) \mapsto \kappa(h_1, h_2).$$

Thus the vector space $\langle \mathcal{H}^* \rangle_{\mathbb{R}}$ endowed with this positive definite form is a Euclidean space, denoted by $\mathcal{V} := \langle \mathcal{H}^* \rangle_{\mathbb{R}}$. For all $r \in \Phi$, $r \in \langle \mathcal{H}^* \rangle_{\mathbb{R}}$. For $r, s \in \Phi$, we set $\langle r, s \rangle := 2\frac{(r,s)}{(r,r)}$. A subset Δ is called a **fundamental system** of roots, if

(i) Δ is a basis of $\mathcal{V}$,

(ii) Every root $s \in \Phi$ can be written as $s = \sum_{r \in \Delta} c_r r$ with integral coefficients c_r all non-negative or non-positive.

The roots of Δ are called **simple**. The **height** of a root $s \in \Phi$ (relative to Δ) is $\mathrm{ht}(s) := \sum_{r \in \Delta} c_r$. If all $c_r \geq 0$ (resp. all $c_r \leq 0$), the root s is called positive (resp. negative). The sets of positive and negative roots (relative to Δ) will be denoted by Φ^+ and Φ^-. Let fundamental roots $\Delta = \{r_1, \ldots, r_n\}$, then the (i, j)-entry $A_{i,j}$ of the Cartan matrix A is

$$A_{r_i,r_j} := A_{i,j} := \langle r_i, r_j \rangle \overset{\text{Def.}}{=} \frac{2\kappa(t_{r_i}, t_{r_j})}{\kappa(t_{r_i}, t_{r_i})} \overset{\text{Def.}}{=} r_j(h_{r_i}) = n_{r_i,r_j} - \tilde{n}_{r_i,r_j}.$$

Now we define a total order for the Euclidean space $\mathcal{V}$.

A.2.9 Definition. *Let $\mathcal{L}$ be a semisimple complex Lie algebra $\mathcal{L}$, $\mathcal{H}$ a Cartan subalgebra of $\mathcal{L}$, the Euclidean space $\mathcal{V} := \langle \mathcal{H}^* \rangle_{\mathbb{R}}$, $\Delta := \{r_1, r_2, \ldots, r_n\}$ be a $\mathbb{R}$-basis of $\mathcal{V}$, and $r := \sum_{i=1}^n x_i r_i \in \mathcal{V}$, $s := \sum_{i=1}^n y_i r_i \in \mathcal{V}$ then we write*

$$r \prec s,$$

if $\sum_{i=1}^n x_i < \sum_{i=1}^n y_i$, or if $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$ and the first non-zero coefficient $x_i - y_i$ is positive.

A.2.10 Definition ([Car72], §4.2). *Let $\mathcal{L}$ be a complex simple Lie algebra with root system Φ, fundamental roots Δ, a Cartan subalgebra $\mathcal{H}$ and*

$$\mathcal{L} = \mathcal{H} \oplus \sum_{r \in \Phi} \mathcal{L}_r$$

be a Cartan decomposition of $\mathcal{L}$. Let $h_r \in \mathcal{H}$ be the co-root corresponding to the root r and $e_r \in \mathcal{L}_r$ for all $r \in \Phi$ such that

$$\begin{aligned} [e_r, e_{-r}] =& h_r, \\ [e_r, e_s] =& N_{r,s} e_{r+s} \quad \textit{if } r + s \in \Phi, \end{aligned}$$

*where $N_{r,s} = \pm(n_{r,s} + 1)$ and $n_{r,s} \in \mathbb{N}$ is the greatest integer for which $s - n_{r,s} r \in \Phi$. Then the set $\{h_r \mid r \in \Delta\} \cup \{e_r \mid r \in \Phi\}$ is a **Chevalley basis** of the Lie algebra $\mathcal{L}$. In particular, $[h_r, e_s] = s(h_r)e_s = \langle r, s \rangle e_s$ for all $r \in \Delta$ and $s \in \Phi$.*

Let $\mathcal{L}$ be a simple Lie algebra over $\mathbb{C}$ with Chevalley basis

$$\{h_r \mid r \in \Delta\} \cup \{e_r \mid r \in \Phi\}.$$

Let $x_{r,\mathbb{C}}(\zeta) := \exp(\zeta \operatorname{ad} e_r)$ for all $r \in \Phi$, $\zeta \in \mathbb{C}$ and $A_{r,\mathbb{C}}(\zeta)$ be the matrix representing $x_{r,\mathbb{C}}(\zeta)$ with respect to the Chevalley basis of $\mathcal{L}$. Observe that the coefficient of $A_{r,\mathbb{C}}(\zeta)$ have the form $a\zeta^i$ where $a \in \mathbb{Z}$ and $0 \leq i \in \mathbb{N}$. Let K be any field with identity 1_K, and $\mathcal{L}_{\mathbb{Z}}$ be the subset of $\mathcal{L}$ of all linear combinations of the basis elements with coefficients in the ring $\mathbb{Z}$ of rational integers. We get a Lie algebra $\mathcal{L}_K := K \otimes \mathcal{L}_{\mathbb{Z}}$ with basis

$$\{1_K \otimes h_r \mid r \in \Delta\} \cup \{1_K \otimes e_r \mid r \in \Phi\}.$$

Let $t \in K$ and $A_{r,K}(t)$ be the matrix obtained from $A_{r,\mathbb{C}}(\zeta)$ by replacing each coefficient $a\zeta^i$ by $\tilde{a}t^i \in K$, where $\tilde{a}$ is the element of the prime field $\mathbb{F}_q$ of Kcorresponding to $a \in \mathbb{Z}$. We define $x_{r,K}(t)$ to be the linear map of $\mathcal{L}_K$ into itself represented by the matrix $A_{r,K}(t)$ with respect to the basis $\{1_K \otimes h_r \mid r \in \Delta\} \cup \{1_K \otimes e_r \mid r \in \Phi\}$ ([Car72], §4.4).

A.2.11 Definition ([Car72], p.64). *The **Chevalley group** of type $\mathcal{L}$ over the field K, denoted by $\mathcal{L}(K)$, is defined to be the group of automorphisms of the Lie algebra $\mathcal{L}_K$ generated by the $x_{r,K}(t)$ for all $r \in \Phi$ and $t \in K$, i.e.*

$$\mathcal{L}(K) = \langle x_{r,K}(t) \mid r \in \Phi, t \in K \rangle.$$

A.2.12 Proposition ([Car72], 4.4.3). *The Chevalley group $\mathcal{L}(K)$ is determined (up to isomorphisms) by the simple Lie algebra $\mathcal{L}$ over $\mathbb{C}$ and the field K.*

A.3. Chevalley basis of a Lie algebra $\mathcal{L}_{D_n}$

Fix $n \in \mathbb{N}^*$, $N = 2n$, and K be a field. We construct a Chevalley basis for the matrix Lie algebra $\mathcal{L}_{D_n}$ (see A.3.8). The general facts can be found in [Car72], [Hum78] and [Car05].

A.3.1 Lemma ([Car72], 11.2.2)**.** *Let $k \in \mathbb{N}^*$, J_k be a $k \times k$ matrix over $\mathbb{C}$. Then the $k \times k$ matrices A satisfying*

$$A^\top J_k + J_k A = 0$$

form a Lie algebra.

A.3.2 Notation/Lemma. *Set*

$$J_N^+ := \sum_{i=1}^{N} e_{i,N+1-i} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}_{N\times N} \in \mathrm{GL}_N(\mathbb{C}).$$

In particular, J_N^+ is the transpose and also the inverse of J_N^+.

Let $J_n^+ := \sum_{i=1}^n e_{i,n+1-i} \in \mathrm{GL}_n(\mathbb{C})$,

$$\pi := \begin{array}{|c|c|}\hline I_n & \\ \hline & J_n^+ \\ \hline \end{array}_{2n\times 2n} \in \mathrm{GL}_N(\mathbb{C}), \quad \text{and} \quad I := \begin{array}{|c|c|}\hline & I_n \\ \hline I_n & \\ \hline \end{array}_{2n\times 2n} \in \mathrm{GL}_N(\mathbb{C}),$$

then $I = \pi^{-1} \cdot J_{2n}^+ \cdot \pi$.

Let

$$\begin{aligned} L_n :=& \{B \in \mathrm{Mat}_{2n\times 2n}(\mathbb{C}) \mid B^\top I + IB = 0\} \\ \mathcal{L}_{D_n} :=& \{A \in \mathrm{Mat}_{2n\times 2n}(\mathbb{C}) \mid A^\top J_{2n}^+ + J_{2n}^+ A = 0\}, \end{aligned}$$

then

$$\begin{aligned} L_n =& \{B \in \mathrm{Mat}_{2n\times 2n}(\mathbb{C}) \mid B^\top \cdot (\pi^{-1} \cdot J_{2n}^+ \cdot \pi) + (\pi^{-1} \cdot J_{2n}^+ \cdot \pi) \cdot B = 0\} \\ =& \{B \in \mathrm{Mat}_{2n\times 2n}(\mathbb{C}) \mid \pi \cdot B^\top \cdot \pi^{-1} \cdot J_{2n}^+ + \cdot J_{2n}^+ \cdot \pi \cdot B \cdot \pi = 0\} \\ =& \{B \in \mathrm{Mat}_{2n\times 2n}(\mathbb{C}) \mid (\pi \cdot B \cdot \pi^{-1})^\top \cdot J_{2n}^+ + \cdot J_{2n}^+ \cdot (\pi \cdot B \cdot \pi^{-1}) = 0\} \end{aligned}$$

so $\pi \cdot L_n \cdot \pi^{-1} = \mathcal{L}_{D_n}$.

A.3.3 Remark. *(1) L_n is the Lie algebra of [Car72, 11.2.3]. The advantadge of $\mathcal{L}_{D_n}$ is that the Sylow p-subgroup arising from $\mathcal{L}_{D_n}$ is upper unitriangular. The following example can show us explicitly.*

(2) *The corresponding groups arising from the Lie algebras L_n and $\mathcal{L}_{D_n}$ are conjugate, so we can apply the properties of [Car72] directly to our groups.*

A.3.4 Definition/Lemma ([Jed13], 1.1.2)**.** *Define the* ***mirror map***

$$\bar{\cdot}: \{1, 2, \ldots, N\} \to \{1, 2, \ldots, N\} : i \mapsto \bar{i} := N + 1 - i.$$

The map mirrors every entry on $\frac{N+1}{2}$. It satisfies $\bar{\bar{i}} = i$.

In [Car72, 11.2.3], the rows and columns are numbered as follows:

$$1, 2, \ldots, n, -1, -2, \ldots, -n.$$

We change the order with $-i \mapsto \overline{n+1-i}$ for all $i = 1, 2, \ldots, n$, i.e.

$$1, 2, \ldots, n, \bar{n}, \overline{n-1}, \ldots, \bar{1}.$$

A.3.5 Example. *Let $B = (b_{i,j}) \in L_4$ and $A = \pi \cdot B \cdot \pi^{-1} \in \mathcal{L}_{D_4}$. The colored the entries (i.e. $b_{i,j}$, $b_{i,i}$, $b_{i,j}$, $b_{i,j}$) of B and A are freely choosen, the rest black entries are dependent. Some entries (e.g $b_{1,3}$) of B, which are corresponding to the positive roots, lie in the upper triangular part (e.g. position $(1, 3)$) and lower trianular part (e.g. position $(-3, -1)$ or $(\bar{2}, \bar{4})$), while all of the entries (e.g $b_{1,3}$) of A, which are corresponding to the positive roots, lie in the upper triangular part (e.g. positions $(1, 3)$ and $(-2, -4)$, or positions $(1, 3)$ and $(\bar{3}, \bar{1})$).*

$$B = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ -1 \\ -2 \\ -3 \\ -4 \end{array} \begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & -1 & -2 & -3 & -4 \\ \hline b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} & 0 & b_{1,-2} & b_{1,-3} & b_{1,-4} \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} & -b_{1,-2} & 0 & b_{2,-3} & b_{2,-4} \\ b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} & -b_{1,-3} & -b_{2,-3} & 0 & b_{3,-4} \\ b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} & -b_{1,-4} & -b_{2,-4} & -b_{3,-4} & 0 \\ \hline 0 & -b_{-2,1} & -b_{-3,1} & -b_{-4,1} & -b_{1,1} & -b_{2,1} & -b_{3,1} & -b_{4,1} \\ b_{-2,1} & 0 & -b_{-3,2} & -b_{-4,2} & -b_{1,2} & -b_{2,2} & -b_{3,2} & -b_{4,2} \\ b_{-3,1} & b_{-3,2} & 0 & -b_{-4,3} & -b_{1,3} & -b_{2,3} & -b_{3,3} & -b_{4,3} \\ b_{-4,1} & b_{-4,2} & b_{-4,3} & 0 & -b_{1,4} & -b_{2,4} & -b_{3,4} & -b_{4,4} \\ \hline \end{array}$$

$$= \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ \bar{4} \\ \bar{3} \\ \bar{2} \\ \bar{1} \end{array} \begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & \bar{4} & \bar{3} & \bar{2} & \bar{1} \\ \hline b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} & 0 & b_{1,3} & b_{1,\bar{2}} & b_{1,\bar{1}} \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} & -b_{1,\bar{3}} & 0 & b_{2,\bar{2}} & b_{2,\bar{1}} \\ b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} & -b_{1,\bar{2}} & -b_{2,\bar{2}} & 0 & b_{3,\bar{1}} \\ b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} & -b_{1,\bar{1}} & -b_{2,\bar{1}} & -b_{3,\bar{1}} & 0 \\ \hline 0 & -b_{\bar{3},1} & -b_{\bar{2},1} & -b_{\bar{1},1} & -b_{1,1} & -b_{2,1} & -b_{3,1} & -b_{4,1} \\ b_{\bar{3},1} & 0 & -b_{\bar{2},2} & -b_{\bar{1},2} & -b_{1,2} & -b_{2,2} & -b_{3,2} & -b_{4,2} \\ b_{\bar{2},1} & b_{\bar{2},2} & 0 & -b_{\bar{1},3} & -b_{1,3} & -b_{2,3} & -b_{3,3} & -b_{4,3} \\ b_{\bar{1},1} & b_{\bar{1},2} & b_{\bar{1},3} & 0 & -b_{1,4} & -b_{2,4} & -b_{3,4} & -b_{4,4} \\ \hline \end{array}$$

$$
A = \begin{array}{c|cccc|cccc|}
 & 1 & 2 & 3 & 4 & -1 & -2 & -3 & -4 \\
\hline
1 & b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} & b_{1,-4} & b_{1,-3} & b_{1,-2} & 0 \\
2 & b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} & b_{2,-4} & b_{2,-3} & 0 & -b_{1,-2} \\
3 & b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} & b_{3,-4} & 0 & -b_{2,-3} & -b_{1,-3} \\
4 & b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} & 0 & -b_{3,-4} & -b_{2,-4} & -b_{1,-4} \\
\hline
-1 & b_{-4,1} & b_{-4,2} & b_{-4,3} & 0 & -b_{4,4} & -b_{3,4} & -b_{2,4} & -b_{1,4} \\
-2 & b_{-3,1} & b_{-3,2} & 0 & -b_{-4,3} & -b_{4,3} & -b_{3,3} & -b_{2,3} & -b_{1,3} \\
-3 & b_{-2,1} & 0 & -b_{-3,2} & -b_{-4,2} & -b_{4,2} & -b_{3,2} & -b_{2,2} & -b_{1,2} \\
-4 & 0 & -b_{-2,1} & -b_{-3,1} & -b_{-4,1} & -b_{4,1} & -b_{3,1} & -b_{2,1} & -b_{1,1} \\
\hline
\end{array}
$$

$$
= \begin{array}{c|cccc|cccc|}
 & 1 & 2 & 3 & 4 & \bar{4} & \bar{3} & \bar{2} & \bar{1} \\
\hline
1 & b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} & b_{1,\bar{1}} & b_{1,\bar{2}} & b_{1,\bar{3}} & 0 \\
2 & b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} & b_{2,\bar{1}} & b_{2,\bar{2}} & 0 & -b_{1,\bar{3}} \\
3 & b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} & b_{3,\bar{1}} & 0 & -b_{2,\bar{2}} & -b_{1,\bar{2}} \\
4 & b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} & 0 & -b_{3,\bar{1}} & -b_{2,\bar{1}} & -b_{1,\bar{1}} \\
\hline
\bar{4} & b_{\bar{1},1} & b_{\bar{1},2} & b_{\bar{1},3} & 0 & -b_{4,4} & -b_{3,4} & -b_{2,4} & -b_{1,4} \\
\bar{3} & b_{\bar{2},1} & b_{\bar{2},2} & 0 & -b_{\bar{1},3} & -b_{4,3} & -b_{3,3} & -b_{2,3} & -b_{1,3} \\
\bar{2} & b_{\bar{3},1} & 0 & -b_{\bar{2},2} & -b_{\bar{1},2} & -b_{4,2} & -b_{3,2} & -b_{2,2} & -b_{1,2} \\
\bar{1} & 0 & -b_{\bar{3},1} & -b_{\bar{2},1} & -b_{\bar{1},1} & -b_{4,1} & -b_{3,1} & -b_{2,1} & -b_{1,1} \\
\hline
\end{array}
$$

A.3.6 Remark. *Let $N := 2n$ and $A = (A_{i,j}) \in \mathrm{Mat}_{N\times N}(K)$. Then $(J_N^+ A^\top J_N^+)_{i,j} = A_{\bar{j},\bar{i}}$ for all $1 \le i,j \le N$. Note that $J_N^+ A^\top J_N^+$ is obtained from A by reflecting A at its anti-diagonal.*

Proof. Let $1 \le i,j \le N$, then $J_N^+ e_{i,j}^\top J_N^+ = \sum_{i=1}^N e_{k,\bar{k}} e_{j,i} J_N^+ = e_{\bar{j},i} J_N^+ = e_{\bar{j},i} \sum_{i=1}^N e_{k,\bar{k}} = e_{\bar{j},\bar{i}}$. $\square$

A.3.7 Notation. *Define the following sets of matrix entry coordinates*

$$
\begin{aligned}
\square &:= \{(i,j) \mid 1 \le i,j \le N\}, \\
\triangledown &:= \{(i,j) \in \square \mid 1 \le i < j \le N\}, \\
\bigtriangledown &:= \{(i,j) \in \square \mid i < j < \bar{i}\}.
\end{aligned}
$$

Referring to [Car72, 11.2.3] and [Car05, §8.2], for a fixed $n \ge 4$, we construct a matrix Lie algebra of type D_n and determine a Chevalley basis.

Let $J_{2n} = J_{2n}^+$, then $J_{2n}^{-1} = J_{2n}^\top = J_{2n}$. We consider the $2n \times 2n$ matrices A satisfying

$$
\begin{aligned}
A^\top J_{2n} + J_{2n} A = 0 &\iff J_{2n} A^\top J_{2n} + A = 0 \\
&\overset{\text{A.3.6}}{\iff} A_{i,j} + A_{\bar{j},\bar{i}} = 0 \quad \text{for all } 1 \le j \le \bar{i} \le 2n \\
&\iff \begin{cases} A_{i,j} + A_{\bar{j},\bar{i}} = 0 & \text{if } 1 \le j < \bar{i} \le 2n \\ A_{i,j} = 0 & \text{if } 1 \le j = \bar{i} \le 2n \end{cases}.
\end{aligned}
$$

Let $\mathcal{L}_{D_n}$ be the Lie algebra of all such matrices and $\mathcal{H}_{D_n}$ be the set of diagonal matrices in $\mathcal{L}_{D_n}$. The elements of $\mathcal{H}_{D_n}$ have the form

$$h = \begin{pmatrix} \lambda_1 & & & & & & & \\ & \lambda_2 & & & & & & \\ & & \ddots & & & & & \\ & & & \lambda_n & & & & \\ & & & & -\lambda_n & & & \\ & & & & & \ddots & & \\ & & & & & & -\lambda_2 & \\ & & & & & & & -\lambda_1 \end{pmatrix}_{N\times N}.$$

We have

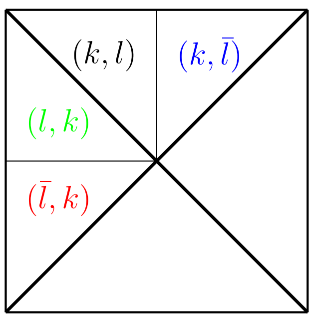
$$\mathcal{L}_{D_n} = \mathcal{H}_{D_n} \oplus \sum_{1\le i<j<\bar{i}\le 2n} \mathbb{C}(e_{i,j} - e_{\bar{j},\bar{i}}) \oplus \sum_{1\le j<i<\bar{j}\le 2n} \mathbb{C}(e_{i,j} - e_{\bar{j},\bar{i}}) = \mathcal{H}_{D_n} \oplus \sum_r \mathbb{C}e_r,$$

where $e_r = e_{i,j} - e_{\bar{j},\bar{i}}$ $(1 \le j < i < \bar{j} \le 2n$ or $1 \le j < i < \bar{j} \le 2n)$.

The diagonal matrix $h := \sum_{k=1}^{2n} h_{k,k}e_{k,k} \in \mathcal{H}_{D_n}$ transforms these root vectors according to the following formulae: for $1 \le j < i < \bar{j} \le 2n$ or $1 \le j < i < \bar{j} \le 2n$, we have

$$\begin{aligned}[][h, e_{i,j} - e_{\bar{j},\bar{i}}] &= \left(\sum_{k=1}^{2n} h_{k,k}e_{k,k}\right)(e_{i,j} - e_{\bar{j},\bar{i}}) - (e_{i,j} - e_{\bar{j},\bar{i}})\left(\sum_{k=1}^{2n} h_{k,k}e_{k,k}\right) \\ &= h_{i,i}e_{i,j} - h_{\bar{j},\bar{j}}e_{\bar{j},\bar{i}} - h_{j,j}e_{i,j} + h_{\bar{i},\bar{i}}e_{\bar{j},\bar{i}} = (h_{i,i} - h_{j,j})(e_{i,j} - e_{\bar{j},\bar{i}}).\end{aligned}$$

Let $1 \le k < l \le n$, then the following picture shows the positions of the coordinates.



So

$$\begin{cases} [h, e_{i,j} - e_{\bar{j},\bar{i}}] = (\lambda_k - \lambda_l)(e_{i,j} - e_{\bar{j},\bar{i}}) & \text{if } (i,j) = (k,l) \text{ (i.e. } 1 \le i < j \le n) \\ [h, e_{i,j} - e_{\bar{j},\bar{i}}] = (\lambda_k + \lambda_l)(e_{i,j} - e_{\bar{j},\bar{i}}) & \text{if } (i,j) = (k,\bar{l}) \text{ (i.e. } n < j < \bar{i} \le 2n) \\ [h, e_{i,j} - e_{\bar{j},\bar{i}}] = -(\lambda_k - \lambda_l)(e_{i,j} - e_{\bar{j},\bar{i}}) & \text{if } (i,j) = (l,k) \text{ (i.e. } 1 \le j < i \le n) \\ [h, e_{i,j} - e_{\bar{j},\bar{i}}] = -(\lambda_k + \lambda_l)(e_{i,j} - e_{\bar{j},\bar{i}}) & \text{if } (i,j) = (\bar{l},k) \text{ (i.e. } n < i < \bar{j} \le 2n) \end{cases}.$$

Thus $\mathcal{L}_{D_n}$ is a Lie simple algebra of type D_n, and the above decomposition is a Cartan decomposition with respect to $\mathcal{H}_{D_n}$ (see [Car05, §8.2]).

Let $\mathcal{V}_n := \mathcal{V}_{D_n} := \langle \mathcal{H}^*_{D_n} \rangle_{\mathbb{R}}$. For $1 \leq i \leq n$, let $\varepsilon_i : h \mapsto \lambda_i \in \mathcal{H}^*_{D_n}$, then the set

$$\Phi_{D_n} := \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\} \subseteq \mathcal{H}^*_{D_n}$$

is a root system of type D_n.

A fundamental system of the root system Φ_{D_n} is

$$\Delta_{D_n} :=\{\varepsilon_1 - \varepsilon_2,\ \ldots,\ \varepsilon_{n-1} - \varepsilon_n,\ \varepsilon_{n-1} + \varepsilon_n\}.$$

The collection of positive roots (relative to Δ_{D_n}) is

$$\Phi^+_{D_n} := \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\}.$$

The co-roots of $\mathcal{L}_{D_n}$ are the elements

$$[e_{i,j} - e_{\bar{j},\bar{i}}, e_{j,i} - e_{\bar{i},\bar{j}}] = (e_{i,i} - e_{\bar{i},\bar{i}}) - (e_{j,j} - e_{\bar{j},\bar{j}})$$

where $1 \leq j < i < \bar{j} \leq 2n$ or $1 \leq j < i < \bar{j} \leq 2n$ Let $1 \leq k < l \leq n$, then

$$\begin{cases} [e_{i,j} - e_{\bar{j},\bar{i}}, e_{j,i} - e_{\bar{i},\bar{j}}] = (e_{k,k} - e_{\bar{k},\bar{k}}) - (e_{l,l} - e_{\bar{l},\bar{l}}) & \text{if } (i,j) = (k,l) \\ [e_{i,j} - e_{\bar{j},\bar{i}}, e_{j,i} - e_{\bar{i},\bar{j}}] = (e_{k,k} - e_{\bar{k},\bar{k}}) + (e_{l,l} - e_{\bar{l},\bar{l}}) & \text{if } (i,j) = (k,\bar{l}) \end{cases}.$$

In this case we have $[h_r, e_r] = 2e_r$.

Let θ be the map of $\mathcal{L}_{D_n}$ into itself given by $\theta(x) = -x^\top$, where $x^\top$ is the tranpose of x. Since

$$\theta([x,y]) = -[x,y]^\top = -(xy-yx)^\top = -(y^\top x^\top - x^\top y^\top) = [x^\top, y^\top] = [\theta(x), \theta(y)],$$

θ is an automorphism of $\mathcal{L}_{D_n}$. Then $\theta(e_r) = -e_r^\top = -e_{-r}$ for all $r \in \Phi_{D_n}$. Now we have

$$[e_r, e_s] = N_{r,s}e_{r+s} \implies \theta([e_r, e_s]) = N_{r,s}\theta(e_{r+s}) \implies [e_{-r}, e_{-s}] = -N_{r,s}e_{-r-s}.$$

Since $[e_{-r}, e_{-s}] = N_{-r,-s}e_{-r-s}$, $N_{-r,-s} = -N_{r,s}$. By [Car72, 4.1.2],

$$N_{r,s}N_{-r,-s} = -(n_{r,s}+1)^2,$$

it follows that $N_{r,s} = \pm(n_{r,s}+1)$. Thus the fundamental co-roots h_r together with e_r, e_{-r} (r positive) form a Chevalley basis.

We have proved the following results:

A.3.8 Proposition (Chevalley basis of $\mathcal{L}_{D_n}$)**.** *Let $n \geq 4$, then the Lie algebra*

$$\mathcal{L}_{D_n} = \{A \in \mathrm{Mat}_{2n\times 2n}(\mathbb{C}) \mid A^\top J_{2n}^+ + J_{2n}^+ A = 0\}$$

is simple of type D_n, $\mathcal{H}_{D_n}$ is a Cartan subalgebra and a Chevalley basis is obtained as follows:

$$\begin{cases} e_r = e_{i,j} - e_{\bar{j},\bar{i}}, & \{r \in \Phi_{D_n}^+\} \wedge \{(i,j) \in \nabla\} \\ e_{-r} = e_r^\top, & r \in \Phi_{D_n}^+ \\ h_r = [e_r, e_{-r}], & r \in \Delta_{D_n} \end{cases}.$$

A.3.9 Corollary. *For all $r \in \Phi_{D_n}$, $e_r^2 = 0$.*

Proof. By A.3.8, $e_r = e_{i,j} - e_{\bar{j},\bar{i}}$, then

$$e_r^2 = (e_{i,j} - e_{\bar{j},\bar{i}})^2 = e_{i,j}^2 - e_{i,j}e_{\bar{j},\bar{i}} - e_{\bar{j},\bar{i}}e_{i,j} + e_{\bar{j},\bar{i}}^2 \overset{i\neq j}{=} -e_{i,j}e_{\bar{j},\bar{i}} - e_{\bar{j},\bar{i}}e_{i,j}.$$

Suppose $i = \bar{i}$, then

$$i = \bar{i} = N + 1 - i \implies i = \frac{N+1}{2} = n + \frac{1}{2} \notin \mathbb{N}.$$

This is a contradiction, so $i \neq \bar{i}$ for all $0 \leq i \leq N$. Thus $e_r^2 = 0$. □

A.3.10 Notation. *Set $h_i := e_{i,i} - e_{\bar{i},\bar{i}}$ for all $1 \leq i \leq n$.*

A.3.11 Proposition. *Let $x = \sum_{i=1}^n x_i h_i \in \mathcal{H}_{D_n}$, $y = \sum_{j=1}^n y_j h_j \in \mathcal{H}_{D_n}$, then*

$$\kappa(x,y) = 4(n-1)\sum_{i=1}^n (x_i y_i).$$

In particular, $\kappa(h_i, h_j) = 4(n-1)\delta_{i,j}$.

Proof. We know

$$\begin{cases} (\mathrm{ad}\, x\, \mathrm{ad}\, y)(e_{\varepsilon_i-\varepsilon_j}) = (x_i - x_j)(y_i - y_j)e_{\varepsilon_i-\varepsilon_j} \\ (\mathrm{ad}\, x\, \mathrm{ad}\, y)(e_{\varepsilon_i+\varepsilon_j}) = (x_i + x_j)(y_i + y_j)e_{\varepsilon_i+\varepsilon_j} \\ (\mathrm{ad}\, x\, \mathrm{ad}\, y)(e_{\varepsilon_i-\varepsilon_j}) = (-x_i - x_j)(-y_i - y_j)e_{-\varepsilon_i-\varepsilon_j} \\ (\mathrm{ad}\, x\, \mathrm{ad}\, y)(e_{\varepsilon_i-\varepsilon_j}) = (-x_i + x_j)(-y_i + y_j)e_{-\varepsilon_i+\varepsilon_j} \\ (\mathrm{ad}\, x\, \mathrm{ad}\, y)(h_r) = O_8 & \text{if } r \in \Delta_{D_n} \end{cases}$$

then

$$\begin{aligned} \kappa(x,y) =& \mathrm{tr}(\mathrm{ad}\, x\, \mathrm{ad}\, y) = \sum_{\mu\in\Phi} \mu(x)\mu(y) \\ =& 2\sum_{1\leq i<j\leq n}(x_i - x_j)(y_i - y_j) + 2\sum_{1\leq i<j\leq n}(x_i + x_j)(y_i + y_j) \\ =& 4\sum_{1\leq i<j\leq n}(x_i y_i + x_j y_j) = 4(n-1)\sum_{i=1}^n (x_i y_i). \end{aligned}$$

□

A.3.12 Proposition. *Let $1 \le i < j \le n$, $\pm\varepsilon_i \pm \varepsilon_j \in \Phi_{D_n}$, then*

$$\begin{cases} t_{\varepsilon_i-\varepsilon_j} = \frac{1}{4(n-1)}(h_i - h_j) \\ t_{\varepsilon_i+\varepsilon_j} = \frac{1}{4(n-1)}(h_i + h_j) \\ t_{-\varepsilon_i-\varepsilon_j} = \frac{1}{4(n-1)}(-h_i - h_j) \\ t_{-\varepsilon_i+\varepsilon_j} = \frac{1}{4(n-1)}(-h_i + h_j) \end{cases}$$

In particular, $(r, r) = \kappa(t_r, t_r) = \frac{2}{4(n-1)}$ for all $r \in \Phi_{D_n}$.

Proof. Let $h = \sum_{i=1}^{n} \lambda_i h_i \in \mathcal{L}_{D_n}$, then

$$\begin{aligned} \kappa\left(\frac{1}{4(n-1)}(h_i - h_j), h\right) =& 4(n-1)\sum_{k=1}^{n}\left(\frac{1}{4(n-1)}(h_i - h_j)_{k,k} \cdot h_{k,k}\right) \\ =& \lambda_i - \lambda_j = (\varepsilon_i - \varepsilon_j)(h), \end{aligned}$$

and

$$(\varepsilon_i - \varepsilon_j, \varepsilon_i - \varepsilon_j) = \kappa\left(\frac{1}{4(n-1)}(h_i - h_j), \frac{1}{4(n-1)}(h_i - h_j)\right) = \frac{2}{4(n-1)}.$$

Similarly, the other formulae are determined. $\square$

A.4. Sylow p-subgroups of $D_n^{syl}(q)$

Fix $n \in \mathbb{N}^*$, $N := 2n$, $p \neq 2$ and K be a field. We determine a matrix Sylow p-subgroup $D_n^{syl}(q)$ of the Chevalley group $D_n(q)$. We mainly refer to [Car72], [Jed13] and [Gro02].

Set the matrix group $\bar{D}_n(q) := \left\langle \exp(t\, e_r) \;\middle|\; r \in \Phi_{D_n},\ t \in \mathbb{F}_q \right\rangle$. The Chevalley group $D_n(q)$ of type $\mathcal{L}_{D_n}$ over the field $\mathbb{F}_q$ is denoted by

$$D_n(q) := \left\langle \exp(t \operatorname{ad} e_r) \;\middle|\; r \in \Phi_{D_n},\ t \in \mathbb{F}_q \right\rangle$$

By the adaption of [Car72, Theorem 11.3.2], $\bar{D}_n(q)$ is a commutator subgroup, denoted by $\Omega_{2n}(q, f_D)$, of the (positive) orthogonal group $O_{2n}(q, f_D)$, where f_D is the quadratic form

$$x_1 x_{\bar{1}} + x_2 x_{\bar{2}} + \cdots + x_n x_{\bar{n}}.$$

By [Car72, §11.3], $D_n(q) \cong \bar{D}_n(q)/Z(\bar{D}_n(q))$, and $D_n(q)$ is isomorphic to the projective group

$$P\Omega_{2n}(q, f_D) = \Omega_{2n}(q, f_D)/Z(O_{2n}(q, f_D)) \cap \Omega_{2n}(q, f_D).$$

We set

$$D_n^{syl}(q) := \left\langle \exp(te_r) \;\middle|\; r \in \Phi^+_{D_n},\ t \in \mathbb{F}_q \right\rangle.$$

Since $D_n^{syl}(q) \cap Z(O_{2n}(q, f_D)) = \{I_{2n}\}$, $D_n^{syl}(q)$ is a Sylow p-subgroup of the Chevalley group $D_n(q)$. Compare the orders of $O_{2n}(q, f_D)$ and $P\Omega_{2n}(q, f_D)$ (see [Gro02, Chapter 9]), $D_n^{syl}(q)$ is also a Sylow p-subgroup of $O_{2n}(q, f_D)$.

We set $x_r(t) := \exp(te_r) = I_{2n} + t \cdot e_r$ for all $r \in \Phi_{D_4}$ and $t \in \mathbb{F}_q$, and the root subgroups $X_r := \{x_r(t) \mid t \in \mathbb{F}_q\}$ for all $r \in \Phi_{D_n}$.

A.4.1 Lemma ([Car65]). *The order of the group $D_n(q)$ is*

$$|D_n(q)| = \frac{1}{(4, q^n - 1)} q^{n(n-1)}(q^n - 1)(q^{2n-2} - 1)(q^{2n-4} - 1) \cdots (q^4 - 1)(q^2 - 1).$$

where $(4, q^n - 1)$ is the greatest common divisor of 4 and $q^n - 1$.

A.4.2 Lemma. *Let $r, s \in \Phi^+_{D_n}$ and $r \neq s$, then*

$$\overline{\operatorname{supp}}(X_r) \cap \overline{\operatorname{supp}}(X_s) = \emptyset.$$

Proof. By 1.1.8, we have $\overline{\operatorname{supp}}(X_r) \cap \overline{\operatorname{supp}}(X_s) = \operatorname{supp}(e_r) \cap \operatorname{supp}(e_s)$. By A.3.8, $\overline{\operatorname{supp}}(e_r) \cap \overline{\operatorname{supp}}(e_s) = \emptyset$. □

The following property is well known.

A.4.3 Lemma. *For $(i,j) \in \daleth$ and $t \in \mathbb{F}_q$, let $\tilde{x}_{i,j}(t) := I_{2n} + te_{i,j}$. Then*

$$A_{2n}(\mathbb{F}_q) = \left\{ \prod_{(i,j)\in\daleth} \tilde{x}_{i,j}(t_{i,j}) \,\middle|\, t_{i,j} \in \mathbb{F}_q \right\}$$

where the product can be taken in an arbitrary, but fixed, order. In particular,

$$\prod_{(i,j)\in\daleth} \tilde{x}_{i,j}(t_{i,j}) = \prod_{(i,j)\in\daleth} \tilde{x}_{i,j}(s_{i,j}) \Longleftrightarrow t_{i,j} = s_{i,j} \text{ for all } (i,j) \in \daleth.$$

A.4.4 Proposition.

$$D_n^{syl}(q) = \left\{ \prod_{r\in\Phi^+_{D_n}} x_r(t_r) \,\middle|\, t_r \in \mathbb{F}_q \right\}.$$

where the product can be taken in an arbitrary, but fixed, order.

Proof. We know $D_n^{syl}(q) \overset{\text{definition}}{\supseteq} \left\{ \prod_{r\in\Phi^+_{D_n}} x_r(t_r) \,\middle|\, t_r \in \mathbb{F}_q \right\}$, where the product is taken in an arbitrary, but fixed, order. Suppose

$$\prod_{r\in\Phi^+_{D_n}} x_r(t_r) = \prod_{r\in\Phi^+_{D_n}} x_r(s_r).$$

We have $x_r(t_r) = (I_{2n} + t_r e_{i,j})(I_{2n} - t_r e_{\bar{j},\bar{i}})$. By A.4.2 and A.4.3, we get $t_r = s_r$ $(r \in \Phi^+_{D_n})$. Thus

$$\left| \left\{ \prod_{r\in\Phi^+_{D_n}} x_r(t_r) \,\middle|\, t_r \in \mathbb{F}_q \right\} \right| = q^{n(n-1)}.$$

Therefore the desired formula is obtained. □

A.4.5 Remark. *With the similar method of this section, we determine matrix Sylow p-subgroups of Chevalley groups of type E_6 and E_7.*

A.5. Structure constants of the Lie algebra $\mathcal{L}_{D_4}$

In this section, we determine the suitable structure constants of the Lie algebra $\mathcal{L}_{D_4}$ such that the construction of the Steinberg triality group as simple as possible.

A.5.1 Notation. *Define the following sets of matrix entry coordinates*

$$\begin{aligned}\square :=&\{(i,j) \mid 1 \leq i,j \leq 8\},\\ \text{◹} :=&\{(i,j) \mid 1 \leq i < j \leq 8\},\\ \triangledown :=&\{(i,j) \in \square \mid i < j < 9-i\}.\end{aligned}$$

A.5.2 Definition/Lemma ([Jed13], 1.1.2)**.** *Define the* ***mirror map***

$$\bar{\cdot}: \{1,2,\ldots,8\} \to \{1,2,\ldots,8\} : i \mapsto \bar{i} := 9-i.$$

The map mirrors every entry on $\frac{9}{2}$*. It satisfies* $\bar{\bar{i}} = i$*.*

A.5.3 Notation/Lemma. *Set*

$$J_8^+ := \sum_{i=1}^{8} e_{i,9-i} = \begin{pmatrix} &&&&&&&1\\ &&&&&&1&\\ &&&&&1&&\\ &&&&1&&&\\ &&&1&&&&\\ &&1&&&&&\\ &1&&&&&&\\ 1&&&&&&& \end{pmatrix}_{8\times 8} \in \mathrm{GL}_8(\mathbb{C}).$$

In particular, J_8^+ *is the transpose and also the inverse of* J_8^+*.*

By A.3, the set

$$\{A \in \mathrm{Mat}_{8\times 8}(\mathbb{C}) \mid A^\top J_8^+ + J_8^+ A = 0\}$$

forms a simple Lie algebra of type D_4.

A.5.4 Notation/Lemma. *For* $1 \leq i \leq 4$*, let* $h_i := e_{i,i} - e_{9-i,9-i} \in \mathrm{Mat}_{8\times 8}(\mathbb{C})$*, then a Cartan subalgebra of* $\mathcal{L}_{D_4}$ *is*

$$\begin{aligned}\mathcal{H}_{D_4} =&\{\sum_{i=1}^{8} \lambda_i h_i \mid \lambda_i \in \mathbb{C}\}\\ =&\left\{ \begin{pmatrix} \lambda_1 &&&&&&&\\ &\lambda_2&&&&&&\\ &&\lambda_3&&&&&\\ &&&\lambda_4&&&&\\ &&&&-\lambda_4&&&\\ &&&&&-\lambda_3&&\\ &&&&&&-\lambda_2&\\ &&&&&&&-\lambda_1 \end{pmatrix}_{8\times 8} \middle| \lambda_i \in \mathbb{C} \right\}.\end{aligned}$$

Let $\mathcal{V}_4 := \mathcal{V}_{D_4} := \langle \mathcal{H}^*_{D_4} \rangle_{\mathbb{R}}$ be a $\mathbb{R}$-vector space (see Appendix A.3) and $h := \sum_{i=1}^{4} \lambda_i h_i$. For $1 \leq i \leq 4$, let $\varepsilon_i \in \mathcal{H}^*_{D_4}$ and $\varepsilon_i \colon h \mapsto \lambda_i$, then the set

$$\Phi_{D_4} = \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq 4\}$$

is a root system of type D_4. The fundamental system of roots of the root system Φ_{D_4} is

$$\Delta_{D_4} = \{\varepsilon_1 - \varepsilon_2,\ \varepsilon_2 - \varepsilon_3,\ \varepsilon_3 - \varepsilon_4,\ \varepsilon_3 + \varepsilon_4\}.$$

We set $r_1 := \varepsilon_1 - \varepsilon_2,\ r_2 := \varepsilon_2 - \varepsilon_3,\ r_3 := \varepsilon_3 - \varepsilon_4,\ r_4 := \varepsilon_3 + \varepsilon_4$. Then $\Delta_{D_4} = \{r_1, r_2, r_3, r_4\}$. The **positive system of roots** of Φ_{D_4} is

$$\Phi^+_{D_4} := \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq 4\}.$$

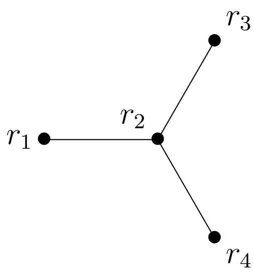
We define the Killing form as follows:

$$\kappa \colon \mathcal{L}_{D_4} \times \mathcal{L}_{D_4} \to \mathbb{C} : (x, y) \mapsto \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y).$$

For all $r \in \Phi_{D_4}$, there exists a unique element $t_r \in \mathcal{H}_{D_4}$ satisfying

$$r(h) = \kappa(t_r, h) \qquad \text{for all } h \in \mathcal{H}_{D_4}.$$

Then $\kappa(h_i, h_j) = 12\delta_{i,j}$ and and $(r, r) = \frac{2}{12}$ for all $r \in \Phi_{D_4}$. The Dynkin diagram of D_4 is



The Cartan matrix M of D_4 is

$$\begin{array}{c c} & \begin{array}{cccc} r_1 & r_2 & r_3 & r_4 \end{array} \\ \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \end{array} & \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix} \end{array}$$

where $M_{i,j} = \langle r_i, r_j \rangle = \frac{2(r_i, r_j)}{(r_i, r_i)}$ for all $r_i, r_j \in \Delta_{D_4}$.

A Chevalley basis is obtained as follows:

$$\begin{cases} e_r = e_{i,j} - e_{\bar{j},\bar{i}}, & \{r \in \Phi^+_{D_4}\} \wedge \{(i,j) \in \nabla\} \\ e_{-r} = e_r^{\top}, & r \in \Phi^+_{D_4} \\ h_r = [e_r, e_{-r}], & r \in \Delta_{D_4} \end{cases}.$$

A.5.5 Definition. *Let* $r := \sum_{i=1}^{4} x_i r_i \in \mathcal{V}_4$*,* $s := \sum_{i=1}^{4} y_i r_i \in \mathcal{V}_4$*, then we write*

$$r \prec s,$$

if $\sum_{i=1}^{4} x_i < \sum_{i=1}^{4} y_i$*, or if* $\sum_{i=1}^{4} x_i = \sum_{i=1}^{4} y_i$ *and the first non-zero coefficient* $x_i - y_i$ *is positive. If* $r = \sum_{i=1}^{4} x_i r_i \in \Phi_{D_4}$*, the* ***height*** *of* r *(relative to* Δ_{D_4}*) is denoted by* $\mathrm{ht}(r) := \sum_{i=1}^{4} x_i$*.*

Let

$$\begin{aligned} r_5 &:= r_1 + r_2 &&= \varepsilon_1 - \varepsilon_3, & r_6 &:= r_2 + r_3 &&= \varepsilon_2 - \varepsilon_4,\\ r_7 &:= r_2 + r_4 &&= \varepsilon_2 + \varepsilon_4, & r_8 &:= r_1 + r_2 + r_3 &&= \varepsilon_1 - \varepsilon_4,\\ r_9 &:= r_1 + r_2 + r_4 &&= \varepsilon_1 + \varepsilon_4, & r_{10} &:= r_2 + r_3 + r_4 &&= \varepsilon_2 + \varepsilon_3,\\ r_{11} &:= r_1 + r_2 + r_3 + r_4 &&= \varepsilon_1 + \varepsilon_3, & r_{12} &:= r_1 + 2r_2 + r_3 + r_4 &&= \varepsilon_1 + \varepsilon_2. \end{aligned}$$

So, $\Phi^+_{D_4} = \{r_i \mid i = 1, 2, \ldots, 12\}$. We determine the following total order of $\Phi^+_{D_4}$

$$0 \prec r_1 \prec r_2 \prec r_3 \prec r_4 \prec r_5 \prec r_6 \prec r_7 \prec r_8 \prec r_9 \prec r_{10} \prec r_{11} \prec r_{12}.$$

An ordered pair $(r,\ s)$ of roots is called a **special pair**, if $r + s \in \Phi_{D_4}$ and $0 \prec r \prec s$. An ordered pair is called **extraspecial** if $(r,\ s)$ is a special pair and if for all special pairs $(r',\ s')$ with $r' + s' = r + s$ we have $r \preccurlyeq r'$. Then every root in $\Phi^+_{D_4}$ which is the sum of two roots in $\Phi^+_{D_4}$ can be expressed uniquely as the sum of an extraspecial pair. The extraspecial pairs in $\Phi^+_{D_4}$ are: $(r_1,\ r_2)$, $(r_2,\ r_3)$, $(r_2,\ r_4)$, $(r_1,\ r_2 + r_3)$, $(r_1,\ r_2 + r_4)$, $(r_2,\ r_3 + r_4)$, $(r_1,\ r_2 + r_3 + r_4)$ and $(r_2,\ r_1 + r_2 + r_3 + r_4)$. The signs of the structure constants $N_{r,s}$ may be chosen arbitrarily for extraspecial pairs $(r,\ s)$, then the structure constants for all pairs are uniquely determined.

A.5.6 Example. *We choose the following signs of the structure constants of the Lie algebra* $\mathcal{L}_{D_4}$*.*

$r_i \in \Phi^+_{D_4} \backslash \Delta_{D_4}$	*extraspecial pairs* (r, s)	$N_{r,s}$
$r_5 = r_1 + r_2$	$(r_1,\ r_2)$	-1
$r_6 = r_2 + r_3$	$(r_2,\ r_3)$	1
$r_7 = r_2 + r_4$	$(r_2,\ r_4)$	1
$r_8 = r_1 + r_2 + r_3$	$(r_1,\ r_2 + r_3) = (r_1,\ r_6)$	1
$r_9 = r_1 + r_2 + r_4$	$(r_1,\ r_2 + r_4) = (r_1,\ r_7)$	1
$r_{10} = r_2 + r_3 + r_4$	$(r_3,\ r_2 + r_4) = (r_3,\ r_7)$	1
$r_{11} = r_1 + r_2 + r_3 + r_4$	$(r_1,\ r_2 + r_3 + r_4) = (r_1,\ r_{10})$	1
$r_{12} = r_1 + 2r_2 + r_3 + r_4$	$(r_2,\ r_1 + r_2 + r_3 + r_4) = (r_2,\ r_{11})$	1

Then the structure constants of the other special pairs of $r_i \in \Phi^+_{D_4} \backslash \Delta_{D_4}$ *are 1.*

According to the structure constants of Lie algebra $\mathcal{L}_{D_4}$ in Example A.5.6, we obtain another Chevalley basis $\{h_r \mid r \in \Delta_{D_4}\} \cup \{e_{\pm r} \mid r \in \Phi^+_{D_4}\}$ of the Lie algebra $\mathcal{L}_{D_4}$ of type D_4 by changing the signs of $e_{r_1+r_2}$ and $e_{-r_1-r_2}$.

A.5.7 Lemma (Chevalley basis of $\mathcal{L}_{D_4}$)**.** *The Lie algebra $\mathcal{L}_{D_4}$ has a Chevalley basis $\{h_r \mid r \in \Delta_{D_4}\} \cup \{e_{\pm r} \mid r \in \Phi^+_{D_4}\}$ as follows:*

$r \in \Phi^+_{D_4}$	h_r	e_r	e_{-r}
r_1	$(e_{1,1}-e_{8,8})-(e_{2,2}-e_{7,7})$	$e_{1,2}-e_{7,8}$	$e_{2,1}-e_{8,7}$
r_2	$(e_{2,2}-e_{7,7})-(e_{3,3}-e_{6,6})$	$e_{2,3}-e_{6,7}$	$e_{3,2}-e_{7,6}$
r_3	$(e_{3,3}-e_{6,6})-(e_{4,4}-e_{5,5})$	$e_{3,4}-e_{5,6}$	$e_{4,3}-e_{6,5}$
r_4	$(e_{3,3}-e_{6,6})+(e_{4,4}-e_{5,5})$	$e_{3,5}-e_{4,6}$	$e_{5,3}-e_{6,4}$
$r_5=r_1+r_2$	$(e_{1,1}-e_{8,8})-(e_{3,3}-e_{6,6})$	$-(e_{1,3}-e_{6,8})$	$-(e_{3,1}-e_{8,6})$
$r_6=r_2+r_3$	$(e_{2,2}-e_{7,7})-(e_{4,4}-e_{5,5})$	$e_{2,4}-e_{5,7}$	$e_{4,2}-e_{7,5}$
$r_7=r_2+r_4$	$(e_{2,2}-e_{7,7})+(e_{4,4}-e_{5,5})$	$e_{2,5}-e_{4,7}$	$e_{5,2}-e_{7,4}$
$r_8=r_1+r_2+r_3$	$(e_{1,1}-e_{8,8})-(e_{4,4}-e_{5,5})$	$e_{1,4}-e_{5,8}$	$e_{4,1}-e_{8,5}$
$r_9=r_1+r_2+r_4$	$(e_{1,1}-e_{8,8})+(e_{4,4}-e_{5,5})$	$e_{1,5}-e_{4,8}$	$e_{5,1}-e_{8,4}$
$r_{10}=r_2+r_3+r_4$	$(e_{2,2}-e_{7,7})+(e_{3,3}-e_{6,6})$	$e_{2,6}-e_{3,7}$	$e_{6,2}-e_{7,3}$
$r_{11}=r_1+r_2+r_3+r_4$	$(e_{1,1}-e_{8,8})+(e_{3,3}-e_{6,6})$	$e_{1,6}-e_{3,8}$	$e_{6,1}-e_{8,3}$
$r_{12}=r_1+2r_2+r_3+r_4$	$(e_{1,1}-e_{8,8})+(e_{2,2}-e_{7,7})$	$e_{1,7}-e_{2,8}$	$e_{7,1}-e_{8,2}$

A.5.8 Notation. *The Chevalley group $D_4(\mathbb{F}_q)$ of type $\mathcal{L}_{D_4}$ over the field $\mathbb{F}_q$ is denoted by $D_4(q) := \left\langle \exp(t\,\mathrm{ad}\,e_r) \,\middle|\, r \in \Phi_{D_4},\ t \in \mathbb{F}_q \right\rangle$. We set*

$$D_4^{syl}(q) := \left\langle \exp(te_r) \,\middle|\, r \in \Phi^+_{D_4},\ t \in \mathbb{F}_q \right\rangle.$$

By A.4, $D_4^{syl}(q)$ is a Sylow p-subgroup of the Chevalley group $D_4(q)$.

A.5.9 Corollary. *We know that $x_r(t) := \exp(te_r) = I_8 + t \cdot e_r$ for all $r \in \Phi_{D_4}$ and $t \in \mathbb{F}_q$ and the root subgroups $X_r := \{x_r(t) \mid t \in \mathbb{F}_q\}$ for all $r \in \Phi_{D_4}$. Let 1 denote the identity matrix I_8 and $t \in \mathbb{F}_q$, then*

$$\begin{aligned}
x_{r_1}(t) &=1+te_{12}-te_{78}, & x_{r_2}(t) &=1+te_{23}-te_{67}, & x_{r_3}(t) &=1+te_{34}-te_{56},\\
x_{r_4}(t) &=1+te_{35}-te_{46}, & x_{r_5}(t) &=1-t(e_{13}-e_{68}), & x_{r_6}(t) &=1+te_{24}-te_{57},\\
x_{r_7}(t) &=1+te_{25}-te_{47}, & x_{r_8}(t) &=1+te_{14}-te_{58}, & x_{r_9}(t) &=1+te_{15}-te_{48},\\
x_{r_{10}}(t) &=1+te_{26}-te_{37}, & x_{r_{11}}(t) &=1+te_{16}-te_{38}, & x_{r_{12}}(t) &=1+te_{17}-te_{28}.
\end{aligned}$$

We get the commutators of $D_4^{syl}(q)$ by calculation.

A.5.10 Lemma. *For all $r_i, r_j \in \Phi^+_{D_4}$ and $t_{r_i}, t_{r_j} \in \mathbb{F}_q$ with $i, j \in \{1, 2, \ldots, 12\}$, define the commutator*

$$[x_{r_i}(t_{r_i}), x_{r_j}(t_j)] := x_{r_i}(t_{r_i})^{-1} x_{r_j}(t_j)^{-1} x_{r_i}(t_{r_i}) x_{r_j}(t_j).$$

Then the non-trivial commutators of $D_4^{syl}(q)$ are determined as follows:

$$
\begin{aligned}
[x_{r_1}(t_{r_1}), x_{r_2}(t_{r_2})] &= x_{r_5}(-t_{r_1}t_{r_2}), & [x_{r_1}(t_{r_1}), x_{r_6}(t_{r_6})] &= x_{r_8}(t_{r_1}t_{r_6}),\\
[x_{r_1}(t_{r_1}), x_{r_7}(t_{r_7})] &= x_{r_9}(t_{r_1}t_{r_7}), & [x_{r_1}(t_{r_1}), x_{r_{10}}(t_{r_{10}})] &= x_{r_{11}}(t_{r_1}t_{r_{10}}),\\
[x_{r_2}(t_{r_2}), x_{r_3}(t_{r_3})] &= x_{r_6}(t_{r_2}t_{r_3}), & [x_{r_2}(t_{r_2}), x_{r_4}(t_{r_4})] &= x_{r_7}(t_{r_2}t_{r_4}),\\
[x_{r_2}(t_{r_2}), x_{r_{11}}(t_{r_{11}})] &= x_{r_{12}}(t_{r_2}t_{r_{11}}), & [x_{r_3}(t_{r_3}), x_{r_5}(t_{r_5})] &= x_{r_8}(t_{r_3}t_{r_5}),\\
[x_{r_3}(t_{r_3}), x_{r_7}(t_{r_7})] &= x_{r_{10}}(t_{r_3}t_{r_7}), & [x_{r_3}(t_{r_3}), x_{r_9}(t_{r_9})] &= x_{r_{11}}(t_{r_3}t_{r_9}),\\
[x_{r_4}(t_{r_4}), x_{r_5}(t_{r_5})] &= x_{r_9}(t_{r_4}t_{r_5}), & [x_{r_4}(t_{r_4}), x_{r_6}(t_{r_6})] &= x_{r_{10}}(t_{r_4}t_{r_6}),\\
[x_{r_4}(t_{r_4}), x_{r_8}(t_{r_8})] &= x_{r_{11}}(t_{r_4}t_{r_8}), & [x_{r_5}(t_{r_5}), x_{r_{10}}(t_{r_{10}})] &= x_{r_{12}}(t_{r_5}t_{r_{10}}),\\
[x_{r_6}(t_{r_6}), x_{r_9}(t_{r_9})] &= x_{r_{12}}(t_{r_6}t_{r_9}), & [x_{r_7}(t_{r_7}), x_{r_8}(t_{r_8})] &= x_{r_{12}}(t_{r_7}t_{r_8}).
\end{aligned}
$$

Let ρ be a linear transformation of $\mathcal{V}_4$ into itself arising from a non-trivial symmetry of the Dynkin diagram of $\mathcal{L}_{D_4}$ sending r_1 to r_3, r_3 to r_4, r_4 to r_1, and fixing r_2, then $\rho^3 = \mathrm{id}_{\mathcal{V}_4}$ and

$$r_1 \overset{\rho}{\mapsto} r_3 \overset{\rho}{\mapsto} r_4, \quad r_2 \overset{\rho}{\mapsto} r_2, \quad r_5 \overset{\rho}{\mapsto} r_6 \overset{\rho}{\mapsto} r_7, \quad r_8 \overset{\rho}{\mapsto} r_{10} \overset{\rho}{\mapsto} r_9, \quad r_{11} \overset{\rho}{\mapsto} r_{11}, \quad r_{12} \overset{\rho}{\mapsto} r_{12}.$$

So $\rho(\Phi_{D_4}) = \Phi_{D_4}$.

A.5.11 Lemma. *Let an automorphism of the Lie algebra $\mathcal{L}_{D_4}$ be determined by*

$$h_r \mapsto h_{\rho(r)}, \quad e_r \mapsto e_{\rho(r)}, \quad e_{-r} \mapsto e_{-\rho(r)}, \qquad (r \in \Delta_{D_4})$$

satisfying $e_r \mapsto \gamma_r e_{\rho(r)}$ $(r \in \Phi_{D_4})$ where $\{h_r \,|\, r \in \Delta_{D_4}\} \cup \{e_r \,|\, r \in \Phi_{D_4}\}$ is a Chevalley basis of $\mathcal{L}_{D_4}$ and $\gamma_r = \pm 1$. Then

$$
\begin{aligned}
&\gamma_r = 1 \quad (r \in \Phi_{D_4})\\
\iff &\begin{cases} N_{r_1,r_2} = -N_{r_2,r_3} = -N_{r_2,r_4} = \pm 1 \text{ (i.e. are all 1 or are all -1)},\\ N_{r_1,r_2+r_3} = N_{r_1,r_2+r_4} = N_{r_2,r_3+r_4} = \pm 1 \text{ (i.e. are all 1 or are all -1)}. \end{cases}
\end{aligned}
$$

Proof. (1) We know $\gamma_r = \gamma_{-r} = 1$ $(r \in \Delta_{D_4})$. The order of ρ is 3, and $\rho(r_{11}) = r_{11}$, so $e_{r_{11}} \mapsto \gamma_{r_{11}} e_{r_{11}}$ where $\gamma_{r_{11}}^3 = 1$. Hence $\gamma_{r_{11}} = 1$. Similarly, $\gamma_{r_{12}} = 1$.

(2) Let $r \in \Phi_{D_4}^+ \backslash \Delta_{D_4}$. Then r is written as $r = a + b$ where $a, b \in \Phi_{D_4}^+$, $\mathrm{ht}(a) < \mathrm{ht}(r)$ and $\mathrm{ht}(b) < \mathrm{ht}(r)$. By induction we have

$$e_a \mapsto \gamma_a e_{\rho(a)}, \quad e_b \mapsto \gamma_b e_{\rho(b)},$$

we know $[e_a, e_b] = N_{a,b} e_r$, so

$$
\begin{aligned}
[\gamma_a e_{\rho(a)}, \gamma_b e_{\rho(b)}] = N_{a,b}\gamma_r e_r &\implies \gamma_a\gamma_b N_{\rho(a),\rho(b)} e_{\rho(r)} = N_{a,b}\gamma_r e_{\rho(r)}\\
&\implies \gamma_r = \frac{\gamma_a\gamma_b N_{\rho(a),\rho(b)}}{N_{a,b}} = \pm 1.
\end{aligned}
$$

(3) Since $[e_r, e_{-r}] = h_r$ $(r \in \Phi_{D_4})$,

$$\gamma_r\gamma_{-r}[e_{\rho(r)}, e_{\rho(-r)}] = h_{\rho(r)} \implies \gamma_r\gamma_{-r} = 1 \implies \gamma_{-r} = \gamma_r.$$

(4) The following γ_{r_i} with $i \in \{5, 6, \dots, 10\}$ are determined

$$\begin{aligned}
\gamma_{r_5} =& \gamma_{r_1+r_2} = \frac{\gamma_{r_1}\gamma_{r_2}N_{r_3,r_2}}{N_{r_1,r_2}},\\
\gamma_{r_6} =& \gamma_{r_2+r_3} = \frac{\gamma_{r_2}\gamma_{r_3}N_{r_2,r_4}}{N_{r_2,r_3}},\\
\gamma_{r_7} =& \gamma_{r_2+r_4} = \frac{\gamma_{r_2}\gamma_{r_4}N_{r_2,r_1}}{N_{r_2,r_4}},\\
\gamma_{r_8} =& \gamma_{r_1+r_2+r_3} = \frac{\gamma_{r_1}\gamma_{r_2+r_3}N_{r_3,r_2+r_4}}{N_{r_1,r_2+r_3}} = \frac{\gamma_{r_3}\gamma_{r_1+r_2}N_{r_4,r_2+r_3}}{N_{r_3,r_1+r_2}},\\
\gamma_{r_9} =& \gamma_{r_2+r_3+r_4} = \frac{\gamma_{r_3}\gamma_{r_2+r_4}N_{r_4,r_1+r_2}}{N_{r_3,r_2+r_4}} = \frac{\gamma_{r_4}\gamma_{r_2+r_3}N_{r_1,r_2+r_4}}{N_{r_4,r_2+r_3}},\\
\gamma_{r_{10}} =& \gamma_{r_1+r_2+r_4} = \frac{\gamma_{r_1}\gamma_{r_2+r_4}N_{r_3,r_1+r_2}}{N_{r_1,r_2+r_4}} = \frac{\gamma_{r_4}\gamma_{r_1+r_2}N_{r_1,r_2+r_3}}{N_{r_4,r_1+r_2}}.
\end{aligned}$$

(5) It is sufficient to prove that

$$\begin{aligned}
&\gamma_{r_i} = 1, \quad i = 5, 6, 7, 8, 9, 10\\
\iff &\begin{cases} N_{r_1,r_2} = -N_{r_2,r_3} = -N_{r_2,r_4} = \pm 1,\\ N_{r_1,r_2+r_3} = N_{r_1,r_2+r_4} = N_{r_2,r_3+r_4} = \pm 1.\end{cases}
\end{aligned}$$

By the relations of the structure constants (see [Car72, Theorem 4.1.2]), we obtain that

$$\begin{aligned}
N_{r_3,r_1+r_2}N_{r_1,r_2+r_3} &= -N_{r_2,r_3}N_{r_1,r_2},\\
N_{r_4,r_2+r_3}N_{r_3,r_2+r_4} &= N_{r_2,r_4}N_{r_2,r_3},\\
N_{r_4,r_1+r_2}N_{r_1,r_2+r_4} &= -N_{r_2,r_4}N_{r_1,r_2}.
\end{aligned}$$

Then the claim (5) is proved by (4).

□

A.5.12 Corollary. *By A.5.6 and A.5.11, the Chevalley basis of $\mathcal{L}_{D_4}$ in 2.1.1 is in such a way that the automorphism of $\mathcal{L}_{D_4}$ determined by*

$$h_r \mapsto h_{\rho(r)}, \quad e_r \mapsto e_{\rho(r)}, \quad e_{-r} \mapsto e_{-\rho(r)}, \qquad (r \in \Delta_{D_4})$$

satisfies $e_r \mapsto \gamma_r e_{\rho(r)}$ $(r \in \Phi_{D_4})$ where $\gamma_r = 1$ for all $r \in \Phi_{D_4}$.

Nomenclature

$(\mathcal{X}, \mathcal{K})$	a supercharacter theory for $G_2^{syl}(q)$, page 154
$(\mathcal{X}, \mathcal{K})$	a supercharacter theory for ${}^2G_2^{syl}(3^{2m+1})$, page 195
$(\mathcal{X}, \mathcal{K})$	a supercharacter theory for ${}^3D_4^{syl}(q^3)$, page 78
$(\mathcal{X}, \mathcal{K})$	a supercharacter theory for a finite group , page 76
(f, κ)	usually denotes a monomial linearisation for a finite group
$-.-$	denotes a truncated row operation: $u.A = \pi(u^{-\top}A)$
$- \circ -$	usually denotes a right action of G (or U) on V: $A \circ g = \pi(Ag)$
$[\,,\,]$	the Lie bracket $(x, y) \mapsto [x, y]$, page 265
$[\,,\,]$	the commutator a group: $[x, y] = x^{-1}y^{-1}xy$, page 82
$[A]$	a basis element of the group algebra $\mathbb{C}U$ corresponding to $A \in V$
$\bar{i}$	the image of a positive integer i under the mirror map , page 272
Δ	the fundamental system (or base) of roots, page 268
η	an fixed element in $\mathbb{F}_{q^2} \backslash \mathbb{F}_q$ such that $\eta^q + \eta = 1$, page 243
η	an fixed element in $\mathbb{F}_{q^3} \backslash \mathbb{F}_q$ such that $\eta^{q^2} + \eta^q + \eta = 1$, page 34
κ	a non-degenerate symmetric bilinear form on V_0 called the trace form, except in §1.2 , page 32
κ	the Killing form of a Lie algebra , page 268
κ_q	equals to $\pi_q \circ \kappa$, a symmetric $\mathbb{F}_q$-bilinear form on $V_0 = \mathrm{Mat}_{8\times 8}(q^2)$, page 244
κ_q	equals to $\pi_q \circ \kappa$, a symmetric $\mathbb{F}_q$-bilinear form on $V_0 = \mathrm{Mat}_{8\times 8}(q^3)$, page 35
$\ker \chi$	the kernel of χ , page 82

λ_a	$\mathbb{C}U$-right endomorphism of $\mathbb{C}U$ given by left multiplication with a
$\langle \mathcal{H}^* \rangle_{\mathbb{R}}$	the Euclidean space spanned by the root system Φ, page 268
$\mathbb{C}$	the complex field , page 7
$\mathbb{C}\mathcal{O}_U([A])$	U-orbit module associate to A , page 17
$\mathbb{F}_q$	the finite field with q elements , page 7
$\mathbb{F}_{q^3}$	the finite field with q^3 elements , page 7
$\mathbb{N}$	the set of all nonnegative integers , page 7
$\mathbb{N}^*$	the set of all positive integers , page 7
$\mathbb{Z}$	the set of all integers , page 7
$\mathcal{K}$	the set of superclasses of $G_2^{syl}(q)$, page 157
$\mathcal{K}$	the set of superclasses of ${}^2G_2^{syl}(3^{2m+1})$, page 196
$\mathcal{K}$	the set of superclasses of ${}^3D_4^{syl}(q^3)$, page 81
$\mathcal{K}$	the set of superclasses of a finite group , page 76
$\mathcal{L}_{D_n}$	a complex Lie algebra of type D_n , page 276
$\mathcal{L}_{E_6}$	a complex Lie algebra of type E_6 , page 203
$\mathcal{L}_{E_7}$	a complex Lie algebra of type E_7 , page 249
$\mathcal{L}_{F_4}$	a complex Lie algebra of type F_4 , page 220
$\mathcal{L}_{G_2}$	a complex Lie algebra of type G_2 , page 115
$\mathcal{L}_r$	the root space of a root r , page 267
$\mathcal{M}$	the set of supermodules of $G_2^{syl}(q)$, page 157
$\mathcal{M}$	the set of supermodules of ${}^2G_2^{syl}(3^{2m+1})$, page 196
$\mathcal{M}$	the set of supermodules of ${}^3D_4^{syl}(q^3)$, page 81
$\mathcal{M}$	the set of supermodules of a finite group, page 76
$\mathcal{V}$	equals to $\langle \mathcal{H}^* \rangle_{\mathbb{R}}$, the Euclidean space spanned by the root system Φ, page 268

$\mathcal{X}$	the set of supercharacters of $G_2^{syl}(q)$, page 157
$\mathcal{X}$	the set of supercharacters of ${}^2G_2^{syl}(3^{2m+1})$, page 196
$\mathcal{X}$	the set of supercharacters of ${}^3D_4^{syl}(q^3)$, page 81
$\mathcal{X}$	the set of supercharacters of a finite group , page 76
$\operatorname{ad} x$	the adjoint map of a Lie algebra: $y \mapsto [x, y]$, page 266
$\operatorname{ConjCl}(G)$	the set of conjugacy classes of a finite group G , page 77
$\operatorname{core}(A)$	the core of $A \in V$, page 50
$\operatorname{ht}(s)$	the height of a root $s \in \Phi$, page 268
$\operatorname{Ind}_H^G \lambda$	the character induced from λ , page 82
$\operatorname{Irr}(\chi)$	the set of all irreducible constituents of the character χ , page 76
$\operatorname{Irr}(G)$	the set of all complex irreducible characters of a finite group G , page 10
$\operatorname{l.main}(A)$	the set of left main conditions of $A \in V$, page 49
$\operatorname{main}(A)$	the set of main conditions of $A \in V$, page 49
$\operatorname{main}_k(A)$	the set of the k-th main condition of $A \in V$, page 50
$\operatorname{Mat}_{M\times N}(K)$	the set of all $M \times N$ matrices with entries in the field K , page 7
$\operatorname{minor}(A)$	the set of minor conditions of $A \in V$, page 50
$\operatorname{Pstab}_U([A])$	the projective stabilizer of $[A]$ in U , page 17
$\operatorname{r.main}(A)$	the set of right main conditions of $A \in V$, page 50
reg_G	the regular character of a group G , page 76
$\operatorname{Res}_H^G \chi$	the restriction of χ to H , page 82
$\operatorname{Stab}_U(A, B)$	the intersection of the stabilizers of A and B , page 17
$\operatorname{supp}(A)$	the support of the matrix A , page 9
triv_G	the trivial character of a group G , page 77
$\operatorname{tr}(A)$	the trace of A , page 8

$\mathrm{verge}(A)$	the verge of $A \in V$, page 50
$\mathrm{verge}_k(A)$	the set of the i-th verge of $A \in V$, page 50
$\overline{\mathrm{supp}}(g)$	the essential support of the group element g , page 9
$\overline{\mathrm{supp}}(H)$	the support of the subgroup $H \leqslant A_N(K)$, page 9
Φ	the set of roots , page 267
ϕ_0	an $\mathbb{F}_q$-epimorphism $\mathbb{F}_{q^2} \to \mathbb{F}_q : t \mapsto t + t^q$, page 243
ϕ_0	an $\mathbb{F}_q$-epimorphism $\mathbb{F}_{q^3} \to \mathbb{F}_q : t \mapsto t + t^q + t^{q^2}$, page 34
π	an $\mathbb{F}_q$-epimorphism from V_0 to V , page 36
π_J	a projection of V_0
$\prec$	a total order of a Euclidean space , page 269
ρ	a linear transformation of $\mathcal{V}_4$ into itself , page 22
$\square$	the set of matrix positions , page 7
$\triangledown$	the set of matrix positions , page 7
$\mathit{\Phi}$	a map $\mathit{\Phi}\colon K^G \to KG : \tau \mapsto \sum_{g\in G} \tau(g)g$ in §1.2
ϑ	a fixed nontrivial complex linear character of the additive group $\mathbb{F}_q^+$, page 15
ζ_u	an $\mathbb{F}_q$-automorphism for a fixed $u \in \mathbb{F}_{q^3}^*$: $\mathbb{F}_{q^3} \to \mathbb{F}_{q^3} : t \mapsto ut^{q^2} + u^q t^q$, page 48
$A^\top$	the transpose of A , page 8
$A_N(K)$	the unipotent linear group of upper unitriangular $N \times N$-matrices over K , page 8
$e_{i,j}$	a matrix with entry 1 at the (i,j)-position and 0 entries otherwise , page 8
f	usually denotes a 1-cocycle of G (or U) in V
G	equals to $A_8(q)$ in §4.2 and §4.3
G	equals to $G_8(q)$ in §3.2 and §3.3
G	equals to $G_8\,(q^3)$ in §2.2 and §2.3

G'	the commutator subgroup of a group G , page 82
$G_2^{syl}(q)$	a Sylow p-subgroup of the Chevalley group $G_2(q)$, page 121
$G_8(q)$	a group such that $G_2^{syl}(q) \leqslant G_8(q) \leqslant A_8(q)$, page 122
$G_8\left(q^3\right)$	a group such that ${}^3D_4^{syl}\left(q^3\right) \leqslant G_8\left(q^3\right) \leqslant A_8\left(q^3\right)$, page 29
$I_G(\lambda)$	the inertia group in G , page 82
$I_G(L)$	the inertia group in G , page 82
I_n	identity $n \times n$-matrix , page 7
J	a subset of $\urcorner$
J^c	a subset of $\urcorner$, page 9
J_{2n}^+	an invertible $2n \times 2n$-matrix with 1s in the anti-diagonal and 0s elsewhere , page 271
K	a field , page 7
K^*	the multiplicative group $K \backslash \{0\}$ of K , page 7
K^+	the additive group of K , page 7
p	a prime number , page 7
q	a fixed power of a prime p , page 7
T^{a^*}	denotes a complete set of coset representatives of $(a^*\mathbb{F}_q^+)$ in $\mathbb{F}_{q^3}^+$ for an fixed $a^* \in \mathbb{F}_{q^3}^*$, page 51
$T_{H\backslash G}$	a complete set of right coset representatives of H in G , page 83
U	equals to $G_2^{syl}(q)$ in §3.2 and §3.3
U	equals to ${}^3D_4^{syl}\left(q^3\right)$ in §2.2 and §2.3
U	equals to ${}^2G_2^{syl}\left(3^{2m+1}\right)$ in §4.2 and §4.3
V	an $\mathbb{F}_q$-subspace of V_J
$V^\perp$	the orthogonal complement of V with respect to κ_q in §2.2.1 , page 37
V^c	a vector subspace of V_0 , page 9
V_0	a K-vector space , page 8

V_0	equals to $\mathrm{Mat}_{8\times 8}(q^3)$ in §2.2 , page 32
V_J	an $\mathbb{F}_q$-subspace of V_0
$V_J^\perp$	the orthogonal complement of V_J with respect to κ
$Z(G)$	the center of G , page 82
${}^2E_6^{syl}(q^2)$	a Sylow p-subgroup of the Steinberg group ${}^2E_6(q^2)$, page 239
${}^2G_2^{syl}(3^{2m+1})$	a Sylow 3-subgroup of the Ree group ${}^2G_2(3^{2m+1})$, page 181
$D_4^{syl}(q)$	a Sylow p-subgroup of the Chevalley group $D_4(q)$, page 22
$E_6^{syl}(q)$	a Sylow p-subgroup of the Chevalley group $E_6(q)$, page 213
$E_7^{syl}(q)$	a Sylow p-subgroup of the Chevalley group $E_7(q)$, page 256
$F_4^{syl}(q)$	a Sylow p-subgroup of the Chevalley group $F_4(q)$, page 228
$GL_N(K)$	the general linear group of all invertible $N \times N$-matrices over K , page 8
${}^3D_4^{syl}(q^3)$	a Sylow p-subgroup of the Steinberg triality group ${}^3D_4(q^3)$, page 26

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