On the semiclassical limit of the Dirichlet Laplace operator: Two-term spectral asymptotics and sharp spectral estimates

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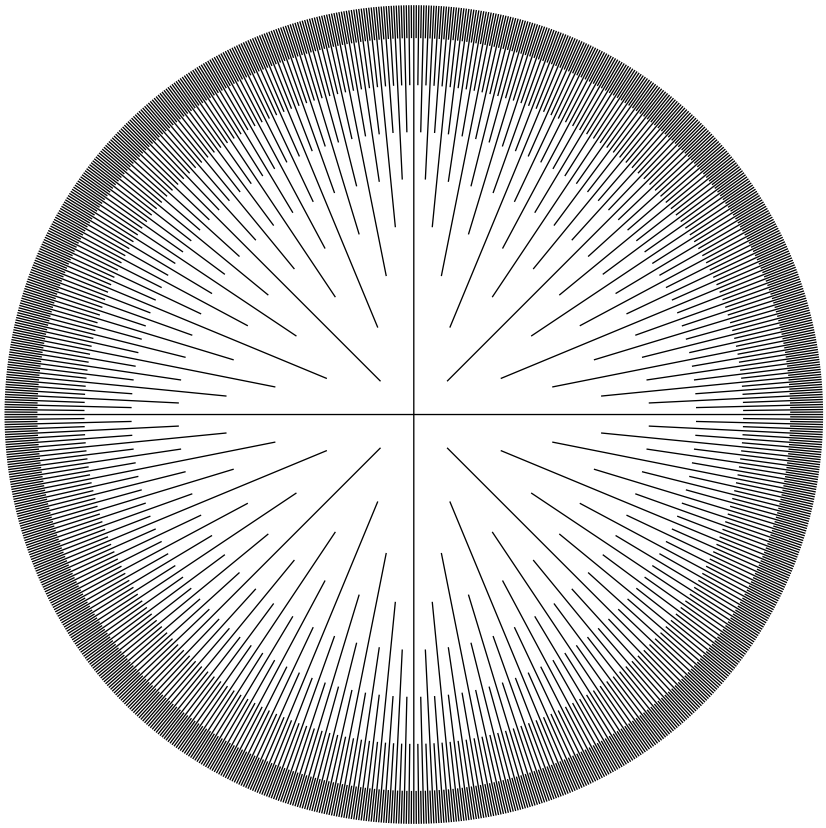

Institut für Analysis, Dynamik und Modellierung
Universität Stuttgart

2011

On the semiclassical limit of the Dirichlet Laplace operator: Two-term spectral asymptotics and sharp spectral estimates

Von der Fakultät Mathematik und Physik der Universität Stuttgart
zur Erlangung der Würde eines Doktors der Naturwissenschaften
(Dr. rer. nat.) genehmigte Abhandlung

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Prüfungsdatum:	17. Oktober 2011

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Bedeutet aber nicht die Statuierung des Ewigen und Unendlichen die logisch-rechnerische Vernichtung alles Begrenzten und Endlichen, eine verhältnismäßige Reduzierung auf Null? Ist im Ewigen ein Nacheinander möglich, im Unendlichen ein Nebeneinander?

Thomas Mann, Der Zauberberg, 1924.

But does not the very positing of eternity and infinity imply the logical, mathematical negation of things limited and finite, their relative reduction to zero? Is a sequence of events possible in eternity, a juxtaposition of objects in infinity?

Thomas Mann, The Magic Mountain, 1924.

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Preface

We know a great deal more about the forces which produce the vibrations of sound than about those which produce the vibrations of light. To find out the different tunes send out by a vibrating system is a problem which may or may not be solvable in certain special cases, but it would baffle the most skilful mathematician to solve the inverse problem and to find out the shape of a bell by means of the sounds which it is capable of sending out. And this is the problem which ultimately spectroscopy hopes to solve in the case of light. In the meantime we must welcome with delight even the smallest step in the desired direction.

Sir Franz Arthur Friedrich Schuster, 1882

Is it possible to find out the shape of a bell by means of the sound it sends out? Or, quoting M. Kac' famous article from 1966, *Can one hear the shape of a drum*? This question – in less picturesque language – is the question of how the oscillating frequencies of a vibrating membrane are related to the geometry of the membrane is a longstanding problem of spectral theory. In the 18th and 19th century physicists and mathematicians such as J. d'Alembert, P. S. Laplace, and J. Fourier investigated how vibrations, for example vibrating strings, oscillating membranes like drums, vibrations in other music instruments producing sound, and in general oscillations of elastic bodies can be described mathematically by differential operators. Their considerations also extend to more subtle oscillations which according to the theory of electromagnetism produce light and radiation. It was discovered that the frequencies of the normal modes of general oscillating systems correspond to the spectrum of differential operators; more precisely they correspond to the eigenvalues. Hence, it became important to study spectral properties of such operators, in particular, how the spectrum of differential operators and thus frequencies of oscillations depend on the geometry of the underlying system.

The first pertinent answer to this question is that one can *hear the size* of a drum. This is an old result – probably not very surprising for drummers – with a rich history. However, this problem was not brought up by musicians but by physicists who studied the laws of electromagnetic radiation. Lord Rayleigh analyzed the frequencies and the intensity of radiation within a black-body, which is an idealized physical body absorbing all radiation. In 1900, he derived the relation between intensity and frequency of the radiation within a black-body using the laws of classical mechanics. To this end he needed to determine the number of standing electromagnetic waves – analogs of the overtones of an organ pipe – at a certain frequency. Roughly speaking this is similar as the problem of determining the number of normal modes of an oscillating membrane at a certain frequency.

Lord Rayleigh used explicit calculations for rectangles and cubes to find that this number depends, at least for high frequencies, mainly on the volume of the body and not on any other geometric quantities. He assumed that this is true for any black-body of arbitrary shape. These considerations lead to the famous Rayleigh-Jeans law [**Ray00, Jea05**] relating the intensity of radiation of a black-body at a certain temperature to the frequency of the radiation.

However, Lord Rayleigh's generalisation from cubes to arbitrarily shaped bodies was still to be justified. This problem was taken up in 1910 by H. A. Lorentz who raised this question during a series of lectures entitled *'Alte und neue Fragen der Physik' ('Old and new problems of physics')* held at Göttingen. He challenged those present to prove that the number of oscillating modes of a membrane at sufficiently high frequencies is independent of the shape of the membrane and is simply proportional to its area. Legend has it that D. Hilbert predicted that this theorem would not be proved within his life time. True or not, this prediction was wrong by many years. Only two years later Hilbert's student H. Weyl, who was present at Lorentz' lectures, proved the theorem in question [**Wey12a**].

It is interesting to note that Weyl's result is related to a much older number-theoretical question. Actually, Weyl's law can be reduced to the problem of finding the number of points with integer coordinates within a ball of given radius. It is easy to see that for large radii this number is well approximated by the volume of the ball. However, it is much more difficult to find good estimates for the error that occurs by making this approximation. This problem was considered by famous mathematicians such as Gauß, Hardy, Wright, Landau and more recently by Walfisz.

Similar considerations concerning estimates for the error in his result led H. Weyl to the conjecture that the law describing the number of normal modes of an oscillating membrane could be refined by a second lower order term depending on the length of the boundary of the membrane. This problem took much longer to be solved than Lorentz' original question and it seems to be related to the geometry of the membrane in a very subtle way. Only in 1980 did V. Ivrii [**Ivr80a, Ivr80b**] give a proof of H. Weyl's conjecture, and questions related to this conjecture still pose substantial mathematical challenges.

H. Weyl's ideas and results were refined by R. Courant and D. Hilbert and were included in their famous book *Methoden der Mathematischen Physik (Methods of Mathematical Physics)* [**CH24**]. This initiated a lot of research around the asymptotic behavior of the spectrum of differential operators and marked the beginning of an important branch of spectral analysis. In this way Lord Rayleigh's initial considerations stimulated various developments in modern mathematical physics.

However, the famous Rayleigh-Jeans law turned out to be wrong. In particular, for high frequencies it is not in accordance with experimental results. Moreover, the energy output predicted by the Rayleigh-Jeans law is infinite as the frequency tends to infinity. This so-called ultraviolet catastrophe is a substantial problem of classical mechanics since it is not due to any error in the derivation of the law but due to conceptual problems in the theory of classical mechanics itself.

The ultraviolet catastrophe does not occur in M. Planck's empirically derived law for black-body radiation and this disagreement ultimately led to a whole new theory, namely quantum mechanics. However, the connection of Lord Rayleigh's and H. Weyl's considerations with quantum mechanics is much deeper: the same differential operators describing the frequencies of an oscillating membrane are postulated to determine the energy levels of particles that are described by the laws of quantum mechanics. Hence, the results relating the spectrum of partial differential operators to the geometry of the underlying system found numerous applications in quantum mechanics and still there are various questions to be answered.

With this thesis I hope to contribute a small step in the desired direction. This work was carried out from 2008 until 2011 at Universität Stuttgart, Imperial College London and Princeton University.

Acknowledgements

First I would like to express my deep gratitude to my supervisor Timo Weidl for teaching me mathematics and analysis and for introducing me to exciting topics in spectral theory and mathematical physics. I had the fortune to profit from his help in various mathematical and personal matters and from his great enthusiasm and devotion to teaching mathematics.

I am indebted to my teachers and co-authors Ari Laptev and Rupert L. Frank for sharing beautiful problems and ideas. It is a great pleasure to learn from them and to work with them and I am looking forward to future collaborations. I am very grateful to Ari Laptev for inviting me to Imperial College London and to Elliott H. Lieb and Rupert L. Frank for inviting me to Princeton University. I would like to thank them for their kind hospitality and for their support and advice.

Special thanks go to Evans Harrell, Jürgen Pöschel and Simone Warzel for their readiness to act as a referee and to Marcel Griesemer and Wolfgang Kühnel for being members of my thesis committee.

Financial support from the German Research Foundation (Deutsche Forschungsgemeinschaft), the German National Academic Foundation (Studienstiftung des deutschen Volkes) and the German Academic Exchange Service (Deutscher Akademischer Austausch Dienst) is gratefully acknowledged.

I am truly grateful to all my teachers, colleagues and friends from Universität Stuttgart, Imperial College London and Princeton University for their encouragement and advice and for many stimulating, fruitful discussions.

Last but not least I would like to express my thanks to all my friends, to my family and to Vanessa for their constant support.

Abstract

In this thesis we study spectral properties of the Dirichlet Laplace operator and related differential and pseudo-differential operators defined on Euclidean domains: the eigenvalues of such operators and properties of functions of the eigenvalues are investigated. In particular, we prove refined asymptotic formulas for sums of eigenvalues in the semiclassical limit and we derive improved uniform bounds on eigenvalues and eigenvalue means.

Let $0 < \lambda_1 \leq \lambda_2 \leq \dots$ denote the eigenvalues of the Dirichlet Laplacian defined on a bounded domain $\Omega \subset \mathbb{R}^d$. In 1912, H. Weyl analysed the function $N(\Lambda) = \sharp\{k \in \mathbb{N} : \lambda_k < \Lambda\}$ counting the number of eigenvalues below $\Lambda > 0$. He calculated the leading term of $N(\Lambda)$ in the semiclassical limit $\Lambda \to \infty$, which is given by the phase-space volume of the problem. On the one hand, this result and its generalizations relating the eigenvalues of a differential operator to the phase-space volume of the respective problem has numerous applications in physics, in the theory of oscillations and radiation, and in quantum mechanics. On the other hand, the work of H. Weyl inspired the development of modern mathematical techniques and raised deep mathematical problems that are still challenging today.

For example, H. Weyl conjectured that there exists a second term of lower order in the semiclassical limit of $N(\Lambda)$ depending on $|\partial\Omega|$, the surface area of the boundary. In 1980, V. Ivrii used a detailed microlocal analysis to prove Weyl's conjecture:

$$N(\Lambda) = \frac{\omega_d}{(2\pi)^d} |\Omega| \Lambda^{d/2} - \frac{1}{4} \frac{\omega_{d-1}}{(2\pi)^{d-1}} |\partial\Omega| \Lambda^{(d-1)/2} + o(\Lambda^{(d-1)/2}), \quad \Lambda \to \infty,$$

where ω_d denotes the volume of the unit ball in $\mathbb{R}^d$. However, this approach requires strong assumptions on the domain, in particular an involved global condition on the geometry of Ω. Therefore the question arises of whether these conditions are necessary for the existence of a second term.

At least for averaged versions of the counting function, a two-term formula exists under weaker conditions. Here we give a new proof for the equivalent of Weyl's conjecture for the sum of the eigenvalues under weak smoothness assumptions on the boundary. In particular, without any global geometric conditions on Ω, we establish that

$$R_\sigma(\Lambda) = \sum_{k\in\mathbb{N}} (\Lambda - \lambda_k)_+^\sigma = L^{cl}_{\sigma,d} |\Omega| \Lambda^{\sigma+d/2} - \frac{1}{4} L^{cl}_{\sigma,d-1} |\partial\Omega| \Lambda^{\sigma+(d-1)/2} + o(\Lambda^{\sigma+(d-1)/2})$$

holds as $\Lambda \to \infty$ for $\sigma \geq 1$, with sharp positive constants $L^{cl}_{\sigma,d}$, $L^{cl}_{\sigma,d-1}$. The sum $R_1(\Lambda)$ is directly related to the sum of the eigenvalues. This quantity describes the energy

of non-interacting, fermionic particles 'trapped' in Ω and plays an important role in physical applications.

This asymptotic formula is extended to fractional powers of the Laplace operator. For these non-local, non-smooth operators the microlocal methods leading to V. Ivrii's result cannot be applied and up to now it was unknown whether a corresponding two-term formula exists. One of the main results of this thesis is a proof of precise spectral asymptotics for the fractional Laplacian with the leading (Weyl) term given by the volume and the second term given by the surface area of the domain.

The second part of this thesis is devoted to improved uniform spectral estimates for the Dirichlet Laplace operator on bounded domains. To deduce information for specific domains Ω, it is necessary to supplement the asymptotic relations mentioned above with uniform bounds on the eigenvalue means. For example, the Berezin-Lieb-Li-Yau inequality

$$R_\sigma(\Lambda) \;\leq\; L^{cl}_{\sigma,d}\,|\Omega|\,\Lambda^{\sigma+d/2}\,, \quad \sigma \geq 1\,,$$

holds for any open set $\Omega \subset \mathbb{R}^d$ and all $\Lambda > 0$. This estimate is sharp: the constant in the bound cannot be improved. However, it is possible to strengthen the estimate with a negative remainder term. Here we present different possibilities to improve sharp semiclassical estimates uniformly with negative remainder terms correctly capturing asymptotic properties of the eigenvalue means.

Another main result of this work gives an improved Berezin inequality for $\sigma \geq 3/2$. It is valid for all $\Lambda > 0$ and reflects precisely the asymptotic and geometric properties of the semiclassical asymptotics. Under certain geometric conditions these results imply new lower bounds on individual eigenvalues that improve the Li-Yau inequality.

Similarly, we derive universal bounds on eigenvalue means and on the trace of the heat kernel of the Dirichlet Laplace operator. Again the bounds show the correct asymptotic behavior in the semiclassical limit and their geometric dependence is expressed in terms of the volume of the domain only. These results improve universal inequalities by Kac and Berezin.

Finally, we use the developed methods to prove sharp spectral estimates in quasi-bounded domains of infinite volume. For such domains, semiclassical spectral estimates based on the phase-space volume, and therefore on the volume of the domain, must fail. Here we present a method how one can nevertheless prove uniform bounds on eigenvalues and eigenvalue means which are sharp in the semiclassical limit and we extend some results to Schrödinger operators.

Zusammenfassung

In dieser Arbeit werden spektrale Eigenschaften des Dirichlet Laplace Operators und verwandter Differential- und Pseudodifferential-Operatoren untersucht, die auf Euklidischen Gebieten definiert sind. Wir analysieren die Eigenwerte solcher Operatoren und Eigenschaften von Funktionen dieser Eigenwerte. Insbesondere beweisen wir genaue asymptotische Formeln für die Summe der Eigenwerte im semiklassischen Grenzwert und leiten verbesserte semiklassische Abschätzungen an Eigenwerte und Eigenwertmittel her.

Seien $0 < \lambda_1 \leq \lambda_2 \leq \dots$ die Eigenwerte des Dirichlet Laplace Operators, der auf einem beschränkten Gebiet $\Omega \subset \mathbb{R}^d$ definiert sei. Im Jahr 1912 analysierte H. Weyl die Funktion $N(\Lambda) = \sharp\{k \in \mathbb{N} : \lambda_k < \Lambda\}$, die die Anzahl der Eigenwerte unterhalb von $\Lambda > 0$ angibt. Er berechnete den führenden Term von $N(\Lambda)$ im semiklassischen Grenzwert $\Lambda \to \infty$, der durch das Phasenraumvolumen des Problems bestimmt wird. Dieses Resultat hat einerseits zahlreiche Anwendungen in der Physik, zum Beispiel in der Theorie der Schwingungen und der elektromagnetischen Strahlung und in der Quantenmechanik. Andererseits regte die Arbeit von H. Weyl die Entwicklung moderner mathematischer Methoden an und es ergaben sich wichtige, zum Teil bis heute ungelöste, mathematische Fragestellungen.

Zum Beispiel vermutete H. Weyl, dass im semiklassischen Grenzwert von $N(\Lambda)$ ein zweiter Term von niedrigerer Ordnung existiert, der von $|\partial\Omega|$, dem Oberflächeninhalt des Randes, abhängt. Im Jahr 1980 konnte V. Ivrii diese Vermutung mit Hilfe einer detaillierten mikrolokalen Analyse beweisen:

$$N(\Lambda) \;=\; \frac{\omega_d}{(2\pi)^d}\,|\Omega|\,\Lambda^{d/2} - \frac{1}{4}\frac{\omega_{d-1}}{(2\pi)^{d-1}}\,|\partial\Omega|\,\Lambda^{(d-1)/2} + o(\Lambda^{(d-1)/2})\,, \quad \Lambda \to \infty\,,$$

wobei ω_d das Volumen des Einheitsballs in $\mathbb{R}^d$ bezeichnet. Dieser Ansatz setzt jedoch starke Bedingungen an das Gebiet voraus, insbesondere eine komplizierte globale Bedingung an die Geometrie von Ω. Dies wirft die Frage auf, ob solche Bedingungen in Euklidischen Gebieten notwendig sind.

Zumindest für gemittelte Versionen der Zählfunktion existiert eine asymptotische Formel mit zwei Termen auch unter schwächeren Voraussetzungen. In dieser Arbeit zeigen wir einen neuen Beweis der Weylschen Vermutung für die Asymptotik der Summe der Eigenwerte unter schwachen Glattheitsbedingungen an den Rand von Ω. Insbesondere existiert der Grenzwert

$$R_\sigma(\Lambda) \;=\; \sum_{k\in\mathbb{N}} (\Lambda - \lambda_k)_+^\sigma \;=\; L^{cl}_{\sigma,d}\,|\Omega|\,\Lambda^{\sigma+d/2} - \frac{1}{4}L^{cl}_{\sigma,d-1}\,|\partial\Omega|\,\Lambda^{\sigma+(d-1)/2} + o(\Lambda^{\sigma+(d-1)/2})$$

für $\Lambda \to \infty$ und alle $\sigma \geq 1$ mit scharfen positiven Konstanten $L^{cl}_{\sigma,d}$, $L^{cl}_{\sigma,d-1}$ ohne globale Voraussetzungen an die Geometrie von Ω.

Dieses Resultat kann auf fraktionelle Potenzen des Laplace Operators verallgemeinert werden. Für diese nicht-lokalen, nicht-glatten Operatoren können die mikrolokalen Methoden, die zu V. Ivriis Resultat führen, nicht angewendet werden und bis jetzt war nicht bekannt, ob eine entsprechende Asymptotik mit zwei Termen existiert. Ein zentrales Resultat dieser Arbeit ist der Beweis einer präzisen Spektralasymptotik für den fraktionellen Laplace Operator, die den führenden (Weyl) Term enthält und den zweiten Term, der vom Oberflächeninhalt des Randes abhängt.

Der zweite Teil dieser Arbeit widmet sich verbesserten, gleichmäßigen Spektralabschätzungen für den Dirichlet Laplace Operator auf beschränkten Gebieten: Die Berezin-Li-Yau Ungleichung zeigt, dass die Abschätzung

$$R_\sigma(\Lambda) \leq L^{cl}_{\sigma,d} |\Omega| \Lambda^{\sigma+d/2}, \quad \sigma \geq 1,$$

für alle offenen Mengen $\Omega \subset \mathbb{R}^d$ und alle $\Lambda > 0$ gilt. Diese Schranke ist scharf: Die Konstante in der Abschätzung kann nicht verbessert werden. Allerdings ist es möglich, die Ungleichung durch einen negativen Restterm zu verbessern. Hier präsentieren wir verschiedene Möglichkeiten, scharfe semiklassische Ungleichungen durch negative Restterme zu verbessern, die im semiklassischen Grenzwert die richtigen Eigenschaften aufweisen.

Ein zweites zentrales Ergebnis dieser Arbeit ist eine verbesserte Berezin Ungleichung für $\sigma \geq 3/2$. Dieses Resultat gilt für alle $\Lambda > 0$ und gibt die asymptotischen und geometrischen Eigenschaften der semiklassischen Asymptotik genau wieder. Unter bestimmten geometrischen Voraussetzungen folgen daraus neue untere Schranken an einzelne Eigenwerte, die die Li-Yau Ungleichung verbessern.

In ähnlicher Weise leiten wir universelle Schranken an Eigenwertmittel und an die Spur des Wärmeleitungskerns des Dirichlet Laplace Operators her. Dabei ist die geometrische Abhängigkeit dieser Schranken lediglich durch das Volumen des Gebiets gegeben. Diese Resultate zeigen wieder das korrekte asymptotische Verhalten im semiklassischen Grenzwert und verbessern universelle Ungleichungen von Kac und Berezin.

Schließlich benutzen wir die entwickelten Methoden, um scharfe Abschätzungen an Eigenwertmittel in quasi-beschränkten Gebieten mit unendlichem Volumen zu beweisen. Für solche Gebiete sind semiklassische Abschätzungen, die auf dem Phasenraumvolumen und damit auf dem Volumen des Gebiets basieren, nicht anwendbar. Hier zeigen wir eine Methode, wie man dennoch scharfe, gleichmäßige Abschätzungen beweisen kann, und verallgemeinern Resultate auf Schrödingeroperatoren.

CHAPTER 1

Introduction

1.1. Weyl's law and the variational principle

1.1.1. The semiclassical limit. One hundred years ago H. Weyl studied the frequencies of oscillating membranes. Stimulated by questions arising from the theory of radiation [**Jea05, Som10**] he analyzed how the frequencies depend on the geometry of the membrane [**Wey12a, Wey12b, Wey12c, Wey13**]. In particular, he was interested in the high-energy limit, corresponding to high frequencies.

It is remarkable that these questions are closely connected with problems in quantum mechanics. The equation determining the frequencies of an oscillating membrane describes a quantum-mechanical particle trapped in a domain corresponding to the membrane. In quantum mechanics, the frequencies are equivalent to the energy levels of the particle under consideration and the high-energy limit corresponds to the semiclassical limit of the quantum mechanical system.

H. Weyl's main result [**Wey12a**] shows that the high-energy limit of the frequencies of an oscillating membrane depends to first order on the area of the membrane, more precisely on the phase-space volume of the respective problem. This result stimulated a lot of research around the semiclassical limit of quantum mechanical systems and the underlying partial differential equations. It marked the starting point for an important branch in mathematical physics and spectral analysis.

Let us briefly review the mathematical setting of the problem. The frequencies of an oscillating membrane $\Omega \subset \mathbb{R}^2$ with fixed boundary are given by the eigenvalues of the Dirichlet Laplace operator defined in the Hilbert space $L^2(\Omega)$. In general, we consider an open set $\Omega \subset \mathbb{R}^d$, $d \geq 1$, and the quadratic form

$$\int_\Omega |\nabla u(x)|^2 \, dx$$

with form domain given by the Sobolev space $H_0^1(\Omega)$. The Dirichlet Laplace operator $-\Delta_\Omega^D$ is defined as the self-adjoint operator in $L^2(\Omega)$ generated by this quadratic form. That means $-\Delta_\Omega^D$ is the Friedrichs extension of the Laplace operator initially defined on $C_0^\infty(\Omega)$, see [**BS87**] for details.

We always assume that the spectrum of $-\Delta_\Omega^D$ is purely discrete. This is equivalent to compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ and is satisfied, for example, if Ω is bounded. Then the spectrum of the Dirichlet Laplace operator consists of positive eigenvalues $0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \dots$ accumulating at infinity only.

In general these eigenvalues cannot be calculated explicitly and especially for large k it is difficult to approximate $\lambda_k(\Omega)$ numerically. Therefore it is interesting to find

a precise formula for $\lambda_k(\Omega)$ in the semiclassical limit $k \to \infty$ and it is important to understand how this limit depends on the geometry of Ω.

As mentioned above the first rigorous step towards this was made by H. Weyl who considered the equivalent problem of analyzing the counting function

$$N(\Lambda;\Omega) \,=\, \sharp\{k \in \mathbb{N} \,:\, \lambda_k(\Omega) < \Lambda\}\,, \quad \Lambda > 0\,,$$

in the semiclassical limit $\Lambda \to \infty$. Let $|\Omega|$ denote the d-dimensional Lebesgue-measure of Ω. Then the limit

$$N(\Lambda;\Omega) \,=\, L^{cl}_{0,d}\,|\Omega|\,\Lambda^{d/2} + o(\Lambda^{d/2}) \tag{1.1}$$

holds as $\Lambda \to \infty$ [**Wey12a**], where $L^{cl}_{0,d}$ denotes the semiclassical constant

$$L^{cl}_{0,d} \,=\, \frac{\omega_d}{(2\pi)^d} \,=\, \frac{1}{(4\pi)^{d/2}\Gamma(\frac{d}{2}+1)}$$

and ω_d denotes the volume of the unit ball in $\mathbb{R}^d$. The identity

$$L^{cl}_{0,d}\,|\Omega|\,\Lambda^{d/2} \,=\, \frac{1}{(2\pi)^d}\int_\Omega\int_{\mathbb{R}^d} \big(|\xi|^2 - \Lambda\big)^0_-\, d\xi\, dx$$

shows that the semiclassical limit is indeed determined by the phase-space volume of the problem under consideration. This agrees with the Bohr-Sommerfeld quantization rule from early quantum mechanics: Each particle state occupies a volume of $(2\pi)^d$ in phase-space.

Note that the leading term of the semiclassical limit is independent of the choice of boundary conditions. Thus (1.1) is also valid, e.g., for the Neumann Laplace operator.

Weyl's law (1.1) can also be generalized to Schrödinger operators

$$H(\lambda V) \,=\, -\Delta + \lambda V\,, \quad \lambda > 0\,,$$

with real-valued potential V given on $\mathbb{R}^d$. Under suitable assumptions on the potential, the operator $H(\lambda V)$ can be defined in $L^2(\mathbb{R}^d)$ again as Friedrichs extension of the respective operator with domain $C^\infty_0(\mathbb{R}^d)$, see [**BS87, RS75**] for details. Let $V_-(x) = \max\{-V(x), 0\}$ denote the negative part of the potential. If $V_- \in L^{d/2}(\mathbb{R}^d)$ then the negative spectrum of $H(\lambda V)$ is discrete and one defines $N(\lambda; V)$ to be the number of negative eigenvalues.

The semiclassical limit of the counting function $N(\lambda; V)$ is again determined by the phase-space volume. As $\lambda \to \infty$ the limit

$$\begin{aligned} N(\lambda;V) \,&=\, \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \big(|\xi|^2 - \lambda V_-(x)\big)^0_-\, d\xi\, dx + o(\lambda^{d/2}) \\ &=\, L^{cl}_{0,d}\int_{\mathbb{R}^d} V_-^{d/2}(x)\, dx\, \lambda^{d/2} + o(\lambda^{d/2}) \end{aligned} \tag{1.2}$$

holds true [**BB71, Mar72, Tam74**]. We point out that by setting $V(x) = -1$ on an open and bounded set $\Omega \subset \mathbb{R}^d$ and $V(x) = +\infty$ on $\mathbb{R}^d \setminus \Omega$ we recover Weyl's law (1.1).

1.1.2. The variational principle. In order to review some basic methods let us outline the main ideas leading to (1.1). More details can be found for example in [**CH24, RS78**].

The starting point is the variational principle. One version is the following characterization of the eigenvalues of a self-adjoint, semi-bounded operator H on a Hilbert space $\mathcal{H}$ known as the min-max principle [**Cou20**]. Let $Q(H) \subset \mathcal{H}$ denote the form domain of the operator H and for $\varphi \in Q(H)$ write $\langle \varphi, H\varphi \rangle$ for the quadratic form corresponding to H. For $k \in \mathbb{N}$ put

$$\mu_k \;=\; \max_{\psi_1,\dots,\psi_{k-1}\in\mathcal{H}} \; \min_{\substack{\varphi\in Q(H);\, \|\varphi\|=1 \\ \varphi\perp[\psi_1,\dots,\psi_{k-1}]}} \langle \varphi, H\varphi \rangle \,. \tag{1.3}$$

Then either there are k eigenvalues below the essential spectrum of H and μ_k is the k-th eigenvalue counted in increasing order with multiplicity, or μ_k is the bottom of the essential spectrum and there are at most $k-1$ eigenvalues below μ_k.

For the Dirichlet Laplace operator the variational principle gives

$$\lambda_k(\Omega) \;=\; \max_{\psi_1,\dots,\psi_{k-1}\in L^2(\Omega)} \; \min_{\substack{\varphi\in H^1_0(\Omega),\, \|\varphi\|=1 \\ \varphi\perp[\psi_1,\dots,\psi_{k-1}]}} \langle \varphi, -\Delta^D_\Omega \varphi \rangle \,, \quad k \in \mathbb{N} \,.$$

Since additional Dirichlet boundary conditions in the interior of Ω reduce the form domain $H^1_0(\Omega)$, adding a Dirichlet condition rises the eigenvalues. On the other hand, adding a Neumann boundary conditions – i.e. a condition $\frac{\partial\varphi}{\partial\nu}\big|_\Gamma = 0$ along a surface $\Gamma \subset \Omega$, where $\frac{\partial}{\partial\nu}$ is the normal derivative with respect to Γ – enlarges the form domain and therefore lowers the eigenvalues of the Laplace operator.

The crucial idea was to approximate Ω by a covering consisting of small cubes and to put additional Dirichlet or Neumann boundary conditions at the sides of the cubes. The point is that one can calculate the eigenvalues of the Dirichlet and Neumann Laplace operator on a cube explicitly. Indeed, if $N^D(\Lambda; Q_l)$ and $N^N(\Lambda; Q_l)$ denote the counting functions of the Dirichlet and Neumann Laplace operator on a cube $Q_l \subset \mathbb{R}^d$ with side length $l > 0$ then

$$N^D(\Lambda; Q_l) \;=\; \sum_{k\in\mathbb{N}^d} \frac{\pi^2}{l^2} \sum_{j=1}^d k_j^2 \;=\; L^{cl}_{0,d}\, l^d\, \Lambda^{d/2} + o(\Lambda^{d/2}) \tag{1.4}$$

$$N^N(\Lambda; Q_l) \;=\; \sum_{k\in\mathbb{N}_0^d} \frac{\pi^2}{l^2} \sum_{j=1}^d k_j^2 \;=\; L^{cl}_{0,d}\, l^d\, \Lambda^{d/2} + o(\Lambda^{d/2}) \tag{1.5}$$

as $\Lambda \to \infty$.

By the variational principle, the counting function of the Dirichlet Laplace operator on Ω can be estimated from below by $N^D(\Lambda; Q_l)$ times the maximal number of disjoint cubes Q_l in the interior of Ω. As the side length l tends to zero the latter number tends to $|\Omega| l^{-d}$, hence from (1.4) it follows that

$$N(\Lambda; \Omega) \;\geq\; N^D(\Lambda; Q_l)\, |\Omega|\, l^{-d} + o(\Lambda^{d/2}) \;=\; L^{cl}_{0,d}\, |\Omega|\, \Lambda^{d/2} + o(\Lambda^{d/2})$$

as $\Lambda \to \infty$. Similarly, approximating Ω by cubes with Neumann boundary conditions and using (1.5) one gets

$$N(\Lambda;\Omega) \;\leq\; N^N(\Lambda;Q_l)\,|\Omega|\,l^{-d} + o(\Lambda^{d/2}) \;=\; L^{cl}_{0,d}\,|\Omega|\,\Lambda^{d/2} + o(\Lambda^{d/2})$$

as $\Lambda \to \infty$, and (1.1) follows.

We remark that the same methods can be applied to derive the limit (1.2) for the number of negative eigenvalues of Schrödinger operators [**CH24, RS78**]. Again the first term of the semiclassical limit can be proved entirely by variational methods.

To conclude this section let us mention that the variational principle implies the following convenient identity for the sum of the negative eigenvalues of a Schrödinger operator $H(V) = -\Delta + V$. By (1.3), for the trace of the negative part of $H(V)$ the identity

$$-\mathrm{Tr}\,(H(V)_-) \;=\; \inf_{0\leq\gamma\leq 1} \mathrm{Tr}(\gamma H(V)) \tag{1.6}$$

holds, where the infimum is taken over all trial density matrices γ, i.e. over all trace class operators $0 \leq \gamma \leq 1$ with range belonging to the form domain of $H(V)$.

1.2. The second term in Weyl's law

After H. Weyl calculated the leading term of the semiclassical limit for the Dirichlet Laplace operator (1.1), he put forward the conjecture that there exists a second term of lower order depending on $|\partial\Omega|$, the surface area of the boundary of Ω [**Wey13**]. Explicit calculations for cubes show that a two-term formula

$$N(\Lambda;\Omega) \;=\; L^{cl}_{0,d}\,|\Omega|\,\Lambda^{d/2} - \frac{1}{4}L^{cl}_{0,d-1}\,|\partial\Omega|\,\Lambda^{(d-1)/2} + o(\Lambda^{(d-1)/2}) \tag{1.7}$$

is plausible as $\Lambda \to \infty$. However, it turned out to be difficult to prove this conjecture in general. One of the difficulties is to handle the discontinuity of the counting function.

1.2.1. Riesz-means and the trace of the heat kernel. As a general principle one can state that things get easier if one does not consider the counting function directly but averaged or smoothed versions. For example one often studies the Riesz-means

$$R_\sigma(\Lambda;\Omega) \;=\; \mathrm{Tr}\left(-\Delta^D_\Omega - \Lambda\right)^\sigma_- \;=\; \sum_{k\in\mathbb{N}} (\Lambda - \lambda_k)^\sigma_+ \,, \quad \Lambda > 0\,, \quad \sigma \geq 0\,.$$

Here and in the following $x_\pm = (|x| \pm x)/2$ denotes the positive and negative part of $x \in \mathbb{R}$ respectively. Apart from the counting function – corresponding to $\sigma = 0$ – the case $\sigma = 1$ is of special interest, since it is directly related to the sum of the eigenvalues, see Section 1.3.2 below. This quantity describes the energy of non-interacting, fermionic particles trapped in Ω and plays an important role in physical applications.

The fact that the behavior of the eigenvalue means R_σ becomes more regular as σ increases, allows one to deduce results for eigenvalue means of higher order from

results for eigenvalue means of lower order. In particular, one can use the identity [**AL78**]

$$R_\sigma(\Lambda;\Omega) \;=\; \frac{1}{B(\gamma+1,\sigma-\gamma)}\int_0^\infty \tau^{\sigma-\gamma-1} R_\gamma(\Lambda-\tau;\Omega)\,d\tau\,, \quad \sigma>\gamma\geq 0\,, \tag{1.8}$$

where B denotes the Beta-function. From (1.7) and (1.8) with $\sigma>\gamma=0$ we see that the equivalent of Weyl's conjecture for Riesz-means reads as

$$R_\sigma(\Lambda;\Omega) \;=\; L^{cl}_{\sigma,d}\,|\Omega|\,\Lambda^{\sigma+d/2} - \frac{1}{4}L^{cl}_{\sigma,d-1}\,|\partial\Omega|\,\Lambda^{\sigma+(d-1)/2} + o(\Lambda^{\sigma+(d-1)/2}) \tag{1.9}$$

with

$$L^{cl}_{\sigma,d} \;=\; \frac{\Gamma(\sigma+1)}{(4\pi)^{d/2}\Gamma(\sigma+\frac{d}{2}+1)}\,.$$

One can smooth the counting function even further and consider the trace of the heat kernel, the so-called partition function,

$$Z(t;\Omega) \;=\; \mathrm{Tr}\left(e^{\Delta^D_\Omega t}\right) \;=\; \sum_k e^{-\lambda_k(\Omega)t}\,, \quad t>0\,.$$

This trace function corresponds to the endpoint of the scale of Riesz-means R_σ as σ tends to infinity. Let

$$\mathcal{L}[f(\cdot)](t) \;=\; \int_0^\infty f(\lambda)e^{-\lambda t}\,d\lambda$$

denote the Laplace transform of a suitable function $f:[0,\infty)\to\mathbb{R}$ and note that the partition function is related to the Riesz-means by the formula

$$Z(t;\Omega) \;=\; \frac{t^{\sigma+1}}{\Gamma(\sigma+1)}\mathcal{L}\left[R_\sigma(\cdot;\Omega)\right](t)\,, \quad \sigma\geq 0\,. \tag{1.10}$$

For this smooth and convex trace function of the Dirichlet Laplace operator one can employ local properties of the heat kernel to study the semiclassical limit which corresponds to the short-time limit $t\to 0+$. The question how this limit depends on the geometry of the domain Ω was addressed by M. Kac in his famous paper *'Can one hear the shape of a drum?'* [**Kac66**]. There he proved that for convex domains $\Omega\subset\mathbb{R}^2$ the limit

$$Z(t;\Omega) \;=\; \frac{|\Omega|}{4\pi t} - \frac{|\partial\Omega|}{4\sqrt{4\pi t}} + o\left(t^{-1/2}\right)$$

holds as $t\to 0+$, see also [**Ple54**]. This result was generalized to a wide class of not necessarily convex domains in higher dimension. Even for Lipschitz domains $\Omega\subset\mathbb{R}^d$, i.e., for domains with boundary given by a Lipschitz continuous curve, one has

$$Z(t;\Omega) \;=\; \frac{|\Omega|}{(4\pi t)^{d/2}} - \frac{\mathscr{H}(\partial\Omega)}{4(4\pi t)^{(d-1)/2}} + o\left(t^{-(d-1)/2}\right) \tag{1.11}$$

as $t\to 0+$ [**Bro93**], where $\mathscr{H}(\partial\Omega)$ denotes the $d-1$-dimensional Hausdorff measure of the boundary. In view of (1.10) this is in agreement with Weyl's conjecture (1.7).

For smooth domains $\Omega\subset\mathbb{R}^d$ there even exists an asymptotic expansion

$$Z(t;\Omega) \;=\; t^{-d/2}\sum_{k\geq 0} C_k\,t^{k/2}$$

as $t \to 0+$ [**Min54, MS67, Gre71**]. The constants C_k are related to the geometry of Ω [**Smi81**], for $k \geq 2$ to properties of the boundary. However, the inverse problem of determining the geometry Ω from the spectrum of $-\Delta_\Omega^D$ cannot be solved uniquely [**GWW92**].

1.2.2. Ivrii's result. Since the trace of the heat kernel is directly related to the counting function and the Riesz-means via the Laplace transform it is evident and tempting to use (1.11) to prove that Weyl's conjecture (1.7) and therefore (1.9) holds true. However, since the Riesz-means and especially the counting function are not smooth, carrying out this conclusion rigorously is very cumbersome.

In general, one can use classical Tauberian theorems, see e.g. [**Sim05**, Theorem 10.3], to deduce asymptotic results for Riesz-means from (1.11). However, this yields only the first term of the semiclassical limit.

Already in 1924 R. Courant used variational methods to prove that for domains $\Omega \subset \mathbb{R}^d$ with smooth boundary the limit

$$N(\Lambda;\Omega) \;=\; L_{0,d}^{cl}\,|\Omega|\,\Lambda^{d/2} + O(\Lambda^{(d-1)/2}\ln\Lambda) \tag{1.12}$$

holds as $\Lambda \to \infty$ [**CH24**]. Recently, Y. Netrusov and Y. Safarov applied variational methods and a very subtle covering to prove that the estimate

$$\left|N(\Lambda;\Omega) - L_{0,d}^{cl}\,|\Omega|\,\Lambda^{d/2}\right| \;\leq\; c_d\,\Lambda^{(d-1)/2}\int_0^\Lambda |\Omega_{t^{-1}}|\,dt$$

holds for all $\Lambda > 0$ with a positive constant $c_d > 0$ [**NS05**]. Here

$$\Omega_{t^{-1}} = \{x \in \Omega \,:\, \operatorname{dist}(x, \mathbb{R}^d \setminus \Omega) < t^{-1}\}$$

denotes the inner parallel set of Ω with distance to the boundary less than t^{-1}. While this result gives the correct asymptotic behavior of the second term for domains Ω with rough boundary – for example for fractals as considered in [**Lap91, FV93, FLV95**] – it does not improve on (1.12) for smooth domains.

Since neither variational methods nor Tauberian theorems are strong enough to prove Weyl's conjecture (1.7) one needs a more elaborate strategy.

The main idea which goes back to T. Carleman [**Car36**] and B. M. Levitan [**Lev52**] is to study another integral transform of the counting function, namely

$$\sigma(t) \;=\; \operatorname{Tr}\cos\left(t\sqrt{-\Delta_\Omega^D}\right) \;=\; \int_{\mathbb{R}} \cos\left(\sqrt{\lambda}t\right) dN(\lambda;\Omega)\,.$$

Since $\sigma(t)$ equals $\int_\Omega u(x,x,t)dx$, where $u(x,y,t)$ solves the wave equation

$$\begin{aligned} \partial_t^2 u - \Delta u \;&=\; 0 \\ u\,|_{t=0} = \delta(x-y)\,,\; \partial_t u|_{t=0} \;&=\; 0\,,\; u \;=\; 0 \text{ on } \partial\Omega\,, \end{aligned}$$

one can analyze local properties of solutions of the wave equation and apply the Fourier cosine transform to gather information about the counting function.

This microlocal approach was employed by R. Seeley to prove

$$N(\Lambda;\Omega) \;=\; L_{0,d}^{cl}\,|\Omega|\,\Lambda^{d/2} + O(\Lambda^{(d-1)/2})\,, \quad \Lambda \to \infty\,,$$

in domains with smooth boundary [**See78**]. In 1980 V. Ivrii [**Ivr80a, Ivr80b**] finally established Weyl's conjecture (1.7) under certain conditions on the geometry of Ω. See also [**Hör85, SV97, Ivr98**] for further results and references.

Here we follow [**Hör85**, Ch. 29.3] to briefly review Ivrii's result and the necessary conditions. Let $T^*(\mathbb{R}\times\Omega\times\Omega)$ be the cotangent bundle of $\mathbb{R}\times\Omega\times\Omega$ and set

$$\begin{aligned}\Gamma \,=\, \{(t,x,y;\tau,\xi,\eta)\in T^*(\mathbb{R}\times\Omega\times\Omega)\setminus 0\ ;\ \tau=|\xi|\ ,\\ \text{and } (t,x,\tau,\xi),(0,y,\tau,\eta) \text{ lie on a generalized bicharacteristic of } \tau^2-|\xi|^2\big\}\ .\end{aligned}$$

Here generalized bicharacteristics - or so called geodesic billiards - are orbits of the Hamiltonian vector field generated by the Hamiltonian $|\xi|$ which are reflected at the boundary, see [**Hör85**, Ch. 24] and also [**SV97**]. Hence, the last assumption means that the Hamiltonian orbit which is at (y,η) when the time parameter is 0 arrives at (x,ξ) when the parameter is t. Thus the condition $(t,x,x;\tau,\xi,\xi)\in\Gamma$ means that the orbit starting at (x,ξ) returns to (x,ξ) at time t. Put

$$\Pi^*(x,\xi) \,=\, \inf\{t>0\,:\,(t,x,x;\tau,\xi,\xi)\in\Gamma\}\,,$$

and define Π^* to be $+\infty$ if no such t exists.

The set Γ is closed in $T^*(\mathbb{R}\times\Omega\times\Omega)$ and so is $\{(t,x,x;\tau,\xi,\xi)\in\Gamma\ ,\ t\neq 0\}$. Hence $\Pi^*(x,\xi)$ is a strictly positive, lower semi-continuous function of degree 0 and has a positive lower bound.

In [**Ivr80a, Ivr80b**], see also [**Hör85**, Thm 29.3.3], it is shown that

$$\begin{aligned}&\limsup_{\Lambda\to\infty}\Lambda^{(1-d)/2}\left|N(\Lambda)-L^{cl}_{0,d}|\Omega|\Lambda^{d/2}+\frac{1}{4}L^{cl}_{0,d-1}|\partial\Omega|\Lambda^{(d-1)/2}\right|\\ &\leq\, C\int_{|\xi|<1}\int_{\Omega}\Pi^*(x,\xi)^{-1}dx\,d\xi\,.\end{aligned}$$

Now we can state V. Ivrii's result proving Weyl's conjecture: Let $\Omega\subset\mathbb{R}^d$ be a bounded domain with smooth boundary and assume that the measure of all periodic geodesic billiards is zero. Then we have $\Pi^*(x,\xi)^{-1}=0$ a.e. and (1.7) follows.

For general manifolds the condition about periodic geodesic billiards is known to be necessary. For example, let us consider the Laplace-Beltrami operator on the sphere $\mathbb{S}^2=\{(\varphi,\theta)\,:\,0\leq\varphi<2\pi,\,0\leq\theta\leq\pi\}$, more precisely on domains

$$\Omega_\beta \,=\, \{(\varphi,\theta)\,:\,0<\varphi<\beta,\,0<\theta<\pi\}\,\subset\,\mathbb{S}^2$$

with $0<\beta<\pi$. Based on explicit calculations the following asymptotics were found in [**Gro66**]. If β/π is irrational then as $\Lambda\to\infty$

$$N(\Lambda) \,=\, \frac{|\Omega_\beta|}{4\pi}\Lambda-\frac{|\partial\Omega_\beta|}{4\pi}\sqrt{\Lambda}+o\left(\sqrt{\Lambda}\right)\,.$$

However, if $\beta/\pi=b/a$ with $a,b\in\mathbb{Z}$ and relatively prime then as $\Lambda\to\infty$

$$N(\Lambda) \,=\, \frac{|\Omega_\beta|}{4\pi}\Lambda-\frac{|\partial\Omega_\beta|}{4\pi}\sqrt{\Lambda}-\frac{1}{a}\left\{b\left(\sqrt{\Lambda+\frac{1}{4}}-\frac{1}{2}\right)\right\}\sqrt{\Lambda}+o\left(\sqrt{\Lambda}\right)\,,$$

where $\{x\} = x - [x] - 1/2$ and $[x]$ denotes the integer part of $x \in \mathbb{R}$. This shows that without further geometric conditions oscillations can occur at the order of the second term.

For planar domains, however, it is conjectured that the measure of all periodic geodesic billiards is always zero. A positive answer to this conjecture was announced in [**Gut06**] for dimension 2. Nevertheless it would be convenient to find a new approach to refined semiclassics on planar domains not requiring any assumptions on geodesic billiards.

1.2.3. Refined semiclassics for Schrödinger operators. Let us briefly turn to Schrödinger operators $H(\lambda V) = -\Delta - \lambda V$ defined for $\lambda > 0$ on $\mathbb{R}^d$ without boundary. In [**HR90**], B. Helffer and D. Robert analyze Riesz-means of the negative eigenvalues

$$R_\sigma(\lambda V) \,=\, \mathrm{Tr}\,(-\Delta + \lambda V)^\sigma_- \,, \quad \sigma \geq 0\,,$$

in the semiclassical limit $\lambda \to \infty$. The asymptotics of the counting function (1.2) and identity (1.8) applied with $\gamma = 0$ imply that

$$R_\sigma(\lambda V) \,=\, L^{cl}_{\sigma,d} \int_{\mathbb{R}^d} V_-^{\sigma+d/2}(x)\, dx\, \lambda^{\sigma+d/2} + o(\lambda^{\sigma+d/2}) \tag{1.13}$$

holds as $\lambda \to \infty$ for potentials with $V_- \in L^{\sigma+d/2}(\mathbb{R}^d)$. In [**HR90**] it is shown that one can refine this limit with more and more lower order terms as σ increases. Again this is due to the increasing regularity of the eigenvalue means R_σ. In particular, for $\sigma > N - 1$, $N \in \mathbb{N}$, an expansion

$$R_\sigma(\lambda V) \,=\, \lambda^{\sigma+d/2} \left(\sum_{k=0}^{N} C_{\sigma,k}\, \lambda^{-k/2} + O(\lambda^{-(\sigma+1)/2}) \right) \tag{1.14}$$

holds as $\lambda \to \infty$. From (1.2) and (1.8) one has

$$C_{\sigma,0} \,=\, L^{cl}_{\sigma,d} \int_{\mathbb{R}^d} V_-^{\sigma+d/2}(x)\, dx\,.$$

Since $H(\lambda V)$ is defined on $\mathbb{R}^d$ without boundary one also has $C_{\sigma,1} = 0$ for all $\sigma > 0$. We point out that for all $\sigma > 0$ the second term of this expansion is of smaller order than $\lambda^{\sigma+(d-1)/2}$ without any assumptions on periodic geodesics.

Even though this result is not directly applicable to the Laplace operator on a bounded domain it supports the conjecture that the condition about periodic geodesics is not necessary to prove Weyl's refined law (1.9) for $\sigma > 0$. Thus the problem arises to find minimal conditions on Ω and its boundary such that the refined limit (1.9) holds.

1.3. Uniform semiclassical estimates

1.3.1. Lieb-Thirring inequalities. The asymptotic relations (1.7), (1.9) and (1.11) for the Laplace operator and (1.2) and (1.14) for Schödinger operators give precise information about the semiclassical limit. However, to study specific domains Ω and specific potentials V for finite energies, it is important to supplement the

asymptotic results with uniform estimates valid for finite values of $\Lambda > 0$ and finite potentials V respectively.

The question whether bounds

$$R_\sigma(V) \;=\; \mathrm{Tr}\,(-\Delta - V)_-^\sigma \;\leq\; L_{\sigma,d} \int_{\mathbb{R}^d} V_-^{\sigma+d/2}(x)\,dx \tag{1.15}$$

hold true for all potentials satisfying $V_- \in L^{\sigma+d/2}(\mathbb{R}^d)$ with a uniform constant $L_{\sigma,d} > 0$ was first addressed by E. Lieb and W. Thirring [**LT76**]. They provide a proof of (1.15) for $\sigma > \max\{0, 1 - d/2\}$.

Note that in dimensions $d \geq 3$ one can deduce (1.15) for all $\sigma > 0$ from (1.8) and the Cwikel-Lieb-Rozenblyum inequality [**Cwi77, Lie76, Roz72a**] that is inequality (1.15) in the case $\sigma = 0$.

Later on the Lieb-Thirring inequalities were also established in dimension $d = 1$ in the critical case $\sigma = 1/2$ [**Wei96**]. Let us mention that (1.15) cannot hold in the remaining cases $d = 2$, $\sigma = 0$ and $d = 1$, $\sigma < 1/2$ due to the existence of at least one negative eigenvalue for any non-vanishing potential $V \leq 0$ and its behavior in the weak coupling limit [**Sim76**].

The Lieb-Thirring inequalities turn out to be important in many physical applications for example in proving the stability of matter in different models in quantum mechanics [**Lie97, LS10**] and in studying fluid mechanics [**Rue82, Lie84, Ily05**].

While the question, whether the Lieb-Thirring inequalities hold with uniform constants $L_{\sigma,d}$, is answered completely, it is still a striking mathematical problem to find the best constants for which (1.15) is valid. In [**LW00**] it is shown that for $d \geq 1$ and $\sigma \geq 3/2$ the Lieb-Thirring inequality

$$R_\sigma(V) \;\leq\; L^{cl}_{\sigma,d} \int_{\mathbb{R}^d} V_-^{\sigma+d/2}(x)\,dx \tag{1.16}$$

holds true with the semiclassical constant $L^{cl}_{\sigma,d}$, see also [**BL00**].

We point out that this estimate is sharp. In view of asymptotics (1.13) this bound corresponds to the leading term of the semiclassical limit, thus the constant $L^{cl}_{\sigma,d}$ in (1.16) cannot be improved. In this case the phase-space volume not only determines the semiclassical limit of the eigenvalue means but also yields a uniform upper bound.

1.3.2. Berezin-Lieb-Li-Yau inequalities. If one studies the Dirichlet Laplace operator on a bounded, open domain $\Omega \subset \mathbb{R}^d$, then uniform semiclassical estimates turn out to be somewhat simpler to find. Such estimates were first obtained for the partition function. In 1951 M. Kac proved that for all open sets $\Omega \subset \mathbb{R}^d$ with finite volume the estimate

$$Z(t;\Omega) \;\leq\; \frac{|\Omega|}{(4\pi t)^{d/2}} \tag{1.17}$$

holds for all $t > 0$ [**Kac51**]. In view of (1.11) we see that the bound is sharp. The proof of this result is based on the principle of not feeling the boundary which allows to use local results for the heat kernel defined on $\mathbb{R}^d$ without boundary.

Even though this principle is very useful when dealing with heat kernels it is not directly applicable if one considers Riesz-means. For $\sigma \geq 3/2$ one can use (1.16) and

set $V \equiv -\Lambda$ on Ω and $V \equiv +\infty$ on $\mathbb{R}^d \setminus \Omega$ to obtain

$$R_\sigma(\Lambda;\Omega) \;\leq\; L^{cl}_{\sigma,d}\,|\Omega|\,\Lambda^{\sigma+d/2} \tag{1.18}$$

for all $\Lambda > 0$. Again this bound coincides with the leading term of the semiclassical limit (1.9). Actually such an estimate holds true for any convex function of the Dirichlet Laplace operator, see [**Ber72b, Lie73**] and Section 1.3.3 below. In [**Ber72a**] it is proved that for the specific case of Riesz-means the sharp Berezin inequality (1.18) holds for all $\Lambda > 0$ whenever $\sigma \geq 1$.

The case $\sigma = 1$ is of special importance since it is directly related to the sum of the first K eigenvalues. Indeed, if $\mathscr{L}[f(\cdot)](w) = \sup_{\Lambda\in\mathbb{R}}(w\Lambda - f(\Lambda))$ denotes the Legendre transform of a real-valued function f then for $K \in \mathbb{N}$

$$\mathscr{L}[R_1(\cdot;\Omega)](K) \;=\; \sup_{\Lambda\in\mathbb{R}}\left(K\Lambda - \sum_{k\in\mathbb{N}}(\Lambda-\lambda_k)_+\right) \;=\; \sum_{k=1}^{K}\lambda_k(\Omega)\,,$$

since the maximizing Λ is determined by $N(\Lambda;\Omega) = K$. The Legendre transform reverses inequalities, hence applying it to both sides of (1.18) with $\sigma = 1$ yields

$$\sum_{k=1}^{K}\lambda_k(\Omega) \;\geq\; C_d\frac{d}{d+2}K^{2/d+1}\,|\Omega|^{-2/d}\,, \tag{1.19}$$

for all $K \in \mathbb{N}$ with $C_d = (2\pi)^2\omega_d^{-2/d}$. This estimate was found by other means by P. Li and S. T. Yau [**LY83**], see Section 1.4.1 below. Hence, estimates (1.18) and (1.19) are often referred to as Berezin-Lieb-Li-Yau inequalities.

Let us briefly recall how these semiclassical estimates imply bounds on individual eigenvalues. One can either use that

$$\lambda_N(\Omega) \;\geq\; \frac{1}{N}\sum_{k=1}^{N}\lambda_k(\Omega) \;\geq\; C_d\frac{d}{d+2}N^{2/d}\,|\Omega|^{-2/d} \tag{1.20}$$

or one can employ the estimate [**Lap97**]

$$N(\Lambda;\Omega) \;\leq\; (\tau\Lambda)^{-\sigma}R_\sigma\left((1+\tau)\Lambda;\Omega\right)\,, \quad \Lambda>0\,,\ \tau>0\,,\ \sigma>0\,, \tag{1.21}$$

with $\sigma = 1$ and insert (1.18). Optimizing the resulting inequality in $\tau > 0$ yields

$$N(\Lambda;\Omega) \;\leq\; \left(1+\frac{2}{d}\right)L^{cl}_{0,d}\,|\Omega|\,\Lambda^{d/2} \tag{1.22}$$

for all $\Lambda > 0$, which is easily seen to be equivalent to (1.20).

The question whether (1.20) and (1.22) hold for all open sets Ω with the sharp constants C_d and $L^{cl}_{0,d}$ respectively is still unsolved. G. Pólya proved the sharp estimate for tiling domains [**Pol61**]. Whether this result extends to all open bounded sets is unclear, see [**Lap97, FLW09**] for extensions and recent developments.

1.3.3. Convex trace functions. To show the importance of convexity for the proof of the Berezin inequality, let us recall the following proof of (1.18) from [**Lap97**]. Actually we prove that for any open set $\Omega \subset \mathbb{R}^d$ with finite volume and any convex function $\Phi : \mathbb{R} \to \mathbb{R}$ the estimate

$$\operatorname{Tr} \Phi(-\Delta_\Omega^D) \leq \frac{|\Omega|}{(2\pi)^d} \int_{\mathbb{R}^d} \Phi(|\xi|^2)\, d\xi \tag{1.23}$$

holds true. Choosing $\Phi(x) = (\Lambda - x)_+^\sigma$, $\sigma \geq 1$, $\Lambda > 0$ yields the Berezin inequality as stated in (1.18).

Let $(\psi_k)_{k\in\mathbb{N}}$ denote the normalized eigenfunctions of $-\Delta_\Omega^D$ corresponding to the eigenvalues $(\lambda_k(\Omega))_{k\in\mathbb{N}}$. Then

$$\operatorname{Tr} \Phi(-\Delta_\Omega^D) = \sum_{k\in\mathbb{N}} \Phi(\lambda_k(\Omega)) = \sum_{k\in\mathbb{N}} \Phi\left(\langle \psi_k, -\Delta_\Omega^D \psi_k\rangle\right) = \sum_{k\in\mathbb{N}} \Phi\left(\int_{\mathbb{R}^d} |\xi|^2 |\hat{\psi}_k(\xi)|^2 d\xi\right).$$

Since $\int_{\mathbb{R}^d} |\hat{\psi}_k(\xi)|^2 d\xi = 1$ by Plancherel's Theorem we can apply Jensen's inequality to get

$$\operatorname{Tr} \Phi(-\Delta_\Omega^D) \leq \int_{\mathbb{R}^d} \Phi\left(|\xi|^2\right) \sum_{k\in\mathbb{N}} |\hat{\psi}_k(\xi)|^2 d\xi\,.$$

The set $(\psi_k)_{k\in\mathbb{N}}$ forms an orthonormal basis in $L^2(\Omega)$, hence, Parseval's identity gives

$$\sum_{k\in\mathbb{N}} |\hat{\psi}_k(\xi)|^2 = \frac{1}{(2\pi)^d} \left\| e^{ik\cdot} \right\|_{L^2(\Omega)}^2 = \frac{|\Omega|}{(2\pi)^d}$$

and (1.23) follows.

1.4. Improving sharp estimates

The semiclassical estimates (1.17) and (1.18) are universal in the sense that they hold for any open set Ω of finite volume, depend only on the volume of the set and are valid for all $t > 0$ and $\Lambda > 0$ respectively. As mentioned above these estimates are sharp. Since they coincide with the leading term of the respective semiclassical limit the constants in (1.17) and (1.18) cannot be improved.

However, the refined asymptotics (1.11) and (1.9) show that the second term of the semiclassical limit is negative. Therefore it might be possible to strengthen the estimates with negative correction terms of lower order and the question arises whether this can be done uniformly for all open sets Ω and all $t > 0$ and $\Lambda > 0$ respectively.

Let us mention that in the case of the discrete Laplace operator defined on a bounded set $\Gamma \subset \mathbb{Z}^d$ a two-term bound on the sum of the eigenvalues holds true with a main term depending on the volume of Γ and a remainder term depending on the analog of the surface area of the boundary [**FLU02**].

Such a result cannot hold for general domains $\Omega \subset \mathbb{R}^d$ in the continuous case. In particular, a direct analog of the first two terms of the semiclassical limit (1.9) cannot yield a uniform upper bound. It is easy to see that by varying the boundary of Ω slightly one can increase the surface area of the boundary more and more without changing the volume of Ω and the eigenvalues of the Laplacian significantly. Hence,

if an estimate with a negative remainder term depending only on $|\partial\Omega|$ were true, one could fix Λ and make the bound negative.

Therefore an appropriate negative correction term must depend on the geometry of Ω in a more subtle way. It is challenging to find a suitable geometric quantity replacing the surface area of the boundary such that improved semiclassical estimates hold uniformly for all $\Lambda > 0$ with a negative correction term reflecting the asymptotic properties of the second term of the refined limit (1.9).

1.4.1. Melás' improvement and decreasing rearrangements. The first result giving an improved Berezin-Li-Yau inequality for the continuous Laplace operator is due to A. Melás [**Mel03**]. His proof as well as the proof of the classical Li-Yau inequality (1.19) relies on the following results from the theory of rearrangements, see e.g. [**LL97**] for more information.

Let $A \subset \mathbb{R}^d$ be a measurable set with finite volume. Then A^*, the symmetric rearrangement of A, is defined to be the open ball centred at the origin with volume $|A|$. For a measurable function $f : \mathbb{R}^d \to \mathbb{R}$ let A_t denote the set $\{x \in \mathbb{R}^d : |f(x)| > t\}$. The symmetric decreasing rearrangement f^* is defined by

$$f^*(x) \;=\; \int_0^\infty \chi_{A_t^*}(x)\, dt\,,$$

where χ_{A^*} denotes the characteristic function of A^*.

Note that f^* is a non-negative, radially symmetric function satisfying $f^*(x) \geq f^*(y)$ for $|x| \leq |y|$. Moreover, one has

$$\int_{\mathbb{R}^d} |f(x)|\, dx \;=\; \int_{\mathbb{R}^d} f^*(x)\, dx \tag{1.24}$$

and if $g : [0,\infty) \to [0,\infty)$ is non-decreasing then

$$\int_{\mathbb{R}^d} |f(x)|\, g(|x|)\, dx \;\geq\; \int_{\mathbb{R}^d} f^*(x)\, g(|x|)\, dx\,. \tag{1.25}$$

Let us first explain how these facts can be used to prove the Li-Yau inequality (1.19). Similar as in Section 1.3.3 one can write

$$\sum_{k=1}^N \lambda_k(\Omega) \;=\; \int_{\mathbb{R}^d} |\xi|^2 \sum_{k=1}^N |\hat\psi_k(\xi)|^2\, d\xi\,, \tag{1.26}$$

where $(\psi_k)_{k\in\mathbb{N}}$ denotes the orthonormal basis of eigenfunctions corresponding to the eigenvalues $(\lambda_k(\Omega))_{k\in\mathbb{N}}$ and $(\hat\psi_k)_{k\in\mathbb{N}}$ denotes its Fourier transform. By Parseval's identity one has

$$F(\xi) \;=\; \sum_{k=1}^N |\hat\psi_k(\xi)|^2 \;\leq\; \frac{1}{(2\pi)^d} \sum_{k\in\mathbb{N}} \left|\langle e^{ik\cdot}, \psi_k\rangle_{L^2(\Omega)}\right|^2 \;\leq\; \frac{|\Omega|}{(2\pi)^d} \tag{1.27}$$

for all $\xi \in \mathbb{R}^d$ and by Plancherel's theorem

$$\int_{\mathbb{R}^d} F(\xi)\, d\xi \;=\; \sum_{k=1}^N \int_{\mathbb{R}^d} |\hat\psi_k(\xi)|^2\, d\xi \;=\; N\,.$$

In view of (1.24) this implies

$$\int_{\mathbb{R}^d} F^*(\xi)\, d\xi \;=\; N\,. \tag{1.28}$$

By definition of the decreasing rearrangement the bound (1.27) gives

$$F^*(\xi) \;\le\; \frac{|\Omega|}{(2\pi)^d} \tag{1.29}$$

for all $\xi \in \mathbb{R}^d$. Finally from (1.26) and (1.25) one gets

$$\sum_{k=1}^{N} \lambda_k(\Omega) \;=\; \int_{\mathbb{R}^d} |\xi|^2 F(\xi)\, d\xi \;\ge\; \int_{\mathbb{R}^d} |\xi|^2 F^*(\xi)\, d\xi\,. \tag{1.30}$$

Minimizing the right-hand side under the restrictions (1.28) and (1.29) yields $F^*(\xi) = |\Omega|/(2\pi)^d$ for $0 \le |\xi| \le r_N$ and $F^*(\xi) = 0$ for $|\xi| > r_N$, where $r_N = 2\pi N^{1/d}(|\Omega|\omega_d)^{-1/d}$. Inserting this into (1.30) proves the Li-Yau inequality (1.19).

A. Melás observed that in addition to (1.27) the inequality

$$\sum_{k=1}^{N} |\nabla_\xi\, \hat{\psi}_k(\xi)|^2 \;\le\; \frac{1}{(2\pi)^d} \int_\Omega |ixe^{ix\xi}|^2\, dx \;=\; \frac{I(\Omega)}{(2\pi)^d}$$

holds. In general,

$$I(\Omega) \;=\; \min_{y\in\mathbb{R}^d} \int_\Omega |x-y|^2\, dx$$

denotes the second moment of the set Ω. Here one can translate Ω to get $I(\Omega) = \int_\Omega |x|^2 dx$. Applying results from the theory of rearrangements one can show that

$$|\nabla_\xi F^*(\xi)| \;\le\; \frac{2}{(2\pi)^d} \sqrt{I(\Omega)|\Omega|}$$

holds for all $\xi \in \mathbb{R}^d$. Taking this additional restriction into account, minimizing (1.30) gives the following improvement of the Li-Yau inequality. For all $N \in \mathbb{N}$ the estimate

$$\sum_{k=1}^{N} \lambda_k(\Omega) \;\ge\; C_d \frac{d}{d+2} N^{2/d+1}\, |\Omega|^{-2/d} + M_d\, \frac{|\Omega|}{I(\Omega)}\, k \tag{1.31}$$

holds with a constant $M_d > 0$ depending only on the dimension [**Mel03**]. See also [**Ily10, Yol10, YY10**] for recent extensions.

Following the discussion in Section 1.3.2 one can apply the Legendre transform to (1.31) and use (1.8) to obtain the improved Berezin inequality

$$R_\sigma(\Lambda;\Omega) \;\le\; L^{cl}_{\sigma,d}\, |\Omega| \left(\Lambda - M_d \frac{|\Omega|}{I(\Omega)}\right)^{\sigma+d/2} \tag{1.32}$$

valid for all $\Lambda > 0$ and $\sigma \ge 1$.

In the limit $\Lambda \to \infty$ the remainder term of this bound is of the order $\Lambda^{\sigma+d/2-1}$. Comparing this to the asymptotics (1.9) shows that the improvement is not of optimal order. We conclude that the effects leading to Melas' improvement are in this sense third order effects and do not capture the influence of the Dirichlet boundary condition properly.

To complete the discussion of Melas' result let us mention that in [**HH07**] it was noted that applying the Laplace transform to (1.32) yields the following improvement of Kac' inequality (1.17). In view of (1.10) the estimate (1.32) implies

$$Z(t;\Omega) \;\leq\; \frac{|\Omega|}{(4\pi t)^{d/2}} \exp\left(-M_d \frac{|\Omega|}{I(\Omega)} t\right)$$

for all $t > 0$.

The simplicity of this improvement of Kac' inequaltiy lead to the conjecture that for all open sets $\Omega \subset \mathbb{R}^d$ with finite volume and all $t > 0$ the universal bound

$$Z(t;\Omega) \;\leq\; \frac{|\Omega|}{(4\pi t)^{d/2}} \exp\left(-\frac{t}{|\Omega|^{d/2}}\right) \tag{1.33}$$

holds [**HH07**]. Asymptotic considerations show that this conjecture is plausible for small $t > 0$ as well as for large t. However, in the same way as (1.32) these improved bounds do not capture the correct order of the second term of the semiclassical limit (1.11).

This was enhanced in [**KVW09**] where an improved Li-Yau inequality was derived in the two-dimensional case. There it is shown that even for the sum of the negative eigenvalues one can choose the order of the correction term arbitrarily close to the correct one. However, the correction depends on the geometry of the set Ω in a rather involved way.

1.4.2. A correction term of correct order. Again it turns out to be easier to remedy these defects if one considers Riesz-means R_σ of higher order. In [**Wei08**] T. Weidl considered the case $\sigma \geq 3/2$ and derived an improved Berezin-Li-Yau inequality with a negative remainder term of correct order compared to the second term of the semiclassical limit (1.9).

To state this result and to explain how it depends on the geometry of the underlying domain $\Omega \subset \mathbb{R}^d$ let us introduce the following notation. Fix a Cartesian coordinate system in $\mathbb{R}^d$ and write $x = (x', t) \in \mathbb{R}^{d-1} \times \mathbb{R}$ for $x \in \mathbb{R}^d$. For $x' \in \mathbb{R}^{d-1}$ consider the one-dimensional sections $\Omega(x') = \{t \in \mathbb{R} \,:\, (x', t) \in \Omega\}$. If not empty, each section $\Omega(x')$ consists of at most countably many open intervals $J_k(x') \subset \mathbb{R}$, $k = 1, \dots, N(x') \leq \infty$.

Fix $\Lambda > 0$ and let $\kappa(x', \Lambda) \subset \mathbb{N}$ denote the subset of all indices for which $J_k(x')$ is larger than $\pi/\sqrt{\Lambda}$. Set

$$\Omega_\Lambda(x') \;=\; \bigcup_{k \in \kappa(x', \Lambda)} J_k(x') \;\subset\; \Omega(x')$$

$$\Omega_\Lambda \;=\; \bigcup_{x' \in \mathbb{R}^{d-1}} \{x'\} \times \Omega_\Lambda(x') \;\subset\; \Omega$$

and let $\chi(x', \Lambda)$ be the number of elements in $\kappa(x', \Lambda)$ that is the number of intervals in $\Omega(x')$ larger than $\pi/\sqrt{\Lambda}$. The function $\chi(x', \Lambda)$ is measurable and finite a.e. if the volume of Ω_Λ is finite [**Gei08**]. For $\mu > 0$ let us define constants

$$\varepsilon_\mu \;=\; \min_{A \geq 1} \left(\frac{A}{2} B\left(1 + \mu, \frac{1}{2}\right) - \sum_{k \in \mathbb{N}} \left(1 - \frac{k^2}{A^2}\right)_+^\mu \right), \tag{1.34}$$

where B denotes the beta-function. In [**Wei08**] it is shown that the improved Berezin-Li-Yau inequality

$$R_\sigma(\Lambda;\Omega) \;\leq\; L^{cl}_{\sigma,d}\,|\Omega_\Lambda|\,\Lambda^{\sigma+d/2} - \varepsilon_{\sigma+(d-1)/2}\,L^{cl}_{\sigma,d-1}\int_{\mathbb{R}^{d-1}}\chi(x',\Lambda)dx'\,\Lambda^{\sigma+(d-1)/2} \qquad (1.35)$$

holds for all $\Lambda > 0$ and $\sigma \geq 3/2$.

Comparing this bound with the original Berezin inequality (1.18) we recognize two improvements. First the volume of Ω is replaced by the smaller volume of the effective domain Ω_Λ consisting of one-dimensional sections larger than $\pi/\sqrt{\Lambda}$. Note that in the case of an open interval $I \subset \mathbb{R}$ on has $N(\Lambda; I) = 0$ for all $\Lambda > \pi^2/|I|^2$, i.e. for $|I| \leq \pi/\sqrt{\Lambda}$. Roughly speaking, one can get rid of those parts of Ω which are too narrow to support eigenvalues less or equal than Λ.

Analyzing the difference $|\Omega| - |\Omega_\Lambda|$ is difficult in general since it depends on the geometry of the domain. However, calculating it for bounded domains, e.g. for the ball, wee see that it is typically of order $1/\Lambda$. As in Melas' improved bound (1.32) we see that this first improvement captures some kind or third order effect.

The second improvement of (1.35) in comparison to (1.18) is due to the negative remainder term. Again considering the integral $\int_{\mathbb{R}^{d-1}}\chi(x',\Lambda)dx'$ on bounded domains, we see that it is a positive and increasing function of order zero in Λ. Hence, the negative remainder term is indeed of correct order in Λ in comparison to the second term of the semiclassical limit (1.9).

Comparing the remainder with the second term of the semiclassical limit (1.9) shows that the quantity $\int_{\mathbb{R}^{d-1}}\chi(x',\Lambda)dx'$ takes the effect of the Dirichlet boundary condition into account. While the improved inequality (1.35) is very precise for specific domains this geometric dependence of the remainder term is rather involved. In particular, in the limit $\Lambda \to \infty$ the quantity $\int_{\mathbb{R}^{d-1}}\chi(x',\Lambda)dx'$ does not necessarily converge to a multiple of the surface area of the boundary. Therefore it is interesting to study the geometric dependence in more detail in order to simplify the bound. It is challenging to find an improved estimate with a remainder term capturing not only the correct order in Λ but also the geometric properties of the second term of the semiclassical limit.

1.4.3. Induction in the dimension and operator-valued Lieb-Thirring inequalities. The proof of (1.35) relies on an induction in the dimension argument lifting one-dimensional estimates to higher dimensions. This lifting technique which goes back to [**GGM78**] and [**Lap97**] allows to reduce the problem of estimating the eigenvalues of $-\Delta^D_\Omega$ on a set $\Omega \subset \mathbb{R}^d$, $d \geq 2$, to eigenvalue bounds for one-dimensional differential operators. The eigenvalues of these operators are known explicitly. Hence one can use elementary calculations to find suitable bounds.

Here the implementation of this method is based on operator-valued Lieb-Thirring inequalities introduced in [**LW00**] to establish the sharp Lieb-Thirring inequality (1.16). In fact, this result is obtained for systems of Schrödinger operators with operator-valued potentials. Let $\mathcal{G}$ be a separable Hilbert space, let $\mathbb{I}_\mathcal{G}$ be the identity

operator in $\mathcal{G}$ and consider the Schrödinger-type operator

$$-\Delta \otimes \mathbb{I}_{\mathcal{G}} + V$$

in $L^2(\mathbb{R}^d, \mathcal{G})$. Assume that $V(x)$, $x \in \mathbb{R}^d$, is a family of self-adjoint non-positive operators in $\mathcal{G}$ such that $\mathrm{Tr} V_- \in L^{\sigma+d/2}(\mathbb{R}^d)$. Then for $\sigma \geq 3/2$ and all $\Lambda > 0$ the estimate

$$\mathrm{Tr}\,(-\Delta \otimes \mathbb{I}_{\mathcal{G}} + V)_-^\sigma \;\leq\; L^{cl}_{\sigma,d} \int_{\mathbb{R}^d} \mathrm{Tr} V_-^{\sigma+d/2}(x)\, dx$$

holds [**LW00**]. Note that this result itself is obtained from the one-dimensional Lieb-Thirring inequality [**LT76**]

$$\mathrm{Tr}\left(-\frac{d^2}{dt^2} + V\right)_-^{3/2} \;\leq\; \frac{3}{16} \int_{\mathbb{R}} V_-(t)\, dt$$

by means of induction in the dimension.

This argument can be applied to the Dirichlet Laplace operator on bounded open sets $\Omega \subset \mathbb{R}^d$, $d \geq 2$. In terms of the notation from Section 1.4.2 the estimate

$$R_\sigma(\Lambda;\Omega) = \mathrm{Tr}\left(-\Delta_\Omega^D - \Lambda\right)_-^\sigma \leq L^{cl}_{\sigma,d-1} \int_{\mathbb{R}^{d-1}} \mathrm{Tr}\left(-\frac{d^2}{dt^2}\Big|_{\Omega(x')} - \Lambda\right)_-^{\sigma+(d-1)/2} dx' \quad (1.36)$$

holds for $\sigma \geq 3/2$ and all $\Lambda > 0$ [**Wei08**]. Since

$$\mathrm{Tr}\left(-\frac{d^2}{dt^2}\Big|_{\Omega(x')} - \Lambda\right)_-^{\sigma+(d-1)/2} = \sum_{k \in \kappa(x',\Lambda)} \sum_{j \in \mathbb{N}} \left(\frac{\pi^2 j^2}{|J_k(x')|^2} - \Lambda\right)_-^{\sigma+(d-1)/2}$$

one can now use explicit calculations to estimate the right-hand side. Indeed, in view of (1.34) one has

$$\begin{aligned}\mathrm{Tr}\left(-\frac{d^2}{dt^2}\Big|_{\Omega(x')} - \Lambda\right)_-^{\sigma+(d-1)/2} &\leq \frac{B\left(\sigma + \frac{d+1}{2}, \frac{1}{2}\right)}{2\pi}\, \Lambda^{\sigma+d/2} \sum_{k \in \kappa(x',\Lambda)} |J_k(x')| \\ &\quad - \varepsilon_{\sigma+(d-1)/2}\, \Lambda^{\sigma+(d-1)/2}\, \chi(x',\Lambda)\,.\end{aligned}$$

Inserting this into (1.36) yields (1.35).

CHAPTER 2

Summary of main results

This thesis is based upon the articles

1. R. L. Frank and L. Geisinger, *Two-term spectral asymptotics of the Dirichlet Laplacian on a bounded domain*, Mathematical Results in Quantum Physics: Proceedings of the Qmath11 Conference (Pavel Exner, ed.), World Scientific Publishing Company, 2011, pp. 138–147.
2. R. L. Frank and L. Geisinger, *Refined semiclassical asymptotics for fractional powers of the Laplace operator*, submitted (2011).
3. L. Geisinger, A. Laptev, and T. Weidl, *Geometrical versions of improved Berezin-Li-Yau inequalities*, Journal of Spectral Theory **1** (2011), no. 1, 87–109.
4. L. Geisinger and T. Weidl, *Universal bounds for traces of the Dirichlet Laplace operator*, J. Lond. Math. Soc. **82** (2010), no. 2, 395–419.
5. L. Geisinger and T. Weidl, *Sharp spectral estimates in domains of infinite volume*, Reviews in Mathematical Physics **23** (2011), no. 6, 615–641.

The work leading to these publications was carried out from 2008 until 2011 at Universität Stuttgart, Imperial College London and Princeton University.

While each of the following chapters corresponds to one of the articles we use this chapter to summarize the main results.

The main subject of this thesis is the analysis of the Dirichlet Laplace operator and related differential and pseudo-differential operators on bounded Euclidean domains. We study the eigenvalues of these operators and their relation to the geometry of the underlying domain. The material presented here is divided into two main parts.

First we study the Dirichlet Laplace operator in the semiclassical limit. We derive refined asymptotic formulas of the type introduced in Section 1.2. We take up the questions raised in Section 1.2.2 and examine under which minimal conditions two-term spectral asymptotics exist for the Dirichlet Laplace operator. We establish the refined limit (1.9) for $\sigma \geq 1$ without assuming any global geometric conditions on the domain. We only require weak smoothness conditions on the boundary. This done in Chapter 3.

In Chapter 4 we generalize this result to fractional powers of the Laplace operator on bounded domains. Up to now it was not known whether a second term exists in the semiclassical limit for eigenvalue means of such non-local, non-smooth operators. We show that our approach is applicable and that a two-term formula analogous to (1.9) holds for $\sigma \geq 1$ for the fractional Laplacian. To avoid confusion, we emphasize

that the fractional Laplacian on a domain is different from the Dirichlet Laplacian raised to the respective fractional power. One of our results is that, while the leading terms in the semiclassical limit coincide for both operators, the second order terms do not. This means, in particular, that our result cannot be obtained from the study of the (local) Dirichlet Laplacian, and that our analysis needs to take into account non-local effects.

The second part of the thesis is devoted to improved uniform bounds on the eigenvalues of the Laplace operator, in particular to improved Berezin-Lieb-Li-Yau inequalities. We analyze how the improvements depend on the geometry of the underlying domain and how the improved estimates are related to the aforementioned asymptotic results.

In Chapter 5 we solve the problem discussed in Section 1.4.2. Based on the methods introduced in Section 1.4.3 we derive an improved version of the Berezin inequality (1.18) with a negative remainder term reflecting the properties of the second term of the semiclassical limit (1.9) accurately. Under certain geometric conditions these results imply new lower bounds on individual eigenvalues which improve the Li-Yau inequality.

Then we simplify the geometric dependence of the bounds in order to derive universal bounds on traces of the Dirichlet Laplace operator: In Chapter 6 we prove uniform estimates on the trace of the heat kernel and on Riesz-means depending only on the volume of the underlying domain. These results strengthen universal bounds from Kac and Berezin and in low dimensions these estimates prove the conjectured inequality (1.33).

Finally, in Chapter 7, we apply the developed methods to prove uniform spectral estimates for the Dirichlet Laplace operator on domains with infinite volume. In this case the classical results (1.1) and (1.18) fail and even the leading order of the asymptotics of the eigenvalue means depends on geometric properties of the domain. Nevertheless, we derive sharp spectral estimates for the Dirichlet Laplace operator and extend some results to Schrödinger operators on bounded domains.

2.1. Two-term spectral asymptotics for the Dirichlet Laplacian on a bounded domain

In Chapter 3 we take up the questions raised in Section 1.2.2. We study the sum of the eigenvalues of the Dirichlet Laplace operator on a bounded domain and give a new proof for the refined semiclassical limit (1.9) under weak smoothness assumptions on the geometry of the domain.

More precisely, we study the sum of the negative eigenvalues of the operator

$$H_\Omega \;=\; -h^2\Delta_\Omega^D - 1$$

defined on an open bounded set $\Omega \subset \mathbb{R}^d$ in the limit $h \to 0+$. This limit coincides with the semiclassical limit of the Dirichlet Laplace operator introduced in Section 1.1.1. Indeed, we have

$$\operatorname{Tr}(H_\Omega)_- \;=\; h^2 \operatorname{Tr}\left(-\Delta_\Omega^D - \frac{1}{h^2}\right) \;=\; h^2\, R_1\left(\frac{1}{h^2}, \Omega\right). \tag{2.1}$$

Our main result holds under weak smoothness assumptions on the boundary. We only require that the boundary of Ω belongs to $C^{1,\alpha}$ for some $\alpha > 0$. That means, we assume that the local charts of Ω are differentiable and that the derivatives are Hölder continuous with exponent α. Without further conditions – in particular without any assumptions on the global geometry of Ω – we prove that the limit

$$\operatorname{Tr}(H_\Omega)_- \;=\; L^{cl}_{1,d}\,\Omega\,h^{-d} - \frac{1}{4}L^{cl}_{1,d-1}\,|\partial\Omega|\,h^{-d+1} + O\left(h^{-d+1+\alpha/(2+\alpha)}\right)$$

holds as $h \to 0+$, see Theorem 3.1 below. In view of (2.1) and (1.8) it is easily seen that this implies the refined semiclassical limit (1.9) for all $\sigma \geq 1$ and for any bounded open set Ω with boundary belonging to $C^{1,\alpha}$.

This work was stimulated by the question whether a similar two-term formula holds for non-local, non-smooth operators. This is unknown since the microlocal methods described in Section 1.2.2 are not applicable to such operators. Therefore it was necessary to find a new approach.

Our proof is based on the variational principle for the sum of the eigenvalues (1.6) and on a precise localization of the Laplace operator inspired by an application of the IMS-formula in [**SS03**]. The key idea is to choose the localization not only depending on the semiclassical parameter h but also depending on the distance to the complement of Ω. Hence, the localization becomes very fine as the boundary of Ω is approached.

This multiscale localization allows to analyze two different situations separately. First one can study the semiclassical limit of H_Ω localized to the interior of the domain where the influence of the boundary is not felt. Thus a simple application of the variational principle and the methods presented in Section 1.3.3 yields precise asymptotics with a second term of order h^{-d+2}.

Close to the boundary we have to take the effects of the Dirichlet condition into account. Due to the fine localization at the boundary we can introduce a local coordinate transform which straightens the boundary and transforms the problem locally to the half-space. There we can further reduce the problem to the analysis of a one-dimensional operator given on a half-line with a boundary condition at zero. This problem can be analyzed by means of the Fourier sine transform.

Let us point out that the same approach can indeed be applied to non-local, non-smooth operators. In Chapter 4 we use the same strategy to prove an analogous formula for the fractional Laplace operator. Hence, Chapter 3 is meant as an exposition where we solve the problem for the Laplace operator itself and outline the main ideas of proof. More details can be found in Chapter 4.

2.2. Refined semiclassical asymptotics for fractional powers of the Laplace operator

Here we generalize the results of the preceding chapter to the fractional Laplace operator on a bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$. For $h > 0$ and $0 < s < 1$ let

$$H_\Omega = (-h^2\Delta)^s - 1$$

be defined as a self-adjoint operator, generated by the quadratic form

$$(v, H_\Omega v) \;=\; \int_{\mathbb{R}^d} \left(|hp|^{2s} - 1\right) |\hat{v}(p)|^2 \, dp$$

with form domain

$$\mathcal{H}^s(\Omega) \;=\; \left\{ v \in H^s(\mathbb{R}^d) \,:\, v \equiv 0 \text{ on } \mathbb{R}^d \setminus \overline{\Omega} \right\} .$$

For $0 < s < 1$ we have the representation

$$(v, H_\Omega v) \;=\; C_{s,d}\, h^{2s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|v(x) - v(y)|^2}{|x-y|^{d+2s}} dx\, dy - \int_\Omega |v(x)|^2 \, dx$$

with a constant $C_{s,d} > 0$ given in (4.5).

We see that in this setting H_Ω is a non-smooth, non-local operator and the microlocal methods described in Section 1.2.2 are not applicable. Hence, it is unknown whether a second term exists in the semiclassical limit of the respective eigenvalue means. Especially for $0 < s < 1/2$ it is not clear whether the second term has a local form and depends only on $|\partial\Omega|$, since in this case the form domain $\mathcal{H}^s(\Omega)$ coincides with the Sobolev space $H^s(\Omega)$.

In Chapter 4 we prove a two-term asymptotic expansion of the sum of the eigenvalues of H_Ω for $0 < s < 1$. Again our results hold under weak smoothness assumptions on the boundary, in particular, without any global conditions on Ω. Let the boundary of Ω satisfy $\partial\Omega \in C^{1,\alpha}$ with $0 < \alpha \leq 1$. Then as $N \to \infty$

$$\frac{1}{N} \sum_{n=1}^{N} \lambda_n^{(s)} = C_{d,s}^{(1)} |\Omega|^{-2s/d} \; N^{2s/d} + C_{d,s}^{(2)} |\partial\Omega| |\Omega|^{-(d-1+2s)/d} \; N^{(2s-1)/d} \; (1 + o(1))$$

with positive, universal constants $C_{d,s}^{(1)}$ and $C_{d,s}^{(2)}$ depending only on d and s.

To prove this result it is convenient to rewrite it in an equivalent form: In Theorem 4.1 below we show that the limit

$$\operatorname{Tr}(H_\Omega)_- \;=\; L_{s,d}^{(1)} \, |\Omega| \, h^{-d} - L_{s,d}^{(2)} \, |\partial\Omega| \, h^{-d+1} + o(h^{-d+1}) \,, \quad h \to 0+ \,, \tag{2.2}$$

holds with positive constants $L_{s,d}^{(1)}$ and $L_{s,d}^{(2)}$ given in (4.7) and (4.28).

Let us briefly summarize the strategy and the main results necessary to prove (2.2). First, we localize the operator H_Ω into balls, whose size varies depending on the distance to the complement of Ω. Then we can analyze the semiclassical limit in the bulk and at the boundary separately.

Let $d(u) = \inf\{|x-u| \,:\, x \notin \Omega\}$ denote the distance of $u \in \mathbb{R}^d$ to the complement of Ω. We set

$$l(u) \;=\; \frac{1}{2} \left(1 + \left(d(u)^2 + l_0^2\right)^{-1/2}\right)^{-1} ,$$

where $0 < l_0 \leq 1$ is a small parameter depending only on h.

In Section 4.5 we construct real-valued functions $\phi_u \in C_0^\infty(\mathbb{R}^d)$ with support in the ball $B_u = \{x \in \mathbb{R}^d \,:\, |x-u| < l(u)\}$. For all $u \in \mathbb{R}^d$ these functions are uniformly bounded and for all $x \in \mathbb{R}^d$ we have

$$\int_{\mathbb{R}^d} \phi_u^2(x) \, l(u)^{-d} \, du \;=\; 1 \,.$$

PROPOSITION. There exits a constant $C > 0$ depending only on s and d such that for all $0 < l_0 \leq 1/2$ and all $0 < h \leq C^{-1} l_0$ the estimates

$$0 \;\leq\; \mathrm{Tr}(H_\Omega)_- - \int_{\mathbb{R}^d} \mathrm{Tr}\,(\phi_u H_\Omega \phi_u)_-\, l(u)^{-d}\, du \;\leq\; C\, h^{-d+2}\, l_0^{-1} R_{\mathrm{loc}}(h, l_0)$$

hold with a remainder $R_{\mathrm{loc}}(h, l_0)$ given explicitly in Proposition 4.3 below.

In view of this result, one can analyze the local asymptotics, i.e., the asymptotic behavior of $\mathrm{Tr}(\phi_u H_\Omega \phi_u)_-$, separately on different parts of Ω. First, in the bulk, where the influence of the boundary is not felt.

PROPOSITION. Assume that $\phi \in C_0^1(\Omega)$ is supported in a ball of radius $l > 0$ and

$$\|\nabla \phi\|_\infty \;\leq\; C l^{-1}\,. \tag{2.3}$$

Then for $h > 0$ the estimates

$$-C l^{d-2} h^{-d+2} \;\leq\; \mathrm{Tr}\,(\phi H_\Omega \phi)_- - L^{(1)}_{s,d} \int_\Omega \phi^2(x)\, dx\, h^{-d} \;\leq\; 0$$

hold, with a constant depending only on the constant in (2.3).

Close to the boundary of Ω, more precisely, if the support of ϕ intersects the boundary, a boundary term of the order h^{-d+1} appears.

PROPOSITION. Assume that $\phi \in C_0^\infty(\mathbb{R}^d)$ is supported in a ball of radius $0 < l \leq 1$ intersecting the boundary of Ω and assume that (2.3) is satisfied. Then for $h > 0$ the estimates

$$-\tilde{R}_{\mathrm{bd}}(l, h) \leq \mathrm{Tr}\,(\phi H_\Omega \phi)_- - L^{(1)}_{s,d} \int_\Omega \phi^2(x) dx h^{-d} + L^{(2)}_{s,d} \int_{\partial\Omega} \phi^2(x) d\sigma(x) h^{-d+1} \leq R_{\mathrm{bd}}(l, h)$$

hold. Here $d\sigma$ denotes the $(d-1)$-dimensional volume element of $\partial\Omega$ and the lower-order remainder terms R_{bd}, $\tilde{R}_{\mathrm{bd}}$ are given explicitly in Proposition 4.5 below.

Based on these results we establish the limit (2.2).

2.3. Geometrical versions of improved Berezin-Li-Yau inequalities

Here we return to the Dirichlet Laplace operator and give answers to the problems discussed in Section 1.4. We supplement the precise asymptotic formula derived in Chapter 3 with uniform bounds valid for all values of the spectral parameter and for any open set $\Omega \subset \mathbb{R}^d$, $d \geq 2$. Our results improve the Berezin-Li-Yau inequality (1.18) for $\sigma \geq 3/2$ with a negative remainder term capturing the asymptotic and geometric properties of the second term of the semiclassical limit.

The proof is based on the induction-in-the-dimension argument outlined in Section 1.4.3. In view of (1.36) the crucial point is to derive suitable eigenvalue estimates in the one-dimensional setting.

Let $I \subset \mathbb{R}$ be an open interval and let $\delta(t) = \mathrm{dist}(t, I^c)$ denote the distance of $t \in \mathbb{R}$ to the complement of I. We consider the operator $-d^2/dt^2 - \Lambda$ on I with

Dirichlet boundary conditions at the endpoints of the interval. In Lemma 5.1 we prove that for $\sigma \geq 1$ and all $\Lambda > 0$ the estimate

$$\mathrm{Tr}\left(-\frac{d^2}{dt^2} - \Lambda\right)_-^\sigma = \sum_{k\in\mathbb{N}} \left(\Lambda - \frac{\pi^2 k^2}{|I|^2}\right)_+^\sigma \leq L^{cl}_{\sigma,1} \int_I \left(\Lambda - \frac{1}{4\delta(t)^2}\right)_+^{\sigma+1/2} dt$$

holds. Comparing this bound with the Berezin inequality (1.18) and the asymptotics (1.9) we see that the improving term $-\frac{1}{4}\delta(t)^{-2}$ captures the effect of the Dirichlet boundary condition.

To generalize this result to higher dimensions we introduce the following notation. Let $\mathbb{S}^{d-1}$ denote the unit sphere in $\mathbb{R}^d$ and for $u \in \mathbb{S}^{d-1}$ and $x \in \Omega$ set

$$\begin{aligned} \theta(x,u) &= \inf\{t > 0 \,:\, x + tu \notin \Omega\}\,, \\ d(x,u) &= \inf\{\theta(x,u), \theta(x,-u)\}\,. \end{aligned}$$

Using (1.36) we prove the following bound valid for any open set $\Omega \subset \mathbb{R}^d$, $d \geq 2$. For any $u \in \mathbb{S}^{d-1}$, $\sigma \geq 3/2$ and all $\Lambda > 0$ the estimate

$$\mathrm{Tr}\left(-\Delta_\Omega^D - \Lambda\right)_-^\sigma \leq L^{cl}_{\sigma,d} \int_\Omega \left(\Lambda - \frac{1}{4\,d(x,u)^2}\right)_+^{\sigma+d/2} dx$$

holds, see Theorem 5.4. This bound improves the Berezin inequality (1.18) by the term $\frac{1}{4}d(x,u)^{-2}$. While the bound is very precise for specific domains it depends on the choice of direction $u \in \mathbb{S}^{d-1}$. Moreover, the relation of the improvement and the second term of the semiclassical asymptotics (1.9) is not immediate.

Hence, we average the bound over all directions $u \in \mathbb{S}^{d-1}$. To state the result let

$$\Omega(x) = \{y \in \Omega \,:\, x + t(y-x) \in \Omega\,,\ \forall\, t \in [0,1]\}$$

be the part of Ω that 'can be seen' from $x \in \Omega$ and let

$$\delta(x) = \inf\left\{|y-x| \,:\, y \notin \overline{\Omega(x)}\right\}$$

denote the distance to the exterior of $\Omega(x)$.

For fixed $\varepsilon > 0$ put

$$A_\varepsilon(x) = \left\{a \in \mathbb{R}^d \setminus \overline{\Omega(x)} \,:\, |x-a| < \delta(x) + \varepsilon\right\}$$

and for $a \in A_\varepsilon(x)$ set $B_x(a) = \{y \in \mathbb{R}^d \,:\, |y-a| < |x-a|\}$ and

$$\rho_a(x) = \frac{|B_x(a) \setminus \overline{\Omega(x)}|}{\omega_d |x-a|^d}\,,$$

where ω_d denotes the volume of the unit ball in $\mathbb{R}^d$. To get a result, independent of a and ε, set

$$\rho(x) = \inf_{\varepsilon>0} \sup_{a\in A_\varepsilon(x)} \rho_a(x)\,.$$

Finally, we define

$$M_\Omega(\Lambda) = \int_{R_\Omega(\Lambda)} \rho(x)\,dx\,,$$

where $R_\Omega(\Lambda) \subset \Omega$ denotes the set $\{x \in \Omega \,:\, \delta(x) < 1/(4\sqrt{\Lambda})\}$.

In Theorem 5.5 we show that for any open set $\Omega \subset \mathbb{R}^d$, $\sigma \geq 3/2$, and all $\Lambda > 0$ the bound

$$\operatorname{Tr}(-\Delta - \Lambda)_-^\sigma \;\leq\; L^{cl}_{\sigma,d}\,|\Omega|\Lambda^{\sigma+d/2} - L^{cl}_{\sigma,d}\,2^{-d+1}\,\Lambda^{\sigma+d/2}\,M_\Omega(\Lambda)$$

holds true. This result solves the problem raised in Section 1.4.2. The negative remainder term not only captures the correct order in Λ in comparison with the second term of the semiclassical asymptotics, it also reflects the geometric properties of this second term. Indeed, the quantity $\sqrt{\Lambda}M_\Omega(\Lambda)$ is an appropriate replacement of the surface area of the boundary. On the one hand, the function $\rho(x)$ depends on local properties of the boundary. On the other hand, if Ω is for example convex or Lipschitz, then we find that $\sqrt{\Lambda}M_\Omega(\Lambda)$ converges to a multiple of $|\partial\Omega|$ as $\Lambda \to \infty$ and we recover the properties of the second term of the semiclassical limit.

Under certain conditions on the geometry of Ω our results improve the Li-Yau inequality (1.19). Set

$$l(x,u) \;=\; \theta(x,u) + \theta(x,-u) \quad \text{and} \quad l_0 \;=\; \inf_{u\in\mathbb{S}^{d-1}} \sup_{x\in\Omega} l(x,u)\,.$$

We show that for any open set $\Omega \subset \mathbb{R}^d$ and any $k \in \mathbb{N}$ the estimate

$$\lambda_k(\Omega) \;\geq\; c_d\,\frac{k^{2/d}}{|\Omega|^{2/d}} + \frac{1}{l_0^2}$$

holds with a positive constant c_d which is, however, smaller than the constant $\frac{d}{d+2}C_d$ from (1.20). Finally, we derive a similar result improving the Li-Yau inequality (1.20) for any convex domain $\Omega \subset \mathbb{R}^2$. For such domains we obtain new lower bounds on $\lambda_k(\Omega)$ for $k \leq 23$.

2.4. Universal bounds for traces of the Dirichlet Laplace operator

The work of Chapter 6 was stimulated by conjecture (1.33), namely by the question whether for any open set Ω of finite volume the improved Kac' inequality

$$Z(t;\Omega) \;\leq\; \frac{|\Omega|}{(4\pi t)^{d/2}}\exp\left(-\frac{t}{|\Omega|^{2/d}}\right)$$

holds true for all $t > 0$.

In Section 1.2.1 we explained that the trace of the heat kernel $Z(t;\Omega)$ is closely connected to the Riesz-means $R_\sigma(\Lambda;\Omega)$ via the Laplace transform. By virtue of (1.10) every bound on the Riesz-means implies a bound on the trace of the heat kernel.

Here we use the induction-in-the-dimension argument outlined in Section 1.4.3 to derive precise uniform bounds on Riesz-means and deduce estimates on the trace of the heat kernel. However, to prove the conjectured inequality (1.33) we need to prove universal bounds, i.e., bounds that depend only on the volume of the underlying domain Ω and not on any other geometric quantity. Hence, we have to analyze and simplify the geometric dependence of the improved inequalities. This is accomplished using symmetric rearrangements, introduced in Section 1.4.1, and isoperimetric results. In particular, we make use of the estimate [**Lut73**]

$$Z(t;\Omega) \;\leq\; Z(t;\Omega^*)\,, \quad t > 0\,, \tag{2.4}$$

where Ω^* denotes the ball in $\mathbb{R}^d$ with the same volume as Ω.

To apply the methods described in Section 1.4.3, fix a coordinate system in $\mathbb{R}^d$ and for $x \in \mathbb{R}^d$ write $(x', t) \in \mathbb{R}^{d-1} \times \mathbb{R}$. Let $|\Omega(x')|$ denote the length of the one-dimensional sections $\Omega(x') = \{t \in \mathbb{R} : (x', t) \in \Omega\}$ and let

$$m_d(\tau; \Omega) \,=\, |\{x' \,:\, |\Omega(x')| > \tau\}| \,, \quad \tau > 0 \,,$$

denote the distribution function of Ω.

Using the inductive argument from [**Wei08**] we prove that the estimate

$$R_\sigma(\Lambda; \Omega) \,\leq\, L^{cl}_{\sigma,d} \int_{\pi/\sqrt{\Lambda}}^\infty m_d(\tau; \Omega) d\tau \, \Lambda^{\sigma + d/2} \tag{2.5}$$

holds for all $\sigma \geq 5/2$ and all $\Lambda > 0$, see Proposition 6.5. Note that m_d is a non-negative function that satisfies

$$\int_0^\infty m_d(\tau; \Omega) d\tau \,=\, |\Omega| \,.$$

Thus (2.5) is an improvement of the Berezin inequality (1.18), however, it depends on the geometry of Ω and on the choice of coordinate system.

Averaging over all directions and using properties of rearrangements we deduce universal bounds on the Riesz-means and on the trace of the heat kernel. In particular, applying (2.4) yields the improved Kac' inequality stated below.

The general result depends on a lower bound on the ground state of $-\Delta^D_\Omega$. In view of the isoperimetric inequality by Rayleigh, Faber and Krahn [**Fab23, Kra25**] we can always choose

$$\tilde{\lambda}(\Omega) \,=\, \frac{\pi \, j^2_{d/2-1,1}}{\Gamma\left(d/2+1\right)^{2/d} |\Omega|^{2/d}} \,\leq\, \lambda_1(\Omega)$$

as a lower bound on $\lambda_1(\Omega)$, where $j_{k,1}$ denotes the first zero of the Bessel-function J_k.

Let $\Gamma(z)$ be the usual Gamma-function and by

$$\hat{\Gamma}(z, s) \,=\, \frac{1}{\Gamma(z)} \int_s^\infty t^{z-1} e^{-t} dt$$

we denote normed incomplete Gamma-functions. For $d \in \mathbb{N}$ put

$$\sigma_d \,=\, \begin{cases} 5/2 & \text{if} \quad d = 2 \\ 2 & \text{if} \quad d = 3 \\ 3/2 & \text{if} \quad d \geq 4 \end{cases} \,.$$

Let $\Omega \subset \mathbb{R}^d$ be an arbitrary open set with finite volume and let $\lambda \in [\tilde{\lambda}(\Omega), \lambda_1(\Omega)]$ be a lower bound on the ground state. In Theorem 6.1 it is stated that for any $t > 0$ the improved Kac' inequality

$$Z(t) \,\leq\, \frac{|\Omega|}{(4\pi t)^{d/2}} \hat{\Gamma}\left(\sigma_d + \frac{d}{2} + 1, \lambda t\right) \,-\, (R(t, \lambda))_+$$

holds true with a remainder term

$$R(t, \lambda) \,=\, c_{1,d} \frac{|\Omega|^{(d-1)/d}}{(4\pi t)^{(d-1)/2}} \hat{\Gamma}\left(\sigma_d + \frac{d+1}{2}, \lambda t\right) - c_{2,d} \frac{|\Omega|^{(d-3)/d}}{(4\pi t)^{(d-3)/2}} \hat{\Gamma}\left(\sigma_d + \frac{d-1}{2}, \lambda t\right)$$

and with explicit positive constants $c_{1,d}$ and $c_{2,d}$ depending only on the dimension.

If we choose $\lambda = \tilde{\lambda}(\Omega)$ we arrive at a universal upper bound on $Z(t)$ depending only on $|\Omega|$ and not including any explicit information on $\lambda_1(\Omega)$. We note that the bound captures the main asymptotic behaviour of $Z(t)$ as t tends to zero: The first term equals the leading term of the short time asymptotics of $Z(t)$ and the second term shows the correct order in t compared with the second term of the asymptotic expansion (1.11).

Moreover, the bound decays exponentially as t tends to infinity. In particular, this result implies the conjectured inequality (1.33) in dimensions $d \leq 633$.

The same approach based on (2.5) can be applied to unbounded domains of infinite volume. While the results of Kac (1.17) and Luttinger (2.4) must fail for such domains, we show that under appropriate conditions on Ω our methods can still be applied and give sharp upper bounds on the trace of the heat kernel.

2.5. Sharp spectral estimates in domains of infinite volume

In [**Wei08**] it is mentioned that the improved Berezin-Li-Yau inequality (1.35) can be applied in domains of infinite volume. In [**GW10**] – see the previous section and Chapter 6 – we refine this approach to derive sharp upper bounds on the trace of the heat kernel in unbounded domains.

In Chapter 7 we consider spectral estimates in domains of infinite volume in more detail. For unbounded domains Ω even the discreteness of the spectrum of the Dirichlet Laplacian on Ω is no longer guaranteed. A necessary condition is the so called quasi-boundedness of Ω [**AF03**] which is satisfied, by definition, if

$$\lim_{\substack{x\in\Omega \\ |x|\to\infty}} \operatorname{dist}(x, \partial\Omega) \,=\, 0\,.$$

But even for quasi-bounded domains the classical Berezin inequality (1.18) and Weyl's asymptotic formula (1.1) must fail if the volume of Ω is infinite. Here we show that one can nevertheless prove uniform bounds on eigenvalue means for certain domains of infinite volume. In this case the leading order of the semiclassical limit depends on the geometry of Ω. Applying the inductive argument described in Section 1.4.3 we can prove estimates valid for all $\Lambda > 0$ and reflecting the correct asymptotic behavior.

First we consider horn-shaped regions, domains stretching to infinity along distinguished directions, see [**vdB92a**] for a general definition. In [**Sim83**] the semiclassical limit of the counting function was calculated for the horn-shaped domains

$$\Omega_\nu \,=\, \left\{(x,y)\in\mathbb{R}^2 \,:\, |x|\cdot|y|^\nu \,<\, 1\right\}\,, \quad \nu \geq 1\,.$$

In view of (1.8) one can immediately generalize these results to get asymptotic formulas for the semiclassical limit of eigenvalues means. For $\sigma \geq 0$ and $\nu > 1$ we obtain

$$R_\sigma(\Lambda;\Omega_\nu) \,=\, \zeta(\nu)\,\left(\frac{2}{\pi}\right)^\nu \frac{B(\frac{\nu}{2}+1,\sigma+1)}{B\left(\sigma+\frac{\nu+3}{2},\frac{1}{2}\right)}\,\Lambda^{\sigma+(\nu+1)/2} + o\left(\Lambda^{\sigma+(\nu+1)/2}\right) \tag{2.6}$$

as $\Lambda\to\infty$, where ζ denotes the zeta function.

In Theorem 7.3 we show that for $\sigma \geq 3/2$ and $\nu > 1$ the uniform bound

$$R_\sigma(\Lambda;\Omega_\nu) \,\leq\, \zeta(\nu)\,\left(\frac{2}{\pi}\right)^\nu \frac{B(\frac{\nu}{2}+1,\sigma+1)}{B\left(\sigma+\frac{\nu+3}{2},\frac{1}{2}\right)}\,\Lambda^{\sigma+(\nu+1)/2}$$

holds for all $\Lambda > 0$. Comparing this bound with the asymptotic relation (2.6) we see that the estimate is sharp: For horn-shaped regions, just as well as for bounded domains, the leading term of the semiclassical limit yields a uniform upper bound.

We generalize this result to higher dimensions and using the estimate (1.21) from [**Lap97**] we deduce order-sharp uniform bounds on the counting function $R_0(\Lambda;\Omega_\nu)$. Moreover, we give similar estimates in the critical case $\nu = 1$.

As a second example for domains with infinite volume we consider so called spiny urchins, radially symmetric domains $\Omega_S \subset \mathbb{R}^2$ with infinite volume.

To construct Ω_S we use polar coordinates $(r,\varphi) \in [0,\infty)\times[0,2\pi)$ and choose an increasing sequence $(r_n)_{n\in\mathbb{N}}$ of positive real numbers and put $r_0 = 0$. For $n \in \mathbb{N}_0$ and $k = 1, 2, \ldots, 2^{n+2}$ let

$$\Gamma_{n,k} \,=\, \left\{(r,\varphi)\,:\, r \geq r_n\,,\, \varphi = \frac{k-1}{2^{n+1}}\pi\right\}$$

be semi-axes and define

$$\Omega_S \,=\, \mathbb{R}^2 \setminus \bigcup_{n,k} \Gamma_{n,k}\,,$$

see Figure 2 in Chapter 7 and the figure on the title page. Note that this domain, though quasi bounded, has empty exterior. However, if $\lim_{n\to\infty} r^n 2^{-n} = 0$, then the spectrum of H_{Ω_S} is discrete.

Again we apply the inductive argument from Section 1.4.3. We adapt it to the radially symmetric situation and derive order-sharp bounds on the eigenvalue means $R_\sigma(\Lambda;\Omega_S)$. Here the leading order of the semiclassical limit depends on the choice of the sequence $(r_n)_{n\in\mathbb{N}}$. To give a flavor of our results let us just state the following examples considered in Corollary 7.9.

Let $\sigma \geq 0$ and assume $r_n = n$. Then for $0 < \Lambda \leq 15/4$ we have $R_\sigma(\Lambda,\Omega_S) = 0$ and for $\Lambda > 15/4$

$$R_\sigma(\Lambda,\Omega_S) \,\leq\, C_\sigma\,\Lambda^{\sigma+1}(\ln\Lambda)^2\,.$$

To give a second example assume $r_n = n^{\delta n}$ with $0 < \delta < 1$. Then for $0 < \Lambda \leq 15\cdot 2^{-2(1+\delta)}$ we have $R_\sigma(\Lambda,\Omega_S) = 0$ and for $\Lambda > 15\cdot 2^{-2(1+\delta)}$

$$R_\sigma(\Lambda,\Omega_S) \,\leq\, C_{\sigma,\delta}\,\Lambda^{\sigma+1/(1-\delta)}\,.$$

We prove that all bounds capture the correct order in Λ as $\Lambda \to \infty$.

Finally, we examine Schrödinger operators

$$H_\Omega(V) \,=\, -\Delta_\Omega^D - V$$

defined on unbounded domains $\Omega \subset \mathbb{R}^d$, $d \geq 2$ with non constant potential $V \geq 0$. If $V \notin L^{\sigma+d/2}(\Omega)$ then the classical bounds (1.15) and (1.16) and the asymptotic formula (1.13) must fail.

Under certain conditions on Ω and on V we can nevertheless prove uniform bounds on the eigenvalue means of $H_\Omega(V)$. However, if the potential V is not constant the

situation is more difficult, since even in the one-dimensional setting the eigenvalue means cannot be calculated explicitly. Therefore we have to adapt the aforementioned arguments and analyze the one-dimensional problem in more detail. In particular, we have to study the effect of different boundary conditions on the eigenvalues.

The result yields an improved version of the semiclassical bound (1.16), see Theorem 7.15. Again, this sharp Lieb-Thirring inequality with remainder term can be applied in situations, where all known results – in particular (1.13) and (1.16) – fail.

2.6. Open problems and further research

One hundred years after H. Weyl proved the fundamental asymptotic law (1.1) and conjectured about the second term in the semiclassical limit (1.7) and fifty years after G. Polya's – still unsolved – conjecture about the sharp constant in the semiclassical estimate on the eigenvalues (1.20) there are still numerous open questions and challenging problems about the spectrum of the Laplace operator on bounded domains. Here we only mention some questions and further considerations arising naturally from the results presented in this thesis.

In Theorem 3.1 and Theorem 4.1 we establish the existence of a second term in the semiclassical limit for the sum of the eigenvalues of the Dirichlet Laplace operator and its fractional powers. The results hold under weak assumptions on Ω, however, they do not give a complete answer to the question under which minimal conditions the second term exists. In view of the short-time limit of the trace of the heat semigroup, see (1.11) and related results in [**BKS09**], one might expect that Theorem 3.1 and Theorem 4.1 can be extended to Lipschitz domains Ω.

The question raised in [**BKS09**] of whether an analogue of Ivrii's two-term asymptotics (1.7) holds for the counting function of the fractional Laplacian remains a challenging open problem.

Considering the Laplace operator itself, it is natural to ask what happens for different boundary conditions. Since the methods presented in Chapter 3 are not specific to the Dirichlet case it is possible to generalize Theorem 3.1 to Neumann or arbitrary Robin boundary conditions.

Comparing Ivrii's result described in Section 1.2.2 with Theorem 3.1 the question arises whether any intermediate results hold. It might be possible to adapt the methods from [**HR90**] to prove the asymptotic formula (1.9) for any $0 < \sigma < 1$ at least for domains with smooth boundary. In the same way as in (1.14) more and more lower order terms should exist in the semiclassical limit of the eigenvalue means $R_\sigma(\Lambda;\Omega)$ as σ tends to infinity.

Concerning uniform bounds on the eigenvalue means the results presented in Chapters 5 to 7 are rather satisfactory for $\sigma \geq 3/2$. In particular, Theorem 5.5 gives a sharp bound valid for all $\Lambda > 0$ and reflects the properties of the semiclassical limit accurately. However, the question whether similar results are valid in the important case $\sigma = 1$ remains open. Such a result could lead to a uniform improvement of (1.20) and (1.22), in particular to better constants.

CHAPTER 3

Two-term spectral asymptotics for the Dirichlet Laplacian on a bounded domain

Rupert L. Frank and L. Geisinger

Dedicated to Ari Laptev on the occasion of his 60th birthday.

Abstract. Let $-\Delta$ denote the Dirichlet Laplace operator on a bounded open set in $\mathbb{R}^d$. We study the sum of the negative eigenvalues of the operator $-h^2\Delta - 1$ in the semiclassical limit $h \to 0+$. We give a new proof that yields not only the first term of the asymptotic formula but also the second term involving the surface area of the boundary of the set. The proof is valid under weak smoothness assumptions on the boundary.

3.1. Introduction and main result

3.1.1. Introduction. Let Ω be a bounded open set in $\mathbb{R}^d$, $d \geq 2$. We consider the Dirichlet Laplace operator $-\Delta_\Omega$ defined as a self-adjoint operator in $L^2(\Omega)$ generated by the form

$$(v, -\Delta_\Omega v) \;=\; \int_\Omega |\nabla v(x)|^2 dx$$

with form domain $H^1_0(\Omega)$. Since Ω is bounded the embedding of $H^1_0(\Omega)$ into $L^2(\Omega)$ is compact and the spectrum of $-\Delta_\Omega$ is discrete. It consists of a series of positive eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \dots$ accumulating at infinity only.

In general, the eigenvalues λ_k cannot be calculated explicitly and especially for large k it is difficult to evaluate them numerically. Therefore it is interesting to describe the asymptotic behavior of λ_k as $k \to \infty$. This is equivalent to the asymptotics of the negative eigenvalues of the operator

$$H_\Omega \;=\; -h^2\Delta_\Omega - 1$$

in the semiclassical limit $h \to 0+$.

The first general result is due to H. Weyl who studied the counting function

$$N_\Omega(h) \;=\; \sharp\{\lambda_k < h^{-2}\} = \operatorname{Tr}(H_\Omega)^0_- \;.$$

In 1912 he showed that the first term of its semiclassical limit is given by the phase-space volume [**Wey12a**]: For any open bounded set $\Omega \subset \mathbb{R}^d$ the limit

$$N_\Omega(h) \;=\; C_d\,|\Omega|\,h^{-d} + o(h^{-d})$$

holds as $h \to 0+$, where

$$C_d \;=\; \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (|p|^2 - 1)_-^0 dp \;=\; \frac{\omega_d}{(2\pi)^d}$$

and ω_d denotes the volume of the unit ball in $\mathbb{R}^d$.

H. Weyl conjectured in [**Wey13**] that this formula can be refined by a second term of order h^{-d+1} depending on the boundary of Ω. This stimulated a detailed analysis of the semiclassical limit of partial differential operators. We refer to the books [**Hör85, SV97, Ivr98**] for general results and an overview over the literature. Eventually, the existence of a second term was proved by V. Ivrii by means of a detailed microlocal analysis [**Ivr80a, Ivr80b**]: If the boundary of Ω is smooth and if the measure of all periodic geodesic billiards is zero then the limit

$$N_\Omega(h) \;=\; C_d\,|\Omega|\,h^{-d} - \frac{1}{4} C_{d-1}\,|\partial\Omega|\,h^{-d+1} + o(h^{-d+1}) \tag{3.1}$$

holds as $h \to 0+$, where $|\partial\Omega|$ denotes the surface area of the boundary.

In this article we are interested in the sum of the negative eigenvalues

$$\mathrm{Tr}(H_\Omega)_- \;=\; \sum_{k\in\mathbb{N}} (h^2\lambda_k - 1)_- \,.$$

This quantity describes the energy of non-interacting, fermionic particles trapped in Ω and plays an important role in physical applications.

The asymptotic relation (3.1) immediately implies a refined formula for the semiclassical limit of $\mathrm{Tr}(H_\Omega)_-$: Suppose that the aforementioned geometric conditions on Ω are satisfied. Then integrating (3.1) yields

$$\mathrm{Tr}(H_\Omega)_- \;=\; L_d\,|\Omega|\,h^{-d} - \frac{1}{4} L_{d-1} |\partial\Omega| h^{-d+1} + o(h^{-d+1}) \tag{3.2}$$

as $h \to 0+$, with

$$L_d \;=\; \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (|p|^2 - 1)_- dp \;=\; \frac{2}{d+2} \frac{\omega_d}{(2\pi)^d} \,.$$

In the following we present a direct approach to derive the semiclassical limit of $\mathrm{Tr}(H_\Omega)_-$. We prove (3.2) without using the result for the counting function. Since we do not apply any microlocal methods the proof works under much weaker conditions.

3.1.2. Main Result. Our main result holds without any global geometric conditions on Ω. We only require weak smoothness conditions on the boundary - namely that the boundary belongs to the class $C^{1,\alpha}$ for some $\alpha > 0$. That means, we assume that the local charts of Ω are differentiable and the derivatives are Hölder continuous with exponent α.

THEOREM 3.1. *Let the boundary of Ω satisfy $\partial\Omega \in C^{1,\alpha}$, $0 < \alpha \leq 1$. Then the asymptotic limit*

$$\mathrm{Tr}(H_\Omega)_- \;=\; L_d\,|\Omega|\,h^{-d} - \frac{1}{4} L_{d-1}\,|\partial\Omega|\,h^{-d+1} + O\left(h^{-d+1+\alpha/(2+\alpha)}\right)$$

holds as $h \to 0+$.

Our work was stimulated by the question whether similar two-term formulae hold for non-local, non-smooth operators. This is unknown, since the microlocal methods leading to (3.1) are not applicable. Therefore it is necessary to use a direct approach.

Indeed, Theorem 3.1 can be extended to fractional powers of the Dirichlet Laplace operator [**FG11a**]. The strategy of the proof is similar but dealing with non-local operators is more difficult and elaborate. In order to give a flavor of our techniques we confine ourselves in this article to the local case.

The question whether the second term of the semiclassical limit of $\mathrm{Tr}(H_\Omega)_-$ exists for Lipschitz domains Ω remains open.

3.1.3. Strategy of the proof. The proof of Theorem 3.1 is divided into three steps: First, we localize the operator H_Ω into balls, whose size varies depending on the distance to the complement of Ω. Then we analyze separately the semiclassical limit in the bulk and at the boundary.

To localize, let $d(u) = \inf\{|x-u| \,:\, x \notin \Omega\}$ denote the distance of $u \in \mathbb{R}^d$ to the complement of Ω. We set

$$l(u) \,=\, \frac{1}{2}\left(1 + \left(d(u)^2 + l_0^2\right)^{-1/2}\right)^{-1} ,$$

where $0 < l_0 \leq 1$ is a parameter depending only on h. Indeed, we will finally choose l_0 proportional to $h^{2/(\alpha+2)}$.

In Section 3.3 we introduce real-valued functions $\phi_u \in C_0^\infty(\mathbb{R}^d)$ with support in the ball $B_u = \{x \in \mathbb{R}^d \,:\, |x-u| < l(u)\}$. For all $u \in \mathbb{R}^d$ these functions satisfy

$$\|\phi_u\|_\infty \,\leq\, C\,, \qquad \|\nabla\phi_u\|_\infty \leq C\, l(u)^{-1} \tag{3.3}$$

and for all $x \in \mathbb{R}^d$

$$\int_{\mathbb{R}^d} \phi_u^2(x)\, l(u)^{-d}\, du \,=\, 1\,. \tag{3.4}$$

Here and in the following the letter C denotes various positive constants that might depend on Ω, but that are independent of u, l_0 and h.

Proposition 3.2. *For $0 < l_0 \leq 1$ and $h > 0$ we have*

$$\left|\mathrm{Tr}(H_\Omega)_- - \int_{\mathbb{R}^d} \mathrm{Tr}\,(\phi_u H_\Omega \phi_u)_-\, l(u)^{-d}\, du\right| \,\leq\, C\, l_0^{-1}\, h^{-d+2}\,.$$

In view of this result, one can analyze the local asymptotics, i. e., the asymptotic behavior of $\mathrm{Tr}(\phi_u H_\Omega \phi_u)_-$ separately on different parts of Ω. First, in the bulk, where the influence of the boundary is not felt.

Proposition 3.3. *Assume that $\phi \in C_0^\infty(\Omega)$ is supported in a ball of radius $l > 0$ and that*

$$\|\nabla\phi\|_\infty \,\leq\, C\, l^{-1} \tag{3.5}$$

is satisfied. Then for $h > 0$ the estimate

$$\left|\mathrm{Tr}\,(\phi H_\Omega \phi)_- - L_d \int_\Omega \phi^2(x)\, dx\, h^{-d}\right| \,\leq\, C\, l^{d-2}\, h^{-d+2}$$

holds, with a constant depending only on the constant in (3.5).

Close to the boundary of Ω, more precisely, if the support of ϕ intersects the boundary, a term of order h^{-d+1} appears:

PROPOSITION 3.4. *Assume that $\phi \in C_0^\infty(\mathbb{R}^d)$ is supported in a ball of radius $l > 0$ intersecting the boundary of Ω and that inequality (3.5) is satisfied.*

Then for all $0 < l \leq 1$ and $0 < h \leq 1$ the estimate

$$\left| \operatorname{Tr} (\phi H_\Omega \phi)_- - L_d \int_\Omega \phi^2(x)\, dx\, h^{-d} + \frac{1}{4} L_{d-1} \int_{\partial\Omega} \phi^2(x) d\sigma(x)\, h^{-d+1} \right| \leq r(l,h)$$

holds. Here $d\sigma$ denotes the $d-1$-dimensional volume element of $\partial\Omega$ and the remainder satisfies

$$r(l,h) \leq C \left(\frac{l^{d-2}}{h^{d-2}} + \frac{l^{2\alpha+d-1}}{h^{d-1}} + \frac{l^{d+\alpha}}{h^d} \right)$$

with a constant depending on Ω, $\|\phi\|_\infty$ and the constant in (3.5).

Based on these propositions we can complete the proof of the main result.

PROOF OF THEOREM 3.1. In order to apply Proposition 3.4 to the operators $\phi_u H_\Omega \phi_u$, we need to estimate $l(u)$ uniformly. Assume that $u \in \mathbb{R}^d$ satisfies $B_u \cap \partial\Omega \neq \emptyset$. Then we have $d(u) \leq l(u)$, which by definition of $l(u)$ implies

$$l(u) \leq l_0/\sqrt{3}\,. \tag{3.6}$$

In view of (3.3) we can therefore apply Proposition 3.3 and Proposition 3.4 to all functions ϕ_u, $u \in \mathbb{R}^d$. Combining these results with Proposition 3.2 we get

$$\begin{aligned} &\left| \operatorname{Tr} (H_\Omega)_- - \frac{L_d}{h^d} \int_{\mathbb{R}^d} \int_\Omega \phi_u^2(x) dx\, \frac{du}{l(u)^d} + \frac{L_{d-1}}{4h^{d-1}} \int_{\mathbb{R}^d} \int_{\partial\Omega} \phi_u^2(x) d\sigma(x)\, \frac{du}{l(u)^d} \right| \\ &\leq C \left(l_0^{-1} h^{-d+2} + \int_{U_1} l(u)^{-2}\, du\, h^{-d+2} + \int_{U_2} r(l(u),h) l(u)^{-d}\, du \right), \end{aligned}$$

where $U_1 = \{u \in \Omega \,:\, B_u \cap \partial\Omega = \emptyset\}$ and $U_2 = \{u \in \mathbb{R}^d \,:\, B_u \cap \partial\Omega \neq \emptyset\}$. Now we change the order of integration and by virtue of (3.4) we obtain

$$\begin{aligned} &\left| \operatorname{Tr} (H_\Omega)_- - L_d\, |\Omega|\, h^{-d} + \frac{1}{4} L_{d-1}\, |\partial\Omega|\, h^{-d+1} \right| \\ &\leq C \left(l_0^{-1} h^{-d+2} + \int_{U_1} l(u)^{-2}\, du\, h^{-d+2} + \int_{U_2} r(l(u),h) l(u)^{-d}\, du \right). \end{aligned} \tag{3.7}$$

It remains to estimate the remainder terms. Note that, by definition of $l(u)$, we have

$$l(u) \geq \frac{1}{4} \min\left(d(u), 1\right) \quad \text{and} \quad l(u) \geq \frac{l_0}{4}$$

for all $u \in \mathbb{R}^d$. Together with (3.6) this implies

$$\int_{U_1} l(u)^{-2} du \leq C l_0^{-1} \quad \text{and} \quad \int_{U_2} l(u)^a du \leq C l_0^a \int_{\{d(u) \leq l_0\}} du \leq C l_0^{a+1} \tag{3.8}$$

for any $a \in \mathbb{R}$. Inserting these estimates into (3.7) we find that the remainder terms are bounded from above by a constant times

$$l_0^{-1} h^{-d+2} + l_0^{2\alpha} h^{-d+1} + l_0^{\alpha+1} h^{-d}\,.$$

Finally, we choose l_0 proportional to $h^{2/(\alpha+2)}$ and conclude that all error terms in (3.7) equal $O(h^{-d+1+\alpha/(2+\alpha)})$ as $h \to 0+$. □

The remainder of the text is structured as follows. In Section 3.2 we analyze the local asymptotics and outline the proofs of Proposition 3.3 and 3.4. In Section 3.3, we perform the localization and, in particular, prove Proposition 3.2.

3.2. Local asymptotics

To prove the propositions we need the following rough estimate, a variant of the Berezin-Lieb-Li-Yau inequality [**Ber72b, Lie73, LY83**].

LEMMA 3.5. *For any $\phi \in C_0^\infty(\mathbb{R}^d)$ and $h > 0$*

$$\operatorname{Tr}(\phi H_\Omega \phi)_- \;\le\; L_d \int_{\mathbb{R}^d} \phi^2(x)\, dx\, h^{-d}\,.$$

PROOF. Let us introduce the operator

$$H_0 = -h^2\Delta - 1\,,$$

defined with form domain $H^1(\mathbb{R}^d)$. The variational principle for sums of eigenvalues implies $\operatorname{Tr}(\phi H_\Omega \phi)_- \le \operatorname{Tr}(\phi (H_0)_- \phi)_-$. Using the Fourier-transform one can derive an explicit expression for the kernel of $(H_0)_-$ and inserting this yields the claim. □

3.2.1. Local asymptotics in the bulk. First we assume $\phi \in C_0^\infty(\Omega)$. Then we have $\operatorname{Tr}(\phi H_\Omega \phi)_- = \operatorname{Tr}(\phi H_0 \phi)_-$, since the form domains of $\phi H_\Omega \phi$ and $\phi H_0 \phi$ coincide. Moreover, by scaling, we can assume $l = 1$. Thus, to prove Proposition 3.3, it suffices to establish the estimate

$$\left| \operatorname{Tr}(\phi H_0 \phi)_- - L_d \int_{\mathbb{R}^d} \phi^2(x)\, dx\, h^{-d} \right| \;\le\; C h^{-d+2}$$

for $h > 0$. The lower bound follows immediately from Lemma 3.5. The upper bound can be derived in the same way as in the proof of Lemma 3.7 below. Indeed, by choosing the trial density matrix $\gamma = \chi (H_0)_-^0 \chi$ we find

$$\operatorname{Tr}(\phi H_0 \phi)_- \;\ge\; L_d \int_{\mathbb{R}^d} \phi^2(x)\, dx - C_d \int_{\mathbb{R}^d} (\nabla\phi)^2(x)\, dx\, h^{-d+2}$$

and the claim follows.

3.2.2. Straightening the boundary. Here we transform the operator H_Ω locally to an operator given on the half-space $\mathbb{R}^d_+ = \{y \in \mathbb{R}^d : y_d > 0\}$. There we define the operator H^+ in the same way as H_Ω, with form domain $H_0^1(\mathbb{R}^d_+)$.

Under the conditions of Proposition 3.4 let B denote the open ball of radius $l > 0$, containing the support of ϕ. Choose $x_0 \in B \cap \partial\Omega$ and let ν_{x_0} be the normed inner normal vector at x_0. We choose a Cartesian coordinate system such that $x_0 = 0$ and $\nu_{x_0} = (0, \dots, 0, 1)$, and we write $x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$ for $x \in \mathbb{R}^d$.

For sufficiently small $l > 0$ one can introduce new local coordinates near the boundary. Let D denote the projection of B on the hyperplane given by $x_d = 0$.

Since the boundary of Ω is compact and in $C^{1,\alpha}$, there exists a constant $c > 0$, such that for $0 < l \leq c$ we can find a real function $f \in C^{1,\alpha}$ given on D, satisfying

$$\partial\Omega \cap B \,=\, \{(x', x_d) \,:\, x' \in D, x_d = f(x')\} \cap B\,.$$

The choice of coordinates implies $f(0) = 0$ and $\nabla f(0) = 0$. Since $f \in C^{1,\alpha}$ and the boundary of Ω is compact we can estimate

$$\sup_{x' \in D} |\nabla f(x')| \,\leq\, C\, l^{\alpha}\,, \tag{3.9}$$

with a constant $C > 0$ depending only on Ω, in particular independent of f.

Now we introduce new local coordinates given by a diffeomorphism $\varphi : D \times \mathbb{R} \to \mathbb{R}^d$. We set $y_j = \varphi_j(x) = x_j$ for $j = 1, \ldots, d-1$ and $y_d = \varphi_d(x) = x_d - f(x')$. Note that the determinant of the Jacobian matrix of φ equals 1 and that the inverse of φ is defined on $\operatorname{ran}\varphi = D \times \mathbb{R}$. There we define $\tilde{\phi} = \phi \circ \varphi^{-1}$ and extend it by zero to $\mathbb{R}^d$, such that $\tilde{\phi} \in C_0^1(\mathbb{R}^d)$ and $\|\nabla\tilde{\phi}\|_\infty \leq Cl^{-1}$ holds.

LEMMA 3.6. *For $0 < l \leq c$ and any $h > 0$ the estimate*

$$\left|\operatorname{Tr}(\phi H_\Omega \phi)_- - \operatorname{Tr}(\tilde{\phi} H^+ \tilde{\phi})_-\right| \,\leq\, C\, l^{d+\alpha}\, h^{-d} \tag{3.10}$$

holds. Moreover, we have

$$\int_\Omega \phi^2(x)\, dx \,=\, \int_{\mathbb{R}^d_+} \tilde{\phi}^2(y)\, dy \tag{3.11}$$

and

$$\left|\int_{\partial\Omega} \phi^2(x)\, d\sigma(x) - \int_{\mathbb{R}^{d-1}} \tilde{\phi}^2(y', 0)\, dy'\right| \,\leq\, C\, l^{d-1+2\alpha}\,. \tag{3.12}$$

PROOF. The definition of $\tilde{\phi}$ and the fact $\det J\varphi = 1$ immediately give (3.11). Using (3.9) we estimate

$$\int_{\partial\Omega} \phi^2(x) d\sigma(x) = \int_{\mathbb{R}^{d-1}} \tilde{\phi}^2(y', 0)\sqrt{1 + |\nabla f|^2} dy' \leq \int_{\mathbb{R}^{d-1}} \tilde{\phi}^2(y', 0) dy' + Cl^{d-1+2\alpha}$$

from which (3.12) follows.

To prove (3.10) fix $v \in H_0^1(\Omega)$ with support in $\overline{B}$. For $y \in \operatorname{ran}\varphi$ put $\tilde{v}(y) = v \circ \varphi^{-1}(y)$ and extend $\tilde{v}$ by zero to $\mathbb{R}^d$. Note that $\tilde{v}$ belongs to $H_0^1(\mathbb{R}^d_+)$.

An explicit calculation shows

$$\left|(\tilde{v}, -\Delta_{\mathbb{R}^d_+}\tilde{v}) - (v, -\Delta_\Omega v)\right| \,\leq C\, l^{\alpha}\, (\tilde{v}, -\Delta_{\mathbb{R}^d_+}\tilde{v})\,.$$

Hence, we find

$$\operatorname{Tr}(\phi H_\Omega \phi)_- \,\leq\, \operatorname{Tr}(\tilde{\phi}(-(1 - Cl^{\alpha})h^2\Delta_{\mathbb{R}^d_+} - 1)\tilde{\phi})_-\,.$$

Set $\varepsilon = 2Cl^{\alpha}$ and assume l to be sufficiently small, so that $0 < \varepsilon \leq 1/2$ holds. Then

$$\begin{aligned}
\operatorname{Tr}(\phi H_\Omega \phi)_- \,&\leq\, \operatorname{Tr}(\tilde{\phi}(-(1 - Cl^{\alpha})h^2\Delta_{\mathbb{R}^d_+} - 1)\tilde{\phi})_- \\
&\leq\, \operatorname{Tr}(\tilde{\phi}(-h^2\Delta_{\mathbb{R}^d_+} - 1)\tilde{\phi})_- + \operatorname{Tr}(\tilde{\phi}(-(\varepsilon - Cl^{\alpha})h^2\Delta_{\mathbb{R}^d_+} - \varepsilon)\tilde{\phi})_- \\
&\leq\, \operatorname{Tr}(\tilde{\phi} H^+ \tilde{\phi})_- + \varepsilon \operatorname{Tr}(\tilde{\phi}(-(h^2/2)\Delta_{\mathbb{R}^d_+} - 1)\tilde{\phi})_-\,.
\end{aligned}$$

By Lemma 3.5 we have $\mathrm{Tr}(\tilde\phi(-(h^2/2)\Delta_{\mathbb{R}^d_+}-1)\tilde\phi)_- \le Cl^d h^{-d}$ and we obtain

$$\mathrm{Tr}(\phi H_\Omega \phi)_- \;\le\; \mathrm{Tr}(\tilde\phi H^+ \tilde\phi)_- + C\, l^{d+\alpha}\, h^{-d}\,.$$

Finally, by interchanging the roles of H_Ω and H^+, we get an analogous upper bound and the proof of Lemma 3.6 is complete. □

3.2.3. Local asymptotics in half-space. In view of Lemma 3.6 we can reduce Proposition 3.4 to a statement concerning the operator H^+, given on the half-space $\mathbb{R}^d_+$. Indeed, to prove Proposition 3.4, it suffices to establish the following result.

LEMMA 3.7. *Assume that $\phi \in C_0^1(\mathbb{R}^d)$ is supported in a ball of radius $l>0$ and that (3.5) is satisfied. Then for $h>0$ the estimate*

$$\left|\mathrm{Tr}\left(\phi H^+\phi\right)_- - \frac{L_d}{h^d}\int_{\mathbb{R}^d_+}\phi^2(x)dx + \frac{L_{d-1}}{4h^{d-1}}\int_{\mathbb{R}^{d-1}}\phi^2(x',0)dx'\right| \;\le\; Cl^{d-2}h^{-d+2}$$

holds with a constant depending only on the constant in (3.5).

PROOF. On $\mathbb{R}^d_+$ we can rescale ϕ and assume $l=1$. In a first step we prove the estimate

$$\left|\mathrm{Tr}\left(\phi H^+\phi\right)_- - \frac{L_d}{h^d}\int_{\mathbb{R}^d_+}\phi^2(x)dx + \int_{\mathbb{R}^d_+}\phi^2(x)\int_{\mathbb{R}^d}\cos(2\xi_d x_d h^{-1})(|\xi|^2-1)_-\frac{d\xi\,dx}{(2\pi h)^d}\right|$$
$$\le\; C\,h^{-d+2}\,. \tag{3.13}$$

To derive a lower bound we use the inequality $\mathrm{Tr}\,(\phi H^+\phi)_- \le \mathrm{Tr}\,(\phi (H^+)_-\phi)$ and diagonalize the operator $(H^+)_-$, applying the Fourier-transform in the x'-coordinates and the sine-transform in the x_d-coordinate. This yields

$$\mathrm{Tr}(\phi H^+\phi)_- \;\le\; \int_{\mathbb{R}^d_+}\phi^2(x)\int_{\mathbb{R}^d} 2\sin^2(\xi_d x_d h^{-1})\left(|\xi|^2-1\right)_-\frac{d\xi\,dx}{(2\pi h)^d}$$

and the lower bound follows from the identity

$$2\sin^2(\xi_d x_d h^{-1}) \;=\; 1-\cos(2\xi_d x_d h^{-1})\,. \tag{3.14}$$

To prove the upper bound, define the operator $\gamma = \chi(H^+)^0_-\chi$ with kernel

$$\gamma(x,y) \;=\; \frac{2}{(2\pi h)^d}\,\chi(x)\int_{|\xi|<1} e^{i\xi'(x'-y')/h}\,\sin(\xi_d x_d h^{-1})\sin(\xi_d y_d h^{-1})d\xi\,\chi(y)\,,$$

where χ denotes the characteristic function of an open ball containing the support of ϕ. Thus, γ is a trace-class operator, satisfying $0\le\gamma\le 1$ and by the variational principle it follows that

$$\begin{aligned}
&\mathrm{Tr}(\phi H^+\phi)_-\\
&\ge\; -\mathrm{Tr}(\gamma\phi H^+\phi)\\
&=\; -2\int_{|\xi|<1}\left(h^2\|\nabla e^{i\xi'\cdot/h}\sin(\xi_d\cdot h^{-1})\phi\|^2_{L^2(\mathbb{R}^d_+)} - \left\|\sin(\xi_d\cdot h^{-1})\phi\right\|^2_{L^2(\mathbb{R}^d_+)}\right)\frac{d\xi}{(2\pi h)^d}\\
&\ge\; \int_{\mathbb{R}^d}\left(|\xi|^2-1\right)_-\int_{\mathbb{R}^d_+}\phi^2(x)\,2\sin^2(\xi_d x_d h^{-1})\frac{dx\,d\xi}{(2\pi h)^d} - Ch^{-d+2}\,.
\end{aligned}$$

In view of (3.14) this gives an upper bound and we established (3.13).

We proceed to analyzing the term in (3.13) which contains the cosine. We substitute $x_d = th$ and write

$$\int_{\mathbb{R}^d_+} \phi^2(x) \int_{\mathbb{R}^d} \cos(2\xi_d x_d h^{-1}) \left(|\xi|^2 - 1\right)_- \frac{d\xi\, dx}{(2\pi h)^d}$$
$$= \frac{1}{(2\pi)^d} \int_0^\infty \int_{\mathbb{R}^{d-1}} \phi^2(x', th) dx' \int_{\mathbb{R}^d} \cos(2\xi_d t) \left(|\xi|^2 - 1\right)_- d\xi\, dt\, h^{-d+1}\,. \quad (3.15)$$

Note that

$$\frac{1}{(2\pi)^d} \int_0^\infty \int_{\mathbb{R}^d} \cos(2\xi_d t) \left(|\xi|^2 - 1\right)_- d\xi\, dt \;=\; \frac{1}{4} L_{d-1}\,. \quad (3.16)$$

Moreover, in [**AS72**, (9.1.20)] it is shown that

$$\int_{\mathbb{R}^d} \cos(2\xi_d t) \left(|\xi|^2 - 1\right)_- d\xi = C \int_0^1 \cos(2\xi_d t)(1 - \xi_d^2)^{(d+1)/2} d\xi_d = C \frac{J_{d/2+1}(2t)}{t^{d/2+1}}\,,$$

where $J_{d/2+1}$ denotes the Bessel function of the first kind. We remark that $|J_{d/2+1}(2t)|$ is proportional to $t^{d/2+1}$ as $t \to 0+$ and bounded by a constant times $t^{-1/2}$ as $t \to \infty$, see [**AS72**, (9.1.7) and (9.2.1)]. It follows that

$$\int_0^\infty t \left| \int_{\mathbb{R}^d} \cos(2\xi_d t) \left(|\xi|^2 - 1\right)_- d\xi \right| dt \;\le\; C \int_0^\infty t^{-d/2} |J_{d/2+1}(2t)|\, dt \;\le\; C\,. \quad (3.17)$$

In view of (3.15), (3.16) and (3.17) we find

$$\left| \int_{\mathbb{R}^d_+} \phi^2(x) \int_{\mathbb{R}^d} \cos(2\xi_d x_d h^{-1}) \left(|\xi|^2 - 1\right)_- \frac{d\xi\, dx}{(2\pi h)^d} - \frac{L_{d-1}}{4h^{d-1}} \int_{\mathbb{R}^{d-1}} \phi^2(x', 0) dx' \right|$$
$$\le C h^{-d+2}\,.$$

Inserting this into (3.13) proves Lemma 3.7. □

Proposition 3.4 is a consequence of Lemma 3.6 and Lemma 3.7.

3.3. Localization

Here we construct the family of localization functions $(\phi_u)_{u\in\mathbb{R}^d}$ and prove Proposition 3.2. The key idea is to choose the localization depending on the distance to the complement of Ω, see [**Hör85**, Theorem 17.1.3] and [**SS03**].

Fix a real-valued function $\phi \in C_0^\infty(\mathbb{R}^d)$ with support in $\{|x| < 1\}$ and $\|\phi\|_2 = 1$. For $u, x \in \mathbb{R}^d$ let $J(x,u)$ be the Jacobian of the map $u \mapsto (x-u)/l(u)$. We define

$$\phi_u(x) \;=\; \phi\left(\frac{x-u}{l(u)}\right) \sqrt{J(x,u)}\, l(u)^{d/2}\,,$$

such that ϕ_u is supported in $\{x : |x-u| < l(u)\}$. According to [**SS03**], the functions ϕ_u satisfy (3.3) and (3.4) for all $u \in \mathbb{R}^d$.

To prove the upper bound in Proposition 3.2, put

$$\gamma \;=\; \int_{\mathbb{R}^d} \phi_u \left(\phi_u H_\Omega \phi_u\right)_-^0 \phi_u\, l(u)^{-d}\, du\,.$$

Obviously, $\gamma \geq 0$ holds and in view of (3.4) also $\gamma \leq 1$. The range of γ belongs to $H_0^1(\Omega)$ and by the variational principle it follows that

$$-\mathrm{Tr}(H_\Omega)_- \;\leq\; \mathrm{Tr}\,\gamma H_\Omega \;=\; -\int_{\mathbb{R}^d} \mathrm{Tr}\,(\phi_u H_\Omega \phi_u)_-\, l(u)^{-d}\, du\,.$$

To prove the lower bound we make use of the IMS-formula

$$\frac{1}{2}\left(f, \phi^2(-\Delta)f\right) + \frac{1}{2}\left(f, -\Delta\phi^2 f\right) \;=\; (f, \phi(-\Delta)\phi f) - \left(f, f(\nabla\phi)^2\right)\,,$$

valid for $\phi \in C_0^\infty(\mathbb{R}^d)$ and $f \in H_0^1(\Omega)$. Combining this identity with (3.4) yields

$$(f, -\Delta f) \;=\; \int_{\mathbb{R}^d} \left((f, \phi_u(-\Delta)\phi_u f) - \left(f, (\nabla\phi_u)^2 f\right)\right) l(u)^{-d}\, du\,. \tag{3.18}$$

Using (3.3) and (3.4) one can show [**SS03**]

$$\int_{\mathbb{R}^d} (\nabla\phi_u)^2(x) l(u)^{-d}\, du \;\leq\; C\int_{\mathbb{R}^d} \phi_u^2(x)\, l(u)^{-d-2}\, du\,.$$

We insert this into (3.18) and deduce

$$\mathrm{Tr}\,(H_\Omega)_- \;\leq\; \int_{\Omega^*} \mathrm{Tr}\left(\phi_u\left(-h^2\Delta - 1 - Ch^2 l(u)^{-2}\right)\phi_u\right)_-\, l(u)^{-d}\, du\,,$$

where $\Omega^* = \{u \in \mathbb{R}^d \,:\, \mathrm{supp}\phi_u \cap \Omega \neq \emptyset\}$. To estimate the localization error we use Lemma 3.5. For any $u \in \mathbb{R}$, let ρ_u be another parameter $0 < \rho_u \leq 1/2$ and estimate

$$\begin{aligned}
&\mathrm{Tr}\left(\phi_u\left(-h^2\Delta - 1 - Ch^2 l(u)^{-2}\right)\phi_u\right)_- \\
&\quad\leq\; \mathrm{Tr}\left(\phi_u(-h^2\Delta - 1)\phi_u\right)_- + C\,\mathrm{Tr}\left(\phi_u\left(-\rho_u h^2\Delta - \rho_u - h^2 l(u)^{-2}\right)\phi_u\right)_- \\
&\quad\leq\; \mathrm{Tr}\,(\phi_u H_\Omega \phi_u)_- + C\, l(u)^d (\rho_u h^2)^{-d/2}\left(\rho_u + h^2 l(u)^{-2}\right)^{1+d/2}\,.
\end{aligned}$$

With ρ_u proportional to $h^2 l(u)^{-2}$ we find

$$\mathrm{Tr}\,(H_\Omega)_- \;\leq\; \int_{\Omega^*} \mathrm{Tr}\,(\phi_u H_\Omega \phi_u)_-\, l(u)^{-d} du + Ch^{-d+2}\int_{\Omega^*} l(u)^{-2} du\,.$$

In view of (3.8) the last integral is bounded by a constant times l_0^{-1} and the proof of Proposition 3.2 is complete.

CHAPTER 4

Refined semiclassical asymptotics for fractional powers of the Laplace operator

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Abstract. We consider the fractional Laplacian on a domain and investigate the asymptotic behavior of its eigenvalues. Extending methods from semi-classical analysis we are able to prove a two-term formula for the sum of eigenvalues with the leading (Weyl) term given by the volume and the subleading term by the surface area. Our result is valid under very weak assumptions on the regularity of the boundary.

4.1. Introduction and main result

4.1.1. Introduction. In this paper we study the asymptotic behavior of eigenvalues for fractional powers of the Laplacian. The operator $(-\Delta)^s$ with $0 < s < 1$ appears in numerous fields of mathematical physics, mathematical biology and mathematical finance and has attracted a lot of attention recently. The key difference between this operator and the usual Laplacian is the non-locality of $(-\Delta)^s$, which allows one to model long-range interactions in applications and leads to challenging mathematical problems.

From a probabilistic point of view, the fractional Laplacian of order s on a domain $\Omega \subset \mathbb{R}^d$ can be defined as the generator of the $2s$-stable process killed upon exiting Ω. A more operator theoretic definition, which we employ here, is in terms of the quadratic form

$$C_{s,d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}} dx\, dy = \int_{\mathbb{R}^d} |p|^{2s} |\hat{u}(p)|^2\, dp\,, \tag{4.1}$$

restricted to functions $u \in H^s(\mathbb{R}^d)$ which satisfy $u \equiv 0$ in $\mathbb{R}^d \setminus \overline{\Omega}$. Here $H^s(\mathbb{R}^d)$ is the Sobolev space of order s, $\hat{u}(p) = (2\pi)^{-d/2} \int e^{-ip\cdot x} u(x)\, dx$ is the Fourier transform of u and $C_{s,d}$ is an explicit constant given in (4.5). The identity in (4.1) is an easy consequence of Plancherel's theorem.

For bounded domains Ω the spectrum of the fractional Laplacian is discrete and we denote its eigenvalues (in increasing order, repeated according to multiplicities) by $\lambda_n^{(s)}$. Our main result in this paper is a two-term asymptotic expansion of the sum of these eigenvalues,

$$\frac{1}{N} \sum_{n=1}^{N} \lambda_n^{(s)} = C_{d,s}^{(1)} |\Omega|^{-2s/d}\ N^{2s/d} + C_{d,s}^{(2)} |\partial\Omega| |\Omega|^{-(d-1+2s)/d}\ N^{(2s-1)/d}\ (1+o(1)) \tag{4.2}$$

as $N \to \infty$. Here $|\Omega|$ and $|\partial\Omega|$ denote the d-dimensional measure of Ω and the $(d-1)$-dimensional surface measure of $\partial\Omega$, respectively, and $C^{(1)}_{d,s}$ and $C^{(2)}_{d,s}$ are positive, universal constants, depending only on d and s, for which we shall obtain explicit expressions. Our result is valid for non-smooth domains, requiring only that $\partial\Omega \in C^{1,\alpha}$ for some (arbitrarily small) $\alpha > 0$. It is remarkable that, despite the fact that we are dealing with a non-local operator, both coefficients in (4.2) have a local form, depending only on Ω and $\partial\Omega$, just like in the case of the Laplacian. This will become clearer from the reformulation given in Theorem 4.1 below.

In order to avoid confusion, we emphasize that the fractional Laplacian of order s on a domain Ω is different from the Dirichlet Laplacian on Ω raised to the s-th power. For the Dirichlet Laplacian, and hence for its fractional powers, asymptotics analogous to (4.2) are well-known. One of our results is that, while the first terms in (4.2) coincide for both operators, the second terms do *not*. This means, in particular, that our result cannot be obtained from the study of the (local) Dirichlet Laplacian, and that our analysis needs to take into account the non-locality inherent in (4.2). For further results about the relation between the fractional Laplacian on a domain and the fractional power of the Dirichlet Laplacian we refer to [**CS05**]; see also Section 4.6 below.

The one-term asymptotics $\lambda^{(s)}_N = \frac{d+2s}{d}\, C^{(1)}_{d,s} |\Omega|^{-2s/d}\, N^{2s/d}(1+o(1))$, which is a fractional version of Weyl's law, is a classical result of Blumenthal and Getoor [**BG59**]. More recently, Bañuelos and Kulczycki [**BK08**] and Bañuelos, Kulczycki and Siudeja [**BKS09**] have shown a two-term asymptotic formula for $\sum_{n=1}^\infty \exp(-t\lambda^{(s)}_n)$ as $t \to 0$. Note that $\sum_{n=1}^\infty \exp(-t\lambda^{(s)}_n)$ and $\frac{1}{N}\sum_{n=1}^N \lambda^{(s)}_n$ correspond to the Abel and Cesàro summation of the sequence $\lambda^{(s)}_n$, respectively. As is well-known, asymptotics of Cesàro means imply asymptotics of Abel means, but not vice versa. Hence for $C^{1,\alpha}$ domains we recover and improve upon the result of [**BK08, BKS09**].

This is, actually, a significant improvement since our asymptotics are no longer derived for the infinitely smooth function e^{-tE} of the fractional Laplacian, but, as we shall see shortly, for the Lipschitz function $(\Lambda - E)_+$. Moreover, since we are no longer able to apply the probabilistic machinery available for the partition function, we have to find new and more robust tools. Our methods also work for the ordinary Dirichlet Laplacian on a bounded domain, and in [**FG11b**] we use the techniques developed here to give an elementary and short proof of two-term asymptotics in that case.

Another point in which we go beyond [**BK08, BKS09**] is that we give an expression for the constant $C^{(2)}_{d,s}$ in (4.2) in terms of a model operator on a *half-line* instead of a model operator on a *half-space*. In this way our expression is similar to familiar two-term formulas in semi-classical analysis; see, for instance, [**SV97**]. This is possible due to some recent beautiful results of Kwaśnicki [**Kwa10b**] about a general class of half-line operators.

We find it convenient to prove (4.2) in an equivalent form, namely

$$\sum_{n=1}^\infty \left(\Lambda - \lambda^{(s)}_n\right)_+ = L^{(1)}_{s,d}\, |\Omega|\, \Lambda^{1+d/2s} - L^{(2)}_{s,d}\, |\partial\Omega|\, \Lambda^{1+(d-1)/2s}(1+o(1)) \quad \text{as } \Lambda \to \infty\,. \tag{4.3}$$

Here $x_+ := \max\{x, 0\}$ denotes the positive part of a number x. (The fact that (4.2) and (4.3) are equivalent is well-known to experts in the field, but we include a short proof in Section 4.7 for the sake of completeness, see Lemma 4.20.) Note also that (4.3) can be rewritten as

$$\sum_{n=1}^{\infty} \left(1 - h^{2s}\lambda_n^{(s)}\right)_+ = L_{s,d}^{(1)} |\Omega|\, h^{-d} - L_{s,d}^{(2)} |\partial\Omega|\, h^{-d+1}(1 + o(1)) \quad \text{as } h \to 0+\,, \tag{4.4}$$

and this is the form in which we shall state and prove our main theorem. The small parameter h has the interpretation of Planck's constant and (4.4) emphasizes the semi-classical nature of the problem.

Our approach extends the multiscale analysis to the fractional setting. By this we mean that we localize simultaneously on different length scales according to the distance from the boundary. Of course, a main difficulty when dealing with our non-local operator comes from the treatment of the localization error. At this point we have to improve upon previous results from [**LY88, SSS10**]. Another major impasse, as compared to the local case, is the analysis of a one-dimensional model operator for which an (almost) explicit diagonalization is far from trivial. This is where Kwaśnicki's work [**Kwa10b**] enters. It requires, however, still substantial work to bring these results into a form which is useful for us. We will explain the strategy of our proof in more detail in Subsection 4.1.3 after a precise statement of our main result.

Throughout this paper we assume that the dimension $d \geq 2$. In the one dimensional case (the fractional Laplacian on an interval) considerably stronger results are known [**KKMS10, Kwa10a**]. The powerful methods developed there are, however, intrinsically one-dimensional and seem of little help in the multi-dimensional case. The question raised in [**BKS09**] of whether an analogue of Ivrii's two-term asymptotics [**Ivr80a**] holds for $\lambda_n^{(s)}$ in $d \geq 2$ without Abel or Cesàro averaging remains a challenging open problem.

4.1.2. Main Result. Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded open set. For $h > 0$ and $0 < s < 1$ let

$$H_\Omega = (-h^2\Delta)^s - 1$$

be the self-adjoint operator in $L^2(\Omega)$ generated by the quadratic form

$$(u, H_\Omega u) \;=\; \int_{\mathbb{R}^d} \left(|hp|^{2s} - 1\right) |\hat{u}(p)|^2\, dp$$

with form domain

$$\mathcal{H}^s(\Omega) \;=\; \left\{u \in H^s(\mathbb{R}^d) \,:\, u \equiv 0 \text{ on } \mathbb{R}^d \setminus \overline{\Omega}\right\}\,.$$

For $0 < s < 1$ we have the representation

$$(u, H_\Omega u) \;=\; C_{s,d}\, h^{2s} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x - y|^{d+2s}} dx\, dy - \int_\Omega |u(x)|^2\, dx$$

with constant

$$C_{s,d} \;=\; 2^{2s-1}\pi^{-d/2}\frac{\Gamma(d/2 + s)}{|\Gamma(-s)|} \;>\; 0\,. \tag{4.5}$$

Our main results hold without any global geometric conditions on Ω. We only require weak smoothness conditions on the boundary - namely that the boundary belongs to the class $C^{1,\alpha}$ for some $\alpha > 0$. That is, the local charts of $\partial\Omega$ are differentiable and the derivatives are Hölder continuous with exponent α.

THEOREM 4.1. *Let $0 < s < 1$ and assume that the boundary of Ω satisfies $\partial\Omega \in C^{1,\alpha}$ with some $0 < \alpha \leq 1$. Then*

$$\mathrm{Tr}(H_\Omega)_- \;=\; L^{(1)}_{s,d}\,|\Omega|\,h^{-d} - L^{(2)}_{s,d}\,|\partial\Omega|\,h^{-d+1} + R_h \tag{4.6}$$

with $R_h = o(h^{-d+1})$ as $h \to 0+$. Here

$$L^{(1)}_{s,d} \;=\; \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \big(|p|^{2s} - 1\big)_-\, dp \tag{4.7}$$

and the positive constant $L^{(2)}_{s,d}$ is given in (4.28).

More precisely, we have the lower bound $R_h \geq -Ch^{-d+1+\epsilon_-}$ for any

$$0 < \epsilon_- \;<\; \begin{cases} \frac{\alpha}{\alpha+2} & \text{if } 1/2 \leq s < 1\,, \\ \frac{2s\alpha}{\alpha+1+2s} & \text{if } 0 < s < 1/2\,, \end{cases}$$

and the upper bound $R_h \leq Ch^{-d+1+\epsilon_+}$ for any

$$0 < \epsilon_+ < \frac{\alpha}{\alpha+2} \qquad \text{if } 1 - d/4 \leq s < 1\,,$$

$$0 < \epsilon_+ \leq \frac{\alpha(2s-1+d/2)}{\alpha+2s+d/2} \qquad \text{if } 0 < s < 1 - d/4\,.$$

We do not claim that our remainder estimates are sharp. They show, however, that our methods are rather explicit and they correctly reflect the intuitive fact that the estimate worsens as the boundary gets rougher. We also mention that for not too small s we (almost) get the same remainder estimate $h^{-d+1+\alpha/(\alpha+2)}$ that our method yields in the local case $s = 1$ [**FG11b**].

In Section 4.6 we will derive several representations of the constant $L^{(2)}_{s,d}$ in (4.6). One of these, which emphasizes the semi-classical nature of the problem, leads to a rewriting of (4.6) as

$$\mathrm{Tr}(H_\Omega)_- = \iint_{T\Omega} \big(|p|^{2s} - 1\big)_- \frac{dpdx}{(2\pi h)^d} - \iint_{T\partial\Omega} \zeta(|p'|^{-2s}) \frac{dp' d\sigma(x)}{(2\pi h)^{d-1}} + R_h\,, \tag{4.8}$$

where $T\Omega = \Omega \times \mathbb{R}^d$ and $T\partial\Omega = \partial\Omega \times \mathbb{R}^{d-1}$ are the cotangent bundles over Ω and $\partial\Omega$, respectively, and where $d\sigma$ is the surface element of $\partial\Omega$. Here ζ is a universal (i.e., depending on s, but independent of Ω or d) function, which has the interpretation of an energy shift (the integral of a spectral shift). It is given in terms of a one-dimensional model operator A^+ on the half-line $\mathbb{R}_+$ and its analogue A on the whole line (see Section 4.3) by

$$\zeta(\mu) \;=\; \mu^{-1} \int_0^\infty \big(a(t,t,\mu) - a^+(t,t,\mu)\big)\, dt\,, \quad \mu > 0\,,$$

where $a(t,u,\mu)$ and $a^+(t,u,\mu)$ denote the integral kernels of $(A-\mu)_-$ and $(A^+-\mu)_-$, respectively. Another representation, derived in Remark 4.6.1, shows that our result is consistent with the result of [**BK08, BKS09**].

In Section 4.6 we also prove that

$$L^{(2)}_{s,d} > 0\,.$$

Moreover, we compare this constant with the one obtained from the corresponding fractional power of the Dirichlet Laplacian.

PROPOSITION 4.2. *Let $0 < s < 1$ and assume that the boundary of Ω satisfies $\partial\Omega \in C^{1,\alpha}$ with some $0 < \alpha \leq 1$. Let $-\Delta_\Omega$ be the Dirichlet Laplacian on Ω. Then*

$$\operatorname{Tr}\left(\left(-h^2\Delta_\Omega\right)^s - 1\right)_- \;=\; L^{(1)}_{s,d}\,|\Omega|\,h^{-d} - \tilde{L}^{(2)}_{s,d}\,|\partial\Omega|\,h^{-d+1} + R_h \tag{4.9}$$

with $R_h = o(h^{-d+1})$ as $h \to 0+$. Here $L^{(1)}_{s,d}$ is the same as in (4.7) and $\tilde{L}^{(2)}_{s,d}$ satisfies

$$L^{(2)}_{s,d} < \tilde{L}^{(2)}_{s,d}\,. \tag{4.10}$$

In other words, the operators H_Ω and $(-h^2\Delta_\Omega)^s - 1$ differ semi-classically to first subleading order.

4.1.3. Strategy of the proof. The proof of Theorem 4.1 is divided into three main steps: First, we localize the operator H_Ω into balls, whose size varies depending on the distance to the complement of Ω. Then we can analyze separately the semiclassical limit in the bulk and at the boundary.

The key idea is to choose the localization depending on the distance to the complement of Ω, see [**Hör85**, Theorem 17.1.3] and [**SS03**]. Let $d(u) = \inf\{|x-u| \,:\, x \notin \Omega\}$ denote the distance of $u \in \mathbb{R}^d$ to the complement of Ω. We set

$$l(u) \;=\; \frac{1}{2}\left(1 + \left(d(u)^2 + l_0^2\right)^{-1/2}\right)^{-1}\,, \tag{4.11}$$

where $0 < l_0 \leq 1/2$ is a small parameter depending only on h. Indeed, we will finally choose l_0 proportional to h^β with suitable $0 < \beta < 1$.

In Section 4.5 we construct real-valued functions $\phi_u \in C^\infty_0(\mathbb{R}^d)$ with support in the ball $B_u = \{x \in \mathbb{R}^d \,:\, |x-u| < l(u)\}$. For all $u \in \mathbb{R}^d$ these functions satisfy

$$\|\phi_u\|_\infty \;\leq\; C\,, \qquad \|\nabla\phi_u\|_\infty \leq C\,l(u)^{-1} \tag{4.12}$$

and for all $x \in \mathbb{R}^d$

$$\int_{\mathbb{R}^d} \phi_u^2(x)\,l(u)^{-d}\,du \;=\; 1\,. \tag{4.13}$$

Here and in the following the letter C denotes various positive constants that are independent of u, l_0 and h.

PROPOSITION 4.3. *There is a constant $C > 0$ depending only on s and d such that for all $0 < l_0 \leq 1/2$ and all $0 < h \leq C^{-1}l_0$ the estimates*

$$0 \;\leq\; \operatorname{Tr}(H_\Omega)_- - \int_{\mathbb{R}^d} \operatorname{Tr}\left(\phi_u H_\Omega \phi_u\right)_-\, l(u)^{-d}\,du \;\leq\; C\,h^{-d+2}\,l_0^{-1} R_{\mathrm{loc}}(h,l_0)$$

hold with a remainder

$$R_{\mathrm{loc}}(h,l_0) \;=\; \begin{cases} 1 & \text{if } 1 - d/4 < s < 1 \\ |\ln(l_0/h)|^{1/2} & \text{if } 0 < s = 1 - d/4 \\ (l_0/h)^{2-2s-d/2} & \text{if } 0 < s < 1 - d/4 \end{cases}\,.$$

In view of this result, one can analyze the local asymptotics, i.e., the asymptotic behavior of $\mathrm{Tr}(\phi_u H_\Omega \phi_u)_-$, separately on different parts of Ω. First, we consider the bulk, where the influence of the boundary is not felt.

PROPOSITION 4.4. *Assume that $\phi \in C_0^1(\Omega)$ is supported in a ball of radius $l > 0$ and that*

$$\|\nabla \phi\|_\infty \leq C l^{-1} . \tag{4.14}$$

Then for all $h > 0$ the estimates

$$-C l^{d-2} h^{-d+2} \leq \mathrm{Tr}\,(\phi H_\Omega \phi)_- - L^{(1)}_{s,d} \int_\Omega \phi^2(x)\, dx\, h^{-d} \leq 0$$

hold with a constant depending only on the constant in (4.14).

Close to the boundary of Ω, more precisely, if the support of ϕ intersects the boundary, a boundary term of the order h^{-d+1} appears.

PROPOSITION 4.5. *Assume that $\phi \in C_0^1(\mathbb{R}^d)$ is supported in a ball of radius $0 < l \leq 1$ intersecting the boundary of Ω and assume that (4.14) is satisfied. Then for all $h > 0$ the estimates*

$$-\tilde{R}_{\mathrm{bd}}(l,h) \leq \mathrm{Tr}\,(\phi H_\Omega \phi)_- - L^{(1)}_{s,d} \int_\Omega \phi^2(x) dx h^{-d} + L^{(2)}_{s,d} \int_{\partial\Omega} \phi^2(x) d\sigma(x) h^{-d+1} \leq R_{\mathrm{bd}}(l,h)$$

hold. Here $d\sigma$ denotes the $(d-1)$-dimensional volume element of $\partial\Omega$ and the remainder terms satisfy for any $0 < \delta_1 < 1$ and $0 < \delta_2 < \min\{1, 2s\}$

$$\begin{aligned} R_{\mathrm{bd}}(l,h) &\leq C_{\delta_1} \left(\frac{l^{d-1-\delta_1}}{h^{d-1-\delta_1}} + \frac{l^{d+\alpha}}{h^d} \right) , \\ \tilde{R}_{\mathrm{bd}}(l,h) &\leq C_{\delta_1,\delta_2} \left(\frac{l^{d-1-\delta_1}}{h^{d-1-\delta_1}} + \frac{l^{d-1-\delta_2}}{h^{d-1-\delta_2}} + \frac{l^{2\alpha+d-1}}{h^{d-1}} + \frac{l^{d+\alpha}}{h^d} \right) , \end{aligned}$$

with constants depending on δ_1, δ_2, Ω, $\|\phi\|_\infty$ and the constant in (4.14).

Based on these propositions we can complete the proof of Theorem 4.1.

PROOF OF THEOREM 4.1. In order to apply Proposition 4.5 to the operators $\phi_u H_\Omega \phi_u$, we need to estimate $l(u)$ uniformly. Let

$$U(\Omega) = \{u \in \mathbb{R}^d \,:\, B_u \cap \partial\Omega \neq \emptyset\}$$

be a small neighborhood of the boundary. For $u \in U(\Omega)$ we have $d(u) \leq l(u)$, which by the definition of $l(u)$ implies

$$l(u) \leq l_0/\sqrt{3} . \tag{4.15}$$

In view of (4.12) and (4.15) we can apply Proposition 4.4 and Proposition 4.5 to all functions ϕ_u, $u \in \mathbb{R}^d$, if l_0 is sufficiently small. Combining these results with

Proposition 4.3 we get

$$
\begin{aligned}
&- C \int_{\Omega\setminus U(\Omega)} l(u)^{-2} du\, h^{-d+2} - \int_{U(\Omega)} \tilde{R}_{\mathrm{bd}}(l(u),h)\, l(u)^{-d} du \\
&\leq \operatorname{Tr}(H_\Omega)_- - L^{(1)}_{s,d} \int_{\mathbb{R}^d}\int_\Omega \phi_u^2(x) dx\, \frac{du}{l(u)^d}\, h^{-d} + L^{(2)}_{s,d} \int_{\mathbb{R}^d}\int_{\partial\Omega} \phi_u^2(x) d\sigma(x)\, \frac{du}{l(u)^d}\, h^{-d+1} \\
&\leq \int_{U(\Omega)} R_{\mathrm{bd}}(l(u),h) l(u)^{-d} du + C h^{-d+2} l_0^{-1} R_{\mathrm{loc}}(l_0,h)\,.
\end{aligned}
$$

Now we change the order of integration and in view of (4.13) we obtain

$$
\begin{aligned}
&- C \int_{\Omega\setminus U(\Omega)} l(u)^{-2} du\, h^{-d+2} - \int_{U(\Omega)} \tilde{R}_{\mathrm{bd}}(l(u),h)\, l(u)^{-d} du \\
&\leq \operatorname{Tr}(H_\Omega)_- - L^{(1)}_{s,d} |\Omega|\, h^{-d} + L^{(2)}_{s,d} |\partial\Omega|\, h^{-d+1} \\
&\leq \int_{U(\Omega)} R_{\mathrm{bd}}(l(u),h) l(u)^{-d} du + C h^{-d+2} l_0^{-1} R_{\mathrm{loc}}(l_0,h)\,. \qquad (4.16)
\end{aligned}
$$

It remains to estimate the error terms.

By definition of $l(u)$ we have

$$
l(u) \geq \frac{1}{4} \min\left(d(u),1\right) \quad \text{and} \quad l(u) \geq \frac{l_0}{4} \qquad (4.17)
$$

for all $u \in \mathbb{R}^d$. For $u \in \Omega \setminus U(\Omega)$, we find $d(u) \geq l(u) \geq l_0/4$. Hence, we can estimate

$$
\int_{\Omega\setminus U(\Omega)} l(u)^{-2} du \leq C \left(1 + \int_{\{d(u)\geq l_0/4\}} d(u)^{-2} du\right) \leq C \left(1 + \int_{l_0/4}^\infty t^{-2} |\partial\Omega_t|\, dt\right),
$$

where $|\partial\Omega_t|$ denotes the surface area of the boundary of $\Omega_t = \{x \in \Omega : d(x) > t\}$. Using the fact that $|\partial\Omega_t|$ is uniformly bounded and that $|\partial\Omega_t| = 0$ for large t, we get

$$
\int_{\Omega\setminus U(\Omega)} l(u)^{-2} du \leq C l_0^{-1}\,. \qquad (4.18)
$$

For $u \in U(\Omega)$ the inequalities (4.15) and (4.17) show that $l(u)$ is proportional to l_0. Since $B_u \cap \partial\Omega \neq \emptyset$ we find $d(u) < l(u) \leq C l_0$ and

$$
\int_{U(\Omega)} l(u)^a du \leq C l_0^a \int_{\{d(u)\leq l_0\}} du \leq C l_0^{a+1}\,, \qquad (4.19)
$$

for any $a \in \mathbb{R}$.

We insert (4.18) and (4.19) into (4.16) and get (using the fact that $h \leq C^{-1} l_0$)

$$
\begin{aligned}
-C\left(l_0^{-\delta_2} h^{\delta_2} + l_0^{2\alpha} + l_0^{\alpha+1} h^{-1}\right) &\leq h^{d-1} \left(\operatorname{Tr}(H_\Omega)_- - L^{(1)}_{s,d} |\Omega|\, h^{-d} + L^{(2)}_{s,d} |\partial\Omega|\, h^{-d+1}\right) \\
&\leq C\left(l_0^{-\delta_1} h^{\delta_1} + l_0^{\alpha+1} h^{-1} + l_0^{-1} h\, R_{\mathrm{loc}}(l_0,h)\right). \qquad (4.20)
\end{aligned}
$$

In order to choose l_0 we need to distinguish several cases. For the lower bound we recall that $0 < \delta_2 < \min\{1, 2s\}$. The stated lower bound on R_h follows with l_0 proportional to h^β, where $\beta = (1+\delta_2)/(1+\alpha+\delta_2)$.

For the upper bound we have $0 < \delta_1 < 1$. If $1 - d/4 < s < 1$, we pick l_0 proportional to h^β, where $\beta = (1+\delta_1)/(1+\alpha+\delta_1)$. If $0 < s \leq 1 - d/4$, we pick h^β, where $\beta = (2s + d/2)/(\alpha + 2s + d/2)$. This completes the proof of Theorem 4.1. □

The remainder of the text is structured as follows. First we analyze the local asymptotics in the bulk and prove Proposition 4.4. This is done in Section 4.2. In Section 4.3 we consider the local asymptotics in the case where Ω is replaced by a half-space. We reduce the problem close to the boundary to the analysis of a one-dimensional model operator given on a half-line and give an analogue of Proposition 4.5 for a half-space. In Section 4.4 we show how Proposition 4.5 follows from the previous considerations by local straightening of the boundary. In Section 4.5, we perform the localization and, in particular, prove Proposition 4.3. In Section 4.8 we provide some technical results about the one-dimensional model operator introduced in Section 4.3.

Notation. We define the positive and negative parts of a real number x by $x_\pm = \max\{0, \pm x\}$. We use a similar notation for the heavy side function, namely, $x_\pm^0 = 1$ if $\pm x \geq 0$ and $x_\pm^0 = 0$ if $\pm x < 0$. For a self-adjoint operator X, the operators $X_\pm$ and $X_\pm^0$ are defined similarly via the spectral theorem.

4.2. Local asymptotics in the bulk

This section is a warm-up dealing with the spectral asymptotics in the boundaryless case. Although the estimates in this case are essentially known, we include a proof for the sake of completeness and in order to introduce the methods that will be important later on. We divide the proof of Proposition 4.4 into two subsections containing the lower and the upper bound, respectively. The operator

$$H_0 = (-h^2\Delta)^s - 1 \qquad \text{in } L^2(\mathbb{R}^d)\,,$$

defined with form domain $H^s(\mathbb{R}^d)$, will appear frequently.

4.2.1. Lower bound. The lower bound is given by a variant of the Berezin-Lieb-Li-Yau inequality, see [**Ber72b, Lie73, LY83**]. For later purposes we record this as

LEMMA 4.6. *For any $\phi \in L^2(\mathbb{R}^d)$ and $h > 0$*

$$\operatorname{Tr}(\phi H_\Omega \phi)_- \;\leq\; L^{(1)}_{s,d} \int_{\mathbb{R}^d} \phi^2(x)\, dx\, h^{-d}\,.$$

PROOF. We apply the variational principle for the sum of the eigenvalues

$$-\operatorname{Tr}(\phi H_\Omega \phi)_- \;=\; \inf_{0\leq\gamma\leq 1} \operatorname{Tr}(\gamma \phi H_\Omega \phi)\;,$$

where the infimum is taken over all trial density matrices, i.e., over all trace-class operators $0 \leq \gamma \leq 1$ with range belonging to the form domain of H_Ω. We apply this twice and find

$$\operatorname{Tr}(\phi H_\Omega \phi)_- \;\leq\; \operatorname{Tr}(\phi H_0 \phi)_- \;\leq\; \operatorname{Tr}\left(\phi\,(H_0)_-\,\phi\right)\;.$$

Applying the Fourier transform to diagonalize the operator $(H_0)_-$ yields the bound

$$\operatorname{Tr}\left(\phi\,(H_0)_-\,\phi\right) \;=\; \frac{1}{(2\pi h)^d} \iint \phi(x)^2 \left(|p|^{2s} - 1\right)_- dp\, dx \;=\; L^{(1)}_{s,d} \int \phi(x)^2\, dx\, h^{-d}\,,$$

as claimed. □

4.2.2. Upper bound. We now assume that ϕ satisfies the conditions of Proposition 4.4. In particular, we assume that ϕ has support in Ω. To derive the upper bound we put $\gamma = (H_0)^0_-$, i.e.,

$$\gamma(x,y) \;=\; (2\pi h)^{-d} \int_{|p|<1} e^{ip\cdot(x-y)/h}\, dp\,,$$

and obtain that

$$\begin{aligned} -\mathrm{Tr}\,(\phi H_\Omega \phi)_- \;&\leq\; \mathrm{Tr}\,(\gamma\phi H_\Omega \phi) \;=\; \mathrm{Tr}\,(\gamma\phi H_0 \phi) \\ &=\; \int_{|p|<1} \left(\|(-h^2\Delta)^{s/2} \phi\, e^{ip\cdot/h}\|_2^2 - \|\phi\|_2^2 \right) \frac{dp}{(2\pi h)^d}\,. \end{aligned} \tag{4.21}$$

Lemma 4.7. *For $\phi \in C_0^\infty(\mathbb{R}^d)$ and $h>0$ we have*

$$\|(-h^2\Delta)^{s/2}\phi\, e^{-ip\cdot/h}\|_2^2 = |p|^{2s}\,\|\phi\|_2^2 + \int \left(\frac{1}{2}\left(|p+h\eta|^{2s} + |p-h\eta|^{2s} \right) - |p|^{2s} \right) |\hat\phi(\eta)|^2 d\eta\,.$$

Proof. By Plancherel's theorem we get

$$\begin{aligned} &\|(-h^2\Delta)^{s/2}\phi\, e^{ip\cdot/h}\|_2^2 \\ &= \iiint |\xi|^{2s}\, \phi(x)\phi(y)\, e^{i(p-\xi)\cdot(x-y)/h} \frac{dxdyd\xi}{(2\pi h)^d} \\ &= \frac{1}{2} \iiint |\xi|^{2s} \left(\phi^2(x) + \phi^2(y) - |\phi(x)-\phi(y)|^2 \right) e^{i(p-\xi)\cdot(x-y)/h}\, \frac{dxdyd\xi}{(2\pi h)^d}\,. \end{aligned} \tag{4.22}$$

In the first two terms we perform the ξ integration and either the x or the y integration to arrive at

$$\frac{1}{2} \iiint |\xi|^{2s} \left(\phi^2(x) + \phi^2(y) \right) e^{i(p-\xi)\cdot(x-y)/h} \frac{dxdyd\xi}{(2\pi h)^d} \;=\; |p|^{2s} \int \phi^2(x)\, dx\,. \tag{4.23}$$

We are left with calculating the third term in (4.22). Again, by Plancherel's theorem we see that it equals

$$\frac{1}{2} \iiint |\xi|^{2s} \left| \hat\phi\left(\frac{\eta}{h}\right) \right|^2 \left| 1 - e^{-iz\cdot\eta/h} \right|^2 e^{i(p-\xi)\cdot z/h}\, \frac{d\eta dz d\xi}{(2\pi)^d h^{2d}}\,.$$

We can write

$$\left| 1 - e^{-iz\cdot\eta/h} \right|^2 \;=\; 2 - e^{iz\cdot\eta/h} - e^{-iz\cdot\eta/h}$$

and perform the integration in z and ξ to obtain

$$\begin{aligned} &\frac{1}{2} \iiint |\xi|^{2s} \left| \phi(x) - \phi(y) \right|^2 e^{i(p-\xi)\cdot(x-y)/h} \frac{dxdyd\xi}{(2\pi h)^d} \\ &= \frac{1}{h^d} \int \left(|p|^{2s} - \frac{1}{2}\left(|p+\eta|^{2s} + |p-\eta|^{2s} \right) \right) \left| \hat\phi\left(\frac{\eta}{h}\right) \right|^2 d\eta\,. \end{aligned} \tag{4.24}$$

Hence, combining (4.22), (4.23) and (4.24) yields the claim. □

In view of identity (4.21) and Lemma 4.7 we conclude

$$\mathrm{Tr}\,(\gamma\phi H_0\phi) \;=\; (2\pi h)^{-d} \int_{|p|<1} \left(|p|^{2s} - 1 \right) dp\, \|\phi\|_2^2 + (2\pi h)^{-d} \int_{|p|<1} R_h(p)\, dp \tag{4.25}$$

with

$$R_h(p) \,=\, \int \left(\frac{1}{2} \left(|p+h\eta|^{2s} + |p-h\eta|^{2s} \right) - |p|^{2s} \right) |\hat{\phi}(\eta)|^2 d\eta \,.$$

We proceed to estimate $R_h(p)$. Note that for any $a > 0$

$$\max_{|t|\leq a} \left((a+t)^s + (a-t)^s \right) \,=\, 2a^s \,.$$

Taking $a = |p|^2 + |\eta|^2$ and $t = 2p \cdot \eta$ we deduce that

$$\frac{1}{2} \left(|p+\eta|^{2s} + |p-\eta|^{2s} \right) - |p|^{2s} \,\leq\, (|p|^2 + |\eta|^2)^s - |p|^{2s} \,.$$

Next, for $0 < s < 1$ concavity implies that $(a+b)^s \leq a^s + sa^{s-1}b$ for $a, b > 0$, from which we learn that

$$(|p|^2 + |\eta|^2)^s - |p|^{2s} \,\leq\, s\, |p|^{2(s-1)}\, |\eta|^2 \,.$$

Hence, replacing η with $h\eta$ and using (4.14) we can estimate

$$R_h(p) \,\leq\, s \int |p|^{-2+2s} |h\eta|^2 |\hat{\phi}(\eta)|^2 d\eta \,=\, s\, |p|^{-2+2s}\, h^2 \int |\nabla \phi|^2 dx \,\leq\, C h^2 |p|^{-2+2s} \,.$$

Thus the upper bound follows from (4.21) and (4.25).

4.3. Asymptotics on the half-space

Our goal in this section is to prove the analogue of Proposition 4.5 in the case where Ω is the half-space $\mathbb{R}^d_+ = \{(x', x_d) : \, x_d > 0\}$. We define the operator H^+ on $L^2(\mathbb{R}^d_+)$, in the same way as H_Ω, with form domain

$$\mathcal{H}^s(\mathbb{R}^d_+) \,=\, \left\{ v \in H^s(\mathbb{R}^d) \,:\, v \equiv 0 \text{ on } \mathbb{R}^d \setminus \overline{\mathbb{R}^d_+} \right\} \,.$$

We shall prove

PROPOSITION 4.8. *Assume that $\phi \in C^1_0(\mathbb{R}^d)$ is supported in a ball of radius $l > 0$ and assume that (4.14) is satisfied. Then for $h > 0$ and any $0 < \delta_1 < 1$ and $0 < \delta_2 < \min\{1, 2s\}$ we have*

$$\begin{aligned} &- C_{\delta_1,\delta_2} \left(l^{d-1-\delta_1} h^{-d+1+\delta_1} + l^{d-1-\delta_2} h^{-d+1+\delta_2} \right) \\ &\leq \, \mathrm{Tr} \left(\phi H^+ \phi \right)_- - L^{(1)}_{s,d} \int_{\mathbb{R}^d_+} \phi^2(x) dx h^{-d} + L^{(2)}_{s,d} \int_{\mathbb{R}^{d-1}} \phi^2(x',0) dx' h^{-d+1} \\ &\leq \, C_{\delta_1} l^{d-1-\delta_1} h^{-d+1+\delta_1} \,. \end{aligned}$$

This result depends on a more or less explicit diagonalization of the operator H^+, which is far from obvious. This is accomplished in Subsections 4.3.1 and 4.3.2, relying crucially on recent results of Kwaśnicki [**Kwa10b**] about non-local operators on a half-line. These results are collected and extended to our needs in Section 4.8.

4.3.1. The model operator on the half-line. In this subsection we collect some facts about the one-dimensional operator

$$A^+ \,=\, \left(-\frac{d^2}{dt^2}+1\right)^s$$

in $L^2(\mathbb{R}_+)$ with form domain $\mathcal{H}^s(\mathbb{R}_+)$, and about the corresponding operator A in $L^2(\mathbb{R})$, defined analogously to A^+, but with form domain $H^s(\mathbb{R})$.

For $\mu > 0$ and $t, u \in \mathbb{R}_+$, let $e^+(t,u,\mu)$ and $a^+(t,u,\mu)$ be the integral kernels of $(A^+ - \mu)^0_-$ and $(A^+ - \mu)_-$, respectively. Similarly, we define $a(t,u,\mu)$ via $(A - \mu)_-$. To simplify notation we abbreviate $a^+(t,\mu) = a^+(t,t,\mu)$. We also note that $a(\mu) = a(t,t,\mu)$ is independent of $t \in \mathbb{R}_+$. The inequality $A^+ \geq 1$ implies that $a^+(t,u,\mu) = e^+(t,u,\mu) = 0$ for $\mu < 1$ and similarly for $a(t,u,\mu)$ and $e(t,u,\mu)$.

The following two results about $e^+(t,\mu)$ and $a^+(t,\mu)$ are rather technical and we defer the proofs to Subsections 4.8.1 and 4.8.2. The first one provides a rough a-priori bound on $e^+(t,u,\mu)$.

LEMMA 4.9. *For any $\mu > 0$ and $t, u \in \mathbb{R}_+$ one has $|e^+(t,u,\mu)| \leq C\mu^{1/2s}$.*

The second result in this subsection quantifies that $a^+(t,\mu)$ is close to $a(\mu)$ for large t.

LEMMA 4.10. *For any $0 \leq \gamma < 1$ there is a constant C_γ such that for all $\mu \geq 1$,*

$$\int_0^\infty t^\gamma |a^+(t,\mu) - a(\mu)|\, dt \leq C_\gamma\, \mu \left((\ln \mu)^2 + 1\right) . \tag{4.26}$$

In particular, the function

$$K(t) \,=\, \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{R}^{d-1}} |\xi'|^{1+2s} \left(a(|\xi'|^{-2s}) - a^+(t|\xi'|, |\xi'|^{-2s})\right) d\xi' \,, \qquad t > 0 \,, \tag{4.27}$$

satisfies for every $0 \leq \gamma < 1$

$$\int_0^\infty t^\gamma \, |K(t)|\, dt \,<\, \infty \,.$$

With this lemma at hand we can now define the constant $L^{(2)}_{s,d}$ which appears in our main theorem by

$$L^{(2)}_{s,d} \,=\, \int_0^\infty K(t)\, dt \,. \tag{4.28}$$

(This integral converges by Lemma 4.10.) Expression (4.28) suffices for the proof of our main result. In Section 4.6, see also (4.71), we will derive different representation for $L^{(2)}_{s,d}$.

4.3.2. Reduction from the half-space to the half-line. Our goal in this subsection is to write the spectral projections of the operator H^+ on the half-space in terms of those of the operator A^+ on the half-line. Before turning to spectral projections we treat resolvents.

LEMMA 4.11. *For $x = (x', x_d) \in \mathbb{R}^d_+$, $y = (y', y_d) \in \mathbb{R}^d_+$ and $z \in \mathbb{C} \setminus [0, \infty)$ the resolvent kernels of H^+s and of A^+ are related by*

$$(H^+ + 1 - z)^{-1}(x, y)$$
$$= \frac{1}{h^d} \int_{\mathbb{R}^{d-1}} |\xi'|^{1-2s} \, e^{i\xi'\cdot(x'-y')/h} \left(A^+ - \frac{z}{|\xi'|^{2s}} \right)^{-1} \left(\frac{x_d|\xi'|}{h}, \frac{y_d|\xi'|}{h} \right) \frac{d\xi'}{(2\pi)^{d-1}} \,.$$

This lemma, together with the representations (see, e.g., [**Kat66**])

$$(H^+)^0_- = -\frac{1}{2\pi i} \int_{\Gamma(1)} (H^+ + 1 - z)^{-1} \, dz$$

and

$$(H^+)_- = -h^{2s} \int_0^{h^{-2s}} \frac{1}{2\pi i} \int_{\Gamma(\nu)} (H^+ + 1 - z)^{-1} \, dz \, d\nu \,,$$

where $\Gamma(\nu) = \{z \in \mathbb{C} : |z| = \nu\}$, implies that

$$(H^+)^0_-(x, y) = \frac{1}{h^d} \int_{\mathbb{R}^{d-1}} |\xi'| e^{i\xi'\cdot(x'-y')/h} \, e^+ \left(\frac{x_d|\xi'|}{h}, \frac{y_d|\xi'|}{h}, \frac{1}{|\xi'|^{2s}} \right) \frac{d\xi'}{(2\pi)^{d-1}} \tag{4.29}$$

and

$$(H^+)_-(x, y) = \frac{1}{h^d} \int_{\mathbb{R}^{d-1}} |\xi'|^{1+2s} \, e^{i\xi'\cdot(x'-y')/h} \, a^+ \left(\frac{x_d|\xi'|}{h}, \frac{y_d|\xi'|}{h}, \frac{1}{|\xi'|^{2s}} \right) \frac{d\xi'}{(2\pi)^{d-1}} \,. \tag{4.30}$$

We now give the

PROOF OF LEMMA 4.11. By scaling we may assume that $h = 1$. Given $f \in L^2(\mathbb{R}^d_+)$ we want to solve $(-\Delta)^s u = zu + f$. Take $\psi \in C_0^\infty(\mathbb{R}^{d-1})$ and $\phi \in C_0^\infty(0, \infty)$. Then $\psi \otimes \phi$ belongs to the form domain of H^+ and therefore the equation implies that

$$\iint (|\xi'|^2 + \xi_d^2)^s \overline{\hat\psi(\xi')\hat\phi(\xi_d)} \hat u(\xi', \xi_d) \, d\xi' \, d\xi_d = z \iint \overline{\hat\psi(\xi')\hat\phi(\xi_d)} \hat u(\xi', \xi_d) \, d\xi' \, d\xi_d + \iint \overline{\hat\psi(\xi')\hat\phi(\xi_d)} \hat f(\xi', \xi_d) \, d\xi' \, d\xi_d \,.$$

Since ψ is arbitrary, this means that for a.e. ξ',

$$\int (|\xi'|^2 + \xi_d^2)^s \overline{\hat\phi(\xi_d)} \hat u(\xi', \xi_d) \, d\xi_d = z \int \overline{\hat\phi(\xi_d)} \hat u(\xi', \xi_d) \, d\xi_d + \int \overline{\hat\phi(\xi_d)} \hat f(\xi', \xi_d) \, d\xi_d \,. \tag{4.31}$$

For fixed ξ' define functions $v_{\xi'}$, $g_{\xi'}$ and $\chi_{\xi'}$ on $(0, \infty)$ by

$$v_{\xi'}(t) := (2\pi)^{-(d-1)/2} \int_{\mathbb{R}^{d-1}} |\xi'|^{-1} u(x', |\xi'|^{-1}t) e^{-i\xi'\cdot x'} \, dx' \,,$$
$$g_{\xi'}(t) := (2\pi)^{-(d-1)/2} \int_{\mathbb{R}^{d-1}} |\xi'|^{-1-2s} f(x', |\xi'|^{-1}t) e^{-i\xi'\cdot x'} \, dx' \,,$$
$$\chi_{\xi'}(t) := |\xi'|^{-1} \phi(|\xi'|^{-1}t) \,.$$

We note that the one-dimensional Fourier transforms of these functions are given by

$$\begin{aligned}\widehat{v_{\xi'}}(k) &= \hat{u}(\xi', |\xi'|k)\,,\\ \widehat{g_{\xi'}}(k) &= |\xi'|^{-2s}\hat{f}(\xi', |\xi'|k)\,,\\ \widehat{\chi_{\xi'}}(k) &= \hat{\phi}(|\xi'|k)\,.\end{aligned}$$

Hence we can rewrite (4.31) as

$$\int (1+k^2)^s \overline{\widehat{\chi_{\xi'}}(k)}\widehat{v_{\xi'}}(k)\,dk \;=\; |\xi'|^{-2s} z \int \overline{\widehat{\chi_{\xi'}}(k)}\widehat{v_{\xi'}}(k)\,dk + \int \overline{\widehat{\chi_{\xi'}}(k)}\widehat{g_{\xi'}}(k)\,dk\,.$$

Note that both $v_{\xi'}$ and $\chi_{\xi'}$ belong to the form domain of A^+ and that the set of all functions $\chi_{\xi'}$ obtained in this way is dense in the form sense (by the definition of A^+). Therefore the equation can be written as

$$A^+ v_{\xi'} = |\xi'|^{-2s} z v_{\xi'} + g_{\xi'}\,.$$

We abbreviate $r_z^+(t,w) = (A^+ - z)^{-1}(t,w)$ and conclude that

$$v_{\xi'}(t) = \int_0^\infty r^+_{z|\xi'|^{-2s}}(t,w) g_{\xi'}(w)\,dw\,.$$

Recalling the definitions of $v_{\xi'}$ and $g_{\xi'}$ this reads

$$\begin{aligned}&\int_{\mathbb{R}^{d-1}} |\xi'|^{-1} u(x', |\xi'|^{-1}t) e^{-i\xi'\cdot x'}\,dx'\\ &\qquad = \int_0^\infty \int_{\mathbb{R}^{d-1}} r^+_{z|\xi'|^{-2s}}(t,w)|\xi'|^{-1-2s} f(x', |\xi'|^{-1}t) e^{-i\xi'\cdot x'}\,dw\,dx'\,.\end{aligned}$$

Multiplying by $|\xi'|$, setting $t = |\xi'|x_d$ and inverting the Fourier transform, we obtain

$$\begin{aligned}u(x) &= (2\pi)^{-d+1}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^{d-1}}\int_0^\infty r^+_{z|\xi'|^{-2s}}(|\xi'|x_d, w)|\xi'|^{-2s} f(y', |\xi'|^{-1}w)\, e^{i\xi'\cdot(x'-y')} dw\,d\xi'\,dy'\\ &= (2\pi)^{-d+1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-1}} r^+_{z|\xi'|^{-2s}}(|\xi'|x_d, |\xi'|y_d)|\xi'|^{1-2s} f(y)\, e^{i\xi'\cdot(x'-y')} d\xi'\,dy\,.\end{aligned}$$

This proves the lemma. □

4.3.3. Proof of Proposition 4.8. Our next step is to state upper and lower bounds on $\operatorname{Tr}(\phi H^+\phi)_-$ in terms of the one-dimensional model operators A and A^+, in particular, in terms of the function $K(t)$ given in (4.27). As explained below, the main result of this section, Proposition 4.8, will be a direct consequence of the following estimates.

PROPOSITION 4.12. *Assume that $\phi \in C_0^1(\mathbb{R}^d)$ is supported in a ball of radius $l = 1$ and assume that (4.14) is satisfied with $l = 1$. Then for any $0 < \delta_2 < \min\{1, 2s\}$ there is a constant C_{δ_2} such that for all $h > 0$ we have*

$$\operatorname{Tr}\left(\phi H^+\phi\right)_- \le L^{(1)}_{s,d}\int_{\mathbb{R}^d_+}\phi^2(x)dx h^{-d} - \int_{\mathbb{R}^d_+}\phi^2(x)\frac{1}{h}K\left(\frac{x_d}{h}\right)dx h^{-d+1}\,, \tag{4.32}$$

$$\operatorname{Tr}\left(\phi H^+\phi\right)_- \ge L^{(1)}_{s,d}\int_{\mathbb{R}^d_+}\phi^2(x)dx h^{-d} - \int_{\mathbb{R}^d_+}\phi^2(x)\frac{1}{h}K\left(\frac{x_d}{h}\right)dx h^{-d+1} - C_{\delta_2}h^{-d+1+\delta_2}\,. \tag{4.33}$$

Assuming Proposition 4.12, we now give the short

PROOF OF PROPOSITION 4.8. To prove the proposition we may rescale ϕ and hence assume $l = 1$. Proposition 4.8 is then an immediate consequence of Proposition 4.12 provided we can show that for any $0 < \delta_1 < 1$ there is a C_{δ_1} such that for all $h > 0$

$$\left| \int_{\mathbb{R}^d_+} \phi^2(x) \frac{1}{h} K\left(\frac{x_d}{h}\right) dx - L^{(2)}_{s,d} \int_{\mathbb{R}^{d-1}} \phi^2(x',0) dx' \right| \leq C_{\delta_1} h^{\delta_1} . \tag{4.34}$$

In order to obtain the latter bound, we substitute $x_d = th$ and write, recalling (4.28),

$$\begin{aligned} &\int_{\mathbb{R}^d_+} \phi^2(x) \frac{1}{h} K\left(\frac{x_d}{h}\right) dx - L^{(2)}_{s,d} \int_{\mathbb{R}^{d-1}} \phi^2(x',0) dx' \\ &= \int_0^\infty K(t) \int_{\mathbb{R}^{d-1}} \int_0^{th} \partial_\tau \phi^2(x',\tau) d\tau dx' dt . \end{aligned}$$

By Hölder's inequality we can further estimate

$$\begin{aligned} \left| \int_{\mathbb{R}^{d-1}} \int_0^{th} \partial_\tau \phi^2(x',\tau) d\tau dx' \right| &\leq \left(\int_0^{th} d\tau \right)^{\delta_1} \left(\int_0^\infty \left| \int_{\mathbb{R}^{d-1}} \partial_\tau \phi^2(x',\tau) dx' \right|^{(1-\delta_1)^{-1}} d\tau \right)^{1-\delta_1} \\ &\leq C t^{\delta_1} h^{\delta_1} . \end{aligned}$$

Since $\int_0^\infty t^{\delta_1} |K(t)| \, dt < \infty$ by Lemma 4.10, we obtain inequality (4.34). □

In the following two subsections we shall prove the lower and the upper bound in Proposition 4.12, respectively.

4.3.4. Lower bound. To prove (4.32) we use that

$$-\mathrm{Tr}\left(\phi H^+ \phi\right)_- \geq -\mathrm{Tr}\left(\phi (H^+)_- \phi\right) .$$

The lower bound follows from this by integrating the identity

$$(H^+)_-(x,x) = h^{-d} L^{(1)}_{s,d} - h^{-d} K\left(\frac{x_d}{h}\right) , \tag{4.35}$$

against ϕ^2. Equation (4.35) is a consequence of (4.30). Indeed, by the same argument as in Subsection 4.3.2 we learn that

$$(H_0)_-(x,x) = \frac{1}{(2\pi)^{d-1}} \frac{1}{h^d} \int_{\mathbb{R}^{d-1}} |\xi'|^{1+2s} \, a\left(|\xi'|^{-2s}\right) d\xi' .$$

On the other hand, by direct diagonalization as in Subsection 4.2.1 we find that

$$(H_0)_-(x,x) = h^{-d} L^{(1)}_{s,d} .$$

Comparing these two identities with (4.30) we arrive at (4.35), thus establishing (4.32).

4.3.5. Upper bound. To prove (4.33) we set $\gamma = (H^+)^0_-$. Its integral kernel is given by (4.29) in terms of the kernel $e^+(\cdot,\cdot,\mu)$ of $(A^+-\mu)^0_-$. By the variational principle it follows that

$$\begin{aligned}-\mathrm{Tr}\left(\phi H^+\phi\right)_- &\le \mathrm{Tr}\left(\phi\gamma\phi H^+\right)\\ &= \frac{1}{h^{2d}}\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^d}|\xi'|e^{i\xi'\cdot(x'-y')/h}\,e^+\left(x_d|\xi'|h^{-1},y_d|\xi'|h^{-1},|\xi'|^{-2s}\right)\\ &\quad\times\left(|p|^{2s}-1\right)e^{ip\cdot(y-x)/h}\,\phi(x)\,\phi(y)\,\frac{dp\,d\xi'\,dx\,dy}{(2\pi)^{2d-1}}\,. \end{aligned}\tag{4.36}$$

We insert the identity

$$\phi(x)\,\phi(y) \;=\; \frac{1}{2}\left(\phi^2(x)+\phi^2(y)-|\phi(x)-\phi(y)|^2\right)\,,$$

use the symmetry in x and y and substitute $q=p_d/|p'|$ to obtain

$$-\mathrm{Tr}\left(\phi H^+\phi\right) \;\le\; I_h[\phi]-R_h[\phi]$$

with the main term

$$\begin{aligned}I_h[\phi] =&\frac{1}{h^{2d}}\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}}|\xi'|e^{i(\xi'-p')\cdot(x'-y')/h}e^+\left(x_d|\xi'|h^{-1},y_d|\xi'|h^{-1},|\xi'|^{-2s}\right)\\ &\times e^{i(y_d-x_d)|p'|q/h}\left((q^2+1)^s-|p'|^{-2s}\right)|p'|^{1+2s}\phi^2(x)\,\frac{dq\,dp'\,d\xi'\,dx\,dy}{(2\pi)^{2d-1}}\end{aligned}$$

and the remainder

$$\begin{aligned}R_h[\phi] =&\frac{1}{h^{2d}}\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^d}|\xi'|e^{i\xi'\cdot(x'-y')/h}\,e^+\left(x_d|\xi'|h^{-1},y_d|\xi'|h^{-1},|\xi'|^{-2s}\right)\\ &\times|p|^{2s}e^{ip\cdot(y-x)/h}\,|\phi(x)-\phi(y)|^2\,\frac{dp\,d\xi'\,dx\,dy}{2(2\pi)^{2d-1}}\,.\end{aligned}$$

Since $\phi\in C^1_0(\mathbb{R}^d)$ we can perform the y'-integration in $I_h[\phi]$. We use the fact that

$$\int_{\mathbb{R}}\int_0^\infty e^+\left(x_d,y_d,\mu\right)\left((q^2+1)^s-\mu\right)e^{i(y_d-z_d)q}\,dy_d\,dq=-a^+(x_d,z_d,\mu)$$

and obtain

$$\begin{aligned}I_h[\phi] =&\frac{1}{h^{d+1}}\int_{\mathbb{R}^d_+}\int_0^\infty\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}}|\xi'|^{2s+2}\,e^+\left(x_d|\xi'|h^{-1},y_d|\xi'|h^{-1},|\xi'|^{-2s}\right)\\ &\times\left((q^2+1)^s-|\xi'|^{-2s}\right)e^{i(y_d-x_d)|\xi'|q/h}\phi^2(x)\,\frac{dq\,d\xi'\,dy_d\,dx}{(2\pi)^d}\\ =&-\frac{1}{h^d}\int_{\mathbb{R}^d_+}\phi^2(x)\int_{\mathbb{R}^{d-1}}|\xi'|^{2s+1}\,a^+\left(x_d|\xi'|h^{-1},|\xi'|^{-2s}\right)\frac{d\xi'\,dx}{(2\pi)^{d-1}}\,.\end{aligned}$$

Using again (4.35) we find that

$$I_h[\phi]=-L^{(1)}_{s,d}\int_{\mathbb{R}^d_+}\phi^2(x)\,dx\,h^{-d}+\int_{\mathbb{R}^d_+}\phi^2(x)\,K\left(\frac{x_d}{h}\right)dx\,h^{-d}\,.\tag{4.37}$$

It remains to study $R_h[\phi]$. We claim that for any $\frac{1}{2}-s<\sigma<\min\{\frac{1}{2},1-s\}$ there is a C_σ such that

$$|R_h[\phi]| \;\le\; C_\sigma h^{-d+2s+2\sigma}\tag{4.38}$$

for all $h > 0$. This, together with (4.37) will complete the proof of (4.33).

In order to show (4.38) we perform the p integration and find that

$$R_h[\phi] \;=\; -\frac{C}{h^{d-2s}} \int_{\mathbb{R}^d_+} \int_{\mathbb{R}^d_+} \int_{\mathbb{R}^{d-1}} |\xi'| e^{i\xi'\cdot(x'-y')/h} e^+ \left(\frac{x_d|\xi'|}{h}, \frac{y_d|\xi'|}{h}, \frac{1}{|\xi'|^{2s}} \right) \times \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{d+2s}} d\xi'\, dx\, dy\,.$$

We insert

$$e^{i\xi'\cdot(x'-y')/h} \;=\; \frac{h^{2\sigma}}{|\xi'|^{2\sigma}} (-\Delta_{x'})^\sigma e^{i\xi'\cdot(x'-y')/h}$$

and integrate by parts to get

$$R_h[\phi] \;=\; -\frac{C}{h^{d-2s-2\sigma}} \int_{\mathbb{R}^d_+} \int_{\mathbb{R}^d_+} \int_{\mathbb{R}^{d-1}} |\xi'|^{1-2\sigma} e^{i\xi'\cdot(x'-y')/h} e^+ \left(\frac{x_d|\xi'|}{h}, \frac{y_d|\xi'|}{h}, \frac{1}{|\xi'|^{2s}} \right) d\xi' \times (-\Delta_{x'})^\sigma \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{d+2s}} dx\, dy\,.$$

By Lemma 4.9 and the fact that $e^+(t,u,\mu) = 0$ for $\mu \le 1$ we arrive at

$$\begin{aligned} |R_h[\phi]| \le & \frac{C}{h^{d-2s-2\sigma}} \int_{\mathbb{R}^d_+} \int_{\mathbb{R}^d_+} \int_{\{\xi'\in\mathbb{R}^{d-1}:|\xi'|<1\}} |\xi'|^{-2\sigma} d\xi' \left| (-\Delta_{x'})^\sigma \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{d+2s}} \right| dxdy \\ \le & \frac{C}{h^{d-2s-2\sigma}} \int_{\mathbb{R}^d_+} \int_{\mathbb{R}^d_+} \left| (-\Delta'_x)^\sigma \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{d+2s}} \right| dxdy\,. \end{aligned}$$

According to Lemma 4.24 this implies (4.38) and hence completes the proof of (4.33).

4.4. Local asymptotics near the boundary

In this section we prove Proposition 4.5. After having analyzed the half-space case in the previous section, we now show how the case of a general domain follows. We shall transform the operator H_Ω locally to an operator given on the half-space $\mathbb{R}^d_+ = \{(y', y_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \,:\, y_d > 0\}$ and we shall quantify the error made by this straightening of the boundary.

Under the conditions of Proposition 4.5, let B denote the open ball of radius $l > 0$, containing the support of ϕ. For $x_0 \in B \cap \partial\Omega$ let ν_{x_0} be the inner normal unit vector at x_0. We choose a Cartesian coordinate system such that $x_0 = 0$ and $\nu_{x_0} = (0, \dots, 0, 1)$, and we write $x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$ for $x \in \mathbb{R}^d$.

For sufficiently small $l > 0$ one can introduce new local coordinates near the boundary. Let D denote the projection of B on the hyperplane given by $x_d = 0$. Since the boundary of Ω is compact and $C^{1,\alpha}$ there is a constant $c > 0$ such that for $0 < l \le c$ we can find a real function $f \in C^{1,\alpha}$ given on D, satisfying

$$\partial\Omega \cap B \;=\; \{(x', x_d) \,:\, x' \in D, x_d = f(x')\} \cap B\,.$$

The choice of coordinates implies $f(0) = 0$ and $\nabla f(0) = 0$. Hence, we can estimate

$$\sup_{x'\in D} |\nabla f(x')| \;=\; \sup_{x'\in D} |\nabla f(x') - \nabla f(0)| \;\le\; C_f\, |x'|^\alpha \;\le\; C_f\, l^\alpha\,.$$

Since the boundary of Ω is compact we can choose a constant $C > 0$, depending only on Ω, in particular independent of f, such that the bound

$$\sup_{x' \in D} |\nabla f(x')| \,\leq\, C l^\alpha \tag{4.39}$$

holds.

We introduce new local coordinates via the diffeomorphism $\varphi \,:\, D \times \mathbb{R} \,\to\, \mathbb{R}^d$, given by

$$y_j \,=\, \varphi_j(x) \,=\, x_j \quad \text{for} \quad j = 1, \ldots, d-1$$

and

$$y_d \,=\, \varphi_d(x) \,=\, x_d - f(x')\,.$$

Note that the determinant of the Jacobian matrix of φ equals 1 and that the inverse of φ is given on $\operatorname{ran}\varphi = D \times \mathbb{R}$. In particular, we get

$$\varphi\left(\partial\Omega \cap B\right) \,\subset\, \partial\mathbb{R}^d_+ \,=\, \{y \in \mathbb{R}^d \,:\, y_d = 0\}\,.$$

Fix $v \in \mathcal{H}^s(\Omega)$ with support in $\overline{B}$. For $y \in \operatorname{ran}\varphi$ put $\tilde{v}(y) = v \circ \varphi^{-1}(y)$ and extend $\tilde{v}$ by zero to $\mathbb{R}^d$.

Lemma 4.13. *The function $\tilde{v}$ belongs to $\mathcal{H}^s(\mathbb{R}^d_+)$ and for $0 < l \leq c$ we have*

$$\left|(\tilde{v}, (-\Delta)^s_{\mathbb{R}^d_+}\tilde{v}) - (v, (-\Delta)^s_\Omega v)\right| \,\leq C\, l^\alpha \,\min\left\{(\tilde{v}, (-\Delta)^s_{\mathbb{R}^d_+}\tilde{v}),\ (v, (-\Delta)^s_\Omega v)\right\}\,.$$

Proof. By definition, $\tilde{v}$ belongs to $H^s(\mathbb{R}^d)$ and for $y \in \mathbb{R}^d \setminus \overline{\mathbb{R}^d_+}$ we find $x_d = y_d + f(y') < f(x')$, thus $\tilde{v}(y) = v(x) = 0$. Therefore $\tilde{v}$ belongs to $\mathcal{H}^s(\mathbb{R}^d_+)$.

Using the new local coordinates we get

$$(v, (-\Delta)^s_\Omega v) = C_{s,d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|v(x) - v(w)|^2}{|x-w|^{d+2s}} dx\, dw = C_{s,d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|\tilde{v}(y) - \tilde{v}(z)|^2}{|x-w|^{d+2s}} dy\, dz\,, \tag{4.40}$$

where $y = \varphi(x)$ and $z = \varphi(w)$, thus $x = (y', y_d + f(y'))$ and $w = (z', z_d + f(z'))$. Let us write

$$\begin{aligned}
&\left|\frac{1}{|y-z|^{d+2s}} - \frac{1}{|x-w|^{d+2s}}\right| \\
&= \,\frac{1}{|y-z|^{d+2s}} \left|1 - \frac{|y-z|^{d+2s}}{[|y'-z'|^2 + (y_d + f(y') - z_d - f(z'))^2]^{d/2+s}}\right|\,.
\end{aligned}$$

After multiplying out, the last fraction equals

$$\left(1 + \frac{(f(y') - f(z'))^2 + 2(y_d - z_d)(f(y') - f(z'))}{|y-z|^2}\right)^{-(d/2+s)}$$

and we can employ (4.39) to estimate

$$\begin{aligned}
&\left|\frac{(f(y') - f(z'))^2 + 2(y_d - z_d)(f(y') - f(z'))}{|y-z|^2}\right| \\
&\leq \,\sup|\nabla f|^2\, \frac{|y'-z'|^2}{|y-z|^2} + 2\,\sup|\nabla f|\, \frac{|y'-z'|\,|y_d - z_d|}{|y-z|^2} \,\leq\, C l^\alpha\,.
\end{aligned}$$

Choosing l small enough we can assume $Cl^\alpha < 1/2$. Then, combining the foregoing relations, we find

$$\left| \frac{1}{|x-w|^{d+2s}} - \frac{1}{|y-z|^{d+2s}} \right| \leq C \frac{l^\alpha}{|y-z|^{d+2s}} \,. \tag{4.41}$$

From (4.40) and (4.41) we conclude

$$\begin{aligned} &\left| (\tilde{v}, (-\Delta)^s_{\mathbb{R}^d_+} \tilde{v}) - (v, (-\Delta)^s_\Omega v) \right| \\ &\leq C_{s,d} \iint |\tilde{v}(y) - \tilde{v}(z)|^2 \left| \frac{1}{|y-z|^{d+2s}} - \frac{1}{|x-w|^{d+2s}} \right| dy\, dz \\ &\leq C\, l^\alpha (\tilde{v}, (-\Delta)^s_{\mathbb{R}^d_+} \tilde{v}) \,. \end{aligned}$$

This proves the first claim of the Lemma. The second claim follows by interchanging the roles of $(-\Delta)^s_{\mathbb{R}^d_+}$ and $(-\Delta)^s_\Omega$. □

On the range of φ we define $\tilde{\phi}_u = \phi_u \circ \varphi^{-1}$ and extend it by zero to $\mathbb{R}^d$ such that $\tilde{\phi}_u \in C^1_0(\mathbb{R}^d)$ and $\|\nabla \tilde{\phi}_u\|_\infty \leq C l^{-1}$ hold. Using Lemma 4.13 we show the following relations.

LEMMA 4.14. *For $0 < l \leq c$ and any $h > 0$ the estimate*

$$\left| \mathrm{Tr}(\phi H_\Omega \phi)_- - \mathrm{Tr}(\tilde{\phi} H^+ \tilde{\phi})_- \right| \leq C\, l^{d+\alpha}\, h^{-d} \tag{4.42}$$

holds. Moreover, we have

$$\int_\Omega \phi^2(x)\, dx = \int_{\mathbb{R}^d_+} \tilde{\phi}^2(y)\, dy \tag{4.43}$$

and

$$0 \leq \int_{\partial\Omega} \phi^2(x)\, d\sigma(x) - \int_{\mathbb{R}^{d-1}} \tilde{\phi}^2(y', 0)\, dy' \leq C\, l^{d-1+2\alpha} \,. \tag{4.44}$$

PROOF. The definition of $\tilde{\phi}$ and the fact that the Jacobian of ϕ equals 1 immediately gives (4.43). Using (4.39) we estimate

$$\int_{\partial\Omega} \phi^2(x)\, d\sigma(x) = \int_{\mathbb{R}^{d-1}} \tilde{\phi}^2(y', 0) \sqrt{1 + |\nabla f|^2}\, dy' \leq \int_{\mathbb{R}^{d-1}} \tilde{\phi}^2(y', 0)\, dy' + C l^{d-1+2\alpha} \,.$$

from which (4.44) follows.

To prove (4.42) we refer to the variational principle once more and note that

$$-\mathrm{Tr}\,(\phi H_\Omega \phi)_- = \inf_{0 \leq \gamma \leq 1} \mathrm{Tr}\,(\phi \gamma \phi H_\Omega) \,,$$

where we can assume that infimum is taken over trial density matrices γ supported in $\overline{B} \times \overline{B}$. Fix such a γ. For y and z from $D \times \mathbb{R}$ set

$$\tilde{\gamma}(y, z) = \gamma\left(\varphi^{-1}(y), \varphi^{-1}(z)\right) \,,$$

so that $0 \leq \tilde{\gamma} \leq 1$ and the range of $\tilde{\gamma}$ belongs to the form domain of $\tilde{\phi} H^+ \tilde{\phi}$. According to Lemma 4.13 it follows that

$$\begin{aligned} \mathrm{Tr}\,(\phi \gamma \phi H_\Omega) &\geq \mathrm{Tr}\left(\tilde{\phi} \tilde{\gamma} \tilde{\phi} \left(h^{2s}(1 - Cl^\alpha)(-\Delta)^s_{\mathbb{R}^d_+} - 1 \right) \right) \\ &\geq -\mathrm{Tr}\left(\tilde{\phi} \left((1 - Cl^\alpha) h^{2s} (-\Delta)^s_{\mathbb{R}^d_+} - 1 \right) \tilde{\phi} \right)_- \end{aligned}$$

and consequently

$$\mathrm{Tr}\,(\phi H_\Omega \phi)_- \;\leq\; \mathrm{Tr}\left(\tilde\phi\left((1-Cl^\alpha)h^{2s}(-\Delta)^s_{\mathbb{R}^d_+}-1\right)\tilde\phi\right)_-\,.$$

Set $\varepsilon = 2Cl^\alpha$ and assume l to be sufficiently small, so that $0<\varepsilon\leq 1/2$. Then

$$\begin{aligned}\mathrm{Tr}\,(\phi H_\Omega \phi)_- &\leq \mathrm{Tr}\left(\tilde\phi\left((1-Cl^\alpha)h^{2s}(-\Delta)^s_{\mathbb{R}^d_+}-1\right)\tilde\phi\right)_-\\ &\leq \mathrm{Tr}\left(\tilde\phi\left((-h^2\Delta)^s_{\mathbb{R}^d_+}-1\right)\tilde\phi\right)_- + \mathrm{Tr}\left(\tilde\phi\left((\varepsilon-Cl^\alpha)h^{2s}(-\Delta)^s_{\mathbb{R}^d_+}-\varepsilon\right)\tilde\phi\right)_-\\ &\leq \mathrm{Tr}(\tilde\phi H^+\tilde\phi)_- + \varepsilon\mathrm{Tr}\left(\tilde\phi\left((h^{2s}/2)(-\Delta)^s_{\mathbb{R}^d_+}-1\right)\tilde\phi\right)_-\,.\end{aligned}$$

Using Lemma 4.6 we estimate $\mathrm{Tr}(\tilde\phi((h^{2s}/2)(-\Delta)^s_{\mathbb{R}^d_+}-1)\tilde\phi)_- \leq Cl^d h^{-d}$ and it follows that

$$\mathrm{Tr}(\phi H_\Omega\phi)_- \;\leq\; \mathrm{Tr}(\tilde\phi H^+\tilde\phi)_- + C\,l^{d+\alpha}\,h^{-d}\,.$$

Finally, by interchanging the roles of H_Ω and H^+, we get an analogous lower bound and the proof of the Lemma is complete. □

We conclude this section by giving the short

Proof of Proposition 4.5. It suffices to combine Lemma 4.14 and Proposition 4.8. □

4.5. Localization

In this section we construct the family of localization functions $(\phi_u)_{u\in\mathbb{R}^d}$ and prove Proposition 4.3. Fix a real-valued function $\phi\in C_0^\infty(\mathbb{R}^d)$ with support in the ball $\{x\in\mathbb{R}^d\,:\,|x|<1\}$ that satisfies $\|\phi\|_2=1$. We recall the definition of the local length scale $l(u)$ from (4.11). For $u,x\in\mathbb{R}^d$ let $J(x,u)$ be the Jacobian of the map $u\mapsto(x-u)/l(u)$. We define

$$\phi_u(x) \;=\; \phi\left(\frac{x-u}{l(u)}\right)\sqrt{J(x,u)}\,l(u)^{d/2}\,,$$

such that ϕ_u is supported in the ball $B_u=\{x\in\mathbb{R}^d\,:\,|x-u|<l(u)\}$.

By definition, the function $l(u)$ is smooth and satisfies $0<l(u)\leq 1/2$ and $\|\nabla l\|_\infty\leq 1/2$. Therefore, according to [**SS03**], the functions ϕ_u satisfy (4.12) and (4.13) for all $u\in\mathbb{R}^d$.

To prove the lower bound in Proposition 4.3 we follow some ideas from [**LY88**]. In particular, we need the following auxiliary results; the first one gives an IMS-type localization formula for the fractional Laplacian.

Lemma 4.15. *For the family of functions $(\phi_u)_{u\in\mathbb{R}^d}$ introduced above and for all $f\in\mathcal{H}^s(\Omega)$ the identity*

$$(f,(-\Delta)^s f) \;=\; \int_{\Omega^*}(\phi_u f,(-\Delta)^s\phi_u f)\,l(u)^{-d}\,du - (f,Lf)$$

holds with $\Omega^*=\{u\in\mathbb{R}^d\,:\,\mathrm{supp}\phi_u\cap\Omega\neq\emptyset\}$. The operator L is of the form

$$L \;=\; \int_{\Omega^*} L_{\phi_u}\,l(u)^{-d}\,du\,, \tag{4.45}$$

where L_{ϕ_u} is a bounded operator with integral kernel

$$L_{\phi_u}(x,y) \;=\; C_{s,d}\,\frac{|\phi_u(x)-\phi_u(y)|^2}{|x-y|^{d+2s}}\chi_\Omega(x)\chi_\Omega(y)\,.$$

Here χ_Ω denotes the characteristic function of Ω.

Lemma 4.15 implies that for any operator γ with range in $\mathcal{H}^s(\Omega)$

$$\operatorname{Tr}\gamma(-\Delta)^s \;=\; \int_{\mathbb{R}^d} \operatorname{Tr}\left(\gamma\phi_u(-\Delta)^s\phi_u\right) l(u)^{-d}\,du - \operatorname{Tr}\gamma L\,. \tag{4.46}$$

The next result allows to estimate the localization error $\operatorname{Tr}\gamma L$.

LEMMA 4.16. *For $u\in\mathbb{R}^d$ and $0<\delta\le 1/2$ we have*

$$\operatorname{Tr}\gamma L_{\phi_u} \;\le\; \operatorname{Tr}\gamma\left(C\delta^{2-2s}l(u)^{-2s}\chi_\delta\chi_\Omega\right) + C\,\|\gamma\|\,l(u)^{-2s}\,\delta^{-d+2-2s}r(\delta)$$

with

$$r(\delta) \;=\; \begin{cases} 1 & \text{if } 1-d/4<s<1 \\ |\ln\delta| & \text{if } 0<s=1-d/4 \\ \delta^{d+4s-4} & \text{if } 0<s<1-d/4 \end{cases}\,.$$

where χ_δ denotes the characteristic function of $\{x\in\mathbb{R}^d\,:\,|x-u|<l(u)(1+\delta)\}$.

PROOF. By translation and scaling we can assume that $u=0$ and $l(u)=1$, and hence $\phi_u=\phi$. (This rescaling changes Ω, but the bound we are going to prove is independent of the domain and therefore not affected by this dilation.) We set

$$L^1_\phi(x,y) \;=\; \begin{cases} L_\phi(x,y)\,\chi_\delta(x)\,\chi_\delta(y) & \text{if } |x-y|<\delta \\ 0 & \text{if } |x-y|\ge\delta \end{cases}\,,$$

$L^0_\phi(x,y)=L_\phi(x,y)-L^1_\phi(x,y)$ and $\theta(x)=\int L^1_\phi(x,y)\,dy$. By a simple adaption of the arguments of [**LY88**, Thm. 10] we find that for any $\epsilon>0$

$$\operatorname{Tr}\gamma L_\phi \;\le\; \operatorname{Tr}\gamma\left(\theta+\varepsilon\,\chi_0\right) + \frac{\|\gamma\|}{2\varepsilon}\operatorname{Tr}\left(L^0_\phi\right)^2\,. \tag{4.47}$$

It remains to bound θ and $\operatorname{Tr}(L^0_\phi)^2$.

We begin by estimating θ. By definition, for $|x|\ge 1+\delta$ we have $L^1_\phi(x,y)=0$ and hence $\theta(x)=0$, and for $|x|<1+\delta$ we get

$$\theta(x)=C_{s,d}\int_{\substack{|x-y|<\delta\\ |y|<1+\delta}}\frac{(\phi(x)-\phi(y))^2}{|x-y|^{d+2s}}\,dy \le C\,\|\nabla\phi\|_\infty^2\int_{|x-y|<\delta}\frac{1}{|x-y|^{d+2s-2}}\,dy\,.$$

Thus, for all $x\in\mathbb{R}^d$

$$\theta(x) \;\le\; C\,\delta^{2-2s}\,\chi_\delta(x)\,. \tag{4.48}$$

Finally, we estimate $\operatorname{Tr}(L^0_\phi)^2$. The symmetry of $L^0_\phi(x,y)$ implies

$$\operatorname{Tr}\left(L^0_\phi\right)^2 \;\le\; C\iint_A\left(\frac{(\phi(x)-\phi(y))^2}{|x-y|^{d+2s}}\right)^2 dx\,dy$$

where A denotes the set $\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d \,:\, |x| < \min(|y|,1)\,,\, |x-y| \geq \delta\}$. Set $A_1 = \{(x,y) \in A \,:\, |y| \geq 2\}$ and $A_2 = \{(x,y) \in A \,:\, |y| < 2\}$. Then

$$\begin{aligned} \operatorname{Tr}\left(L^0_\phi\right)^2 &\leq C \iint_{A_1} \left(\frac{\phi^4(x)}{|x-y|^{2d+4s}}\right) dx\, dy + C\, \|\nabla\phi\|_\infty^4 \iint_{A_2} \frac{1}{|x-y|^{2d+4s-4}}\, dx\, dy \\ &\leq C\, r(\delta)\,. \end{aligned} \tag{4.49}$$

Choosing $\varepsilon = \delta^{2-2s}$ and combining (4.47) with (4.48) and (4.49) yields the claimed result. □

Proof of Proposition 4.3. We apply Lemma 4.16 with a parameter $0 < \delta_u \leq 1/2$ to be specified later. For ease of notation we write χ_u instead of χ_{δ_u}. Identities (4.45) and (4.46) and the estimate from Lemma 4.16 imply

$$\begin{aligned} \operatorname{Tr}\gamma(-\Delta)^s \geq \int_{\Omega^*} \operatorname{Tr}\gamma\left(\phi_u(-\Delta)^s\phi_u - C\delta_u^{2-2s} l(u)^{-2s}\chi_u(x)\,\chi_\Omega(x)\right) l(u)^{-d}\, du \\ - C\,\|\gamma\| \int_{\Omega^*} \delta_u^{-d+2-2s} r(\delta_u) l(u)^{-d-2s}\, du\,. \end{aligned} \tag{4.50}$$

If the supports of χ_u and $\phi_{u'}$ overlap, we have $|u-u'| \leq (3/2)l(u) + l(u')$. It follows that $l(u') - l(u) \leq \|\nabla l\|_\infty\,((3/2)l(u) + l(u'))$. Since $\|\nabla l\|_\infty \leq 1/2$ we find $l(u') \leq Cl(u)$ and $l(u)^{-1} \leq Cl(u')^{-1}$. Similarly, we get $l(u) \leq Cl(u')$. We assume now that δ_u satisfies

$$\delta_u \leq C\delta_{u'} \quad \text{if } |u-u'| \leq (3/2)(l(u)+l(u'))\,. \tag{4.51}$$

Using these locally uniform bounds on $l(u)/l(u')$ and $\delta_u/\delta_{u'}$, together with (4.13), we can deduce the pointwise bound for all $x \in \mathbb{R}^d$

$$\begin{aligned} &\int_{\Omega^*} \delta_u^{2-2s} l(u)^{-2s}\,\chi_u(x)\,\chi_\Omega(x)\,\frac{du}{l(u)^d} \\ &= \int_{\Omega^*} \delta_u^{2-2s} l(u)^{-2s}\,\chi_u(x)\,\chi_\Omega(x) \left(\int \phi^2_{u'}(x) \frac{du'}{l(u')^d}\right) \frac{du}{l(u)^d} \\ &\leq C \int_{\Omega^*} \phi_{u'}(x)\,\delta_{u'}^{2-2s} l(u')^{-2s}\,\phi_{u'}(x)\,\frac{du'}{l(u')^d}\,. \end{aligned}$$

Rewriting the last integral with u as integration variable, in view of (4.50), we find

$$\begin{aligned} \operatorname{Tr}\gamma(-\Delta)^s \geq \int_{\Omega^*} \operatorname{Tr}\gamma\left(\phi_u\left((-\Delta)^s - \frac{C\delta_u^{2-2s}}{l(u)^{2s}}\right)\phi_u\right)\frac{du}{l(u)^d} \\ - C\|\gamma\| \int_{\Omega^*} \delta_u^{-d+2-2s} r(\delta_u) \frac{du}{l(u)^{d+2s}}\,. \end{aligned}$$

By the variational principle it follows that

$$\begin{aligned} \operatorname{Tr}(H_\Omega)_- &= -\inf_{0\leq\gamma\leq 1} \operatorname{Tr}\gamma\left((-h^2\Delta)^s - 1\right) \\ &\leq \int_{\Omega^*} \operatorname{Tr}\left(\phi_u\left((-h^2\Delta)^s - 1 - Ch^{2s}\delta_u^{2-2s} l(u)^{-2s}\right)\phi_u\right)_- \frac{du}{l(u)^d} \\ &\quad + Ch^{2s} \int_{\Omega^*} \delta_u^{-d+2-2s} r(\delta_u) \frac{du}{l(u)^{d+2s}}\,. \end{aligned} \tag{4.52}$$

To bound the first term, we use Lemma 4.6. For any $u \in \mathbb{R}^d$, let ρ_u be another parameter satisfying $0 < \rho_u \leq 1/2$ and estimate

$$\begin{aligned}
&\operatorname{Tr}\left(\phi_u \left((-h^2\Delta)^s - 1 - Ch^{2s}\delta_u^{2-2s}\, l(u)^{-2s}\right)\phi_u\right)_- \\
&\quad\leq\ \operatorname{Tr}\left(\phi_u H_\Omega \phi_u\right)_- + C\operatorname{Tr}\left(\phi_u\left(\rho_u h^{2s}(-\Delta)^s - \rho_u - h^{2s}\delta_u^{2-2s} l(u)^{-2s}\right)\phi_u\right)_- \\
&\quad\leq\ \operatorname{Tr}\left(\phi_u H_\Omega \phi_u\right)_- + C\, l(u)^d (\rho_u h^{2s})^{-d/(2s)} \left(\rho_u + h^{2s}\delta_u^{2-2s} l(u)^{-2s}\right)^{1+d/(2s)}\,.
\end{aligned}$$

We pick $\rho_u = h^{2s}\,\delta_u^{2-2s}\, l(u)^{-2s}$. By (4.17) and our assumption that $\delta_u \leq 1/2$, we see that $\rho_u \leq (h/l_0)^{2s} 2^{6s-2}$. We assume now that $h \leq C^{-1} l_0$ (with a possibly large constant C) in order to guarantee that $\rho_u \leq 1/2$. With this choice we find

$$\operatorname{Tr}\left(\phi_u\left((-h^2\Delta)^s - 1 - \frac{Ch^{2s}\delta_u^{2-2s}}{l(u)^{2s}}\right)\phi_u\right)_- \ \leq\ \operatorname{Tr}\left(\phi_u H_\Omega \phi_u\right)_- + C\,\frac{\delta_u^{2-2s}\, l(u)^{d-2s}}{h^{d-2s}}\,. \tag{4.53}$$

Combining (4.52) and (4.53) we obtain

$$\operatorname{Tr}(H_\Omega)_- \leq \int_{\Omega^*} \operatorname{Tr}\left(\phi_u H_\Omega \phi_u\right)_- \frac{du}{l(u)^d} + C\int_{\Omega^*}\left(\frac{\delta_u^{2-2s}}{h^{d-2s} l(u)^{2s}} + \frac{h^{2s}\delta_u^{-d+2-2s} r(\delta_u)}{l(u)^{d+2s}}\right) du\,. \tag{4.54}$$

At this point we choose δ_u in order to minimize the second integrand, which we shall denote by I_u. We pick

$$\delta_u \ =\ \begin{cases} h/l(u) & \text{if } 1 - d/4 < s < 1 \\ (h/l(u))|\ln(l(u)/h)|^{1/(4-4s)} & \text{if } 0 < s = 1 - d/4 \\ (h/l(u))^{d/(4-4s)} & \text{if } 0 < s < 1 - d/4 \end{cases}$$

and note that $\delta_u \leq 1/2$ if $h \leq C^{-1} l_0$ by (4.17). Moreover, (4.51) is an easy consequence of the corresponding estimate for $l(u)/l(u')$. With this choice we arrive at the bounds

$$I_u \ \leq\ C\begin{cases} h^{-d+2} l(u)^{-2} & \text{if } 1 - d/4 < s < 1 \\ h^{-d+2} l(u)^{-2} |\ln(l(u)/h)|^{1/2} & \text{if } 0 < s = 1 - d/4 \\ h^{-d/2+2s} l(u)^{-d/2-2s} & \text{if } 0 < s < 1 - d/4 \end{cases}.$$

Finally, we integrate with respect to u. The same arguments that lead to (4.18) and (4.19) yield

$$\int_{\Omega^*} I_u\, du \ \leq\ C\begin{cases} h^{-d+2} l_0^{-1} & \text{if } 1 - d/4 < s < 1 \\ h^{-d+2} l_0^{-1} |\ln(l_0/h)|^{1/2} & \text{if } 0 < s = 1 - d/4 \\ h^{-d/2+2s} l_0^{-d/2-2s+1} & \text{if } 0 < s < 1 - d/4 \end{cases}.$$

This completes the proof of the lower bound with the remainder stated in Proposition 4.3.

To prove the upper bound we put

$$\gamma \ =\ \int_{\mathbb{R}^d} \phi_u\ \left(\phi_u H_\Omega \phi_u\right)_-^0\ \phi_u\, l(u)^{-d}\, du\,.$$

Obviously, $\gamma \geq 0$ holds and in view of (4.13) also $\gamma \leq 1$. The range of γ belongs to $\mathcal{H}^s(\Omega)$ and by the variational principle it follows that

$$-\operatorname{Tr}(H_\Omega)_- \ \leq\ \operatorname{Tr}\gamma H_\Omega \ =\ -\int_{\mathbb{R}^d} \operatorname{Tr}\left(\phi_u H_\Omega \phi_u\right)_- l(u)^{-d}\, du\,.$$

This yields the upper bound and finishes the proof of Proposition 4.3. □

4.6. Discussion of the second term

4.6.1. Representations for the second constant. In this section we study the second term of (4.6) in more detail. First we derive representation (4.8).

PROPOSITION 4.17. *One has*

$$L^{(2)}_{s,d} = \int_{\mathbb{R}^{d-1}} \zeta(|p'|^{-2s})\, \frac{dp'}{(2\pi)^{d-1}} = \frac{|\mathbb{S}^{d-2}|}{(2\pi)^{d-1}} \frac{2s}{(d-1)(d-1+2s)} \operatorname{Tr}\left[\chi A^{-(d-1)/2s}\chi - (A^+)^{-(d-1)/2s}\right]. \tag{4.55}$$

Here χ is the characteristic function of $\mathbb{R}_+$ and

$$\zeta(\mu) = \mu^{-1} \int_0^\infty \big(a(\mu) - a^+(t,\mu)\big)\, dt\,. \tag{4.56}$$

PROOF. The first identity follows immediately from (4.27) and (4.28). The second identity follows from the fact that

$$\int_{\mathbb{R}^{d-1}} |p'|^{2s}(E - |p'|^{-2s})_- \frac{dp'}{(2\pi)^{d-1}} = \frac{|\mathbb{S}^{d-2}|}{(2\pi)^{d-1}} \frac{2s}{(d-1)(d-1+2s)} E^{-(d-1)/2s}$$

for any $E > 0$, which by the spectral theorem implies that

$$\int_{\mathbb{R}^{d-1}} |p'|^{2s} a^+(t, |p'|^{-2s}) \frac{dp'}{(2\pi)^{d-1}} = \frac{|\mathbb{S}^{d-2}|}{(2\pi)^{d-1}} \frac{2s}{(d-1)(d-1+2s)} (A^+)^{-(d-1)/2s}(t,t)$$

and similarly for A. □

REMARK. There is another representation, namely,

$$L^{(2)}_{s,d} = \frac{2s}{d-1+2s} \int_{\mathbb{R}^{d-1}} \xi(|p'|^{-2s})\, \frac{dp'}{(2\pi)^{d-1}}\,, \tag{4.57}$$

where

$$\xi(\mu) = \int_0^\infty \big(e(\mu) - e^+(t,\mu)\big)\, dt\,. \tag{4.58}$$

Here $e(\mu)$ and $e^+(t,\mu)$ are the diagonals of the integral kernels of the spectral projectors $(A-\mu)^0_-$ and $(A^+-\mu)^0_-$, respectively. We have not shown that the integral in (4.58) converges, since we will not use (4.57) in the remainder of this paper. Identity (4.57) is an easy consequence of (4.55) and the fact that

$$a(\mu) = \int_0^\mu e(\tau)\, d\tau \qquad a^+(t,\mu) = \int_0^\mu e^+(t,\tau)\, d\tau$$

which follows by the spectral theorem from $(E-\mu)_- = \int_0^\mu (E-\tau)^0_-\, d\tau$. Representation (4.57) is natural since in terms of this function the conjectured formula for the *number* of negative eigenvalues of H_Ω takes the form

$$\iint_{T\Omega} \big(|p|^{2s} - 1\big)^0_- \frac{dpdx}{(2\pi h)^d} - \iint_{T\partial\Omega} \xi(|p'|^{-2s}) \frac{dp' d\sigma(x)}{(2\pi h)^{d-1}} + o(h^{-d+1})\,,$$

which is the analogue of well-known two-term semi-classical formulas in the local case; see, for instance, [**Ivr80a, SV97**]. The function ξ plays the role of a spectral

shift. Note that we avoided to write (4.56) and (4.58) in terms of a trace. While the integrals on the diagonals converge, we do not expect the operators to be trace class, see [**Pus08**].

REMARK. Yet another representation is

$$h^{-d+1}L^{(2)}_{s,d} = \int_0^\infty \big(H_-(x,x) - (H^+)_-(x,x)\big)\, dx_d\,.$$

(Note that the right side is independent of x'.) This follows from (4.30) and the corresponding formula for H. Using this representation one sees that our asymptotic formula coincides with the one obtained in [**BK08, BKS09**].

Finally, we refer to (4.71) in Section 4.8 for a representation of $L^{(2)}_{s,d}$ in terms of generalized eigenfunctions of A^+.

4.6.2. Positivity of the constant. Here we shall prove

PROPOSITION 4.18. *For any $0 < s < 1$ and $d \geq 2$, one has $L^{(2)}_{s,d} > 0$.*

PROOF. We use the second representation in (4.55) and the fact that

$$E^{-(d-1)/2s} = \frac{1}{\Gamma((d-1)/2s)} \int_0^\infty e^{-\beta E} \beta^{(d-1)/(2s)-1} d\beta$$

for every $E > 0$ to see that

$$\begin{aligned} L^{(2)}_{s,d} =& \frac{|\mathbb{S}^{d-2}|}{(2\pi)^{d-1}} \frac{2s}{(d-1)(d-1+2s)\Gamma((d-1)/2s)} \\ &\times \int_0^\infty \int_0^\infty \Big(e^{-\beta A}(t,t) - e^{-\beta A^+}(t,t)\Big)\, \beta^{(d-1)/(2s)-1} dt\, d\beta\,. \end{aligned}$$

By means of the Trotter product formula (see also [**BK08**] for a probabilistic derivation) it is easy to see that

$$\exp(-\beta A)(t,u) \geq \exp(-\beta A^+)(t,u)$$

for every $\beta > 0$ and every $t, u > 0$. This, together with the fact that $A \not\equiv A^+$, proves the proposition. □

4.6.3. Comparison with a fractional power of the Dirichlet Laplacian. It is well-known that the Dirichlet Laplacian $-\Delta_\Omega$ on Ω satisfies

$$\operatorname{Tr}\big(-h^2\Delta_\Omega - 1\big)_- \;=\; L^{(1)}_{1,d}\,|\Omega|\,h^{-d} - L^{(2)}_{1,d}\,|\partial\Omega|\,h^{-d+1} + o(h^{-d+1})\,,$$

see, e.g., [**FG11b**] for a proof under the sole assumption that $\partial\Omega \in C^{1,\alpha}$ for some $0 < \alpha \leq 1$. Here

$$L^{(1)}_{1,d} \;=\; \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \big(|p|^2 - 1\big)_-\, dp$$

and, by an argument similar to that in our Proposition 4.17, one can bring the second constant in the form

$$L^{(2)}_{1,d} \;=\; \frac{|\mathbb{S}^{d-2}|}{(2\pi)^{d-1}} \frac{2}{(d-1)(d+1)} \operatorname{Tr}\big[\chi B^{-(d-1)/2}\chi - (B^+)^{-(d-1)/2}\big]$$

where $B = -\frac{d^2}{dt^2} + 1$ in $L^2(\mathbb{R})$ and $B^+ = -\frac{d^2}{dt^2} + 1$ with Dirichlet boundary conditions in $L^2(\mathbb{R}_+)$. A short computation, using the fact that

$$(E^s - 1)_- = s(1-s) \int_0^1 (E - \tau)_- \tau^{s-2} \, d\tau + s(E-1)_- \,,$$

gives

$$\begin{aligned} \operatorname{Tr}\left(\left(-h^2\Delta_\Omega\right)^s - 1\right)_- &= L^{(1)}_{1,d} \, |\Omega| \, h^{-d} \, s \left((1-s) \int_0^1 \tau^{d/2+s-1} \, d\tau + 1 \right) \\ &\quad - L^{(2)}_{1,d} \, |\partial\Omega| \, h^{-d+1} \, s \left((1-s) \int_0^1 \tau^{(d-1)/2+s-1} \, d\tau + 1 \right) \\ &\quad + o(h^{-d+1}) \\ &= L^{(1)}_{s,d} \, |\Omega| \, h^{-d} - \frac{s(d+1)}{d-1+2s} \, L^{(2)}_{1,d} \, |\partial\Omega| \, h^{-d+1} + o(h^{-d+1}) \,, \end{aligned}$$

that is,

$$\tilde{L}^{(2)}_{s,d} = \frac{s(d+1)}{d-1+2s} L^{(2)}_{1,d} = \frac{|\mathbb{S}^{d-2}|}{(2\pi)^{d-1}} \frac{2s}{(d-1)(d-1+2s)} \operatorname{Tr}\left[\chi B^{-(d-1)/2} \chi - (B^+)^{-(d-1)/2}\right] .$$

Since

$$B^{-(d-1)/2}(t,t) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{(1+p^2)^{(d-1)/2}} \, dp = A^{-(d-1)/2s}(t,t)$$

we find that

$$\tilde{L}^{(2)}_{s,d} - L^{(2)}_{s,d} = \frac{|\mathbb{S}^{d-2}|}{(2\pi)^{d-1}} \frac{2s}{(d-1)(d-1+2s)} \operatorname{Tr}\left[(A^+)^{-(d-1)/2s} - (B^+)^{-(d-1)/2}\right] .$$

We now apply Lemma 4.19 below with $B = -d^2/dt^2 + 1$ in $L^2(\mathbb{R})$, with P being the projector onto $L^2(\mathbb{R}_+)$ and with $\phi(E) = E^s$. Then $\phi(PBP) = (B^+)^s$ and $P\phi(B)P = A^+$, and therefore (4.59) yields

$$(B^+)^s \geq A^+ \,.$$

Since $E \mapsto E^{-(d-1)/2s}$ is strictly monotone and since the operators A^+ and $(B^+)^s$ are not identical, we conclude that

$$\operatorname{Tr}\left((A^+)^{-(d-1)/2s} - (B^+)^{-(d-1)/2}\right) > 0 \,.$$

This shows that $\tilde{L}^{(2)}_{s,d} - L^{(2)}_{s,d} > 0$ and completes the proof of Proposition 4.2. □

In the previous proof we used

LEMMA 4.19. *Let B be a non-negative operator with $\ker B = \{0\}$ and let P be an orthogonal projection. Then for any complete Bernstein function*

$$\phi(PBP) \geq P\phi(B)P \,. \tag{4.59}$$

We recall (see, e.g., [**SSV10**]) that complete Bernstein functions (also known as operator-monotone functions) are characterized by the representation

$$\phi(E) = a + bE + \int_0^\infty \frac{E}{\tau(E+\tau)} d\rho(\tau) \tag{4.60}$$

with $a \in \mathbb{R}$, $b \geq 0$ and a positive measure ρ satisfying

$$\int_0^\infty \frac{1}{\tau(1+\tau)} d\rho(\tau) < \infty \,.$$

PROOF. We first prove that

$$B^{-1} \geq (PBP)^{-1} \,. \tag{4.61}$$

Let us write $P^\perp = 1 - P$, so that

$$B = PBP + P^\perp BP^\perp + P^\perp BP + PBP^\perp \,.$$

By the Schwarz inequality we have

$$P^\perp BP + PBP^\perp \leq \varepsilon PBP + \frac{1}{\varepsilon} P^\perp BP^\perp$$

for any $\varepsilon > 0$, and hence

$$B \leq (1+\varepsilon) PBP + \left(1 + \frac{1}{\varepsilon}\right) P^\perp BP^\perp \,.$$

Since P and $P^\perp$ are orthogonal we can invert this inequality and obtain

$$B^{-1} \geq \frac{1}{1+\varepsilon} (PBP)^{-1} + \frac{\varepsilon}{1+\varepsilon} (P^\perp BP^\perp)^{-1} \,.$$

Thus (4.61) follows by taking $\varepsilon \to 0+$.

Now if ϕ is of the form (4.60) then

$$\phi(PBP) - P\phi(B)P = -\int_0^\infty \left((PBP+\tau)^{-1} - P(B+\tau)^{-1}P \right) d\rho(\tau) \,.$$

By (4.61) with B replaced by $B + \tau$, this is non-negative. □

4.7. Equivalence of (4.2) and (4.3)

For the sake of completeness we include a short proof of

LEMMA 4.20. *Let $(\lambda_k)_{k\in\mathbb{N}}$ be a non-decreasing sequence of real numbers and let $A, C > 0$, $B, D \in \mathbb{R}$ and $-1 < a - 1 < b < a$ be related by*

$$C = A^{-1/a} a (a+1)^{-(1+a)/a} \,, \qquad D = B(A(a+1))^{-(1+b)/a} \,.$$

Then the asymptotic formula

$$\sum_{k=1}^N \lambda_k = AN^{a+1} + BN^{b+1}(1 + o(1)) \,, \quad N \to \infty \,, \tag{4.62}$$

is equivalent to

$$\sum_{k\in\mathbb{N}} (\Lambda - \lambda_k)_+ = C\Lambda^{(1+a)/a} - D\Lambda^{(1+b)/a}(1 + o(1)) \,, \quad \Lambda \to \infty \,. \tag{4.63}$$

PROOF. This lemma is a consequence of Hardy, Littlewood and Pólya's majorization theorem, which says that for any non-decreasing sequences $\{a_k\}$ and $\{b_k\}$

$$\sum_{k=1}^{N} a_k \leq \sum_{k=1}^{N} b_k \qquad \text{for all } N \in \mathbb{N} \tag{4.64}$$

is equivalent to

$$\sum_{k=1}^{\infty} (\Lambda - a_k)_+ \geq \sum_{k=1}^{\infty} (\Lambda - b_k)_+ \qquad \text{for all } \Lambda \in \mathbb{R}\,;$$

see, e.g., [**MO79**, Prop. 4.B.4]. As usual, we will denote property (4.64) by $\{a_k\} \prec \{b_k\}$.

We fix $\epsilon > 0$ and set $\beta_k^{\pm} = A(a+1)k^a + (B \pm \epsilon)(b+1)k^b$. Note that the assumptions on a and b imply

$$\sum_{k=1}^{N} \beta_k^{\pm} = AN^{a+1} + (B \pm \epsilon)N^{b+1}(1 + o(1))\,, \quad N \to \infty\,, \tag{4.65}$$

and

$$\sum_{k\in\mathbb{N}} (\Lambda - \beta_k^{\pm})_+ = \frac{aA}{(A(a+1))^{1+1/a}} \Lambda^{(1+a)/a} - \frac{B \pm \epsilon}{(A(a+1))^{(1+b)/a}} \Lambda^{(1+b)/a}(1 + o(1))\,, \tag{4.66}$$

as $\Lambda \to \infty$. First, we assume that (4.62) holds. Then, by (4.62) and (4.65) there is an $N_\epsilon \in \mathbb{N}$ such that for all $N \geq N_\epsilon$

$$\sum_{k=1}^{N} \beta_k^- \leq \sum_{k=1}^{N} \lambda_k \leq \sum_{k=1}^{N} \beta_k^+\,.$$

We put $\alpha_k^{\pm} = \beta_k^{\pm}$ for $k \geq N_\epsilon$ and $\alpha_k^+ = \max(\beta_k^+, \lambda_k)$, $\alpha_k^- = \min(\beta_k^-, \lambda_k)$ for $k < N_\epsilon$. Thus $\{\alpha_k^-\} \prec \{\lambda_k\} \prec \{\alpha_k^+\}$, and therefore

$$\sum_{k\in\mathbb{N}} (\Lambda - \alpha_k^+)_+ \leq \sum_{k\in\mathbb{N}} (\Lambda - \lambda_k)_+ \leq \sum_{k\in\mathbb{N}} (\Lambda - \alpha_k^-)_+ \quad \text{for all } \Lambda \in \mathbb{R}\,.$$

Since $\sum_{k\in\mathbb{N}} (\Lambda - \alpha_k^{\pm})_+ = \sum_{k\in\mathbb{N}} (\Lambda - \beta_k^{\pm})_+ + O(1)$, the assertion (4.63) follows from (4.66). The converse implication is proved similarly. □

4.8. The one-dimensional model operator

Here we outline the calculations that are necessary to complete the analysis of the model operator A^+ introduced in Section 4.3. The results depend on the following spectral representation of the operator A^+ found in [**Kwa10b**].

THEOREM 4.21. *For $E > 0$ let*

$$\psi(E) = (E+1)^s - 1$$

and for $\lambda > 0$ *put* $\gamma_\lambda(\xi) = 0$ *if* $0 < \xi < 1$ *and*

$$\gamma_\lambda(\xi) = \frac{1}{\pi} \frac{\lambda\, \psi'(\lambda^2)\, \sin(\pi s)\, (\xi^2-1)^s}{\psi(\lambda^2)^2 + (\xi^2-1)^s - 2\psi(\lambda^2)(\xi^2-1)\cos(\pi s)} \times \exp\left(-\frac{1}{\pi} \int_0^\infty \frac{\xi}{\xi^2+\zeta^2} \ln \frac{\psi'(\lambda^2)(\lambda^2-\zeta^2)}{\psi(\lambda^2)-\psi(\zeta^2)} d\zeta \right)$$

if $\xi \geq 1$. *Moreover, define a phase-shift*

$$\vartheta_\lambda = \frac{1}{\pi} \int_0^\infty \frac{\lambda}{\zeta^2-\lambda^2} \ln \frac{\psi'(\lambda^2)(\lambda^2-\zeta^2)}{\psi(\lambda^2)-\psi(\zeta^2)}\, d\zeta \tag{4.67}$$

and functions

$$F_\lambda(x) = \sin(\lambda x + \vartheta_\lambda) + \int_0^\infty e^{-x\xi}\, \gamma_\lambda(\xi)\, d\xi\,, \quad x > 0\,. \tag{4.68}$$

Then

$$\Phi f(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\, F_\lambda(x)\, dx$$

defines a unitary operator from $L^2(\mathbb{R}_+)$ *to* $L^2(\mathbb{R}_+)$.

This operator diagonalizes A^+ *in the sense that a function* $f \in L^2(\mathbb{R}_+)$ *is in the domain of* A^+ *if and only if* $(\lambda^2+1)^s \Phi f(\lambda)$ *is in* $L^2(\mathbb{R}_+)$, *and in this case*

$$\Phi A^+ f(\lambda) = (\lambda^2+1)^s \Phi f(\lambda)\,.$$

According to [**Kwa10b**] the Laplace transform of γ_λ is a completely monotone function bounded by one. From (4.68) it follows that for all $t \geq 0$

$$|F_\lambda(t)| \leq 2\,. \tag{4.69}$$

Theorem 4.21 states that the functions F_λ are generalized eigenfunctions of the operator A^+. Hence, we can write

$$e^+(t,u,\mu) = \frac{2}{\pi} \int_0^\infty \left((\lambda^2+1)^s - \mu\right)_-^0 F_\lambda(t) F_\lambda(u)\, d\lambda\,. \tag{4.70}$$

From (4.28) and (4.8) it follows that

$$L^{(2)}_{s,d} = \frac{4s}{(d-1+2s)(d-1)} \frac{|\mathbb{S}^{d-2}|}{(2\pi)^d} \int_0^\infty \int_0^\infty \left(1 - 2F_\lambda^2(t)\right) \left(\lambda^2+1\right)^{-(d-1)/2} d\lambda\, dt\,. \tag{4.71}$$

4.8.1. Proof of Lemma 4.9. Lemma 4.9 is an immediate consequence of (4.70). In view of (4.69) we estimate

$$\left|e^+(t,u,\mu)\right| \leq C \int_0^{(\mu^{1/s}-1)_+^{1/2}} d\lambda \leq C\mu^{1/(2s)}\,.$$

This proves the lemma.

4.8.2. Proof of Lemma 4.10. First we need the following technical result about ϑ_λ.

LEMMA 4.22. *The phase-shift ϑ_λ is monotone increasing and twice differentiable in $\lambda > 0$. It satisfies*

$$\vartheta_0 \,=\, 0 \qquad \text{and} \qquad \vartheta_\lambda \to \frac{\pi}{4}(1-s) \quad \text{as} \quad \lambda \to \infty\,.$$

The first and second derivatives are bounded and one has, as $\lambda \to \infty$,

$$\frac{d\vartheta_\lambda}{d\lambda} \,=\, \frac{d^2\vartheta_\lambda}{d\lambda^2} \,=\, O\left(\frac{1}{\lambda}\right)\,.$$

PROOF. Following [**Kwa10b**], we substitute $\zeta = \lambda z$ for $\zeta \in (0,1)$ and $\zeta = \lambda/z$ for $\zeta \in (1,\infty)$ in the definition of ϑ_λ and obtain

$$\vartheta_\lambda \,=\, \frac{1}{\pi}\int_0^1 \frac{1}{1-z^2}\ln\left(\frac{1}{z^2}\frac{\psi(\lambda^2)-\psi(\lambda^2 z^2)}{\psi(\lambda^2/z^2)-\psi(\lambda^2)}\right)dz.$$

Note that the function

$$\frac{1}{z^2}\frac{\psi(\lambda^2)-\psi(\lambda^2 z^2)}{\psi(\lambda^2/z^2)-\psi(\lambda^2)} \,=\, \frac{1}{z^2}\frac{(1+\lambda^2)^s-(1+\lambda^2 z^2)^s}{(1+\lambda^2/z^2)^s-(1+\lambda^2)^s}$$

equals 1 for $\lambda = 0$ and that for all $z \in (0,1)$ it is increasing in $\lambda > 0$ and tends to z^{2s-2} as λ tends to infinity. By Lebesgue's dominated convergence we find $\vartheta_0 = 0$ and

$$\lim_{\lambda\to\infty}\vartheta_\lambda \,=\, \frac{1}{\pi}\int_0^1 \frac{1}{1-z^2}\ln(z^{2s-2})\,dz \,=\, \frac{\pi}{4}(1-s)\,.$$

By (4.67), we also have

$$\vartheta_\lambda \,=\, \frac{1}{\pi}\int_0^\infty b_\lambda(\zeta)\,d\zeta$$

with

$$b_\lambda(\zeta) \,=\, \frac{\lambda}{\zeta^2-\lambda^2}\,\ln\left(\frac{s(1+\lambda^2)^{s-1}(\lambda^2-\zeta^2)}{(\lambda^2+1)^s-(\zeta^2+1)^s}\right)\,.$$

We remark that

$$|\partial_\lambda b_\lambda(\zeta)| \,\le\, \partial_\lambda b_\lambda(\zeta)|_{\lambda=0} \,=\, \frac{1}{\zeta^2}\ln\left(\frac{s\zeta^2}{(1-\zeta^2)^s-1}\right)\,.$$

for all $\zeta \in (0,\infty)$. Since the last expression is integrable in $\zeta \in (0,\infty)$ it follows that

$$\frac{d\vartheta_\lambda}{d\lambda} \,=\, \frac{1}{\pi}\int_0^\infty \partial_\lambda b_\lambda(\zeta)\,d\lambda$$

is bounded and, in particular, we obtain

$$\left.\frac{d\vartheta_\lambda}{d\lambda}\right|_{\lambda=0} \,=\, \frac{1}{\pi}\int_0^\infty \frac{1}{\zeta^2}\ln\left(\frac{s\zeta^2}{(1+\zeta^2)^s-1}\right)d\zeta\,. \tag{4.72}$$

Similarly, we can show existence and boundedness of the second derivative and decay of the derivatives as $\lambda \to \infty$ by explicit calculations and Lebesgue's dominated convergence theorem. □

To simplify notation we put

$$\psi_\lambda(E) \;=\; \frac{1-E/\lambda^2}{1-\psi(E)/\psi(\lambda^2)}$$

for $E>0$. Moreover, we write G_λ for the Laplace transform of γ_λ and g_λ for the Laplace transform of G_λ. According to [**Kwa10b**] we have

$$g_\lambda(t) \;=\; \frac{\lambda\cos\vartheta_\lambda + t\sin\vartheta_\lambda}{\lambda^2+t^2} - \lambda^2\sqrt{\frac{\psi'(\lambda^2)}{\psi(\lambda^2)}}\,\frac{\varphi_\lambda(t)}{\lambda^2+t^2}\,, \quad t>0\,, \tag{4.73}$$

with

$$\varphi_\lambda(t) \;=\; \exp\left(\frac{1}{\pi}\int_0^\infty \frac{t}{t^2+\zeta^2}\ln\big(\psi_\lambda(\zeta^2)\big)\,d\zeta\right).$$

To prove Lemma 4.10 we need the following properties of φ_λ.

LEMMA 4.23. *The function $t\mapsto\varphi_\lambda(t)$ is differentiable in $t>0$ and its derivative satisfies*

$$\begin{aligned}\varphi_\lambda'(0) \;&=\; o(1) \quad \text{as } \lambda\to\infty\,,\\ \varphi_\lambda'(0) \;&=\; \frac{d\vartheta_\lambda}{d\lambda}\bigg|_{\lambda=0} + O(\lambda) \quad \text{as } \lambda\to 0\,.\end{aligned}$$

PROOF. For fixed $\zeta\in(0,\infty)$ the function $\lambda\mapsto\psi_\lambda(\zeta^2)$ is non-increasing in $\lambda>0$ and tends to 1 as $\lambda\to\infty$. Moreover,

$$\frac{1}{\zeta^2}\ln\big(\psi_0(\zeta^2)\big) \;=\; \frac{1}{\zeta^2}\ln\left(\frac{s\zeta^2}{(\zeta^2+1)^s-1}\right)$$

is integrable with respect to $\zeta\in(0,\infty)$. Hence we find that

$$\varphi_\lambda'(0) \;=\; \frac{1}{\pi}\int_0^\infty \frac{1}{\zeta^2}\ln\big(\psi_\lambda(\zeta^2)\big)\,d\zeta$$

and $\varphi_\lambda'(0)=o(1)$ as $\lambda\to\infty$ by Lebesgue's theorem.

In view of (4.72)

$$\varphi_\lambda'(0)|_{\lambda=0} \;=\; \frac{1}{\pi}\int_0^\infty \frac{1}{\zeta^2}\ln\big(\psi_0(\zeta^2)\big)\,d\zeta \;=\; \frac{d\vartheta_\lambda}{d\lambda}\bigg|_{\lambda=0}\,.$$

The second claim now follows from the fact that the derivative of $\lambda\mapsto\varphi_\lambda'(0)$ is bounded. □

PROOF OF LEMMA 4.10. In view of Theorem 4.21 we can write

$$a(\mu)-a^+(t,\mu) \;=\; \frac{1}{\pi}\int_0^\infty \big((\lambda^2+1)^s-\mu\big)_-\,\big(1-2F_\lambda^2(t)\big)\,d\lambda$$

and by (4.68)

$$1-2F_\lambda(t)^2 \;=\; \cos(2\lambda t+2\vartheta_\lambda) - 4\sin(\lambda t+\vartheta_\lambda)\,G_\lambda(t) - 2G_\lambda(t)^2\,.$$

We get

$$\int_0^\infty t^\gamma|a(\mu)-a^+(t,\mu)|dt \;\le\; R_1(\mu)+R_2(\mu)$$

with

$$R_1(\mu) = \int_0^\infty t^\gamma \left| \int_0^{(\mu^{1/s}-1)_+^{1/2}} \left(\mu - (\lambda^2+1)^s\right) \cos(2\lambda t + 2\vartheta_\lambda)\, d\lambda \right| dt\,,$$

$$R_2(\mu) = \int_0^\infty t^\gamma \left| \int_0^{(\mu^{1/s}-1)_+^{1/2}} \left(\mu - (\lambda^2+1)^s\right) \left(2\sin(\lambda t + \vartheta_\lambda)\, G_\lambda(t) + G_\lambda(t)^2\right) d\lambda \right| dt\,.$$

To estimate $R_1(\mu)$ we split the integration in t and integrate over $t \in [0,1]$ first. We assume $0 < \gamma < 1$. The proof for $\gamma = 0$ follows similarly.

We write

$$\cos(2\lambda t + 2\vartheta_\lambda) \;=\; \frac{1}{2t}\,\frac{d}{d\lambda}\sin(2\lambda t + 2\vartheta_\lambda) - \frac{\cos(2\lambda t + 2\vartheta_\lambda)}{t}\,\frac{d\vartheta_\lambda}{d\lambda}$$

and insert this identity in the expression for $R_1(\mu)$. After integrating by parts in the λ-integral one can estimate

$$\int_0^1 t^\gamma \left| \int_0^{(\mu^{1/s}-1)_+^{1/2}} \left(\mu - (\lambda^2+1)^s\right) \cos(2\lambda t + 2\vartheta_\lambda)\, d\lambda \right| dt \;\le\; C\mu\left((\ln\mu)^2 + 1\right)\,.$$

To estimate the integral over $t \in [1,\infty)$ we proceed similarly. We integrate by parts twice and get

$$\int_1^\infty t^\gamma \left| \int_0^{(\mu^{1/s}-1)_+^{1/2}} \left(\mu - (\lambda^2+1)^s\right) \cos(2\lambda t + 2\vartheta_\lambda)\, d\lambda \right| dt \;\le\; C\mu(\ln\mu + 1)\,.$$

We conclude

$$R_1(\mu) \;\le\; C\mu\left((\ln\mu)^2 + 1\right)$$

and turn to estimating $R_2(\mu)$.

Since G_λ is non-negative and uniformly bounded, we have

$$R_2(\mu) \;\le\; C\int_0^{(\mu^{1/s}-1)_+^{1/2}} \left(\mu - (\lambda^2+1)^s\right) \int_0^\infty t^\gamma\, G_\lambda(t) du\, d\lambda\,. \tag{4.74}$$

Identity (4.73) implies $\int_0^\infty G_\lambda(t)dt = g_\lambda(0)$ and $\int_0^\infty tG_\lambda(t)dt = g'_\lambda(0)$. We note that

$$g_\lambda(0) \;=\; \frac{\cos\vartheta_\lambda}{\lambda} - \sqrt{\frac{\psi'(\lambda^2)}{\psi(\lambda^2)}}$$

and apply Lemma 4.22 to estimate $\int_0^\infty G_\lambda(t)dt \le C\,(\lambda \wedge \lambda^{-1})$. Moreover, by (4.73),

$$g'_\lambda(0) \;=\; \frac{\sin\vartheta_\lambda}{\lambda^2} - \sqrt{\frac{\psi'(\lambda^2)}{\psi(\lambda^2)}}\,\varphi'_\lambda(0)$$

and we apply Lemma 4.22 and Lemma 4.23 to estimate $\int_0^\infty tG_\lambda(t)dt \le C\,(1 \wedge \lambda^{-1})$. It follows that

$$\int_0^\infty t^\gamma\, G_\lambda(t)\, dt \;\le\; C\left(1 \wedge \lambda^{-1}\right)\,.$$

Thus, by (4.74), we arrive at

$$R_2(\mu) \;\le\; C\int_0^{(\mu^{1/s}-1)_+^{1/2}} \left(\mu - (\lambda^2+1)^s\right)\left(1 \wedge \lambda^{-1}\right)\, d\lambda \;\le\; C\,\mu\,(\ln\mu + 1)\,.$$

This finishes the first part of the proof of Lemma 4.10.

In order to prove the assertion about $K(t)$, we bound

$$\int_0^\infty t^\gamma \, |K(t)| \, dt \le \int_{|\xi'|<1} |\xi'|^{1+2s} \int_0^\infty t^\gamma |a^+(t|\xi'|, |\xi'|^{-2s}) - a(|\xi'|^{-2s})| \, dt \, d\xi' \, .$$

Here we also used that, since $a(\mu) = a^+(t,\mu) = 0$ for $\mu \le 1$, we can restrict the integration in the definition of K to $|\xi'| < 1$. On the other hand, from (4.26) we know that

$$\int_0^\infty t^\gamma |a^+(t\mu^{-1/2s}, \mu) - a(\mu)| \, dt \le C_\gamma \mu^{1+(\gamma+1)/(2s)} \left((\ln \mu)^2 + 1 \right) \, .$$

Combining these two bounds and using that $\gamma < 1 \le d-1$ we obtain the second part of Lemma 4.10. □

4.8.3. A remainder estimate. The following technical lemma was needed in the proof of the upper bound near the boundary.

LEMMA 4.24. *Assume that $\phi \in C_0^1(\mathbb{R}^d)$ is supported in a ball of radius $l = 1$ and that (4.14) is satisfied with $l = 1$. Then for any $\frac{1}{2} - s < \sigma < \min\{\frac{1}{2}, 1-s\}$ one has*

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| (-\Delta_{x'})^\sigma \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{d+2s}} \right| dx dy \, \le \, C \tag{4.75}$$

PROOF. For $x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$ and $y = (y', y_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$ put

$$F_{x_d,y}(x') \, = \, \frac{\left(\phi(x', x_d) - \phi(y', y_d)\right)^2}{\left(|x'-y'|^2 + (x_d - y_d)^2\right)^{d/2+s}} \, .$$

To establish (4.75) we use that

$$\left| (-\Delta_{x'})^\sigma \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{d+2s}} \right| \, \le \, C \int_{\mathbb{R}^{d-1}} \frac{|F_{x_d,y}(x') - F_{x_d,y}(z')|}{|x'-z'|^{d-1+2\sigma}} \, dz' \tag{4.76}$$

and split the integration in $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$ in four parts. First we assume that x and y are in B_1. Then we have to show that

$$\begin{aligned} &\int_{B_1} \int_{B_1} \int_{\mathbb{R}^{d-1}} \frac{|F_{x_d,y}(x') - F_{x_d,y}(z')|}{|x'-z'|^{d-1+2\sigma}} \, dz' \, dx \, dy \, = \\ &\int_{B_1} \int_{B_1} \int_{|x'-z'|<|x-y|/2} \frac{|F_{x_d,y}(x') - F_{x_d,y}(z')|}{|x'-z'|^{d-1+2\sigma}} \, dz' \, dx \, dy \\ &+ \int_{B_1} \int_{B_1} \int_{|x'-z'|\ge|x-y|/2} \frac{|F_{x_d,y}(x') - F_{x_d,y}(z')|}{|x'-z'|^{d-1+2\sigma}} \, dz' \, dx \, dy \end{aligned} \tag{4.77}$$

is bounded from above.

To estimate the first integral over $|x'-z'| < |x-y|/2$ we use that

$$F(z') - F(x') \, = \, \sum_{j=1}^{d-1} \frac{(z_j - x_j)}{|x'-z'|} \int_0^{|x'-z'|} (\partial_j F) \left(x' + t \frac{(z'-x')}{|x'-z'|} \right) \, dt \, .$$

For $j = 1, \ldots, d-1$ we have

$$(\partial_j F_{x_d,y})(x') \, = \, \frac{2(\phi(x', x_d) - \phi(y))(\partial_j \phi(x))}{|x-y|^{d+2s}} - (d+2s)(x_j - y_j) \frac{(\phi(x) - \phi(y))^2}{|x-y|^{d+2s+2}} \, ,$$

thus

$$|(\partial_j F_{x_d,y})(x')| \leq C\,|x-y|^{-d+1-2s}\,.$$

Hence, we obtain

$$\begin{aligned}&|F_{x_d,y}(z') - F_{x_d,y}(x')| \\ &\leq C|x'-z'|^\alpha \left(\int_0^{|x'-z'|} \left(\left|x' + t\frac{(z'-x')}{|x'-z'|} - y'\right|^2 + (x_d-y_d)^2\right)^\beta dt\right)^{1-\alpha}\,, \end{aligned} \tag{4.78}$$

with $0<\alpha<1$ and $\beta = (\frac{d-1}{2}+s)/(\alpha-1)$, by applying Hölder's inequality. Note that

$$\begin{aligned}\left|x'-y'+t\frac{(z'-x')}{|x'-z'|}\right|^2 + (x_d-y_d)^2 &= |x-y|^2 + t^2 + 2t\frac{(x'-y')\cdot(z'-x')}{|x'-z'|} \\ &\geq (|x-y|-t)^2\,.\end{aligned}$$

Inserting this into (4.78) we get for $|x'-z'|<|x-y|/2$

$$\begin{aligned}|F_{x_d,y}(z') - F_{x_d,y}(x')| &\leq C|x'-z'|^\alpha \left(\int_0^{|x-y|/2} (|x-y|-t)^{2\beta} dt\right)^{1-\alpha} \\ &\leq C|x'-z'|^\alpha |x-y|^{(2\beta+1)(1-\alpha)}\,,\end{aligned}$$

where $(2\beta+1)(1-\alpha) = -d-2s+2-\alpha$. We conclude that for any $2\sigma<\alpha<1$ and $\sigma<1-s$

$$\begin{aligned}&\int_{B_1}\int_{B_1}\int_{|x'-z'|<|x-y|/2} \frac{|F_{x_d,y}(x') - F_{x_d,y}(z')|}{|x'-z'|^{d-1+2\sigma}}\,dz'\,dx\,dy \\ &\leq C\int_{B_1}\int_{B_1}\int_{|x'-z'|<|x-y|/2} |x'-z'|^{-d+1-2\sigma+\alpha} dz'\,|x-y|^{-d-2s+2-\alpha}\,dx\,dy \\ &\leq C\,.\end{aligned} \tag{4.79}$$

Now we turn to the second integral in (4.77) over $|x'-z'|\geq|x-y|/2$. Since

$$0 \leq F_{x_d,y}(x') \leq |x-y|^{-d-2s+2} \tag{4.80}$$

and $\sigma<1-s$ we have

$$\int_{B_1}\int_{B_1}\int_{|x'-z'|\geq|x-y|/2} \frac{F_{x_d,y}(x')}{|x'-z'|^{d-1+2\sigma}} dz'dxdy \leq C\int_{B_1}\int_{B_1} \frac{1}{|x-y|^{d+2s-2+2\sigma}} \leq C. \tag{4.81}$$

Moreover,

$$\int_{|x'-z'|\geq|x-y|/2} \frac{F_{x_d,y}(z')}{|x'-z'|^{d-1+2\sigma}} dz' \leq C|x-y|^{1-d-2\sigma+(d-1)/p} \left(\int_{|x'-z'|\geq|x-y|/2} F^q_{x_d,y}(z')dz'\right)^{\frac{1}{q}}$$

with $\frac{1}{p}+\frac{1}{q}=1$, by Hölder's inequality. Since $\sigma > \frac{1}{2}-s$ we can choose $p > \frac{d-1}{2\sigma}$ and $q > \frac{d-1}{d+2s-2}$. By (4.80), we have

$$\begin{aligned}\left(\int_{|x'-z'|\geq|x-y|/2} F_{x_d,y}^q(z')dz'\right)^{\frac{1}{q}} &\leq C\left(\int_{\mathbb{R}^{d-1}} \left(|z'-y'|^2+(x_d-y_d)^2\right)^{-q(d/2+s-1)} dz'\right)^{\frac{1}{q}} \\ &\leq C\,|x_d-y_d|^{-d-2s+2+(d-1)/q}\,.\end{aligned}$$

It follows that

$$\begin{aligned}&\int_{B_1}\int_{B_1}\int_{|x'-z'|\geq|x-y|/2} \frac{F_{x_d,y}(z')}{|x'-z'|^{d-1+2\sigma}}\,dz'\,dx\,dy \\ &\leq C\int_{B_1}\int_{B_1} |x-y|^{-d+1-2\sigma+(d-1)/p}\,|x_d-y_d|^{-d-2s+2+(d-1)/q}\,dx\,dy \\ &\leq C\int_0^2 t^{-d-2s+2+(d-1)/q}\int_0^2 r^{d-2}\left(r^2+t^2\right)^{(-d+1-2\sigma)/2+(d-1)/(2p)}\,dr\,dt\,,\end{aligned}$$

where we substituted $t=|x_d-y_d|$ and $r=|x'-y'|$. Since $p>\frac{d-1}{2\sigma}$ and $\sigma<1-s$ we find

$$\int_{B_1}\int_{B_1}\int_{|x'-z'|\geq|x-y|/2} \frac{F_{x_d,y}(z')}{|x'-z'|^{d-1+2\sigma}}\,dz'\,dx\,dy \;\leq\; C\int_0^2 t^{1-2s-2\sigma}dt \;\leq\; C\,. \tag{4.82}$$

The estimates (4.81) and (4.82) show that

$$\int_{B_1}\int_{B_1}\int_{|x'-z'|\geq|x-y|/2} \frac{|F_{x_d,y}(x')-F_{x_d,y}(z')|}{|x'-z'|^{d-1+2\sigma}}\,dz'\,dx\,dy \;\leq\; C \tag{4.83}$$

and from (4.76), (4.79), and (4.83) it follows that

$$\int_{B_1}\int_{B_1}\left|(-\Delta_{x'})^\sigma \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{d+2s}}\right| dxdy \;\leq\; C\,.$$

The proof that the respective integrals over $B_1\times(\mathbb{R}^d\setminus B_1)$, $(\mathbb{R}^d\setminus B_1)\times B_1$, and $(\mathbb{R}^d\setminus B_1)\times(\mathbb{R}^d\setminus B_1)$ are finite is similar but easier, since $\mathrm{supp}\phi\subset B_1$ and we only have to handle one singularity at a time. □

CHAPTER 5

Geometrical versions of improved Berezin-Li-Yau inequalities

Leander Geisinger, Ari Laptev, and Timo Weidl

Abstract. We study the eigenvalues of the Dirichlet Laplace operator on an arbitrary bounded, open set in $\mathbb{R}^d$, $d \geq 2$. In particular, we derive upper bounds on Riesz means of order $\sigma \geq 3/2$, that improve the sharp Berezin inequality by a negative second term. This remainder term depends on geometric properties of the boundary of the set and reflects the correct order of growth in the semi-classical limit.
Under certain geometric conditions these results imply new lower bounds on individual eigenvalues, which improve the Li-Yau inequality.

5.1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be an open set and let $-\Delta$ denote the Dirichlet Laplace operator on $L^2(\Omega)$, defined as a self-adjoint operator with form domain $H^1_0(\Omega)$. We assume that the volume of Ω, denoted by $|\Omega|$, is finite. Then the embedding $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$ is compact and the spectrum of $-\Delta$ is discrete: It consists of positive eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$$

accumulating at infinity only.

Here we are interested in upper bounds on the Riesz means

$$\sum_k (\Lambda - \lambda_k)^\sigma_+ = \operatorname{Tr}(-\Delta - \Lambda)^\sigma_- , \quad \sigma \geq 0 ,$$

where we use the notation $x_\pm = (|x| \pm x)/2$. In 1972 Berezin proved that convex eigenvalue means are bounded uniformly by the corresponding phase-space volume, see [**Ber72b**]: For any open set $\Omega \subset \mathbb{R}^d$, $\sigma \geq 1$, and all $\Lambda > 0$

$$\operatorname{Tr}(-\Delta - \Lambda)^\sigma_- \leq \frac{1}{(2\pi)^d} \iint_{\Omega \times \mathbb{R}^d} \left(|p|^2 - \Lambda\right)^\sigma_- dp\, dx = L^{cl}_{\sigma,d} |\Omega| \, \Lambda^{\sigma + d/2} , \tag{5.1}$$

see also [**LY83**], where the problem is treated from a different point of view. Here $L^{cl}_{\sigma,d}$ denotes the so-called Lieb-Thirring constant

$$L^{cl}_{\sigma,d} = \frac{\Gamma(\sigma+1)}{(4\pi)^{d/2}\,\Gamma(\sigma+1+d/2)} .$$

The Berezin inequality (5.1) captures, in particular, the well-known asymptotic limit that goes back to Hermann Weyl [**Wey12a**]: For $\Omega \subset \mathbb{R}^d$ and $\sigma \geq 0$ the asymptotic identity

$$\operatorname{Tr}(-\Delta - \Lambda)^\sigma_- = L^{cl}_{\sigma,d} |\Omega| \, \Lambda^{\sigma + d/2} + o\left(\Lambda^{\sigma + d/2}\right) \tag{5.2}$$

holds true as $\Lambda \to \infty$. From this it follows that the Berezin inequality is sharp, in the sense that the constant in (5.1) cannot be improved. However, Hermann Weyl's work stimulated further analysis of the asymptotic formula and (5.2) was gradually improved by studying the second term, see [**CH24, Hör68, Ivr80a, Ivr80b, Mel80, SV97, Ivr98**] and references within. The precise second term was found by Ivrii [**Ivr80a, Ivr80b**]: Under appropriate conditions on the set Ω and its boundary $\partial\Omega$ the relation

$$\mathrm{Tr}(-\Delta - \Lambda)^{\sigma}_{-} \;=\; L^{cl}_{\sigma,d}\,|\Omega|\,\Lambda^{\sigma+d/2} - \frac{1}{4}\,L^{cl}_{\sigma,d-1}\,|\partial\Omega|\,\Lambda^{\sigma+(d-1)/2} + o\left(\Lambda^{\sigma+(d-1)/2}\right) \qquad (5.3)$$

holds as $\Lambda \to \infty$. To simplify notation we write $|\Omega|$ for the volume (the d-dimensional Lebesgue measure) of Ω, as well as $|\partial\Omega|$ for the $d-1$-dimensional surface area of its boundary. Since the second term of this semi-classical limit is negative, the question arises, whether the Berezin inequality (5.1) can be improved by a negative remainder term.

Recently, several such improvements have been found, initially for the discrete Laplace operator, see [**FLU02**]. The first result for the continuous Laplace operator is due to Melas [**Mel03**]. From his work follows that

$$\mathrm{Tr}\,(-\Delta - \Lambda)^{\sigma}_{-} \;\leq\; L^{cl}_{\sigma,d}\,|\Omega|\left(\Lambda - M_d\frac{|\Omega|}{I(\Omega)}\right)^{\sigma+d/2}_{+}, \quad \Lambda > 0\,, \quad \sigma \geq 1\,, \qquad (5.4)$$

where M_d is a constant depending only on the dimension and $I(\Omega)$ denotes the second moment of Ω, see also [**Ily10, Yol10, YY10**] for further generalisations. One should mention, however, that these corrections do not capture the correct order in Λ from the second term of the asymptotics (5.3). This was improved in the two-dimensional case in [**KVW09**], where it is shown that one can choose the order of the correction term arbitrarily close to the correct one.

In this paper we are interested in the case $\sigma \geq 3/2$. For these values of σ it is known, [**Wei08**], that one can strengthen the Berezin inequality for any open set $\Omega \subset \mathbb{R}^d$ with a negative remainder term reflecting the correct order in Λ in comparison to the second term of (5.3). However, since one can increase $|\partial\Omega|$ without changing the individual eigenvalues λ_k significantly, a direct analog of the first two terms of the asymptotics (5.3) cannot yield a uniform bound on the eigenvalue means. Therefore - without further conditions on Ω - any uniform improvement of (5.1) must invoke other geometric quantities.

In the result from [**Wei08**] the remainder term involves certain projections on $d-1$-dimensional hyperplanes. In [**GW10**] a universal improvement of (5.1) was found, that holds for $\sigma \geq 3/2$ with a correction term of correct order, depending only on the volume of Ω.

The proof of the aforementioned results relies on operator-valued Lieb-Thirring inequalities [**LW00**] and an inductive argument, that allows to reduce the problem to estimating traces of the one-dimensional Dirichlet Laplace operator on open intervals.

In this paper we use the same approach, but with new estimates in the one-dimensional case, in order to make the dependence on the geometry more transparent. The new one-dimensional bounds involve the distance to the boundary of the interval

in question and are related to Hardy-Lieb-Thirring inequalities for Schrödinger operators, see [**EF06**]. There it is shown that for $\sigma \geq 1/2$ and potentials $V \in L^{\sigma+1/2}(\mathbb{R}_+)$, given on the half-line $\mathbb{R}_+ = (0, \infty)$, the inequality

$$\operatorname{Tr}\left(-\frac{d^2}{dt^2} - V\right)_-^\sigma \leq L_\sigma \int_0^\infty \left(V(t) - \frac{1}{4t^2}\right)_+^{\sigma+1/2} dt$$

holds true, with a constant L_σ independent of V. For further developments see [**FLS08, Fra09**].

We start this paper by analysing the special case of the Dirichlet Laplace operator given on a finite interval $I \subset \mathbb{R}$, with the constant potential $V \equiv \Lambda$. For $\sigma \geq 1$ we establish that the estimate

$$\operatorname{Tr}\left(-\frac{d^2}{dt^2} - \Lambda\right)_-^\sigma \leq L^{cl}_{\sigma,1} \int_I \left(\Lambda - \frac{1}{4\delta(t)^2}\right)_+^{\sigma+1/2} dt$$

is valid with the sharp constant $L^{cl}_{\sigma,1}$, where $\delta(t)$ denotes the distance to the boundary of I. This is done in Section 5.2.

Then we can use results from [**LW00, Wei08**] to deduce bounds in higher dimensions: In Section 5.3 we first derive improvements of (5.1), which are valid for any open set $\Omega \subset \mathbb{R}^d$, $d \geq 2$. These improvements depend on the geometry of Ω. In view of the asymptotic result (5.3) one might expect, that this geometric dependence can be expressed in terms of the boundary of Ω. To see this, we adapt methods, which were used in [**Dav95, Dav99, HHL02**] to derive geometric versions of Hardy's inequality. Here the result gives an improved Berezin inequality with a correction term of correct order depending on geometric properties of the boundary.

If Ω is convex and smooth this dependence can be expressed only in terms of $|\Omega|$, $|\partial\Omega|$ and the curvature of the boundary. In particular the first remainder term of the estimate is very similar to the second term of the semiclassical asymptotics (5.3): it shows the same order in Λ and it depends only on the surface area of the boundary.

In Section 5.4 we return to the general case, where $\Omega \subset \mathbb{R}^d$ is not necessarily convex or smooth, and obtain lower bounds on individual eigenvalues λ_k. Under certain conditions on the geometry of Ω these results improve the estimate

$$\lambda_k \geq C_d \frac{d}{d+2} \left(\frac{k}{|\Omega|}\right)^{2/d}, \tag{5.5}$$

from [**LY83**], where C_d denotes the semi-classical constant $4\pi\,\Gamma(d/2+1)^{2/d}$.

Finally, in Section 5.5, we specialise to the two-dimensional case, where we can use the foregoing results and refined geometric considerations to further improve and generalise the inequalities. In particular, we avoid dependence on curvature, thus we do not require smoothness of the boundary.

The question whether such improved estimates can be generalised to $1 \leq \sigma < 3/2$ remains open.

5.2. One-dimensional considerations

Let us consider an open interval $I \subset \mathbb{R}$ of length $l > 0$. For $t \in I$ let

$$\delta(t) \,=\, \inf\{|t-s| \,:\, s \notin I\}$$

be the distance to the boundary. The eigenvalues of $-d^2/dt^2$ subject to Dirichlet boundary conditions at the endpoints of I are given by $\lambda_k = k^2\pi^2/l^2$. Therefore the Riesz means equal

$$\operatorname{Tr}\left(-\frac{d^2}{dt^2}-\Lambda\right)_-^\sigma \,=\, \sum_k \left(\Lambda - \frac{k^2\pi^2}{l^2}\right)_+^\sigma .$$

To find precise bounds on the Riesz means in the one-dimensional case, it suffices to analyse this sum explicitly. Our main observation is

LEMMA 5.1. *Let $I \subset \mathbb{R}$ be an open interval and $\sigma \geq 1$. Then the estimate*

$$\operatorname{Tr}\left(-\frac{d^2}{dt^2}-\Lambda\right)_-^\sigma \,\leq\, L^{cl}_{\sigma,1}\int_I \left(\Lambda - \frac{1}{4\delta(t)^2}\right)_+^{\sigma+1/2} dt\,,$$

holds true for all $\Lambda > 0$. The constant $1/4$ cannot be improved.

The remainder of this section deals with the proof of this estimate. First we need two rather technical results. The proof is elementary but not trivial and therefore will be given in Section 5.6.

LEMMA 5.2. *For all $A \geq 1/\pi$*

$$\sum_k \left(1-\frac{k^2}{A^2}\right)_+ \,\leq\, \frac{2}{3\pi}\int_1^{\pi A}\left(1-\frac{1}{s^2}\right)^{3/2} ds\,. \tag{5.6}$$

LEMMA 5.3. *Let $I \subset \mathbb{R}$ be an open interval of length $l > 0$. Then for $\sigma \geq 1$ and $c > 0$*

$$L^{cl}_{\sigma,1}\int_I \left(\Lambda - \frac{c}{\delta(t)^2}\right)_+^{\sigma+1/2} dt - \sum_k \left(\Lambda - \frac{\pi^2k^2}{l^2}\right)_+^\sigma \,=\, \left(\frac{1}{2}-\sqrt{c}\right)\Lambda^\sigma + o\left(\Lambda^\sigma\right)$$

holds as $\Lambda \to \infty$.

PROOF OF LEMMA 5.1. Note that one can always assume $I = (0,l)$, where $l > 0$ denotes the length of the interval. First we deduce the estimate for $\sigma = 1$ from Lemma 5.2. Assume $\Lambda \geq l^{-2}$ and apply Lemma 5.2 with $A = l\sqrt{\Lambda}/\pi$ to get

$$\operatorname{Tr}\left(-\frac{d^2}{dt^2}-\Lambda\right)_- \,=\, \Lambda\sum_k\left(1-\frac{\pi^2k^2}{l^2\Lambda}\right)_+ \,\leq\, \Lambda\,\frac{2}{3\pi}\int_1^{l\sqrt{\Lambda}}\left(1-\frac{1}{s^2}\right)^{3/2} ds\,.$$

Substituting $s = 2t\sqrt{\Lambda}$, we find that

$$\operatorname{Tr}\left(-\frac{d^2}{dt^2}-\Lambda\right)_- \,\leq\, \frac{4}{3\pi}\int_{1/(2\sqrt{\Lambda})}^{l/2}\left(\Lambda-\frac{1}{4t^2}\right)^{3/2} dt \,=\, 2L^{cl}_{1,1}\int_0^{l/2}\left(\Lambda-\frac{1}{4t^2}\right)_+^{3/2} dt$$

holds for all $\Lambda \geq l^{-2}$. Note that this inequality is trivially true for $0 < \Lambda < l^{-2}$, since the left hand side is zero. Finally, we use the identities

$$\int_0^{l/2} \left(\Lambda - \frac{1}{4t^2}\right)_+^{3/2} dt \;=\; \int_{l/2}^{l} \left(\Lambda - \frac{1}{4(l-t)^2}\right)_+^{3/2} dt \;=\; \frac{1}{2}\int_0^{l} \left(\Lambda - \frac{1}{4\delta(t)^2}\right)_+^{3/2} dt$$

to finish the proof for $\sigma = 1$.

To deduce the claim for $\sigma > 1$ we can apply a method from [**AL78**]. Writing

$$\sum_k (\Lambda - \lambda_k)_+^\sigma \;=\; \frac{1}{B(2,\sigma-1)} \int_0^\infty \tau^{\sigma-2} \sum_k (\Lambda - \tau - \lambda_k)_+ \, d\tau \,,$$

we estimate

$$\begin{aligned} \operatorname{Tr}\left(-\frac{d^2}{dt^2} - \Lambda\right)_-^\sigma \;&\leq\; \frac{1}{B(2,\sigma-1)} L^{cl}_{1,1} \int_I \int_0^\infty \tau^{\sigma-2} \left(\Lambda - \frac{1}{4\delta(t)^2} - \tau\right)_+^{3/2} d\tau \, dt \\ &=\; L^{cl}_{1,1} \frac{B(5/2,\sigma-1)}{B(2,\sigma-1)} \int_I \left(\Lambda - \frac{1}{4\delta(t)^2}\right)_+^{\sigma+1/2} dt \end{aligned}$$

and the result follows from the identity $L^{cl}_{1,1} B(5/2,\sigma-1) = L^{cl}_{\sigma,1} B(2,\sigma-1)$.

The claim, that the constant $1/4$ cannot be improved, follows from Lemma 5.3: For $c = 1/4$ the leading term of the asymptotics in Lemma 5.3 vanishes. For any constant $c > 1/4$ the leading term is negative, thus the estimate from Lemma 5.1 must fail in this case, for large values of Λ. □

Figure 1 illustrates the results of Lemma 5.1 and Lemma 5.3 for $l = \pi^2$ and $\sigma = 1$ with the sharp constant $c = 1/4$: The function

$$f(\Lambda) \;=\; L^{cl}_{1,1} \int_0^\pi \left(\Lambda - \frac{1}{4\,\delta(t)^2}\right)_+^{3/2} dt - \sum_k \left(\Lambda - k^2\right)_+$$

is plotted for $1 < \Lambda < 112$, so that the first ten minima are shown.

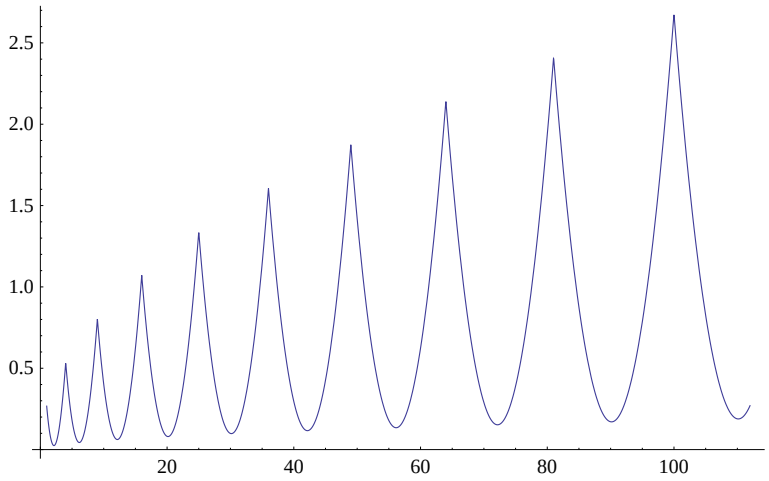

FIGURE 1. The function $f(\Lambda)$, illustrating the results from Section 5.2.

5.3. Results in higher dimensions

In this section we use the one-dimensional result to prove uniform eigenvalue estimates for the Dirichlet Laplace operator in bounded open sets in higher dimensions. These estimates - refinements of the Berezin inequality (5.1) - depend on the geometry of the set, in particular on properties of the boundary.

5.3.1. Arbitrary open sets. First we provide general estimates, valid for any open subset $\Omega \subset \mathbb{R}^d$, $d \geq 2$. Let

$$\mathbb{S}^{d-1} \;=\; \big\{x \in \mathbb{R}^d \,:\, |x| = 1\big\}$$

denote the unit-sphere in $\mathbb{R}^d$. For an arbitrary direction $u \in \mathbb{S}^{d-1}$ and $x \in \Omega$ set

$$\begin{aligned} \theta(x,u) &= \inf\{t>0 \,:\, x+tu \notin \Omega\}\,, \\ d(x,u) &= \inf\{\theta(x,u), \theta(x,-u)\} \quad \text{and} \\ l(x,u) &= \theta(x,u)+\theta(x,-u)\,. \end{aligned}$$

THEOREM 5.4. *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $u \in \mathbb{S}^{d-1}$ and $\sigma \geq 3/2$. Then for all $\Lambda > 0$ the estimate*

$$\operatorname{Tr}(-\Delta-\Lambda)^\sigma_- \;\leq\; L^{cl}_{\sigma,d} \int_\Omega \left(\Lambda - \frac{1}{4\,d(x,u)^2}\right)^{\sigma+d/2}_+ dx \tag{5.7}$$

holds true.

REMARK. Let us define

$$l_0 \;=\; \inf_{u\in\mathbb{S}^{d-1}} \sup_{x\in\Omega} l(x,u)\,. \tag{5.8}$$

Then Theorem 5.4 implies the following improvement of Melas-type: For $\sigma \geq 3/2$ and all $\Lambda > 0$ the estimate

$$\operatorname{Tr}(-\Delta-\Lambda)^\sigma_- \;\leq\; L^{cl}_{\sigma,d}\,|\Omega| \left(\Lambda - \frac{1}{l_0^2}\right)^{\sigma+d/2}_+ \tag{5.9}$$

holds. In convex domains l_0 is the minimal width of the domain, see [**BF48**]. In this case $1/l_0^2$ is bounded from below by a multiple of $|\Omega|^{-2/d}$, [**YB61**], while no such bound holds for the improving term $|\Omega|/I(\Omega)$ in Melas' inequality (5.4).

The proof of Theorem 5.4 relies on a lifting technique, which was introduced in [**Lap97**], see also [**LW00, ELW04, Wei08, FL08**] for further developments and applications.

PROOF OF THEOREM 5.4. We apply the argument used in [**Wei08**] to reduce the problem to one-dimensional estimates. Fix a Cartesian coordinate system in $\mathbb{R}^d$, such that the given direction u corresponds to the vector $(0,\dots,0,1)$.

For $x \in \mathbb{R}^d$ write $x = (x',t) \in \mathbb{R}^{d-1} \times \mathbb{R}$ and let ∇' and $-\Delta'$ denote the gradient and the Laplace operator in the first $d-1$ dimensions. Each section $\Omega(x') = \{t \in \mathbb{R} \,:\, (x',t) \in \Omega\}$ consists of at most countably many open intervals $J_k(x') \subset \mathbb{R}$, $k = 1,\dots,N(x') \leq \infty$.

We consider the quadratic form of $-\Delta - \Lambda$ on functions φ from the form core $C_0^\infty(\Omega)$ and write

$$\|\nabla\varphi\|^2_{L^2(\Omega)} - \Lambda \|\varphi\|^2_{L^2(\Omega)} \;=\; \|\nabla'\varphi\|^2_{L^2(\Omega)} + \int_{\mathbb{R}^{d-1}} dx' \int_{\Omega(x')} \left(|\partial_t\varphi|^2 - \Lambda|\varphi|^2\right) dt\,.$$

The functions $\varphi(x',\cdot)$ satisfy Dirichlet boundary conditions at the endpoints of each interval $J_k(x')$ forming $\Omega(x')$, hence

$$\begin{aligned}\int_{\Omega(x')} \left(|\partial_t\varphi|^2 - \Lambda|\varphi|^2\right) dt \;&=\; \sum_{k=1}^{N(x')} \int_{J_k(x')} \left(|\partial_t\varphi|^2 - \Lambda|\varphi|^2\right) dt \\ &\geq\; -\sum_{k=1}^{N(x')} \langle V_k(x')\varphi(x',\cdot), \varphi(x',\cdot)\rangle_{L^2(J_k(x'))}\,,\end{aligned}$$

where the bounded, non-negative operators $V_k(x',\Lambda) = (-\partial_t^2 - \Lambda)_-$ are the negative parts of the Sturm-Liouville operators $-\partial_t^2 - \Lambda$ with Dirichlet boundary conditions on $J_k(x')$. Let

$$V(x',\Lambda) = \bigoplus_{k=1}^{N(x')} V_k(x',\Lambda)$$

be the negative part of $-\partial_t^2 - \Lambda$ on $\Omega(x')$ subject to Dirichlet boundary conditions on the endpoints of each interval $J_k(x')$, $k = 1,\ldots,N(x')$, that is on $\partial\Omega(x')$. Then

$$\int_{\Omega(x')} \left(|\partial_t\varphi|^2 - \Lambda|\varphi|^2\right) dt \;\geq\; -\langle V(x',\Lambda)\varphi(x',\cdot), \varphi(x',\cdot)\rangle_{L^2(\Omega(x'))}$$

and consequently

$$\|\nabla\varphi\|^2_{L^2(\Omega)} - \Lambda \|\varphi\|^2_{L^2(\Omega)} \;\geq\; \|\nabla'\varphi\|^2_{L^2(\Omega)} - \int_{\mathbb{R}^{d-1}} dx' \,\langle V\varphi(x',\cdot), \varphi(x',\cdot)\rangle_{L^2(\Omega(x'))}.$$

Now we can extend this quadratic form by zero to $C_0^\infty\left(\mathbb{R}^d \setminus \partial\Omega\right)$, which is a form core for $(-\Delta_{\mathbb{R}^d\setminus\Omega}) \oplus (-\Delta_\Omega - \Lambda)$. This operator corresponds to the left hand side of the equality above, while the semi-bounded form on the right hand side is closed on the larger domain $H^1\left(\mathbb{R}^{d-1}, L^2(\mathbb{R})\right)$, where it corresponds to the operator

$$-\Delta' \otimes \mathbb{I} - V(x',\Lambda) \quad \text{on} \quad L^2\left(\mathbb{R}^{d-1}, L^2(\mathbb{R})\right)\,. \tag{5.10}$$

Due to the positivity of $-\Delta_{\mathbb{R}^d\setminus\Omega}$ we can use the variational principle to deduce that for any $\sigma \geq 0$

$$\begin{aligned}\operatorname{Tr}(-\Delta_\Omega - \Lambda)^\sigma_- \;&=\; \operatorname{Tr}\left(\left(-\Delta_{\mathbb{R}^d\setminus\Omega}\right) \oplus (-\Delta_\Omega - \Lambda)\right)^\sigma_- \\ &\leq\; \operatorname{Tr}(-\Delta' \otimes \mathbb{I} - V(x',\Lambda))^\sigma_-\,.\end{aligned}$$

Now we apply sharp Lieb-Thirring inequalities [**LW00**] to the Schrödinger operator (5.10) with the operator-valued potential $-V(x',\Lambda)$ and obtain that for $\sigma \geq 3/2$

$$\operatorname{Tr}(-\Delta_\Omega - \Lambda)^\sigma_- \;\leq\; L^{cl}_{\sigma,d-1} \int_{\mathbb{R}^{d-1}} \operatorname{Tr} V(x',\Lambda)^{\sigma+(d-1)/2}\, dx'\,. \tag{5.11}$$

To estimate the trace of the one-dimensional differential operator $V(x',\Lambda)$ we apply Lemma 5.1. Our choice of coordinate system implies that for $x = (x',t) \in J_k(x')$

the distance of t to the boundary of the interval $J_k(x')$ is given by $d(x,u)$. Hence, Lemma 5.1 implies

$$\begin{aligned} \mathrm{Tr} V(x',\Lambda)^{\sigma+(d-1)/2} &= \sum_{k=1}^{N(x')} \mathrm{Tr}\left(-\frac{d^2}{dt^2}\Big|_{J_k(x')} - \Lambda \right)_-^{\sigma+(d-1)/2} \\ &\leq L^{cl}_{\sigma+(d-1)/2,1} \int_{\Omega(x')} \left(\Lambda - \frac{1}{4\, d((x',t),u)} \right)_+^{\sigma+d/2} dt \end{aligned}$$

and the result follows from (5.11) and the identity $L^{cl}_{\sigma,d-1}\, L^{cl}_{\sigma+(d-1)/2,1} = L^{cl}_{\sigma,d}$. □

We proceed to analysing the geometric properties of (5.7). Note that the left hand side of (5.7) is independent of the choice of direction $u \in \mathbb{S}^{d-1}$, while the right hand side depends on u and therefore on the geometry of Ω. For a given set Ω one can minimise the right hand side in $u \in \mathbb{S}^{d-1}$. The result, however, depends on the geometry of Ω in a rather tricky way. In order to make this geometric dependence more transparent, we average the right hand side of (5.7) over $u \in \mathbb{S}^{d-1}$. Even though the resulting bound is - in general - not as precise as (5.7), it allows a more appropriate geometric interpretation.

To analyse the effect of the boundary, one would like to estimate $d(x,u)$ in terms of the distance to the boundary. See [**Dav95, Dav99, HHL02**], where this approach is used to derive geometrical versions of Hardy's inequality. To avoid complications that arise, for example, if the complement of Ω contains isolated points, we use slightly different notions: For $x \in \Omega$ let

$$\Omega(x) \,=\, \{y \in \Omega \,:\, x + t(y-x) \in \Omega\,,\ \forall\, t \in [0,1]\}$$

be the part of Ω that "can be seen" from x and let

$$\delta(x) \,=\, \inf\Big\{ |y-x| \,:\, y \notin \overline{\Omega(x)} \Big\}$$

denote the distance to the exterior of $\Omega(x)$.

For fixed $\varepsilon > 0$ put

$$A_\varepsilon(x) \,=\, \Big\{ a \in \mathbb{R}^d \setminus \overline{\Omega(x)} \,:\, |x-a| < \delta(x) + \varepsilon \Big\}$$

and for $a \in A_\varepsilon(x)$ set $B_x(a) = \{y \in \mathbb{R}^d \,:\, |y-a| < |x-a|\}$ and

$$\rho_a(x) \,=\, \frac{|B_x(a) \setminus \overline{\Omega(x)}|}{\omega_d |x-a|^d}\,,$$

where ω_d denotes the volume of the unit ball in $\mathbb{R}^d$. To get a result, independent of a and ε, set

$$\rho(x) \,=\, \inf_{\varepsilon>0} \sup_{a \in A_\varepsilon(x)} \rho_a(x)\,.$$

Note that $\mathbb{R}^d \setminus \overline{\Omega(x)}$ is open, hence $\rho_a(x) > 0$ and $\rho(x) > 0$ hold for $x \in \Omega$. Finally, we define

$$M_\Omega(\Lambda) \,=\, \int_{R_\Omega(\Lambda)} \rho(x)\, dx\,,$$

where $R_\Omega(\Lambda) \subset \Omega$ denotes the set $\{x \in \Omega : \delta(x) < 1/(4\sqrt{\Lambda})\}$. The main result of this section allows a geometric interpretation of the remainder term:

THEOREM 5.5. *Let $\Omega \subset \mathbb{R}^d$ be an open set with finite volume and $\sigma \geq 3/2$. Then for all $\Lambda > 0$ we have*

$$\operatorname{Tr}(-\Delta - \Lambda)_-^\sigma \;\leq\; L^{cl}_{\sigma,d}\,|\Omega|\Lambda^{\sigma+d/2} - L^{cl}_{\sigma,d}\,2^{-d+1}\,\Lambda^{\sigma+d/2}\,M_\Omega(\Lambda)\,. \tag{5.12}$$

The function $\rho(x)$ depends on the behaviour of the boundary close to $x \in \Omega$. For example, $\rho(x)$ is small close to a cusp. On the other hand $\rho(x)$ is larger than $1/2$ in a convex domain. By definition, the function $M_\Omega(\Lambda)$ gives an average of this behaviour over $R_\Omega(\Lambda)$, which is like a tube of width $1/(4\sqrt{\Lambda})$ around the boundary.

Note that $M_\Omega(\Lambda)$ tends to zero as Λ tends to infinity. This decay in Λ is of the order $(\delta_M - d)/2$, where δ_M denotes the interior Minkowski dimension of the boundary, see e.g. [**Lap91, FV93, FLV95**] for definition and examples. If $d-1 \leq \delta_M < d$ and if the upper Minkowski content of the boundary is finite, then the second term of the asymptotic limit of the Riesz means equals $O\left(\Lambda^{\sigma+\delta_M/2}\right)$ as $\Lambda \to \infty$, see [**Lap91**]. Therefore the remainder term in (5.12) reflects the correct order of growth in the asymptotic limit.

In particular, if the dimension of the boundary equals $d-1$, we find

$$M_\Omega(\Lambda) = |\partial\Omega|\,\Lambda^{-1/2} + o(\Lambda^{-1/2})$$

as $\Lambda \to \infty$ and the second term in (5.12) is in close correspondence with the asymptotic formula (5.3).

PROOF OF THEOREM 5.5. We start from the result of Theorem 5.4 and average over all directions to get

$$\operatorname{Tr}(-\Delta - \Lambda)_-^\sigma \;\leq\; L^{cl}_{\sigma,d}\,\Lambda^{\sigma+d/2} \int_\Omega \int_{\mathbb{S}^{d-1}} \left(1 - \frac{1}{4\,\Lambda\,d(x,u)^2}\right)_+^{\sigma+d/2} d\nu(u)\,dx\,, \tag{5.13}$$

where $d\nu(u)$ denotes the normed measure on $\mathbb{S}^{d-1}$.

For $x \in \Omega$ and $a \notin \overline{\Omega(x)}$ let $\Theta(x,a) \subset \mathbb{S}^{d-1}$ be the subset of all directions $u \in \mathbb{S}^{d-1}$, satisfying $x + su \in B_x(a) \setminus \overline{\Omega(x)}$ for some $s > 0$. For such s we have

$$s \;\leq\; 2\,|x-a|\,. \tag{5.14}$$

By definition of $\rho_a(x)$ and $\Theta(x,a)$ we find

$$\rho_a(x)\,\omega_d|x-a|^d \;=\; |B_x(a) \setminus \overline{\Omega(x)}| \;\leq\; \int_{\Theta(x,a)} d\nu(u)\,\omega_d(2|x-a|)^d\,,$$

hence

$$\int_{\Theta(x,a)} d\nu(u) \;\geq\; 2^{-d}\,\rho_a(x)\,. \tag{5.15}$$

Using (5.14) we also see that for $u \in \Theta(x,a)$ the estimate $d(x,u) \leq s \leq 2\,|x-a|$ holds.

Now fix $\Lambda > 0$ and choose $0 < \varepsilon < 1/(4\sqrt{\Lambda})$ and $a \in A_\varepsilon(x)$. By definition of $A_\varepsilon(x)$ it follows that for all $u \in \Theta(x,a)$

$$d(x,u) \;\leq\; 2|x-a| \;<\; 2(\delta(x)+\varepsilon)\,. \tag{5.16}$$

The set $\Theta(x,a)$ must be contained in one hemisphere of $\mathbb{S}^{d-1}$ which we denote by $\mathbb{S}^{d-1}_+$. Using that $d(x,u)=d(x,-u)$ we estimate

$$\begin{aligned}\int_{\mathbb{S}^{d-1}} \left(1-\frac{1}{4\Lambda d(x,u)^2}\right)_+^{\sigma+d/2} d\nu(u) \;&=\; 2\int_{\mathbb{S}^{d-1}_+} \left(1-\frac{1}{4\Lambda d(x,u)^2}\right)_+^{\sigma+d/2} d\nu(u)\\ &\leq\; 1-2\int_{\left\{u\in\mathbb{S}^{d-1}_+\,:\,d(x,u)\leq 1/(2\sqrt{\Lambda})\right\}} d\nu(u)\,.\end{aligned}$$

Assume that $\delta(x)\leq 1/(4\sqrt{\Lambda})-\varepsilon$. From (5.16) it follows that

$$\Theta(x,a)\subset\left\{u\in\mathbb{S}^{d-1}_+\;:\;d(x,u)\leq 1/(2\sqrt{\Lambda})\right\}\,,$$

hence, using (5.15), we conclude

$$\int_{\mathbb{S}^{d-1}} \left(1-\frac{1}{4\Lambda d(x,u)^2}\right)_+^{\sigma+d/2} d\nu(u) \;\leq\; 1-2\int_{\Theta(x,a)} d\nu(u) \;\leq\; 1-2^{1-d}\rho_a(x)\,.$$

Since $a\in A_\varepsilon(x)$ was arbitrary we arrive at

$$\int_{\mathbb{S}^{d-1}} \left(1-\frac{1}{4\Lambda d(x,u)^2}\right)_+^{\sigma+d/2} d\nu(u) \;\leq\; 1-2^{1-d}\rho(x)\,,$$

for all $x\in\Omega$ with $\delta(x)\leq 1/(4\sqrt{\Lambda})-\varepsilon$ and we can take the limit $\varepsilon\to 0$.

It follows that

$$\int_\Omega\int_{\mathbb{S}^{d-1}} \left(1-\frac{1}{4\Lambda d(x,u)^2}\right)_+^{\sigma+d/2} d\nu(u)\,dx \;\leq\; |\Omega|-2^{1-d}\int_{\left\{x\in\Omega\,:\,\delta(x)<1/(4\sqrt{\Lambda})\right\}}\rho(x)\,dx$$

and inserting this into (5.13) yields the claimed result. □

5.3.2. Convex domains. If $\Omega\subset\mathbb{R}^d$ is convex, we have $\Omega(x)=\Omega$ and

$$\rho(x)\;>\;1/2 \tag{5.17}$$

for all $x\in\Omega$. Thus we can simplify the remainder term, by estimating $M_\Omega(\Lambda)$.

Corollary 5.6. *Let $\Omega\subset\mathbb{R}^d$ be a bounded, convex domain and let $\Omega_t=\{x\in\Omega\,:\,\delta(x)>t\}$ be the inner parallel set of Ω. Then for $\sigma\geq 3/2$ and all $\Lambda>0$ we have*

$$\operatorname{Tr}\left(-\Delta-\Lambda\right)_-^\sigma \;\leq\; L^{cl}_{\sigma,d}\,|\Omega|\Lambda^{\sigma+d/2}-L^{cl}_{\sigma,d}\,2^{-d-2}\,|\partial\Omega_{1/(4\sqrt{\Lambda})}|\,\Lambda^{\sigma+(d-1)/2}\,.$$

Proof. Inserting (5.17) into the definition of $M_\Omega(\Lambda)$ yields

$$M_\Omega(\Lambda)\;>\;\frac{1}{2}\int_{R_\Omega(\Lambda)} dx \;=\; \frac{1}{2}\int_{\left\{x\in\Omega\,:\,\delta(x)<1/(4\sqrt{\Lambda})\right\}} dx \;=\; \frac{1}{2}\int_0^{1/(4\sqrt{\Lambda})} |\partial\Omega_t|\,dt\,. \tag{5.18}$$

In view of (5.12) estimating

$$\int_0^{1/(4\sqrt{\Lambda})} |\partial\Omega_t|\,dt \;\geq\; \frac{1}{4\sqrt{\Lambda}}|\partial\Omega_{1/(4\sqrt{\Lambda})}|$$

proves the claim. □

If more is known about the geometry of Ω we can refine the estimate.

COROLLARY 5.7. *Let $\Omega \subset \mathbb{R}^d$ be a bounded, convex domain with smooth boundary and assume that the curvature of $\partial\Omega$ is bounded from above by $1/R$. Then for $\sigma \geq 3/2$ and all $\Lambda > 0$ we have*

$$\operatorname{Tr}(-\Delta - \Lambda)_-^\sigma \;\leq\; L^{cl}_{\sigma,d}\,|\Omega|\Lambda^{\sigma+d/2} - L^{cl}_{\sigma,d}\,2^{-d-2}\,|\partial\Omega|\,\Lambda^{\sigma+(d-1)/2}\int_0^1 \left(1 - \frac{d-1}{4R\sqrt{\Lambda}}\,s\right)_+ ds\,.$$

Note that one can estimate the integral in the remainder term to get the simplified bound

$$\operatorname{Tr}(-\Delta - \Lambda)_-^\sigma \;\leq\; L^{cl}_{\sigma,d}\,|\Omega|\Lambda^{\sigma+d/2} - L^{cl}_{\sigma,d}\,2^{-d-2}\,|\partial\Omega|\,\Lambda^{\sigma+(d-1)/2}\left(1 - \frac{d-1}{8R\sqrt{\Lambda}}\right)_+ .$$

PROOF OF COROLLARY 5.7. To estimate $M_\Omega(\Lambda)$ we can use Steiner's Theorem, see [**Gug77, vdB84b**], namely

$$|\partial\Omega_t| \;\geq\; \left(1 - \frac{d-1}{R}\,t\right)_+ |\partial\Omega|\,. \tag{5.19}$$

From (5.18) it follows that

$$M_\Omega(\Lambda) \;>\; \frac{1}{2}|\partial\Omega|\int_0^{1/(4\sqrt{\Lambda})}\left(1 - \frac{d-1}{R}\,t\right)_+ dt \;=\; \frac{|\partial\Omega|}{8\sqrt{\Lambda}}\int_0^1 \left(1 - \frac{d-1}{4R\sqrt{\Lambda}}\,s\right)_+ ds\,.$$

Inserting this into (5.12) completes the proof. □

5.4. Lower bounds on individual eigenvalues

In order to further estimate the remainder terms, in particular to show that the remainder is negative for all $\Lambda \geq \lambda_1(\Omega)$ one needs suitable bounds on the ground state $\lambda_1(\Omega)$. We point out the following consequence of the proof of Theorem 5.4 which might be of independent interest.

COROLLARY 5.8. *For any open set $\Omega \subset \mathbb{R}^d$ the estimate*

$$\lambda_1(\Omega) \;\geq\; \frac{\pi^2}{l_0^2}$$

holds, where l_0 is given in (5.8).

REMARK. One can compare this bound with

$$\lambda_1(\Omega) \;\geq\; \inf_{x\in\Omega}\left(\frac{d}{4m(x)^2}\right),$$

see [**Dav95**]. Here $m(x)^{-2}$ denotes the average of $d(x,u)^{-2}$ over all directions $u \in \mathbb{S}^{d-1}$. The relation between these bounds depends on the geometry of Ω. In particular, the bound from Corollary 5.8 is stronger, if the width of Ω is small along an arbitrary direction $u \in \mathbb{S}^{d-1}$ such that l_0 is small enough.

PROOF OF COROLLARY 5.8. Fix $\varepsilon > 0$ and choose a direction $u_0 \in \mathbb{S}^{d-1}$, such that $\sup_{x\in\Omega} l(x,u_0) < l_0 + \varepsilon$. We write $x = (x',t) \in \mathbb{R}^{d-1} \times \mathbb{R}$, where the t-axes is

chosen in the direction of u_0. Let us recall inequality (5.11) from the proof of Theorem 5.4: For any $\sigma \geq 3/2$

$$\operatorname{Tr}(-\Delta - \Lambda)_-^\sigma \;\leq\; L^{cl}_{\sigma,d-1} \int_{\mathbb{R}^{d-1}} \operatorname{Tr} V(x', \Lambda)^{\sigma+(d-1)/2}\, dx' \,,$$

where $V(x', \Lambda)$ denotes the negative part of the operator $-\partial_t^2 - \Lambda$ on

$$\Omega(x') = \bigcup_{k=1}^{N(x')} J_k(x')$$

with Dirichlet boundary conditions at the endpoints of each interval $J_k(x')$. This inequality can be rewritten as

$$\operatorname{Tr}(-\Delta - \Lambda)_-^\sigma \;\leq\; L^{cl}_{\sigma,d-1} \int_{\mathbb{R}^{d-1}} \sum_{k=1}^{N(x')} \sum_{j\in\mathbb{N}} \left(\Lambda - \frac{\pi^2\, j^2}{|J_k(x')|^2} \right)_+^{\sigma+(d-1)/2} dx' \,.$$

Our choice of coordinate system implies $|J_k(x')| \leq \sup_{x\in\Omega} l(x, u_0) < l_0 + \varepsilon$ for all $k = 1, \ldots, N(x')$ and all $x' \in \mathbb{R}^{d-1}$. It follows that the right hand side of the inequality above is zero for all $\Lambda \leq \pi^2/(l_0 + \varepsilon)^2$. Thus by taking the limit $\varepsilon \to 0$ we find

$$\sum_{n\in\mathbb{N}} (\Lambda - \lambda_n)_+^\sigma \;=\; \operatorname{Tr}(-\Delta - \Lambda)_-^\sigma \;=\; 0 \,,$$

for all $\Lambda \leq \pi^2/l_0^2$ and $\lambda_1 \geq \pi^2/l_0^2$ follows. □

From (5.9) we obtain similar bounds on higher eigenvalues using a method introduced in [**Lap97**].

Corollary 5.9. *For any open set $\Omega \subset \mathbb{R}^d$ with finite volume and any $k \in \mathbb{N}$ the estimate*

$$\lambda_k(\Omega) \;\geq\; C_d \left(\frac{12}{\pi} \right)^{1/d} \frac{d}{(d+3)^{1+1/d}} \left(\frac{\Gamma\left((d+3)/2\right)}{\Gamma(d/2+1)} \right)^{2/d} \frac{k^{2/d}}{|\Omega|^{2/d}} + \frac{1}{l_0^2}$$

holds, with

$$C_d \;=\; 4\pi\, \Gamma\left(d/2+1\right)^{2/d} \,.$$

Proof. Let $N(\Lambda) = \operatorname{Tr}(-\Delta - \Lambda)_-^0$ denote the counting function of the eigenvalues below $\Lambda > 0$. In [**Lap97**] it is shown that for $\sigma > 0$, and all $\Lambda > 0$, $\tau > 0$

$$N(\Lambda) \;\leq\; (\tau\Lambda)^{-\sigma}\, \operatorname{Tr}(-\Delta - (1+\tau)\Lambda)_-^\sigma \,. \tag{5.20}$$

If we apply this inequality with $\sigma = 3/2$, we can use (5.9) to estimate

$$N(\Lambda) \;\leq\; L^{cl}_{3/2,d}\, |\Omega|\, \Lambda^{d/2}\, \frac{(1+\tau)^{(d+3)/2}}{\tau^{3/2}} \left(1 - \frac{1}{\Lambda(1+\tau) l_0^2} \right)_+^{(d+3)/2} \,.$$

Minimising the right hand side in $\tau > 0$ yields $\tau_{\min} = 3(\Lambda l_0^2 - 1)/(d\Lambda l_0^2)$ and inserting this we find

$$N(\Lambda) \;\leq\; L^{cl}_{3/2,d}\, |\Omega|\, \frac{(d+3)^{(d+3)/2}}{3^{3/2}\, d^{d/2}} \left(\Lambda - \frac{1}{l_0^2} \right)_+^{d/2} \,.$$

This is equivalent to the claimed result. □

REMARK. Applying the same method to (5.1) with $\sigma = 1$ we recover the Li-Yau inequality (5.5). In the proof of Corollary 5.9 we have to start from $\sigma = 3/2$, therefore the result is not strong enough to improve (5.5) in general, i. e. the coefficient of $k^{2/d}/|\Omega|^{2/d}$ in Corollary 5.9 is less than $C_d \cdot d/(d+2)$ for all $d \geq 2$. However, one gets an improvement of (5.5) for low eigenvalues whenever l_0 is small.

5.5. Further improvements in dimension 2

In this section we further improve Corollary 5.7 and generalise it to a large class of bounded convex domains $\Omega \subset \mathbb{R}^2$. Here we do not require smoothness, therefore we cannot use (5.19) to estimate inner parallels of the boundary. To find a suitable substitute let w denote the minimal width of Ω and note that for l_0 given in (5.8) the identity

$$w \,=\, l_0$$

holds true, see e.g. [**BF48**]. In the remainder of this section we assume that for all $t > 0$

$$|\partial\Omega_t| \,\geq\, \left(1 - \frac{3t}{w}\right)_+ |\partial\Omega| \,. \tag{5.21}$$

This is true for a large class of convex domains, including the circle, regular polygons and arbitrary triangles and equality holds for the equilateral triangle. Actually we conjecture that (5.21) holds true for all bounded convex domains in $\mathbb{R}^2$.

Furthermore we need a lower bound on the ground state. From Corollary 5.8 we obtain that for all convex domains $\Omega \subset \mathbb{R}^2$

$$\lambda_1(\Omega) \,\geq\, \frac{\pi^2}{w^2} \tag{5.22}$$

holds. One should mention that this follows immediately from the variational principle and the fact that $\lambda_1(\Omega) = \pi^2/w^2$ if Ω is the infinite strip with width w. Moreover, the same estimate can be obtained from the inequality $\lambda_1(\Omega) \geq \pi^2/(4r_{in}^2)$, see [**Oss77**], where r_{in} is the inradius of Ω.

Using (5.21) and (5.22) we can derive a more precise version of Theorem 5.5, where the correction term depends only on $|\partial\Omega|$ and is apart from that independent of the geometry of Ω.

THEOREM 5.10. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, convex domain, satisfying (5.21). Then for $\sigma \geq 3/2$ we have*

$$\begin{aligned} &\mathrm{Tr}\,(-\Delta - \Lambda)_-^\sigma \,=\, 0 && \text{if } \Lambda \leq \pi^2/w^2 \text{ and} \\ &\mathrm{Tr}\,(-\Delta - \Lambda)_-^\sigma \,\leq\, L^{cl}_{\sigma,2}\,|\Omega|\,\Lambda^{\sigma+1} - C_{co}\,L^{cl}_{\sigma,1}\,|\partial\Omega|\,\Lambda^{\sigma+1/2} && \text{if } \Lambda > \pi^2/w^2 \,, \end{aligned}$$

with a constant

$$C_{co} \,\geq\, \frac{11}{9\pi^2} - \frac{3}{20\pi^4} - \frac{2}{5\pi^2}\ln\left(\frac{4\pi}{3}\right) \,>\, 0.0642\,.$$

PROOF. The first claim follows directly from (5.22), thus we can assume $\Lambda > \pi^2/w^2$. First we prove the result for $\sigma = 3/2$. Again we can start from (5.13) and we need to estimate $d(x,u)$ in terms of $\delta(x)$, which is just the distance to the boundary, since Ω is convex.

Fix $x \in \Omega$. Since Ω is convex and smooth we can choose $u_0 \in \mathbb{S}^{d-1}$, such that $d(x,u_0) = \delta(x)$. We can assume $u_0 = (1,0,\dots,0)$ and put $\mathbb{S}^{d-1}_+ = \big\{u \in \mathbb{R}^d \,:\, u_1 > 0\big\}$.

Let a be the intersection point of the semi-axes $\{x + tu_0, t > 0\}$ with $\partial\Omega$ and for arbitrary $u \in \mathbb{S}^{d-1}_+$ let b_u be the intersection point of $\{x + tu, t > 0\}$ with the plane through a, orthogonal to u_0. We find $d(x,u) \leq |x - b_u|$ and if θ_u denotes the angle between u_0 and u, we find

$$d(x,u) \,\leq\, |x - b_u| \,=\, \frac{|x-a|}{\cos\theta_u} \,=\, \frac{\delta(x)}{\cos\theta_u}\,.$$

Using (5.13) and taking into account that $d(x,u) = d(x,-u)$ we get

$$\mathrm{Tr}\,(-\Delta - \Lambda)^{3/2}_- \,\leq\, L^{cl}_{3/2,2}\,\Lambda^{5/2}\int_\Omega \frac{2}{\pi}\int_{\theta_0}^{\pi/2}\left(1 - \frac{\cos^2(\theta)}{4\Lambda\delta(x)^2}\right)^{5/2} d\theta\,dx\,,$$

where $\theta_0 = 0$ if $\delta(x) \geq 1/(2\sqrt{\Lambda})$ and $\theta_0 = \arccos(2\delta(x)\sqrt{\Lambda})$ if $\delta(x) < 1/(2\sqrt{\Lambda})$. To obtain a simple bound one could estimate

$$\left(1 - \frac{\cos^2(\theta)}{4\Lambda\delta(x)^2}\right)^{5/2} \,\leq\, 1 - \frac{\cos^2(\theta)}{4\Lambda\delta(x)^2}\,.$$

However, to get a result as precise as possible we use

$$\left(1 - \frac{\cos^2(\theta)}{4\Lambda\delta(x)^2}\right)^{5/2} \,\leq\, 1 - \frac{\cos^2(\theta)}{2\Lambda\delta(x)^2} + \frac{\cos^4(\theta)}{16\Lambda^2\delta(x)^4}$$

and get

$$\mathrm{Tr}\,(-\Delta - \Lambda)^{3/2}_- \,\leq\, \frac{\Lambda^{5/2}}{5\pi^2}\int_\Omega\int_{\theta_0}^{\pi/2}\left(1 - \frac{\cos^2(\theta)}{2\Lambda\delta(x)^2} + \frac{\cos^4(\theta)}{16\Lambda^2\delta(x)^4}\right) d\theta\,dx\,. \tag{5.23}$$

Set $u_0 = \min(1, 2\delta(x)\sqrt{\Lambda})$ and calculate

$$\begin{aligned}&\int_{\theta_0}^{\pi/2}\left(1 - \frac{\cos^2(\theta)}{2\Lambda\delta(x)^2} + \frac{\cos^4(\theta)}{16\Lambda^2\delta(x)^4}\right) d\theta\\ &\quad=\; \frac{\pi}{2} - \theta_0 - \frac{\arcsin(u_0) - u_0\sqrt{1-u_0^2}}{4\Lambda\delta(x)^2} + \frac{3\arcsin(u_0) - u_0\,(2u_0^2+3)\,\sqrt{1-u_0^2}}{128\Lambda^2\delta(x)^4}\,.\end{aligned}$$

Inserting this back into (5.23) yields

$$\mathrm{Tr}\,(-\Delta - \Lambda)^{3/2}_- \,\leq\, \frac{1}{10\pi}\,|\Omega|\,\Lambda^{5/2} - \Lambda^{5/2}\,(I_1 + I_2)\,, \tag{5.24}$$

where

$$I_1 \,=\, \int_{\{\delta(x)\geq 1/(2\sqrt{\Lambda})\}} \frac{1}{10\pi}\left(\frac{1}{4\Lambda\delta(x)^2} - \frac{3}{128\Lambda^2\delta(x)^4}\right) dx$$

and

$$\begin{aligned}I_2 \;=\; &\int_{\{\delta(x)<1/(2\sqrt{\Lambda})\}} \frac{1}{5\pi^2}\left(\arccos(2\sqrt{\Lambda}\delta(x)) + \frac{\arcsin(2\sqrt{\Lambda}\delta(x))}{4\Lambda\delta(x)^2} - \frac{\sqrt{1-4\Lambda\delta(x)^2}}{2\sqrt{\Lambda}\delta(x)}\right.\\ &\left. - \frac{3\arcsin(2\sqrt{\Lambda}\delta(x))}{128\Lambda^2\delta(x)^4} + \frac{(8\Lambda\delta(x)^2+3)\sqrt{1-4\Lambda\delta(x)^2}}{64\Lambda^{3/2}\delta(x)^3}\right) dx\,.\end{aligned}$$

First we turn to

$$I_1 \;=\; \frac{1}{10\pi}\int_{1/(2\sqrt{\Lambda})}^{\infty} |\partial\Omega_t| \left(\frac{1}{4\Lambda t^2} - \frac{3}{128\Lambda^2 t^4}\right) dt\,.$$

Note that the term in brackets is positive, thus after substituting $s = 2\sqrt{\Lambda}t$ we can use (5.21) and $\Lambda > \pi^2/w^2$ to obtain

$$\begin{aligned} I_1 \;&\geq\; \frac{1}{20\pi}\frac{|\partial\Omega|}{\sqrt{\Lambda}} \int_1^{\infty} \left(1 - \frac{3s}{2\pi}\right)_+ \left(\frac{1}{s^2} - \frac{3}{8s^4}\right) ds \\ &=\; \frac{1}{20\pi}\frac{|\partial\Omega|}{\sqrt{\Lambda}} \left(\frac{7}{8} - \frac{39}{32\pi} - \frac{27}{128\pi^3} - \frac{3}{2\pi}\ln\left(\frac{2\pi}{3}\right)\right)\,. \end{aligned}$$

Similarly we can treat I_2 and get

$$I_2 \;\geq\; \frac{1}{10\pi^2}\frac{|\partial\Omega|}{\sqrt{\Lambda}} \left(\frac{557}{192} - \frac{7\pi}{16} - \frac{3}{2\pi}\int_0^1 \frac{\arcsin(s)}{s} ds\right)\,.$$

In view of $L^{cl}_{3/2,1} = 3/16$ we can write

$$I_1 + I_2 \;\geq\; L^{cl}_{3/2,1}\frac{|\partial\Omega|}{\sqrt{\Lambda}} \left(\frac{11}{9\pi^2} - \frac{3}{20\pi^4} - \frac{2}{5\pi^2}\ln\left(\frac{2\pi}{3}\right) - \frac{2}{5\pi^2}\ln(2)\right)$$

and inserting this into (5.24) yields the claim in the case $\sigma = 3/2$.

To prove the estimate for $\sigma > 3/2$ we again refer to [**AL78**] and use the identity

$$\operatorname{Tr}\left(-\Delta - \Lambda\right)_-^{\sigma} \;=\; \frac{1}{B(\sigma - 3/2, 5/2)}\int_0^{\infty} \tau^{\sigma - 5/2} \operatorname{Tr}\left(-\Delta - (\Lambda - \tau)\right)_-^{3/2} d\tau\,,$$

from which the general result follows. □

Now we can apply the same arguments that lead to Corollary 5.9 to derive lower bounds on individual eigenvalues.

Corollary 5.11. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, convex domain, satisfying (5.21). Then for $k \in \mathbb{N}$ and any $\alpha \in (0,1)$ the estimate*

$$\frac{\lambda_k(\Omega)}{1-\alpha} \geq 10\pi\alpha^{3/2}\frac{k}{|\Omega|} + \frac{15\pi C_{co}}{8}\frac{|\partial\Omega|}{|\Omega|}\sqrt{10\pi\alpha^{3/2}\frac{k}{|\Omega|} + \frac{225\pi^2 C_{co}^2}{256}\frac{|\partial\Omega|^2}{|\Omega|^2}} + \frac{225\pi^2 C_{co}^2}{128}\frac{|\partial\Omega|^2}{|\Omega|^2}$$

holds, with the constant C_{co} given in Theorem 5.10.

Proof. Applying (5.20) and Theorem 5.10 with $\sigma = 3/2$ yields

$$N(\Lambda) \;\leq\; L^{cl}_{3/2,2}\,|\Omega|\,\Lambda\,\frac{(1+\tau)^{5/2}}{\tau^{3/2}} - C_{co}\,L^{cl}_{3/2,1}\,|\partial\Omega|\,\sqrt{\Lambda}\,\frac{(1+\tau)^2}{\tau^{3/2}}$$

for any $\tau > 0$ and $\Lambda > \pi^2/w^2$. With $\tau = \alpha/(1-\alpha)$, $\alpha \in (0,1)$, this is equivalent to the claimed estimate. □

Remark. Given a fixed ratio $|\partial\Omega|/|\Omega|$ one can optimise the foregoing estimate in $\alpha \in (0,1)$, depending on $k \in \mathbb{N}$. As mentioned in the remark after Corollary 5.9, the result cannot improve the Li-Yau inequality (5.5) in general, since we have to apply (5.20) with $\sigma = 3/2$ instead of $\sigma = 1$. However, the estimates obtained from Corollary 5.11 are stronger than (5.5) for low eigenvalues and the improvements depend on the ratio $|\partial\Omega|/|\Omega|$.

In particular, one can use the isoperimetric inequality, namely that

$$|\partial\Omega| \geq 2(\pi|\Omega|)^{1/2}$$

for all $\Omega \subset \mathbb{R}^2$, to derive general improvements of the Li-Yau inequality (5.5) for low eigenvalues. From (5.5) we get

$$\lambda_k(\Omega) \geq \frac{2\pi k}{|\Omega|},$$

and one can supplement these estimates with the Krahn-Szegö inequality [**Hen06**]

$$\lambda_2(\Omega) \geq \frac{2\pi j_{0,1}^2}{|\Omega|}. \tag{5.25}$$

Optimising the estimate of Corollary 5.11 with $|\partial\Omega| = 2(\pi|\Omega|)^{1/2}$, we find that for any convex domain, satisfying (5.21),

$$\begin{aligned}
&\lambda_6(\Omega) > \frac{40.50}{|\Omega|} > \frac{12\pi}{|\Omega|} > \frac{2\pi j_{0,1}^2}{|\Omega|}, \\
&\lambda_7(\Omega) > \frac{46.74}{|\Omega|} > \frac{14\pi}{|\Omega|}, \quad \dots, \qquad \lambda_{23}(\Omega) > \frac{144.58}{|\Omega|} > \frac{46\pi}{|\Omega|}.
\end{aligned}$$

In this way we can improve (5.5) and (5.25) in convex domains for all eigenvalues $\lambda_k(\Omega)$ with $6 \leq k \leq 23$.

Finally let us make a remark about the square $Q_l = (0,l) \times (0,l) \subset \mathbb{R}^2$, $l > 0$. Using the methods introduced in section 5.2 one can establish the following two-dimensional version of Lemma 5.1: Choose a coordinate system $(x_1, x_2) \in \mathbb{R}^2$ with axes parallel to the sides of the square and for $x \in Q_l$ put

$$\delta(x_i) = \min(x_i, l - x_i), \quad i = 1, 2.$$

Then for $\sigma \geq 1$ and all $\Lambda > 0$ the estimate

$$\begin{aligned}
\operatorname{Tr}(-\Delta - \Lambda)_-^\sigma &= \sum_{m,n\in\mathbb{N}} \left(\Lambda - \frac{\pi^2}{l^2}\left(n^2 + m^2\right)\right)_+^\sigma \\
&\leq L_{\sigma,2}^{cl} \int_0^l \int_0^l \left(\Lambda - C_{sq}\left(\frac{1}{\delta(x_1)} + \frac{1}{\delta(x_2)}\right)^2\right)_+^{\sigma+1} dx_1\, dx_2
\end{aligned}$$

holds with a constant $C_{sq} > 1/10$.

5.6. Proof of Lemma 5.2 and Lemma 5.3

5.6.1. Proof of Lemma 5.2. For $A \in \mathbb{R}$ let $\overline{A}$ and $\tilde{A}$ denote the integer and fractional part of A respectively. Then we can calculate

$$\begin{aligned}
\sum_k \left(1 - \frac{k^2}{A^2}\right)_+ &= \sum_{k=1}^{\overline{A}} \left(1 - \frac{k^2}{A^2}\right) = \overline{A} - \frac{1}{A^2}\left(\frac{\overline{A}^3}{3} + \frac{\overline{A}^2}{2} + \frac{\overline{A}}{6}\right) \\
&= \frac{2A}{3} - \frac{1}{2} - \frac{1}{6A} + \tilde{A}(1-\tilde{A})\frac{1}{A} + \tilde{A}\left(1 - 3\tilde{A} + 2\tilde{A}^2\right)\frac{1}{6A^2}.
\end{aligned}$$

From $0 \le \tilde{A} < 1$ we conclude $\tilde{A}(1-\tilde{A}) \le 1/4$ and $\tilde{A}\left(1 - 3\tilde{A} + 2\tilde{A}^2\right) \le \sqrt{3}/18$ and we get

$$\sum_k \left(1 - \frac{k^2}{A^2}\right)_+ \le \frac{2A}{3} - \frac{1}{2} + \frac{1}{12A} + \frac{\sqrt{3}}{108A^2}\,. \tag{5.26}$$

To estimate the right hand side of (5.6) note that

$$\int \left(1 - \frac{1}{s^2}\right)^{3/2} ds \;=\; \left(1 + \frac{1}{2s^2}\right)\sqrt{s^2-1} + \frac{3}{2}\arctan\left(\frac{1}{\sqrt{s^2-1}}\right),$$

thus

$$\frac{2}{3\pi}\int_1^{\pi A}\left(1 - \frac{1}{s^2}\right)^{3/2} ds \;=\; \frac{2\pi^2A^2+1}{3\pi^2 A}\frac{\sqrt{\pi^2A^2-1}}{\pi A} + \frac{1}{\pi}\arctan\left(\frac{1}{\sqrt{\pi^2A^2-1}}\right) - \frac{1}{2}\,.$$

Now we can insert the elementary estimates

$$\arctan\left(\frac{1}{\sqrt{\pi^2A^2-1}}\right) \ge \frac{1}{\pi A} \qquad \text{and} \qquad \frac{\sqrt{\pi^2A^2-1}}{\pi A} \ge 1 - \frac{1}{2\pi^2A^2} - \frac{a}{A^4}$$

both valid for $A \ge 2$, where we write $a = 16 - 2/\pi^2 - 8\sqrt{4\pi^2-1}/\pi$ for simplicity. We get

$$\frac{2}{3\pi}\int_1^{\pi A}\left(1 - \frac{1}{s^2}\right)^{3/2} ds \;\ge\; \frac{2A}{3} - \frac{1}{2} + \frac{1}{\pi^2 A} - \frac{1}{6\pi^4A^3} - \frac{a}{3}\left(\frac{2}{A^3} + \frac{1}{\pi^2A^5}\right) \tag{5.27}$$

for all $A \ge 2$. From (5.26) and (5.27) we deduce that (5.6) holds true for all $A \ge 2$, since

$$\left(\frac{1}{\pi^2} - \frac{1}{12}\right)A^4 - \frac{\sqrt{3}}{108}A^3 - \left(\frac{1}{6\pi^4} + \frac{2a}{3}\right)A^2 - \frac{a}{3\pi^2} \;\ge\; 0$$

for all $A \ge 2$.

Note that (5.6) is trivial for $1/\pi \le A \le 1$, since the left hand side equals zero. The remaining case $1 \le A \le 2$ can be checked by hand.

5.6.2. Proof of Lemma 5.3. We assume $I = (0, l)$, substitute $t = s\sqrt{c/\Lambda}$ and write

$$\int_0^l \left(\Lambda - \frac{c}{\delta(t)^2}\right)_+^{\sigma+1/2} dt \;=\; 2\sqrt{c}\,\Lambda^\sigma \int_1^{l\sqrt{\Lambda}/(2\sqrt{c})}\left(1 - \frac{1}{s^2}\right)^{\sigma+1/2} ds\,.$$

The claim of the Lemma follows, if we show that

$$2\sqrt{c}L^{cl}_{\sigma,1}\int_1^{l\sqrt{\Lambda}/(2\sqrt{c})}\left(1 - \frac{1}{s^2}\right)^{\sigma+1/2} ds - \sum_k \left(1 - \frac{\pi^2k^2}{\Lambda\, l^2}\right)_+^\sigma \;=\; \frac{1}{2} - \sqrt{c} + o\,(1)$$

as $\Lambda \to \infty$. With $A = l\sqrt{\Lambda}/\pi$ this is equivalent to

$$2\sqrt{c}\,L^{cl}_{\sigma,1}\int_1^{\pi A/(2\sqrt{c})}\left(1 - \frac{1}{s^2}\right)^{\sigma+1/2} ds - \sum_k \left(1 - \frac{k^2}{A^2}\right)_+^\sigma \;=\; \frac{1}{2} - \sqrt{c} + o\,(1) \tag{5.28}$$

as $A \to \infty$.

It is easy to see that

$$\sum_k \left(1 - \frac{k^2}{A^2}\right)_+^\sigma \;=\; \frac{A}{2}B\left(\sigma + 1, \frac{1}{2}\right) - \frac{1}{2} + o(1)$$

as $A \to \infty$. Moreover, we claim

$$\int_1^{\pi A/(2\sqrt{c})} \left(1 - \frac{1}{s^2}\right)^{\sigma+1/2} ds \;=\; \frac{\pi A}{2\sqrt{c}} + \frac{1}{2} B\left(-\frac{1}{2}, \sigma + \frac{3}{2}\right) + o(1) \tag{5.29}$$

as $A \to \infty$ and (5.28) follows from $L^{cl}_{\sigma,1} = B(\sigma+1, 1/2)/(2\pi)$, if we can establish (5.29). Let us write

$$\left(1 - \frac{1}{s^2}\right)^m \;=\; \sum_{k\geq 0} (-1)^k \binom{m}{k} s^{-2k} \;=\; \sum_{k\geq 0} \binom{k-m-1}{k} s^{-2k}$$

for $m \geq 1$ and note that the sum is finite if $m \in \mathbb{N}$, while the sum converges uniformly on $s \in [1, \infty)$ if $m \notin \mathbb{N}$. Hence we have

$$\int_1^y \left(1 - \frac{1}{s^2}\right)^m ds \;=\; y + \sum_{k\geq 0} \binom{k-m-1}{k} \frac{1}{2k-1} + o(1)$$

as $y \to \infty$. Using that

$$\sum_{k\geq 0} \binom{k-m-1}{k} \frac{1}{2k-1} \;=\; \frac{1}{2} B\left(-\frac{1}{2}, m+1\right)$$

we obtain

$$\int_1^y \left(1 - \frac{1}{s^2}\right)^m ds \;=\; y + \frac{1}{2} B\left(-\frac{1}{2}, m+1\right) + o(1)$$

as $y \to \infty$, which is equivalent to (5.29). This finishes the proof of Lemma 5.3.

CHAPTER 6

Universal bounds for traces of the Dirichlet Laplace operator

Leander Geisinger and Timo Weidl

Abstract. We derive upper bounds for the trace of the heat kernel $Z(t)$ of the Dirichlet Laplace operator in an open set $\Omega \subset \mathbb{R}^d$, $d \geq 2$. In domains of finite volume the result improves an inequality of Kac. Using the same methods we give bounds on $Z(t)$ in domains of infinite volume.
For domains of finite volume the bound on $Z(t)$ decays exponentially as t tends to infinity and it contains the sharp first term and a correction term reflecting the properties of the short time asymptotics of $Z(t)$. To prove the result we employ refined Berezin-Li-Yau inequalities for eigenvalue means.

6.1. Introduction and main results

Let Ω be an open subset of $\mathbb{R}^d$, $d \geq 2$. Consider the Laplace operator $-\Delta_\Omega$ on Ω subject to Dirichlet boundary conditions defined in the form sense on the form domain $H_0^1(\Omega)$. If the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, e.g. if the volume of Ω is finite, the spectrum of $-\Delta_\Omega$ is discrete and consists of a monotone sequence of positive eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$ accumulating at infinity. We count these eigenvalues according to their multiplicity.

The main goal of this paper is to derive some new universal upper bounds for the trace of the heat kernel

$$Z(t) \;=\; \operatorname{Tr}\left(e^{+\Delta_\Omega t}\right) \;=\; \sum_k e^{-\lambda_k t}$$

which are valid for arbitrary open sets $\Omega \subset \mathbb{R}^d$ with finite volume $|\Omega|$ and for all $t > 0$. The first and most fundamental bound of this type is due to M. Kac [**Kac51**]. He proved that for any open domain $\Omega \subset \mathbb{R}^d$ and all $t > 0$ the estimate

$$Z(t) \;\leq\; \frac{|\Omega|}{(4\pi t)^{d/2}} \tag{6.1}$$

holds true. This bound is sharp in the sense that it reflects the leading term of the short time asymptotics of the function $Z(t)$, see [**Min54, Kac66**]

$$Z(t) \;\sim\; \frac{|\Omega|}{(4\pi t)^{d/2}} \quad \text{as} \quad t \to 0+\,. \tag{6.2}$$

Several improvements of (6.1) are known, e.g. see [**vdB84b, FLV95, Dav85, Dav89, Sim83, vdB84a**] and further references therein. For example, M. van den

Berg proved in [**vdB87**], that if Ω is a connected region with a smooth boundary $\partial\Omega$ and a surface area $|\partial\Omega|$, then

$$\left| Z(t) - \frac{|\Omega|}{(4\pi t)^{d/2}} + \frac{|\partial\Omega|}{4\,(4\pi t)^{(d-1)/2}} \right| \;\leq\; \frac{d^4}{\pi^{d/2}} \frac{|\Omega|}{t^{d/2-1} R^2}, \quad t > 0,$$

where the constant R depends on properties of $\partial\Omega$. This estimate contains even the second term of the short time asymptotic expansion of $Z(t)$, see [**MS67, Smi81, BC86**] and [**Bro93**]. Most of these results are based on a probabilistic approach and implement local estimates for the heat kernel. Therefore one has to impose appropriate conditions on Ω and on its boundary $\partial\Omega$.

We use a different approach based on some refined spectral estimates for the Riesz means

$$R_\sigma(\Lambda) \;=\; \mathrm{Tr}\,(-\Delta_\Omega - \Lambda)_-^\sigma \;=\; \sum_k (\Lambda - \lambda_k)_+^\sigma, \quad \Lambda > 0.$$

For these objects the fundamental bounds are given by the Berezin-Li-Yau inequalities

$$R_\sigma(\Lambda) \;\leq\; L^{cl}_{\sigma,d}\,|\Omega|\,\Lambda^{\sigma+d/2}, \quad \sigma \geq 1\,,\ \Lambda > 0, \tag{6.3}$$

where

$$L^{cl}_{\sigma,d} \;=\; \frac{\Gamma(\sigma+1)}{(4\pi)^{d/2}\Gamma\,(\sigma+d/2+1)}.$$

This result is sharp as well, in the sense that the bound captures the first term of the high energy asymptotics

$$R_\sigma(\Lambda) \;=\; L^{cl}_{\sigma,d}\,|\Omega|\,\Lambda^{\sigma+d/2} + o\left(\Lambda^{\sigma+d/2}\right) \quad \text{as} \quad \Lambda \to +\infty.$$

Via Laplace transformation - and reversely via Tauberian theorems - this asymptotic formula is closely connected with (6.2). On the level of uniform inequalities one can deduce Kac' inequality on $Z(t)$ from Berezin-Li-Yau bounds. Reversely, to recover sharp Berezin-Li-Yau bounds from Kac' inequality one needs some additional information. For example, in [**HH07**] Harrell and Hermi formally deduced Berezin-Li-Yau bounds for $\sigma \geq 2$ from Kac' inequality based on a monotonicity result by Harrell and Stubbe.[1] Similar arguments fail for $\sigma < 2$.

While both (6.3) and (6.1) are sharp in the sense that they capture the main asymptotic behaviour and therefore constants in these inequalities cannot be improved, one can expect that more subtle bounds might invoke additional lower order correction terms. Indeed, we know that under certain conditions on the geometry of Ω the asymptotics

$$R_\sigma(\Lambda) \;=\; L^{cl}_{\sigma,d}\,|\Omega|\,\Lambda^{\sigma+d/2} - \frac{1}{4}\,L^{cl}_{\sigma,d-1}\;|\partial\Omega|\;\Lambda^{\sigma+(d-1)/2} + o\left(\Lambda^{\sigma+(d-1)/2}\right)$$

holds true as $\Lambda \to \infty$, see [**Ivr98**]. Recently there have been several results on semiclassical inequalities improving (6.3) with negative correction terms of lower order, reflecting the effect of the second term of the asymptotics, see [**Mel03, Wei08, KVW09**], and [**FLU02**] for discrete operators.

[1] One should mention, that in fact, due to Weyl's asymptotic law, the monotonicity result implies sharp Berezin-Li-Yau bounds for $\sigma \geq 2$ on its own.

Let us first point out a result of Melas. In [**Mel03**] he effectively showed that[2]

$$R_1(\Lambda) \;\leq\; L^{cl}_{1,d}\,|\Omega|\left(\Lambda - M_d\frac{|\Omega|}{I(\Omega)}\right)_+^{1+d/2} \tag{6.4}$$

holds for $\Lambda > 0$. Again applying Laplace transformation Harrell and Hermi deduced an improvement of Kac' inequality [**HH07**]

$$Z(t) \;\leq\; \frac{|\Omega|}{(4\pi t)^{d/2}}\,\exp\left(-M_d\frac{|\Omega|}{I(\Omega)}\,t\right), \tag{6.5}$$

where $I(\Omega) = \min_{a\in\mathbb{R}^d}\int_\Omega |x-a|^2\,dx$ and M_d is a constant depending only on the dimension. This improvement holds true for all $t > 0$ and any open set Ω with finite volume - without any conditions on the boundary $\partial\Omega$. These authors conjecture also that (6.5) can be improved to

$$Z(t) \;\leq\; \frac{|\Omega|}{(4\pi t)^{d/2}}\,\exp\left(-\frac{t}{|\Omega|^{2/d}}\right) \tag{6.6}$$

for all $t > 0$ and all open sets Ω of finite volume. Asymptotic considerations show that this conjecture is plausible for small t as well as for large t. However, one should mention, that neither the correction term in (6.4) is of the expected order for high energies, nor is the improvement (6.5) or even the conjecture (6.6) of correct order for small $t > 0$.

To derive universal bounds on $Z(t)$ like (6.6) depending only on the volume of Ω and not including any further geometrical information one can employ an isoperimetric result due to Luttinger [**Lut73**]. He shows that Steiner-symmetrization of an open set Ω increases the trace of the heat kernel in this set. Thus for any open set $\Omega \subset \mathbb{R}^d$ with finite volume the inequality

$$Z(t) \;\leq\; Z^*(t) \tag{6.7}$$

holds true for all $t > 0$, where $Z^*(t)$ denotes the trace of the heat kernel in the ball $B \subset \mathbb{R}^d$ with the same volume as Ω.

Here we prove a refined universal bound on $Z(t)$ reflecting the correct asymptotic properties. To this end we shall follow the approach in [**Wei08**]. There a Berezin-Li-Yau type bound on R_σ for $\sigma \geq 3/2$ with a correction term of the expected order has been found, see inequality (6.18) below. Using the same method we prove a refined Berezin-Li-Yau inequality, see Proposition 6.5, that gives rise to an improved bound on $Z(t)$ applicable to any open set Ω with finite volume. This bound decays exponentially as t tends to infinity and contains a negative correction term of correct order as t tends to zero.

Moreover, we can consider unbounded domains $\Omega \subset \mathbb{R}^d$ with infinite volume. While the results of Kac and Luttinger must fail for such domains, we show that under appropriate conditions on Ω our refined inequalities can still be applied and give sharp upper bounds.

[2]This inequality is in fact the Legendre transform of Melas' result.

This paper is structured as follows: In Section 6.2 we state the main results. Then in Section 6.3 we provide some auxiliary notation and auxiliary results including improved Berezin-Li-Yau inequalities. In Section 6.4 we prove Theorem 6.1 and compare this result to other bounds on $Z(t)$. In Section 6.5 we discuss some applications to unbounded domains and domains with infinite volume. Finally, in Section 6.6 we apply a method by M. Aizenman and E. H. Lieb [**AL78**] to the results from Section 6.3 in order to prove refined bounds on the eigenvalue means $R_\sigma(\Lambda)$.

We thank Rupert L. Frank for helpful discussions and in particular for indicating the result of J. M. Luttinger.

6.2. Main Results

To state the main result we have to introduce some auxiliary notation. Let $\Gamma(z)$ be the usual Gamma-function and by

$$\tilde{\Gamma}(z, s_1, s_2) = \int_{s_1}^{s_2} s^{z-1} e^{-s} ds / \Gamma(z)$$

we denote normed incomplete Gamma-functions. If $s_1 = 0$ we write $\tilde{\Gamma}(z, s) = \tilde{\Gamma}(z, 0, s)$ and $\hat{\Gamma}(z, s) = 1 - \tilde{\Gamma}(z, s) = \tilde{\Gamma}(z, s, +\infty)$. Note that for $a > 0$ we have

$$\tilde{\Gamma}(a, t) = \frac{t^a}{a\,\Gamma(a)} + O\left(t^{a+1}\right) \quad \text{as} \quad t \to 0+ \text{ and} \tag{6.8}$$

$$\hat{\Gamma}(a, t) = \frac{t^{a-1}}{\Gamma(a)} \exp(-t) + O\left(t^{a-2} \exp(-t)\right) \quad \text{as} \quad t \to \infty. \tag{6.9}$$

Furthermore, let $B(\alpha, \beta)$ be the usual Beta-function. By

$$\tilde{B}(s_1, s_2, \alpha, \beta) = \int_{s_1}^{s_2} s^{\alpha-1} (1-s)^{\beta-1} ds / B(\alpha, \beta)$$

we denote normed incomplete Beta-functions and for $s_1 = 0$ we write in short $\tilde{B}(s, \alpha, \beta) = \tilde{B}(0, s, \alpha, \beta)$ and $\hat{B}(s, \alpha, \beta) = 1 - \tilde{B}(s, \alpha, \beta) = \tilde{B}(s, 1, \alpha, \beta)$. Note that for $\alpha, \beta > 0$ we have

$$B(0, t, \alpha, \beta) = \frac{1}{\alpha}\, t^\alpha + O\left(t^{\alpha+1}\right) \quad \text{as} \quad t \to 0+ \,. \tag{6.10}$$

Next we remark that in view of the isoperimetric inequality by Rayleigh, Faber and Krahn [**Fab23, Kra25**] on the ground state λ_1 we can always choose

$$\tilde{\lambda} = \frac{\pi\, j_{d/2-1,1}^2}{\Gamma\left(d/2+1\right)^{2/d} |\Omega|^{2/d}} \;\leq\; \lambda_1 \tag{6.11}$$

as a lower bound on λ_1, where $j_{k,1}$ denotes the first zero of the Bessel-function J_k.

For $r \in \mathbb{R}$ put $(r)_+ = \max\{r, 0\}$ and for $d \in \mathbb{N}$ let

$$\sigma_d \;=\; \begin{cases} 5/2 & \text{if} \quad d = 2 \\ 2 & \text{if} \quad d = 3 \\ 3/2 & \text{if} \quad d \geq 4 \end{cases} \,. \tag{6.12}$$

Finally, let $\Omega \subset \mathbb{R}^d$ be an arbitrary open set with finite volume $|\Omega|$.

THEOREM 6.1. *Let $\lambda \in [\tilde{\lambda}, \lambda_1]$. For any $t > 0$ the bound*

$$Z(t) \;\leq\; \frac{|\Omega|}{(4\pi t)^{d/2}} \hat{\Gamma}\left(\sigma_d + d/2 + 1, \lambda t\right) \;-\; (R(t,\lambda))_+$$

holds true with a remainder term

$$R(t,\lambda) \;=\; c_{1,d}\, \frac{|\Omega|^{(d-1)/d}}{(4\pi t)^{(d-1)/2}}\, \hat{\Gamma}\left(\sigma_d + \frac{d+1}{2}, \lambda t\right) - c_{2,d}\, \frac{|\Omega|^{(d-3)/d}}{(4\pi t)^{(d-3)/2}}\, \hat{\Gamma}\left(\sigma_d + \frac{d-1}{2}, \lambda t\right),$$

where

$$\begin{aligned} c_{1,d} \;&=\; \frac{1}{2}\, B\left(\frac{1}{2}, \sigma_d + \frac{d+1}{2}\right) \frac{\Gamma\left(d/2+1\right)^{(d-1)/d}}{\Gamma\left(\frac{d+1}{2}\right)} \\ \textit{and}\quad c_{2,d} \;&=\; \frac{\pi^2 (d-1)}{96(2\sigma_d + d - 1)}\, B\left(\frac{1}{2}, \sigma_d + \frac{d+1}{2}\right) \frac{\Gamma\left(d/2+1\right)^{(d-3)/d}}{\Gamma\left(\frac{d+1}{2}\right)}. \end{aligned}$$

REMARK. Because of (6.8) Theorem 6.1 can then be read as

$$Z(t) \;\leq\; \frac{|\Omega|}{(4\pi t)^{d/2}} - c_{1,d}\, \frac{|\Omega|^{(d-1)/d}}{(4\pi t)^{(d-1)/2}} - r(t) \tag{6.13}$$

with an explicit remainder term $r(t) = O(t^{-(d-3)/2})$ as $t \to 0+$. We note that the bound captures the main asymptotic behaviour of $Z(t)$ as t tends to zero: The first term equals the leading term of the short time asymptotics of $Z(t)$ and the second term shows the correct order in t compared with the second term of the asymptotic expansion.

Moreover, note that in view of (6.9) the bound from Theorem 6.1 decays exponentially as t tends to infinity. More precisely, it follows that the bound is of order $O(t^{\sigma_d+1} \exp(-\lambda\, t))$ as $t \to \infty$.

REMARK. If we choose $\lambda = \tilde{\lambda}$ introduced in (6.11) we arrive at a universal upper bound on $Z(t)$ depending only on $|\Omega|$ and not including any explicit information on λ_1. For the explicit statement see Corollary 6.9 in Section 6.4. This result implies the conjectured inequality (6.6) for dimensions $d \leq 633$.

As stated above, our proof of Theorem 6.1 relies on improved bounds for Riesz means of eigenvalues. Let us state the corresponding result.

THEOREM 6.2. *Let $\lambda \in [\tilde{\lambda}, \lambda_1]$ and $\sigma > \sigma_d$ and put $\tau_\Omega = \pi^2\, d^2\, |\Omega|^{-2/d}$. Then the estimate*

$$R_\sigma(\Lambda) \;\leq\; L^{cl}_{\sigma,d}\, |\Omega|\, \hat{B}\left(\frac{\lambda}{\Lambda}, \sigma_d + d/2 + 1, \sigma - \sigma_d\right) \Lambda^{\sigma + d/2} \;-\; (S(\Lambda,\lambda))_+$$

holds true for all $\Lambda \geq \lambda$, where

$$S(\Lambda,\lambda) \;=\; L^{cl}_{\sigma,d-1}\, |\Omega|^{(d-1)/d}\, \Lambda^{\sigma+(d-1)/2}\, \frac{B\left(\frac{1}{2}, \sigma_d + \frac{d+1}{2}\right)}{2} \hat{B}\left(\frac{\lambda}{\Lambda}, \sigma_d + \frac{d+1}{2}, \sigma - \sigma_d\right) \tag{6.14}$$

if $\lambda \geq \tau_\Omega$,

$$S(\Lambda,\lambda) \;=\; L^{cl}_{\sigma,d}\, |\Omega|\, \Lambda^{\sigma+d/2}\, \frac{1}{d}\, \hat{B}\left(\frac{\lambda}{\Lambda}, \sigma_d + d/2 + 1, \sigma - \sigma_d\right) \tag{6.15}$$

if $\lambda < \tau_\Omega$ *and* $\Lambda < \tau_\Omega$, *or*

$$S(\Lambda,\lambda) = L^{cl}_{\sigma,d-1}\,|\Omega|^{(d-1)/d}\Lambda^{\sigma+(d-1)/2}\frac{B\left(\frac{1}{2},\sigma_d+\frac{d+1}{2}\right)}{2}\hat{B}\left(\frac{\tau_\Omega}{\Lambda},\sigma_d+\frac{d+1}{2},\sigma-\sigma_d\right) + L^{cl}_{\sigma,d}\,|\Omega|\,\Lambda^{\sigma+d/2}\,\frac{1}{d}\,\tilde{B}\left(\frac{\lambda}{\Lambda},\frac{\tau_\Omega}{\Lambda},\sigma_d+d/2+1,\sigma-\sigma_d\right), \tag{6.16}$$

if $\lambda < \tau_\Omega$ *and* $\Lambda \geq \tau_\Omega$.

Remark. Again we can choose λ as in (6.11) and we arrive at a universal bound depending only on $|\Omega|$.

Remark. In view of (6.10) Theorem 6.2 can be read as

$$R_\sigma(\Lambda) \leq L^{cl}_{\sigma,d}\,|\Omega|\,\Lambda^{\sigma+d/2} - \frac{1}{2}\,B\left(\frac{1}{2},\sigma_d+\frac{d+1}{2}\right)\,L^{cl}_{\sigma,d-1}\,|\Omega|^{(d-1)/d}\,\Lambda^{\sigma+(d-1)/2} + s(\Lambda)$$

with an explicit remainder term $s(\Lambda) = O\left(\Lambda^{-1}\right)$ as $\Lambda \to \infty$.

6.3. Notation and auxiliary results

Fix a Cartesian coordinate system in $\mathbb{R}^d$ and write $x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$ for $x \in \mathbb{R}^d$. For a given $\Lambda > 0$ define

$$l_\Lambda = \pi\Lambda^{-1/2}.$$

Now consider an open set $\Omega \subset \mathbb{R}^d$. Each section $\Omega(x') = \{x_d \in \mathbb{R} : (x', x_d) \in \Omega\}$ is a one-dimensional open set and consists of at most countably many open disjoint intervals $J_k(x')$, $k = 1, \dots, N(x') \leq \infty$. Let $\kappa(x', \Lambda) \subset \mathbb{N}$ be the set of all those indices k, for which the corresponding interval $J_k(x')$ is strictly longer than l_Λ. The number these indices is denoted by $\chi(x', \Lambda)$. Put

$$\Omega_\Lambda(x') = \bigcup_{k\in\kappa(x',\Lambda)} J_k(x') \quad \text{and} \quad \Omega_\Lambda = \bigcup_{x'\in\mathbb{R}^{d-1}} \{x'\} \times \Omega_\Lambda(x')\,.$$

Obviously the set Ω_Λ is the subset of Ω, where Ω is "wide enough" in x_d-direction. The quantity

$$d_\Lambda(\Omega) = \int_{\mathbb{R}^{d-1}} \chi\left(x', \Lambda\right) dx'$$

is an effective area of the projection of Ω_Λ onto the $(d-1)$-dimensional hyperplane $(x', 0)$ counting also the multiplicities of the sufficiently long intervals $J_k(x')$.

Moreover, for $\mu \geq 2$ put

$$\varepsilon(\mu) = \inf_{A\geq 1}\left(\frac{A}{2}B\left(\frac{1}{2},\mu+1\right) - \sum_{k\geq 1}\left(1-\frac{k^2}{A^2}\right)_+^\mu\right) > 0\,. \tag{6.17}$$

We are now in the position to state the improved Berezin-Li-Yau bound from [**Wei08**]:

Proposition 6.3. *For any open domain* $\Omega \subset \mathbb{R}^d$, $\sigma \geq 3/2$ *and all* $\Lambda > 0$ *the bound*

$$R_\sigma(\Lambda) \leq L^{cl}_{\sigma,d}\,|\Omega_\Lambda|\,\Lambda^{\sigma+d/2} - \varepsilon\left(\sigma+\frac{d-1}{2}\right)\,L^{cl}_{\sigma,d-1}d_\Lambda(\Omega)\,\Lambda^{\sigma+(d-1)/2} \tag{6.18}$$

holds true.

Let us state also the following result on the explicit values of $\varepsilon(\mu)$.

LEMMA 6.4. *For all $\mu \geq 3$ we have*

$$\varepsilon(\mu) = \frac{1}{2} B\left(\frac{1}{2}, \mu + 1\right).$$

PROOF. In view of definition (6.17) and the identity

$$\int_0^A \left(1 - \frac{t^2}{A^2}\right)_+^\mu dt = \frac{A}{2} B\left(\frac{1}{2}, \mu + 1\right)$$

we have to show that

$$\sum_{k \geq 1} \left(1 - \frac{k^2}{A^2}\right)_+^\mu \leq \frac{A-1}{2} B\left(\frac{1}{2}, \mu + 1\right)$$

holds true for $\mu \geq 3$ and $A \geq 1$.

For $\mu = 3$ the claim can be checked by elementary analytic methods, since there is an explicit expression for the sum in terms of A and its integer part.

To deduce the estimate for $\mu > 3$, we start with the identity [**AL78**]

$$\sum_{k \geq 1} \left(1 - \frac{k^2}{A^2}\right)_+^\mu = \frac{1}{A^{2\mu}} \frac{1}{B(4, \mu - 3)} \int_0^{A^2 - 1} \tau^{\mu - 4} \left(A^2 - \tau\right)^3 \sum_{k \geq 1} \left(1 - \frac{k^2}{A^2 - \tau}\right)_+^3 d\tau$$

and estimate

$$\begin{aligned} \sum_{k \geq 1} \left(1 - \frac{k^2}{A^2}\right)_+^\mu &\leq \frac{1}{A^{2\mu}} \frac{B(1/2, 4)}{B(4, \mu - 3)} \int_0^{A^2 - 1} \tau^{\mu - 4} \left(A^2 - \tau\right)^3 \frac{\left(A^2 - \tau\right)^{1/2} - 1}{2} d\tau \\ &= \frac{1}{2A^{2\mu}} \frac{B(1/2, 4)}{B(4, \mu - 3)} \int_0^{A^2 - 1} \tau^{\mu - 4} \left(\left(A^2 - \tau\right)^{7/2} - \left(A^2 - \tau\right)^3\right) d\tau . \end{aligned}$$

If we substitute $s = \tau / A^2$, we see that the last integral equals

$$\begin{aligned} &A^{2\mu} \int_0^{1 - A^{-2}} s^{\mu - 4} \left(A(1 - s)^{7/2} - (1 - s)^3\right) ds \\ &= A^{2\mu} \left(AB(\mu - 3, 9/2) - B(4, \mu - 3) - \int_{1 - A^{-2}}^1 s^{\mu - 4} \left(A(1 - s)^{7/2} - (1 - s)^3\right) ds\right). \end{aligned}$$

Now we can use the identity $B(\mu - 3, 9/2)\, B(1/2, 4) / B(4, \mu - 3) = B(1/2, \mu + 1)$ and substitute $t = 1 - s$ to conclude

$$\begin{aligned} \sum_{k \geq 1} \left(1 - \frac{k^2}{A^2}\right)_+^\mu &\leq \frac{A}{2} B(1/2, \mu + 1) - \frac{B(1/2, 4)}{2B(4, \mu - 3)} \\ &\quad \times \left(B(4, \mu - 3) - \int_0^{A^{-2}} (1 - t)^{\mu - 4} t^3 \left(1 - A\sqrt{t}\right) dt\right). \end{aligned}$$

It remains to remark that the inequality

$$\frac{B(1/2, 4)}{B(4, \mu - 3)} \left(B(4, \mu - 3) - \int_0^{A^{-2}} (1 - t)^{\mu - 4} t^3 \left(1 - A\sqrt{t}\right) dt\right) \geq B\left(\frac{1}{2}, \mu + 1\right),$$

holds true for all $A \geq 1$, since we have equality in the case $A = 1$ and since the left hand side is non-decreasing in $A \geq 1$. □

In fact, we shall need a modified version of Proposition 6.3.

Let $p_d(x';\Omega) = |\Omega(x')|_1$ be the one-dimensional Lebesgue measure of $\Omega(x')$, that is the aggregated length of all intervals forming $\Omega(x')$. Since Ω is open, the function $p_d(x';\Omega)$ is Lebesgue measurable, and we can define the distribution function [3]

$$m_d(\tau;\Omega) = |\{x' : p_d(x';\Omega) > \tau\}|_{d-1}\,, \quad \tau > 0.$$

It is non-negative, non-increasing, continuous from the right and it satisfies the identity

$$\int_0^\infty m_d(\tau;\Omega)\,d\tau = |\Omega|. \tag{6.19}$$

We interchange now the roles of x_d and x_i for $i = 1,\dots,d-1$ and introduce in the same way the distribution functions $m_i(\cdot;\Omega)$ for Ω measured along the x_i-axes. Finally, put

$$M_i(y;\Omega) \;=\; \int_0^y m_i(\tau;\Omega)\,d\tau \quad \text{for} \quad i = 1,\dots,d.$$

With this notation we can formulate a result similar to (6.18):

Proposition 6.5. *For any open domain $\Omega \subset \mathbb{R}^d$, $\sigma \geq 3/2$ and all $\Lambda > 0$*

$$R_\sigma(\Lambda) \;\leq\; L^{cl}_{\sigma,d} \int_{l_\Lambda}^\infty m_i(\tau;\Omega)\,d\tau\,\Lambda^{\sigma+d/2} + \delta_{\sigma,d}\; m_i\left(l_\Lambda;\Omega\right)\;\Lambda^{\sigma+(d-1)/2}$$

holds true for $i = 1,\dots,d$ with $\delta_{\sigma,d} = \pi L^{cl}_{\sigma,d} - \varepsilon\left(\sigma + \frac{d-1}{2}\right) L^{cl}_{\sigma,d-1}$.

Remark. Note that in the case of $\varepsilon\left(\sigma + \frac{d-1}{2}\right) = \frac{1}{2}\,B\left(\sigma + \frac{d+1}{2}, \frac{1}{2}\right)$ we have $\delta_{\sigma,d} = 0$. In view of Lemma 6.4 this occurs, if $\sigma + \frac{d-1}{2} \geq 3$, in particular, if $\sigma = \sigma_d$ with σ_d introduced in (6.12).

Remark. For domains Ω with finite volume (6.19) yields

$$\int_{l_\Lambda}^\infty m_i(\tau;\Omega)\,d\tau = |\Omega| - M_i\left(l_\Lambda;\Omega\right)\,.$$

Thus we arrive at

$$R_\sigma(\Lambda) \;\leq\; L^{cl}_{\sigma,d}\left(|\Omega| - M_i\left(l_\Lambda;\Omega\right)\right)\Lambda^{\sigma+d/2} + \delta_{\sigma,d}\, m_i\left(l_\Lambda;\Omega\right)\;\Lambda^{\sigma+(d-1)/2}$$

for $i = 1,\dots,d$. Averaging over all directions one claims

$$R_\sigma(\Lambda) \;\leq\; L^{cl}_{\sigma,d}\left(|\Omega| - M\left(l_\Lambda;\Omega\right)\right)\Lambda^{\sigma+d/2} + \delta_{\sigma,d}\, m\left(l_\Lambda;\Omega\right)\;\Lambda^{\sigma+(d-1)/2}, \tag{6.20}$$

where

$$\begin{aligned} m(t;\Omega) &= \frac{1}{d}\left(m_1(t;\Omega) + \dots + m_d(t;\Omega)\right)\,, \\ M(y;\Omega) &= \frac{1}{d}\left(M_1(y;\Omega) + \dots + M_d(y;\Omega)\right) = \int_0^y m(t;\Omega)dt\,. \end{aligned}$$

[3] Here $|\cdot|_{d-1}$ stands for the Lebesgue measure in the dimension $d-1$.

Although Proposition 6.5 is, in general, not as sharp as (6.18), we cannot deduce it directly quoting Proposition 6.3, but we have to modify the respective proof from [**Wei08**], which relies on operator-valued Lieb-Thirring inequalities from [**LW00**].

Proof of Proposition 6.5. Consider the quadratic form

$$\|\nabla u\|^2_{L^2(\Omega)} - \Lambda \|u\|^2_{L^2(\Omega)} = \|\nabla' u\|^2_{L^2(\Omega)} + \int_{\mathbb{R}^{d-1}} dx' \int_{\Omega(x')} \left(|\partial_{x_d} u|^2 - \Lambda |u|^2\right) dx_d$$

on functions u from the form core $C_0^\infty(\Omega)$. Here ∇' and Δ' denote the gradient and the Laplace operator in the first $d-1$ directions. The functions $u(x', \cdot)$ satisfy Dirichlet boundary conditions at the endpoints of each interval $J_k(x')$ forming $\Omega(x')$. Let the bounded, non-negative operators $W_k(x', \Lambda)$ be the negative parts[4] of the Sturm-Liouville Operators $-\partial^2_{x_d, J_k(x')} - \Lambda$ with Dirichlet boundary conditions on $J_k(x')$. Then

$$W(x', \Lambda) = \bigoplus_{k=1}^{N(x')} W_k(x', \Lambda)$$

is the negative part of

$$-\partial^2_{x_d, \Omega(x')} - \Lambda = \bigoplus_{k=1}^{N(x')} \left(-\partial^2_{x_d, J_k(x')} - \Lambda\right)$$

subject to Dirichlet boundary conditions on the endpoints of the intervals $J_k(x')$, $k = 1, \dots, N(x')$, that is on $\partial\Omega(x')$. Then

$$\int_{\Omega(x')} \left(|\partial_{x_d} u|^2 - \Lambda |u|^2\right) dx_d \geq -\langle W u(x', \cdot), u(x', \cdot)\rangle_{L^2(\Omega(x'))}.$$

and consequently

$$\|\nabla u\|^2_{L^2(\Omega)} - \Lambda \|u\|^2_{L^2(\Omega)} \geq \|\nabla' u\|^2_{L^2(\Omega)} - \int_{\mathbb{R}^{d-1}} dx' \langle W u(x', \cdot), u(x', \cdot)\rangle_{L^2(\Omega(x'))}. \quad (6.21)$$

Now we can extend this quadratic form by zero to $C_0^\infty\left(\mathbb{R}^d \setminus \partial\Omega\right)$, which is a form core for $\left(-\Delta_{\mathbb{R}^d \setminus \Omega}\right) \oplus (-\Delta_\Omega - \Lambda)$. This operator corresponds to the left hand side of (6.21), while the semi-bounded form on the right hand side is closed on the larger domain $H^1\left(\mathbb{R}^{d-1}, L^2(\mathbb{R})\right)$, where it corresponds to the operator

$$-\Delta' \otimes \mathbb{I} - W(x', \Lambda) \quad \text{on} \quad L^2\left(\mathbb{R}^{d-1}, L^2(\mathbb{R})\right). \quad (6.22)$$

Due to the positivity of $-\Delta_{\mathbb{R}^d \setminus \Omega}$ the variational principle implies that for any $\sigma \geq 0$

$$\begin{aligned} \operatorname{Tr}(-\Delta_\Omega - \Lambda)^\sigma_- &= \operatorname{Tr}\left(\left(-\Delta_{\mathbb{R}^d \setminus \Omega}\right) \oplus (-\Delta_\Omega - \Lambda)\right)^\sigma_- \\ &\leq \operatorname{Tr}(-\Delta' \otimes \mathbb{I} - W(x', \Lambda))^\sigma_-. \end{aligned}$$

We can now apply a sharp Lieb-Thirring inequality to the Schrödinger operator (6.22) with the operator-valued potential $-W(x', \Lambda)$, see [**LW00**], and claim that

$$\operatorname{Tr}(-\Delta' \otimes \mathbb{I} - W(x', \Lambda))^\sigma_- \leq L^{cl}_{\sigma, d-1} \int_{\mathbb{R}^{d-1}} \operatorname{Tr} W^{\sigma + (d-1)/2}(x', \Lambda)\, dx', \quad \sigma \geq \frac{3}{2}.$$

[4]The negative part of a real number r is given by $r_- = (|r| - r)/2 \geq 0$. For operators we use the same convention in the spectral sense.

Now let $p_d(x') = \sum_k |J_k(x')|_1$ be the total length of all intervals $J_k(x')$. Then shifting these intervals and dropping intermediate Dirichlet conditions by a variational argument we see that the j-th eigenvalue of $-\partial^2_{x_d,\Omega(x')} - \Lambda$ is not smaller than the j-th eigenvalue of $-\partial^2_{x_d,L(x')} - \Lambda$ on the interval $L(x') = [0, p_d(x')]$ subject to Dirichlet conditions at the endpoint of this one interval only. Thus,

$$\operatorname{Tr} W^{\sigma+(d-1)/2}(x', \Lambda) \leq \operatorname{Tr} \tilde{W}^{\sigma+(d-1)/2}(x', \Lambda),$$

where $\tilde{W}(x', \Lambda)$ is the negative part of $-\partial^2_{x_d,L(x')} - \Lambda$. The nonzero eigenvalues of $\tilde{W}(x', \Lambda)$ are given explicitly by

$$\mu_j = \Lambda - \frac{\pi^2 j^2}{p_d^2(x')} = \Lambda \left(1 - \frac{l_\Lambda^2 j^2}{p_d^2(x')}\right) \quad \text{for} \quad j = 1, \dots, \left[\frac{p_d(x')}{l_\Lambda}\right].$$

From this we conclude that

$$\operatorname{Tr} (-\Delta_\Omega - \Lambda)^\sigma_- \;\leq\; \Lambda^{\sigma+(d-1)/2}\, L^{cl}_{\sigma,d-1} \int_{\mathbb{R}^{d-1}} \sum_{j\geq 1} \left(1 - \frac{l_\Lambda^2 j^2}{p_d^2(x')}\right)^{\sigma+(d-1)/2}_+ dx'. \tag{6.23}$$

Note that the right hand side of this bound vanishes if $p_d(x') \leq l_\Lambda$. For $p_d(x') > l_\Lambda$ from (6.17) we get

$$\sum_{j\geq 1} \left(1 - \frac{l_\Lambda^2 j^2}{p_d^2(x')}\right)^{\sigma+(d-1)/2}_+ \;\leq\; \frac{p_d(x')}{2\, l_\Lambda}\, B\left(\sigma + \frac{d+1}{2}, \frac{1}{2}\right) - \varepsilon \left(\sigma + \frac{d-1}{2}\right)$$

and in view of $l_\Lambda = \pi \Lambda^{-1/2}$ we find

$$\begin{aligned} \operatorname{Tr} (-\Delta_\Omega - \Lambda)^\sigma_- \;\leq\; & \frac{1}{2\pi}\, B\left(\sigma + \frac{d+1}{2}, \frac{1}{2}\right) \Lambda^{\sigma+d/2} L^{cl}_{\sigma,d-1} \int_{x':p_d(x')>l_\Lambda} p_d(x')\, dx' \\ & - \varepsilon \left(\sigma + \frac{d-1}{2}\right) \Lambda^{\sigma+(d-1)/2}\, L^{cl}_{\sigma,d-1} \int_{x':p_d(x')>l_\Lambda} dx'. \end{aligned} \tag{6.24}$$

Note that

$$\int_{x':p_d(x')>l_\Lambda} dx' = m_d\,(l_\Lambda; \Omega)$$

and

$$\int_{x':p_d(x')>l_\Lambda} p_d(x')\, dx' \;=\; m_d\,(l_\Lambda; \Omega)\, l_\Lambda + \int_{l_\Lambda}^\infty m_d\,(\tau; \Omega)\, d\tau\,.$$

Moreover, using

$$\frac{1}{2\pi} B\left(\sigma + \frac{d+1}{2}, \frac{1}{2}\right) L^{cl}_{\sigma,d-1} = L^{cl}_{\sigma,d},$$

we insert the identities above into (6.24) and arrive at

$$\begin{aligned} R_\sigma(\Lambda) \;=\; \operatorname{Tr}(-\Delta - \Lambda)^\sigma_- \;\leq\; & L^{cl}_{\sigma,d}\, \Lambda^{\sigma+d/2} \left(m_d\,(l_\Lambda; \Omega)\, l_\Lambda + \int_{l_\Lambda}^\infty m_d(\tau; \Omega) d\tau\right) \\ & - \varepsilon \left(\sigma + \frac{d-1}{2}\right) L^{cl}_{\sigma,d-1}\, m_d\,(l_\Lambda; \Omega)\, \Lambda^{\sigma+(d-1)/2}\,. \end{aligned}$$

In view of $l_\Lambda = \pi \Lambda^{-1/2}$ this yields

$$R_\sigma(\Lambda) \leq L^{cl}_{\sigma,d} \int_{l_\Lambda}^\infty m_d(\tau; \Omega) d\tau\, \Lambda^{\sigma+d/2} + \delta_{\sigma,d}\, m_d\,(l_\Lambda; \Omega)\, \Lambda^{\sigma+(d-1)/2}, \quad \sigma \geq \frac{3}{2}.$$

Interchanging the roles of x_d and x_i we find the respective inequalities for any direction $i = 1, \dots, d$. □

In order to derive universal bounds on R_σ independent from M, in particular to prove Theorem 6.2, one needs bounds on $M(y)$. Identity (6.19) immediately implies

$$M(y;\Omega) = \int_0^y m(\tau;\Omega)d\tau \le \int_0^\infty m(\tau;\Omega)d\tau = |\Omega| \quad \text{for all} \quad 0 < y < \infty\,.$$

To prove a lower bound we first need an auxiliary result concerning rearrangements of Ω. For $\Omega \subset \mathbb{R}^d$, $d \ge 2$, fix a Cartesian coordinate system $(x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$. Again put

$$p_d(x';\Omega) = |\{x_d : (x', x_d) \in \Omega\}|_1 = |\Omega(x')|_1$$

and for $\tau > 0$

$$\Omega^*(\tau) = \{x' : p_d(x';\Omega) > \tau\} \subset \mathbb{R}^{d-1}.$$

This is a non-increasing set function, that means $\Omega^*(\tau_1) \supset \Omega^*(\tau_2)$ for $0 < \tau_1 \le \tau_2$. Let

$$\Omega^* = \cup_{\tau>0}\, \Omega^*(\tau) \times \{\tau\} \subset \mathbb{R}^d \tag{6.25}$$

be a non-increasing rearrangement of Ω in the direction of the x_d-coordinate. Then we have

LEMMA 6.6. *For all $i = 1, \dots, d$ and all $y > 0$*

$$M_i(y;\Omega) \;\ge\; M_i(y;\Omega^*).$$

PROOF. First note that in the case $i = d$ we have by construction $p_d(x';\Omega) = p_d(x';\Omega^*)$ and consequently

$$m_d(\tau;\Omega) = m_d(\tau;\Omega^*) = |\Omega^*(\tau)|_{d-1}\,,$$

what implies $M_d(y;\Omega) = M_d(y;\Omega^*)$.

Assume now that $j = 1, \dots, d-1$. Put

$$x'' = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{d-1}) \in \mathbb{R}^{d-2}$$

and

$$p_j(x'', x_d;\Omega) = |\{x_j : (x', x_d) \in \Omega\}|_1\,.$$

By definition

$$m_j(s;\Omega) = |\{(x'', x_d) : p_j(x'', x_d;\Omega) > s\}|_{d-1} = \int_{\mathbb{R}^{d-2}} \hat{m}_j(x'', s;\Omega)\, dx''$$

where

$$\hat{m}_j(x'', s;\Omega) = |\{x_d : p_j(x'', x_d;\Omega) > s\}|_1\,, \quad j = 1, \dots, d-1\,.$$

Hence,

$$M_j(y;\Omega) = \int_0^y m_j(s;\Omega)ds = \int_{\mathbb{R}^{d-2}} \int_0^y \hat{m}_j(x'', s;\Omega)\, ds\, dx''.$$

Applying the same notation to Ω^* yields

$$M_j(y;\Omega^*) = \int_0^y m_j(s;\Omega^*)\, ds = \int_{\mathbb{R}^{d-2}} \int_0^y \hat{m}_j(x'', s;\Omega^*)\, ds\, dx''.$$

If we can show that for $x'' \in \mathbb{R}^{d-2}$ and all $y > 0$ the inequality

$$\int_0^y \hat{m}_j(x'', s; \Omega) ds \geq \int_0^y \hat{m}_j(x'', s; \Omega^*) ds \tag{6.26}$$

holds true, the assertion is proven.

To establish (6.26) we consider for fixed $x'' \in \mathbb{R}^{d-2}$ the two-dimensional sets

$$\hat{\Omega} = \{(x_j, x_d) : (x', x_d) \in \Omega\} \quad \text{and} \quad \hat{\Omega}^* = \{(x_j, x_d) : (x', x_d) \in \Omega^*\} .$$

Note that

$$p_d(x'; \Omega) = |\{x_d : (x', x_d) \in \Omega\}|_1 = \left|\{x_d : (x_j, x_d) \in \hat{\Omega}\}\right|_1 =: \hat{p}_d(x_j; \hat{\Omega}) .$$

As above we get

$$\hat{p}_d(x_j; \hat{\Omega}) = \hat{p}_d(x_j; \hat{\Omega}^*) . \tag{6.27}$$

In the jth direction we have

$$p_j(x'', x_d; \Omega) = |\{x_j : (x', x_d) \in \Omega\}|_1 = \left|\left\{x_j : (x_j, x_d) \in \hat{\Omega}\right\}\right|_1 =: \hat{p}_j(x_d; \hat{\Omega})$$

and

$$\hat{m}_j(x'', s; \Omega) = |\{x_d : p_j(x'', x_d; \Omega) > s\}|_1 = \left|\left\{x_d : \hat{p}_j(x_d, \hat{\Omega}) > s\right\}\right|_1 =: \hat{m}_j(s; \hat{\Omega}) .$$

We also use the corresponding notions with respect to the domains Ω^* and $\hat{\Omega}^*$. In contrast to the preservation of length in the dth direction the values of $\hat{p}_j(x_d; \hat{\Omega})$ and $\hat{p}_j(x_d; \hat{\Omega}^*)$ (and thus of $\hat{m}_j(s; \hat{\Omega})$ and $\hat{m}_j(s; \hat{\Omega}^*)$) do not coincide in general.

Lets examine the functions $\hat{p}_j(x_d; \hat{\Omega}^*)$ and $\hat{m}_j(s; \hat{\Omega}^*)$ in more detail. By construction of $\hat{\Omega}^*$, the set function $\hat{\Omega}^*(x_d) = \{x_j : (x_j, x_d) \in \hat{\Omega}\}$ is non-increasing in $x_d > 0$ and by definition

$$\hat{p}_j\left(x_d; \hat{\Omega}^*\right) = \left|\hat{\Omega}^*(x_d)\right|_1 .$$

Moreover, $\hat{m}_j(s; \hat{\Omega}^*)$ is the distribution function of $\hat{p}_j(x_d; \hat{\Omega}^*)$. Hence,

$$\int_0^y \hat{m}_j(s; \hat{\Omega}^*)\, ds = \int_{\left\{x_d : \hat{p}_j(x_d; \hat{\Omega}^*) < y\right\}} \hat{p}_j(x_d; \hat{\Omega}^*)\, dx_d + y \left|\left\{x_d : \hat{p}_j(x_d; \hat{\Omega}^*) \geq y\right\}\right|_1 .$$

The monotonicity of the set function $\hat{\Omega}^*(x_d)$ implies that we can choose $I_y \subset \mathbb{R}$ with total length y satisfying $I_y \subset \hat{\Omega}^*(x_d)$, wherever $\hat{p}_j(x_d; \hat{\Omega}^*) \geq y$. Again, by the monotonicity of $\hat{\Omega}^*(x_d)$ the reverse inclusion $\hat{\Omega}^*(x_d) \subset I_y$ holds for all $x_d > 0$ with $\hat{p}_j(x_d; \hat{\Omega}^*) < y$. Put

$$\hat{\Omega}_y^* = \bigcup_{x_d > 0} \left(\hat{\Omega}^*(x_d) \cap I_y\right) \times \{x_d\} \quad \text{and} \quad \hat{\Omega}_y = \bigcup_{x_d > 0} \left(\hat{\Omega}(x_d) \cap I_y\right) \times \{x_d\} .$$

From the above representation for $\int_0^y \hat{m}_j(s; \hat{\Omega}^*)\, ds$ we deduce

$$\int_0^y \hat{m}_j(s; \hat{\Omega}^*) ds = \left|\hat{\Omega}_y^*\right| . \tag{6.28}$$

Moreover, note that for $x_j \in I_y$

$$\begin{aligned}\left\{x_d : (x_j, x_d) \in \hat{\Omega}_y^*\right\} &= \left\{x_d : (x_j, x_d) \in \hat{\Omega}^*\right\} \quad \text{and} \\ \left\{x_d : (x_j, x_d) \in \hat{\Omega}_y\right\} &= \left\{x_d : (x_j, x_d) \in \hat{\Omega}\right\}.\end{aligned}$$

In view of (6.27) we get

$$\hat{p}_d(x_j; \hat{\Omega}_y) = \hat{p}_d(x_j; \hat{\Omega}) = \hat{p}_d(x_j; \hat{\Omega}^*) = \hat{p}_d(x_j; \hat{\Omega}_y^*)$$

and we conclude

$$\left|\hat{\Omega}_y^*\right| = \left|\hat{\Omega}_y\right|. \tag{6.29}$$

Finally, we analyse $\hat{m}_j(s, \hat{\Omega})$. The inclusion $\hat{\Omega}_y \subset \hat{\Omega}$ implies

$$\int_0^y \hat{m}_j(s; \hat{\Omega})\, ds \geq \int_0^y \hat{m}_j(s; \hat{\Omega}_y)\, ds\,. \tag{6.30}$$

Moreover, by construction of $\hat{\Omega}_y$ we have

$$\hat{p}_j(x_d; \hat{\Omega}_y) \leq |I_y| = y$$

for all $x_d > 0$ and consequently $m_j(s; \hat{\Omega}_y) = 0$ for all $s \geq y$. Using (6.30) we conclude

$$\int_0^y \hat{m}_j(s; \hat{\Omega})\, ds \geq \int_0^y \hat{m}_j(s; \hat{\Omega}_y)\, ds \;= \int_0^\infty \hat{m}_j(s; \hat{\Omega}_y)\, ds = \left|\hat{\Omega}_y\right|.$$

In view (6.29) and (6.28) we arrive at

$$\int_0^y \hat{m}_j(s; \hat{\Omega})\, ds \geq \left|\hat{\Omega}_y\right| = \left|\hat{\Omega}_y^*\right| = \int_0^y \hat{m}_j(s; \hat{\Omega}^*)\, ds\,.$$

This shows that (6.26) holds true and the proof is complete. □

Now we can give a lower bound on $M(y; \Omega)$:

LEMMA 6.7. *For all open sets $\Omega \subset \mathbb{R}^d$ and all $y > 0$*

$$M(y; \Omega) \;\geq\; \min\left(\frac{|\Omega|}{d},\, |\Omega|^{(d-1)/d}\, y\right). \tag{6.31}$$

PROOF. We use induction in the dimension. For $d = 1$ and an interval of length $|\Omega|$ we get

$$m(\tau; \Omega) \;=\; \begin{cases} 1 & |\Omega| > \tau \\ 0 & |\Omega| \leq \tau \end{cases}$$

and therefore $M(y; \Omega) = \int_0^y m(\tau; \Omega) d\tau = \min(y, |\Omega|)$ for all $y > 0$.

Now assume $\Omega \subset \mathbb{R}^d$, $d \geq 2$. For any given $j = 1, \ldots, d-1$ put

$$x'' = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{d-1}) \in \mathbb{R}^{d-2}$$

and let $\tilde{m}_j(s; \tilde{\Omega}) = |\{x'' : \tilde{p}_j(x''; \tilde{\Omega}) > s\}|_{d-2}$ be the distribution function of a set $\tilde{\Omega} \subset \mathbb{R}^{d-1}$ with respect to the j-th direction, where $\tilde{p}_j(x''; \tilde{\Omega}) = |\{x_j : x' \in \tilde{\Omega}\}|_1$ is

the total length of the section through $\tilde{\Omega}$ at x'' in the direction of the x_j-coordinate. Applying these notions to Ω^* given in (6.25) we get

$$\begin{aligned} m_j(s;\Omega^*) &= \left|\{(x'',\tau)\in\mathbb{R}^{d-1} : p_j\,(x'',\tau;\Omega^*) > s\}\right|_{d-1} \\ &= \int_0^\infty |\{x''\in\mathbb{R}^{d-2} : \tilde{p}_j\,(x'';\Omega^*(\tau)) > s\}|_{d-2}\,d\tau \\ &= \int_0^\infty \tilde{m}_j(s;\Omega^*(\tau))\,d\tau\,, \qquad j=1,\dots,d-1\,. \end{aligned} \tag{6.32}$$

Put $\tilde{m}(s;\tilde{\Omega}) = (d-1)^{-1}\sum_{j=1}^{d-1}\tilde{m}_j(s;\tilde{\Omega})$. By induction assumption we have

$$\tilde{M}(y;\tilde{\Omega}) = \int_0^y \tilde{m}(s;\tilde{\Omega})\,ds \geq \min\left(\frac{|\tilde{\Omega}|_{d-1}}{d-1},\, |\tilde{\Omega}|_{d-1}^{(d-2)/(d-1)}\,y\right)\,, \quad y>0. \tag{6.33}$$

Next note that in view of (6.32)

$$\begin{aligned} d\cdot M(y;\Omega^*) &= M_1(y;\Omega^*)+\dots+M_{d-1}(y;\Omega^*)+M_d(y;\Omega^*) \\ &= \int_0^y (m_1(s;\Omega^*)+\dots+m_{d-1}(s;\Omega^*))\,ds + \int_0^y m_d(s;\Omega^*)\,ds \\ &= (d-1)\int_0^y\int_0^\infty \tilde{m}(s;\Omega^*(\tau))\,d\tau\,ds + \int_0^y m_d(s;\Omega^*)\,ds \\ &= (d-1)\int_0^\infty \tilde{M}(y;\Omega^*(\tau))\,d\tau + \int_0^y m_d(s;\Omega^*)\,ds\,. \end{aligned}$$

Using (6.33) we claim

$$M(y;\Omega^*) \geq \frac{d-1}{d}\int_0^\infty \min\left(\frac{|\Omega^*(\tau)|_{d-1}}{d-1},\, |\Omega^*(\tau)|_{d-1}^{(d-2)/(d-1)}\,y\right)d\tau + \frac{1}{d}\int_0^y m_d(s;\Omega^*)ds\,.$$

We point out that $|\Omega^*(\tau)|_{d-1} = m_d\,(\tau,\Omega^*)$ for $\tau>0$. Put

$$\tau^* = \inf\left\{\tau>0 : m_d(\tau;\Omega^*) \leq (d-1)^{d-1}y^{d-1}\right\}\,.$$

Then

$$\begin{aligned} M(y;\Omega^*) \;\geq\; \frac{1}{d}\int_{\tau^*}^\infty m_d\,(\tau,\Omega^*)\,d\tau + \frac{1}{d}\int_0^y m_d(\tau;\Omega^*)d\tau \\ + \frac{d-1}{d}y\int_0^{\tau^*} m_d\,(\tau;\Omega^*)^{(d-2)/(d-1)}\,d\tau\,. \end{aligned}$$

By (6.19) we have $\int_0^\infty m_d\,(\tau;\Omega^*)\,d\tau = \int_0^\infty m_d\,(\tau;\Omega)\,d\tau = |\Omega|$ and using Lemma 6.6 we estimate

$$\begin{aligned} M(y;\Omega) \;\geq\; M(y;\Omega^*) \;\geq\; \frac{|\Omega|}{d} - \frac{1}{d}\int_0^{\tau^*} m_d\,(\tau;\Omega^*)\,d\tau + \frac{1}{d}\int_0^y m_d(\tau;\Omega^*)d\tau \\ + \frac{d-1}{d}\,y\int_0^{\tau^*} m_d\,(\tau;\Omega^*)^{(d-2)/(d-1)}\,d\tau\,. \end{aligned} \tag{6.34}$$

In particular, in the case of $\tau^*\leq y$ we see from the previous bound that

$$M(y;\Omega) \geq d^{-1}\,|\Omega|$$

and the assertion is proven. Hence, let us consider the remaining case $\tau^* > y$ in more detail. For $\tau^* > y$ we have

$$m_d(y;\Omega^*) \geq (d-1)^{d-1} y^{d-1}\,.$$

Because of the monotonicity of m_d we conclude

$$\int_0^y m_d(\tau;\Omega^*)^{(d-2)/(d-1)} d\tau \;\geq\; y\, m_d(y;\Omega^*)^{(d-2)/(d-1)}\,,$$

$$\int_0^y m_d(\tau;\Omega^*) d\tau \;\geq\; y\, m_d(y;\Omega^*)\,.$$

Let us rewrite inequality (6.34) as follows

$$\begin{aligned} M(y;\Omega) \geq \frac{|\Omega|}{d} \;&+\; \frac{d-1}{d}\, y \int_0^y m_d(\tau;\Omega^*)^{(d-2)/(d-1)} d\tau \\ &+\; \frac{d-1}{d}\, y \int_y^{\tau^*} m_d(\tau;\Omega^*)^{(d-2)/(d-1)} d\tau - \frac{1}{d}\int_y^{\tau^*} m_d(\tau;\Omega^*) d\tau\,. \end{aligned}$$

Put $A = \int_y^{\tau^*} m_d\,(\tau;\Omega^*)\, d\tau$. Then

$$0 < A = \int_0^{\tau^*} m_d\,(\tau;\Omega^*)\, d\tau - \int_0^y m_d\,(\tau;\Omega^*)\, d\tau \leq |\Omega| - y\, m_d(y;\Omega^*)\,. \tag{6.35}$$

Moreover,

$$M(y;\Omega) \geq \frac{|\Omega|}{d} + \frac{d-1}{d}\, y^2\, m_d(y;\Omega^*)^{(d-2)/(d-1)} + \frac{d-1}{d}\, y \int_y^{\tau^*} m_d(\tau;\Omega^*)^{(d-2)/(d-1)} d\tau - \frac{A}{d}\,.$$

Due to the monotonicity of m_d we have, in particular, $m_d(\tau;\Omega^*) \leq m_d(y;\Omega^*)$ for $y \leq \tau$ and

$$\begin{aligned} \int_y^{\tau^*} m_d(\tau;\Omega^*)^{(d-2)/(d-1)} d\tau \;&=\; m_d(y;\Omega^*)^{(d-2)/(d-1)} \int_y^{\tau^*} \left(\frac{m_d(\tau;\Omega^*)}{m_d(y;\Omega^*)}\right)^{(d-2)/(d-1)} d\tau \\ &\geq\; m_d(y;\Omega^*)^{(d-2)/(d-1)} \int_y^{\tau^*} \frac{m_d(\tau;\Omega^*)}{m_d(y;\Omega^*)}\, d\tau \\ &=\; m_d(y;\Omega^*)^{-1/(d-1)}\, A\,. \end{aligned}$$

Thus,

$$M(y;\Omega) \geq \frac{|\Omega|}{d} + \frac{d-1}{d}\, y^2\, m_d(y;\Omega^*)^{(d-2)/(d-1)} - \frac{1}{d}\left(1 - (d-1)\, y\, m_d(y;\Omega^*)^{-1/(d-1)}\right) A\,.$$

For $\tau^* > y$ we have $1 - (d-1)\, y\, m_d(y;\Omega^*)^{-1/(d-1)} > 0$ and we can insert (6.35) in this estimate and arrive at

$$\begin{aligned} M(y;\Omega) \;&\geq\; \frac{|\Omega|}{d} + \frac{d-1}{d}\, y^2\, m_d(y;\Omega^*)^{(d-2)/(d-1)} \\ &\quad -\frac{1}{d}\left(1 - (d-1)\, y\, m_d(y;\Omega^*)^{-1/(d-1)}\right)\left(|\Omega| - y\, m_d(y;\Omega^*)\right) \\ &\geq\; \frac{y}{d}\left((d-1)|\Omega|\, m_d(y;\Omega^*)^{-1/(d-1)} + m_d(y;\Omega^*)\right)\,. \end{aligned}$$

Since the function $f(m) = (d-1)|\Omega|\, m^{-1/(d-1)} + m$ takes its minimal value for positive arguments at $m = |\Omega|^{(d-1)/d}$, we arrive for $y < \tau^*$ at

$$M(y;\Omega) \geq \frac{y}{d} f(m_d(y;\Omega^*)) \geq \frac{y}{d} f(|\Omega|^{(d-1)/d}) = y\,|\Omega|^{(d-1)/d}\,.$$

This completes the proof. □

6.4. Proof of Theorem 6.1 and remarks

Let

$$\mathcal{L}[f(\cdot)](t) = \int_0^\infty f(\Lambda) e^{-\Lambda t} d\Lambda$$

be the Laplace transformation of a suitable function $f : (0, +\infty) \to \mathbb{R}$. For real values of t it is monotone, that means a pointwise estimate $f_1(\Lambda) \leq f_2(\Lambda)$ for all $\Lambda > 0$ implies $\mathcal{L}[f_1](t) \leq \mathcal{L}[f_2](t)$ for any $t \in \mathbb{R}$, for which both transformations are defined. In particular, for $\lambda \geq 0$ and $\sigma > 0$ one has

$$\mathcal{L}[(\Lambda - \lambda)_+^\sigma](t) = \int_\lambda^\infty (\Lambda - \lambda) e^{-\Lambda t} d\Lambda = e^{-\lambda t} t^{-\sigma-1} \Gamma(\sigma+1)\,, \quad t > 0\,.$$

In view of the linearity of the Laplace transformation one finds for $t > 0$ and $\sigma > 0$ the well-known identity

$$Z(t) = \operatorname{Tr} e^{+\Delta_\Omega t} = \sum_k e^{-\lambda_k t} = \sum_k \frac{t^{\sigma+1}}{\Gamma(\sigma+1)} \mathcal{L}[(\Lambda - \lambda_k)_+^\sigma](t) = \frac{t^{\sigma+1}}{\Gamma(\sigma+1)} \mathcal{L}[R_\sigma(\Lambda)]\,.$$

Therefore, any bound on the Riesz means of the type

$$R_\sigma(\Lambda) \leq f(\Lambda, \Omega) \quad \text{for all} \quad \Lambda > 0 \tag{6.36}$$

implies a bound on the heat kernel

$$Z(t) \leq \frac{t^{\sigma+1}}{\Gamma(\sigma+1)} \mathcal{L}[f(\cdot,\Omega)](t) \tag{6.37}$$

valid for all $t > 0$, for which the r.h.s. is defined. For example, this way one can deduce (6.1) from (6.3) with any $\sigma \geq 1$.

Next note that in view of $R_\sigma(\Lambda) = 0$ for $0 < \Lambda \leq \lambda_1$ we have in fact

$$\Gamma(\sigma+1) t^{-\sigma-1} Z(t) = \mathcal{L}[R_\sigma](t) = \mathcal{L}[R_\sigma, \lambda](t) \quad \text{for any} \quad 0 \leq \lambda \leq \lambda_1\,,$$

where

$$\mathcal{L}[f,\lambda](t) = \int_\lambda^\infty f(\Lambda) e^{-\Lambda t} d\Lambda = e^{-\lambda t} \mathcal{L}[f(\cdot + \lambda)](t)\,, \quad \lambda \geq 0\,,$$

is the reduced Laplace transformation of a suitable function f. This transformation preserves pointwise inequalities as well and from (6.36) one can deduce an improved version of (6.37)

$$Z(t) \leq \frac{t^{\sigma+1}}{\Gamma(\sigma+1)} \mathcal{L}[f(\cdot,\Omega),\lambda](t) \quad \text{for arbitrary} \quad 0 \leq \lambda \leq \lambda_1\,.$$

Applying this bound to (6.3) one gets the estimate

$$Z(t) \leq \frac{|\Omega|}{(4\pi t)^{d/2}} \hat{\Gamma}\left(\sigma + d/2 + 1, \lambda t\right)\,, \quad t > 0\,,\ \sigma \geq 1\,,\ 0 \leq \lambda \leq \lambda_1\,, \tag{6.38}$$

which already contains an exponential decay for large t. Instead of referring to the classical Berezin-Li-Yau-bound (6.3) we can apply this idea also directly to the improved bound (6.18) and claim

$$\begin{aligned} Z(t) &\leq \frac{t^{\sigma+1}}{\Gamma(\sigma+1)} L^{cl}_{\sigma,d} \int_{\lambda_1}^{\infty} |\Omega_\Lambda| \, \Lambda^{\sigma+d/2} e^{-\Lambda t} \, d\Lambda \\ &\quad - \frac{t^{\sigma+1}}{\Gamma(\sigma+1)} L^{cl}_{\sigma,d-1} \, \varepsilon \left(\sigma + \frac{d-1}{2}\right) \int_{\lambda_1}^{\infty} d_\Lambda(\Omega) \Lambda^{\sigma+(d-1)/2} e^{-\Lambda t} \, d\Lambda, \end{aligned} \tag{6.39}$$

where $t > 0$ and $\sigma \geq \frac{3}{2}$. This bound is even sharper than the estimates presented below. But the geometric properties of Ω enter in a rather tricky way and cannot be simplified in a straightforward manner. Therefore we prefer to present also a slightly weaker, but sometimes more convenient version of this bound. For that end we choose $\sigma = \sigma_d$ given in (6.12) and apply the reduced Laplace transformation to (6.20). Thus we get the following estimate valid for $\lambda \in [\tilde{\lambda}, \lambda_1]$ and $t > 0$:

$$Z(t) \leq \frac{|\Omega|}{(4\pi t)^{d/2}} \hat{\Gamma}\left(\sigma_d + d/2 + 1, \lambda t\right) - \frac{t^{\sigma_d+1}}{\Gamma(\sigma_d+1)} L^{cl}_{\sigma_d,d} \int_{\lambda}^{\infty} M\left(l_\Lambda\right) \Lambda^{\sigma_d+d/2} e^{-\Lambda t} d\Lambda \,. \tag{6.40}$$

We are now in the position to provide bounds on $Z(t)$ depending only on the volume of Ω. To this end we use inequality (6.7) and calculate $M\left(l_\Lambda\right)$ explicitly on the ball.

PROPOSITION 6.8. *Let $\lambda \in [\tilde{\lambda}, \lambda_1]$. For any open set $\Omega \subset \mathbb{R}^d$ and any $t > 0$ the bound*

$$\begin{aligned} Z(t) &\leq \frac{|\Omega|}{(4\pi t)^{d/2}} \, \hat{\Gamma}\left(\sigma_d + d/2 + 1, \lambda t\right) \\ &\quad - \frac{|\Omega|}{(4\pi t)^{d/2} \, \Gamma\left(\sigma_d + d/2 + 1\right)} \int_{\lambda t}^{\infty} e^{-s} s^{\sigma_d + d/2} \tilde{B}\left(\frac{\pi^2 t}{4R^2 s}, \frac{1}{2}, \frac{d+1}{2}\right) ds \end{aligned}$$

holds true, where $R = R\left(|\Omega|\right)$ is the radius of the ball $B_R \subset \mathbb{R}^d$ with $|B_R| = |\Omega|$.

PROOF. Lets consider the ball B_R and apply (6.40) to estimate $Z^*(t)$, i.e. $Z(t)$ on B_R. Note that $m_i(\tau; B_R) = m(\tau; B_R)$ for $i = 1, \ldots, d$ and we can choose an arbitrary coordinate system $(x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$.

For $\tau < 2R$ the set $\{x' : p_d(x', B_R) > \tau\}$ is itself a ball in $\mathbb{R}^{d-1}$ with radius $\left(R^2 - \tau^2/4\right)^{1/2}$. Thus we find

$$m(\tau; B_R) = \left|\{x' : p_d(x', B_R) > \tau\}\right|_{d-1} \, = \, R^{d-1} \, \frac{\pi^{(d-1)/2}}{\Gamma\left(\frac{d+1}{2}\right)} \left(1 - \frac{\tau^2}{4R^2}\right)_+^{(d-1)/2}$$

and

$$M(y; B_R) = \frac{\pi^{(d-1)/2}}{\Gamma\left(\frac{d+1}{2}\right)} R^d \, B\left(0, \frac{y^2}{4R^2}, \frac{1}{2}, \frac{d+1}{2}\right) = |B_R| \, \tilde{B}\left(\frac{y^2}{4R^2}, \frac{1}{2}, \frac{d+1}{2}\right) .$$

We insert this estimate into (6.40) and arrive at

$$\begin{aligned} Z^*(t) &\leq \frac{|B_R|}{(4\pi t)^{d/2}} \hat{\Gamma}\left(\sigma_d + d/2 + 1, \lambda t\right) \\ &\quad - \frac{t^{\sigma_d+1}}{\Gamma(\sigma_d+1)} L^{cl}_{\sigma_d,d} \, |B_R| \int_{\lambda}^{\infty} \tilde{B}\left(\frac{\pi^2}{4R^2\Lambda}, \frac{1}{2}, \frac{d+1}{2}\right) \Lambda^{\sigma_d+d/2} e^{-\Lambda t} \, d\Lambda \,. \end{aligned}$$

The assumption $\lambda \geq \tilde{\lambda}$ implies $\frac{\pi^2}{4R^2} < \lambda$ and in view of (6.7) and $|B_R| = |\Omega|$ the claimed result follows by simplifying the right hand side. □

We can now derive Theorem 6.1 from Proposition 6.8:

Proof of Theorem 6.1. The inequality

$$(1-u)^{(d-1)/2} \geq 1 - \frac{d-1}{2}u\,, \quad 0 \leq u \leq 1\,,$$

implies the estimate

$$\begin{aligned}\tilde{B}\left(\frac{\pi^2 t}{4R^2 s}, \frac{1}{2}, \frac{d+1}{2}\right) &\geq \frac{1}{B\left(\frac{1}{2}, \frac{d+1}{2}\right)} \int_0^{(\pi^2 t)/(4R^2 s)} u^{-1/2}\left(1 - \frac{d-1}{2}u\right) du \\ &= \frac{\Gamma\left(d/2+1\right)}{\Gamma\left(\frac{d+1}{2}\right)}\left(\frac{\sqrt{\pi t}}{R\sqrt{s}} - \frac{(d-1)\pi^{5/2}t^{3/2}}{24R^3 s^{3/2}}\right).\end{aligned}$$

Therefore we claim

$$\begin{aligned}&\int_{\lambda t}^{\infty} e^{-s} s^{\sigma_d + d/2} \tilde{B}\left(\frac{\pi^2\, t}{4R^2\, s}, \frac{1}{2}, \frac{d+1}{2}\right) ds \\ &\quad \geq \frac{\Gamma\left(d/2+1\right)}{\Gamma\left(\frac{d+1}{2}\right)}\left(\frac{\sqrt{\pi t}}{R}\,\Gamma\left(\sigma_d + \frac{d+1}{2}, \lambda t\right) - \frac{(d-1)\pi^{5/2}t^{3/2}}{24R^3}\Gamma\left(\sigma_d + \frac{d-1}{2}, \lambda t\right)\right).\end{aligned}$$

Inserting the last estimate into the bound from Proposition 6.8 yields

$$Z(t) \leq \frac{|\Omega|}{(4\pi t)^{d/2}}\,\hat{\Gamma}\left(\sigma_d + d/2 + 1, \lambda t\right) - R(t,\lambda)$$

with $R(t,\lambda) = r_1(t,\lambda) - r_2(t,\lambda)$ and

$$\begin{aligned} r_1(t,\lambda) &= \frac{|\Omega|}{(4\pi t)^{d/2}}\,\frac{\Gamma\left(d/2+1\right)}{\Gamma\left(\frac{d+1}{2}\right)}\,\frac{\sqrt{\pi t}}{R}\,\frac{\Gamma\left(\sigma_d + \frac{d+1}{2}, \lambda t\right)}{\Gamma\left(\sigma_d + d/2 + 1\right)} \\ r_2(t,\lambda) &= \frac{|\Omega|}{(4\pi t)^{d/2}}\,\frac{\Gamma\left(d/2+1\right)}{\Gamma\left(\frac{d+1}{2}\right)}\,\frac{(d-1)\pi^{5/2}\,t^{3/2}}{24R^3}\,\frac{\Gamma\left(\sigma_d + \frac{d-1}{2}, \lambda t\right)}{\Gamma\left(\sigma_d + d/2 + 1\right)}\,.\end{aligned}$$

From $|B_R| = |\Omega|$ we deduce

$$R = \frac{|\Omega|^{1/d}}{\sqrt{\pi}}\Gamma\left(d/2+1\right)^{1/d}\,, \tag{6.41}$$

and get

$$\begin{aligned} r_1(t,\lambda) &= \frac{|\Omega|^{(d-1)/d}}{(4\pi t)^{(d-1)/2}}\,\frac{B\left(\frac{1}{2}, \sigma_d + \frac{d+1}{2}\right)}{2}\,\frac{\Gamma\left(d/2+1\right)^{(d-1)/d}}{\Gamma\left(\frac{d+1}{2}\right)}\hat{\Gamma}\left(\sigma_d + \frac{d+1}{2}, \lambda t\right) \\ r_2(t,\lambda) &= \frac{|\Omega|^{(d-3)/d}}{(4\pi t)^{(d-3)/2}} \\ &\quad \times \frac{\pi^2(d-1)B\left(\frac{1}{2}, \sigma_d + \frac{d+1}{2}\right)}{96(2\sigma_d + d - 1)}\,\frac{\Gamma\left(d/2+1\right)^{(d-3)/d}}{\Gamma\left(\frac{d+1}{2}\right)}\hat{\Gamma}\left(\sigma_d + \frac{d-1}{2}, \lambda t\right).\end{aligned}$$

To complete the proof it remains to note that in view of (6.38) we can always estimate the remainder term $R(t,\lambda)$ from above by zero. □

According to (6.11) we can choose

$$\lambda \,=\, \tilde{\lambda} \,=\, \frac{\pi\, j_{d/2-1,1}^2}{\Gamma\left(d/2+1\right)^{2/d} |\Omega|^{2/d}}$$

as a suitable lower bound on λ_1. With this special choice of parameter we find

COROLLARY 6.9. *For any open set $\Omega \subset \mathbb{R}^d$ with finite volume and all $t > 0$*

$$Z(t) \,\leq\, \frac{|\Omega|}{(4\pi t)^{d/2}}\, \hat{\Gamma}\left(\sigma_d + d/2 + 1, \tilde{\lambda}\, t\right) \,-\, (R(t))_+ \tag{6.42}$$

holds true with

$$R(t) = c_{1,d} \frac{|\Omega|^{(d-1)/d}}{(4\pi t)^{(d-1)/2}} \hat{\Gamma}\left(\sigma_d + \frac{d+1}{2}, \tilde{\lambda} t\right) - c_{2,d} \frac{|\Omega|^{(d-3)/d}}{(4\pi t)^{(d-3)/2}} \hat{\Gamma}\left(\sigma_d + \frac{d-1}{2}, \tilde{\lambda} t\right)$$

and constants $c_{1,d}$, $c_{2,d}$ given explicitly in Theorem 6.1.

Now we can apply (6.7) to known estimates on $Z(t)$ and compare the resulting universal bounds with the result from Corollary 6.9.

To analyse the asymptotics of $Z(t)$ for $t \to 0+$ on convex domains van den Berg proved [**vdB84b**] that for all convex domains $D \subset \mathbb{R}^d$ and all $t > 0$

$$Z(t) \leq \frac{|D|}{(4\pi t)^{d/2}} - \frac{|\partial D|}{4(4\pi t)^{(d-1)/2}} + \frac{(d-1)\,|\partial D|\, t}{2\,(4\pi t)^{d/2}\, R},$$

where ∂D denotes the boundary of D and at each point of ∂D the curvature is bounded by $1/R$. To prove bounds for general domains Ω we can apply this bound to the ball. Note that

$$|\partial B_R| \,=\, d\,\pi^{d/2} \frac{R^{d-1}}{\Gamma\left(d/2+1\right)}\,.$$

In view of (6.7) and (6.41) we find

COROLLARY 6.10. *For any open domain $\Omega \subset \mathbb{R}^d$ and any $t > 0$*

$$Z(t) \leq \frac{|\Omega|}{(4\pi t)^{d/2}} - \frac{d\sqrt{\pi}}{\Gamma\left(d/2+1\right)^{1/d}} \frac{|\Omega|^{(d-1)/d}}{4(4\pi t)^{(d-1)/2}} + \frac{d(d-1)}{\Gamma\left(d/2+1\right)^{2/d}} \frac{|\Omega|^{(d-2)/d}}{8\,(4\pi t)^{(d-2)/2}}\,.$$

REMARK. The bounds from Corollary 6.9 and Corollary 6.10 both capture the main asymptotic behaviour of $Z(t)$ as t tends to zero. Moreover, they contain order-sharp remainder terms. Actually, in the regime $t \to 0+$ the bound form Corollary 6.10 is stronger than (6.42). On the other hand the bound from Corollary 6.10 does not show an exponential decay as t tends to infinity.

Finally, one can use the ideas of [**Mel03**] and [**HH07**] to derive universal bounds on $Z(t)$. We can employ inequality (6.5) and the result of Luttinger (6.7). For the ball $B_R \subset \mathbb{R}^d$ with $|B_R| = |\Omega|$ the second moment $I\left(B_R\right)$ can be calculated explicitly. If we insert the result into (6.5) we find

$$Z(t) \,\leq\, \frac{|\Omega|}{(4\pi t)^{d/2}}\, \exp\left(-\tilde{M}_d \frac{t}{|\Omega|^{2/d}}\right)\,, \tag{6.43}$$

with a constant $\tilde{M}_d = \frac{d+2}{d}\,\pi\,\Gamma\left(d/2+1\right)^{-2/d} M_d$. For example, in dimension $d=2$ we have $M_2 = \frac{1}{32}$, see [**KVW09**], and we get

$$Z(t) \;\leq\; \frac{|\Omega|}{4\pi t}\,\exp\left(-\frac{\pi}{16}\frac{t}{|\Omega|}\right)\,.$$

In general we have $\tilde{M}_d < 1$ and the estimate (6.43) is not strong enough to imply the conjectured inequality (6.6).

But one can employ Corollary 6.9 to prove (6.6) at least in low dimensions. To analyse the asymptotic behaviour of the bound from Corollary 6.9 we refer to the inequalities

$$\begin{array}{llll} j_{0,1} & > 2.4 & > \pi^{-1/2} & \text{if } d=2 \\ j_{\frac{1}{2},1} & > 3.1 & > \Gamma\left(5/2\right)^{1/3}\,\pi^{-1/2} & \text{if } d=3 \\ j_{d/2-1,1} & > d/2-1 & > \Gamma\left(d/2+1\right)^{1/d}\,\pi^{-1/2} & \text{if } d\geq 4\,, \end{array}$$

see [**AS72**]. We find

$$\tilde{\lambda} = \frac{\pi\, j_{d/2-1,1}^2}{\Gamma\left(2/d+1\right)^{2/d}\,|\Omega|^{2/d}} > \frac{1}{|\Omega|^{2/d}}\,.$$

In view of (6.9) we deduce that (6.42) is stronger than (6.6) in the limit $t\to\infty$. Moreover one can employ (6.13) to show that this relation holds true also in the limit $t\to 0+$. Finally one can compare the bounds for finite values of t numerically and find that (6.42) is stronger than (6.6) for all $t>0$ if $d\leq 633$ and that in these dimensions conjecture (6.6) holds true.

On the other hand numerical evaluations show that for dimensions $d>633$ there exist $t>0$ so that the bound in (6.6) is smaller than the bound in (6.42). Since the conjecture (6.6) does not show the expected asymptotic properties we confine ourselves to this numerical discussion.

6.5. Heat kernel estimates in unbounded domains

Here we use the methods introduced in Section 6.3 to prove sharp upper bounds on $Z(t)$ in unbounded domains, in particular in domains with infinite volume. In such domains, not much is known about universal bounds on $Z(t)$, see [**Dav85, Dav89**] for results valid in a very general setting. As an example for unbounded domains $\Omega\subset\mathbb{R}^2$, so called "horn-shaped" regions were introduced [**Sim83, vdB84a**]: Assume $f:[0,\infty)\to[0,\infty)$ is a non-increasing function with $\lim_{s\to\infty} f(s)=0$ and put

$$\Omega_f = \left\{(x,y)\in\mathbb{R}^2 : x>0\,,\,0<y<f(x)\right\}\,. \tag{6.44}$$

Then Ω_f is "horn-shaped". Lets state some examples, where the short time asymptotics of $Z_f(t)$ can be computed explicitly. Assume $f_\mu(s)=s^{-1/\mu}$, $\mu\geq 1$. Then for $t\to 0+$ we get

$$Z(t;\Omega_{f_\mu}) \;=\; \frac{\Gamma\left(1+\mu/2\right)\zeta(\mu)}{2\pi^{\mu+1/2}}\,t^{-(\mu+1)/2} + o\left(t^{-(\mu+1)/2}\right) \tag{6.45}$$

if $\mu>1$ and

$$Z(t;\Omega_{f_1}) \;=\; -\frac{\ln t}{4\pi t} + \frac{1+\gamma-2\ln(2\pi)}{4\pi t} + O\left(t^{-1/2}\right)\,, \tag{6.46}$$

where $\zeta(\mu)$ is the Zeta function and γ denotes Euler's constant, see [**Sim83**] and [**ST90**] for refined results. Moreover one can choose $f_e(s) = \exp(-2s)$ and find

$$Z(t;\Omega_{f_e}) = \frac{1}{4\pi t} + \frac{\ln t}{4\sqrt{\pi t}} + O\left(t^{-1/2}\right) \tag{6.47}$$

as $t \to 0+$, see [**vdB87**] and [**ST90**].

In order to derive universal bounds on $Z(t)$ in unbounded domains, let us first note that all results mentioned in the previous sections, in particular Theorem 6.1 and Corollary 6.9 remain valid for unbounded domains $\Omega \subset \mathbb{R}^d$, as long as $|\Omega|$ is finite. Even if the volume of Ω is infinite, the estimate (6.39) holds true, as long as Ω_Λ is finite. Moreover, one can use Proposition 6.5 to estimate $R_\sigma(\Lambda)$ and $Z(t)$, as long as

$$\int_{l_\Lambda}^{\infty} m_i(\tau;\Omega)\, d\tau < \infty \tag{6.48}$$

for all $\Lambda > 0$. This condition is satisfied for $i = d$ and a suitable choice of coordinate system $(x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$, whenever

$$m_d(\tau;\Omega) = o\left(\tau^{-1}\right) \ , \quad \tau \to \infty\,.$$

For example we can apply Proposition 6.5 to horn-shaped regions introduced in (6.44) with $f_\mu(s) = s^{-1/\mu}$, $\mu > 0$. First we derive bounds that decay exponentially as t tends to infinity:

THEOREM 6.11. *For $\mu > 0$ and all $t > 0$*

$$\begin{aligned} Z(t;\Omega_{f_\mu}) \leq & \frac{4}{105\pi^{3/2}} \frac{1}{\mu - 1} \left(\frac{2}{\pi^2}\right)^{(\mu-1)/2} t^{-(\mu+1)/2}\, \hat{\Gamma}\left(\frac{\mu}{2} + 4, \frac{\pi^2}{2} t\right) + \frac{1}{4\pi t}\, \hat{\Gamma}\left(\frac{9}{2}, \frac{\pi^2}{2} t\right) \\ & + \frac{4}{105\pi^{3/2}} \frac{\mu}{1 - \mu} \left(\frac{2}{\pi^2}\right)^{(1-\mu)/(2\mu)} t^{-(1+\mu)/(2\mu)}\, \hat{\Gamma}\left(\frac{1}{2\mu} + 4, \frac{\pi^2}{2} t\right) \end{aligned}$$

if $\mu \neq 1$ and

$$Z(t;\Omega_{f_1}) \leq \left(-\frac{\ln t}{4\pi t} - \frac{1}{4\pi t}(2\ln\pi - \ln 2)\right) \hat{\Gamma}\left(\frac{9}{2}, \frac{\pi^2}{2} t\right) + \frac{4}{105\pi^{3/2} t} \int_{\pi^2 t/2}^{\infty} s^{7/2} e^{-s} \ln s\, ds\,.$$

PROOF. In order to apply Proposition 6.5, choose a coordinate system $(x_1, x_2) \in \mathbb{R}^2$ rotated by $\frac{\pi}{4}$ with respect to the coordinate system (x, y) used in definition (6.44). Then for $x_1 = 0$ we have

$$p_2\left(0;\Omega_{f_\mu}\right) = \left|\left\{x_2 : (0, x_2) \in \Omega_{f_\mu}\right\}\right|_1 = \sqrt{2}$$

and we find that $m_d\left(\tau;\Omega_{f_\mu}\right) = 0$ for all $\tau \geq \sqrt{2}$. Moreover, we can estimate

$$p_2\left(x_1;\Omega_{f_\mu}\right) \ \leq \ \sqrt{2}\ f_\mu\left(\sqrt{2}\, x_1\right)$$

if $x_1 > 0$ and

$$p_2\left(x_1;\Omega_{f_\mu}\right) \ \leq \ \sqrt{2}\ f_\mu^{-1}\left(\sqrt{2}\, |x_1|\right)$$

if $x_1 < 0$, hence

$$m_d\left(\tau;\Omega_{f_\mu}\right) \ \leq \ 2^{(\mu-1)/2}\tau^{-\mu} + 2^{(1-\mu)/(2\mu)}\tau^{-1/\mu}$$

for all $0 < \tau < \sqrt{2}$. Inserting these estimates into the inequality from Proposition 6.5 with $\sigma = \sigma_2 = 5/2$ yields

$$R_{5/2}(\Lambda) \,=\, 0 \quad \text{for all } 0 < \Lambda \leq \frac{\pi^2}{2}$$

and for $\Lambda > \frac{\pi^2}{2}$ we get

$$R_{5/2}(\Lambda) \,\leq\, \frac{1}{14\pi}\left(1 - \frac{1}{1-\mu}\,\pi^{1-\mu}\,(2\Lambda)^{(\mu-1)/2} - \frac{\mu}{\mu-1}\,\pi^{1-1/\mu}\,(2\Lambda)^{(1-\mu)/(2\mu)}\right)\Lambda^{7/2}$$

if $\mu \neq 1$ and

$$R_{5/2}(\Lambda) \,\leq\, \frac{1}{14\pi}\,(\ln\Lambda + \ln 2 - 2\ln\pi)\,\Lambda^{7/2}$$

if $\mu = 1$. Finally by applying the Laplace transformation to these inequalities and simplifying the resulting estimates on $Z(t)$ we arrive at the claimed results. □

Remark. Comparing these bounds with the asymptotic results (6.45) and (6.46) we see that the main terms capture the correct order in t as $t \to 0+$. In the case $\mu = 1$ the first term contains the sharp constant and even the second term is of correct order.

If one is mainly interested in the short time behaviour, one can improve the leading term of the bound for $\mu > 1$, to get the sharp constant as well; see Theorem 6.12 and the subsequent remark, below.

To generalise these considerations to higher dimensions we use slightly different notions. Assume a non-negative function $m(\tau)$ is given for $\tau > 0$, right-continuous, non-increasing and satisfying $m(\tau) = o(\tau^{-1})$ as $t \to \infty$. Choose

$$f(s) = \inf\left\{\tau > 0 : m(\tau) \leq \omega_{d-1}s^{d-1}\right\},$$

where $\omega_{d-1} = \pi^{(d-1)/2}\,\Gamma\left(\frac{d+1}{2}\right)^{-1}$ denotes the volume of the unit ball in $\mathbb{R}^{d-1}$, and put

$$\tilde{\Omega}_f = \left\{(x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} : |x_d| < \frac{1}{2}f\,(|x'|)\right\}.$$

Then $\tilde{\Omega}_f$ represents an example of a domain with the distribution function

$$m_d(\tau; \tilde{\Omega}_f) \,=\, m(\tau)\,.$$

In this case, to study explicit examples, we choose $f_\mu(s) = \left(\omega_{d-1}s^{d-1}\right)^{-1/\mu}$.

Theorem 6.12. *For any $\mu > 1$ and all $t > 0$*

$$Z(t; \tilde{\Omega}_{f_\mu}) \,\leq\, \frac{\Gamma(\mu/2+1)\,\zeta(\mu)}{(d-1)\,2^{d-1}\,\pi^{\mu+(d-1)/2}}\,t^{-(\mu+d-1)/2}\,.$$

Remark. In dimension $d = 2$, according to [**Sim83, vdB84a, ST90**], the asymptotics (6.45), (6.46) and (6.47) are valid for $\tilde{\Omega}_f$ as well. For $d = 2$ Theorem 6.12 reads as

$$Z(t; \tilde{\Omega}_{f_\mu}) \,\leq\, \frac{\Gamma(\mu/2+1)\,\zeta(\mu)}{2\,\pi^{\mu+1/2}}\,t^{-(\mu+1)/2}$$

and from (6.45) we see, that this bound coincides with the leading asymptotic term.

PROOF OF THEOREM 6.12. The definition of $\tilde{\Omega}_{f_\mu}$ and the choice of f_μ implies $m_d(\tau, \Omega_{f_\mu}) = \tau^{-\mu}$. Hence, one could immediately employ Proposition 6.5 to derive bounds on $R_\sigma(\Lambda)$.

To get a more precise result, we start from (6.23) and use the notation introduced in Section 6.3. By definition of $\tilde{\Omega}_{f_\mu}$, we have $p_d(x') = f_\mu(|x'|)$ for $x' \in \mathbb{R}^{d-1}$ and from (6.23) we obtain

$$\begin{aligned} R_\sigma(\Lambda) &\leq \Lambda^{\sigma+(d-1)/2}\, L^{cl}_{\sigma,d-1} \int_{\mathbb{R}^{d-1}} \sum_{j\geq 1} \left(1 - \frac{l_\Lambda^2 j^2}{f_\mu(|x'|)^2}\right)_+^{\sigma+(d-1)/2} dx' \\ &= \Lambda^{\sigma+(d-1)/2}\, L^{cl}_{\sigma,d-1}\, \omega_{d-1} \sum_{j\geq 1} \int_0^\infty \left(1 - \frac{\pi^2 j^2}{\Lambda\, f_\mu(s)^2}\right)_+^{\sigma+(d-1)/2} s^{d-2}\, ds \end{aligned}$$

for any $\sigma \geq 3/2$. Now we insert $f_\mu(s) = \left(\omega_{d-1} s^{d-1}\right)^{-1/\mu}$ and substitute

$$t \;=\; \frac{\pi^2\, j^2}{\Lambda}\, w_{d-1}^{2/\mu}\, s^{2(d-1)/\mu}$$

to get

$$\begin{aligned} R_\sigma(\Lambda) &\leq \Lambda^{\sigma+(d-1)/2}\, L^{cl}_{\sigma,d-1}\, \omega_{d-1} \sum_{j\geq 1} \left(\frac{\Lambda}{\pi^2 j^2}\right)^{\mu/2} \\ &\quad \times \frac{\mu}{2(d-1)\omega_{d-1}} \int_0^\infty (1-t)_+^{\sigma+(d-1)/2}\; t^{\mu/2-1}\, dt \\ &= L^{cl}_{\sigma,d}\, \frac{\Gamma(\sigma + d/2 + 1)\, \Gamma(\mu/2)}{\Gamma(\sigma + (d+1+\mu)/2)}\, \frac{\mu\, \zeta(\mu)}{(d-1)\, \pi^{\mu-1/2}}\, \Lambda^{\sigma+(d+\mu-1)/2}\,. \end{aligned}$$

To this inequality we can apply the Laplace transformation and simplify the resulting bound on $Z(t)$ to arrive at the claimed result. □

In order to state an example for unbounded domains with finite volume, choose $f_e(s) = \exp\left(-\omega_{d-1} s^{d-1}\right)$. In the same way as above one can show

THEOREM 6.13. *For all $t > 0$ the estimate*

$$\begin{aligned} Z(t, \tilde{\Omega}_{f_e}) &\leq \frac{1}{(4\pi t)^{d/2}} \hat{\Gamma}\left(\sigma_d + d/2 + 1, \pi^2 t\right) \\ &\quad + (\ln t + 2\ln\pi - 2)\, \frac{\sqrt{\pi}}{4(4\pi t)^{(d-1)/2}}\, \frac{\hat{\Gamma}\left(\sigma_d + \frac{d+1}{2}, \pi^2 t\right)}{\Gamma\left(\sigma_d + d/2 + 1\right)} \\ &\quad + \frac{\sqrt{\pi}}{4(4\pi t)^{(d-1)/2}}\, \frac{1}{\Gamma\left(\sigma_d + d/2 + 1\right)} \int_{\pi^2 t}^\infty s^{\sigma_d + d/2} \ln s\, e^{-s}\, ds \end{aligned}$$

holds true.

REMARK. In view of (6.2) the first term of this bound is sharp in the limit $t \to 0+$ since $|\Omega_{f_e}| = 1$. In dimension $d = 2$ we can use (6.8) and (6.47) to point out that even the second term of the bound captures the right order in t as t tends to zero.

6.6. Proof of Theorem 6.2

Here we use the results from Section 6.3 to derive universal bounds with correction terms on the Riesz means $R_\sigma(\Lambda)$. First we note that Proposition 6.5 and Lemma 6.7 immediately imply the following estimates (6.49) and (6.50). Recall that $\tau_\Omega = d^2\,\pi^2\,|\Omega|^{-2/d}$ and let $\sigma \geq \frac{3}{2}$ satisfy $\sigma + \frac{d-1}{2} \geq 3$, hence $\delta_{\sigma,d} = 0$. Then for any open domain $\Omega \subset \mathbb{R}^d$ and all $\Lambda > 0$ we find

$$R_\sigma(\Lambda) \;\leq\; L^{cl}_{\sigma,d}\,\frac{d-1}{d}\,|\Omega|\,\Lambda^{\sigma+d/2} \tag{6.49}$$

if $\Lambda < \tau_\Omega$ and

$$R_\sigma(\Lambda) \;\leq\; L^{cl}_{\sigma,d}\,|\Omega|\,\Lambda^{\sigma+d/2} - \pi\,L^{cl}_{\sigma,d}\,|\Omega|^{(d-1)/d}\Lambda^{\sigma+(d-1)/2} \tag{6.50}$$

if $\Lambda \geq \tau_\Omega$.

Next we discuss how a trick by Aizenman and Lieb [**AL78**] can be applied to inequalities for eigenvalue means $R_\gamma(\Lambda)$ with remainder terms.

LEMMA 6.14. *Let $\gamma > \sigma \geq \frac{3}{2}$, $\lambda_1 \geq \lambda \geq 0$ and $\Lambda \geq \lambda$. Then*

$$\begin{aligned} R_\gamma(\Lambda) \;\leq\; & L^{cl}_{\gamma,d}\,|\Omega|\,\Lambda^{\gamma+d/2}\,\hat{B}\left(\frac{\lambda}{\Lambda}, \sigma + d/2 + 1, \gamma - \sigma\right) \\ & - \frac{L^{cl}_{\sigma,d}}{B\,(\sigma+1,\gamma-\sigma)}\int_0^{\Lambda-\lambda} \tau^{\gamma-\sigma-1}\,M\,(l_{\Lambda-\tau};\Omega)\;(\Lambda-\tau)^{\sigma+d/2}\,d\tau \\ & + \frac{\delta_{\sigma,d}}{B\,(\sigma+1,\gamma-\sigma)}\int_0^{\Lambda-\lambda} \tau^{\gamma-\sigma-1}\,m\,(l_{\Lambda-\tau};\Omega)\;(\Lambda-\tau)^{\sigma+(d-1)/2}\,d\tau. \end{aligned} \tag{6.51}$$

PROOF. We start from the well-known identity [**AL78**]

$$\begin{aligned} R_\gamma(\Lambda) \;&=\; \frac{1}{B\,(\sigma+1,\gamma-\sigma)}\int_0^{\infty} \tau^{\gamma-\sigma-1}\,R_\sigma(\Lambda-\tau)\,d\tau \\ &=\; \frac{1}{B\,(\sigma+1,\gamma-\sigma)}\int_0^{\Lambda-\lambda} \tau^{\gamma-\sigma-1}\,R_\sigma(\Lambda-\tau)\,d\tau. \end{aligned} \tag{6.52}$$

Here we have taken into account that $R_\sigma(\tilde{\Lambda}) = 0$ for $\tilde{\Lambda} \leq \lambda \leq \lambda_1$. Now we can apply Proposition 6.5 and find

$$\begin{aligned} R_\gamma(\Lambda) \;\leq\; & \frac{L^{cl}_{\sigma,d}\,|\Omega|}{B\,(\sigma+1,\gamma-\sigma)}\int_0^{\Lambda-\lambda} \tau^{\gamma-\sigma-1}\;(\Lambda-\tau)^{\sigma+d/2}\;d\tau \\ & - \frac{L^{cl}_{\sigma,d}}{B\,(\sigma+1,\gamma-\sigma)}\int_0^{\Lambda-\lambda} \tau^{\gamma-\sigma-1}\,M\,(l_{\Lambda-\tau};\Omega)\;(\Lambda-\tau)^{\sigma+d/2}\,d\tau \\ & + \frac{\delta_{\sigma,d}}{B\,(\sigma+1,\gamma-\sigma)}\int_0^{\Lambda-\lambda} \tau^{\gamma-\sigma-1}\,m\,(l_{\Lambda-\tau};\Omega)\;(\Lambda-\tau)^{\sigma+(d-1)/2}\,d\tau. \end{aligned}$$

Finally let us evaluate the first term on the right hand side of this expression. A substitution of the integration variable $s = \frac{\tau}{\Lambda}$ gives

$$\begin{aligned}
&\frac{L^{cl}_{\sigma,d}|\Omega|}{B\left(\sigma+1,\gamma-\sigma\right)}\,\Lambda^{\gamma+d/2}\int_0^{1-\lambda/\Lambda} s^{\gamma-\sigma-1}\,(1-s)^{\sigma+d/2}\,ds \\
&\quad= |\Omega|\,\Lambda^{\gamma+d/2}L^{cl}_{\sigma,d}\frac{B(\gamma-\sigma,\sigma+d/2+1)}{B(\sigma+1,\gamma-\sigma)}\left(1-\frac{\int_0^{\lambda/\Lambda}(1-t)^{\gamma-\sigma-1}t^{\sigma+d/2}dt}{B(\gamma-\sigma,\sigma+d/2+1)}\right) \\
&\quad= |\Omega|\,\Lambda^{\gamma+d/2}L^{cl}_{\gamma,d}\left(1-\tilde{B}\left(\frac{\lambda}{\Lambda},\sigma+d/2+1,\gamma-\sigma\right)\right).
\end{aligned}$$

□

If we apply this Lemma with $\sigma = \sigma_d$ then because of $\delta_{\sigma_d,d} = 0$ the last term on the right hand side of (6.51) vanishes. This enables us to finish the proof of Theorem 6.2.

Proof of Theorem 6.2. Inequality (6.51) with $\gamma > \sigma = \sigma_d$ and with a substitution $y = l_{\Lambda-\tau} = \frac{\pi}{\sqrt{\Lambda-\tau}}$ gives

$$\begin{aligned}
R_\gamma(\Lambda) \;\le\; & L^{cl}_{\gamma,d}\,|\Omega|\,\Lambda^{\gamma+d/2}\hat{B}\left(\frac{\lambda}{\Lambda},\sigma_d+d/2+1,\gamma-\sigma_d\right) \\
& -\frac{2\pi^{2\sigma_d+d+2}L^{cl}_{\sigma_d,d}}{B\left(\sigma_d+1,\gamma-\sigma_d\right)}\int_{l_\Lambda}^{l_\lambda}\left(\Lambda-\frac{\pi^2}{y^2}\right)^{\gamma-\sigma_d-1} M(y;\Omega)\,y^{-2\sigma_d-d-3}\,dy
\end{aligned}$$

for all $\Lambda \ge \lambda$. First we assume $\lambda \ge \tau_\Omega$. Then we have $y\,|\Omega|^{(d-1)/d} \le \frac{1}{d}|\Omega|$ for all $l_\Lambda \le y \le l_\lambda$ and in view of Lemma 6.7 we get

$$\begin{aligned}
R_\gamma(\Lambda) \;\le\; & L^{cl}_{\gamma,d}\,|\Omega|\,\Lambda^{\gamma+d/2}\hat{B}\left(\frac{\lambda}{\Lambda},\sigma_d+d/2+1,\gamma-\sigma_d\right) \\
& -\frac{2\pi^{2\sigma_d+d+2}L^{cl}_{\sigma_d,d}}{B\left(\sigma_d+1,\gamma-\sigma_d\right)}\,|\Omega|^{(d-1)/d}\int_{l_\Lambda}^{l_\lambda}\left(\Lambda-\frac{\pi^2}{y^2}\right)^{\gamma-\sigma_d-1} y^{-2\sigma_d-d-2}\,dy\,.
\end{aligned}$$

If we substitute $s = \frac{\pi^2}{y^2\Lambda}$ and simplify the expression of the remainder term we arrive at

$$R_\gamma(\Lambda) \;\le\; L^{cl}_{\gamma,d}\,|\Omega|\,\Lambda^{\gamma+d/2}\hat{B}\left(\frac{\lambda}{\Lambda},\sigma_d+d/2+1,\gamma-\sigma_d\right) - S(\Lambda,\lambda)$$

with $S(\Lambda,\lambda)$ as stated in (6.14).

Next we assume $\lambda < \tau_\Omega$ and proceed in two steps. If at the same time $\Lambda < \tau_\Omega$ we have $y\,|\Omega|^{(d-1)/d} > \frac{1}{d}|\Omega|$ for all $l_\Lambda < y < l_\lambda$ and by similar calculations as above we arrive at

$$\begin{aligned}
R_\gamma(\Lambda) \;\le\; & L^{cl}_{\gamma,d}\,|\Omega|\,\Lambda^{\gamma+d/2}\hat{B}\left(\frac{\lambda}{\Lambda},\sigma_d+d/2+1,\gamma-\sigma_d\right) \\
& -\frac{2\pi^{2\sigma_d+d+2}L^{cl}_{\sigma_d,d}}{B\left(\sigma_d+1,\gamma-\sigma_d\right)}\,\frac{|\Omega|}{d}\int_{l_\Lambda}^{l_\lambda}\left(\Lambda-\frac{\pi^2}{y^2}\right)^{\gamma-\sigma_d-1} y^{-2\sigma_d-d-3}\,dy
\end{aligned}$$

and we obtain the claimed inequality with $S(\Lambda,\lambda)$ given in (6.15). On the other hand, if $\lambda < \tau_\Omega \le \Lambda$ we get

$$\begin{aligned} R_\gamma(\Lambda) \;&\le\; L^{cl}_{\gamma,d}\,|\Omega|\,\Lambda^{\gamma+d/2}\hat{B}\left(\frac{\lambda}{\Lambda},\sigma_d+d/2+1,\gamma-\sigma_d\right) \\ &\quad - \frac{2\pi^{2\sigma_d+d+2}L^{cl}_{\sigma_d,d}}{B\left(\sigma_d+1,\gamma-\sigma_d\right)}\,|\Omega|^{(d-1)/d}\int_{l_\Lambda}^{(1/d)|\Omega|^{1/d}}\left(\Lambda-\frac{\pi^2}{y^2}\right)^{\gamma-\sigma_d-1} y^{-2\sigma_d-d-2}\,dy \\ &\quad - \frac{2\pi^{2\sigma_d+d+2}L^{cl}_{\sigma_d,d}}{B\left(\sigma_d+1,\gamma-\sigma_d\right)}\,\frac{|\Omega|}{d}\int_{(1/d)|\Omega|^{1/d}}^{l_\lambda}\left(\Lambda-\frac{\pi^2}{y^2}\right)^{\gamma-\sigma_d-1} y^{-2\sigma_d-d-3}\,dy. \end{aligned}$$

In this case after a simplification we arrive at $S(\Lambda,\lambda)$ as stated in (6.16). Finally, if we apply (6.52) directly to (6.3) we claim

$$R_\gamma(\Lambda) \;\le\; L^{cl}_{\gamma,d}\,|\Omega|\,\Lambda^{\gamma+d/2}\,\hat{B}\left(\frac{\lambda}{\Lambda},\sigma_d+d/2+1,\gamma-\sigma_d\right).$$

Hence, in the final bound $S(\Lambda,\lambda)$ can be replaced by its positive part $(S(\Lambda,\lambda))_+$. □

Again in view of (6.11) we can choose

$$\lambda \;=\; \tilde{\lambda} \;=\; \frac{\pi\, j^2_{d/2-1,1}}{\Gamma\left(d/2+1\right)^{2/d}\,|\Omega|^{2/d}}$$

as a lower bound on λ_1. Thus we find

Corollary 6.15. *Let $\Omega\subset\mathbb{R}^d$ be an open set with finite volume. Then for $\gamma>\sigma_d$ the estimate*

$$R_\gamma(\Lambda) \;\le\; L^{cl}_{\gamma,d}\,|\Omega|\,\hat{B}\left(\frac{\tilde{\lambda}}{\Lambda},\sigma_d+d/2+1,\gamma-\sigma_d\right)\Lambda^{\gamma+d/2} \;-\; (S(\Lambda))_+$$

holds for all $\Lambda\ge\tilde{\lambda}$, where

$$S(\Lambda) \;=\; L^{cl}_{\gamma,d}\,|\Omega|\,\Lambda^{\gamma+d/2}\,\frac{1}{d}\,\hat{B}\left(\frac{\tilde{\lambda}}{\Lambda},\sigma_d+d/2+1,\gamma-\sigma_d\right)$$

if $\Lambda<\tau_\Omega$ and

$$\begin{aligned} S(\Lambda) \;&=\; L^{cl}_{\gamma,d-1}\,|\Omega|^{(d-1)/d}\Lambda^{\gamma+(d-1)/2}\frac{B\left(\frac{1}{2},\sigma_d+\frac{d+1}{2}\right)}{2}\,\hat{B}\left(\frac{\tau_\Omega}{\Lambda},\sigma_d+\frac{d+1}{2},\gamma-\sigma_d\right) \\ &\quad + L^{cl}_{\gamma,d}\,|\Omega|\,\Lambda^{\gamma+d/2}\,\frac{1}{d}\,\tilde{B}\left(\frac{\tilde{\lambda}}{\Lambda},\frac{\tau_\Omega}{\Lambda},\sigma_d+d/2+1,\gamma-\sigma_d\right) \end{aligned}$$

if $\Lambda\ge\tau_\Omega$.

Remark. We can now compare this result with estimate (6.18) from Proposition 6.3. In both bounds the high energy asymptotics $\Lambda\to\infty$ is dominated by the sharp first term. In view of (6.10) also the remainder terms show the correct order as Λ tends to infinity. In this limit the bound from Corollary 6.15 is stronger than (6.18) whenever

$$\varepsilon\left(\gamma+\frac{d-1}{2}\right)d_\Lambda(\Omega) < \frac{1}{2}\,B\left(\frac{1}{2},\sigma_d+\frac{d+1}{2}\right)|\Omega|^{(d-1)/d} \tag{6.53}$$

holds true. We remark that the right hand side is independent of γ while $\varepsilon\left(\gamma + \frac{d-1}{2}\right)$ tends to zero as γ tends to infinity and $d_\Lambda(\Omega)$ is bounded from above by the diameter of Ω. Hence condition (6.53) will be satisfied for large enough γ.

Moreover, the bound from Corollary 6.15 contains the factor $\hat{B}(\tilde{\lambda}/\Lambda, \sigma_d + d/2 + 1, \gamma - \sigma_d)$ which decays exponentially if $\Lambda \to \tilde{\lambda}+$ and which improves the bound from Theorem 6.2 in comparison to (6.18) for values of Λ close to $\tilde{\lambda}$.

CHAPTER 7

Sharp spectral estimates in domains of infinite volume

Leander Geisinger and Timo Weidl

Abstract. We consider the Dirichlet Laplace operator on open, quasi-bounded domains of infinite volume. For such domains semiclassical spectral estimates based on the phase-space volume – and therefore on the volume of the domain – must fail. Here we present a method how one can nevertheless prove uniform bounds on eigenvalues and eigenvalue means which are sharp in the semiclassical limit.
We give examples in horn-shaped regions and so-called spiny urchins. Some results are extended to Schrödinger operators defined on quasi-bounded domains with Dirichlet boundary conditions.

7.1. Introduction

Let $V(x)$ be a non-negative function on an open set $\Omega \subset \mathbb{R}^d$, $d \geq 1$. In this article we study the negative spectrum of Schrödinger operators

$$H_\Omega \,=\, -\Delta - V$$

defined in $L^2(\Omega)$ with Dirichlet conditions on the boundary of Ω. More precisely, one defines H_Ω to be the self-adjoint operator generated by the quadratic form

$$\langle u, H_\Omega u\rangle \,=\, \int_\Omega |\nabla u(x)|^2\,dx - \int_\Omega V(x)\,|u(x)|^2\,dx\,,$$

with form domain $H_0^1(\Omega)$, see [**BS87**] for details. We always assume that H_Ω has purely discrete spectrum. Then the negative spectrum of H_Ω, if not empty, consists of finitely many eigenvalues $-\lambda_1 \leq -\lambda_2 \leq \cdots - \lambda_N < 0$, $N < \infty$, counted with multiplicity. In general, these eigenvalues cannot be calculated explicitly and for large N it is difficult to approximate them numerically. Hence, to deduce information about the eigenvalues one studies also the Riesz means

$$R_\sigma(V;\Omega) \,=\, \mathrm{Tr}(H_\Omega)_-^\sigma \,=\, \sum_{k=1}^N \lambda_k^\sigma$$

of order $\sigma \geq 0$ and their dependence on Ω and V.

The first rigorous step in this direction dates back to H. Weyl, R. Courant and D. Hilbert [**Wey12a, CH24**] who calculated the semiclassical limit of the eigenvalues in the case of a constant potential. To state the general result let us introduce a

scaling parameter $\lambda > 0$ and replace the potential V by λV. Then for $\sigma \geq 0$ and $V \in L^{\sigma+d/2}(\Omega)$ the limit

$$R_\sigma(\lambda V;\Omega) \;=\; L^{cl}_{\sigma,d} \int_\Omega V(x)\,dx\,\lambda^{\sigma+d/2} + o(\lambda^{\sigma+d/2})\,, \quad \lambda\to\infty\,, \tag{7.1}$$

holds with the semiclassical constant

$$L^{cl}_{\sigma,d} \;=\; \frac{\Gamma(\sigma+1)}{(4\pi)^{d/2}\Gamma(\sigma+\frac{d}{2}+1)}\,,$$

see e.g. [**RS78**]. To get information about finite potentials one needs to supplement this asymptotic result with uniform estimates. In [**LT76**] it was shown that for $\Omega=\mathbb{R}^d$ and $\sigma > \max\{0, 1-d/2\}$ the estimate

$$R_\sigma(V;\mathbb{R}^d) \;\leq\; L_{\sigma,d} \int_{\mathbb{R}^d} V(x)^{\sigma+d/2}\,dx$$

holds with certain positive constants $L_{\sigma,d}$. These inequalities have many important applications, for example, in proving the stability of matter [**Lie97, LS10**].

Finding the best constants for which the Lieb-Thirring inequalities hold poses a substantial mathematical challenge. In [**LW00**] the inequalities were established for $\sigma \geq 3/2$ with the sharp constants $L_{\sigma,d} = L^{cl}_{\sigma,d}$. This result immediately implies that for any open set $\Omega\subset\mathbb{R}^d$, $\sigma\geq 3/2$, and any non-negative potential $V\in L^{\sigma+d/2}(\Omega)$

$$R_\sigma(V;\Omega) \;\leq\; L^{cl}_{\sigma,d} \int_\Omega V(x)^{\sigma+d/2}\,dx\,. \tag{7.2}$$

If $V\in L^{\sigma+d/2}(\Omega)$ then both (7.1) and (7.2) hold and we see that the bound (7.2) is sharp: It shows the correct power of V and holds with the sharp constant.

In this article we are interested in the case $V\notin L^{\sigma+d/2}(\Omega)$, where the bound (7.2) and even the asymptotics (7.1) must fail and one needs to find a new approach to get sharp uniform bounds on eigenvalues means. If $V\notin L^{\sigma+d/2}(\Omega)$ the leading order of the semiclassical limit depends on the potential V and on the geometry of Ω and it is challenging to find estimates that take these dependencies into account.

Let us discuss the case of a constant potential $V\equiv\Lambda>0$ on Ω in more detail. If Ω is bounded then the semiclassical limit (7.1) reads as

$$R_\sigma(\Lambda;\Omega) \;=\; L^{cl}_{\sigma,d}\,|\Omega|\,\Lambda^{\sigma+d/2} + o(\Lambda^{\sigma+d/2})\,, \quad \sigma\geq 0\,, \quad \Lambda\to\infty\,, \tag{7.3}$$

where $|\Omega|$ denotes the volume of Ω. In this case the asymptotic results are supplemented by the Berezin-Lieb-Li-Yau inequality [**Ber72b, Lie73, LY83**]: For $\sigma\geq 1$

$$R_\sigma(\Lambda;\Omega) \;\leq\; L^{cl}_{\sigma,d}\,|\Omega|\,\Lambda^{\sigma+d/2}\,, \quad \Lambda>0\,. \tag{7.4}$$

Again, the constant in this inequality is sharp and cannot be improved. However, under certain conditions on the geometry of Ω a negative second term exists in the semiclassical limit (7.3), see [**Ivr80a, Ivr80b, Hör85, SV97, Ivr98, FG11b**], and the question arises whether (7.4) can be improved by an additional negative correction term. Recently, several results have been found giving answer to this question [**FLU02, Mel03, Wei08, KVW09, GW10, GLW11**]. In [**FLU02**] the corresponding sharp estimate for the discrete Laplacian was improved by a negative remainder term capturing the properties of the second term of the semiclassical

limit. The first uniform improvement for the continuous Laplacian is due to A. Melás [**Mel03**]. He improved the estimate (7.4) for $\sigma \geq 1$, however, the remainder does not reflect the correct order of the second term of the semiclassical limit.

In [**Wei08**] this was improved in the case $\sigma \geq 3/2$. Using an inductive argument based on operator-valued Lieb-Thirring inequalities [**LW00**] the Berezin inequality (7.4) was strengthened by a negative remainder term of correct order in comparison with the second term of the semiclassical limit. Here we are not concerned with the remainder term but we apply the same inductive argument to derive sharp spectral inequalities in domains of infinite volume.

However, for unbounded domains Ω even the discreteness of the spectrum of the Dirichlet Laplacian is no longer guaranteed. A necessary condition is the so called quasi-boundedness of Ω [**AF03**] which is satisfied, by definition, if

$$\lim_{\substack{x\in\Omega \\ |x|\to\infty}} \operatorname{dist}(x, \partial\Omega) \;=\; 0\,.$$

But even for quasi-bounded domains (7.3) and (7.4) must fail if the volume of Ω is infinite. In this article we show that one can nevertheless prove uniform bounds on the eigenvalue means for certain domains with infinite volume. In this case the leading order of the semiclassical limit depends on the geometry of Ω, see e.g. [**Fle78, Sim83**]. However, applying the induction-in-the-dimension argument from [**Wei08**] we can prove sharp estimates valid for all $\Lambda > 0$ that capture the correct asymptotic behavior.

If the potential V is not constant the situation is more difficult. The same inductive argument can still be used to reduce the problem to one dimension. But in contrast to the case of constant potential the eigenvalues of the resulting one-dimensional operator cannot be calculated explicitly. Therefore we have to study the one-dimensional problem in more detail. In particular, we have to analyze the effect of different boundary conditions on the eigenvalues. The result yields an improved version of the semiclassical bound (7.2). Again, this sharp Lieb-Thirring inequality with remainder term can be applied in situations, where all known results – in particular (7.1) and (7.2) – fail.

The remainder of the article is structured as follows. First we mention some key ingredients of the proofs. In particular, we review the induction-in-the-dimension argument from [**Wei08**] and adapt it to our needs here. This is done in Section 7.2.

In Section 7.3 we consider constant potentials on domains with infinite volume. We give examples, where the leading order of the semiclassical limit depends on the geometry of the domain Ω. In these examples we derive sharp upper bounds on the eigenvalue means.

The last part of the article is devoted to the general setting of non-constant potentials. In Section 7.4 we first analyze the effect of different boundary conditions on the eigenvalues of one-dimensional Schrödinger operators. We find an improvement of (7.2) that can be generalized to higher dimensions. Finally, we give an example to show that the result applies to certain potentials $V \notin L^{\sigma+d/2}(\Omega)$.

7.2. Induction in the dimension

In this section we prove an inequality reducing estimates for eigenvalue means of the operator H_Ω to estimates for one-dimensional Schrödinger operators. The proof relies on a lifting technique from [**Lap97**] and uses operator-valued Lieb-Thirring inequalities [**LW00**]. Here we follow the proof from [**Wei08**], where this approach of induction-in-the-dimension is employed to derive improvements of (7.4) for constant potentials.

Fix a Cartesian coordinate system in $\mathbb{R}^d$ and for $x \in \mathbb{R}^d$ write $x = (x', t) \in \mathbb{R}^{d-1} \times \mathbb{R}$. For $x' \in \mathbb{R}^{d-1}$ consider one-dimensional sections $\Omega(x') = \{t \in \mathbb{R} : (x', t) \in \Omega\}$. If not empty, each section $\Omega(x')$ consists of at most countably many open intervals $J_k(x') \subset \mathbb{R}$, $k = 1, \dots, N(x') \leq \infty$.

For $x = (x', t) \in \Omega$ put $V_{x'}(t) = V(x)$ and let the one-dimensional Schrödinger operators

$$H_k(x') = -\frac{d^2}{dt^2} - V_{x'}, \quad k = 1, \dots, N(x'),$$

be defined in $L^2(J_k(x'))$ with Dirichlet boundary conditions. Finally let

$$W(x', V) = \bigoplus_{k=1}^{N(x')} H_k(x')_- \tag{7.5}$$

be the negative part of the Schrödinger operator $-d^2/dt^2 - V_{x'}$ given on $\Omega(x')$ with Dirichlet boundary conditions at the endpoints of each interval forming $\Omega(x')$, that is, on the boundary of $\Omega(x')$.

Applying operator-valued Lieb-Thirring inequalities one can estimate eigenvalue means of H_Ω in terms of $W(x', V)$.

Proposition 7.1. *For $\sigma \geq 3/2$ we have*

$$R_\sigma(V; \Omega) \leq L^{cl}_{\sigma, d-1} \int_{\mathbb{R}^{d-1}} \mathrm{Tr} W(x', V)^{\sigma + (d-1)/2} \, dx' .$$

Remark. In the case of a constant potential $V \equiv \Lambda > 0$ the trace of $W(x', \Lambda)$ can be evaluated explicitly. If Ω is bounded, a detailed analysis of the resulting estimate leads to improved Berezin-Li-Yau inequalities with a remainder term capturing the properties of the second term of the semiclassical limit [**Wei08, GW10, GLW11**].

Proof of Proposition 7.1. We consider the quadratic form $\langle u, H_\Omega u \rangle$ and evaluate it on functions u from the form core $C_0^\infty(\Omega)$. We get

$$\begin{aligned} \langle u, H_\Omega u \rangle_{L^2(\Omega)} &= \|\nabla u\|^2_{L^2(\Omega)} - \int_\Omega V |u|^2 \, dx \\ &= \|\nabla' u\|^2_{L^2(\Omega)} + \int_{\mathbb{R}^{d-1}} dx' \int_{\Omega(x')} \Big(|\partial_t u(x', t)|^2 - V_{x'}(t) \, |u(x', t)|^2 \Big) \, dt , \end{aligned}$$

where ∇' denotes the gradient in the first $(d-1)$-coordinates.

For fixed $x' \in \mathbb{R}^{d-1}$ the functions $u(x', \cdot)$ belong to $C_0^\infty(\Omega(x'))$ and therefore to the form core of $W(x', V)$. It follows that

$$\langle u, H_\Omega u \rangle_{L^2(\Omega)} \geq \|\nabla' u\|^2_{L^2(\Omega)} - \int_{\mathbb{R}^{d-1}} \langle u(x', \cdot), W(x', V) u(x', \cdot) \rangle_{L^2(\Omega(x'))} \, dx' . \tag{7.6}$$

To apply operator-valued Lieb-Thirring inequalities we need to extend these forms to $\mathbb{R}^d$. More precisely, we extend both sides of (7.6) by zero to $C_0^\infty(\mathbb{R}^d \setminus \partial\Omega)$ which is a form core of $(-\Delta_{\mathbb{R}^d\setminus\Omega}) \oplus H_\Omega$. This operator corresponds to the left-hand side of (7.6), while the semi-bounded form on the right-hand side is closed on the larger domain $H^1(\mathbb{R}^{d-1}, L^2(\mathbb{R}))$, where it corresponds to the operator

$$-\Delta' \otimes \mathbb{I} - W(x', V) \tag{7.7}$$

defined in $L^2(\mathbb{R}^{d-1}, L^2(\mathbb{R}))$. Due to the positivity of $(-\Delta_{\mathbb{R}^d\setminus\Omega})$ the variational principle implies

$$R_\sigma(V;\Omega) \;=\; \mathrm{Tr}\left(-\Delta_{\mathbb{R}^d\setminus\Omega} \oplus H_\Omega\right)_-^\sigma \;\leq\; \mathrm{Tr}\left(-\Delta' \otimes \mathbb{I} - W(x', V)\right)_-^\sigma . \tag{7.8}$$

Now we apply sharp Lieb-Thirring inequalities [**LW00**] to the Schrödinger operator (7.7) with operator-valued potential $W(x', V)$. For $\sigma \geq 3/2$ we obtain

$$\mathrm{Tr}\left(-\Delta' \otimes \mathbb{I} - W(x', V)\right)_-^\sigma \;\leq\; L^{cl}_{\sigma,d-1} \int_{\mathbb{R}^{d-1}} \mathrm{Tr} W(x', V)^{\sigma+(d-1)/2}\, dx' \tag{7.9}$$

and the claim follows from (7.8) and (7.9). □

7.3. Constant potentials

In this section we assume $V \equiv \Lambda > 0$ on quasi-bounded open sets $\Omega \subset \mathbb{R}^d$, $d \geq 2$. First we remark the following relations between the eigenvalue means. For $0 \leq \gamma < \sigma$ we have [**AL78**]

$$R_\sigma(\Lambda;\Omega) \;=\; \frac{1}{B(\gamma+1, \sigma-\gamma)} \int_0^\infty \tau^{\sigma-\gamma-1} R_\gamma((\Lambda-\tau)_+;\Omega)\, d\tau\,, \tag{7.10}$$

where B denotes the Beta-function. Hence one can use bounds or asymptotic results for R_γ to deduce the respective results for R_σ with $\sigma > \gamma \geq 0$. Conclusions from eigenvalue means of higher order to means of lower order are more cumbersome since eigenvalue means of lower order are less smooth. To derive uniform bounds on the counting function, that is, on $R_0(\Lambda;\Omega)$ one can make use of the estimate [**Lap97**]

$$R_0(\Lambda;\Omega) \;\leq\; (\tau\Lambda)^{-\sigma} R_\sigma((1+\tau)\Lambda;\Omega)\,, \quad \tau > 0\,, \quad \Lambda > 0\,, \quad \sigma > 0\,, \tag{7.11}$$

and optimize the right hand side in $\tau > 0$. In general, sharp constants are lost but usually the correct order of growth in Λ is preserved.

In the following we consider specific domains with infinite volume. The discreteness of the spectrum of the Dirichlet Laplace operator defined on these domains can be deduced from various sufficient conditions. For example, one can refer to the following result from [**Ada70**].

LEMMA 7.2. *Let Ω be an open subset of $\mathbb{R}^d$ and for $h > 0$ let Q_h be a cube with sides parallel to the coordinate axes. Let $\mu_{d-1}(Q_h, \Omega)$ denote the maximum of the $(d-1)$-dimensional measure of $P(Q_h \setminus \Omega)$, where the maximum is taken over all projections P onto $(d-1)$-dimensional faces of Q_h.*

Assume that for every $\epsilon > 0$ there exist $h \leq 1$ and $r \geq 0$ such that for every cube Q_h of side length h with sides parallel to the coordinate axes and with $Q_h \cap \{x \in \mathbb{R}^d : |x| > r\} \neq \emptyset$ we have

$$\frac{\mu_{d-1}(Q_h, \Omega)}{h^{d+1}} \geq \frac{1}{\epsilon}\,.$$

Then the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact.

In the following examples the trace of the operator $W(x', \Lambda)$ given in (7.5) can be calculated explicitly and we find that Proposition 7.1 yields sharp estimates on eigenvalue means.

7.3.1. Horn-shaped regions. First we consider horn-shaped regions, domains stretching to infinity along distinguished directions, see [**vdB92a**] for a general definition. These regions were first examined by Rozenbljum [**Roz71, Roz72b, Roz76**] and turn out to be of interest in different situations, see e.g. [**Sim83, vdB84a, DS92, vdB92a, MM06, Lun10**]. In [**Sim83**] the semiclassical limit of the counting function was calculated for the horn-shaped regions

$$\Omega_\nu \,=\, \left\{(x,y) \in \mathbb{R}^2 \,:\, |x| \cdot |y|^\nu \,<\, 1\right\}\,, \quad \nu \geq 1\,, \tag{7.12}$$

see Figure 1. Note that discreteness of the spectrum of H_{Ω_ν} can be deduced from Lemma 7.2.

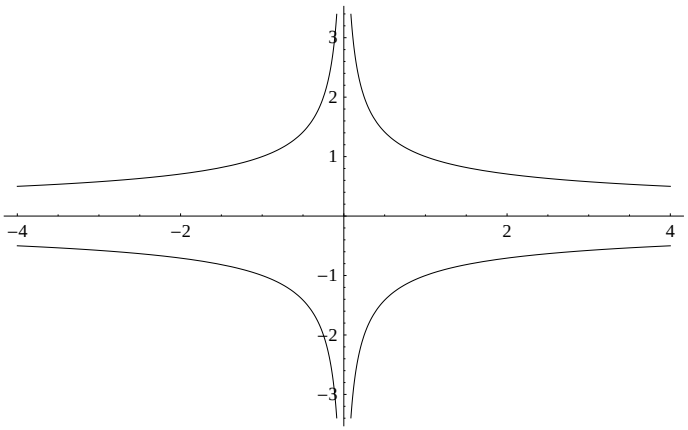

FIGURE 1. The set Ω_2.

In [**GW10**] it was shown that the methods introduced in Section 7.2 yield sharp upper bounds on the trace of the heat kernel of the Dirichlet Laplacian on various horn-shaped regions. Here we derive sharp bounds on eigenvalue means and order-sharp bounds on the counting function.

Let us recall the following asymptotic results from [**Sim83**]. For $\nu > 1$ the limit

$$R_0(\Lambda; \Omega_\nu) \,=\, \zeta(\nu)\, \left(\frac{2}{\pi}\right)^\nu \frac{\Gamma\left(\frac{\nu}{2}+1\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\nu+3}{2}\right)}\, \Lambda^{(\nu+1)/2} + o\left(\Lambda^{(\nu+1)/2}\right)\,, \quad \Lambda \to \infty\,,$$

holds, where $\zeta(\nu)$ denotes the Zeta function. Moreover, for $\nu = 1$

$$R_0(\Lambda; \Omega_1) \,=\, \frac{1}{\pi}\, \Lambda \ln \Lambda + o\left(\Lambda \ln \Lambda\right)\,, \quad \Lambda \to \infty\,.$$

These formulas were improved and extended to higher dimensions in [**vdBL01**]. For more general geometries we refer to [**Lia00**].

To obtain asymptotics for general eigenvalue means, we apply (7.10) with $\gamma = 0$ and for $\sigma > 0$ and $\nu > 1$ we get

$$R_\sigma(\Lambda;\Omega_\nu) \,=\, \zeta(\nu)\,\left(\frac{2}{\pi}\right)^\nu \frac{B(\frac{\nu}{2}+1,\sigma+1)}{B\left(\sigma+\frac{\nu+3}{2},\frac{1}{2}\right)}\,\Lambda^{\sigma+(\nu+1)/2} + o\left(\Lambda^{\sigma+(\nu+1)/2}\right)\,, \quad \Lambda\to\infty\,, \tag{7.13}$$

and for $\nu = 1$

$$R_\sigma(\Lambda;\Omega_1) \,=\, \frac{1}{\pi\,(\sigma+1)}\,\Lambda^{\sigma+1}\,\ln\Lambda + o\left(\Lambda^{\sigma+1}\,\ln\Lambda\right)\,, \quad \Lambda\to\infty\,. \tag{7.14}$$

In order to treat domains in higher dimensions we generalize the notions from [**Sim83**] in the following way:

$$\Omega_\nu \,=\, \left\{(x',x_d)\in\mathbb{R}^{d-1}\times\mathbb{R}\,:\,|x'|\cdot|x_d|^{\nu/(d-1)}\,<\,1\right\}\,, \quad d\geq 2\,, \quad \nu>1\,.$$

For these domains of infinite volume an application of Proposition 7.1 yields sharp spectral estimates.

THEOREM 7.3. *For $\sigma \geq 3/2$, $\nu > 1$, and all $\Lambda > 0$ the estimate*

$$R_\sigma(\Lambda;\Omega_\nu) \,\leq\, \frac{\zeta(\nu)}{2^{d-1}(d-1)}\left(\frac{2}{\pi}\right)^\nu \frac{\Gamma(\frac{\nu}{2}+1)\Gamma(\sigma+1)}{\Gamma(\frac{d+1}{2})\Gamma\left(\sigma+\frac{d+1+\nu}{2}\right)}\,\Lambda^{\sigma+(d-1+\nu)/2}$$

holds.

REMARK. For $d = 2$ we conclude that the bound

$$R_\sigma(\Lambda;\Omega_\nu) \,\leq\, \zeta(\nu)\,\left(\frac{2}{\pi}\right)^\nu \frac{B(\frac{\nu}{2}+1,\sigma+1)}{B\left(\sigma+\frac{\nu+3}{2},\frac{1}{2}\right)}\,\Lambda^{\sigma+(\nu+1)/2}$$

holds for $\sigma \geq 3/2$ and all $\Lambda > 0$. Comparing this bound with the asymptotic relation (7.13) we see that the estimate is sharp: For horn-shaped regions, just as well as for bounded domains, the leading term of the semiclassical limit yields a uniform upper bound.

PROOF OF THEOREM 7.3. In this setting the section $\Omega_\nu(x')$ consists of one open interval

$$(-|x'|^{(1-d)/\nu}, |x'|^{(1-d)/\nu})\,.$$

Since $V \equiv \Lambda$, the trace of the operator-valued potential $W(x',\Lambda)$ defined in (7.5) can be evaluated explicitly. We find

$$\mathrm{Tr}W(x',\Lambda) \,=\, \sum_{j\in\mathbb{N}}\left(\Lambda - \frac{\pi^2 j^2}{4|x'|^{2(1-d)/\nu}}\right)_+\,.$$

Applying Proposition 7.1 yields

$$\begin{aligned} R_\sigma(\Lambda;\Omega_\nu) \,&\leq\, L^{cl}_{\sigma,d-1}\int_{\mathbb{R}^{d-1}}\sum_{j\in\mathbb{N}}\left(\Lambda - \frac{\pi^2 j^2}{4|x'|^{2(1-d)/\nu}}\right)_+^{\sigma+(d-1)/2} dx' \\ &=\, L^{cl}_{\sigma,d-1}\omega_{d-1}\sum_{j\in\mathbb{N}}\int_0^\infty \left(1 - \frac{\pi^2 j^2}{4\Lambda r^{2(1-d)/\nu}}\right)_+^{\sigma+(d-1)/2} r^{d-2}dr\,\Lambda^{\sigma+(d-1)/2}\,, \end{aligned}$$

where ω_{d-1} denotes the volume of the unit sphere in $\mathbb{R}^{d-1}$. We substitute

$$s \;=\; \frac{\pi^2 j^2}{4\Lambda} r^{2(d-1)/\nu}$$

and get

$$\begin{aligned} R_\sigma(\Lambda;\Omega_\nu) &\le L^{cl}_{\sigma,d-1}\omega_{d-1} \sum_{j\in\mathbb{N}} \frac{\nu}{2(d-1)} \left(\frac{2}{\pi j}\right)^\nu \int_0^\infty (1-s)_+^{\sigma+(d-1)/2} s^{\nu/2-1} ds\, \Lambda^{\sigma+(d-1+\nu)/2} \\ &= L^{cl}_{\sigma,d-1}\omega_{d-1}\zeta(\nu) \left(\frac{2}{\pi}\right)^\nu \frac{\nu B\left(\sigma+\frac{d+1}{2},\frac{\nu}{2}\right)}{2(d-1)} \Lambda^{\sigma+(d-1+\nu)/2}\,. \end{aligned}$$

Now we insert the identity

$$L^{cl}_{\sigma,d-1}\,\omega_{d-1}\,\frac{\nu B\left(\sigma+\frac{d+1}{2},\frac{\nu}{2}\right)}{2(d-1)} \;=\; \frac{\Gamma(\sigma+1)\Gamma(\frac{\nu}{2}+1)}{2^{d-1}(d-1)\Gamma(\frac{d+1}{2})\Gamma(\sigma+\frac{d+1+\nu}{2})}$$

and arrive at the claimed estimate. □

Now we apply (7.11) to deduce order-sharp bounds on the counting function.

Corollary 7.4. *For $\nu > 1$ and all $\Lambda > 0$ the estimate*

$$R_0(\Lambda;\Omega_\nu) \;\le\; C_{d,\nu}\,\Lambda^{(d-1+\nu)/2}$$

holds with a constant

$$C_{d,\nu} \;\le\; \frac{(d+\nu+2)^{(d+\nu+2)/2}}{3^{3/2}(d+\nu-1)^{(d+\nu-1)/2}} \frac{\zeta(\nu)}{2^{d-1}(d-1)} \left(\frac{2}{\pi}\right)^\nu \frac{\Gamma(\frac{\nu}{2}+1)\Gamma(\frac{5}{2})}{\Gamma(\frac{d+1}{2})\Gamma\left(\frac{d+\nu}{2}+2\right)}\,.$$

Proof. We use (7.11) with $\sigma = 3/2$ and apply Theorem 7.3 to obtain

$$\begin{aligned} R_0(\Lambda;\Omega_\nu) &\le \frac{1}{(\tau\Lambda)^{3/2}} \frac{\zeta(\nu)}{2^{d-1}(d-1)} \left(\frac{2}{\pi}\right)^\nu \frac{\Gamma(\frac{\nu}{2}+1)\Gamma(\frac{5}{2})}{\Gamma(\frac{d+1}{2})\Gamma\left(\frac{d+\nu}{2}+2\right)} \left((1+\tau)\Lambda\right)^{(d+\nu)/2+1} \\ &= \frac{(1+\tau)^{(d+\nu)/2+1}}{\tau^{3/2}} \frac{\zeta(\nu)}{2^{d-1}(d-1)} \left(\frac{2}{\pi}\right)^\nu \frac{\Gamma(\frac{\nu}{2}+1)\Gamma(\frac{5}{2})}{\Gamma(\frac{d+1}{2})\Gamma\left(\frac{d+\nu}{2}+2\right)} \Lambda^{(d-1+\nu)/2}\,. \end{aligned}$$

Minimizing in $\tau > 0$ yields $\tau_{\min} = 3/(d+\nu-1)$ and inserting this we obtain the claimed result. □

Let us now consider the critical case $\nu = 1$ in dimension $d = 2$. Here the domain yields two equally strong singularities and we cannot distinguish one direction. However, choosing an intermediate direction we obtain a sharp estimate with a remainder term.

Theorem 7.5. *Let $\sigma \ge 3/2$. Then for $\Lambda \le \pi^2/16$ we have $R_\sigma(\Lambda;\Omega_1) = 0$ and for $\Lambda > \pi^2/16$ the estimate*

$$R_\sigma(\Lambda;\Omega_1) \;\le\; \frac{1}{\pi\,(\sigma+1)}\Lambda^{\sigma+1}\ln\Lambda + \frac{C}{\sigma+1}\Lambda^{\sigma+1}$$

holds with a constant

$$C \;<\; \frac{33+16\ln(\frac{4}{\pi})}{8\pi} \;<\; 1.47\,.$$

REMARK. Again, comparing this inequality with the asymptotics (7.14), we see that the main term of the bound is sharp.

PROOF OF THEOREM 7.5. Since the function $|\Omega_1(x)| = \frac{1}{x}$ has non-integrable singularities at zero and at infinity we have to choose a coordinate system $(x_1, x_2) \in \mathbb{R}^2$ rotated by $\frac{\pi}{4}$ with respect to the coordinate system $(x, y) \in \mathbb{R}^2$ which was used in definition (7.12). We get

$$\Omega_1(x_1) \,=\, \left\{x_2 \in \mathbb{R} \,:\, |x_2| \,\leq\, \sqrt{|x_1|^2 + 2}\right\}$$

for $|x_1| \leq \sqrt{2}$ and

$$\Omega_1(x_1) \,=\, \left\{x_2 \in \mathbb{R} \,:\, \sqrt{|x_1|^2 - 2} \,\leq\, |x_2| \,\leq\, \sqrt{|x_1|^2 + 2}\right\}$$

for $|x_1| > \sqrt{2}$. To simplify the following calculations and the resulting bound we confine ourselves to rough estimates which are nevertheless sufficient to prove the sharp constant in the leading term. First, we estimate $|\Omega_1(x_1)| \leq 4$ for $|x_1| \leq 2$ and

$$|\Omega_1(x_1)| \,\leq\, 2\left(\sqrt{|x_1|^2 + 2} - \sqrt{|x_1|^2 - 2}\right) \,\leq\, \frac{4}{|x_1|} + \frac{2}{|x_1|^3}$$

for $|x_1| > 2$.

Suppose that $\Lambda \leq \pi^2/16$. Since $|\Omega_1(x_1)| \leq 4$ for all $x_1 \in \mathbb{R}$ we get

$$\mathrm{Tr} W(x_1, \Lambda) \,=\, \sum_{j \in \mathbb{N}} \left(\Lambda - \frac{\pi^2 j^2}{|\Omega(x_1)|^2}\right)_+ \,=\, 0$$

for all $x_1 \in \mathbb{R}$. From Proposition 7.1 it follows that $R_\sigma(\Lambda; \Omega_1) = 0$ for $\Lambda \leq \pi^2/16$. On the other hand, if $\Lambda > \pi^2/16$ Proposition 7.1 implies

$$\begin{aligned} R_\sigma(\Lambda; \Omega_1) \,&\leq\, L^{cl}_{\sigma,1} \int_{\mathbb{R}} \sum_{j \in \mathbb{N}} \left(\Lambda - \frac{\pi^2 j^2}{|\Omega(x_1)|^2}\right)_+^{\sigma+1/2} dx_1 \\ &\leq\, 4 L^{cl}_{\sigma,1} \sum_{j \in \mathbb{N}} \left(\Lambda - \frac{\pi^2 j^2}{16}\right)_+^{\sigma+1/2} + 2 L^{cl}_{\sigma,1} \int_2^\infty \sum_{j \in \mathbb{N}} \left(\Lambda - \frac{\pi^2 j^2}{l(x_1)^2}\right)_+^{\sigma+1/2} dx_1 \,, \end{aligned} \tag{7.15}$$

with

$$l(x_1) \,=\, \frac{4}{|x_1|} + \frac{2}{|x_1|^3} \,.$$

Note that for $A > 0$ and $\gamma > 0$ we have

$$\sum_{j \in \mathbb{N}} \left(1 - \frac{j^2}{A^2}\right)_+^\gamma \,\leq\, \frac{A}{2} B\left(\frac{1}{2}, \gamma + 1\right) \,, \tag{7.16}$$

thus

$$4 L^{cl}_{\sigma,1} \sum_{j \in \mathbb{N}} \left(\Lambda - \frac{\pi^2 j^2}{16}\right)_+^{\sigma+1/2} \,\leq\, \frac{8}{\pi} L^{cl}_{\sigma,1} B\left(\frac{1}{2}, \sigma + \frac{3}{2}\right) \Lambda^{\sigma+1} \,=\, \frac{4}{\pi} \frac{1}{\sigma+1} \Lambda^{\sigma+1} \,. \tag{7.17}$$

Now we turn to the second term in (7.15). Put $x(\Lambda) = (4\sqrt{\Lambda})/\pi + \pi/(4\sqrt{\Lambda})$. For $x_1 \geq x(\Lambda)$ we have $l(x_1) \leq \pi/\sqrt{\Lambda}$, hence

$$\sum_{j\in\mathbb{N}} \left(\Lambda - \frac{\pi^2 j^2}{l(x_1)^2}\right)_+ \;=\; 0\,.$$

In view of (7.16) it follows that

$$\begin{aligned} 2L^{cl}_{\sigma,1} \int_2^\infty \sum_{j\in\mathbb{N}} \left(\Lambda - \frac{\pi^2 j^2}{l(x_1)^2}\right)_+^{\sigma+1/2} dx_1 \;&\leq\; \frac{1}{\pi} L^{cl}_{\sigma,1} B\left(\frac{1}{2}, \sigma + \frac{3}{2}\right) \int_2^{x(\Lambda)} l(x_1) dx_1\, \Lambda^{\sigma+1} \\ &= \frac{1}{2\pi}\frac{1}{\sigma+1} \int_2^{x(\Lambda)} l(x_1) dx_1\, \Lambda^{\sigma+1}\,. \end{aligned} \tag{7.18}$$

By definition of $x(\Lambda)$ and $l(x_1)$,

$$\begin{aligned} \int_2^{x(\Lambda)} l(x_1) dx_1 \;&=\; \int_2^{x(\Lambda)} \left(\frac{4}{x_1} + \frac{2}{x_1^3}\right) dx_1 \\ &\leq\; 2\ln\Lambda + 4\ln\left(\frac{4}{\pi} + \frac{\pi}{4\Lambda}\right) - 4\ln 2 + \frac{1}{4} \\ &\leq\; 2\ln\Lambda + 4\ln\left(\frac{4}{\pi}\right) + \frac{1}{4} \end{aligned} \tag{7.19}$$

for $\Lambda > \pi^2/16$. Inserting (7.17), (7.18) and (7.19) into (7.15) finishes the proof. □

Again we can apply (7.11) to deduce order-sharp bounds on the counting function.

COROLLARY 7.6. *For $\Lambda \leq \pi^2/16$ we have $R_0(\Lambda;\Omega_1) = 0$ and for $\Lambda > \pi^2/16$ the estimate*

$$R_0(\Lambda;\Omega_1) \;\leq\; \left(\frac{5}{3}\right)^{3/2} \frac{1}{\pi}\,\Lambda\ln\Lambda + C\Lambda\,,$$

holds, with a constant

$$C \;<\; \sqrt{\frac{5}{3}}\,\frac{825 + 400\ln\left(\frac{4}{\pi}\right) + 360\pi\ln\left(\frac{5}{3}\right)}{72\pi} \;<\; 8.56\,.$$

7.3.2. Spiny urchins. In this subsection we study the eigenvalues of the Dirichlet Laplacian on so called spiny urchins, radially symmetric domains $\Omega_S \subset \mathbb{R}^2$ with infinite volume, which were introduced in [**Cla67**].

To construct Ω_S we use polar coordinates $(r,\varphi) \in [0,\infty) \times [0,2\pi)$ and choose an increasing sequence $(r_n)_{n\in\mathbb{N}}$ of positive real numbers and put $r_0 = 0$. For $n \in \mathbb{N}_0$ and $k = 1, 2, \ldots, 2^{n+2}$ let

$$\Gamma_{n,k} \;=\; \left\{(r,\varphi) \;:\; r \geq r_n\,,\; \varphi = \frac{k-1}{2^{n+1}}\pi\right\}$$

be semi-axes and define

$$\Omega_S \;=\; \mathbb{R}^2 \setminus \bigcup_{n,k} \Gamma_{n,k}\,,$$

see Figure 2. Note that this domain, though quasi bounded, has empty exterior. However, if

$$\lim_{n\to\infty} r_n\, 2^{-n} \;=\; 0\,, \tag{7.20}$$

then discreteness of the spectrum of H_{Ω_S} can be deduced from Lemma 7.2, see also [**vdB92b**].

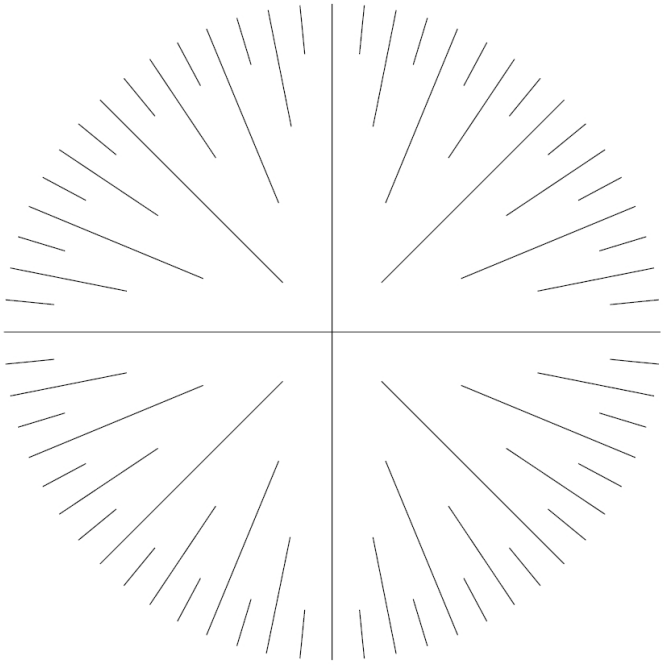

FIGURE 2. The set Ω_S.

For $r_n = n$ the domain Ω_S was analyzed in [**Fle78**], where the leading term of the semiclassical limit was calculated: For $r_n = n$ the asymptotic relation

$$R_0(\Lambda;\Omega_S) \;=\; C\,\Lambda(\ln\Lambda)^2 + o\left(\Lambda(\ln\Lambda)^2\right)\;, \quad \Lambda\to\infty\,,$$

holds with a constant $C > 0$.

The general setting of an arbitrary increasing sequence $(r_n)_{n\in\mathbb{N}_0}$ was studied in [**vdB92b**]: If $r_0 > 0$ and (7.20) is satisfied then for all $\Lambda > 2^{14} r_0^{-2}$ the bound

$$R_0(\Lambda;\Omega_S) \;\le\; 50(8^{-1}+8\pi)^2\,\Lambda\, r^2_{K(\Lambda)}$$

holds with $K(\Lambda) = \max\{n\in\mathbb{N} \,:\, r_n 2^{-n} > (32)^{-1}\sqrt{\Lambda}\}$. Moreover, there is a similar lower bound. Here we extend the upper bound: We derive order-sharp estimates on the eigenvalue means of H_{Ω_S} valid for all $\Lambda > 0$.

First, we need to adapt Proposition 7.1 to the radially symmetric situation. For $r\in(0,\infty)$ put

$$\Omega_S(r) \;=\; \{\varphi\in[0,2\pi) \,:\, (r,\varphi)\in\Omega_S\}\;.$$

Then $\Omega_S(r)$ consists of finitely many open intervals $I_k(r)$, $k = 1,\dots,N(r)$. Choose $u\in C_0^\infty(\Omega_S)$ and consider the quadratic form

$$\begin{aligned}\langle u, H_{\Omega_S} u\rangle_{L^2(\Omega_S)} \;&=\; \int_\Omega \overline{u(x)}\,(-\Delta u(x) - \Lambda u(x))\,dx \\ &=\; \int_0^\infty \int_{\Omega(r)} \overline{u(r,\varphi)}\left(-\partial_r^2 - \frac{1}{r}\partial_r - \frac{1}{r^2}\partial_\varphi^2 - \Lambda\right) u(r,\varphi)\,d\varphi\, r\,dr\,.\end{aligned} \tag{7.21}$$

For fixed $r > 0$ the function $u_r(\varphi) = u(r,\varphi)$ belongs to $C_0^\infty(\Omega_S(r))$. It satisfies Dirichlet boundary conditions at the endpoints of the intervals $I_k(r)$, $k = 1, \dots, N(r)$.

To rewrite the form in the ground state representation put $v(r,\varphi) = \sqrt{r}\, u(r,\varphi)$. Then again $v(r,\varphi)$ belongs to $C_0^\infty(\Omega_S)$ and for fixed $r > 0$, we have $v_r(\varphi) = v(r,\varphi) \in C_0^\infty(\Omega_S(r))$. Moreover,

$$\int_0^\infty \int_{\Omega(r)} |u(r,\varphi)|^2 d\varphi\, r\, dr \;=\; \int_0^\infty \int_{\Omega(r)} |v(r,\varphi)|^2 d\varphi\, dr$$

and

$$\left(-\partial_r^2 - \frac{1}{r}\partial_r - \frac{1}{r^2}\partial_\varphi^2\right) u(r,\varphi) \;=\; \frac{1}{\sqrt{r}} \left(-\partial_r^2 - \frac{1}{4r^2} - \frac{1}{r^2}\partial_\varphi^2\right) v(r,\varphi)\,.$$

Inserting this into (7.21) we obtain

$$\langle u, H_{\Omega_S} u\rangle_{L^2(\Omega_S)} \;=\; \int_0^\infty \int_{\Omega_S(r)} \left(|\partial_r v|^2 + \frac{1}{r^2}|\partial_\varphi v|^2 - \left(\frac{1}{4r^2} + \Lambda\right)|v|^2\right) d\varphi\, dr\,. \quad (7.22)$$

In this setting, we define the Schrödinger-type operators

$$H_k(r) = -\frac{1}{r^2}\frac{d^2}{d\varphi^2} - \left(\frac{1}{4r^2} + \Lambda\right)\,, \quad k = 1, \dots, N(r)\,,$$

in $L^2(I_k(r))$ with Dirichlet boundary conditions at the endpoints of $I_k(r)$. In the same way as in (7.5) let

$$W(r,\Lambda) \;=\; \bigoplus_{k=1}^{N(r)} H_k(r)_-$$

be the negative part of the operator

$$-\frac{1}{r^2}\frac{d^2}{d\varphi^2} - \left(\frac{1}{4r^2} + \Lambda\right)$$

in $L^2(\Omega_S(r))$ with Dirichlet boundary conditions. In view of (7.22) we can apply Proposition 7.1 and for $\sigma \geq 3/2$ we get

$$R_\sigma(\Lambda;\Omega_S) \;\leq\; L^{cl}_{\sigma,1} \int_0^\infty \mathrm{Tr} W(r,\Lambda)^{\sigma+1/2} dr\,. \quad (7.23)$$

To estimate the right hand side and to derive bounds on the eigenvalues means we assume that (7.20) is satisfied and that

$$r_{n+1} \;\leq\; 2r_n \quad (7.24)$$

holds for all $n \in \mathbb{N}$. Then the sequence

$$\frac{2^{2n}}{r_n^2} - \frac{1}{4r_n^2}\,, \quad n \in \mathbb{N}\,,$$

is increasing and for all $\Lambda > 15/4 \cdot r_1^{-2}$ and there is a unique index $\hat{n}(\Lambda) \in \mathbb{N}$ satisfying

$$\Lambda \;>\; \frac{2^{2n}}{r_n^2} - \frac{1}{4r_n^2} \text{ for all } n \leq \hat{n}(\Lambda) \quad \text{and} \quad \Lambda \;\leq\; \frac{2^{2n}}{r_n^2} - \frac{1}{4r_n^2} \text{ for all } n > \hat{n}(\Lambda)\,. \quad (7.25)$$

To simplify notation we put $\hat{r}(\Lambda) = r_{\hat{n}(\Lambda)}$.

LEMMA 7.7. *Let $\sigma \geq 3/2$ and assume that (7.20) and (7.24) are satisfied. Then for $\Lambda \leq 15/4 \cdot r_1^{-2}$ we have $R_\sigma(\Lambda;\Omega_S) = 0$ and for $\Lambda > 15/4 \cdot r_1^{-2}$ the estimate*

$$R_\sigma(\Lambda;\Omega_S) \,\leq\, L^{cl}_{\sigma,2}\,\pi\hat{r}(\Lambda)^2\Lambda^{\sigma+1} + C_\sigma\,\Lambda^\sigma \ln\left(\Lambda\hat{r}(\Lambda)\right)\,.$$

holds with a constant $C_\sigma > 0$ depending only on σ.

REMARK. If we compare the main term of this bound with the Berezin inequality (7.4) we see that the effective domain that enters into the bound is a disk with radius $\hat{r}(\Lambda)$.

PROOF OF LEMMA 7.7. In view of (7.23) we have to estimate

$$\mathrm{Tr}W(r,\Lambda) \,=\, \mathrm{Tr}\left(-\frac{1}{r^2}\frac{d^2}{d\varphi^2} - \Lambda - \frac{1}{4r^2}\right)_- \,=\, \sum_{k=1}^{N(r)}\sum_{j\in\mathbb{N}}\left(\Lambda + \frac{1}{4r^2} - \frac{\pi^2 j^2}{r^2|I_k(r)|^2}\right)_+\,.$$

Fix $r > 0$ and $n_0 \in \mathbb{N}$ such that $r_{n_0-1} < r \leq r_{n_0}$. Then the section $\Omega_S(r) \subset [0,2\pi)$ consists of 2^{n_0+1} identical open intervals of length $|I_k(r)| = \pi/2^{n_0}$. Hence,

$$\mathrm{Tr}W(r,\Lambda) \,=\, 2^{n_0+1}\sum_{j\in\mathbb{N}}\left(\Lambda + \frac{1}{4r^2} - \frac{2^{2n_0}j^2}{r^2}\right)_+\,.$$

Note that for all $j \in \mathbb{N}$

$$\frac{2^{2n_0}j^2}{r^2} - \frac{1}{4r^2} \,\geq\, \frac{2^{2n_0+2}-1}{4r^2} \,\geq\, \frac{15}{4r_1^2}\,.$$

For $\Lambda \leq 15/4 \cdot r_1^{-2}$ we obtain $\mathrm{Tr}W(r,\Lambda) = 0$ and by (7.23) also $R_\sigma(\Lambda;\Omega_S) = 0$.

Hence, we can assume $\Lambda > 15/4 \cdot r_1^{-2}$. Suppose that $r > \hat{r}(\Lambda)$ thus $n_0 > \hat{n}(\Lambda)$. From (7.25) we get

$$\frac{2^{2n_0}j^2}{r^2} - \frac{1}{4r^2} \,\geq\, \frac{2^{2n_0+2}-1}{4r_{n_0}^2} \,\geq\, \Lambda$$

for all $j \in \mathbb{N}$ and it follows that $\mathrm{Tr}W(r,\Lambda) = 0$ for $r > \hat{r}(\Lambda)$. Moreover, if $r^2 \leq 15/(4\Lambda)$ we have $r \leq r_1$ and

$$\frac{4j^2}{r^2} - \frac{1}{4r^2} \,\geq\, \frac{15}{4r^2} \,\geq \Lambda$$

for all $j \in \mathbb{N}$. Again it follows that $\mathrm{Tr}W(r,\Lambda) = 0$ and it remains to consider $\sqrt{15}/(2\sqrt{\Lambda}) < r < \hat{r}(\Lambda)$.

For such r we apply (7.16) to estimate

$$\begin{aligned}\mathrm{Tr}W(r,\Lambda)^{\sigma+1/2} \,&=\, 2^{n_0+1}\sum_{j\in\mathbb{N}}\left(\Lambda + \frac{1}{4r^2} - \frac{2^{2n_0}j^2}{r^2}\right)_+^{\sigma+1/2}\\ &\leq\, r\left(\Lambda + \frac{1}{4r^2}\right)^{\sigma+1} B\left(\frac{1}{2}, \sigma+\frac{3}{2}\right)\,.\end{aligned}$$

From (7.23) we conclude

$$\begin{aligned} R_\sigma(\Lambda;\Omega_S) \;&\le\; L^{cl}_{\sigma,1} B\left(\frac{1}{2},\sigma+\frac{3}{2}\right)\int_{\sqrt{15}/(2\sqrt{\Lambda})}^{\hat r(\Lambda)} r\left(\Lambda+\frac{1}{4r^2}\right)^{\sigma+1} dr \\ &=\; \frac{1}{16(\sigma+1)}\Lambda^\sigma\int_{15}^{4\Lambda\hat r(\Lambda)^2}\left(1+\frac{1}{s}\right)^{\sigma+1} ds \\ &\le\; \frac{1}{4(\sigma+1)}\hat r(\Lambda)^2\,\Lambda^{\sigma+1} \;+\; \frac{16^{\sigma-1}}{15^\sigma}\Lambda^\sigma\,\ln\left(4\Lambda\hat r(\Lambda)^2\right) \end{aligned}$$

and the claim of the lemma follows from the identity $4\pi(\sigma+1)L^{cl}_{\sigma,2}=1$. □

Before we give examples we supplement Lemma 7.7 with the following lower bound on $R_\sigma(\Lambda;\Omega_S)$.

LEMMA 7.8. *Assume there exists $N_0\in\mathbb{N}$ such that $r_{n-1}<(1-2^{-n})r_n$ is satisfied for all $n\ge N_0$. Then for $\sigma\ge 0$ there exist positive constants C and μ independent of Λ such that*

$$R_\sigma(\Lambda;\Omega_S)\;\ge\; C\sum_{n=N_0}^{\hat n(\mu\Lambda)} r_n\,(r_n-r_{n-1})\,\Lambda^{\sigma+1}$$

holds for $\Lambda>0$ with $\hat n(\mu\Lambda)>N_0$.

PROOF. For $n\ge N_0$ and $k\in\{1,\dots,2^{n+1}\}$ consider a segment $\Omega_{n,k}\subset\Omega_S$, i.e., a region between $r=r_{n-1}$, $r=r_n$ and two adjacent semi-axes $\Gamma_{n,k}$ and $\Gamma_{n,k+1}$. Note that there are 2^{n+1} identical segments $\Omega_{n,k}$. Let $\tau(n)$ denote the maximal number of disjoint squares Q_{l_n} with side length $l_n=r_n/2^{n+1}$ that can be placed in the interior of $\Omega_{n,k}$. From the definition of Ω_S it follows that

$$\tau(n)\;\ge\; C\,\frac{r_n-r_{n-1}}{l_n}\,,\quad n\ge N_0\,.^1$$

Hence, the variational principle implies

$$R_\sigma(\Lambda;\Omega_S)\;\ge\;\sum_{n\ge N_0} 2^{n+1}\,\tau(n)\,R_\sigma(\Lambda;Q_{l_n})\;\ge\; C\sum_{n\ge N_0} 2^{n+1}\,\frac{r_n-r_{n-1}}{l_n}\,R_\sigma(\Lambda;Q_{l_n})\,. \tag{7.26}$$

To estimate $R(\Lambda;Q_{l_n})$ from below, we first consider the square Q_1 with side length 1. From Weyl's asymptotic law (7.3) we know that there are positive constants C and Λ_0, such that $R_\sigma(\Lambda;Q_1)\;\ge\; C\,\Lambda^{\sigma+1}$ holds for all $\Lambda\ge\Lambda_0$. By scaling, we deduce that

$$R_\sigma(\Lambda;Q_{l_n})\;\ge\; C\,l_n^2\,\Lambda^{\sigma+1} \tag{7.27}$$

holds for all $\Lambda\ge\Lambda_0/l_n^2$.

Fix $\Lambda>0$. From (7.25) we deduce that

$$\frac{\Lambda_0}{l_n^2}\;=\;4\Lambda_0\frac{2^{2n}}{r_n^2}\;\le\;8\Lambda_0\left(\frac{2^{2n}}{r_n^2}-\frac{1}{4r_n^2}\right)\;\le\;\Lambda$$

[1] Here and in the following the letter C denotes various positive constants that are independent of Λ.

holds if $n \leq \hat{n}(\Lambda/(8\Lambda_0))$. Denoting $\mu = 1/(8\Lambda_0)$ we find that (7.27) is valid for all squares Q_{l_n} with $n \leq \hat{n}(\mu\Lambda)$.

In view of (7.26) it follows that

$$R_\sigma(\Lambda;\Omega_S) \;\geq\; C \sum_{n=N_0}^{\hat{n}(\mu\Lambda)} 2^{n+1}\, \frac{r_{n+1}-r_n}{l_n}\, l_n^2\, \Lambda^{\sigma+1} \;\geq\; C \sum_{n=N_0}^{\hat{n}(\mu\Lambda)} r_n (r_n - r_{n-1})\, \Lambda^{\sigma+1}$$

and the proof is complete. □

Let us state some examples to show that the bounds capture the correct order in Λ and that choosing different sequences $(r_n)_{n\in\mathbb{N}}$ leads to different behavior in the semiclassical limit.

COROLLARY 7.9. *Let $\sigma \geq 0$.*

(1) *Assume $r_n = n$. Then for $0 < \Lambda \leq 15/4$ we have $R_\sigma(\Lambda;\Omega_S) = 0$ and for $\Lambda > 15/4$*

$$R_\sigma(\Lambda;\Omega_S) \;\leq\; C_\sigma\, \Lambda^{\sigma+1} (\ln \Lambda)^2 \,.$$

(2) *Assume $r_n = 2^{\delta n}$ with $0 < \delta < 1$. Then for $0 < \Lambda \leq 15 \cdot 2^{-2(1+\delta)}$ we have $R_\sigma(\Lambda;\Omega_S) = 0$ and for $\Lambda > 15 \cdot 2^{-2(1+\delta)}$*

$$R_\sigma(\Lambda;\Omega_S) \;\leq\; C_{\sigma,\delta}\, \Lambda^{\sigma+1/(1-\delta)} \,.$$

All bounds capture the correct order in Λ as $\Lambda \to \infty$.

PROOF. To prove the bounds for $\sigma \geq 3/2$ we can apply Lemma 7.7 and it remains to estimate $\hat{r}(\Lambda)$. By definition, $\hat{r}(\Lambda) = r_{\hat{n}(\Lambda)}$ and by (7.25) $r_{\hat{n}(\Lambda)}$ satisfies

$$\frac{2^{2\hat{n}(\Lambda)}}{r_{\hat{n}(\Lambda)}^2} - \frac{1}{4 r_{\hat{n}(\Lambda)}^2} \;\leq\; \Lambda \,.$$

It follows that $\hat{r}(\Lambda) \leq C \ln \Lambda$ in the case $r_n = n$ and $\hat{r}(\Lambda) \leq C_\delta \Lambda^{\delta/(2(1-\delta))}$ in the case $r_n = 2^{\delta n}$ and the bounds for $\sigma \geq 3/2$ follow from Lemma 7.7. To deduce the claimed estimates for $0 \leq \sigma < 3/2$ we apply (7.11) and finally (7.10).

It remains to prove that the estimates are of correct order in Λ. Note that in the case $r_n = n$ the assumptions of Lemma 7.8 are satisfied with $N_0 = 1$. Hence, we have

$$\sum_{n=N_0}^{\hat{n}(\mu\Lambda)} r_n \,(r_n - r_{n-1}) \;=\; \sum_{n=1}^{\hat{n}(\mu\Lambda)} n \;\geq\; C \hat{n}(\mu\Lambda)^2 \;=\; C \hat{r}(\mu\Lambda)^2 \,.$$

In the case $r_n = 2^{\delta n}$ we find for sufficiently large Λ that

$$\sum_{n=N_0}^{\hat{n}(\mu\Lambda)} r_n \,(r_n - r_{n-1}) = \sum_{n=N_0}^{\hat{n}(\mu\Lambda)} 2^{\delta n} \left(2^{\delta n} - 2^{\delta(n-1)}\right) \geq C \sum_{n=N_0}^{\hat{n}(\mu\Lambda)} 2^{2\delta n} \geq C 2^{2\delta \hat{n}(\mu\Lambda)} = C \hat{r}(\mu\Lambda)^2 \,,$$

holds. In both cases, we insert this into Lemma 7.8 and get

$$R_\sigma(\Lambda;\Omega_S) \;\geq\; C \Lambda^{\sigma+1}\, \hat{r}(\mu\Lambda)^2 \,. \tag{7.28}$$

For Λ large enough the relations (7.25) imply

$$\hat{r}(\mu\Lambda) \;\geq\; C \ln(\mu\Lambda) \;\geq\; C \ln \Lambda$$

if $r_n = n$ and

$$\hat{r}(\mu\Lambda) \,\geq\, C(\mu\Lambda)^{\delta/(2(1-\delta))} \,\geq\, C\Lambda^{\delta/(2(1-\delta))}$$

if $r_n = 2^{\delta n}$. As $\Lambda \to \infty$ we obtain from (7.28) that $R_\sigma(\Lambda;\Omega_S) = O(\Lambda^{\sigma+1}(\ln\Lambda)^2)$ in the case $r_n = n$ and $R_\sigma(\Lambda;\Omega_S) = O(\Lambda^{\sigma+1/(1-\delta)})$ in the case $r_n = 2^{\delta n}$. Thus the bounds on $R_\sigma(\Lambda,\Omega_S)$ show the correct order in Λ. □

Let us state one more example, where one encounters exponential growth of the eigenvalue means.

COROLLARY 7.10. *Assume $\sigma \geq 3/2$ and $r_n = 2^n/\sqrt{n}$. Then for $0 < \Lambda < 15/16$ we have $R_\sigma(\Lambda;\Omega_S) = 0$ and for $\Lambda > 15/16$*

$$R_\sigma(\Lambda;\Omega_S) \,\leq\, C_\sigma\, 2^{2\Lambda}\, \Lambda^\sigma\,.$$

This bound follows from Lemma 7.7 similar as in Corollary 7.9.

7.4. Non-constant potentials

In this section we consider Schrödinger operators H_Ω with non-constant potentials $V \geq 0$ on open sets $\Omega \subset \mathbb{R}^d$. Since we define H_Ω with Dirichlet boundary conditions the variational principle implies that the sharp Lieb-Thirring inequality (7.2) holds. In fact, the Dirichlet condition gives rise to an improvement of this bound. In this section we use this to derive sharp Lieb-Thirring inequalities with remainder term.

7.4.1. One-dimensional considerations. As in Section 7.3 we apply Proposition 7.1 to reduce the problem to one dimension. However, for non-constant potentials V the trace of the operator-valued potential $W(x',V)$ defined in (7.5) cannot be calculated explicitly. Therefore we need to study the one-dimensional situation in more detail to derive the following improvement of (7.2).

THEOREM 7.11. *Let $I \subset \mathbb{R}$ be an open interval of length $l < \infty$ and assume $\sigma \geq 3/2$ and $V \in L^{\sigma+1/2}(I)$ such that*

$$A \,=\, l \int_I V(t)\,dt \,<\, \infty\,.$$

Then for $A \leq 2\ln 3$ we have $R_\sigma(V;I) = 0$ and for $A > 2\ln 3$

$$R_\sigma(V;I) \,\leq\, L^{cl}_{\sigma,1} \int_I V(t)^{\sigma+1/2}\,dt - \left(\frac{2\left(\int_I V(t)\,dt\right)^2}{\exp(A)-1}\right)^\sigma\,.$$

The remainder of Section 7.4.1 is devoted to the proof of this result. In particular, we study the effect of different boundary conditions on the eigenvalues. First we assume $I = (0,l)$ and $V \in C_0^\infty(I)$. Recall that

$$H_I \,=\, -\frac{d^2}{dt^2} - V$$

is defined in $L^2(I)$ as self-adjoint operator generated by the quadratic form

$$\langle u, H_I u\rangle \,=\, \int \left(|u'(t)|^2 - V(t)|u(t)|^2\right) dt\,, \tag{7.29}$$

with form domain $H_0^1(I)$. Moreover, we define the operator

$$H_{\mathbb{R}} \;=\; -\frac{d^2}{dt^2} - V$$

in $L^2(\mathbb{R})$ generated by the form (7.29) with form domain $H^1(\mathbb{R})$.

We assume that the negative spectrum of H_I consists of N eigenvalues $(-\lambda_k)_{k=1}^N$, $N \in \mathbb{N}$, and denote the negative eigenvalues of $H_{\mathbb{R}}$ by $(-\mu_k)_{k=1}^M$. The variational principle implies $M \geq N$ and $-\mu_k \leq -\lambda_k$ for each $k = 1, \ldots, N$.

In order to derive relations between the eigenvalues of H_I and $H_{\mathbb{R}}$ we define

$$H_I^{(\alpha,\beta)} \;=\; -\frac{d^2}{dt^2} - V\,, \quad 0 \leq \alpha, \beta \leq \frac{\pi}{2}\,,$$

as self-adjoint operators generated by the form

$$\langle u, H_I^{(\alpha,\beta)} u\rangle \;=\; \int |u'(t)|^2 dt - \int V(t)|u(t)|^2 dt + (\cot\alpha)\,|u(0)|^2 + (\cot\beta)\,|u(l)|^2$$

with form domain $H^1(I)$. Note that eigenfunctions of $H_I^{(\alpha,\beta)}$ satisfy boundary conditions of the third kind: $u'(0) = (\cot\alpha)u(0)$ and $u'(l) = -(\cot\beta)u(l)$. For $\alpha, \beta \in \left[0, \frac{\pi}{2}\right]$ the negative spectrum of $H_I^{(\alpha,\beta)}$ consists of eigenvalues $(-\nu_k(\alpha,\beta))_{k=1}^{N(\alpha,\beta)}$. We point out that for $\alpha = \beta = 0$ we recover Dirichlet boundary conditions:

$$H_I^{(0,0)} \;=\; H_I\,, \quad N(0,0) \;=\; N\,, \quad \text{and} \quad (\nu_k(0,0))_{k=1}^{N(0,0)} \;=\; (\lambda_k)_{k=1}^N\,. \tag{7.30}$$

We need the following result from [**Wei03**] about the behavior of the eigenvalues of $H_I^{(\alpha,\beta)}$. For $\alpha \in \left[0, \frac{\pi}{2}\right]$ and $\nu > 0$ let $u(t;\nu,\alpha)$ to be the unique solution of

$$\begin{aligned} -u''(t) - V(t)u(t) \;&=\; -\nu\, u(t)\,, \quad t \in I\,, \\ u(0;\nu,\alpha) \;&=\; \sin\alpha\,, \\ u'(0;\nu,\alpha) \;&=\; \cos\alpha\,. \end{aligned} \tag{7.31}$$

LEMMA 7.12. *Fix $\beta \in \left[0, \frac{\pi}{2}\right]$. Then for $\alpha \in (0, \frac{\pi}{2})$ the map $\alpha \mapsto \nu_k(\alpha,\beta)$ is monotone increasing and differentiable and we have*

$$\frac{d\nu_k(\alpha,\beta)}{d\alpha} \;=\; \|u(\cdot;\nu_k(\alpha,\beta),\alpha)\|_{L^2(I)}^{-2}\,.$$

Because of the symmetry of the eigenvalue problem (7.31) a corresponding result holds for fixed $\alpha \in \left[0, \frac{\pi}{2}\right]$ and the map $\beta \mapsto \nu_k(\alpha,\beta)$, $\beta \in \left[0, \frac{\pi}{2}\right]$. For $k = 1, \ldots N$ it follows that

$$-\nu_k(\alpha,\alpha) \leq -\nu_k(0,0) = -\lambda_k < 0$$

for all $\alpha \in \left[0, \frac{\pi}{2}\right]$.

For $k = 1, \ldots N$ put

$$\omega_k \;=\; \operatorname{arccot}\sqrt{\mu_k} \;\in\; \left[0, \frac{\pi}{2}\right]. \tag{7.32}$$

Then we have $N(\omega_k,\omega_k) \geq N$ and both $-\mu_k$ and $-\nu_k(\omega_k,\omega_k)$ exist as negative eigenvalues of $H_{\mathbb{R}}$ and $H_I^{(\omega_k,\omega_k)}$ respectively.

PROPOSITION 7.13. *For $k = 1, \ldots, N$ the eigenvalues of $H_{\mathbb{R}}$ and $H_I^{(\omega_k,\omega_k)}$ satisfy*

$$-\mu_k \;=\; -\nu_k(\omega_k,\omega_k)\,.$$

PROOF. For arbitrary $k \in \{1, \dots, N\}$ let Φ_k denote the eigenfunction of $H_{\mathbb{R}}$ corresponding to $-\mu_k$. Then supp $V \subset I = (0, l)$ implies

$$\begin{aligned} \Phi_k(t) &= c_1 \exp\left(-\sqrt{\mu_k}t\right) \quad \text{for } t \geq l \quad \text{and} \\ \Phi_k(t) &= c_2 \exp\left(+\sqrt{\mu_k}t\right) \quad \text{for } t \leq 0 \end{aligned}$$

with suitable constants $c_1, c_2 \in \mathbb{R}$. From (7.32) it follows that $\Phi_k'(0) = (\cot \omega_k)\Phi_k(0)$ and $\Phi_k'(l) = -(\cot \omega_k)\Phi_k(l)$. Put $\tilde{\Phi}_k = \Phi_k|_{(0,l)}$. Since $\tilde{\Phi}_k$ belongs to the domain of $H_I^{(\omega_k, \omega_k)}$ we find that $-\mu_k$ is an eigenvalue of $H_I^{(\omega_k, \omega_k)}$. Note that Φ_k has $k-1$ zeros in the interior of I. Therefore $\tilde{\Phi}_k$ has $k-1$ zeros as well and we conclude $-\mu_k = -\nu_k(\omega_k, \omega_k)$. □

Similar as in (7.31) let $\tilde{u}(t; \nu, \beta)$, $\beta \in \left[0, \frac{\pi}{2}\right]$, $\nu > 0$, be the unique solution of

$$\begin{aligned} -\tilde{u}''(t) - V(t)\tilde{u}(t) &= -\nu\, \tilde{u}(t)\,, \quad t \in I\,, \\ \tilde{u}(l; \nu, \beta) &= \sin\beta\,, \\ \tilde{u}'(l; \nu, \beta) &= -\cos\beta\,. \end{aligned}$$

Due to the symmetry of the eigenvalue problem (7.31) there is a result analogous to Lemma 7.12 relating the derivative of the map $\beta \mapsto \nu_k(\alpha, \beta)$ to the L^2-norm of $\tilde{u}(\cdot; \nu_k(\alpha, \beta), \beta)$.

In view of (7.30) and Proposition 7.13 we have

$$\mu_k - \lambda_k = \nu_k(\omega_k, \omega_k) - \nu_k(0,0) = \nu_k(\omega_k, \omega_k) - \nu_k(0, \omega_k) + \nu_k(0, \omega_k) - \nu_k(0,0)\,.$$

Hence, applying Lemma 7.12 and its analog for the map $\beta \mapsto \nu_k(0, \beta)$ yields

$$\mu_k - \lambda_k = \int_0^{\omega_k} \|u(\cdot; \nu_k(\alpha, \omega_k), \alpha)\|_{L^2(I)}^{-2}\, d\alpha + \int_0^{\omega_k} \|\tilde{u}(\cdot, \nu_k(0, \beta), \beta)\|_{L^2(I)}^{-2}\, d\beta \tag{7.33}$$

for $k = 1, \dots, N$.

In the remainder of this subsection we use this identity to complete the proof of Theorem 7.11. In order to get a result valid without further assumptions on the potential V we have to restrict ourselves to considering the ground states.

LEMMA 7.14. *Let $I \subset \mathbb{R}$ be an open interval of length l and $V \in C_0^\infty(I)$. Then the inequality*

$$\mu_1 - \lambda_1 \geq \frac{2\left(\int V(t)dt\right)^2}{\exp\left(l \int V(t)dt\right) - 1}$$

holds. Moreover, if $l \int V(t)\, dt \leq 2 \ln 3$ then $-\lambda_1 \geq 0$ and we have $R_\sigma(V; I) = 0$ for $\sigma \geq 0$.

PROOF. First we remark that it suffices to prove the result for $I = (0, l)$. To apply (7.33) we have to analyze the functions $u(\cdot; \nu_1(\alpha, \omega_1), \alpha)$ and $\tilde{u}(\cdot; \nu_1(0, \beta), \beta)$ for $0 < \alpha, \beta < \omega_1$.

By definition, the function u is the first eigenfunction of $H_I^{(\alpha,\omega_1)}$ thus it is non-negative on I. As a solution of (7.31) u solves the integral equation

$$\begin{aligned} u(t;\nu,\alpha) &= \frac{1}{2}(\sin\alpha)\left(e^{\sqrt{\nu}t}+e^{-\sqrt{\nu}t}\right)+\frac{1}{2}(\cos\alpha)\left(\frac{e^{\sqrt{\nu}t}-e^{-\sqrt{\nu}t}}{\sqrt{\nu}}\right) \\ &\quad -\int_0^t \frac{\sinh\left(\sqrt{\nu}(t-s)\right)}{\sqrt{\nu}}\,V(s)\,u(s;\nu,\alpha)\,ds\,. \end{aligned} \tag{7.34}$$

The first two summands are non-decreasing in $\nu > 0$. For $\alpha \in [0,\omega_1]$, Lemma 7.12 and Proposition 7.13 imply $\nu_1(\alpha,\omega_1) \leq \mu_1$. Since the integrand in (7.34) is positive it follows that

$$\begin{aligned} u(t;\nu_1(\alpha,\omega_1),\alpha) &\leq \frac{1}{2}(\sin\alpha)\left(e^{\sqrt{\mu_1}t}+e^{-\sqrt{\mu_1}t}\right)+\frac{1}{2}(\cos\alpha)\left(\frac{e^{\sqrt{\mu_1}t}-e^{-\sqrt{\mu_1}t}}{\sqrt{\mu_1}}\right) \\ &= \frac{1}{2}e^{\sqrt{\mu_1}t}\left(\sin\alpha+\frac{\cos\alpha}{\sqrt{\mu_1}}\right)+\frac{1}{2}e^{-\sqrt{\mu_1}t}\left(\sin\alpha-\frac{\cos\alpha}{\sqrt{\mu_1}}\right)\,. \end{aligned}$$

Now we use that $\sin\alpha - \cos\alpha/\sqrt{\mu_1} \leq 0$ holds for $\alpha \in [0,\omega_1]$ and conclude

$$0 < u(t;\nu_1(\alpha,\omega_1),\alpha) \leq \frac{1}{2}e^{\sqrt{\mu_1}t}\left(\sin\alpha+\frac{\cos\alpha}{\sqrt{\mu_1}}\right)\,.$$

By explicit calculations it follows that

$$\int_0^{\omega_1} \|u(\cdot;\nu_1(\alpha,\omega_1),\alpha)\|^{-2}\,d\alpha \geq \frac{4\mu_1}{\exp\left(2l\sqrt{\mu_1}\right)-1}\,.$$

Similarly, we find

$$\int_0^{\omega_1} \|\tilde{u}(\cdot;\nu_1(0,\beta),\beta)\|^{-2}\,d\beta \geq \frac{4\mu_1}{\exp\left(2l\sqrt{\mu_1}\right)-1}$$

and (7.33) implies

$$\mu_1-\lambda_1 \geq \frac{8\mu_1}{\exp\left(2l\sqrt{\mu_1}\right)-1}\,. \tag{7.35}$$

For $l\sqrt{\mu_1} \leq \ln 3$ it follows that $-\lambda_1 \geq 0$. Since the right hand side of (7.35) is non-increasing the estimate [**HLT98**]

$$\sqrt{\mu_1} \leq \frac{1}{2}\int_I V(t)\,dt$$

implies the claimed result. □

The proof of Theorem 7.11 is an immediate consequence of the results above:

Proof of Theorem 7.11. Using convexity of the map $\lambda \mapsto \lambda^\sigma$ and the Lieb-Thirring inequality (7.2) we estimate

$$R_\sigma(V;I) = \sum_{k=1}^N \lambda_k^\sigma \leq \sum_{k=1}^N \mu_k^\sigma - (\mu_1^\sigma-\lambda_1^\sigma) \leq L^{cl}_{\sigma,1}\int_I V(t)^{\sigma+1/2}\,dt - (\mu_1-\lambda_1)^\sigma\,.$$

Hence, for $V \in C_0^\infty(I)$ the claim follows from Lemma 7.14. A standard approximation argument allows us to prove the claim for all non-negative potentials $V \in L^{\sigma+1/2}(I)$. □

7.4.2. A sharp Lieb-Thirring inequality with remainder term. Let us now consider general Schrödinger operators H_Ω on bounded or quasi-bounded open sets $\Omega \subset \mathbb{R}^d$ with Dirichlet boundary conditions. To apply the inductive argument introduced in Section 7.2, fix a coordinate system in $\mathbb{R}^d$. For $x \in \Omega$ we write $x = (x', t) \in \mathbb{R}^{d-1} \times \mathbb{R}$ and assume that $V_{x'} \in L^{\sigma+d/2}(\Omega(x'))$, a.e. in $x' \in \mathbb{R}^{d-1}$. We use the notation introduced in Section 7.2 and put

$$A_k(x') \;=\; |J_k(x')| \int_{J_k(x')} V_{x'}(t)\, dt\,,$$

$$B_k(x') \;=\; \int_{J_k(x')} V_{x'}(t)\, dt\,.$$

Let $\kappa(x', V) \subset \mathbb{N}$ be the subset of all indices k with $A_k(x') > 2 \ln 3$ and put

$$\Omega_V(x') \;=\; \bigcup_{k \in \kappa(x',V)} J_k(x') \subset \mathbb{R} \quad \text{and} \quad \Omega_V \;=\; \bigcup_{x' \in \mathbb{R}^{d-1}} \{x'\} \times \Omega_V(x') \subset \Omega\,.$$

The results from Section 7.2 and Section 7.4.1 imply the following sharp Lieb-Thirring inequality with remainder term.

THEOREM 7.15. *Let Ω be an open set in $\mathbb{R}^d$, $d \geq 2$, and assume $\sigma \geq 3/2$. Then the estimate*

$$R_\sigma(V;\Omega) \;\leq\; L^{cl}_{\sigma,d} \int_{\Omega_V} V(x)^{\sigma+d/2}\, dx - L^{cl}_{\sigma,d-1} \int_{\mathbb{R}^{d-1}} \rho(x', V)\, dx'$$

holds with a remainder

$$\rho(x', V) \;=\; \sum_{k \in \kappa(x',V)} \left(\frac{2B_k(x')^2}{\exp\left(A_k(x')\right) - 1} \right)^{\sigma+(d-1)/2}\,.$$

PROOF. In view of Proposition 7.1 we have to estimate

$$\mathrm{Tr} W(x', V)^{\sigma+(d-1)/2} \;=\; \sum_{k=1}^{N(x')} \mathrm{Tr} H_k(x')_-^{\sigma+(d-1)/2} \;=\; \sum_{k=1}^{N(x')} R_{\sigma+(d-1)/2}(V_{x'}; J_k(x'))\,.$$

The potential $V_{x'}$ satisfies the conditions of Theorem 7.11, a.e. in $x' \in \mathbb{R}^{d-1}$. For $k \notin \kappa(x', V)$ we have $|J_k(x')| \int_{J_k(x')} V_{x'} dt \leq 2 \ln 3$ and Theorem 7.11 yields $\mathrm{Tr} H_k(x')_- = 0$. Hence,

$$\begin{aligned}
&\mathrm{Tr} W(x', V)^{\sigma+(d-1)/2} \\
&= \sum_{k \in \kappa(x',V)} R_{\sigma+(d-1)/2}(V_{x'}; J_k(x')) \\
&\leq \sum_{k \in \kappa(x',V)} \left(L^{cl}_{\sigma+(d-1)/2,1} \int_{J_k(x')} V_{x'}(t)^{\sigma+d/2} dt - \left(\frac{2B_k(x')^2}{\exp\left(A_k(x')\right) - 1} \right)^{\sigma+(d-1)/2} \right).
\end{aligned}$$

Thus the claim follows from Proposition 7.1 using the identities

$$\int_{\mathbb{R}^{d-1}} \sum_{k \in \kappa(x',V)} \int_{J_k(x')} V_{x'}(t)^{\sigma+d/2} dt\, dx' \;=\; \int_{\Omega_V} V(x)^{\sigma+d/2}\, dx$$

and $L^{cl}_{\sigma,d-1}\, L^{cl}_{\sigma+(d-1)/2,1} = L^{cl}_{\sigma,d}$. □

7.4.3. An example with $V \notin L^{\sigma+d/2}$. Let us illustrate Theorem 7.15 by an example of a Schrödinger operator defined on a horn-shaped region with a potential such that the classical Lieb-Thirring inequality (7.2) fails. As in Section 7.3.1 set

$$\Omega_1 \,=\, \big\{(x,y)\in\mathbb{R}^2 \,:\, |x|\cdot|y| \,<\, 1\big\}$$

and put $V_\alpha(x,y) = |x|^\alpha |y|^{-\alpha}$ with $\alpha > 0$. Again, we introduce a scaling parameter $\lambda > 0$ and study the operator

$$H_\alpha \,=\, -\Delta - \lambda V_\alpha\,,$$

defined in $L^2(\Omega_1)$ with Dirichlet boundary conditions. Since $V_\alpha \notin L^{\sigma+1}(\Omega_1)$ the classical results (7.2) and (7.1) fail and it is not clear whether the spectrum of H_α is discrete and whether eigenvalue means are bounded in terms of the potential λV_α.

While our methods do not answer this question in general, we can apply Theorem 7.15 to get upper bounds on $R_\sigma(\lambda V_\alpha;\Omega_1)$ for $0<\alpha<2/5$ and $3/2\le\sigma<(1-\alpha)/\alpha$. Indeed, for any $x\in\mathbb{R}$ the section $\Omega_1(x)$ consists of one open interval $(-x^{-1},x^{-1})$ and

$$A_1(x) \,=\, \frac{4}{|x|}\int_0^{|x|^{-1}} \lambda|x|^\alpha|y|^{-\alpha}\,dy \,=\, \frac{4\lambda}{1-\alpha}|x|^{2(\alpha-1)}\,.$$

Since $\alpha<1$ we find that $A_1(x)$ tends to zero as $|x|$ tends to infinity. Thus $A_1(x)\le 2\ln 3$ holds for

$$|x| \ge \left(\frac{2\lambda}{(1-\alpha)\ln 3}\right)^{1/(2-2\alpha)} \,=\, x_\alpha(\lambda)\,.$$

From Theorem 7.15 it follows that for $\sigma\ge 3/2$

$$R_\sigma(\lambda V_\alpha;\Omega_1) \,\le\, 4L^{cl}_{\sigma,2}\int_0^{x_\alpha(\lambda)}\int_0^{x^{-1}} x^{\alpha(\sigma+1)}\,y^{-\alpha(\sigma+1)}\,dy\,dx\,\lambda^{\sigma+1}\,.$$

Note that the right hand side is of this inequality is finite if $\sigma<(1-\alpha)/\alpha$. Thus the estimate

$$R_\sigma(\lambda V_\alpha;\Omega_1) \,\le\, L^{cl}_{\sigma,2}\,\frac{4}{2\alpha(\sigma+1)(1-\alpha(\sigma+1))}\left(\frac{2}{(1-\alpha)\ln 3}\right)^{\alpha(\sigma+1)/(1-\alpha)}\lambda^{(\sigma+1)/(1-\alpha)}$$

holds for $3/2\le\sigma<(1-\alpha)/\alpha$ and for all $\lambda>0$.

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July 2004	Abitur (A-levels) as one of the best 13 students in Baden-Württemberg
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Current Position and Academic Experience

since 2008	PhD student of Mathematics at Universität Stuttgart Advisor: Timo Weidl
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2008 - 2009	Collaboration on the project 'Spectral Estimates in Mathematical Physics', supported by the German Research Foundation (Deutsche Forschungsgemeinschaft)
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September 2008	Research stay at the Royal Institute of Technology Stockholm
October 2009 – March 2010	Research stay at Imperial College London supported by the German Academic Exchange Service (Deutscher Akademischer Austausch Dienst)
February 2011 – April 2011	Research stay at Princeton University supported by the German National Academic Foundation (Studienstiftung des deutschen Volkes)

Conference Participation and Talks

2009 Workshop 'Low Eigenvalues of Laplace and Schrödinger Operators', Mathematisches Forschungsinstitut Oberwolfach
A sharp Lieb-Thirring inequality with a remainder term

7^{th} Congress of the International Society of Analysis, its Applications and Computing (ISAAC), Imperial College London
Universal Bounds for the Trace of the Heat Kernel

16^{th} International Congress of Mathematical Physics (ICMP), Prague

2010 'Queen Dido Conference on the Isoperimetric Problem', Carthage
Universal Bounds on Traces of the Dirichlet Laplace Operator

Summer School on 'Current Topics in Mathematical Physics', Aarhus

Invited speaker at 'QMath 11', Hradez Králové
Refined semiclassical asymptotics of the Laplace operator on bounded domains

Workshop 'Mathematical Physics', Jena
Semiclassical asymptotics of the Dirichlet Laplacian and its fractional powers

2011 Workshop 'Selected Topics in Spectral Theory', ESI, Wien
Two-term spectral asymptotics for the Dirichlet Laplacian and its fractional powers

Invited Seminar Talks

2009 Imperial College London

2010 Universität Stuttgart, Technische Universität München

2011 Georgia Institute of Technology, Atlanta