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MATHEMATISCHE NACHRICHTEN

ORIGINAL ARTICLE

Approximation of a two-dimensional Gross–Pitaevskii equation with a periodic potential in the tight-binding limit

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Abstract

The Gross–Pitaevskii (GP) equation is a model for the description of the dynamics of Bose–Einstein condensates. Here, we consider the GP equation in a two-dimensional setting with an external periodic potential in the x-direction and a harmonic oscillator potential in the y-direction in the so-called tight-binding limit. We prove error estimates which show that in this limit the original system can be approximated by a discrete nonlinear Schrödinger equation. The paper is a first attempt to generalize the results from [19] obtained in the one-dimensional setting to higher space dimensions and more general interaction potentials. Such a generalization is a non-trivial task due to the oscillations in the external periodic potential which become singular in the tight-binding limit and cause some irregularity of the solutions which are harder to handle in higher space dimensions. To overcome these difficulties, we work in anisotropic Sobolev spaces. Moreover, additional non-resonance conditions have to be satisfied in the two-dimensional case.

KEYWORDS
discrete NLS equation, error estimates

1 | INTRODUCTION

A Bose–Einstein condensate is a state of matter that is formed when a gas of bosons at low densities is cooled to temperatures very close to absolute zero. For such condensates in the mean-field limit, the N-particle Schrödinger equation can be described by a single nonlinear Gross–Pitaevskii (GP) equation, cf. [2, 8, 17]. Looking at the condensate in an external periodic field, such as optical gaps, is very common in experimental and in theoretical research. From a mathematical point of view, we have to deal with a nonlinear Schrödinger equation with an external periodic potential. The book [15] gives a good overview on the analysis of such systems. The GP equation is of the form

$$i\partial_t u = -\Delta u + Vu + N(u)$$

with $t \in \mathbb{R}$, $r \in \mathbb{R}^d$ for $d = 1, 2, 3$, where V is the external potential and $u = u(r, t) \in \mathbb{C}$ the wavefunction. The nonlinear terms $N(u)$ model the interaction of the involved molecules. Local hard sphere interactions are modeled with a term

$\sigma|u|^2u$, whereas a term βuv with v satisfying $\Delta v = |u|^2$ models a long range interaction. In the existing literature also dipole interactions are considered.

Here, we consider the GP equation

$$i\partial_t u = -\Delta u + Vu + \sigma|u|^2u, \quad t \in \mathbb{R}, \quad r \in \mathbb{R}^2, \tag{1}$$

in a two-dimensional setting with an external periodic potential V in the x-direction and a harmonic oscillator potential in the y-direction in the so-called tight-binding limit. Tight-binding limits in mathematical physics have been considered in various papers, cf. [1, 9].

Bose–Einstein condensates, respectively the GP equation, with external periodic potential V in the x-direction and a harmonic oscillator potential in the y-direction has been considered for instance in [4]. Herein, $u(t,r) : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$ decays to zero sufficiently fast as $|r| \to \infty$ and $V = V(r)$ is given by a periodic sequence of potential wells with a height of order $\mathcal{O}(\varepsilon^{-2})$ in the x-direction, where $0 < \varepsilon \ll 1$.

For $\varepsilon = 0$ the potential wells are separated and for each potential well a countable number of eigenfunctions exist. For $\varepsilon > 0$ nearby these eigenfunctions there are the combined functions $\widehat{\phi}_{n,m}(x)\psi_j(y)$ where the indices $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ of the so-called Wannier functions $\widehat{\phi}_{n,m}(x)$ denote the nth eigenfunction in the mth potential well in the x-direction and $j \in \mathbb{N}_0$ the jth eigenfunction in the y-direction. These combined functions $\widehat{\phi}_{n,m}(x)\psi_j(y)$ decay with a rate $\mathcal{O}(\mathrm{e}^{-r|x-m|/\varepsilon})$ for an $r > 0$. Hence, the influence of the Wannier functions of the m_1th well to the m_2th well is exponentially small w.r.t. $\varepsilon > 0$, in detail $\mathcal{O}(\mathrm{e}^{-r|m_1-m_2|/\varepsilon})$-small. Suppose for a moment that the solutions, in which we are interested in, are linear combinations of the lowest order combined functions $\widehat{\phi}_{1,m}(x)\psi_0(y)$. We would like to describe such solutions by the amplitudes of these functions alone. In order to obtain a system where the linear and nonlinear terms are of the same order also the amplitude has to be exponentially small w.r.t. $\varepsilon > 0$. As a consequence also, the dynamics will happen on an exponentially long timescale. Therefore, in order to obtain a non-trivial equation for the amplitudes we make the ansatz

$$u(t,r) = \mu^{1/2}\Psi_0(t,r) + \text{higher-order terms},$$

with

$$\mu^{1/2}\Psi_0(t,r) = \mu^{1/2}\varphi_0(T,r)\mathbf{E}(t), \quad \varphi_0(T,r) = \sum_{m\in\mathbb{Z}} a_m(T)\widehat{\phi}_{1,m}(x)\psi_0(y), \tag{2}$$

small parameter $\mu = \varepsilon\mathrm{e}^{-a/\varepsilon}$, with $a \in (0, 2\pi)$ defined in Equation (4), slow time variable $T = \mu t$, and $\mathbf{E}(t) = \mathrm{e}^{-i(\hat{E}_{1,0}+\omega_0)t}$ where $\hat{E}_{1,0} + \omega_0$ is the energy associated with $\widehat{\phi}_{1,m}(x)\psi_0(y)$. We find that in the limit $\varepsilon \to 0$ the dynamics of the original system can formally be described by a discrete nonlinear Schrödinger equation (dNLS) which is a system of infinitely many coupled ordinary differential equation (ODEs)

$$i\partial_T a_m = \alpha(a_{m-1} + a_{m+1}) + \sigma\beta|a_m|^2a_m, \tag{3}$$

with explicitly computable coefficients α and β, cf. Equation (31), and where the amplitude functions $a_m(T)$ are associated with the mth potential well and evolve with respect to the slow time $T = \mu t$ with $\mu = \mu(\varepsilon) > 0$. More details about Wannier functions and the derivation of the dNLS equation which is not restricted to $(n, j) = (1, 0)$ are given in the subsequent sections. For larger amplitudes no linear terms and for smaller amplitudes no non-linear terms appear in the limit equation. The paper is a first attempt to generalize the approximation results from [19] for small $0 < \varepsilon \ll 1$ obtained in the one-dimensional setting to higher space dimensions and to more general interaction potentials. In contrast to the one-dimensional problem in [19], for the two-dimensional model the new difficulty occurs that higher regularity is needed in order to control the nonlinearity of Equation (1) in $\mathbb{R}^2$. To overcome these difficulties, we work in anisotropic Sobolev spaces $\mathcal{H}^{1,2}(\mathbb{R}^2)$. Moreover, additional non-resonance conditions have to be satisfied in the 2D case.

The paper is organized as follows. We start in Section 2 with a short introduction of the model and focus on the definition of the function spaces which we need to prove our approximation result. We also give a short recap of the spectral situation in general and on the theory of Wannier function decomposition for the one-dimensional Schrödinger operator with a periodic potential. The main result is formulated in Section 3. In Section 4.1, we use a multiple scaling expansion to derive the effective amplitude equation. Moreover, we construct an improved approximation and estimate the associated residual. In Section 4.2, we estimate the error by using the variation of constant formula and Gronwall's inequality. Since

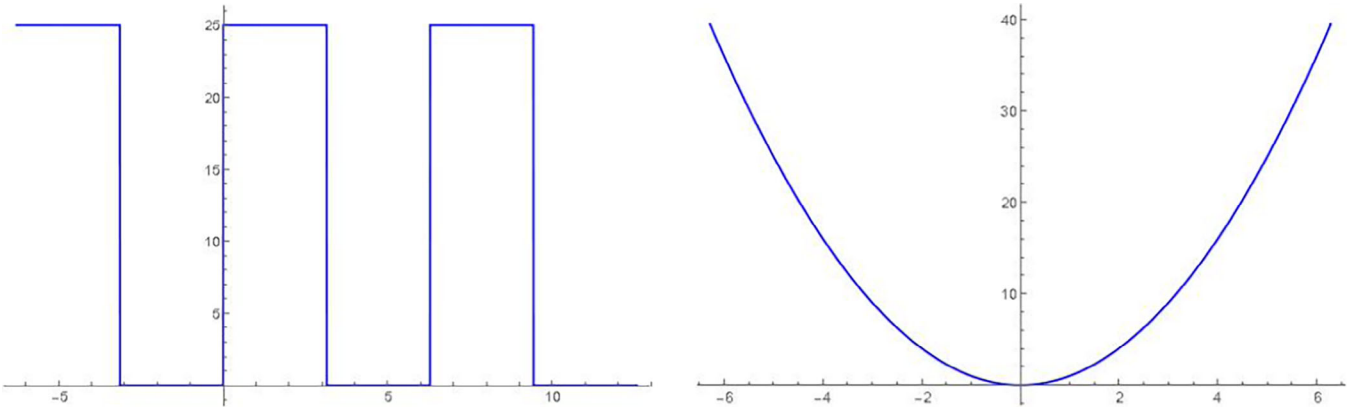

FIGURE 1 The potential function $V_x(x)$ with $a = \pi$ and $\varepsilon = 0.2$ (left) and the potential function $V_y(y) = y^2$ (right).

our overall analysis for the deterministic case follows very closely [19] for the one-dimensional case, we concentrate on the parts which we have to adjust for the two-dimensional setting.

Notation. We denote with $C_b^0(\mathbb{R}^2)$ the space of uniformly continuous and bounded functions $u : \mathbb{R}^2 \to \mathbb{C}$ equipped with the sup-norm $\|\cdot\|_{C_b^0}$. We use the standard notations for the Sobolev space $H^s(\mathbb{R}^2)$ and the Lebesgue space $L^p(\mathbb{R}^2)$ for $s \geq 0$ and $p \geq 1$. The anisotropic Sobolev space $H^{1,2}(\mathbb{R}^2)$ is equipped with the norm

$$\|u\|_{H^{1,2}} = \left(\|\partial_x u\|_{L^2}^2 + \|\partial_y^2 u\|_{L^2}^2 + \|u\|_{L^2}^2\right)^{1/2}.$$

For the complex-valued sequence spaces we write $l^1(\mathbb{Z})$ and $l^2(\mathbb{Z})$ with the corresponding norms $\|\vec{a}\|_{l^1} = \sum_{m\in\mathbb{Z}} |a_m|$ and $\|\vec{a}\|_{l^2} = \left(\sum_{m\in\mathbb{Z}} |a_m|^2\right)^{1/2}$.

2 | THE MODEL

We consider the GP Equation (1) with a periodic potential in two space dimensions, where the potential $V(r) = V_x(x) + V_y(y)$ is given by a piecewise-constant periodic potential $V_x(x)$ in the x-direction and a harmonic oscillator potential of the form $V_y(y) = y^2$ in the y-direction. Analogous to the one-dimensional case in [19], the potential V_x is bounded, real-valued, 2π-periodic and defined by

$$V_x(x) = \begin{cases} \varepsilon^{-2}, & x \in (0, a) \bmod(2\pi), \\ 0, & x \in (a, 2\pi) \bmod(2\pi), \end{cases} \tag{4}$$

where $a \in (0, 2\pi)$ is fixed and $0 < \varepsilon \ll 1$ is a small perturbation parameter. Both potentials are shown in Figure 1. The parameter $\sigma = \pm 1$ in Equation (1) is normalized for convenience.

Remark 2.1. For our analysis, it is essential that the potential is separable, but the special form of the potentials V_x and V_y is not essential as long as they have similar properties as the chosen ones, that is, V_x should be chosen for instance in such a way that the subsequent Lemma 2.9 holds. Other examples can be found for instance in [16, Section 7]. Almost all physical experiments about Bose–Einstein condensates use separable potentials, cf. [18].

In order to prove our result stated in Theorem 3.3, we need a suitable Banach space to bound the error made by the approximation. As in the one-dimensional case, we have $\|V_x\|_{C_b^0} \to \infty$ for $\varepsilon \to 0$. Since $V_x \geq 0$ for all $x \in \mathbb{R}$, it is convenient for our analysis to work in the anisotropic function space $\mathcal{H}^{1,2}(\mathbb{R}^2)$ equipped with the norm

$$\|u\|_{\mathcal{H}^{1,2}}^2 = \langle (L_x + I)u, u\rangle_{L^2} + \langle (L_y + I)^2 u, u\rangle_{L^2} + \langle u, u\rangle_{L^2}, \tag{5}$$

where I is the identity operator and where the occurring scalar products are defined by the linear Schrödinger operators

$$L_x = -\partial_x^2 + V_x \qquad \text{and} \qquad L_y = -\partial_y^2 + V_y.$$

This space will allow us to control the nonlinearity of the original problem (1) in two space dimensions, too.

Remark 2.2. Note that it is not possible to use the space $\mathcal{H}^1(\mathbb{R}^2)$ with $\|u\|_{\mathcal{H}^1}^2 = \langle (L+I)u, u\rangle_{L^2}$ in our analysis due to Sobolev's embedding theorem in two space dimensions where $L = L_x + L_y = -\Delta + V$. For the same reason with the present approach, the 2D case with a potential periodic in both space directions cannot be handled so far in the tight binding limit. Also, the function space $\mathcal{H}^2(\mathbb{R}^2)$ with $\|u\|_{\mathcal{H}^2}^2 = \langle (L+I)^2 u, u\rangle_{L^2}$ seems to be inappropriate for this problem because, as far as we can see, this space is not closed under point-wise multiplication with an ε-independent constant C_B in Theorem 2.7 for the potential $V = V_x + V_y$. Moreover, we would like to mention that the difficulties do not come from the non-smoothness of V_x but from the scaling properties of V_x.

2.1 | The function space $\mathcal{H}^{1,2}$

Anisotropic Sobolev spaces were introduced, for instance, in [13]. See also [7].

As mentioned above, we need the anisotropic function space $\mathcal{H}^{1,2}(\mathbb{R}^2)$ to be closed under point-wise multiplication without a loss of powers of ε when multiplying two functions of order $\mathcal{O}(1)$, cf. Theorem 2.7. Therefore, we first define the space $H^{1,2}(\mathbb{R}^2)$ and equip it with the norm

$$\|u\|_{H^{1,2}} = \left(\|\partial_x u\|_{L^2}^2 + \|\partial_y^2 u\|_{L^2}^2 + \|u\|_{L^2}^2\right)^{1/2}. \tag{6}$$

It is obvious that $\|u\|_{H^1} \le C\|u\|_{H^{1,2}}$. The following lemmas now allow us to control two more norms with Equation (6), which we need in the proof of Theorem 2.7.

Lemma 2.3. *There exists a positive constant $C_C > 0$ such that*

$$\|u\|_{C_b^0} \le C_C \|u\|_{H^{1,2}} \tag{7}$$

for all $u \in H^{1,2}(\mathbb{R}^2)$.

Proof. In this proof, we use the Fourier transform $\mathcal{F}$ and its inverse, denoted by $\mathcal{F}^{-1}$. The continuity of $\mathcal{F}^{-1} : L^1(\mathbb{R}^2) \to C_b^0(\mathbb{R}^2)$ and the Cauchy–Schwarz inequality lead to

$$\|u\|_{C_b^0} \le \|\widehat{u}\|_{L^1} = \|\widehat{u}\rho\rho^{-1}\|_{L^1} \le \|\widehat{u}\rho\|_{L^2}\|\rho^{-1}\|_{L^2},$$

where $\rho(k,l) = 1 + |k| + |l|^2$ and $\widehat{u} = \mathcal{F}(u)$. In fact, the Fourier transform $\mathcal{F}$ is an isometric isomorphism from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$ and by using the weight ρ we get

$$\|\widehat{u}\rho\|_{L^2} \le \|u\|_{H^{1,2}}.$$

Hence, it remains to show that $\|\rho^{-1}\|_{L^2} < \infty$ in order to obtain the bound (7). We find

$$\begin{aligned}\|\rho^{-1}\|_{L^2}^2 &= \int_{l=-\infty}^{\infty}\int_{k=-\infty}^{\infty} \left(1 + |k| + |l|^2\right)^{-2} dk dl \\ &= 4\int_{l=0}^{\infty}\int_{k=0}^{\infty} \left(1 + |k| + |l|^2\right)^{-2} dk dl = 4\int_{l=0}^{\infty} \left(1 + |l|^2\right)^{-1} dl = 2\pi\end{aligned}$$

and the inequality (7) holds. □

Remark 2.4.

a) Embedding theorems for anisotropic Sobolev spaces can be found in the existing literature. However, for completeness we give proofs of the estimates related to the anisotropy of the problem.
b) The reason why we restrict ourselves to two space dimensions is that this embedding just holds in two space dimensions. In the three-dimensional case, the integral

$$\int_{l_y=1}^{\infty}\int_{l_z=1}^{\infty}\int_{k=1}^{\infty}\left(k+l_y^2+l_z^2\right)^{-2}dkdl_zdl_y=\int_{k=1}^{\infty}\int_{r=1}^{\infty}\left(k+r^2\right)^{-2}rdrdk$$

is divergent and we cannot bound the norm $\|\rho^{-1}\|_{L^2}$ of the weight function $\rho(k,l_y,l_z)$. Since the analysis of the linear operator heavily relies on its self-adjointness, we have to use L^2-based spaces and are not allowed to change to L^p-based spaces on the right-hand side of the inequality (7) at this point. We expect that the three-dimensional case can be handled in a space $\mathcal{H}^{1,3,3}(\mathbb{R}^3)$.
c) The case of periodic potentials in both directions cannot be handled with our analysis for similar reasons. This will be the subject of future research.

With the same approach, we can also estimate $\|\partial_y u\|_{L^4}$ with the norm of the Sobolev space $H^{1,2}$ in two space dimensions.

Lemma 2.5. *There exists a positive constant $C_L>0$ such that*

$$\|\partial_y u\|_{L^4}\le C_L\|u\|_{H^{1,2}} \tag{8}$$

for all $u\in H^{1,2}(\mathbb{R}^2)$.

Proof. The inverse Fourier transform $\mathcal{F}^{-1}$ is a continuous mapping from L^q to L^p with $1/p+1/q=1$ for $q\le 2$ due to the Hausdorff–Young inequality. For $p=4$ and $q=4/3$, this leads to the inequality

$$\|\partial_y u\|_{L^4}\le\|il\widehat{u}\|_{L^{4/3}}=\|il\widehat{u}\rho\rho^{-1}\|_{L^{4/3}}\le\|\widehat{u}\rho\|_{L^2}\|il\rho^{-1}\|_{L^4} \tag{9}$$

with the weight $\rho=1+|k|+|l|^2$ and $il\widehat{u}=\mathcal{F}(\partial_y u)$. We also applied the generalized Hölder inequality $\|uv\|_{L^r}\le\|u\|_{L^p}\|v\|_{L^q}$ with $1/p+1/q=1/r$ to obtain the last inequality of Equation (9). Using analogous calculations as in the proof of Lemma 2.3, we find that

$$\begin{aligned}\|il\rho^{-1}\|_{L^4}^4&=\int_{l=-\infty}^{\infty}\int_{k=-\infty}^{\infty}|l|^4\cdot\left(1+|k|+|l|^2\right)^{-4}dkdl\\&=4\int_{l=0}^{\infty}\int_{k=0}^{\infty}|l|^4\cdot\left(1+|k|+|l|^2\right)^{-4}dkdl\\&=\frac{4}{3}\int_{l=0}^{\infty}|l|^4\cdot\left(1+|l|^2\right)^{-3}dl<\infty\end{aligned}$$

is bounded and with $\|\widehat{u}\rho\|_{L^2}\le\|u\|_{H^{1,2}}$ we complete the proof. □

Remark 2.6. Moreover, we use $\|u\|_{L^4}\le\widetilde{C}_L\|u\|_{H^1}$ for all $u\in H^1(\mathbb{R}^2)$ which is a consequence of classical Sobolev's embedding theorem. It can be proved with the same strategy as the proof of Lemma 2.5, using the weight function $\rho=\left(1+k^2+l^2\right)^{1/2}$.

With these two embedding results, the following inequality holds.

Theorem 2.7. *The space $\mathcal{H}^{1,2}(\mathbb{R}^2)$ is closed under point-wise multiplication with*

$$\|u\|_{H^{1,2}} \leq \|u\|_{\mathcal{H}^{1,2}} \qquad \text{and} \qquad \|uv\|_{\mathcal{H}^{1,2}} \leq C_B \|u\|_{\mathcal{H}^{1,2}} \|v\|_{\mathcal{H}^{1,2}}$$

for every $u, v \in \mathcal{H}^{1,2}(\mathbb{R}^2)$ and some positive constant $C_B > 0$ independent of the small parameter $0 < \varepsilon \ll 1$.

Proof. We first show that $\|u\|_{H^{1,2}} \leq \|u\|_{\mathcal{H}^{1,2}}$. Expanding the norm defined in Equation (5), we get

$$\|u\|^2_{\mathcal{H}^{1,2}} = \langle L_x u, u\rangle_{L^2} + \langle L_y^2 u, u\rangle_{L^2} + 2\langle L_y u, u\rangle_{L^2} + 3\langle u, u\rangle_{L^2}.$$

Now, we look at the occurring terms separately. We obtain

$$\langle L_y^2 u, u\rangle_{L^2} = \|\partial_y^2 u\|^2_{L^2} + \|V_y u\|^2_{L^2} + \langle -\partial_y^2 u, V_y u\rangle_{L^2} + \langle V_y u, -\partial_y^2 u\rangle_{L^2} \tag{10}$$

and

$$\langle L_y u, u\rangle_{L^2} = \|\partial_y u\|^2_{L^2} + \|V_y^{1/2} u\|^2_{L^2}, \qquad \langle L_x u, u\rangle_{L^2} = \|\partial_x u\|^2_{L^2} + \|V_x^{1/2} u\|^2_{L^2}, \tag{11}$$

respectively. Using Equations (10) and (11), we rewrite the norm as

$$\begin{aligned} \|u\|^2_{\mathcal{H}^{1,2}} = \|u\|^2_{H^{1,2}} + \|V_y u\|^2_{L^2} + \langle -\partial_y^2 u, V_y u\rangle_{L^2} + \langle V_y u, -\partial_y^2 u\rangle_{L^2} \\ + 2\|\partial_y u\|^2_{L^2} + 2\|V_y^{1/2} u\|^2_{L^2} + \|V_x^{1/2} u\|^2_{L^2} + 2\|u\|^2_{L^2} \end{aligned} \tag{12}$$

and simplify the scalar products by integration by parts to get for $V_y = y^2$ that

$$\begin{aligned} \langle -\partial_y^2 u, V_y u\rangle + \langle V_y u, -\partial_y^2 u\rangle_{L^2} &= \langle \partial_y u, \partial_y (V_y u)\rangle_{L^2} + \langle \partial_y (V_y u), \partial_y u\rangle_{L^2} \\ &= -2\|u\|^2_{L^2} + 2\|V_y^{1/2}(\partial_y u)\|^2_{L^2}, \end{aligned} \tag{13}$$

where $V_y, \partial_y V_y, \partial_y^2 V_y \in \mathbb{R}$ and $\partial_y^2 V_y = 2$. Inserting Equation (13) into Equation (12), we directly obtain

$$\begin{aligned} \|u\|^2_{\mathcal{H}^{1,2}} = \|u\|^2_{H^{1,2}} + \|V_y u\|^2_{L^2} + 2\|V_y^{1/2}(\partial_y u)\|^2_{L^2} \\ + 2\|\partial_y u\|^2_{L^2} + 2\|V_y^{1/2} u\|^2_{L^2} + \|V_x^{1/2} u\|^2_{L^2}, \end{aligned} \tag{14}$$

and thus the bound

$$\|u\|^2_{H^{1,2}} \leq \|u\|^2_{\mathcal{H}^{1,2}} \tag{15}$$

holds.

Now, we come to the proof of the second estimate and write

$$\begin{aligned} \|uv\|^2_{\mathcal{H}^{1,2}} = \|\partial_y^2(uv)\|^2_{L^2} + \|V_y(uv)\|^2_{L^2} + 2\|V_y^{1/2}(\partial_y(uv))\|^2_{L^2} \\ + 2\|\partial_y(uv)\|^2_{L^2} + 2\|V_y^{1/2}(uv)\|^2_{L^2} + \|\partial_x(uv)\|^2_{L^2} \\ + \|V_x^{1/2}(uv)\|^2_{L^2} + \|uv\|^2_{L^2}. \end{aligned} \tag{16}$$

We look at the terms on the right-hand side of Equation (16) separately. For the first term, the triangle inequality yields

$$\begin{aligned} \|\partial_y^2(uv)\|_{L^2} &\leq \|(\partial_y^2 u)v\|_{L^2} + 2\|(\partial_y u)(\partial_y v)\|_{L^2} + \|u(\partial_y^2 v)\|_{L^2} \\ &\leq \|\partial_y^2 u\|_{L^2} \|v\|_{C_b^0} + 2\|\partial_y u\|_{L^4} \|\partial_y v\|_{L^4} + \|u\|_{C_b^0} \|\partial_y^2 v\|_{L^2} \end{aligned}$$

and with the bounds (7), (8), and (15), we obtain

$$\|\partial_y^2(uv)\|_{L^2} \le 2C_C\|u\|_{H^{1,2}}\|v\|_{H^{1,2}} + 2C_L^2\|u\|_{H^{1,2}}\|v\|_{H^{1,2}} \le C\|u\|_{\mathcal{H}^{1,2}}\|v\|_{\mathcal{H}^{1,2}}.$$

Using the representation (14) and the bound (7), we can estimate the second term $\|V_y(uv)\|_{L^2}^2$ as follows:

$$\|V_y(uv)\|_{L^2} \le \|v\|_{C_b^0}\|V_y u\|_{L^2} \le C_C\|v\|_{H^{1,2}}\|u\|_{\mathcal{H}^{1,2}} \le C_C\|v\|_{\mathcal{H}^{1,2}}\|u\|_{\mathcal{H}^{1,2}}.$$

The estimates for the remaining terms follow with the same ideas as in the calculations above. In total, we get the inequality

$$\|uv\|_{\mathcal{H}^{1,2}}^2 \le C_B^2\|u\|_{\mathcal{H}^{1,2}}^2\|v\|_{\mathcal{H}^{1,2}}^2$$

and so the function space $\mathcal{H}^{1,2}(\mathbb{R}^2)$ is closed under point-wise multiplication with a constant $C_B = \mathcal{O}(1)$ for $\varepsilon \to 0$. □

2.2 | The unitary semigroup in $\mathcal{H}^{1,2}(\mathbb{R}^2)$

By definition we have

$$\|u\|_{\mathcal{H}^{1,2}}^2 = \langle (L_x + I)u, u\rangle_{L^2} + \langle (L_y + I)^2 u, u\rangle_{L^2} + \langle u, u\rangle_{L^2},$$

with $L_x = -\partial_x^2 + V_x(x)$ and $L_y = -\partial_y^2 + V_y(y)$. These operators are self-adjoint in $L^2(\mathbb{R}^2)$ and thus, by the classical semigroup theory, cf. [14], we get

$$\langle \mathrm{e}^{-iL_x t}u, \mathrm{e}^{-iL_x t}u\rangle_{L^2} = \langle u, u\rangle_{L^2} \quad \text{and} \quad \langle \mathrm{e}^{-iL_y t}u, \mathrm{e}^{-iL_y t}u\rangle_{L^2} = \langle u, u\rangle_{L^2}.$$

Since $L_xL_y = L_yL_x$, we have that L_x, L_y, $\mathrm{e}^{-iL_x t}$, and $\mathrm{e}^{-iL_y t}$ all commute. Therefore, we find

$$\begin{aligned}
\|\mathrm{e}^{-iLt}u\|_{\mathcal{H}^{1,2}}^2 &= \langle (L_x + I)\mathrm{e}^{-iLt}u, \mathrm{e}^{-iLt}u\rangle_{L^2} + \langle (L_y + I)^2\mathrm{e}^{-iLt}u, \mathrm{e}^{-iLt}u\rangle_{L^2} + \langle \mathrm{e}^{-iLt}u, \mathrm{e}^{-iLt}u\rangle_{L^2} \\
&= \langle \mathrm{e}^{-iLt}(L_x + I)u, \mathrm{e}^{-iLt}u\rangle_{L^2} + \langle \mathrm{e}^{-iLt}(L_y + I)^2 u, \mathrm{e}^{-iLt}u\rangle_{L^2} + \langle \mathrm{e}^{-iLt}u, \mathrm{e}^{-iLt}u\rangle_{L^2} \\
&= \langle (L_x + I)u, u\rangle_{L^2} + \langle (L_y + I)^2 u, u\rangle_{L^2} + \langle u, u\rangle_{L^2} \\
&= \|u\|_{\mathcal{H}^{1,2}}^2,
\end{aligned}$$

and so e^{-iLt} forms a unitary group in $\mathcal{H}^{1,2}(\mathbb{R}^2)$ with

$$\|\mathrm{e}^{-iLt}u\|_{\mathcal{H}^{1,2}} = \|u\|_{\mathcal{H}^{1,2}}. \tag{17}$$

2.3 | The Bloch wave expansion

The form of our potential makes it reasonable to use an approximation ansatz which decomposes the original solutions $u(t, x)$ of the GP equation (1) into a linear combination of the eigenfunctions of the linear problems corresponding to the one-dimensional operators $L_x = -\partial_x^2 + V_x$ and $L_y = -\partial_y^2 + V_y$. Therefore, we first study the spectrum of these operators separately.

i) The spectral problem of the harmonic oscillator in the y-direction is well documented in the literature. The eigenvalue problem is solved by the eigenfunctions

$$\psi_j(y) = \frac{1}{(\pi)^{1/4}(2^j j!)^{1/2}} H_j(y)\mathrm{e}^{-y^2/2} \tag{18}$$

with corresponding eigenvalues $\omega_j = 1 + 2j$ for $j \in \mathbb{N}_0$, where the function H_j represents the jth Hermite polynomial.

ii) Next, we focus on the one-dimensional eigenvalue problem $L_x\varphi = E\varphi$ which is solved by the Bloch wave ansatz

$$\varphi(x) = \mathrm{e}^{i\ell x}\phi(\ell, x), \quad \ell,\ x \in \mathbb{R}.$$

The functions ϕ satisfy

$$\phi(\ell, x) = \phi(\ell, x + 2\pi), \quad \phi(\ell, x) = \phi(\ell + 1, x)e^{ix}, \quad \ell,\ x \in \mathbb{R}, \tag{19}$$

and therefore we can restrict the definition of $\phi(\ell, x)$ to $x \in \mathbb{T}_{2\pi} = \mathbb{R}/(2\pi\mathbb{Z})$ and $\ell \in \mathbb{T}_1 = \mathbb{R}/\mathbb{Z}$. With this ansatz, $\phi(\ell, \cdot)$ is a solution to the eigenvalue problem

$$\widetilde{L}_x(\ell, x)\phi(\ell, x) = E(\ell)\phi(\ell, x)$$

with the linear operator $\widetilde{L}_x(\ell, x) := -(\partial_x + i\ell)^2 + V(x)$. We have cf. [10].

Lemma 2.8. *For fixed $\ell \in \mathbb{T}_1$, the operator $\widetilde{L}_x(\ell, x)$ is a self-adjoint, positive semi-definite operator in $L^2(\mathbb{T}_{2\pi})$.*

By Lemma 2.8 and the spectral theorem for self-adjoint operators with compact resolvent for each $\ell \in \mathbb{T}_1$, there exists a Schauder base $\{\phi_n(\ell, \cdot)\}_{n\in\mathbb{N}}$ of $L^2(\mathbb{T}_{2\pi})$ consisting of eigenfunctions of $\widetilde{L}_x(\ell, x)$ with positive eigenvalues $\{E_n(\ell)\}_{n\in\mathbb{N}}$ ordered as $E_n(\ell) \le E_{n+1}(\ell)$.

Hence, we define the corresponding Bloch wavefunction in $L^2(\mathbb{R})$ to the eigenvalue $E_n(\ell)$ by $\widetilde{\phi}_n(\ell, x) := \mathrm{e}^{i\ell x}\phi_n(\ell, x)$ and each of these pairs solves the spectral problem

$$L_x\widetilde{\phi}_n(\ell, x) = E_n(\ell)\widetilde{\phi}_n(\ell, x) \tag{20}$$

for every $n \in \mathbb{N}$. By Equation (19), the condition $\widetilde{\phi}_n(\ell, x + 2\pi) = \widetilde{\phi}_n(\ell, x)\mathrm{e}^{i2\pi\ell}$ holds for all $x \in \mathbb{R}$, and also all Bloch wavefunctions $\widetilde{\phi}_n(\ell, \cdot)$ satisfy the following orthogonality and normalization relation:

$$\langle\widetilde{\phi}_n(\ell, \cdot), \widetilde{\phi}_{n'}(\ell', \cdot)\rangle_{L^2(\mathbb{R})} = \delta_{n,n'}\delta(l - l'), \quad n, n' \in \mathbb{N}, \quad \ell, \ell' \in \mathbb{T}_1.$$

In order to normalize the phase factors of the Bloch wavefunctions as in [16], we set $\overline{\widetilde{\phi}}_n(\ell, x) = \widetilde{\phi}_n(-\ell, x)$ as eigenfunctions for the eigenvalues $E_n(\ell) = \overline{E}_n(\ell) = E_n(-\ell)$.

2.4 | The Wannier functions

In the limit we are interested in, it is more useful to use the Wannier function decomposition. This allows us to approximate the continuous partial differential equation (PDE) (1) with the lattice equation (3). In the following, we summarize the main properties of the Wannier functions. This is done similarly to [19], following the more detailed approaches in [15, 16].

The band function $E_n(\ell)$ and the Bloch wavefunction $\phi_n(\ell, x)$ introduced above are 1-periodic with respect to $\ell \in \mathbb{T}_1$ for any $n \in \mathbb{N}$. Therefore, we can represent them by the Fourier series

$$E_n(\ell) = \sum_{m\in\mathbb{Z}} \widehat{E}_{n,m}\mathrm{e}^{i2\pi m\ell}, \qquad \ell \in \mathbb{T}_1, \tag{21}$$

and

$$\widetilde{\phi}_n(\ell, x) = \sum_{m\in\mathbb{Z}} \widehat{\phi}_{n,m}(x)\mathrm{e}^{i2\pi m\ell}, \qquad \ell \in \mathbb{T}_1, \quad x \in \mathbb{R}. \tag{22}$$

The Fourier coefficients in Equations (21) and (22) are defined by the integrals

$$\widehat{E}_{n,m} = \int_{\mathbb{T}_1} E_n(\ell) \mathrm{e}^{-i2\pi m\ell}\, d\ell, \qquad m \in \mathbb{Z},$$

and

$$\widehat{\phi}_{n,m}(x) = \int_{\mathbb{T}_1} \widetilde{\phi}_n(\ell, x) e^{-i2\pi m\ell}\, d\ell, \qquad m \in \mathbb{Z}, \quad x \in \mathbb{R},$$

and satisfy

$$\widehat{E}_{n,m} = \overline{\widehat{E}}_{n,-m} = \widehat{E}_{n,-m}, \qquad \widehat{\phi}_{n,m}(x) = \overline{\widehat{\phi}}_{n,m}(x), \qquad \forall m \in \mathbb{Z},\ \forall n \in \mathbb{N},\ \forall x \in \mathbb{R}.$$

Because of the quasi-periodicity, $\widetilde{\phi}_n(\ell, x + 2\pi) = \widetilde{\phi}_n(\ell, x)\mathrm{e}^{i2\pi\ell}$ for any $x \in \mathbb{R}$ and $\ell \in \mathbb{T}_1$, we obtain the property

$$\widehat{\phi}_{n,m}(x) = \widehat{\phi}_{n,m-1}(x - 2\pi) = \widehat{\phi}_{n,0}(x - 2\pi m), \qquad \forall m \in \mathbb{Z},\ \forall n \in \mathbb{N},\ \forall x \in \mathbb{R},$$

and the so-defined real-valued functions $\widehat{\phi}_{n,m}$ are called Wannier functions.

Although the Wannier functions are no eigenfunctions of the Schrödinger operator L_x, we can substitute the Fourier series representations (21) and (22) into the one-dimensional linear problem (20) for a fixed $n \in \mathbb{N}$. Thereby, in every spectral band $\mathcal{E}_n$ the set of functions $\{\widehat{\phi}_{n,m}\}_{m\in\mathbb{Z}}$ satisfies the system

$$L_x \widehat{\phi}_{n,m} = \sum_{m'\in\mathbb{Z}} \widehat{E}_{n,m-m'} \widehat{\phi}_{n,m'}, \qquad \forall m \in \mathbb{Z}, \tag{23}$$

with the corresponding coefficients $\{\widehat{E}_{n,m}\}_{m\in\mathbb{Z}}$. Orthogonality and normalization of Wannier functions, given by

$$\langle \widehat{\phi}_{n,m}(\cdot), \widehat{\phi}_{n',m'}(\cdot)\rangle_{L^2(\mathbb{R})} = \delta_{n,n'}\delta_{m,m'}, \quad n, n' \in \mathbb{N}, \quad m, m' \in \mathbb{Z},$$

directly follow from the relation (21) for Bloch functions.

We complete this part with the following two lemmas from [16], which summarize important properties of the band functions $\widehat{E}_n(\ell)$ and the Wannier functions $\widehat{\phi}_{n,m}(x)$ of the linear one-dimensional spectral problem $L_x\varphi = E\varphi$. For an introduction into the theory, see [12, 20] or the textbooks [6, 11].

Lemma 2.9. *Let V_x be given by Equation (4) and $\mu = \varepsilon e^{-a/\varepsilon}$. For any fixed $n_0 \in \mathbb{N}$, there exist strictly positive constants $\varepsilon_0, C_s, E_0, C_1^{\pm}, C_2, C_0, C_m > 0$, such that, for any $\varepsilon \in [0, \varepsilon_0)$, the band functions and the Wannier functions of the operator $L_x = -\partial_x^2 + V_x$ satisfy the properties.*

(i) (Band separation)

$$\min_{n\in\mathbb{N}\setminus\{n_0\}} \inf_{\ell\in\mathbb{T}_1} |E_n(\ell) - \widehat{E}_{n_0,0}| \geq C_s \tag{24}$$

(ii) (Band boundedness)

$$|\hat{E}_{n_0,0}| \leq E_0 \tag{25}$$

(iii) (Tight-binding approximation)

$$\begin{aligned} &C_1^- \mu \leq |\hat{E}_{n_0,1}| \leq C_1^+ \mu, \\ &|\hat{E}_{n_0,m}| \leq C_2 \mu^2 \qquad \text{for } m \geq 2 \end{aligned} \tag{26}$$

(iv) (Compactly supported approximation)

$$|\widehat{\phi}_{n_0,0}(x) - \widehat{\phi}_0| \leq C_0\varepsilon, \quad \forall x \in [0, 2\pi], \tag{27}$$

where

$$\widehat{\phi}_0(x) = \begin{cases} 0, & \forall x \in [0,a], \\ \frac{\sqrt{2}}{\sqrt{2\pi-a}} \sin\left(\frac{\pi n_0(2\pi-x)}{2\pi-a}\right), & \forall x \in [a,2\pi]. \end{cases}$$

(v) (Exponential decay)

$$|\widehat{\phi}_{n_0,0}(x)| \leq C_m \mu^m, \quad \forall x \in [-2\pi m, -2\pi(m-1)] \cup [2\pi m, 2\pi(m+1)], m \in \mathbb{N} \tag{28}$$

Proof. The proof of *(i)–(iii)* is given in [16, Appendix B] and *(iv) and (v)* are shown in [16, Appendix C]. □

Lemma 2.10. *Let $\mathcal{E}_n$ be the invariant closed subspace of $L^2(\mathbb{R})$ associated with the nth spectral band and assume that $\mathcal{E}_n \cap \mathcal{E}_{n'} = \{0\}$ for a fixed $n \in \mathbb{N}$ and all $n' \neq n$. Then, $\langle \widehat{\phi}_{n,m}, \widehat{\phi}_{n,m'} \rangle_{L^2(\mathbb{R})} = \delta_{m,m'}$ for any $m, m' \in \mathbb{Z}$ and there exists constants $\eta_n > 0$ and $C_n > 0$, such that*

$$|\widehat{\phi}_{n,m}(x)| \leq C_n e^{-\eta_n |x-2\pi m|} \tag{29}$$

Proof. Here, we refer to the proof of [16, Proposition 2]. □

Remark 2.11. The exponential decay (29) is proven in [16] for the spectrum of an operator $L = -\partial_x^2 + V(x)$ consisting of the union of disjoint spectral bands. However, for the proof of Lemma 2.10 we do not need the assumption that all spectral bands are disjoint. It is sufficient that the particular nth spectral band is disjoint from the other spectral bands of the linear operator. According to property (24), this condition is satisfied for the spectrum of the operator L_x for small $\varepsilon > 0$, cf. [19].

3 | MAIN RESULT

We represent an approximate solution of the given GP equation (1) by the formal asymptotic expansion

$$u(t,r) = \mu^{1/2}\Psi_0(t,r) + \text{higher-order terms},$$

with

$$\mu^{1/2}\Psi_0(t,r) = \mu^{1/2}\varphi_0(T,r)\mathbf{E}(t), \quad \varphi_0(T,r) = \sum_{m\in\mathbb{Z}} a_m(T)\widehat{\phi}_{n,m}(x)\psi_j(y) \tag{30}$$

and $\mathbf{E}(t) = \mathrm{e}^{-i(\hat{E}_{n,0}+\omega_j)t}$, for a fixed $n \in \mathbb{N}$ and $j \in \mathbb{N}_0$, where $T = \mu t$ with $\mu = \varepsilon e^{-a/\varepsilon} \ll 1$ is the slow time variable and $r = (x,y) \in \mathbb{R}^2$. The set of Wannier functions $\{\widehat{\phi}_{n,m}\}_{m\in\mathbb{Z}}$ belongs to the subspace of $L^2(\mathbb{R})$ associated with the nth spectral band of the one-dimensional Schrödinger operator L_x. The function ψ_j represents the eigenfunction of the operator L_y with the corresponding energy ω_j.

Substituting the ansatz (30) into the original equation (1) shows that the amplitudes $\{a_m(T)\}_{m\in\mathbb{Z}}$ satisfy the dNLS equation (3), where the parameters are given by

$$\alpha = \lim_{\mu\to 0} \alpha_\mu, \qquad \text{and} \qquad \beta = \lim_{\mu\to 0} \beta_\mu \tag{31}$$

where

$$\alpha_\mu = \frac{\widehat{E}_{n,1}}{\mu} = \alpha + \mathcal{O}(\mu) \qquad \text{and} \qquad \beta_\mu = \|\widehat{\phi}_{n,0}\psi_j\|_{L^4}^4 = \beta + \mathcal{O}(\mu).$$

The values of both constants α_μ and β_μ are uniformly bounded and nonzero as $\mu \to 0$. For a bound on the constant α we refer to (26) and for a bound on the constant β we refer to Remark 3.5. Properties (26) and (27) guarantee that $\alpha, \beta \neq 0$ for $0 < \mu \ll 1$. The initial-value problem for the dNLS equation (3) is locally well-posed in $C^1([0,T_0], l^1(\mathbb{Z}))$ for a $T_0 > 0$.

Lemma 3.1. *Let $\vec{a}_0 \in l^1(\mathbb{Z})$. Then, there exists a $T_0 > 0$ and a unique solution $\vec{a}(T) \in C^1([0,T_0], l^1(\mathbb{Z}))$ of the dNLS equation (3) with the initial data $\vec{a}(0) = \vec{a}_0$.*

Proof. See the proof of [19, Theorem 2]. □

Remark 3.2. The solutions a of the dNLS equation (3) with coefficients α and β and the solutions a_μ of the dNLS equation (3) with coefficients α_μ and β_μ obviously stay $\mathcal{O}(\mu)$-close on an $\mathcal{O}(1)$-timescale with respect to T.

The mathematical justification of the effective amplitude equation (3) by means of error estimates is the main purpose of this section. The approximation result is given by the following theorem.

Theorem 3.3. *Pick $n \in \mathbb{N}$, $j \in \mathbb{N}_0$ such that the following non-resonance condition is satisfied for all $\ell \in \mathbb{T}_1$:*

$$\left(E_{n'}(\ell) - \widehat{E}_{n,0}\right) + 2\left(j' - j\right) \neq 0, \quad \text{for every} \quad n' \in \mathbb{N}\backslash\{n\} \quad \text{and} \quad j' \in \mathbb{N}_0. \tag{32}$$

Let $\vec{a}(T) \in C^1\left([0,T_0], l^1(\mathbb{Z})\right)$ be a solution of the dNLS equation (3) with initial data $\vec{a}(0) = \vec{a}_0$ satisfying the bound

$$\left\| u_0 - \mu^{1/2} \sum_{m\in\mathbb{Z}} a_m(0)\widehat{\phi}_{n,m}(x)\psi_j(y) \right\|_{\mathcal{H}^{1,2}} \leq C_0\mu^{3/2}$$

for some $C_0 > 0$. Then, for any $\mu \in (0,\mu_0)$ with sufficiently small $\mu_0 > 0$, there exists a μ-independent constant $C > 0$ such that the GP equation (1) admits a mild solution $u(t) \in C^0\left([0,T_0/\mu], \mathcal{H}^{1,2}\left(\mathbb{R}^2\right)\right)$ satisfying the bound

$$\left\| u(\cdot,t) - \mu^{1/2} \sum_{m\in\mathbb{Z}} a_m(\mu t)\widehat{\phi}_{n,m}(x)\psi_j(y)e^{-i(\hat{E}_{n,0}+\omega_j)t} \right\|_{\mathcal{H}^{1,2}} \leq C\mu^{3/2}$$

for all $t \in [0,T_0/\mu]$.

Remark 3.4. Since $\mu = \varepsilon e^{-a/\varepsilon}$, the finite time interval $[0,T_0/\mu]$ is exponentially long with respect to ε, cf. [5, 19].

Remark 3.5. Using the embedding result from Remark 2.6 yields

$$\|\widehat{\phi}_{n,0}\psi_j\|_{L^4} \leq C\|\widehat{\phi}_{n,0}\psi_j\|_{H^1} \leq C\|\widehat{\phi}_{n,0}\psi_j\|_{H^{1,2}} \leq C\|\widehat{\phi}_{n,0}\psi_j\|_{\mathcal{H}^{1,2}} \tag{33}$$

and with‘

$$\begin{aligned}
\|\widehat{\phi}_{n,0}\psi_j\|^2_{\mathcal{H}^{1,2}} &= \langle (L_x+I)\widehat{\phi}_{n,0}\psi_j, \widehat{\phi}_{n,0}\psi_j\rangle_{L^2} + \langle (L_y+I)^2\widehat{\phi}_{n,0}\psi_j, \widehat{\phi}_{n,0}\psi_j\rangle_{L^2} \\
&\quad + \langle \widehat{\phi}_{n,0}\psi_j, \widehat{\phi}_{n,0}\psi_j\rangle_{L^2} \\
&= \widehat{E}_{n,0} + 1 + \omega_j^2 + 2\omega_j + 1 + 1 \\
&\leq (E_0+1) + \left(\omega_j+1\right)^2 + 1 < \infty
\end{aligned} \tag{34}$$

we immediately get that $\beta = \|\widehat{\phi}_{n,0}\psi_j\|^4_{L^4} < \infty$.

Remark 3.6. The non-resonance condition (32) in Theorem 3.3 is new compared to the one-dimensional result. This additional condition is needed due to the orthogonal projection on the spectral bands which is introduced in Lemma 4.2 used for the construction of a higher order approximation.

Remark 3.7. The method of reducing a continuous equation to a nonlinear lattice problem as an effective equation in the tight-binding limit for the description of a wavefunction has been applied formally in the physics literature, cf. [3]. Other rigorous approaches on the reduction of a one-dimensional GP equation to a dNLS equation can be found in [2, 5, 16].

4 | PROOF OF THEOREM 3.3

4.1 | Estimates for the residual

Our next goal is to compute and estimate the so-called residual

$$\mathrm{Res(u)} = -i\partial_t u - \Delta u + Vu + \sigma|u|^2 u, \tag{35}$$

which contains the remaining terms which do not cancel after inserting the approximation $u = \mu^{1/2}\Psi_0$ into the GP equation (1). Substituting the ansatz (30) into Equation (35) leads to

$$\begin{aligned}\mathrm{Res}(\mu^{1/2}\Psi_0) = -i\mu^{3/2}\sum_{m\in\mathbb{Z}} \partial_T a_m(T)\widehat{\phi}_{n,m}(x)\psi_j(y)\mathbf{E}(t)\\ - \mu^{1/2}\sum_{m\in\mathbb{Z}} a_m(T)\sum_{\substack{m'\in\mathbb{Z}\\ m\neq m'}} \widehat{E}_{n,m-m'}\widehat{\phi}_{n,m'}(x)\psi_j(y)\mathbf{E}(t)\\ + \sigma\mu^{3/2}|\varphi_0(T,r)|^2\varphi_0(T,r)\mathbf{E}(t).\end{aligned}$$

Using the expansion (23), $(-\partial_y^2 + V_y(y) - \omega_j)\psi_j(y) = 0$, and the fact that the amplitude functions $\{a_m(T)\}_{m\in\mathbb{Z}}$ satisfy the dNLS equation (3) with the coefficients α and β displayed in Equation (31), yields

$$\mathrm{Res}(\mu^{1/2}\Psi_0) = \mathrm{s}_1 + \mathrm{s}_2$$

where

$$s_1 = \mu^{1/2}\sum_{m\in\mathbb{Z}}\sum_{\substack{m'\in\mathbb{Z}\\ m'\neq\{m-1,m,m+1\}}} \widehat{E}_{n,m'-m}a_{m'}(T)\widehat{\phi}_{n,m}(x)\psi_j(y)\mathbf{E}(t),$$

$$s_2 = \sigma\mu^{3/2}\left(|\varphi_0(T,r)|^2\varphi_0(T,r) - \sum_{m\in\mathbb{Z}}\beta|a_m(T)|^2 a_m(T)\widehat{\phi}_{n,m}(x)\psi_j(y)\right)\mathbf{E}(t).$$

The term s_1 is of order $\mathcal{O}(\mu^{5/2})$ due the tight-binding assumption (26). The remaining terms in s_2 are of order $\mathcal{O}(\mu^{3/2})$. However, for our approach it is necessary that the residual will be of order $\mathcal{O}(\mu^{5/2})$. In order to achieve this order we add higher-order terms $\mu^{3/2}\Psi_\mu$ to the ansatz (30) and thus define the improved approximation as

$$\mu^{1/2}\Psi(t,r) = \mu^{1/2}\Psi_0(t,r) + \mu^{3/2}\Psi_\mu(t,r). \tag{36}$$

Due to the $\sigma|u|^2u$-nonlinearity, as above, the higher order terms can be chosen as

$$\mu^{3/2}\Psi_\mu = \mu^{3/2}\varphi_\mu(T,r)\mathbf{E}(t).$$

Inserting the improved approximation (36) into the GP equation (1) yields

$$\mathrm{Res}(\mu^{1/2}\Psi) = \quad s_1 + s_3 + s_4, \tag{37}$$

with s_1 from above, and

$$\begin{aligned}s_3 = \sigma\mu^{3/2}\Bigg(\frac{1}{\sigma}\big(L - \big(\widehat{E}_{n,0} + \omega_j\big)\big)\varphi_\mu(T,r) + |\varphi_0(T,r)|^2\varphi_0(T,r)\\ - \sum_{m\in\mathbb{Z}}\beta|a_m(T)|^2 a_m(T)\widehat{\phi}_{n,m}(x)\psi_j(y)\Bigg)\mathbf{E}(t),\end{aligned}$$

$$s_4 = \sigma\mu^{5/2}\left(-\frac{i}{\sigma}\partial_T\varphi_\mu(T,r) + 2|\varphi_0(T,r)|^2\varphi_\mu(T,r) + \overline{\varphi}_\mu(T,r)\varphi_0^2(T,r)\right)\mathbf{E}(t)$$
$$+\sigma\mu^{7/2}\left(2\left|\varphi_\mu(T,r)\right|^2\varphi_0(T,r)\mathbf{E}(t) + \overline{\varphi}_0(T,r)\varphi_\mu^2(T,r)\right)\mathbf{E}(t)$$
$$+\sigma\mu^{9/2}\left|\varphi_\mu(T,r)\right|^2\varphi_\mu(T,r)\mathbf{E}(t).$$

Above we already found that the term s_1 is of order $\mathcal{O}(\mu^{5/2})$. By a pure counting of powers of μ we also find that $s_4 = \mathcal{O}(\mu^{5/2})$. In order to control the terms in s_3 we use the orthogonal projection $\Pi_{n,j} : L^2(\mathbb{R}^2) \to \mathcal{E}_{n,j}$, where $\mathcal{E}_{n,j} \subset L^2(\mathbb{R}^2)$ is defined as the direct product of the nth spectral band $\mathcal{E}_n$ in the x-direction and the eigenspace U_j corresponding to the eigenvalue ω_j of the harmonic oscillator problem. We set

$$s_{3a} = \Pi_{n,j}s_3 \qquad \text{and} \qquad s_{3b} = (I - \Pi_{n,j})s_3.$$

By construction we have

$$s_{3a} = \Pi_{n,j}\left(\sigma\mu^{3/2}\left(\frac{1}{\sigma}(L - (\widehat{E}_{n,0} + \omega_j))\varphi_\mu(T,r)\right)\right) = \mathcal{O}(\mu^{7/2})$$

as in s_1 due to tight-binding assumption (26). In s_{3b}, the linear terms in φ_μ will be used to cancel the nonlinear terms in φ_0. In order to do so, we need the following two lemmas.

Lemma 4.1. *Let $\vec{a}(T) \in l^1(\mathbb{Z})$ for a fixed $T \in \mathbb{R}$ and let $\varphi_0(r) = \sum_{m\in\mathbb{Z}} a_m\widehat{\phi}_{n,m}(x)\psi_j(y)$ for a fixed $n \in \mathbb{N}$, then $\varphi_0 \in \mathcal{E}_{n,j} \subset L^2(\mathbb{R}^2)$ and $\varphi_0 \in \mathcal{H}^{1,2}(\mathbb{R}^2)$.*

Proof. We have

$$\|\varphi_0\|_{\mathcal{H}^{1,2}}^2 \le \|\vec{a}\|_{l^1}^2 \sup_{m\in\mathbb{Z}} \|\widehat{\phi}_{n,m}\psi_j\|_{\mathcal{H}^{1,2}}^2 \tag{38}$$

by using the triangle inequality and the bound (34). □

Lemma 4.2. *For any $f \in L^2(\mathbb{R}^2)$ there exists a unique solution $\varphi \in \mathcal{H}^{1,2}(\mathbb{R}^2)$ of the inhomogeneous equation*

$$\left(L - \left(\widehat{E}_{n,0} + \omega_j\right)\right)\varphi = \left(I - \Pi_{n,j}\right)f,$$

with $\langle\varphi,\vartheta\rangle_{L^2} = 0$ for every $\vartheta \in \mathcal{E}_{n,j}$ and φ satisfying

$$\|\varphi\|_{\mathcal{H}^{1,2}} \le C\|f\|_{L^2} \tag{39}$$

for some $C > 0$ uniformly in $0 < \mu \ll 1$.

Proof. We find $\widehat{E}_{n,0} \notin \sigma(L_x)|_{\mathcal{E}_n^\perp}$ by the band separation property (24) of Lemma 2.9 and $\omega_j \notin \sigma(L_y)|_{U_j^\perp}$. Using the same arguments as in [15], if φ is a solution of

$$\left((L_x - \widehat{E}_{n,0}) + (L_y - \omega_j)\right)\varphi = \left(I - \Pi_{n,j}\right)f,$$

then φ can be represented by

$$\varphi(r) = \int_{\mathbb{T}_1}\int_{y'=-\infty}^{\infty}\sum_{n'\in\mathbb{N}\setminus\{n\}}\sum_{j'\in\mathbb{N}\setminus\{j\}}\frac{\widetilde{f}_{n'}(\ell,y')\overline{\psi}_{j'}(y')\widetilde{\phi}_{n'}(\ell,x)}{(E_{n'}(\ell) - \widehat{E}_{n,0}) + (\omega_{j'} - \omega_j)}dy'd\ell\cdot\psi_j(y), \tag{40}$$

which implies that there is a constant $C_N > 0$ such that

$$\begin{aligned}\|\varphi\|_{L^2}^2 &\leq \int_{\mathbb{T}_1}\int_{y'=-\infty}^{\infty}\sum_{n'\in\mathbb{N}\setminus\{n\}}\sum_{j'\in\mathbb{N}\setminus\{j\}}\frac{\left|\widetilde{f}_{n'}(\ell,y')\psi_{j'}(y')\right|^2}{\left((E_{n'}(\ell)-\widehat{E}_{n,0})+(\omega_{j'}-\omega_j)\right)^2}dy'd\ell\\ &\leq C_N\int_{\mathbb{T}_1}\int_{y'=-\infty}^{\infty}\sum_{n'\in\mathbb{N}\setminus\{n\}}\sum_{j'\in\mathbb{N}\setminus\{j\}}\left|\widetilde{f}_{n'}(\ell,y')\psi_{j'}(y')\right|^2dy'd\ell\\ &\leq C_N\|f\|_{L^2}^2.\end{aligned} \tag{41}$$

due to the non-resonance condition (32) assumed in Theorem 3.3 such that $\varphi \in L^2(\mathbb{R}^2)$ for any $f \in L^2(\mathbb{R}^2)$. For obtaining the bound (39), we use again the representation (40) and the definition of the $\mathcal{H}^{1,2}$-norm in Equation (40) leading to similar estimates as in Equation (41). □

Using Lemmas 4.1 and 4.2 allows us to define $\varphi_\mu \in \mathcal{E}_{n,j}^{\perp}$ as a unique solution of the system

$$\left(L-\left(\widehat{E}_{n,0}+\omega_j\right)\right)\varphi_\mu = -\sigma\left(I-\Pi_{n,j}\right)|\varphi_0|^2\varphi_0. \tag{42}$$

By this choice we have $s_{3,b} = 0$ such that in the residual only terms of order $\mathcal{O}(\mu^{5/2})$ remain. Hence, we have proved the following lemma.

Lemma 4.3. *Let $\vec{a}(T) \in C^1\left([0,T_0], l^1(\mathbb{Z})\right)$ be a solution of the dNLS equation (3) and let $\mu^{1/2}\Psi$ be the extended approximation defined in Equation (36) associated with this solution. Then, there exists a C_{res} such that for all $\mu \in (0,1]$ we have*

$$\sup_{[0,T_0/\mu]}\|\mathrm{Res}(\mu^{1/2}\Psi)\|_{\mathcal{H}^{1,2}} \leq C_{\mathrm{res}}\mu^{5/2}.$$

4.2 | The error estimates

We derive an evolution equation for the error and estimate its solutions with energy estimates and a simple application of Gronwall's inequality.

We write a solution u of the GP equation (1) as a sum of the improved approximation $\mu^{1/2}\Psi(T,r) = \mu^{1/2}\Phi_\mu(T,r)\mathbf{E}(t)$ and the error term $\mu^{3/2}R(t,r) = \mu^{3/2}\rho(t,r)\mathbf{E}(t)$, that is,

$$u(t,r) = \mu^{1/2}\Phi_\mu(T,r)\mathbf{E}(t) + \mu^{3/2}\rho(t,r)\mathbf{E}(t).$$

The error satisfies

$$i\partial_t\rho = \left(L-\left(\widehat{E}_{n,0}+\omega_j\right)\right)\rho + \mu^{-3/2}N(\Phi_\mu,\rho) + \mu^{-3/2}\mathbf{E}^{-1}(t)\mathrm{Res}(\mu^{1/2}\Psi) \tag{43}$$

with

$$\begin{aligned}\mu^{-3/2}N(\Phi_\mu,\rho) = \mu^{-3/2}\sigma(&|\mu^{1/2}\Phi_\mu(T,r)+\mu^{3/2}\rho(t,r)|^2(\mu^{1/2}\Phi_\mu(T,r)+\mu^{3/2}\rho(t,r))\\ &-|\mu^{1/2}\Phi_\mu(T,r)|^2\mu^{1/2}\Phi_\mu(T,r)).\end{aligned}$$

According to Theorem 2.7, this term obeys the estimate

$$\|\mu^{-3/2}N(\Phi_\mu,\rho)\|_{\mathcal{H}^{1,2}} \leq C(\mu\|\Phi_\mu\|_{\mathcal{H}^{1,2}}^2\|\rho\|_{\mathcal{H}^{1,2}} + \mu^2\|\Phi_\mu\|_{\mathcal{H}^{1,2}}\|\rho\|_{\mathcal{H}^{1,2}}^2 + \mu^3\|\rho\|_{\mathcal{H}^{1,2}}^3). \tag{44}$$

We consider the variation of constant formula associated with Equation (43), namely

$$\rho(t) = i \int_0^t \mathrm{e}^{i(L-(\widehat{E}_{n,0}+\omega_j))(t-\tau)}(\mu^{-3/2}N(\Phi_\mu,\rho) + \mu^{-3/2}\mathbf{E}^{-1}(t)\mathrm{Res}(\mu^{1/2}\Psi))(\tau)d\tau. \tag{45}$$

Using the previous estimates, namely Equation (17), Lemma 4.3, and Equation (44), yields

$$\|\rho(t)\|_{\mathcal{H}^{1,2}} \leq C \int_0^t (\mu\|\Phi_\mu\|^2_{\mathcal{H}^{1,2}}\|\rho\|_{\mathcal{H}^{1,2}} + \mu^2\|\Phi_\mu\|_{\mathcal{H}^{1,2}}\|\rho\|^2_{\mathcal{H}^{1,2}} + \mu^3\|\rho\|^3_{\mathcal{H}^{1,2}})(\tau) + C_{res}\mu d\tau$$

which allows us to estimate the function $\rho(t,\cdot)$ in the $\mathcal{H}^{1,2}(\mathbb{R}^2)$-norm for $t \in [0, T_0/\mu]$ by Gronwall's inequality. In detail, if we assume

$$\mu\|\Phi_\mu\|_{\mathcal{H}^{1,2}}\|\rho\|_{\mathcal{H}^{1,2}} + \mu^2\|\rho\|^2_{\mathcal{H}^{1,2}} \leq 1$$

we get the bound

$$\|\rho(t)\|_{\mathcal{H}^{1,2}} \leq C \int_0^t (\mu(\|\Phi_\mu\|^2_{\mathcal{H}^{1,2}} + 1)\|\rho\|_{\mathcal{H}^{1,2}})(\tau) + C_{\mathrm{res}}\mu d\tau.$$

With

$$\sup_{\tau\in[0,T_0/\mu]} \|\Phi_\mu(\mu\tau,\cdot)\|_{\mathcal{H}^{1,2}} = \sup_{T\in[0,T_0]} \|\Phi_\mu(T,\cdot)\|_{\mathcal{H}^{1,2}} = C_\Phi = \mathcal{O}(1)$$

Gronwall's inequality leads to

$$\|\rho(t)\|_{\mathcal{H}^{1,2}} \leq C_{\mathrm{res}}\mu t\mathrm{e}^{\mu(C_\Phi^2+1)t} \leq C_{\mathrm{res}}T_0\mathrm{e}^{(C_\Phi^2+1)T_0} =: M_\rho$$

for all $t \in [0; T_0/\mu]$. By choosing $\mu_0 > 0$ so small that

$$\mu_0 C_\Phi M_\rho + \mu_0^2 M_\rho^2 \leq 1,$$

we are done with the proof of Theorem 3.3.

Remark 4.4. Local existence and uniqueness of solutions of Equation (1) in $\mathcal{H}^{1,2}$ is straightforward with the previous estimates. Following the arguments of Section 2.2 the operator $e^{-i(L-(\widehat{E}_{n,0}+\omega_j))t}$ forms a unitary group in $\mathcal{H}^{1,2}(\mathbb{R}^2)$ with

$$\|e^{-i(L-(\widehat{E}_{n,0}+\omega_j))t}u\|^2_{\mathcal{H}^{1,2}} = \|u\|^2_{\mathcal{H}^{1,2}}.$$

Since the nonlinear terms are smooth in $\mathcal{H}^{1,2}(\mathbb{R}^2)$ the local existence and uniqueness of solutions of Equation (1) in $\mathcal{H}^{1,2}$ follows by applying the contraction mapping principle to the right-hand side of the variation of constant formula.

Remark 4.5. The local existence and uniqueness of solutions of Equation (1) in $\mathcal{H}^{1,2}$ together with the error estimates as a priori estimates guarantees the long time existence of the solutions as long as the error estimates hold.

5 | DISCUSSION

We close the paper with a discussion about the same situation but with additive white noise located in the potential wells and with a possible extension of the previous approximation result to more than one band.

We consider the GP equation

$$i\partial_t u = -\Delta u + Vu + \sigma|u|^2u + \xi, \quad t \in \mathbb{R}, \quad r \in \mathbb{R}^2, \tag{46}$$

where

$$\xi(x,t) = \sum_{n\in\mathbb{Z}} c_n \xi_n(t) Y(x + 2\pi n, y),$$

with coefficients $c_n \geq 0$, Y a spatially localized function in the first potential well and where the $\xi_n = \partial_t W_n$ are independent white noise processes defined as distributional derivatives of independent Wiener processes W_n. Formally, a system of infinitely many coupled ODEs

$$i\partial_T a_m = \alpha(a_{m-1} + a_{m+1}) + \sigma\beta|a_m|^2 a_m + \widetilde{c}_m \widetilde{\xi}_m, \tag{47}$$

with coefficients $\widetilde{c}_m$ and rescaled white noise processes $\widetilde{\xi}_m = \widetilde{\xi}_m(T)$ is obtained.

Remark 5.1. In the Wiener processes $W_n(t)$, we can rescale time by using the existence of a rescaled Wiener process $\widetilde{W}_n(T)$ with $T = \mu t$ and $\mu^{-1/2}\widetilde{W}_n(\mu t) = W_n(t)$ having the same probability distribution. Thus, we have for the rescaled white noise that $\mu^{1/2}\widetilde{\xi}_n(\mu t) = \xi_n(t)$.

Hence, we need $c_n = \mathcal{O}(\mu)$ in order to have the rescaled process $\widetilde{\xi}_m$ being of the sane order as the other terms in Equation (47). Thus, the noise terms have to be exponentially small in terms of ε. It will be subject of future research to justify amplitude equations like Equation (47) for the GP equation with noise terms like ξ.

It is an easy exercise to extend the tight binding approximation to more than one band. In case of two bands, we chose the formal asymptotic expansion

$$u(t,r) = \mu^{1/2}\Psi_{0,1}(t,r) + \mu^{1/2}\Psi_{0,2}(t,r) + \text{higher-order terms},$$

with

$$\mu^{1/2}\Psi_{0,p}(t,r) = \mu^{1/2}\varphi_{0,p}(T,r)\mathbf{E}_{n_p,j_p}(t), \quad \varphi_{0,p}(T,r) = \sum_{m\in\mathbb{Z}} a_{p,m}(T)\widehat{\phi}_{n_p,m}(x)\psi_{j_p}(y) \tag{48}$$

for $p \in \{1,2\}$ and $\mathbf{E}_{n,j}(t) = \mathrm{e}^{-i(\hat{E}_{n,0}+\omega_j)t}$. By equating the coefficients in front of $\mathbf{E}_{n_1,j_1}$ and $\mathbf{E}_{n_2,j_2}$ to zero yields a system of the form

$$i\partial_T a_{1,m} = \alpha_1(a_{1,m-1} + a_{1,m+1}) + \sigma\beta_{1,1}|a_{1,m}|^2 a_{1,m} + \sigma\beta_{1,2}|a_{2,m}|^2 a_{1,m}, \tag{49}$$

$$i\partial_T a_{2,m} = \alpha_1(a_{2,m-1} + a_{1,m+1}) + \sigma\beta_{2,1}|a_{1,m}|^2 a_{2,m} + \sigma\beta_{2,2}|a_{2,m}|^2 a_{2,m}, \tag{50}$$

with coefficients α_p, β_{p_1,p_2} for $p, p_1, p_2 \in \{1,2\}$. With some modified non-resonance condition, similar to Equation (32), which allows to compute the improved approximation, cf. Lemma 4.2, an approximation theorem, similar to Theorem 3.3, can be established by modifying the previous proof only slightly.

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