Logical Fragments for Mazurkiewicz Traces: Expressive Power and Algebraic Characterizations

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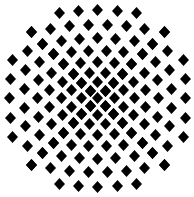

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Contents

Deutsche Zusammenfassung

Spuren wurden erstmals von Keller als Verallgemeinerung von Wörtern eingeführt [Kel73]. Später wurde das Konzept nochmals von Mazurkiewicz als Modell für Nebenläufigkeit aufgegriffen [Maz77]. Spuren entstehen dadurch, dass man partielle Kommutation zwischen einzelnen Buchstaben erlaubt. Seit ihrer Einführung wurden viele Aspekte von Spuren und Spursprachen untersucht [DR95].

Über Wörtern hat sich der Ansatz als sehr nützlich erwiesen, erkennbare Sprachen mit Hilfe von sogenannten Varietäten von endlichen Monoiden zu klassifizieren. Wir verfolgen diesen Ansatz um logische Fragmente über Spuren zu untersuchen. Den Schwerpunkt unserer Untersuchungen bildet die Varietät **DA** endlicher Monoide. Wir geben verschiedene neue sprachtheoretische und logische Charakterisierungen für die Varietät im Zusammenhang mit Spuren an.

Die erste sprachtheoretische Charakterisierung ist, dass das syntaktische Monoid einer Spursprache $L \subseteq \mathbb{M}(\Sigma, I)$ genau dann in **DA** liegt, wenn die Sprache L und ihr Komplement $\mathbb{M}(\Sigma, I) \setminus L$ beide Polynome sind. Eine Sprache ist ein *Polynom*, falls sie eine endliche Vereinigung von Sprachen der folgenden Form ist:

$$A_0^* a_1 A_1^* a_2 A_2^* \cdots a_n A_n^*$$

mit $n \in \mathbb{N}$, $a_i \in \Sigma$ für $1 \leq i \leq n$ und $A_i \subseteq \Sigma$ für $0 \leq i \leq n$.

Eine weitere sprachtheoretische Charakterisierung ist, dass das syntaktische Monoid von $L \subseteq \mathbb{M}(\Sigma, I)$ genau dann in **DA** liegt, wenn sich L aus Sprachen der Form A^* mit $A \subseteq \Sigma$ durch boolesche Operationen und eindeutige Produkte darstellen lässt. Die Definition von eindeutigen Produkten ist bei Spuren etwas technisch, da Spuren im Allgemeinen mehrere Linearisierungen besitzen. Ein Produkt $L_0 a L_1$, $a \in \Sigma$, von Spursprachen L_0 und L_1 ist *eindeutig*, falls es linkseindeutig oder rechtseindeutig ist.

Das Produkt $L = L_0 a L_1$ ist *linkseindeutig*, wenn jede Spur $t \in L$ genau eine Position ν mit folgenden Eigenschaften besitzt:

- die Beschriftung von ν ist a und
- der Präfix von t aller Positionen vor ν ist eine Spur in L_0 und
- der Suffix von t, der aus allen Positionen parallel zu ν oder nach ν besteht, ist eine Spur in L_1.

Rechtseindeutige Produkte werden symmetrisch definiert, indem man die Knoten, die parallel zu ν sind, dem Präfix zuordnet.

Der wichtigste Schritt bei diesen Ergebnissen ist die Untersuchung von allgemeineren Polynomen über kommutativen Sprachen, d.h. über Sprachen, bei denen die Zugehörigkeit einer Spur nicht von der Reihenfolge der Buchstaben abhängt.

Des weiteren beweisen wir drei Charakterisierungen durch temporallogische Fragmente von Spursprachen, deren syntaktisches Monoid zu **DA** gehört. Die erste solche Charakterisierung ist die Klasse derjenigen Spursprachen, die sich durch eine temporallogische Formel ausdrücken lassen, bei der neben den booleschen Operatoren nur die einfachen temporallogischen Operatoren XF und YP vorkommen.

- $\mathsf{XF}\varphi$ gilt an allen Positionen, an denen in der Zukunft eine Position existiert, an der φ gilt.
- $\mathsf{YP}\varphi$ gilt an allen Positionen, an denen in der Vergangenheit eine Position existiert, an der φ gilt.

Als zweite Charakterisierung haben wir gezeigt, dass man zu diesen beiden Operatoren den temporallogischen Operator M hinzunehmen kann, ohne dass sich die Ausdrucksstärke der Logik erhöht.

- $\mathsf{M}\varphi$ gilt an allen Positionen, falls irgendwo eine Position existiert, an der φ gilt.

Dieser Operator erlaubt es, in eine andere Zusammenhangskomponente innerhalb einer Spur zu wechseln. Die dritte temporallogische Charakterisierung ist durch temporallogische Formeln, bei denen neben den booleschen Verknüpfungen nur zwei Sorten von temporallogischen Operatoren erlaubt sind: X_a und Y_a mit $a \in \Sigma$.

- $\mathsf{X}_a\varphi$ gilt an allen Positionen, bei denen an der Position mit dem nächsten Vorkommen des Buchstaben a in der Zukunft die Formel φ gilt. Die Formel gilt nicht an Positionen, bei denen der Buchstabe a in der Zukunft gar nicht vorkommt.
- $\mathsf{Y}_a\varphi$ gilt an allen Positionen, bei denen an der Position mit dem letzten Vorkommen des Buchstaben a in der Vergangenheit die Formel φ gilt. Die Formel gilt nicht an Positionen, bei denen der Buchstabe a in der Vergangenheit gar nicht vorkommt.

Die folgenden beiden Fragmente der Logik erster Stufe stimmen ebenfalls mit der Varietät **DA** in dem Sinne überein, dass eine Sprache genau dann in einer dieser Logiken ausdrückbar ist, wenn ihr syntaktisches Monoid zu **DA** gehört. Die erste Charakterisierung ist durch Formeln in Logik erster Stufe mit nur zwei verschiedenen Variablen. Bei diesen Formeln sind Gleichheit $=$ und die Abhängigkeitsrelation $\rightarrowtriangle$ die einzigen vorkommenden binären Relationen. Dieses Fragment bezeichnen wir mit $\mathrm{FO}^2[\rightarrowtriangle]$. Für zwei Positionen ν und χ einer Spur gilt $\nu \rightarrowtriangle \chi$, falls χ nach ν kommt und die Label der beiden Positionen von einander abhängig sind. Wir sprechen bei $\mathrm{FO}^2[\rightarrowtriangle]$ auch von FO^2 über *Abhängigkeitsgraphen*. Das zweite logische Fragment ist ebenfalls über Abhängigkeitsgraphen. Eine Spursprache L ist in $\Delta_2[\rightarrowtriangle]$, falls zwei Formeln in Logik erster Stufe mit folgenden Eigenschaften existieren. Beide Formeln besitzen nur eine Quantorenalternierung und beide beschreiben die Spursprache L. Die erste dieser Formeln soll mit einem Block von Existenzquantoren beginnen; darauf folgt dann ein Block von Allquantoren. Die zweite Formel beginnt mit einem Block von Allquantoren gefolgt von einem Block von Existenzquantoren.

Beide Fragmente der Logik erster Stufe sind natürlich. Die Frage, welche Eigenschaften man mit zwei Variablen ausdrücken kann, ist interessant, da drei Variablen bereits ausreichend sind, um alle Eigenschaften zu formulieren, die sich überhaupt in Logik erster Stufe ausdrücken lassen [GM02a]. Das Fragment Δ_2 ist die größte unter Komplementbildung abgeschlossene Unterklasse derjenigen Spursprachen, die sich mit nur einer Quantorenalternierung ausdrücken lassen, also mit zwei Blöcken von Quantoren, wobei der erste Block

aus Existenzquantoren besteht. Symmetrisch dazu ist Δ_2 auch die größte unter Komplementbildung abgeschlossene Unterklasse derjenigen Spursprachen, die sich mit nur einer Quantorenalternierung ausdrücken lassen, wobei der erste Block ein Block von Allquantoren ist.

Aus einer algebraischen Charakterisierung von Polynomen durch Mal'cev Produkte ergibt sich durch den Einsatz von Ehrenfeucht-Fraïssé Spielen auf Spuren eine Charakterisierung von Polynomen durch Formeln erster Stufe über Abhängigkeitsgraphen mit einer Quantorenalternierung, beginnend mit Existenzquantoren. Dieses Fragment ist nicht unter Komplementbildung abgeschlossen.

Da die Zugehörigheit zur Varietät **DA** entscheidbar ist, folgt daraus, dass die Zugehörigkeit für alle besprochenen Charakterisierungen von **DA** ebenfalls entscheidbar ist.

Des weiteren untersuchen wir die Fragmente FO^2 und Δ_2 über Spuren als Halbordnungen. Beide Fragmente haben über Halbordnungen eine höhere Ausdrucksstärke als ihre Entsprechungen über Halbordnungen. Diese zusätzliche Ausdrucksstärke entsteht durch die Möglichkeit, dass man in beiden Fragmenten über Halbordnungen die Parallelität von zwei Positionen formulieren kann. Wenn wir zusätzlich zu den temporallogischen Operatoren XF und YP auch noch den Operator Eco für Parallelität hinzunehmen, erhalten wir eine Charakterisierung von FO^2 über Halbordnungen durch Temporallogik.

- $\mathsf{Eco}\varphi$ gilt an allen Positionen, an denen parallel eine Position existiert, an der φ gilt.

Wir zeigen, dass es ist nicht möglich, mit FO^2 die Parallelität von drei Positionen auszudrücken. Dies ist mit Δ_2 jedoch möglich. Außerdem zeigen wir, dass für jedes $n \in \mathbb{N}$ ein Spurmonoid $\mathbb{M}(\Sigma, I)$ und eine Spursprache $L \subseteq \mathbb{M}(\Sigma, I)$ mit der Eigenschaft existieren, dass L in FO^2 ausdrückbar ist aber nicht in Δ_n. Dies ist überraschend, denn zu jeder Spursprache L, die sich durch Logik erster Stufe beschreiben lässt, existiert ein $n \in \mathbb{N}$ so, dass L in der Klasse Δ_n liegt. Über Wörtern haben die Fragmente FO^2 und Δ_2 die selbe Ausdrucksstärke, da hier die Halbordnung und die Abhängigkeitsrelation übereinstimmen.

Alle Ergebnisse von der Form, dass sich gewisse Eigenschaften *nicht* in einem bestimmten Fragment formulieren lassen, werden mit Hilfe von geeigneten Ehrenfeucht-Fraïssé Spielen auf Spuren erzielt.

Chapter 1

Introduction

The concept of partially commutative free monoids has been introduced by Keller as a generalization of free monoids in order to describe the behavior of parallel programs [Kel73]. Later, Mazurkiewicz established the notion of *trace* monoids for theses structures [Maz77, Maz87]. Since then traces, the elements of trace monoids, have become a rather popular setting to study concurrency. A lot of aspects of traces and trace languages have been researched, see *The Book of Traces* [DR95] for an overview.

It has turned out that finite semigroups are a powerful technique to refine the class of recognizable languages [Eil76, Lal79, Pin86, Alm94, PP04]. One of the first and most famous results in this direction is due to Schützenberger: a recognizable language is star-free if and only if its syntactic monoid is aperiodic [Sch65].

Two natural operations on classes of languages are the polynomial closure and the unambiguous polynomial closure. Another result of Schützenberger is, that a recognizable language is the disjoint union of unambiguous polynomials if and only if its syntactic monoid is in the variety **DA** [Sch76]. For particular classes of languages, so called language varieties, it has been shown that there exist algebraic counterparts in terms of the Mal'cev product [PST88, PW97]. We will show that this correspondence between the Mal'cev product and the polynomial closure (resp. the unambiguous polynomial closure) for restricted varieties also holds over traces. Furthermore we give descriptions for particular polynomial closures and unambiguous polynomial closures in terms of logical fragments.

Temporal Logic for traces has been introduced by Ebinger [Ebi92, Ebi94, EM93]. There exist several characterizations of trace languages whose syntactic monoid is aperiodic. For a recognizable trace language L, the following are equivalent:

- The syntactic monoid of L is aperiodic.
- L is star-free [GRS92].
- L is expressible by a first-order formula, see e.g. [DM97].
- L is expressible by a first-order formula with three variables [GM02a].
- L is expressible in global linear temporal logic [DG00, DG02, TW97].
- L is expressible in local linear temporal logic [DG05, GM02b, GMN03].

This makes temporal logic a suitable formalism for specifying properties of concurrent systems, see e.g. [APP95, AS02, MT96, Nie95, Pen93, Ram96, Thi94, Thi95, Wal02]. However, the satisfiability problem for global logics is non-elementary [Wal98] whereas, for fixed alphabets, all local temporal logics over traces where the modalities are definable in monadic second order logic are decidable in PSPACE [GK03].

We contribute to a better comprehension of the relation between local temporal logic and first-order logic over traces. Therefore, we compare the expressivity of certain fragments. As a tool for this purpose, we will use semigroup theory and Ehrenfeucht-Fraïssé (EF) games [Ehr61]. Originally, EF games were applied to the first-order logic over relational structures in general. In [EW96] a modified version was presented in order to characterize fragments of temporal logic over words. We continue this approach and alter EF games for words in order to capture specific logical fragments over traces.

The class **DA** of finite monoids will play an important role. We give two language theoretic characterizations of the class of trace languages whose syntactic monoid is in **DA** and four characterizations in terms of logical fragments. All these characterizations are known to hold for words [Tes, TT02]. Since the variety **DA** is decidable, it follows that expressibility over traces within these fragments is decidable.

In particular, we consider first-order formulae with two variables. This is a natural restriction, because three variables are already sufficient to express

all first-order properties [GM02a]. Traces can be given either as labeled partially ordered sets or as dependence graphs, i.e., labeled sets with directed arcs only between dependent letters. Dependence graphs are no restriction for the full first-order fragment, see e.g. [DG05]. We explain how the distinction between partial orders and dependence graphs affects the expressivity of restricted first-order fragments and show a connection to temporal logic with and without an operator for parallelism. Surprisingly, there are simple fragments that have more exhausting characterizations on dependence graphs than on partial orders, although the latter representation would seem to be more natural.

Most results of this thesis have been published in [Kuf06] and [HK06]. The latter is joint work with Martin Horsch.

Chapter 2

Semigroups, Monoids and Varieties

2.1 Semigroups and Monoids

A *semigroup* is a pair $(S, \cdot)$, where S is a set and $\cdot$ is a binary operation $S \times S \to S$ satisfying $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in S$, i.e., $\cdot$ is *associative*. In the following, we write S for both, the set S and the semigroup $(S, \cdot)$. We say a semigroup S is *finite* if S is a finite set.

A *monoid* is a triple $(M, \cdot, 1)$, where $(M, \cdot)$ is a semigroup and $1 \in M$ satisfies $1 \cdot x = x = x \cdot 1$ for all $x \in M$, i.e., 1 is a *neutral element*. As for semigroups, we write M for both, the set M and the monoid $(M, \cdot, 1)$. We can consider monoids as special semigroups. Therefore, terms and definitions for semigroups can be carried over to monoids, e.g. we say a monoid M is *finite* if M is a finite set. To shorten notation, we omit the operation symbol and write ab instead of $a \cdot b$ if it is clear from the context which operation we mean. As usual, by a^n we denote the n-fold product $a \cdots a$. By $\mathbf{S}$ we denote the class of all finite semigroups and by $\mathbf{M}$ we denote the class of all finite monoids.

Example 2.1 Consider the set $Q = \{a, b, ab, ba, 0\}$ equipped with the binary operation $\cdot$ defined by $a \cdot b = ab$, $b \cdot a = ba$, $a \cdot b \cdot a = a$, $b \cdot a \cdot b = b$, $a^2 = b^2 = 0$ and $x \cdot 0 = 0 = 0 \cdot x$ for all $x \in Q$. This defines a semigroup $(Q, \cdot)$. We call this semigroup B_2. It is the syntactic semigroup of the subset $(ab)^+$ of the free semigroup $\{a, b\}^+$. If we set $b^2 = b$ (instead of $b^2 = 0$) we again obtain a semigroup. We denote this semigroup by U. It is the syntactic semigroup of the subset $\{a, b\}^* aa \{a, b\}^*$ of $\{a, b\}^+$. $\diamond$

An *ordered semigroup* $(S, \leq)$ is a semigroup S equipped with a partial order relation $\leq$ such that $a \leq b$ implies $ca \leq cb$ and $ac \leq bc$ for all $a, b, c \in S$. Every semigroup S forms also an ordered semigroup $(S, =)$. An *ordered monoid* is an ordered semigroup $(S, \leq)$ such that S is a monoid.

A subset T of a semigroup S is a *subsemigroup* if it is closed under multiplication. Analogously, we say that a subset N of a monoid M is a *submonoid* if it is closed under multiplication and contains the neutral element. We write $T \leq S$ if T is a subsemigroup of a semigroup S. We also use the symbol $\leq$ to denote the submonoid relation. An *ordered subsemigroup* $(T, \preceq)$ of an ordered semigroup $(S, \leq)$ is a subsemigroup with the additional property that $\preceq$ is the restriction of $\leq$ to T.

A mapping $\eta : S \to T$ between two semigroups S and T is a (semigroup-) *homomorphism* if $\forall a, b \in S\colon \eta(ab) = \eta(a)\eta(b)$. For *monoid-homomorphisms*, we additionally require $\eta(1) = 1$ and homomorphisms between ordered semigroups $\eta : (S, \leq) \to (T, \preceq)$ have to satisfy $a \leq b \Rightarrow \eta(a) \preceq \eta(b)$ for all $a, b \in S$.

A semigroup T is a *quotient* of a semigroup S if T is the surjective image of S under a homomorphism. In the case of ordered semigroups, for example we have that $(S, \leq)$ is a quotient of $(S, =)$. A semigroup T is a *divisor* of a semigroup S if T is the quotient of a subsemigroup of S.

As usual, the *product* $T = \prod_{i \in I} S_i$ over an index set I of semigroups S_i, $i \in I$, is defined by component-wise multiplication. In the case of ordered semigroups, we take the *product order* as the partial order relation of $(T, \preceq)$, i.e., $(a_i)_{i \in I} \preceq (b_i)_{i \in I}$ if $a_i \leq b_i$ for all $i \in I$. For two homomorphisms $\eta : S \to T$ and $\nu : S \to R$ we define their *product* $\eta \times \nu : S \to T \times R : a \mapsto (\eta(a), \nu(a))$.

The monoid S^1 associated with a semigroup S is S itself if S contains a neutral element; otherwise it is $S \cup \{1\}$ where the operation on S is extended by setting $1 \cdot x = x = x \cdot 1$ for all $x \in S \cup \{1\}$.

Let S be a semigroup. An element $e \in S$ is *idempotent* if $e^2 = e$. An element a of S is *regular* if there exists an element $b \in S$ such that $aba = a$.

If a semigroup element a has an idempotent power this idempotent element is denoted by a^ω. In particular, in finite semigroups every element has an idempotent power, see e.g. [Pin86].

2.2 Green's Relations

Let I be a subset of a semigroup S.

- I is a *right ideal* if $IS^1 \subseteq I$.
- I is a *left ideal* if $S^1I \subseteq I$.
- I is an *ideal* if $S^1IS^1 \subseteq I$.

We define the relations $\mathcal{R}$, $\mathcal{L}$, $\mathcal{J}$ on a semigroup S as:

- $a \,\mathcal{R}\, b \Leftrightarrow aS^1 = bS^1$, i.e., the right ideals generated by a and b are equal.
- $a \,\mathcal{L}\, b \Leftrightarrow S^1a = S^1b$, i.e., the left ideals generated by a and b are equal.
- $a \,\mathcal{J}\, b \Leftrightarrow S^1aS^1 = S^1bS^1$, i.e., the ideals generated by a and b are equal.

From these three relations we derive two more relations on S:

- $a \,\mathcal{H}\, b \Leftrightarrow a \,\mathcal{R}\, b \;\&\; a \,\mathcal{L}\, b$, i.e., $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$.
- $a \,\mathcal{D}\, b \Leftrightarrow \exists c \in S\colon a \,\mathcal{R}\, c \;\&\; c \,\mathcal{L}\, b$, i.e., $\mathcal{D} = \mathcal{R}\mathcal{L}$.

These five relations are called *Green's relations*.

Lemma 2.2 $\mathcal{R}\mathcal{L} = \mathcal{L}\mathcal{R}$.

The proof of the previous lemma is omitted. As all other omitted proofs of this chapter it can be found in [Pin86]. The previous lemma implies that Green's relations are equivalence relations with

$$\mathcal{H} \;=\; \mathcal{L} \cap \mathcal{R} \;\subseteq\; \mathcal{L}, \mathcal{R} \;\subseteq\; \mathcal{L} \vee \mathcal{R} \;=\; \mathcal{D} \;\subseteq\; \mathcal{J}$$

where $\mathcal{L} \vee \mathcal{R}$ is the *join* of the two equivalence relations $\mathcal{L}$ and $\mathcal{R}$, i.e., the least equivalence relation containing both, $\mathcal{L}$ and $\mathcal{R}$.

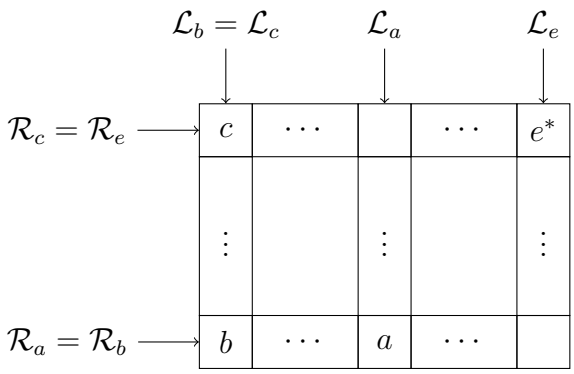


Figure 2.1: Egg-box diagram of $\mathcal{D}_a$ with $a \mathcal{R} b \mathcal{L} c \mathcal{R} e = e^2$

By $\mathcal{R}_a$ (resp. $\mathcal{L}_a$, $\mathcal{J}_a$, $\mathcal{H}_a$, $\mathcal{D}_a$) we denote the $\mathcal{R}$-class of a (resp. $\mathcal{L}$-, $\mathcal{J}$-, $\mathcal{H}$-, $\mathcal{D}$-class of a).

Semigroups can be represented as *egg-box* diagrams. In an egg-box diagram $\mathcal{D}$-classes are represented by rectangular schemes, $\mathcal{R}$-classes correspond to rows, and $\mathcal{L}$-classes correspond to columns in these schemes. Their intersections are $\mathcal{H}$-classes. By drawing the symbol $*$ in a box corresponding to an $\mathcal{H}$-class we indicate that this $\mathcal{H}$-class contains an idempotent element. Figure 2.1 shows a $\mathcal{D}$-class $\mathcal{D}_a$ with $a \mathcal{R} b$ and $b \mathcal{L} c$ and $c \mathcal{R} e = e^2$.

Lemma 2.3 (Green's lemma) *Let S be a semigroup and $a, b, c \in S$.*

(a) *If $a \mathcal{R} b$, then there exist $p, q \in S$ such that $ap = b$ and $bq = a$ and such that the mappings $\cdot p : \mathcal{L}_a \to \mathcal{L}_b : x \mapsto xp$ and $\cdot q : \mathcal{L}_b \to \mathcal{L}_a : x \mapsto xq$ are inverse bijections which preserve $\mathcal{H}$-classes.*

(b) *If $b \mathcal{L} c$, then there exist $r, s \in S$ such that $rb = c$ and $sc = b$ and such that the mappings $r \cdot : \mathcal{R}_b \to \mathcal{R}_c : x \mapsto rx$ and $s \cdot : \mathcal{R}_c \to \mathcal{R}_b : x \mapsto sx$ are inverse bijections which preserve $\mathcal{H}$-classes.*

The situation with $a \mathcal{R} b$ and $b \mathcal{L} c$ is illustrated in Figure 2.2.

Lemma 2.4 *For finite semigroups, we have $\mathcal{D} = \mathcal{J}$.*

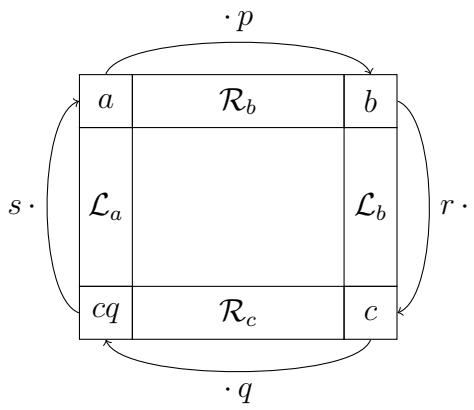


Figure 2.2: Green's lemma with $a \,\mathcal{R}\, b \,\mathcal{L}\, c$

In fact, the previous lemma holds for all semigroups in which every element has an idempotent power.

In analogy to Green's relations we define

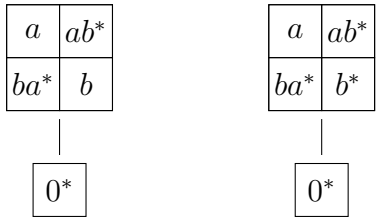
- $a \leq_{\mathcal{R}} b \Leftrightarrow aS^1 \subseteq bS^1$,
- $a \leq_{\mathcal{L}} b \Leftrightarrow S^1a \subseteq S^1b$,
- $a \leq_{\mathcal{J}} b \Leftrightarrow S^1aS^1 \subseteq S^1bS^1$,
- $a \leq_{\mathcal{H}} b \Leftrightarrow a \leq_{\mathcal{R}} b \;\&\; a \leq_{\mathcal{L}} b$.

The relation $\leq_{\mathcal{R}}$ (resp. $\leq_{\mathcal{L}}$, $\leq_{\mathcal{J}}$, $\leq_{\mathcal{H}}$) induces a partial order on the $\mathcal{R}$-classes (resp. $\mathcal{L}$-, $\mathcal{J}$-, $\mathcal{H}$-classes) of a semigroup.

Figure 2.3(a) shows the egg-box representation of the semigroup B_2. The semigroup U is represented in Figure 2.3(b). In both figures the line between the two $\mathcal{D}$-classes $\mathcal{D}_a$ and $\mathcal{D}_0$ indicates that the upper one lies $\leq_{\mathcal{J}}$-above the lower one, i.e., $\mathcal{D}_0 = \mathcal{J}_0 \leq_{\mathcal{J}} \mathcal{J}_a = \mathcal{D}_a$.

In the following, we recapitulate some basic facts for finite semigroups.

Lemma 2.5 *Let S be a finite semigroup. Then for all elements $a, b, c \in S$ and for all idempotents $e^2 = e \in S$ we have:*



(a) B_2 (b) U

Figure 2.3: Egg-box representation of B_2 and U, see Example 2.1

(a) $a \leq_{\mathcal{R}} e \Leftrightarrow ea = a$.

(b) $a \leq_{\mathcal{L}} e \Leftrightarrow ae = a$.

(c) $a \leq_{\mathcal{J}} ab \Rightarrow a \,\mathcal{R}\, ab$.

(d) $a \leq_{\mathcal{J}} ba \Rightarrow a \,\mathcal{L}\, ba$.

(e) $a \leq_{\mathcal{R}} abc \Rightarrow a \,\mathcal{R}\, ab \,\mathcal{R}\, abc$.

(f) $a \leq_{\mathcal{L}} cba \Rightarrow a \,\mathcal{L}\, ba \,\mathcal{L}\, cba$.

(g) $a \leq_{\mathcal{R}} b \;\&\; a \,\mathcal{J}\, b \Rightarrow a \,\mathcal{R}\, b$.

(h) $a \leq_{\mathcal{L}} b \;\&\; a \,\mathcal{J}\, b \Rightarrow a \,\mathcal{L}\, b$.

Lemma 2.6 *Let S be a finite semigroup and let $a, b \in S$. Then*

$$ab \in \mathcal{R}_a \cap \mathcal{L}_b \Leftrightarrow \exists e^2 = e \in \mathcal{L}_a \cap \mathcal{R}_b.$$

Lemma 2.7 *Let e and f be two idempotent elements of a finite semigroup S. Then $e \mathcal{D} f$ if and only if there exists $a, b \in S$ such that $e = ab$ and $f = ba$, i.e., e and f are conjugate.*

Lemma 2.8 *Let S be a finite semigroup and H be an $\mathcal{H}$-class of S. Then the following are equivalent:*

(a) $\exists e \in H : e = e^2$.

(b) $\exists a, b \in H : ab \in H$.

(c) *$H \leq S$ is a maximal group.*

It follows, that there is at most one idempotent element in any $\mathcal{H}$-class. Also note that, if an $\mathcal{H}$-class H is a subgroup of a semigroup, then the idempotent element in H is the neutral element. We say that a $\mathcal{D}$-class is *regular* if all its elements are regular.

Lemma 2.9 *Let D be $\mathcal{D}$-class of a finite semigroup. Then the following are equivalent:*

(a) *D is regular.*

(b) *D contains a regular element.*

(c) *Each $\mathcal{R}$-class of D contains an idempotent.*

(d) *Each $\mathcal{L}$-class of D contains an idempotent.*

(e) *D contains an idempotent.*

2.3 Varieties

A *variety* $\mathbf{V}$ of semigroups (resp. monoids) is a class of finite semigroups (resp. finite monoids) closed under taking subsemigroups (resp. submonoids), quotients, and finite products. A *positive variety* $\mathbf{V}$ is a class of finite semigroups (resp. finite monoids) closed under taking ordered subsemigroups, ordered quotients, and finite products. The attribute *positive* arises from the fact that $(S, \leq) \in \mathbf{V}$ does not imply $(S, \geq) \in \mathbf{V}$. We will see later, that in some sense $(S, \geq)$ is the negation of $(S, \leq)$. We call $(S, \geq)$ the *dual* of $(S, \leq)$.

Varieties are sometimes called *pseudo-varieties* to indicate that only finite products are allowed.

Let Ω be a countable alphabet and let $u, v \in \Omega^+$ (resp. Ω^* for monoids; in this case we denote the empty word by 1). We say that a semigroup S satisfies the *equation* $u = v$ if for all homomorphisms $\eta : \Omega^+ \to S$ we have $\eta(u) = \eta(v)$ in S. Let $u_1, u_2, \ldots, v_1, v_2, \ldots \in \Omega^+$. We say that a semigroup *ultimately satisfies* the equations $(u_i = v_i)_{i \geq 1}$ if there exists an index j such that S satisfies the equations $(u_i = v_i)_{i \geq j}$. A variety $\mathbf{V}$ is *ultimately defined*

by the equations $(u_i = v_i)_{i\geq 1}$ if $\mathbf{V}$ consists of all finite semigroups ultimately satisfying the equations $(u_i = v_i)_{i\geq 1}$. It is easy to see that every class of semigroups ultimately defined by a sequence of equations forms a variety. The converse is also true:

Theorem 2.10 (Reiterman [Rei82]) *Every variety is ultimately defined by a sequence of equations.*

By replacing all equality symbols $=$ by $\leq$ we get a sequence of inequalities. Every sequence of inequalities ultimately defines a positive variety. The converse is shown in [PW96b]:

Theorem 2.11 (Pin/Weil, 1996) *Every positive variety is ultimately defined by a sequence of inequalities.*

It is also possible to use the ω-notation in equations to denote the idempotent generated by an element. This can be realized using the factorial, for example the equation $x^\omega y x^\omega = x^\omega$ can be represented by the sequence of equations $\left(x^{i!}yx^{i!} = x^{i!}\right)_{i\geq 1}$. Let $u_1 = v_1, \ldots, u_n = v_n$ be a finite set of equations over Ω (possibly with ω-terms). By $[\![\, u_1 = v_1, \ldots, u_n = v_n \,]\!]$ we denote the variety of semigroups (resp. monoids) satisfying those equations. These notions are extended to positive varieties, e.g. $[\![\, x^\omega y x^\omega \leq x^\omega \,]\!]$ is the class of all finite ordered semigroups $(S, \leq)$ such that for all $a, b \in S$ we have $a^\omega b a^\omega \leq a^\omega$. A more recent approach to equations is by using topological notions or universal algebra, see e.g. [Alm94, PP04, Wei02]. For our purpose, finite sets of equations or inequalities with ω-terms will be sufficient.

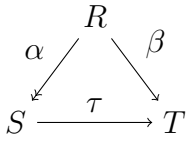
2.4 Operations on Varieties

Let S and T be two semigroups. A relation $\tau \subseteq S \times T$ is a *(relational) morphism* if it satisfies

- $\forall a \in S\colon \tau(a) \neq \emptyset$, where $\tau(a) = \{c \in T \mid (a, c) \in \tau\}$.
- $\forall a, b \in S\colon \tau(a)\tau(b) \subseteq \tau(ab)$.

For monoids we additionally require that $1 \in \tau(1)$. We also use the notation $\tau : S \to T$ for relational morphisms. As for homomorphisms, we define the *product* of two relational morphisms $\tau_1 : S \to T_1$ and $\tau_2 : S \to T_2$ via $\tau_1 \times \tau_2 : S \to T_1 \times T_2 : a \mapsto (\tau_1(a), \tau_2(a))$.

Every relational morphism $\tau : S \to T$ has a *canonical factorization* into two homomorphisms $\alpha : R \to S$ and $\beta : R \to T$, where $R \subseteq S \times T$ is the graph of τ and α, β are projections. The first condition in the definition of relational morphisms implies that α is surjective. We then have $\tau = \alpha^{-1}\beta$.



Lemma 2.12 *Let S and T be semigroups and let $\tau : S \to T$ be a relational morphism. For each idempotent $e \in T$ we have $\tau^{-1}(e) \leq S$.*

Definition 2.13 *The* Mal'cev product $\mathbf{W} \textcircled{M} \mathbf{V}$ *of two varieties* $\mathbf{V}$ *and* $\mathbf{W}$ *consists of all finite semigroups (resp. monoids) S for which there exists a semigroup $T \in \mathbf{V}$ and a relational morphism $\tau : S \to T$ such that for each idempotent $e \in T$ we have $\tau^{-1}(e) \in \mathbf{W}$. We also allow* $\mathbf{W}$ *to be a positive variety.*

For varieties $\mathbf{V}$ and $\mathbf{W}$, their Mal'cev product $\mathbf{W} \textcircled{M} \mathbf{V}$ is again a variety [Pin86, PW96a]. Note that $\mathbf{W}$ might be a positive variety, but it is always a class of semigroups. For $\mathbf{V}$ we will use full varieties of either semigroups or monoids.

Lemma 2.14 *Let $\mathbf{V}$ be a variety and let $\mathbf{W}_1$ and $\mathbf{W}_2$ be (positive) varieties. Then*

$$(\mathbf{W}_1 \textcircled{M} \mathbf{V}) \cap (\mathbf{W}_2 \textcircled{M} \mathbf{V}) \;=\; (\mathbf{W}_1 \cap \mathbf{W}_2) \textcircled{M} \mathbf{V}.$$

Proof: Since $(\mathbf{W}_1 \cap \mathbf{W}_2) \textcircled{M} \mathbf{V}$ is a subset of both, $\mathbf{W}_1 \textcircled{M} \mathbf{V}$ and $\mathbf{W}_2 \textcircled{M} \mathbf{V}$, we have $(\mathbf{W}_1 \cap \mathbf{W}_2) \textcircled{M} \mathbf{V} \subseteq (\mathbf{W}_1 \textcircled{M} \mathbf{V}) \cap (\mathbf{W}_2 \textcircled{M} \mathbf{V})$. Let $S \in (\mathbf{W}_1 \textcircled{M} \mathbf{V}) \cap (\mathbf{W}_2 \textcircled{M} \mathbf{V})$. Then there exist relational morphisms $\tau_1 : S \to T_1$ and $\tau_2 : S \to T_2$ with $T_1, T_2 \in \mathbf{V}$ such that for all idempotent elements $e_i \in T_i$, $i = 1, 2$, we have $\tau_i^{-1}(e_i) \in \mathbf{W}_i$. Consider the relational morphism $\tau = \tau_1 \times \tau_2 : S \to T_1 \times T_2$. Since $\mathbf{V}$ is a variety we have $T_1 \times T_2 \in \mathbf{V}$. Let $(e_1, e_2) \in T_1 \times T_2$ be idempotent. Then we have that the components $e_1 \in T_1$ and $e_2 \in T_2$ are idempotent. Since $\tau^{-1}(e_1, e_2) = \tau_1^{-1}(e_1) \cap \tau_2^{-1}(e_2)$, we have $\tau^{-1}(e_1, e_2) \in \mathbf{W}_1 \cap \mathbf{W}_2$ and therefore $S \in (\mathbf{W}_1 \cap \mathbf{W}_2) \textcircled{M} \mathbf{V}$. □

This lemma has been proven for arbitrary intersections (possibly infinite) in [PW96a].

2.5 Some Selected Varieties

In this section we present selected varieties and some of their characterizations.

By **Com** we denote the class of finite *commutative* monoids $[\![\, xy = yx \,]\!]$. We define the variety $\mathbf{J_1}$ of finite *semilattices* as $\mathbf{J_1} = [\![\, x^2 = x, xy = yx \,]\!]$. Let Σ be a finite set. Then $(2^\Sigma, \cup)$ is a semigroup in $\mathbf{J_1} = [\![\, x^2 = x \,]\!] \cap \mathbf{Com}$.

We define the variety **LI** of *locally trivial* semigroups as $\mathbf{LI} = [\![\, x^\omega y x^\omega = x^\omega \,]\!]$.

The variety **A** of *aperiodic semigroups* is the class of finite semigroups S satisfying $a^{\omega+1} = a^\omega$ for all $a \in S$, i.e., $\mathbf{A} = [\![\, x^{\omega+1} = x^\omega \,]\!]$.

Lemma 2.15 *For every finite semigroup S the following are equivalent:*

(a) $S \in \mathbf{A}$.

(b) $\exists n \in \mathbb{N}\ \forall a \in S\colon a^{n+1} = a^n$.

(c) *S is $\mathcal{H}$-trivial, i.e., every $\mathcal{H}$-class contains exactly one element.*

(d) *All groups in S are trivial, i.e., if a subsemigroup of S is a group then it contains only the neutral element.*

Let **V** be a semigroup-variety. By **DV** we denote the class of finite semigroups such that every regular $\mathcal{D}$-class is a semigroup in **V**. In particular, **DS** is the class of finite semigroups, whose regular $\mathcal{D}$-classes are closed under multiplication.

Lemma 2.16 *For every finite semigroup S the following are equivalent:*

(a) $S \in \mathbf{DS}$.

(b) $S \in [\![\, ((xy)^\omega (yx)^\omega (xy)^\omega)^\omega = (xy)^\omega \,]\!]$.

(c) $\forall e, f \in S\colon e^2 = e \;\&\; f^2 = f \;\&\; e \,\mathcal{J}\, f \;\Rightarrow\; (ef)^\omega e = e.$

(d) $\forall e \in S\colon e^2 = e \;\Rightarrow\; \{a \in S \mid e \leq_{\mathcal{J}} a\} \leq S.$

(e) $\forall a, b \in S\colon a \leq_{\mathcal{J}} b \;\&\; a \text{ is regular} \;\Rightarrow\; a \,\mathcal{J}\, ab \,\mathcal{J}\, ba.$

(f) $\forall a, b \in S\colon ab \,\mathcal{R}\, a \;\Rightarrow\; \mathcal{R}_a b \subseteq \mathcal{R}_a.$

(g) $\forall a, b \in S\colon ba \,\mathcal{L}\, a \;\Rightarrow\; b\mathcal{L}_a \subseteq \mathcal{L}_a.$

(h) $\forall a \in S\colon a \text{ is regular} \;\Rightarrow\; a \,\mathcal{H}\, a^\omega.$

(i) *Neither B_2 nor U is a divisor of S.*

Proof: (b) $\Rightarrow$ (c): Let e and f be idempotent elements such that $e \,\mathcal{J}\, f$. Then by Lemma 2.7 there exist $a, b \in S$ such that $e = ab$ and $f = ba$. From (b) it follows that

$$\begin{aligned}(ef)^\omega e &= (ab \cdot ba)^\omega ab \\ &= ((ab)^\omega (ba)^\omega)^\omega (ab)^\omega \\ &= (ab)^\omega = ab = e.\end{aligned}$$

(c) $\Rightarrow$ (b): Since

$$(xy)^\omega = (xy)^{\omega-1}x \cdot (yx)^\omega \cdot y$$

and

$$(yx)^\omega = (yx)^{\omega-1}y \cdot (xy)^\omega \cdot y,$$

we have $(xy)^\omega \,\mathcal{J}\, (yx)^\omega$ and therefore by (c) we conclude

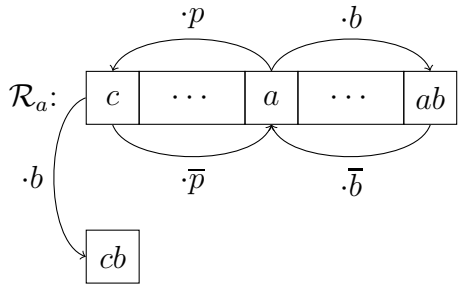
$$((xy)^\omega (yx)^\omega (xy)^\omega)^\omega = ((xy)^\omega (yx)^\omega)^\omega (xy)^\omega = (xy)^\omega.$$

(a) $\Rightarrow$ (c): Let $e^2 = e$, $f^2 = f$ and $e \,\mathcal{J}\, f$. Since $S \in \mathbf{DS}$, arbitrary products of e and f are in the $\mathcal{J}$-class of e and f, in particular $(ef)^\omega e \,\mathcal{J}\, e$. Since $(ef)^\omega e \leq_{\mathcal{R}} e$ and $(ef)^\omega e \leq_{\mathcal{L}} e$ we can conclude by Lemma 2.5 that $(ef)^\omega e \mathcal{H} e$ and since $(ef)^\omega e (ef)^\omega e = (ef)^\omega e$ is idempotent and since by Lemma 2.8 there is a most one idempotent in every $\mathcal{H}$-class we conclude $(ef)^\omega e = e$.

(c) $\Rightarrow$ (d): Let $e^2 = e$ and $a, b \in S$ such that $e \leq_{\mathcal{J}} a$ and $e \leq_{\mathcal{J}} b$. Let $p, q, r, s \in S$ such that $paq = e = rbs$. We then have $(qpa)^\omega \,\mathcal{J}\, e \,\mathcal{J}\, (bsr)^\omega$ and by (c) we conclude $e \,\mathcal{J}\, (qpa)^\omega = (qpa)^\omega (bsr)^\omega)^\omega (qpa)^\omega = c \cdot ab \cdot d$ for $c, d \in S$ and therefore $e \leq_{\mathcal{J}} ab$.

(d) $\Rightarrow$ (e): Let $e^2 = e \,\mathcal{J}\, a \leq_{\mathcal{J}} b$. In particular we have $e \leq_{\mathcal{J}} a$ and by (d) we conclude $e \leq_{\mathcal{J}} ab$. Since $a \leq_{\mathcal{J}} e$ and $ab \leq_{\mathcal{J}} a$ we have $a \,\mathcal{J}\, ab$ and similarly $a \,\mathcal{J}\, ba$.

(e) $\Rightarrow$ (f): Let $ab \mathcal{R} a \mathcal{R} c$. Let $\overline{b}, p, \overline{p} \in S$ such that $ab\overline{b} = a$, $ap = c$ and $c\overline{p} = a$.



By (e) we conclude $((b\overline{b}p\overline{p})^\omega p(b\overline{b}p\overline{p})^\omega)^\omega \mathcal{J} (b\overline{b}p\overline{p})^\omega$. Hence $((b\overline{b}p\overline{p})^\omega p(b\overline{b}p\overline{p})^\omega)^\omega \mathcal{H} (b\overline{b}p\overline{p})^\omega$ and $((b\overline{b}p\overline{p})^\omega p(b\overline{b}p\overline{p})^\omega)^\omega = (b\overline{b}p\overline{p})^\omega$. Now we have

$$\begin{aligned} a &= a(b\overline{b}p\overline{p})^\omega \\ &= a((b\overline{b}p\overline{p})^\omega p(b\overline{b}p\overline{p})^\omega)^\omega \\ &= c(b\overline{b}p\overline{p})^\omega((b\overline{b}p\overline{p})^\omega p(b\overline{b}p\overline{p})^\omega)^{\omega-1} \\ &= cb \cdot q \end{aligned}$$

for some $q \in S$. This shows $a \leq_{\mathcal{R}} cb$ and together with $a \,\mathcal{R}\, c$ is follows $a \,\mathcal{R}\, cb$ and hence $\mathcal{R}_a b \subseteq \mathcal{R}_a$.

(e) $\Rightarrow$ (g): is symmetric to (e) $\Rightarrow$ (f).

(f) $\Rightarrow$ (h): Let $a \in S$ be regular. Then by Lemma 2.9 there exists $e^2 = e \in S$ such that $e \,\mathcal{R}\, a$. It holds that $e = e^2 \leq_{\mathcal{R}} ea \leq_{\mathcal{R}} e$. Hence $ea \,\mathcal{R}\, e$ and by (f) we conclude $a^2 \,\mathcal{R}\, a$ and therefore $a^\omega \,\mathcal{R}\, a$. From $a^\omega \leq_{\mathcal{L}} a$ together with Lemma 2.5 it follows that $a^\omega \,\mathcal{H}\, a$.

(g) $\Rightarrow$ (h): is symmetric to (f) $\Rightarrow$ (h).

(h) $\Rightarrow$ (a): Let a and b be elements of the same regular $\mathcal{D}$-class D. It follows from (h) that $\mathcal{L}_a \cap \mathcal{R}_b$ contains an idempotent and with Lemma 2.6 we conclude that $ab \in D$.

(a) $\Rightarrow$ (i): Suppose that B_2 or U is a divisor of S. Since **DS** is a variety. $B_2 \in$ **DS** or $U \in$ **DS**. But both semigroups contain a regular element that is not $\mathcal{H}$-equivalent to its idempotent power, contradicting (h). Hence, neither B_2 nor U is a divisor of S.

(i) $\Rightarrow$ (h): By contraposition, suppose a is regular and $a^\omega \,\mathcal{H}\, a$ does not hold. Then by Lemma 2.9 there exist idempotent elements e and f such that $e \,\mathcal{R}\, a \,\mathcal{L}\, f$. Depending on whether $\mathcal{L}_e \cap \mathcal{R}_f$ contains an idempotent or not, we find U or B_2 as a divisor. □

The previous lemma also holds, if **DS** is interpreted as a variety of monoids. In this case, we have to substitute B_2 and U by their monoid counterparts B_2^1 and U^1, respectively.

By **DA** we denote the class of finite monoids, whose regular $\mathcal{D}$-classes are semigroups in **A**, i.e., monoids whose regular $\mathcal{D}$-classes are aperiodic semigroups.

Theorem 2.17 *For every finite monoid M the following are equivalent:*

(a) $M \in \mathbf{DA}$.

(b) $M \in \mathbf{DS} \cap \mathbf{A}$.

(c) $M \in [\![(xyz)^\omega y (xyz)^\omega = (xyz)^\omega]\!]$.

(d) $M \in [\![(xy)^\omega x (xy)^\omega = (xy)^\omega]\!]$.

(e) $M \in [\![(xy)^\omega y (xy)^\omega = (xy)^\omega]\!]$.

(f) $M \in \mathbf{LI} \circledcirc{M} \mathbf{J_1}$.

(g) $\forall e, f \in M : e = e^2 \;\&\; e \,\mathcal{J}\, f \;\Rightarrow\; f = f^2$.

(h) $\forall e, f \in M : e = e^2 \;\&\; e \,\mathcal{R}\, f \;\Rightarrow\; f = f^2$.

(i) $\forall e, f \in M : e = e^2 \;\&\; e \,\mathcal{L}\, f \;\Rightarrow\; f = f^2$.

(j) $\forall e, a \in M : e = e^2 \;\&\; e \leq_{\mathcal{J}} a \;\Rightarrow\; eae = e$.

(k) $\forall e, a \in M : e = e^2 \;\&\; e \leq_{\mathcal{R}} a \;\Rightarrow\; eae = e$.

(l) $\forall e, a \in M : e = e^2 \;\&\; e \leq_{\mathcal{L}} a \;\Rightarrow\; eae = e$.

(m) $\forall a \in M :$ *a is regular* $\Rightarrow$ *a is idempotent.*

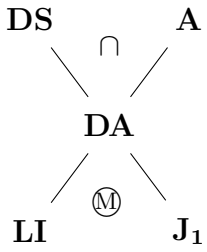


Figure 2.4: The varieties **LI**, $\mathbf{J_1}$, **DA**, **DS** and **A**

Proof: (a) $\Leftrightarrow$ (b): We have $\mathbf{DA} \subseteq \mathbf{DS}$ by definition and $\mathbf{DA} \subseteq \mathbf{A}$ holds due to Lemma 2.15. Hence $\mathbf{DA} \subseteq \mathbf{DS} \cap \mathbf{A}$. The converse also follows by definition of **DS** and Lemma 2.15. The reason for this is that every subgroup is a subset of a regular $\mathcal{D}$-class.

The proof for the equivalence of (a), (c), (f), (g) and (j) can be found in [TT02]. Conditions (d) and (e) are immediate conclusions of (c); conditions (h) and (i) are immediate conclusions of (g) and the conditions (k) and (l) follow from (j). The conditions (j), (k) and (l) are only re-formulations of (c), (d) and (e), respectively. It remains to show that (h), (i), (k) and (l) are sufficient characterizations of **DA**. By symmetry, it suffices to show that (h) and (k) are sufficient and that (m) characterizes **DA**.

(h) $\Rightarrow$ (k): Let $e^2 = e \,\mathcal{R}\, a$. Then there exists $p \in M$ such that $e = ap$. Hence $e = e^2 = eap$ and therefore $e \,\mathcal{R}\, ea$. By (h) we conclude $eaea = ea$. Now, multiplication with p from right yields $eae = e$.

(k) $\Rightarrow$ (m): Let a be regular. Then by Lemma 2.9 there exists $e^2 = e \in M$ such that $e \mathcal{R} a$. From (k) we conclude that ea is idempotent. Finally, Lemma 2.5 implies $ea = a$.

(m) $\Rightarrow$ (b): We have $M \in \mathbf{DS}$ by Lemma 2.16 and $M \in \mathbf{A}$ follows by Lemma 2.15, since all subgroups are trivial. $\square$

Chapter 3

Mazurkiewicz Traces

3.1 Trace Monoids

Let Σ be a finite alphabet and $I \subseteq \Sigma \times \Sigma$ be a symmetric and irreflexive relation. The congruence $\sim_I$ on Σ^* is the reflexive and transitive closure of the relation $\approx_I$ defined by

$$\forall w, v \in \Sigma^* \, \forall a, b \in \Sigma \colon wabv \approx_I wbav \;\Leftrightarrow\; a \, I \, b.$$

The set of equivalence classes $[w]_{\sim_I} = \{ v \in \Sigma^* \mid v \sim_I w \}$ with respect to that congruence forms the free partially commutative monoid generated by (Σ, I). It is denoted by $\mathbb{M}(\Sigma, I)$. By definition, for $v, w \in \Sigma^*$ we have $v \sim_I w$ if and only if v can be transformed into w using only commutations of contiguous letters a and b with $(a, b) \in I$.

The elements of $\mathbb{M}(\Sigma, I)$ are called *(Mazurkiewicz) traces* and $\mathbb{M}(\Sigma, I)$ itself a *trace monoid* following a convention from [Maz77]. The relation I is called the *independence relation* and $D = \Sigma^2 \setminus I$ is the *dependence relation*. By $\pi_I : \Sigma^* \to \mathbb{M}(\Sigma, I) : w \mapsto [w]_I$ we denote the canonical projection from Σ^* to $\mathbb{M}(\Sigma, I)$.

It is common to depict a trace monoid by the graph of its dependence relation, in which the loops resulting from the reflexivity are omitted.

Since $w \sim_I v$ implies that the words w and v contain the same letters, possibly in a different order, the *length* and the *alphabet* of w are invariant within an equivalence class. Therefore, these notions can also be applied to

traces, and for $[w]_{\sim_I} = t \in \mathbb{M}(\Sigma, I)$ we denote the length of the trace t by $|t| = |w|$ and its alphabet is $\mathrm{alph}(t) = \{\, a \in \Sigma \mid a \text{ occurs in } w \,\}$. The word w is called a *word representative* of the trace t.

Another point of view is the following: A *trace* over the alphabet Σ and the dependence relation D is a relational structure $t = (X, \mathrm{label}, \rightarrowtriangle, <)$ with a finite set of positions X, a mapping $\mathrm{label} : X \to \Sigma$, and binary relations $\rightarrowtriangle$ and $<$ over the set of positions X such that:

- $\rightarrowtriangle$ is acyclic and irreflexive,
- for all $\nu, \chi \in X$ with $\nu \neq \chi$, we have that $\mathrm{label}(\nu)\, D\, \mathrm{label}(\chi)$ holds if and only if either $\nu \rightarrowtriangle \chi$ or $\chi \rightarrowtriangle \nu$ holds,
- the relation $<$ is the transitive closure of $\rightarrowtriangle$.

The relation $<$ is a (strict) partial order on X. The word $w = \mathrm{label}(\nu_1) \cdot \mathrm{label}(\nu_2) \cdots \mathrm{label}(\nu_n) \in \Sigma^*$ with $\{\, \nu_1, \ldots, \nu_n \,\} = X_t$ is a word representative of t if and only if $\nu_m < \nu_\ell$ implies $m < \ell$ for all $1 \leq m, \ell \leq n$, i.e., if and only if w is a linearization of $(X, <)$. The relations $\rightarrowtriangle$ and $<$ contain the same information: $<$ is the transitive closure of $\rightarrowtriangle$, and for any two elements $\nu, \chi \in X$ we have

$$\nu \rightarrowtriangle \chi \;\Leftrightarrow\; \nu < \chi \wedge \mathrm{label}(\nu)\, D\, \mathrm{label}(\chi).$$

Therefore $\rightarrowtriangle$ can be reconstructed if only $<$ is given. For this reason, it is equally valid to describe a trace using only one of these relations and to write $t = (X, \mathrm{label}, \rightarrowtriangle)$ or $t = (X, \mathrm{label}, <)$. We refer to $(X, \mathrm{label}, \rightarrowtriangle)$ as the *dependence graph* of t and to $(X, \mathrm{label}, <)$ as partial order.

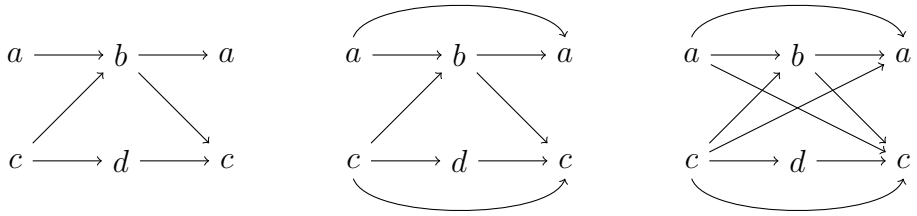
From [DG05, Lemma 5] we can conclude the following lemma.

Lemma 3.1 *The partial order relation $<$ equals* $\bigcup_{1 \leq i < |\Sigma|} (\rightarrowtriangle)^i$.

The parallelism relation $\parallel$ is the complement of the symmetric and reflexive closure of $<$, i.e., two positions are *parallel* (or *concurrent*) if they are incomparable with respect to $<$. By abuse of notation we will identify the word representative w, the trace t and its graph representations $(X, \mathrm{label}, \rightarrowtriangle)$ and $(X, \mathrm{label}, <)$. Let $\prec \subseteq X \times X$ be minimal such that $<$ is the transitive closure of $\prec$. The directed labeled graph $(X, \mathrm{label}, \prec)$ is called the *Hasse diagram* of t.

Example 3.2 Let $D = a-b-c-d$. Consider the trace $t = acdbca$. We have $acdbca \sim_I cabadc$. The trace t has the following graphical presentations:

Hasse diagram: Dependence graph $\rightarrowtriangle$: Partial order $<$:



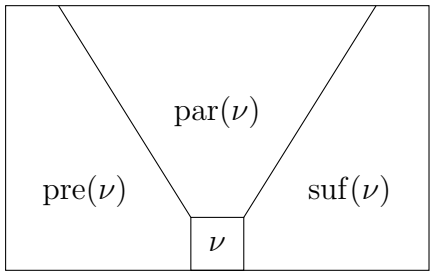
In t, the node labeled with d is parallel to all nodes labeled with either a or b. ◇

We say that a trace t_1 is a *factor* of a trace t_2 if there exist traces s_1 and s_2 such that $t_2 = s_1 t_1 s_2$.

For a position ν of $t \in \mathbb{M}(\Sigma, I)$ we define the following factors:

$$\begin{aligned} \mathrm{pre}(\nu) &= \{\, \chi \in t \mid \chi < \nu \,\} && \text{is the past of } \nu, \\ \mathrm{par}(\nu) &= \{\, \chi \in t \mid \nu \parallel \chi \,\} && \text{is the parallel part of } \nu, \\ \mathrm{suf}(\nu) &= \{\, \chi \in t \mid \nu < \chi \,\} && \text{is the future of } \nu. \end{aligned}$$

By abuse of notation, we now have the following two factorizations $t = \mathrm{pre}(\nu)\,\mathrm{label}(\nu)\,\mathrm{par}(\nu)\,\mathrm{suf}(\nu)$ and $t = \mathrm{pre}(\nu)\,\mathrm{par}(\nu)\,\mathrm{label}(\nu)\,\mathrm{suf}(\nu)$.



3.2 Trace Languages

We are interested in the interplay between classes of finite monoids and classes of recognizable subsets of infinite monoids. In particular, these infinite

monoids will be trace monoids. The connection between them is the syntactic congruence. Let L be a subset of a monoid $\mathbb{M}$. Then the *syntactic congruence* $\sim_L \subseteq \mathbb{M} \times \mathbb{M}$ of L is defined by

$$p \sim_L q \;\Leftrightarrow\; (\forall u, v \in \mathbb{M}\colon upv \in L \;\Leftrightarrow\; uqv \in L)\,.$$

The natural homomorphism $\mu_L : \mathbb{M} \to \mathbb{M}/\!\sim_L : p \mapsto [p]_{\sim_L}$ is called the *syntactic homomorphism* of L. The monoid $M(L) = \mathbb{M}/\!\sim_L$ is called the *syntactic monoid* of L. The *syntactic quasiordering* $\leq_L$ of L is defined by $p \leq_L q \;\Leftrightarrow\; (\forall u, v \in \mathbb{M}\colon uqv \in L \Rightarrow upv \in L)$. The relation $\leq_L$ induces a partial order on $M(L)$ such that $(M(L), \leq_L)$ forms an ordered monoid. It is called the *syntactic ordered monoid* of L. See [Pin95] for more details.

Let $\mathbb{M}(\Sigma, I)$ be a trace monoid. We say that a finite monoid M *recognizes* a trace language $L \subseteq \mathbb{M}(\Sigma, I)$ if there exists a homomorphism $\eta : \mathbb{M}(\Sigma, I) \to M$ such that $\eta^{-1}\eta(L) = L$. We say that a trace language L is *recognizable* if it is recognized by some finite monoid. Every recognizable trace language is recognized by its syntactic monoid via the syntactic homomorphism. If a finite monoid M recognizes a trace language L then the syntactic monoid of L is a divisor of M. In contrast to words, the converse is not true. The reason for this is the lack of the universal property of trace monoids.

We say that a set $\mathcal{V}$ of trace languages over $\mathbb{M}(\Sigma, I)$ *corresponds* to a class of monoids $\mathbf{V}$, if $\mathcal{V} = \{\, L \subseteq \mathbb{M}(\Sigma, I) \mid M(L) \in \mathbf{V} \,\}$, or $\mathcal{V} = \{\, L \subseteq \mathbb{M}(\Sigma, I) \mid (M(L), \leq_L) \in \mathbf{V} \,\}$ for ordered monoids $\mathbf{V}$, respectively.

A *language variety* $\mathcal{V}$ is a class of trace languages closed under boolean operations, under inverse homomorphism and under quotients. A *left quotient* of L is $a^{-1}L = \{\, t \mid at \in L \,\}$ for $a \in \Sigma$. Right quotients are symmetric: A *right quotient* of L is $La^{-1} = \{\, t \mid ta \in L \,\}$ for $a \in \Sigma$. A language variety over words is called $*$-*variety*.

Eilenberg showed in [Eil76, page 186f] that if $\mathcal{V}$ corresponds to a variety of monoids, then $\mathcal{V}$ is a language variety. Note that Eilenberg did not only prove this fact for $*$-varieties, but for recognizable subsets of arbitrary monoids.

For a set $\mathcal{V}$ of trace languages over a trace monoid $\mathbb{M}(\Sigma, I)$ we define the *polynomials* Pol$\mathcal{V}$ over $\mathcal{V}$ as the set of trace languages that are finite unions of languages of the form

$$L_0 a_1 L_1 \cdots a_n L_n,$$

where $n \in \mathbb{N}$, $a_i \in \Sigma$ and $L_i \in \mathcal{V}$ for all $0 \leq i \leq n$. By co-Pol$\mathcal{V}$ we denote the set of trace languages L whose complement $\overline{L} = \mathbb{M}(\Sigma, I) \setminus L$ is in Pol$\mathcal{V}$.

We say that a product $L = L_1 a L_2$ of trace languages $L_1, L_2 \subseteq \mathbb{M}(\Sigma, I)$, $a \in \Sigma$ is *left unambiguous* if for all $t \in L$ there exists a unique position ν in t such that

- $\mathrm{label}(\nu) = a$ and
- $\mathrm{pre}(\nu) \in L_1$ and $\mathrm{par}(\nu)\,\mathrm{suf}(\nu) \in L_2$.

Right unambiguous products are defined symmetrically, i.e., the parallel part $\mathrm{par}(\nu)$ is related to L_1. A product $L_1 a L_2$ is *unambiguous* if it is left unambiguous *or* right unambiguous. Let $\mathcal{V}$ be a set of trace languages. Then we define $\mathrm{UPol}\mathcal{V}$ as the closure of $\mathcal{V}$ under boolean operations and unambiguous products. Note that the unambiguous product for traces is not associative.

3.2.1 Polynomial Closure

In the following, we fix the trace monoid $\mathbb{M}(\Sigma, I)$ over a non-empty finite alphabet Σ.

Lemma 3.3 *Let* $\mathbf{V}$ *be a variety of monoids with* $\mathbf{J_1} \subseteq \mathbf{V}$ *and let* $M_0, \ldots, M_n$ *be monoids in* $\mathbf{V}$. *For all* $i \in \{0, \ldots, n\}$ *let* $\mu_i : \mathbb{M}(\Sigma, I) \to M_i$ *be homomorphisms. Then there exists a monoid* $N \in \mathbf{V}$ *and a homomorphism* $\eta : \mathbb{M}(\Sigma, I) \to N$ *such that for all* $x, y \in \mathbb{M}(\Sigma, I)$ *satisfying* $\eta(x) = \eta(y)$ *the following conditions hold:*

(a) *For all homomorphisms* μ_i, $0 \leq i \leq n$, *we have* $\mu_i(x) = \mu_i(y)$.

(b) $\mathrm{alph}(x) = \mathrm{alph}(y)$.

(c) *Let* x' *and* y' *be connected components of* x *and of* y *respectively such that* $\mathrm{alph}(x') = \mathrm{alph}(y')$. *Then we have* $\eta(x') = \eta(y')$.

(d) *If* $\eta(x)$ *is idempotent then* $\eta(x')$ *is idempotent for every connected component* x' *of* x.

Proof: The power set 2^Σ of Σ using union $\cup$ as operation forms a commutative and idempotent monoid. We set

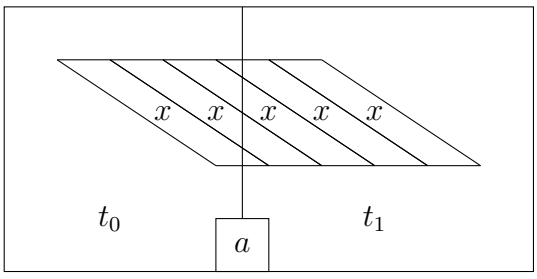
$$N = 2^\Sigma \times \prod_{\Gamma \subseteq \Sigma} M_0 \times \cdots \times M_n.$$

Since $2^\Sigma \in \mathbf{J_1} \subseteq \mathbf{V}$ and since $\mathbf{V}$ is a variety, we have $N \in \mathbf{V}$. Next we define

$$\eta = \mathrm{alph} \times \prod_{\Gamma \subseteq \Sigma} ((\mu_0 \times \cdots \times \mu_n) \circ \pi_\Gamma) : \mathbb{M}(\Sigma, I) \to N,$$

where π_Γ is the natural projection $\mathbb{M}(\Sigma, I) \to \mathbb{M}(\Gamma, \Gamma^2 \cap I) : x \mapsto \pi_\Gamma(x)$. Note that $\mathbb{M}(\Gamma, \Gamma^2 \cap I) \subseteq \mathbb{M}(\Sigma, I)$. Condition (a) is verified in the components of N and η corresponding to $\Gamma = \Sigma$ and condition (b) is fulfilled by reason of the first component. Let x' and y' be connected components of x and of y with $\mathrm{alph}(x') = \mathrm{alph}(y')$. Let $\Gamma \subseteq \Sigma$ and let $i \in \{0, \ldots, n\}$. To conclude (c) we have to show $\mu_i(\pi_\Gamma(x')) = \mu_i(\pi_\Gamma(y'))$. Since $x' = \pi_{\mathrm{alph}(x')}(x)$ we have $\pi_\Gamma(x') = \pi_\Gamma(\pi_{\mathrm{alph}(x')}(x)) = \pi_{\Gamma'}(x)$ with $\Gamma' = \Gamma \cap \mathrm{alph}(x')$. A similar argument for y' and y together with $\eta(x) = \eta(y)$ implies $\mu_i(\pi_\Gamma(x')) = \mu_i(\pi_{\Gamma'}(x)) = \mu_i(\pi_{\Gamma'}(y)) = \mu_i(\pi_\Gamma(y'))$. Now let $\eta(x)$ be idempotent. This means that every component of $\eta(x)$ is idempotent and since every component of $\eta(x')$ occurs as a component of $\eta(x)$, we have that $\eta(x')$ is also idempotent. □

Lemma 3.4 *Let $a \in \Sigma$, let $t_0, t_1 \in \mathbb{M}(\Sigma, I)$ and let $x \in \mathbb{M}(\Sigma, I)$ be connected. If $x^{|\Sigma|+m}$ is a factor of $t_0 a t_1$ for $m \in \mathbb{N}$ then there exist $m_0, m_1 \in \mathbb{N}$ such that $m_0 + m_1 = m$ and x^{m_i} is a factor of t_i for $i = 0$ and $i = 1$.*



Proof: The proof is similar to the proof that x^* is recognizable if $x \in \mathbb{M}(\Sigma, I)$ is connected [Och95, Proposition 6.3.11].

Since x is connected, between any two letters of $\mathrm{alph}(x)$ we have an undirected path in the dependence graph (Σ, D) of length at most $|\Sigma|$ such that

all vertices on this path are in $\mathrm{alph}(x)$. Directed paths following the same labels also exist in $x^{|\Sigma|}$ between all vertices of the first x and all vertices of the last x in this product. There could be some x's that have vertices in t_0 as well as vertices in at_1. The above argument shows that starting with the first x with this property we could lose at most $|\Sigma| - 1$ many of the x's of $x^{|\Sigma|+m}$ as factors of t_0 or at_1. The letter a could be a factor of one x. It follows that there remain m many x's as factors of either t_0 or t_1. □

Theorem 3.5 *Let* $\mathbf{V}$ *be a variety of monoids such that* $\mathbf{J_1} \subseteq \mathbf{V}$ *and let* $\mathcal{V}$ *be the set of trace languages corresponding to* $\mathbf{V}$*. Then the syntactic ordered monoid of every language* $L \in \mathrm{Pol}\mathcal{V}$ *is in* $[\![\, x^\omega y x^\omega \leq x^\omega \,]\!] \circledcirc{M}\ \mathbf{V}$*.*

Proof: We modify the proof for words presented in [PW97]. Let $L = L_0 a_1 L_1 \cdots a_n L_n$, where $L_i \in \mathcal{V}$ for all $0 \leq i \leq n$. Let $\eta : \mathbb{M}(\Sigma, I) \to N$ be as in Lemma 3.3 with $M_i = M(L_i)$ for $i \in \{\, 0, \ldots, n \,\}$. Let $(M(L), \leq_L)$ be the syntactic ordered monoid of L and $\mu : \mathbb{M}(\Sigma, I) \to M(L)$ its syntactic homomorphism. We obtain the relational morphism $\tau = \eta \circ \mu^{-1} : M(L) \to N$.

Let $e \in N$ be idempotent, let $x, y, u, v \in \mathbb{M}(\Sigma, I)$ such that $\eta(x) = e = \eta(y)$ and let $m \geq n\,|\Sigma| + 1$. The trace x can be decomposed into connected components $x = x_1 \cdots x_\ell$ such that $\mathrm{alph}(x_i) \times \mathrm{alph}(x_j) \subseteq I$ for all $1 \leq i \neq j \leq \ell$. Lemma 3.3 (b) implies $\mathrm{alph}(x) = \mathrm{alph}(y)$. Hence, the trace y can also be decomposed into connected components $y = y_1 \cdots y_\ell$ such that $\mathrm{alph}(y_j) = \mathrm{alph}(x_j)$ for all $1 \leq j \leq \ell$. Suppose $ux^m v \in L$. By applying Lemma 3.4 up to n times we can conclude that for every $j \in \{\, 1, \ldots, \ell \,\}$ there exists $i \in \{\, 0, \ldots, n \,\}$ and a factorization $ux^m v = z_0 z_1 x_j z_2 z_3$ such that

$$\begin{aligned} z_0 &\in L_0 a_1 L_1 \cdots L_{i-1} a_i \\ z_1 x_j z_2 &\in L_i \\ z_3 &\in a_{i+1} L_{i+1} \cdots a_n L_n. \end{aligned}$$

By Lemma 3.3 we have that $\mu_i(x_j) = \mu_i(y_j)$ is idempotent and therefore we have $z_1 x_j x_j^{k_1} y_j x_j^{k_2} z_2 \in L_i$ for all $k_1, k_2 \in \mathbb{N}$. By applying this pumping argument to all connected components of x, by a suitable choice of the exponents we can conclude $ux^m y x^m v \in L$.

Thus for all $m \geq n\,|\Sigma| + 1$ we have $\mu(x^m y x^m) \leq_L \mu(x^m)$ and therefore $\tau^{-1}(e) \in [\![\, x^\omega y x^\omega \leq x^\omega \,]\!]$. This shows $(M(L), \leq_L) \in [\![\, x^\omega y x^\omega \leq x^\omega \,]\!] \circledcirc{M}\ \mathbf{V}$. This Mal'cev product forms a variety of ordered semigroups. Language classes corresponding to varieties are closed under finite unions. Hence, we can conclude that $\mathrm{Pol}\mathcal{V}$ is a subset of the trace languages corresponding to $[\![\, x^\omega y x^\omega \leq x^\omega \,]\!] \circledcirc{M}\ \mathbf{V}$. □

Lemma 3.6 *Let $\eta : \Sigma^* \to M$ be a homomorphism from Σ^* to a commutative monoid M. Then there exists a unique homomorphism $\nu : \mathbb{M}(\Sigma, I) \to M$ such that $\nu \circ \pi_I = \eta$.*

The proof of the following lemma can be found in [Eil76] as a special case of Proposition 1.1, page 186.

Lemma 3.7 *Let $L \subseteq \mathbb{M}(\Sigma, I)$ be a trace language and let $\eta : \Sigma^* \to M$ be the syntactic homomorphism of $\pi_I^{-1}(L)$. Then M is isomorphic to the syntactic monoid of L.*

As for sets of trace languages, we define Pol$\mathcal{V}$ as the (word) languages that are finite unions of languages of the form $K_0 a_1 K_1 \cdots a_n K_n$, where $n \in \mathbb{N}$ and $K_i \in \mathcal{V}$ for all $0 \leq i \leq n$. For a partial converse of Theorem 3.5 we will use the following theorem from [PW97].

Theorem 3.8 (Pin/Weil, 1997) *Let $\mathbf{V}$ be a variety of finite monoids, let $\mathcal{V}$ be the corresponding $*$-variety and let $K \subseteq \Sigma^*$. If the syntactic ordered monoid of K is in $[\![x^\omega y x^\omega \leq x^\omega]\!] \, \textcircled{M} \, \mathbf{V}$, then $K \in \mathrm{Pol}\mathcal{V}$.*

For commutative varieties we can state this theorem for traces.

Theorem 3.9 *Let $\mathbf{V} \subseteq \mathbf{Com}$ be a variety of finite commutative monoids, let $\mathcal{V}$ be the set of trace languages corresponding to $\mathbf{V}$ and let $L \subseteq \mathbb{M}(\Sigma, I)$ be a trace language. If the syntactic ordered monoid of L is in the positive variety $[\![x^\omega y x^\omega \leq x^\omega]\!] \, \textcircled{M} \, \mathbf{V}$, then $L \in \mathrm{Pol}\mathcal{V}$.*

Proof: Let $K = \pi_I^{-1}(L)$. By Theorem 3.8 we can conclude that

$$K = \bigcup_{1 \leq i \leq m} K_{i,0} \; a_{i,1} \; K_{i,1} \cdots a_{i,n_i} \; K_{i,n_i}$$

for $m, n_1, \ldots, n_m \in \mathbb{N}$, $a_{i,j} \in \Sigma$ and $K_{i,j} \subseteq \Sigma^*$ such that $M(K_{i,j}) \in \mathbf{V}$. By Lemma 3.6 we have $\pi_I^{-1} \pi_I(K_{i,j}) = K_{i,j}$ and by Lemma 3.7 we can conclude that the syntactic monoid of $L_{i,j} = \pi_I(K_{i,j})$ is in $\mathbf{V}$. Hence

$$\begin{aligned} L = \pi_I(K) &= \pi_I \left(\bigcup_{1 \leq i \leq m} K_{i,0} \; a_{i,1} \; K_{i,1} \cdots a_{i,n_i} \; K_{i,n_i} \right) \\ &= \bigcup_{1 \leq i \leq m} \pi_I(K_{i,0}) \;\; a_{i,1} \; \pi_I(K_{i,1}) \cdots a_{i,n_i} \; \pi_I(K_{i,n_i}) \\ &= \bigcup_{1 \leq i \leq m} L_{i,0} \; a_{i,1} \; L_{i,1} \cdots a_{i,n_i} \; L_{i,n_i} \in \mathrm{Pol}\mathcal{V}. \end{aligned}$$

□

Corollary 3.10 *Let* $\mathbf{J_1} \subseteq \mathbf{V} \subseteq \mathbf{Com}$ *be a variety of finite monoids and let* $\mathcal{V}$ *be the corresponding variety of trace languages. Then* Pol$\mathcal{V}$ *corresponds to* $[\![\, x^\omega y x^\omega \leq x^\omega \,]\!]\, \textcircled{M}\, \mathbf{V}$.

Since the syntactic ordered monoid of the complement $\overline{L}$ of a trace language L is the dual $(M(\overline{L}), \leq_{\overline{L}}) = (M(L), \geq_L)$ of the syntactic ordered monoid $(M(L), \leq_L)$ of L we obtain the following corollary.

Corollary 3.11 *Let* $\mathbf{J_1} \subseteq \mathbf{V} \subseteq \mathbf{Com}$ *be a variety of finite monoids and let* $\mathcal{V}$ *be the corresponding trace languages. Then* coPol$\mathcal{V}$ *corresponds to* $[\![\, x^\omega y x^\omega \geq x^\omega \,]\!]\, \textcircled{M}\, \mathbf{V}$.

By Lemma 2.14 we have:

$$[\![\, x^\omega y x^\omega \leq x^\omega \,]\!]\, \textcircled{M}\, \mathbf{V} \cap [\![\, x^\omega y x^\omega \geq x^\omega \,]\!]\, \textcircled{M}\, \mathbf{V} = [\![\, x^\omega y x^\omega = x^\omega \,]\!]\, \textcircled{M}\, \mathbf{V}$$

and since $[\![\, x^\omega y x^\omega = x^\omega \,]\!]\, \textcircled{M}\, \mathbf{V} = \mathbf{LI}\, \textcircled{M}\, \mathbf{V}$ is a variety if $\mathbf{V}$ is a variety [PW96a, PW97], due to the closure properties of language varieties we obtain the following corollary.

Corollary 3.12 *Let* $\mathbf{J_1} \subseteq \mathbf{V} \subseteq \mathbf{Com}$ *be a variety of finite monoids and let* $\mathcal{V}$ *be the corresponding variety of trace languages. Then* Pol$\mathcal{V} \cap$ coPol$\mathcal{V}$ *is a (trace) language variety that corresponds to the variety* $\mathbf{LI}\, \textcircled{M}\, \mathbf{V}$.

Note that the syntactic monoid $M(L)$ of a trace language L is in the (full) variety $[\![\, x^\omega y x^\omega = x^\omega \,]\!]\, \textcircled{M}\, \mathbf{V}$ if and only if the syntactic ordered monoid of this language $(M(L), \leq_L)$ is in $[\![\, x^\omega y x^\omega = x^\omega \,]\!]\, \textcircled{M}\, \mathbf{V}$ as a positive variety.

3.2.2 Unambiguous Polynomial Closure

Theorem 3.13 *Let* $\mathbf{V}$ *be a variety of monoids such that* $\mathbf{J_1} \subseteq \mathbf{V}$ *and let* $\mathcal{V}$ *be the set of trace languages corresponding to* $\mathbf{V}$. *Then the syntactic monoid of every trace language in* UPol$\mathcal{V}$ *is in* $\mathbf{LI}\, \textcircled{M}\, \mathbf{V}$.

Proof: By definition we have $\mathbf{LI} = [\![x^\omega y x^\omega = x^\omega]\!]$. Let $L = L_1 a L_2$ be a left unambiguous product of L_1 and L_2 and let their syntactic monoids be $M_1, M_2 \in \mathbf{LI} \mathbin{Ⓜ} \mathbf{V}$. Let $\eta : \mathbb{M}(\Sigma, I) \to N$ be as in Lemma 3.3 and let $M(L)$ be the syntactic monoid of L and $\mu : \mathbb{M}(\Sigma, I) \to M(L)$ its syntactic homomorphism. We obtain the relational morphism $\tau = \eta \circ \mu^{-1} : M(L) \to N$. Since $\mathbf{LI} \mathbin{Ⓜ} \mathbf{V}$ forms a variety [Pin86] and since $\mathbf{LI} \mathbin{Ⓜ} (\mathbf{LI} \mathbin{Ⓜ} \mathbf{V}) = \mathbf{LI} \mathbin{Ⓜ} \mathbf{V}$, see [PST88], it is sufficient to show that for all idempotents $e \in N$ we have $\tau^{-1}(e) \in \mathbf{LI}$. The theorem then follows by left-right symmetry and from the fact that classes of languages corresponding to varieties of monoids are closed under boolean operations [Eil76].

Let $e^2 = e \in N$, let $x, y, u, v \in \mathbb{M}(\Sigma, I)$ such that $\eta(x) = e = \eta(y)$ and let $m \geq |\Sigma| + 1$ such that $\mu(x)^m$ is idempotent. We will show that $ux^m v \in L$ if and only if $ux^m y x^m v \in L$. The direction from left to right is the same as in Theorem 3.5. Suppose $ux^m y x^m v \in L$. Then there exists a left unambiguous factorization $ux^m y x^m v = z_1 a z_2$ with $z_1 \in L_1$ and $z_2 \in L_2$. Let $x = x_1 \cdots x_\ell$ and $y = y_1 \cdots y_\ell$ be factorizations into connected components such that $\mathrm{alph}(x_j) = \mathrm{alph}(y_j)$ for all $1 \leq j \leq \ell$. Suppose the connected component x_1 of x from the left x^m block matches with a factor of z_1 and the same connected component x_1 of x from the right x^m block matches with a factor of z_2. Since $\eta(x_1) = \eta(y_1)$ is idempotent, we can arbitrarily pump x_1 and y_1 at these two positions of x_1 without changing membership to L. The possibility of pumping at both position leads to two different factorizations of $ux^m y x^m y_1 x_1^m v \in L$. This contradicts the choice of L_1 and L_2 such that L is left unambiguous. The same argument holds for all connected components of x.

Together with Lemma 3.4 it follows that for every index $j \in \{ 1, \ldots, \ell \}$ of a connected component there exists $i \in \{ 1, 2 \}$ such that the last occurrence of x_j in the left x^m block and the first occurrence of x_j in the right x^m block in $ux^m y x^m v$ are factors of z_i. Hence, the component y_j of y lies between these two occurrences of x_j, i.e., $x_j y_j x_j$ is a factor of z_i. Since $\eta(x_j y_j x_j) = \eta(x_j x_j)$, we can remove all connected components of y without changing membership to L. Therefore, $ux^m x^m v \in L$ and by idempotency of $\mu(x)^m$ we can conclude $ux^m v \in L$. □

Next, we will present a converse of the previous theorem for the variety $\mathbf{V} = \mathbf{J_1}$. Rememeber that $\mathbf{LI} \mathbin{Ⓜ} \mathbf{J_1} = \mathbf{DA}$.

A factorization $t = t_- a t_+$ is a *left factorization* if $a \notin \mathrm{alph}(t_-)$ and if $t_- = sb$ implies $(a, b) \in D$, i.e., in this factorization a is the first occurrence of the letter a in t and all minimal elements of t_- are dependent of a. Symmetrically,

we say that a factorization $t = t_- a t_+$ is a *right factorization* if $a \notin \mathrm{alph}(t_+)$ and if $t_+ = bs$ implies $(a, b) \in D$.

Definition 3.14 *We define the relation* $\equiv_{A,k} \subseteq \mathbb{M}(\Sigma, I)^2$ *for* $A \subseteq \Sigma$ *and* $k \in \mathbb{N}$*:*

- $t \equiv_{A,0} s$ *if* $\mathrm{alph}(t) \subsetneq A \supsetneq \mathrm{alph}(s)$ *or* $\mathrm{alph}(t) \subseteq A \supseteq \mathrm{alph}(s)$.
- $t \equiv_{A,k} s$ *for* $k > 0$ *if* $\mathrm{alph}(t) \subsetneq A \supsetneq \mathrm{alph}(s)$ *or the following three conditions hold:*
  - $\mathrm{alph}(t) = \mathrm{alph}(s) \subseteq A$.
  - *For all* $a \in \mathrm{alph}(t)$ *and all left factorizations* $t = t_- a t_+$ *and* $s = s_- a s_+$ *the conditions* $t_- \equiv_{A\setminus\{a\},k-1} s_-$ *and* $t_+ \equiv_{A,k-1} s_+$ *hold.*
  - *For all* $a \in \mathrm{alph}(t)$ *and all right factorizations* $t = t_- a t_+$ *and* $s = s_- a s_+$ *the conditions* $t_- \equiv_{A,k-1} s_-$ *and* $t_+ \equiv_{A\setminus\{a\},k-1} s_+$ *hold.*

It is clear that for all $A \subseteq \Sigma$ and all $k \in \mathbb{N}$ the relation $\equiv_{A,k}$ is an equivalence relation of finite index. The analog of the following lemma in the case of words was shown in [TT02].

Lemma 3.15 *Let* $\gamma \subseteq \mathbb{M}(\Sigma, I)^2$ *be a congruence of finite index such that the monoid* $\mathbb{M}(\Sigma, I)/\gamma$ *is in* **DA**. *Then there exists* $k \in \mathbb{N}$ *such that* $\equiv_{\Sigma,k} \subseteq \gamma$.

Proof: We adapt the proof of a theorem in [TT02]. By $[t]$ we denote the equivalence class of t with respect to γ. Let $M = \mathbb{M}(\Sigma, I)/\gamma$. A factorization $t = t_0 a_1 t_1 \cdots a_n t_n$ is called $\mathcal{R}$*-decomposition* if the following conditions are satisfied:

- $[t_0] = 1 \in M$.
- $[t_0 a_1 t_1 \cdots a_i] \,\mathcal{R}\, [t_0 a_1 t_1 \cdots a_i t_i]$ for all $1 \leq i \leq n$.
- $[t_0 a_1 t_1 \cdots a_i t_i] \,\overline{\mathcal{R}}\, [t_0 a_1 t_1 \cdots a_i t_i a_{i+1}]$ for all $1 \leq i < n$.
- $|t_i|$ is minimal with the above properties for all $0 \leq i \leq n$ and where $|t_i|$ is always minimized before $|t_{i+1}|$.

Here $\overline{\mathcal{R}} = M^2 \setminus \mathcal{R}$. From $M \in \mathbf{DA} \subseteq \mathbf{DS}$ we can conclude $a_i \notin \mathrm{alph}(t_{i-1})$ for all $1 \leq i \leq n$. The $\mathcal{R}$-decomposition of a trace t is not unique. Let m be the maximum of the number of $\mathcal{R}$-classes and the number of $\mathcal{L}$-classes of M. By induction on $|A|$ we will proof that for all $A \subseteq \Sigma$ and all $t, s \in \mathbb{M}(\Sigma, I)$ the following implication holds: if $\mathrm{alph}(t) \subseteq A \supseteq \mathrm{alph}(s)$ and $t \equiv_{A,|A|m} s$ then $[t] = [s]$. If $|A| = 0$ then $A = \emptyset$ and $t = \varepsilon = s$ and hence $[t] = [s]$.

Suppose $|A| > 0$ and w.l.o.g. let $\emptyset \neq \mathrm{alph}(t) = \mathrm{alph}(s) \subseteq A$. Let $t = t_0 a_1 t_1 \cdots a_n t_n$ be an $\mathcal{R}$-decomposition. Then $n < m$ holds. We define $r_i = t_i a_i t_{i+1} \cdots a_n t_n$. From the minimality of $|t_i|$ it follows that $r_i = t_i a_{i+1} r_{i+1}$ is a left factorization. From $t \equiv_{A,|A|m} s$ we can conclude that there exists a factorization $s = s_0 a_1 s_1 \cdots a_n s_n$ satisfying $t_i \equiv_{A\setminus\{a_{i+1}\},|A|m-(i+1)} s_i$ for all $0 \leq i < n$. Now $i+1 \leq n < m$ implies $t_i \equiv_{A\setminus\{a_{i+1}\},|A\setminus\{a_{i+1}\}|m} s_i$ and therefore by induction $[t_i] = [s_i]$ for all $0 \leq i < n$. Hence $[t]\,\mathcal{R}\,[t_0 a_1 t_1 \cdots a_n] = [s_0 a_1 s_1 \cdots a_n]$ and thus $[s]\,M \subseteq [s_0 a_1 s_1 \cdots a_n]\,M = [t]\,M$. Starting with an $\mathcal{R}$-decomposition of s we can analogously conclude $[t]\,M \subseteq [s]\,M$ and therefore $[t]\,\mathcal{R}\,[s]$. A symmetric reasoning with $\mathcal{L}$-decompositions shows $[t]\,\mathcal{L}\,[s]$ and hence $[t]\,\mathcal{H}\,[s]$. Since $M \in \mathbf{DA} \subseteq \mathbf{A}$ is $\mathcal{H}$-trivial we have $[t] = [s]$. This shows $\equiv_{\Sigma,k} \subseteq \gamma$ for $k \geq |\Sigma|\,m$. □

It follows that every trace language L with $M(L) \in \mathbf{DA}$ is the disjoint union of $\equiv_{A,k}$-classes. We define the set of trace languages $\mathcal{A} = \{\, A^* \mid A \subseteq \Sigma \,\}$. Clearly, $\mathcal{A}$ is a subset of the trace languages corresponding to $\mathbf{J_1}$.

Lemma 3.16 *Let $A \subseteq \Sigma$ and $k \in \mathbb{N}$. Every equivalence class of $\equiv_{A,k}$ is in* $\mathrm{UPol}\mathcal{A}$.

Proof: For $a \in \Sigma$, $t \in \mathbb{M}(\Sigma, I)$, $A \subseteq \Sigma$ and $k \in \mathbb{N}$, we define the following languages:

$$\begin{aligned}
L(a) &= \overline{\bigcup_{(a,b)\in I} \Sigma^* b} \cap (\Sigma \setminus \{a\})^* \\
R(a) &= \overline{\bigcup_{(a,b)\in I} b\Sigma^*} \cap (\Sigma \setminus \{a\})^* \\
B(\Gamma) &= \{\, s \in \mathbb{M}(\Sigma, I) \mid \mathrm{alph}(s) = \Gamma \,\} \\
&= \Gamma^* \cap \bigcap_{b\in\Gamma} L(b)\, b\, \Sigma^* \\
E(A, k, t) &= [t]_{\equiv_{A,k}}.
\end{aligned}$$

Clearly, we have $L(a), R(a), B(t) \in \mathrm{UPol}\mathcal{A}$. The set $L(a)$ contains all traces t such that $\mathrm{alph}(t) \subseteq \Sigma \setminus \{a\}$ and a is the unique maximal element of ta. The set $R(a)$ is symmetric. By induction on k we proof $E(A,k,t) \in \mathrm{UPol}\mathcal{A}$. The equivalence classes of $\equiv_{A,0}$ are A^* and its complement. For $k > 0$ the language $E(A,k,t)$ is $\overline{A^*}$ or

$$B(\mathrm{alph}(t)) \cap \bigcap_{\substack{t = t_- a t_+ \text{ is} \\ \text{left factorization}}} \big(L(a) \cap E(A \setminus \{a\}, k-1, t_-)\big) \cdot a \cdot E(A, k-1, t_+)$$

$$\cap \bigcap_{\substack{t = t_- a t_+ \text{ is} \\ \text{right factorization}}} E(A, k-1, t_-) \cdot a \cdot \big(R(a) \cap E(A \setminus \{a\}, k-1, t_+)\big)$$

Note that all products are unambiguous and all intersections are finite. □

For every set of letters $A \subseteq \Sigma$ we have

$$\{\, t \in \mathbb{M}(\Sigma, I) \mid \mathrm{alph}(t) = A \,\} = \bigcup_{\{a_1,\ldots,a_n\}=A} A^* a_1 A^* \cdots a_n A^*,$$

where $n = |A|$. Monoids in $\mathbf{J_1}$ can only recognize alphabetical constraints. Therefore, the language variety corresponding to $\mathbf{J_1}$ is contained in $\mathrm{Pol}\mathcal{A}$. Together with Corollary 3.12 and Theorem 3.13 we can conclude:

Corollary 3.17 *The language variety* $\mathrm{UPol}\mathcal{A} = \mathrm{Pol}\mathcal{A} \cap \mathrm{coPol}\mathcal{A}$ *corresponds to the variety* **DA**.

Chapter 4

Logic for Mazurkiewicz Traces

4.1 Temporal Logic

In this section we introduce two characterizations of **DA** with temporal logic. In this context, a *temporal formula* is a term of the form

$$\varphi ::= a \mid \neg\varphi \mid (\varphi_1 \vee \varphi_2) \mid (\varphi_1 \wedge \varphi_2) \mid \mathsf{XF}\varphi \mid \mathsf{YP}\varphi \mid \mathsf{Eco}\varphi \mid \mathsf{M}\varphi \mid \mathsf{X}_a\varphi \mid \mathsf{Y}_a\varphi$$

where $a \in \Sigma$. The operators XF, YP, Eco, M, X_a and Y_a are called *temporal operators*. The letter X comes from the word neXt, Y stands for Yesterday, F for Future, P for Past, Eco stands for Exists concurrent position and M for soMetime. In general temporal formulae, there exist other more powerful operators such as U for Until, see e.g. [MP91]. Next we define, when a trace

$t = (X, \text{label}, <)$ at position $\nu \in X$ models a temporal formula:

$$\begin{aligned}
t,\nu \models a \quad &\Leftrightarrow\ \text{label}(\nu) = a, \quad \text{for } a \in \Sigma.\\
t,\nu \models \neg\varphi \quad &\Leftrightarrow\ \text{not } \ t,\nu \models \varphi.\\
t,\nu \models \varphi_1 \vee \varphi_2 \quad &\Leftrightarrow\ t,\nu \models \varphi_1 \ \text{ or } \ t,\nu \models \varphi_2.\\
t,\nu \models \varphi_1 \wedge \varphi_2 \quad &\Leftrightarrow\ t,\nu \models \varphi_1 \ \text{ and } \ t,\nu \models \varphi_2.\\
t,\nu \models \mathsf{XF}\varphi \quad &\Leftrightarrow\ \exists \chi \in X\colon\ \nu < \chi \ \text{ and } \ t,\chi \models \varphi.\\
t,\nu \models \mathsf{YP}\varphi \quad &\Leftrightarrow\ \exists \chi \in X\colon\ \chi < \nu \ \text{ and } \ t,\chi \models \varphi.\\
t,\nu \models \mathsf{Eco}\varphi \quad &\Leftrightarrow\ \exists \chi \in X\colon\ \chi \parallel \nu \ \text{ and } \ t,\chi \models \varphi.\\
t,\nu \models \mathsf{M}\varphi \quad &\Leftrightarrow\ \exists \chi \in X\colon\ t,\chi \models \varphi.\\
t,\nu \models \mathsf{X}_a\varphi \quad &\Leftrightarrow\ \exists \chi \in X\colon\ \nu < \chi \ \text{ and } \ t,\chi \models a \wedge \varphi \ \text{ and}\\
&\qquad \big(\forall \xi \in X\colon\ \nu < \xi < \chi \ \Rightarrow\ \text{label}(\xi) \neq a\big).\\
t,\nu \models \mathsf{Y}_a\varphi \quad &\Leftrightarrow\ \exists \chi \in X\colon\ \chi < \nu \ \text{ and } \ t,\chi \models a \wedge \varphi \ \text{ and}\\
&\qquad \big(\forall \xi \in X\colon\ \chi < \xi < \nu \ \Rightarrow\ \text{label}(\xi) \neq a\big).
\end{aligned}$$

The usage of "alphabetic filters" (as in X_a and Y_a) with $a \in \Sigma$ has been introduced in [GM02b] for local temporal logic over traces.

An *outer* temporal formula is a boolean combination of formulae of the form $\mathsf{XF}\varphi$, $\mathsf{YP}\varphi$, $\mathsf{Eco}\varphi$, $\mathsf{M}\varphi$, $\mathsf{X}_a\varphi$ or $\mathsf{Y}_a\varphi$ where φ is an arbitrary temporal formula. Next we define when a trace $t = (X, \text{label}, <)$ models an outer temporal formula. Boolean combinations are defined straightforwardly.

$$\begin{aligned}
t \models \mathsf{XF}\varphi \quad &\Leftrightarrow\ \exists \nu \in X\colon\ t,\nu \models \varphi.\\
t \models \mathsf{YP}\varphi \quad &\Leftrightarrow\ \exists \nu \in X\colon\ t,\nu \models \varphi.\\
t \models \mathsf{Eco}\varphi \quad &\Leftrightarrow\ \exists \nu \in X\colon\ t,\nu \models \varphi.\\
t \models \mathsf{M}\varphi \quad &\Leftrightarrow\ \exists \nu \in X\colon\ t,\nu \models \varphi.\\
t \models \mathsf{X}_a\varphi \quad &\Leftrightarrow\ \exists \nu \in X\colon\ t,\nu \models a \wedge \varphi \ \text{ and}\\
&\qquad \big(\forall \xi \in X\colon\ \xi < \nu \ \Rightarrow\ \text{label}(\xi) \neq a\big).\\
t \models \mathsf{Y}_a\varphi \quad &\Leftrightarrow\ \exists \nu \in X\colon\ t,\nu \models a \wedge \varphi \ \text{ and}\\
&\qquad \big(\forall \xi \in X\colon\ \nu < \xi \ \Rightarrow\ \text{label}(\xi) \neq a\big).
\end{aligned}$$

Note that $\mathsf{XF}\varphi$, $\mathsf{YP}\varphi$, $\mathsf{Eco}\varphi$ and $\mathsf{M}\varphi$ as outer formulae are equivalent. The idea is that when evaluating XF and X_a we start at a position in front of the trace and when evaluating YP and Y_a we start at a position behind the trace. We say that φ is a $\mathrm{TL}[\mathsf{Op}_1, \ldots, \mathsf{Op}_n]$ *formula*, if φ contains no temporal operators apart from $\mathsf{Op}_1, \ldots, \mathsf{Op}_n$. The trace language generated by an outer temporal formula φ is

$$L(\varphi) = \{\, t \in \mathbb{M}(\Sigma, I) \mid t \models \varphi \,\}.$$

We denote the set of all trace languages $L = L(\varphi)$ where φ is an outer TL[$\mathsf{Op}_1, \ldots, \mathsf{Op}_n$] formula by TL[$\mathsf{Op}_1, \ldots, \mathsf{Op}_n$]. We also say that L is *expressible* by a TL[$\mathsf{Op}_1, \ldots, \mathsf{Op}_n$] formula. Hence, e.g. 'TL[XF, YP]' is a class of trace languages whereas by 'TL[XF, YP] formulae' we mean a class of formulae.

4.1.1 Temporal Logic and DA

Lemma 4.1 *Let φ be an outer* TL[$\mathsf{XF}, \mathsf{YP}, \mathsf{M}$] *formula. Then the syntactic monoid of $L(\varphi)$ is in* **DA**.

Proof: Let m be the number of (nested) temporal operators in φ. Let $x, y, p, q \in \mathbb{M}(\Sigma, I)$, let $n > m\,|\Sigma|$ and let $t = p(xy)^n x(xy)^n q$ and $s = p(xy)^n (xy)^n q$.

We define a function $g : X_t \to X_s$ from the positions in t to the positions in s by mapping all positions of the prefix $p(xy)^n$ of t to the corresponding prefix of s and all positions of the suffix $(xy)^n q$ of t to the corresponding suffix of s. The positions of x in t between this prefix and this suffix are mapped to the corresponding positions of the first x of the suffix $(xy)^n q$ of s. Note that g is onto.

For $\ell < m$ we define a partial function $next_\ell : X_s \to X_s$ by mapping all positions of all xy in $(xy)^{(m-\ell)|\Sigma|}(xy)^{(m-\ell)|\Sigma|}$ in the center of

$$s = p(xy)^{n'} \cdot (xy)^{(m-\ell)|\Sigma|}(xy)^{(m-\ell)|\Sigma|} \cdot (xy)^{n'} q$$

with $n' = n - (m - \ell)\,|\Sigma|$ to the corresponding positions in the consecutive occurrence of xy in s.

We will show that for every temporal formula ζ with ℓ temporal operators, $\ell < m$, we have

$$t, \nu \models \zeta \;\Leftrightarrow\; s, g(\nu) \models \zeta \;\text{ for all } \nu \in X_t \;\text{ and} \tag{4.1}$$

$$s, \chi \models \zeta \;\Leftrightarrow\; s, next_\ell(\chi) \models \zeta \;\text{ for all } \chi \text{ in the domain of } next_\ell. \tag{4.2}$$

For $\ell = 0$ this is true since all positions are mapped to positions with the same label. Suppose $\zeta = \mathsf{XF}\psi$ and $t, \nu \models \zeta$. Then there exists a position ξ such that $\nu <_t \xi$ and $t, \xi \models \psi$. By induction we have $s, g(\xi) \models \psi$. If $g(\nu) <_s g(\xi)$ we are done for this direction. Otherwise ν is a position of x in

the center of t. In this cases we have $g(\nu) <_s next_{\ell-1}(g(\xi))$ and by induction $s, g(\xi) \models \psi$ implies $s, next_{\ell-1}(g(\xi)) \models \psi$. Hence $s, g(\nu) \models \mathsf{XF}\psi$. The other direction of (4.1) follows similarly by using (4.2) if ν is a position of x in the center of t.

For (4.2) suppose $s, \chi \models \mathsf{XF}\psi$ for χ in the domain of $next_\ell$. Then there exists $\xi \in X_s$ such that $\chi <_s \xi$ and $s, \xi \models \psi$. If $next_\ell(\chi) <_s \xi$ we are done. Otherwise we can apply induction hypothesis to conclude $s, next_{\ell-1}(\xi) \models \psi$ since ξ is in the domain of $next_{\ell-1}$. Note that this is the reason for the factor $|\Sigma|$ in the above exponents in the definition of $next_\ell$. Since $next_\ell(\chi) <_s next_{\ell-1}(\xi)$ we have $s, next_\ell(\chi) \models \mathsf{XF}\psi$. The other direction of (4.2) is similar.

The case $\zeta = \mathsf{YP}\psi$ is symmetric and the case $\zeta = \mathsf{M}\psi$ is trivial since no constraint is required for the subsequent position. Boolean operations are straightforward by induction on the size of the formula. It follows that $t \in L(\varphi)$ if and only if $s \in L(\varphi)$ and hence the syntactic monoid of $L(\varphi)$ is in $\mathbf{DA} = [\![\, (xy)^\omega x (xy)^\omega = (xy)^\omega \,]\!]$. □

Lemma 4.2 *Let φ be an outer* TL$[\mathsf{X}_a, \mathsf{Y}_a]$ *formula. Then the syntactic monoid of $L(\varphi)$ is in* **DA**.

Proof: As in the proof of Lemma 4.1, let m be the number of (nested) temporal operators in φ. Let $x, y, p, q \in \mathbb{M}(\Sigma, I)$, let $n > m\,|\Sigma|$ and let $t = p(xy)^n x (xy)^n q$ and $s = p(xy)^n (xy)^n q$. The reason that φ cannot distinguish between t and s is that is not possible to reach a position in x in the middle of t since $\mathrm{alph}(x) \subseteq \mathrm{alph}(xy)$. Therefore, all positions that are taken into account by a formula φ with m (nested) operators in $\{\, \mathsf{X}_a, \mathsf{Y}_a \,\}$ lie within the identical prefixes $p(xy)^{m|\Sigma|}$ and suffixes $(xy)^{m|\Sigma|} q$ of t and s. Hence the syntactic monoid of $L(\varphi)$ is in **DA**. □

We will show that all trace languages L with $M(L) \in \mathbf{DA}$ can be expressed by a formula in TL$[\mathsf{XF}, \mathsf{YP}]$. By Lemma 3.15 it suffices to show that all equivalence classes of $\equiv_{A,k}$ are expressible in TL$[\mathsf{XF}, \mathsf{YP}]$.

Lemma 4.3 *Let $A \subseteq \Sigma$ and $k \in \mathbb{N}$. Every equivalence class of $\equiv_{A,k}$ is in* TL$[\mathsf{XF}, \mathsf{YP}]$.

Proof: We introduce some macro formulae.

$$\begin{aligned}
B^*(\Gamma) &= \bigwedge_{b \notin \Gamma} \neg \mathsf{XF} b && \text{for } \Gamma \subseteq \Sigma \\
B(\Gamma) &= \bigwedge_{b \in \Gamma} \mathsf{XF} b \wedge B^*(\Gamma) && \text{for } \Gamma \subseteq \Sigma \\
\mathrm{LL}(a) &= \mathsf{XF}(a \wedge \neg \mathsf{YP} a) && \text{for } a \in \Sigma \\
\mathrm{LR}(a) &= \neg \mathrm{LL}(a) \wedge \neg(a \wedge \neg \mathsf{YP} a) && \text{for } a \in \Sigma \\
\mathrm{RR}(a) &= \mathsf{YP}(a \wedge \neg \mathsf{XF} a) && \text{for } a \in \Sigma \\
\mathrm{RL}(a) &= \neg \mathrm{RR}(a) \wedge \neg(a \wedge \neg \mathsf{XF} a) && \text{for } a \in \Sigma.
\end{aligned}$$

A trace t models $B^*(\Gamma)$ if and only if $\mathrm{alph}(t) \subseteq \Gamma$ and it models $B(\Gamma)$ if and only if $\mathrm{alph}(t) = \Gamma$. The first letter of the name in the formulae of the form $\mathrm{XY}(a)$ indicates the type of the factorization (*L*eft or *R*ight) and the second letter refers to the side within this factorization. These four formulae will be used to restrict positions to the left or to the right part of a left or a right factorization. For example, $\mathrm{LR}(a)$ is true at positions that are not before the first a and that are not the first position labeled by a and $\mathrm{RR}(a)$ is true at all positions after the last a.

For each equivalence class $[t]_{\equiv_{A,k}}$ we will show that there exists a formula $\varphi(A, k, t)$ such that $L(\varphi(A, k, t)) = [t]_{\equiv_{A,k}}$. For $k = 0$, we have $A^* = L(B^*(A))$ and $\overline{A^*}$ is expressed by the negation of this formula. For $a \in \Sigma$ we define the four transformations $\mathrm{Tr}^a_{\mathrm{XY}}$ for $\mathrm{XY} \in \{\, \mathrm{LL}, \mathrm{LR}, \mathrm{RL}, \mathrm{RR} \,\}$ on formulae in $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}]$:

$$\begin{aligned}
\mathrm{Tr}^a_{\mathrm{XY}}[\, b \,] &= b \wedge \mathrm{XY}(a) \quad \text{for } b \in \Sigma \\
\mathrm{Tr}^a_{\mathrm{XY}}[\, \neg\varphi \,] &= \neg \mathrm{Tr}^a_{\mathrm{XY}}[\, \varphi \,] \\
\mathrm{Tr}^a_{\mathrm{XY}}[\, \varphi_1 \vee \varphi_2 \,] &= \mathrm{Tr}^a_{\mathrm{XY}}[\, \varphi_1 \,] \vee \mathrm{Tr}^a_{\mathrm{XY}}[\, \varphi_2 \,] \\
\mathrm{Tr}^a_{\mathrm{XY}}[\, \varphi_1 \wedge \varphi_2 \,] &= \mathrm{Tr}^a_{\mathrm{XY}}[\, \varphi_1 \,] \wedge \mathrm{Tr}^a_{\mathrm{XY}}[\, \varphi_2 \,] \\
\mathrm{Tr}^a_{\mathrm{XY}}[\, \mathsf{XF}\varphi \,] &= \mathsf{XF}\big(\mathrm{Tr}^a_{\mathrm{XY}}[\, \varphi \,] \wedge \mathrm{XY}(a)\big) \\
\mathrm{Tr}^a_{\mathrm{XY}}[\, \mathsf{YP}\varphi \,] &= \mathsf{YP}\big(\mathrm{Tr}^a_{\mathrm{XY}}[\, \varphi \,] \wedge \mathrm{XY}(a)\big).
\end{aligned}$$

The transformations restrict formulae to special regions. For example $\mathrm{Tr}^a_{\mathrm{LL}}[\varphi]$ restricts φ to all positions before the first occurrence of a. For $k > 0$ the

formula for $[t]_{\equiv_{A,k}}$ is $\varphi(A,k,t) = \neg B^*(A)$ or

$$\begin{aligned}\varphi(A,k,t) \;=\; & B(\mathrm{alph}(t)) \;\wedge \\ & \bigwedge_{\substack{t = t_- a t_+ \text{ is} \\ \text{left factorization}}} \left\{ \begin{array}{l} \mathrm{Tr}^a_{\mathrm{LL}}\big[\,\varphi(A \setminus \{a\}, k-1, t_-)\,\big] \;\wedge \\ \mathrm{Tr}^a_{\mathrm{LR}}\big[\,\varphi(A, k-1, t_+)\,\big] \end{array} \right\} \wedge \\ & \bigwedge_{\substack{t = t_- a t_+ \text{ is} \\ \text{right factorization}}} \left\{ \begin{array}{l} \mathrm{Tr}^a_{\mathrm{RL}}\big[\,\varphi(A, k-1, t_-)\,\big] \;\wedge \\ \mathrm{Tr}^a_{\mathrm{RR}}\big[\,\varphi(A \setminus \{a\}, k-1, t_+)\,\big] \end{array} \right\}.\end{aligned}$$

□

Lemma 4.4 *Let $A \subseteq \Sigma$ and $k \in \mathbb{N}$. Every equivalence class of $\equiv_{A,k}$ is in* $\mathrm{TL}[\mathsf{X}_a, \mathsf{Y}_a]$.

Proof: The proof is analogous to that of Lemma 4.3 but the transformations of formulae will be more involved since we do not only have to ensure that we don't leave some restriction of a trace but also that we are able to reach this restriction. For this task we have to distinguish nested temporal operators between outermost and inner operators. When dealing with the outermost operators we will solve the task of reaching some part of the trace and when we are treating inner temporal operators, we will ensure that they stay within this part. The parts under examination are determined by left and right factorizations.

Analogously to the proof of Lemma 4.3 we introduce some macro formulae. These macros will enable us to check for alphabetical constraints and allow us to restrict formulae to positions relative to the first or last occurrences of a letter $a \in \Sigma$.

$$\begin{aligned} B^*(\Gamma) &= \bigwedge_{b \notin \Gamma} \neg \mathsf{X}_b b && \text{for } \Gamma \subseteq \Sigma \\ B(\Gamma) &= \bigwedge_{b \in \Gamma} \mathsf{X}_b b \wedge B^*(\Gamma) && \text{for } \Gamma \subseteq \Sigma \\ \mathrm{LL}(a) &= \mathsf{X}_a \neg \mathsf{Y}_a a && \text{for } a \in \Sigma \\ \mathrm{LR}(a) &= \neg \mathrm{LL}(a) \wedge \neg(a \wedge \neg \mathsf{Y}_a a) && \text{for } a \in \Sigma \\ \mathrm{LP}(a) &= \neg \mathrm{LL}(a) \wedge \neg \mathsf{Y}_a a \wedge \neg a && \text{for } a \in \Sigma \\ \mathrm{RR}(a) &= \mathsf{Y}_a \neg \mathsf{X}_a a && \text{for } a \in \Sigma \\ \mathrm{RL}(a) &= \neg \mathrm{RR}(a) \wedge \neg(a \wedge \neg \mathsf{X}_a a) && \text{for } a \in \Sigma \\ \mathrm{RP}(a) &= \neg \mathsf{X}_a a \wedge \neg \mathrm{RR}(a) \wedge \neg a && \text{for } a \in \Sigma. \end{aligned}$$

The letter P in the transformations of the form XY(a) stands for *P*arallel. For example, LP(a) is true at positions that are not before the first a and that are not after a position labeled by a and that are not labeled by a.

For $a \in \Sigma$ and for XY $\in \{\,\mathrm{LL}, \mathrm{LR}, \mathrm{RL}, \mathrm{RR}\,\}$ we define the inner transformation $\mathrm{Inner}^a_{\mathrm{XY}}$ on formulae in $\mathrm{TL}[\mathsf{X}_a, \mathsf{Y}_a]$:

$$\begin{aligned}
\mathrm{Inner}^a_{\mathrm{XY}}[\,b\,] &= b \wedge \mathrm{XY}(a) \quad \text{for } b \in \Sigma \\
\mathrm{Inner}^a_{\mathrm{XY}}[\,\mathsf{X}_b\varphi\,] &= \mathsf{X}_b\big(\mathrm{Inner}^a_{\mathrm{XY}}[\,\varphi\,] \wedge \mathrm{XY}(a)\big) \\
\mathrm{Inner}^a_{\mathrm{XY}}[\,\mathsf{Y}_b\varphi\,] &= \mathsf{Y}_b\big(\mathrm{Inner}^a_{\mathrm{XY}}[\,\varphi\,] \wedge \mathrm{XY}(a)\big).
\end{aligned}$$

The description of the transformation of boolean combinations is omitted. Next ,we will define the four transformations $\mathrm{Outer}^a_{\mathrm{LL}}$, $\mathrm{Outer}^a_{\mathrm{RR}}$, $\mathrm{Outer}^a_{\mathrm{LR}}$ and $\mathrm{Outer}^a_{\mathrm{RL}}$ of the outermost temporal operators:

$$\begin{aligned}
\mathrm{Outer}^a_{\mathrm{LL}}[\,\mathsf{X}_b\varphi\,] &= \mathsf{X}_b\big(\mathrm{Inner}^a_{\mathrm{LL}}[\,\varphi\,] \wedge \mathrm{LL}(a)\big) \\
\mathrm{Outer}^a_{\mathrm{LL}}[\,\mathsf{Y}_b\varphi\,] &= \mathsf{X}_a\mathsf{Y}_b\big(\mathrm{Inner}^a_{\mathrm{LL}}[\,\varphi\,] \wedge \mathrm{LL}(a)\big) \\
\mathrm{Outer}^a_{\mathrm{RR}}[\,\mathsf{X}_b\varphi\,] &= \mathsf{Y}_a\mathsf{X}_b\big(\mathrm{Inner}^a_{\mathrm{RR}}[\,\varphi\,] \wedge \mathrm{RR}(a)\big) \\
\mathrm{Outer}^a_{\mathrm{RR}}[\,\mathsf{Y}_b\varphi\,] &= \mathsf{Y}_b\big(\mathrm{Inner}^a_{\mathrm{RR}}[\,\varphi\,] \wedge \mathrm{RR}(a)\big) \\
\mathrm{Outer}^a_{\mathrm{LR}}[\,\mathsf{X}_b\varphi\,] &= \left\{ \begin{array}{l}
\mathsf{X}_b\big(\mathrm{Inner}^a_{\mathrm{LR}}[\,\varphi\,] \wedge \mathrm{LP}(a)\big) \;\vee \\
\mathsf{X}_a\mathsf{Y}_b\mathsf{X}_b\big(\mathrm{Inner}^a_{\mathrm{LR}}[\,\varphi\,] \wedge \mathrm{LP}(a)\big) \;\vee \\
\big(\neg\mathsf{X}_b\big(\mathrm{LP}(a)\big) \wedge \neg\mathsf{X}_a\mathsf{Y}_b\mathsf{X}_b\big(\mathrm{LP}(a)\big) \wedge \\
\mathsf{X}_a\mathsf{X}_b\big(\mathrm{Inner}^a_{\mathrm{LR}}[\,\varphi\,] \wedge \mathrm{LR}(a)\big)\big)
\end{array} \right\} \\
\mathrm{Outer}^a_{\mathrm{LR}}[\,\mathsf{Y}_b\varphi\,] &= \mathsf{Y}_b\big(\mathrm{Inner}^a_{\mathrm{LR}}[\,\varphi\,] \wedge \mathrm{LR}(a)\big) \\
\mathrm{Outer}^a_{\mathrm{RL}}[\,\mathsf{X}_b\varphi\,] &= \mathsf{X}_b\big(\mathrm{Inner}^a_{\mathrm{RL}}[\,\varphi\,] \wedge \mathrm{RL}(a)\big) \\
\mathrm{Outer}^a_{\mathrm{RL}}[\,\mathsf{Y}_b\varphi\,] &= \left\{ \begin{array}{l}
\mathsf{Y}_b\big(\mathrm{Inner}^a_{\mathrm{RL}}[\,\varphi\,] \wedge \mathrm{RP}(a)\big) \;\vee \\
\mathsf{Y}_a\mathsf{X}_b\mathsf{Y}_b\big(\mathrm{Inner}^a_{\mathrm{RL}}[\,\varphi\,] \wedge \mathrm{RP}(a)\big) \;\vee \\
\big(\neg\mathsf{Y}_b\big(\mathrm{RP}(a)\big) \wedge \neg\mathsf{Y}_a\mathsf{X}_b\mathsf{Y}_b\big(\mathrm{RP}(a)\big) \wedge \\
\mathsf{Y}_a\mathsf{Y}_b\big(\mathrm{Inner}^a_{\mathrm{RL}}[\,\varphi\,] \wedge \mathrm{RL}(a)\big)\big)
\end{array} \right\}.
\end{aligned}$$

For example, when transforming $\mathsf{X}_b\varphi$ by $\mathrm{Outer}^a_{\mathrm{LR}}$ we have to ensure that we reach the first position that is labeled by b and that is not before the first a. This is done by a case distinction. The first case is that the first b is parallel to the first a. The next case is that there is a b before the first a and the first b that is not before the first a is parallel to the first a. The third case is that there does not exist a b that is parallel to the first a. Therefore, the first b that is not before the first a is after this first a. All cases are disjoint.

Using the above transformations, we inductively define formulae $\varphi(A,k,t)$ in $\mathrm{TL}[\mathsf{X}_a,\mathsf{Y}_a]$ expressing $[t]_{\equiv_{A,k}}$. For $k=0$ the formula is $B^*(A)$ or its negation. For $k>0$ we have $\varphi(A,k,t)=\neg B^*(A)$ or

$$\begin{aligned}\varphi(A,k,t) \;=\; & B(\mathrm{alph}(t)) \;\wedge \\ & \bigwedge_{\substack{t=t_-at_+ \text{ is} \\ \text{left factorization}}} \left\{ \begin{array}{l} \mathrm{Outer}^a_{\mathrm{LL}}\big[\,\varphi(A\setminus\{a\},k-1,t_-)\,\big] \;\wedge \\ \mathrm{Outer}^a_{\mathrm{LR}}\big[\,\varphi(A,k-1,t_+)\,\big] \end{array} \right\} \wedge \\ & \bigwedge_{\substack{t=t_-at_+ \text{ is} \\ \text{right factorization}}} \left\{ \begin{array}{l} \mathrm{Outer}^a_{\mathrm{RL}}\big[\,\varphi(A,k-1,t_-)\,\big] \;\wedge \\ \mathrm{Outer}^a_{\mathrm{RR}}\big[\,\varphi(A\setminus\{a\},k-1,t_+)\,\big] \end{array} \right\}.\end{aligned}$$

□

In the next theorem we summarize the characterizations of this chapter of trace languages whose syntactic monoid is in the variety **DA**.

Theorem 4.5 *Let $L\subseteq\mathbb{M}(\Sigma,I)$. Then the following are equivalent:*

(a) $M(L)\in\mathbf{DA}$.

(b) $L\in\mathrm{UPol}\mathcal{A}$.

(c) $L\in\mathrm{Pol}\mathcal{A}$ *and* $\overline{L}\in\mathrm{Pol}\mathcal{A}$.

(d) $L\in\mathrm{TL}[\mathsf{XF},\mathsf{YP}]$.

(e) $L\in\mathrm{TL}[\mathsf{XF},\mathsf{YP},\mathsf{M}]$.

(f) $L\in\mathrm{TL}[\mathsf{X}_a,\mathsf{Y}_a]$.

Proof: The equivalence of *(a)*, *(b)* and *(c)* is Corollary 3.17. The direction "*(a)* ⇒ *(d)*" follows from Lemma 3.15 and Lemma 4.3. Since $\mathrm{TL}[\mathsf{XF},\mathsf{YP}]\subseteq\mathrm{TL}[\mathsf{XF},\mathsf{YP},\mathsf{M}]$ we have "*(d)* ⇒ *(e)*". The implication "*(e)* ⇒ *(a)*" is Lemma 4.1. The direction "*(a)* ⇒ *(f)*" follows from Lemma 3.15 and Lemma 4.4 and the implication "*(f)* ⇒ *(a)*" is Lemma 4.2. □

4.1.2 Games for Temporal Logic

In [EW96] an EF game is defined for linear temporal logic for words with three operators: until, eventually and next. We will adapt this game in order to characterize simple fragments of temporal logic for traces.

Definition 4.6 *(EF game for* $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}, \mathsf{Eco}]$*)* *The EF game for the fragment* $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}, \mathsf{Eco}]$ *with n rounds is played on two traces $t_0, t_1 \in \mathbb{M}(\Sigma, I)$ using one pebble per trace. The two players of the game are called Spoiler and Duplicator. A configuration of the game is a pair of positions $(\nu_0, \nu_1) \in X_{t_0} \times X_{t_1}$ currently occupied by the pebbles. In each round, Spoiler selects a side $\sigma \in \{0, 1\}$ and one of the moves* XF*,* YP *and* Eco*.*

XF*: The pebble is moved from ν to a position χ such that $\chi > \nu$.*
YP*: The pebble is moved from ν to a position χ such that $\chi < \nu$.*
Eco*: The pebble is moved from ν to a position χ such that $\chi \parallel \nu$.*

First, Spoiler moves the pebble on t_σ and then, Duplicator carries out the same type of move on $t_{1-\sigma}$. If no other starting configuration is indicated we assume that initially, both pebbles are placed beside the board at an unlabeled position we will refer to as $\iota \notin X_{t_0} \cup X_{t_1}$. Any move starting from ι consist in placing the pebble on an arbitrary position of the respective trace.

Spoiler wins if Duplicator cannot move his pebble to the indicated direction or if, initially or after the move, the pebbles lie on differently labeled nodes. Duplicator wins if this never occurs.

The *depth* of a formula φ in temporal logic is the maximal number of nested temporal operators occurring within φ (cf. [EVW02]).

Lemma 4.7 *The following propositions are equivalent:*

(a) *Duplicator has a winning strategy for the game from Definition 4.6 with $n \in \mathbb{N}$ rounds played on the traces $t, s \in \mathbb{M}(\Sigma, I)$.*

(b) *The traces t and s are models of exactly the same* $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}, \mathsf{Eco}]$ *formulae with a maximal operator depth of n.*

Proof: We first show that t, ν_t and s, ν_s are models of the same n-depth formulae if and only if Duplicator has a winning strategy for the n-round game starting at the configuration $(\nu_t, \nu_s) \in X_t \times X_s$. For $n = 0$ this is true, because at two positions the same 0-depth formulae hold if and only if they are equally labeled. Let $n > 0$.

Let $\mathsf{XF}\varphi$ be an n-depth formula such that (without loss of generality) $t, \nu_t \models \mathsf{XF}\varphi$, whereas $s, \nu_s \not\models \mathsf{XF}\varphi$. In the game with n rounds starting with the configuration (ν_t, ν_s), Spoiler can select the XF move and a position $\chi_t > \nu_t$ with $t, \chi_t \models \psi$, as opposed to Duplicator, who will find no analogous position $\chi_s > \nu_s$. By induction, Spoiler wins the game. The other temporal operators are analogous.

Now suppose that Spoiler wins the game starting with the configuration (ν_t, ν_s) within n rounds. If the positions have different labels, then with $\varphi = \text{label}(\nu_t)$ it follows that $t, \nu_t \models \varphi$ and $s, \nu_s \not\models \varphi$. Otherwise Spoiler does his first move. Without loss of generality let this first move be XF on t. Spoiler moves his pebble to χ_t. By induction, for every position $\chi \in X_s$ with $\chi > \nu_s$ there exists an $(n-1)$-depth formula φ_χ such that $t, \chi_t \models \varphi_\chi$ and $s, \chi \not\models \varphi_\chi$. Let $\varphi = \bigwedge_{\chi > \nu_s} \varphi_\chi$. Then by construction $t, \nu_t \models \mathsf{XF}\varphi$ and $s, \nu_s \not\models \mathsf{XF}\varphi$.

Let $t, \iota \models \varphi$ be equivalent to $t \models \varphi$. Now, the case that the game starts at (ι, ι) works similarly. □

By omitting the Eco operator we obtain:

Corollary 4.8 *The following propositions are equivalent:*

(a) *Duplicator has a winning strategy for the EF game on the traces t, s for* TL *with $n \in \mathbb{N}$ rounds and in which the move corresponding to* Eco *is not allowed.*

(b) *The traces t and s are models of exactly the same* TL[XF, YP] *formulae with a maximal operator depth of n.*

4.2 First Order Logic

In *first order* logic, allowed logical formulae contain only quantifiers $\exists$ and $\forall$ that bind variables representing single elements. First-order logic for traces

is called FO[<] when it is applied to the representation of a trace as $t = (X, \text{label}, <)$, and FO[$\rightarrowtriangle$] if the predicate $\rightarrowtriangle$ is used instead. In addition to the binary predicates $=$ for equality and $<$ or $\rightarrowtriangle$, the formulae may contain the unary predicates $a(\cdot)$ for $a \in \Sigma$ that hold at all positions labeled with that letter.

Let a first-order formula φ contain the free variables $\text{free}(\varphi)$. Then, it can only be assigned a truth value on a trace t when an interpretation $g : \text{free}(\varphi) \to X_t$ of these variables is indicated. Whenever $\text{free}(\varphi)$ is $\{ x_1, \ldots, x_n \}$, interpretations will be denoted by sequences $w \in X_t^n$. We write $(t, w) \models \varphi$ if φ using the interpretation w is true on t. We will discuss the following fragments of first-order logic for traces:

- $\text{FO}^m[<]$ contains all properties that can be expressed with m variables.
- $\Sigma_n^m[<]$ consists of the properties expressible by $\Phi(n, m)$ formulae without free variables. Intuitively, n describes the number of quantifier blocks and m the number of nested variables. More formally: the formulae without quantifiers constitute $\Phi(0, 0)$. A formula φ with free variables $x_1, \ldots, x_j$ is in $\Phi(n, m)$ if and only if for some $k \in \mathbb{N}$ it can be written as $\bigvee_{1 \le i \le k} \exists x_{j+1} \cdots \exists x_{j+\ell(i)} \neg \psi_i$ with $\ell(i) \in \mathbb{N}$, $\psi_i \in \Phi(n-1, m-\ell(i))$ for all $1 \le i \le k$.
- $\Pi_n^m[<]$ contains all properties expressible by a formula $\neg\varphi$ with $\varphi \in \Phi(n, m)$.

  We also write $\Sigma_n[<] = \bigcup_{i \in \mathbb{N}} \Sigma_n^i[<]$ and $\Pi_n[<] = \bigcup_{i \in \mathbb{N}} \Pi_n^i[<]$.
- $\Delta_n[<]$ is defined as the intersection $\Sigma_n[<] \cap \Pi_n[<]$.

By applying the same restrictions on FO[$\rightarrowtriangle$], analogous logical fragments such as $\text{FO}^n[\rightarrowtriangle]$ and $\Sigma_n[\rightarrowtriangle]$ are obtained. Particular attention will be paid to the special cases with $n = 2$.

4.2.1 Characterizations with Temporal Logic

In [EVW02] it is proven that over words, LTL with the operators XF and YP is equally expressive as $\text{FO}^2[<]$. The proof from that paper can be adapted in order to show analogous results for FO^2 over traces. In contrast to words, in the case of $\text{FO}^2[<]$ two positions might be parallel. This case can be covered using the Eco operator.

Lemma 4.9 $\text{FO}^2[<] \subseteq \text{TL}[\mathsf{XF}, \mathsf{YP}, \mathsf{Eco}]$.

Proof: Let $\varphi(x) \in \mathrm{FO}^2[<]$ be a formula with one free variable and without universal quantifiers. The only variables of φ are x and y. We show by induction on the quantifier depth and the size of the formula that there exists a formula $\widetilde{\varphi}$ in $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}, \mathsf{Eco}]$ such that

$$\forall \nu \in X_t \colon (t, \nu) \models \varphi(x) \;\Leftrightarrow\; t, \nu \models \widetilde{\varphi}.$$

If $\varphi(x) \equiv \top$ then we set $\widetilde{\varphi} \equiv a \vee \neg a$. Constant formulae will arise later in the construction. The atomic formula are translated as follows:

$$\begin{array}{lcl} \varphi(x) \equiv a(x) \text{ for } a \in \Sigma & \rightsquigarrow & \widetilde{\varphi} \equiv a \\ \varphi(x) \equiv x < x & \rightsquigarrow & \widetilde{\varphi} \equiv a \wedge \neg a \\ \varphi(x) \equiv x = x & \rightsquigarrow & \widetilde{\varphi} \equiv a \vee \neg a. \end{array}$$

Inductively, boolean operators are translated as follows:

$$\begin{array}{lcl} \varphi(x) \equiv \psi_1(x) \vee \psi_2(x) & \rightsquigarrow & \widetilde{\varphi} \equiv \widetilde{\psi_1} \vee \widetilde{\psi_2} \\ \varphi(x) \equiv \neg\psi(x) & \rightsquigarrow & \widetilde{\varphi} \equiv \neg\widetilde{\psi}. \end{array}$$

If $\varphi(x)$ is of the form $\exists x \colon \psi(x)$, in an intermediate step it is transformed into $\varphi'(x) \equiv \exists y \colon \psi(y)$ by interchanging x and y. If $\varphi(x) \equiv \exists y \colon \psi(y)$ then it can be interpreted as $\varphi(x) \equiv \exists y \colon \psi(x, y)$ where x is a dummy variable in ψ. We now consider the general case $\varphi(x) \equiv \exists y \colon \psi(x, y)$. First, we will transform $\varphi(x)$ into an equivalent formula $\varphi''(x)$ of the same quantifier depth. Let

$$\psi(x, y) \;\equiv\; \zeta\big(\, x = y, x < y, y < x,\, \xi_1(x), \ldots, \xi_n(x),\, \zeta_1(y), \ldots, \zeta_m(y) \,\big)$$

where ζ is a propositional formula and $\xi_i(x)$, $\zeta_j(y)$ are atomic formulae or existential formulae with smaller quantifier depth. The first step in the transformation of $\varphi(x)$ is to *guess* the values of $\xi_i(x)$ before the quantification of y. We use the constants $\top$ and $\bot$ for *true* and *false*, respectively. We set $\varphi'(x) \equiv$

$$\bigvee_{\overline{\gamma} \in \{\top, \bot\}^n} \Bigg(\bigwedge_{1 \le i \le n} (\xi_i(x) \leftrightarrow \gamma_i) \;\wedge\; \exists y \colon \zeta(x = y, x < y, y < x,\, \gamma_1, \ldots, \gamma_n,\, \zeta_1(y), \ldots, \zeta_m(y)\,) \Bigg).$$

The next step is to *guess* the relation τ that holds between x and y in advance. The possible relations are $x = y$, $x < y$, $x > y$ or none of them and then x

and y correspond to parallel positions, $x \parallel y$. Hence we choose τ from the set $\{=, <, >, \parallel\}$. We set $\varphi''(x) \equiv$

$$\bigvee_{\overline{\gamma} \in \{\top,\bot\}^n} \Big(\bigwedge_{1 \le i \le n} (\xi_i(x) \leftrightarrow \gamma_i) \wedge \bigvee_{\tau \in \{=,<,>,\parallel\}} \exists y \colon (x \tau y \wedge \zeta(y)) \Big)$$

with $\zeta(y) = \zeta((x = y)^\tau, (x < y)^\tau, (y < x)^\tau, \overline{\gamma}, \zeta_1(y), \ldots, \zeta_m(y))$. Note that the first $3 + n$ arguments are constant boolean values at this point. After this transformation of $\varphi(x)$ it remains to show how to translate formulae of the form $\exists y \colon (x \tau y \wedge \zeta(y))$ with $\tau \in \{=, <, >, \parallel\}$:

$$\begin{aligned}
\exists y \colon (x = y \wedge \zeta(y)) &\quad\rightsquigarrow\quad \widetilde{\zeta} \\
\exists y \colon (x < y \wedge \zeta(y)) &\quad\rightsquigarrow\quad \mathsf{XF}\widetilde{\zeta} \\
\exists y \colon (x > y \wedge \zeta(y)) &\quad\rightsquigarrow\quad \mathsf{YP}\widetilde{\zeta} \\
\exists y \colon (x \parallel y \wedge \zeta(y)) &\quad\rightsquigarrow\quad \mathsf{Eco}\widetilde{\zeta}.
\end{aligned}$$

Let now $\varphi \in \mathrm{FO}^2[<]$ be a formula without free variables. Next, we show how to construct an outer temporal formula $\widehat{\varphi} \in \mathrm{TL}[\mathsf{XF}, \mathsf{YP}, \mathsf{Eco}]$ such that

$$t \models \varphi \;\Leftrightarrow\; t \models \widehat{\varphi}.$$

For this we use the following transformation scheme:

$$\begin{aligned}
\varphi \equiv \psi_1 \vee \psi_2 &\quad\rightsquigarrow\quad \widehat{\varphi} \equiv \widehat{\psi_1} \vee \widehat{\psi_2} \\
\varphi \equiv \neg\psi &\quad\rightsquigarrow\quad \widehat{\varphi} \equiv \neg\widehat{\psi} \\
\varphi \equiv \exists x \colon \psi(x) &\quad\rightsquigarrow\quad \widehat{\varphi} \equiv \mathsf{XF}\widetilde{\psi}.
\end{aligned}$$

Formulæ of the form $\varphi \equiv \exists y \colon \psi(y)$ are in an intermediate step transformed into $\exists x \colon \psi(x)$ by interchanging x and y. □

Lemma 4.10 $\mathrm{FO}^2[\rightarrowtriangle] \subseteq \mathrm{TL}[\mathsf{XF}, \mathsf{YP}]$.

The proof is analogous to Lemma 4.9. The only difference is that we use the operator M if none of $x = y$, $x \rightarrowtriangle y$ or $y \rightarrowtriangle x$ holds. In this case, we additionally have to check that the labels of x and y are independent. Afterwards the M operator can be removed by Theorem 4.5.

4.2.2 Games for First Order Logic

Ehrenfeucht-Fraïssé (EF) games [Ehr61, Fra54] are often used to characterize logical fragments. For the boolean closure of $\Sigma_n[<]$, Thomas presented an EF game [Tho84] that can be modified in order to describe $\Sigma_n[<]$. The main difference consists of the fact that $\Sigma_n[<]$ is not closed under complementation. If we want to capture this fragment, it is therefore insufficient to determine whether two traces t and s are equivalent or not. Instead, we will ask if t models *at least the same* $\Sigma_n[<]$ properties as s does. By limiting the number of pebbles to $m \in \mathbb{N}$, an idea introduced in [IK87], it is possible to characterize the $\Sigma_n^m[<]$ fragments.

Definition 4.11 *(EF game for $\Sigma_n^m[<]$) The set of* configurations *for the EF game corresponding to the fragment $\Sigma_n^m[<]$ played on the traces t_0 and t_1 with position sets X_0 and X_1 is $X_0^* \times X_1^* \times \{0,1\}$ with the restriction that the size of the first two components is equal and does not exceed m. The first two components of the configuration are interpreted as a distribution of pebbles on the two traces: a pebble labeled with x_i lies at position $\nu \in X_j$ whenever ν is the i-th character of the word corresponding to $j \in \{0,1\}$. The third component contains the number of the trace where Spoiler will carry out his next move.*

Let (w_0, w_1, σ) with $|w_0| = |w_1| = i \leq m$ be the current configuration, then the next turn is carried out as follows:

- *Spoiler takes $j \leq m-i$ pebbles labeled with $x_{i+1}, \ldots, x_{i+j}$ and distributes them on trace t_σ by assigning a position $\nu_\sigma(k) \in X_\sigma$ to each x_{i+k}.*
- *Duplicator places identically labeled pebbles on nodes of the other trace, such that every x_{i+k} is assigned some $\nu_{1-\sigma}(k) \in X_{1-\sigma}$.*
- *The new configuration is $(w_0\nu_0(1)\cdots\nu_0(j), w_1\nu_1(1)\cdots\nu_1(j), 1-\sigma)$.*

The game for $\Sigma_n^m[<]$ consists of n rounds. Duplicator wins if and only if initially and after each of these rounds, the partial mapping $X_0 \to X_1 : w_0(k) \mapsto w_1(k)$ with $1 \leq k \leq |w_0| = |w_1|$ induces an isomorphism with respect to the labels and the relation $<$.

For $t_0, t_1 \in \mathbb{M}(\Sigma, I)$ we write $t_0 \preceq^{\Sigma[<]}_{(n,\,m)} t_1$ if and only if Duplicator has a winning strategy in the EF game for the logical fragment $\Sigma^m_n[<]$, played on t_0 and t_1 and starting with the initial configuration $(\varepsilon, \varepsilon, 0)$. This relation is extended to intermediate configurations of the game by replacing the initial configuration, i.e., $(t_0, w_0) \preceq^{\Sigma[<]}_{(n,\,m)} (t_1, w_1)$ whenever Duplicator has a winning strategy in the game on t_0 and t_1 starting with the configuration $(w_0, w_1, 0)$.

Lemma 4.12 *Let $n, m, j \in \mathbb{N}$ and $t, s \in \mathbb{M}(\Sigma, I)$ with sequences $w_t \in X_t^j$ and $w_s \in X_s^j$. Then $(t, w_t) \preceq^{\Sigma[<]}_{(n,\,m)} (s, w_s)$ holds if and only if for all $\varphi \in \Phi(n, m)$ with j free variables we have that $(t, w_t) \models \varphi$ implies $(s, v_s) \models \varphi$. In particular:*

$$t \preceq^{\Sigma[<]}_{(n,\,m)} s \;\Leftrightarrow\; \big(\forall L \in \Sigma^m_n[<]\colon t \in L \;\Rightarrow\; s \in L\big).$$

Proof: The lemma holds for $n = 0$, as without any rounds Duplicator wins the game if and only if in the initial configuration, the j pebbles are isomorphically distributed on both traces, which amounts to saying that the same $\Sigma^j_0[<]$ formulae hold for (t, w_t) and (s, w_s). Suppose $n > 0$. Let $w_t \in X_t^j$, $w_s \in X_s^j$ be interpretations such that the partial mapping $X_t \to X_s : w_t(i) \mapsto w_s(i)$, $1 \leq i \leq j$ induces an isomorphism with respect to the order relation and the label function.

($\Rightarrow$) Let $\varphi \in \Phi(n, m)$ be a formula with the free variables $x_1, \ldots, x_j$. Without loss of generality, we assume that $\exists$ (and not $\vee$) is the outermost junctor, i.e., $\varphi = \exists x_{j+1} \cdots \exists x_{j+\ell}\, \neg\psi$ with $\ell \leq m$ and $\psi \in \Phi(n-1, m-\ell)$. Suppose that Duplicator has a winning strategy in the EF game with n rounds starting from the configuration $(w_t, w_s, 0)$. Let (w_t, t) be a model of φ. Now let Spoiler distribute ℓ pebbles on positions $\nu_1, \ldots, \nu_\ell$ of X_t such that $\neg\psi$ holds on t with the interpretation $v_t = w_t \nu_1 \nu_2 \cdots \nu_\ell$. If Duplicator proceeds according to his winning strategy, he obtains positions χ_i for all $1 \leq i \leq \ell$. We set $v_s = w_s \chi_1 \chi_2 \cdots \chi_\ell$. We have

$$(s, v_s) \preceq^{\Sigma[<]}_{(n-1,\,m-\ell)} (t, v_t).$$

By induction hypothesis the implication $(s, v_s) \models \psi \;\Rightarrow\; (t, v_t) \models \psi$ holds for all $\psi \in \Phi(n-1, m-\ell)$. Hence, from $(t, v_t) \models \neg\psi$ we can conclude $(s, v_s) \models \neg\psi$ and therefore $(s, w_s) \models \varphi$.

($\Leftarrow$) Assume that Spoiler has a winning strategy starting from the configuration $(w_t, w_s, 0)$. Let his first move according to this strategy consist in

placing $\ell \leq m$ pebbles on t. Let v_t be the new configuration on t. Now, after every possible response of Duplicator, Spoiler has a winning strategy with at most $n-1$ rounds starting on trace s. By induction, for each $v \in w_s X_s^\ell$ there exists a formula $\psi_v \in \Phi(n, m-\ell)$ such that $(s, v) \models \psi_v$ and $(t, v_t) \not\models \psi_v$. Since the range of possible values for v is finite, we can construct $\psi^* = \bigvee_v \psi_v$, which in turn is a $\Phi(n, m-\ell)$ formula. By construction it follows that

$$(t, w_t) \models \exists x_1 \cdots \exists x_\ell \, \neg\psi^* \quad \text{whereas} \quad (s, w_s) \not\models \exists x_1 \cdots \exists x_\ell \, \neg\psi^*.$$

The lemma now follows by contraposition. □

Corollary 4.13 *By replacing $<$ with $\rightarrowtriangle$ we obtain relations $\preceq^{\Sigma[\rightarrowtriangle]}_{(n,m)}$ and EF games for $\Sigma^m_n[\rightarrowtriangle]$. An analogous proof shows that for the game $\Sigma^m_n[\rightarrowtriangle]$ starting with $(w_t, w_s, 0)$, Duplicator has a winning strategy if and only if $(t, w_t) \preceq^{\Sigma[\rightarrowtriangle]}_{(n,j)} (s, w_s)$ holds.*

4.2.3 Characterization with Polynomials

Lemma 4.14 $\mathrm{Pol}\mathcal{A} \subseteq \Sigma_2[\rightarrowtriangle]$.

Proof: It is easy to see that languages of the form A^* for $A \subseteq \Sigma$ are in $\Sigma_2[\rightarrowtriangle]$. Since $\mathrm{Pol}\mathcal{A}$ and $\Sigma_2[\rightarrowtriangle]$ are both closed under union, it suffices to show that $L_0 a L_1 \in \Sigma_2[\rightarrowtriangle]$ if L_0 and L_1 are in $\Sigma_2[\rightarrowtriangle]$. Suppose L_0 and L_1 are expressed by the $\Sigma_2[\rightarrowtriangle]$ formulae φ_0 and φ_1, respectively. We will construct a $\Sigma_2[\rightarrowtriangle]$ formula φ expressing $L = L_0 a L_1$. We have $t \in L$ if and only if there exists a factorization $t = t_0 a t_1$ with $t_0 \in L_0$ and $t_1 \in L_1$. Our aim is to check whether there exists such a factorization. This will be done by determining for each letter in t_0 (resp. t_1) its last occurrence (resp. its first occurrence). We will then use these positions to restrict φ_0 (resp. φ_1) to the left (resp. right) factor by means of the relation $\rightarrowtriangle$. We introduce some macro formulae:

$$\begin{aligned}
\mathrm{label}(x) = \mathrm{label}(y) \quad &\equiv \quad \bigwedge_{a \in \Sigma} a(x) \leftrightarrow a(y) \\
\mathrm{path}(x_1, \ldots, x_n) \quad &\equiv \quad \bigwedge_{1 \leq i < n} (x_i = x_{i+1} \vee x_i \rightarrowtriangle x_{i+1}) \\
x \parallel y \quad &\equiv \quad \forall x_1 \ldots \forall x_{|\Sigma|} : \\
&\quad \big(\neg\mathrm{path}(x_1, \ldots, x_{|\Sigma|}) \vee x \neq x_1 \vee y \neq x_{|\Sigma|}\big) \wedge \\
&\quad \big(\neg\mathrm{path}(x_1, \ldots, x_{|\Sigma|}) \vee x \neq x_{|\Sigma|} \vee y \neq x_1\big)
\end{aligned}$$

$$\begin{aligned} x < y \quad &\equiv \quad x \neq y \wedge \\ &\qquad \exists x_2 \ldots \exists x_{|\Sigma|-1} \colon \mathrm{path}(x, x_2, \ldots, x_{|\Sigma|-1}, y). \end{aligned}$$

By Lemma 3.1 it follows that this definition of $<$ is equivalent with the transitive closure of $\rightarrowtriangle$. For alphabets $A_0 = \{ a_1, \ldots, a_n \}$ and $A_1 = \{ b_1, \ldots, b_m \}$ we define a $\Sigma_2[\rightarrowtriangle]$ formula $\psi_{A_0,a,A_1}(x_1, \ldots, x_n, z_1, y_1, \ldots, y_m)$ with $n + 1 + m$ free variables. Consider a factorization $t = t_0 a t_1$. The formula ψ_{A_0,a,A_1} is true if each variable x_i, $1 \leq i \leq n$ is interpreted at the last occurrence of the letter a_i in t_0 and the alphabet of t_0 is A_0. The last m variables y_i represent the first positions of letters of t_1 whose alphabet is A_1 and z_1 corresponds to the position of a in this factorization. We set $\psi_{A_0,a,A_1}(x_1, \ldots, x_n, z_1, y_1, \ldots, y_m) =$

$$\begin{aligned} &\bigwedge_{1 \leq i \leq n} a_i(x_i) \ \wedge \ a(z_1) \ \wedge \ \bigwedge_{1 \leq i \leq m} b_i(y_i) \ \wedge \\ &\bigwedge_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} \left\{ \begin{array}{l} (x_i \parallel z_1 \ \vee \ x_i < z_1) \ \wedge \\ (x_i \parallel y_j \ \vee \ x_i < y_j) \ \wedge \\ (z_1 \parallel y_j \ \vee \ z_1 < y_j) \end{array} \right\} \ \wedge \\ &\forall z \colon \left\{ \begin{array}{l} z = z_1 \ \vee \\ \displaystyle\bigvee_{1 \leq i \leq n} \big(\mathrm{label}(z) = \mathrm{label}(x_i) \ \wedge \ (z = x_i \ \vee \ z \rightarrowtriangle x_i) \big) \ \vee \\ \displaystyle\bigvee_{1 \leq j \leq m} \big(\mathrm{label}(z) = \mathrm{label}(y_j) \ \wedge \ (y_j = z \ \vee \ y_j \rightarrowtriangle z) \big) \end{array} \right\}. \end{aligned}$$

Let $x_1, \ldots, x_n$ be variables. The restriction of $\varphi_0 = \exists \overline{y} \forall \overline{z} \, \psi_0(\overline{y}, \overline{z})$ to the past of $x_1, \ldots, x_n$ is

$$\begin{aligned} \varphi_0^- \ = \exists y_1 \ldots \exists y_m \forall z_1 \ldots \forall z_\ell \colon & \\ \bigwedge_{1 \leq j \leq m} \left(\bigvee_{1 \leq i \leq n} (y_j = x_i \ \vee \ y_j \rightarrowtriangle x_i) \right) \ \wedge & \\ \left(\bigvee_{1 \leq k \leq \ell} \left(\bigwedge_{1 \leq i \leq n} \neg \, (z_k = x_i \ \vee \ z_j \rightarrowtriangle x_i) \right) \ \vee \ \psi_0(\overline{y}, \overline{z}) \right). & \end{aligned}$$

Similarly, we can define the restriction φ_1^+ of φ_1 to the future of $y_1, \ldots, y_m$. Using these restrictions, we define

$$\varphi = \bigvee_{A_0, A_1 \subseteq \Sigma} \exists x_1 \ldots \exists x_{|A_0|} \exists z_1 \exists y_1 \ldots \exists y_{|A_1|} \colon \big(\psi_{A_0,a,A_1} \ \wedge \ \varphi_0^- \ \wedge \ \varphi_1^+ \big).$$

Note that φ is a $\Sigma_2[\rightarrowtriangle]$ formula. It expresses the language L. $\square$

Lemma 4.15 $\Sigma_2[\rightarrow\!\triangleright] \subseteq \mathrm{Pol}\mathcal{A}$.

Proof: Let $L \in \Sigma_2[\rightarrow\!\triangleright]$. By Corollary 3.10 it suffices to show that the syntactic ordered monoid of L is in $[\![\, x^\omega y x^\omega \leq x^\omega \,]\!]\, \textcircled{\scriptsize M}\, \mathbf{J_1}$. Therefore, we show that for each $t \in \mathbb{M}(\Sigma, I)$ there is an $n' \in \mathbb{N}$ such that, for all $n \geq n'$ and $s \in \mathbb{M}(\Sigma, I)$ with $\mathrm{alph}(t) = \mathrm{alph}(s)$, the following implication holds:

$$\forall r, p \in \mathbb{M}(\Sigma, I)\colon rt^n p \in L \;\Rightarrow\; rt^n s t^n p \in L.$$

Let $m \in \mathbb{N}$ with $L \in \Sigma_2^m[\rightarrow\!\triangleright]$. Consider the EF game for this fragment, played on the traces $q = rt^n p$ and $u = rt^n s t^n p$ with $n \geq (m+1)^2$ starting from the initial configuration $(\varepsilon, \varepsilon, 0)$. The interior sections of both traces, i.e., t^n on q and $t^n s t^n$ on u, are composed of factors t and s. We refer to the n factors t of q as blocks 1 to n.

In the first round, Spoiler places up to m pebbles on positions of q. Due to the large number of blocks in comparison to the number of pebbles, he must necessarily leave a big gap: there is an $\ell < n-m$ such that on trace q, Spoiler places no pebbles on any of the blocks $\ell+1, \ldots, \ell+m$. We decompose q into three factors

$$q \;=\; rt^\ell \cdot t^m \cdot t^{n-\ell-m} p.$$

All pebbles lie on positions of the left and of the right factor. We analogously factorize u:

$$u \;=\; rt^\ell \cdot t^{n-\ell} s t^{\ell+m} \cdot t^{n-\ell-m} p.$$

Duplicator responds by copying the distribution of the pebbles on the two identical outer factors.

In the second round, Spoiler has to distribute his remaining pebbles on u. Duplicator can carry over the distribution of the new pebbles on the outer factors of u to the corresponding positions of the outer factors of q. It remains to show how Duplicator can answer the j pebbles, $j \leq m$, on the intermediate factor $t^{n-\ell} s t^{\ell+m}$ of u. Let $\nu_1, \nu_2, \ldots, \nu_j \in X_u$ be a linearization of the positions with pebbles on this intermediate factor such that $\nu_i \rightarrow\!\triangleright\, \nu_k$ implies $i < k$. Now, Duplicator uses the gap of size m beginning at block $\ell+1$ of q and the assumption that $\mathrm{alph}(t) = \mathrm{alph}(s)$. For each position ν_i, $1 \leq i \leq j$, Duplicator places a pebble of the same type on an arbitrary position χ_i of block $\ell + i$ in trace q such that $\mathrm{label}(\nu_i) = \mathrm{label}(\chi_i)$. By construction, we have $\nu_i \rightarrow\!\triangleright\, \nu_k$ if and only if $\chi_i \rightarrow\!\triangleright\, \chi_k$. Hence, Duplicator has a winning strategy for this EF game. Therefore,

$$rt^n p \;\preceq^{\Sigma[\rightarrow\!\triangleright]}_{(2,\,m)}\; rt^n s t^n p$$

holds due to Corollary 4.13, which shows that $rt^n p \in L$ implies $rt^n s t^n p \in L$. □

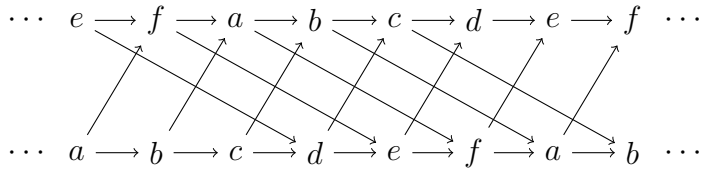


Figure 4.1: Hasse diagram of trace r (see proof of Lemma 4.17)

4.3 Comparison of Logical Fragments

Lemma 4.16 *It is not expressible in* TL[XF, YP] *whether two actions occur in parallel, whereas this property is expressible in* $\Sigma_1^2[<]$.

Proof: Let $\mathbb{M}(\Sigma, I)$ be the trace monoid given by the dependence relation $D = a - b - c$. We formulate the property $L =$ 'the letters a and c occur in parallel' that coincides with the semantics of the $\Sigma_1^2[<]$ formula $\exists x \exists y \colon x \parallel y$. Let $t = abc$ and $s = b$. Then for all $n > 0$ we have $(ts)^n \in L$ whereas $(ts)^n t (ts)^n \notin L$ since in the factor tt the letters a and c occur in parallel. Hence $M(L) \notin \mathbf{DA}$ and by Theorem 4.5 we conclude that $L \notin$ TL[XF, YP]. □

Lemma 4.17 *It is not expressible in* TL[XF, YP, Eco] *whether three actions occur in parallel, whereas this property is expressible in* $\Sigma_1^3[<]$.

Proof: Let $\mathbb{M}(\Sigma, I)$ be the trace monoid with $\Sigma = \{a, b, c, d, e, f, \#\}$ such that among the letters $\{a, b, c, d, e, f\}$ we have the circular dependencies $a - b - c - d - e - f - a$ and all these letters are dependent of $\#$. The maximal sets of independent letters are $\{a, c, e\}$ and $\{b, d, f\}$. In $\Sigma_1^3[<]$, we can express whether three actions occur in parallel by the formula

$$\varphi \;=\; \exists x \exists y \exists z \colon\; (x \parallel y \wedge y \parallel z \wedge z \parallel x)\,.$$

Now, we show that $L(\varphi)$ is not in TL[XF, YP, Eco]. Let $q = acbdcedfeafb$. For an arbitrarily chosen $n \in \mathbb{N}$ consider the traces $r = q^{2n+1}$ and $p = acbedf$. We combine them in order to build the larger traces $t = (\#r)^{2n+1} \not\models \varphi$ and $s = (\#r)^n \#p (\#r)^n \models \varphi$, see Figures 4.1 and 4.2. We say that r and p are *segments* of t and s. We numerate the segments of t and s with numbers running from $-n$ to n from left to right. All segments of t and s consist of r,

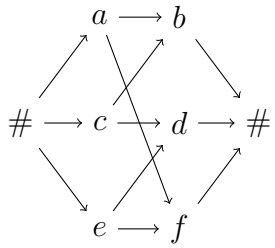


Figure 4.2: Hasse diagram of trace $\#p\#$ (see proof of Lemma 4.17)

except for segment 0 of trace s, which is p. Every segment that corresponds to r is further subdivided into $2n+1$ *blocks* with numbers running from $-n$ to n such that all blocks consist of q.

Consider the n-round EF game for $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}, \mathsf{Eco}]$ played on the traces t and s. In all blocks, except the outermost ones $-n$ and n, every position has parallel occurrences of all letters that are independent of their own labels. Although there do not exist three parallel positions in t, we will use this fact to mimic three parallel positions in order to construct a winning strategy for Duplicator.

The main strategy of Duplicator is to copy all moves of Spoiler. Since only the behavior on the segments 0 of t and s is not evident, we will describe the strategy of Duplicator for this case. Whenever Spoiler accesses segment 0 of trace t, Duplicator avoids placing the pebble on segment 0 of trace s by putting his pebble to the corresponding position of either segment -1 or 1 of trace s. In the up to $n-1$ remaining rounds it is possible for Duplicator to keep a maximal difference of 1 between the numbers of the segments. If Spoiler moves his pebble on segment 0 of trace s, Duplicator responds by placing the other pebble on an identically labeled position of block 0 of either segment -1, 0 or 1 of trace s. From this position he can mimic all possible Eco-moves of Spoiler on s. Each Eco-move of Spoiler could force Duplicator to increase or decrease the block number by 1. Since there occur at most $n-1$ of these moves and there are n blocks to the left as well as to the right, Duplicator wins the game. It follows that $L(\varphi) \notin \mathrm{TL}[\mathsf{XF}, \mathsf{YP}, \mathsf{Eco}]$. □

Lemma 4.18 *For all $n \in \mathbb{N}$, we have* $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}, \mathsf{Eco}] \not\subseteq \Sigma_n[<]$.

Proof: Let $m \in \mathbb{N}$ be arbitrary, and let a series of trace monoids $\mathbb{M}(\Gamma_i, I_i,)$, $i \geq 1$, be given inductively by the alphabets $\Gamma_1 = \{a_1\}$ and $\Gamma_i = \Gamma_{i-1} \cup \{ a_i, b_i \}$ with the dependence graph $D_i = D_{i-1} - b_i - a_i$ or, more formally, $D_1 = \{(a_1, a_1)\}$ and $D_i = D_{i-1} \cup \Gamma_{i-1} \times \{b_i\} \cup \{b_i\} \times \Gamma_{i-1} \cup \{ a_i, b_i \}^2$. This means that each Γ_i with $i \geq 2$ introduces a letter a_i, which is independent of all preceding letters, and a letter b_i that depends on all letters. For $i \in \mathbb{N}$, let $\ell_i = (m+1)^i$. By induction we also define the traces $t_1 = \varepsilon$, $s_1 = a_1$, and

$$\begin{aligned} t_i &= (b_i a_i s_{i-1})^{\ell_i} \\ s_i &= (b_i a_i s_{i-1})^{\ell_i} \cdot b_i a_i t_{i-1} \cdot (b_i a_i s_{i-1})^{\ell_i} \end{aligned}$$

such that $t_i, s_i \in \mathbb{M}(\Gamma_i, I_i,)$. The b_i's partition these traces into *blocks*. We number these blocks from 1 to ℓ_i on t_i and from $-\ell_i$ to ℓ_i on s_i. Within each block, the position labeled with a_i is parallel to all other positions.

We define the formulae $\varphi_1 = a_1$, $\psi_i = \neg \mathsf{Eco} \varphi_i$ and $\varphi_i = a_i \wedge \psi_{i-1}$. By induction, φ_i holds at the position of a_i in $b_i a_i t_{i-1}$ but not in $b_i a_i s_{i-1}$. It follows that s_i, which contains a factor t_{i-1}, does not model ψ_i, as opposed to t_i, which models ψ_i. Hence for all $n \in \mathbb{N}$, there is a TL[$\mathsf{XF}, \mathsf{YP}, \mathsf{Eco}$] property modeled by t_n but not by s_n.

Now consider the EF game for $\Sigma_n^m[<]$ played on the traces t_n and s_n, i.e., there are n rounds, m pebbles, and Spoiler places his first pebbles on t_n. Using induction, we describe a winning strategy of Duplicator for this game. The case for $n = 1$ is trivial: Spoiler cannot place any pebbles and Duplicator responds by doing the same. Assume $n > 1$, then in the first round, Spoiler places $m' \leq m$ pebbles on t_n. Because this trace consists of ℓ_n blocks, there must remain a big continuous gap of $\ell' = (m+1)^{n-1}$ blocks without any pebbles. Let such a gap start after the k-th block and consider the following factorizations of t_n and s_n:

$$\begin{aligned} t_n &= r^k \cdot r^{\ell'} \cdot r^{\ell_n - k - \ell'} \\ s_n &= r^k \cdot r^{\ell_n - k}\, p\, r^{k+\ell'} \cdot r^{\ell_n - k - \ell'} \end{aligned}$$

where $r = b_n a_n s_{n-1}$ and $p = b_n a_n t_{n-1}$. Duplicator can react by placing corresponding pebbles on the respective positions of the left and right factor of s_n. On both traces, the factor in the middle contains no pebbles. For the remaining rounds, we can ignore the outermost factors of both traces because they are identical. By induction we know

$$t_{n-1} \;\; \preceq^{\Sigma[<]}_{(n-1,\, m)} \;\; s_{n-1},$$

i.e., for the rest of the game, the blocks r and p cannot be distinguished. Both middle factors consist of at least $(m+1)^{n-1}$ blocks, and there are $n-1$ rounds to play. This allows Duplicator to win the game. We conclude that

$$t_n \preceq^{\Sigma[<]}_{(n,m)} s_n.$$

Hence for all $n \in \mathbb{N}$, no $\Sigma^m_n[<]$ property is modeled by t_n but not by s_n. As this holds for all $m \in \mathbb{N}$, it follows that $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}, \mathsf{Eco}]$ is not a subset of $\Sigma_n[<]$. □

Theorem 4.19 *We have the following relations:*

(a) $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}] = \mathrm{FO}^2[\rightarrowtriangle] = \Delta_2[\rightarrowtriangle]$.

(b) $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}, \mathsf{Eco}] = \mathrm{FO}^2[<]$.

(c) $\mathrm{FO}^2[\rightarrowtriangle] \subsetneq \mathrm{FO}^2[<]$.

(d) $\Delta_2[\rightarrowtriangle] \subsetneq \Delta_2[<]$.

(e) *For all $n \geq 2$, the fragments $\Delta_n[<]$ and $\mathrm{FO}^2[<]$ are not comparable.*

(f) $\mathrm{Pol}\mathcal{A} = \Sigma_2[\rightarrowtriangle] \subsetneq \Sigma_2[<]$.

(g) $\mathrm{coPol}\mathcal{A} = \Pi_2[\rightarrowtriangle] \subsetneq \Pi_2[<]$.

Proof: The relation $\mathrm{FO}^2[\rightarrowtriangle] \subseteq \mathrm{TL}[\mathsf{XF}, \mathsf{YP}]$ is Lemma 4.10 and $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}] \subseteq \mathrm{FO}^2[\rightarrowtriangle]$ follows since the definition of the temporal operators can be seen as macros using only two variables. By Lemma 3.1, the relation $<$ in those macros can be simulated by $\rightarrowtriangle$.

The relation $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}, \mathsf{Eco}] \subseteq \mathrm{FO}^2[<]$ follows similarly and Lemma 4.9 is $\mathrm{FO}^2[<] \subseteq \mathrm{TL}[\mathsf{XF}, \mathsf{YP}, \mathsf{Eco}]$. The relations $\Sigma_2[\rightarrowtriangle] = \mathrm{Pol}\mathcal{A}$ and $\Pi_2[\rightarrowtriangle] = \mathrm{coPol}\mathcal{A}$ follow from Lemma 4.14 and Lemma 4.15, the latter one by complementation. Now, $\mathrm{Pol}\mathcal{A} \cap \mathrm{coPol}\mathcal{A} = \mathrm{TL}[\mathsf{XF}, \mathsf{YP}]$ implies $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}] = \Delta_2[\rightarrowtriangle]$.

The strictness of the inclusions $\mathrm{FO}^2[\rightarrowtriangle] \subsetneq \mathrm{FO}^2[<]$ and $\Delta_2[\rightarrowtriangle] \subsetneq \Delta_2[<]$ and $\Sigma_2[\rightarrowtriangle] \subsetneq \Sigma_2[<]$ and $\Pi_2[\rightarrowtriangle] \subsetneq \Pi_2[<]$ all follow from Lemma 4.16. The fact, that for all $n \geq 2$, neither $\Delta_n[<] \subseteq \mathrm{FO}^2[<]$ nor $\mathrm{FO}^2[<] \subseteq \Delta_n[<]$ holds follows from Lemma 4.17 and Lemma 4.18, respectively. □

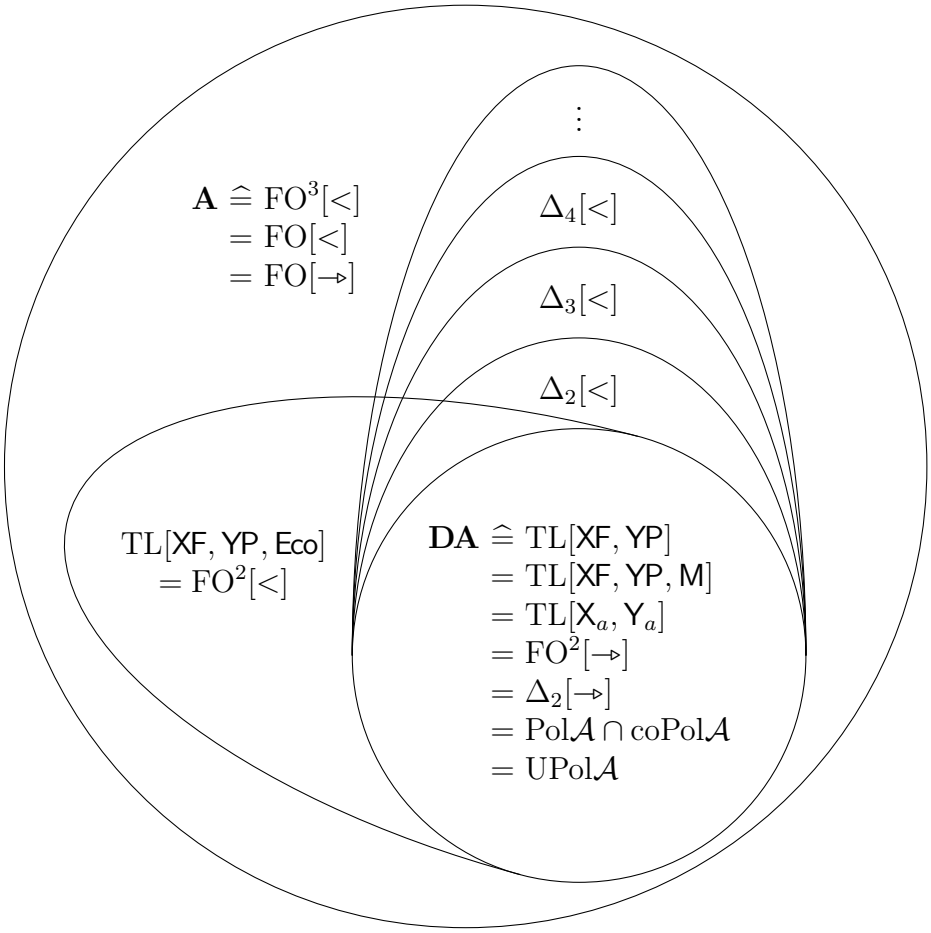


Figure 4.3: The big picture.

Chapter 5

Conclusion and Open Problems

The focus of our investigations has been the variety of monoids **DA**. We have given several new language theoretic and logical characterizations for this variety in the context of Mazurkiewicz traces.

The first language theoretic characterization is that the syntactic monoid of a trace language $L \subseteq \mathbb{M}(\Sigma, I)$ is in **DA** if and only if both, L is a polynomial and its complement $\overline{L}$ is also a polynomial. Here, a polynomial means a finite union of languages of the form

$$A_0^* a_1 A_1^* a_2 A_2^* \cdots a_n A_n^*$$

with $n \in \mathbb{N}$, $a_i \in \Sigma$ for all $1 \leq i \leq n$ and $A_i \subseteq \Sigma$ for all $0 \leq i \leq n$.

Another language theoretic characterization is that the syntactic monoid of $L \subseteq \mathbb{M}(\Sigma, I)$ is in **DA** if and only if L can be combined from languages of the form A^* with $A \subseteq \Sigma$ using only boolean operations and unambiguous products. The definition of unambiguous products over traces is a little bit technical since in general traces have more than one linearization. We say that a product $L_0 a L_1$, $a \in \Sigma$, of trace languages L_0 and L_1 is unambiguous if it is left unambiguous or right unambiguous.

The product $L = L_0 a L_1$ is left unambiguous if for all $t \in L$ there exists a unique position ν in t such that

- $\mathrm{label}(\nu) = a$ and
- $\mathrm{pre}(\nu) \in L_1$ and $\mathrm{par}(\nu)\,\mathrm{suf}(\nu) \in L_2$.

This means that every trace $t \in L$ has to have exactly one position ν labeled with a such that

- the prefix of t that consists of all nodes that are smaller than ν with respect to $<$ is a member of L_0 and
- the suffix of t that consist of all nodes that are larger than or incomparable to ν with respect to $<$ is a trace in L_1.

Right unambiguous products are defined symmetrically by relating the incomparable nodes to the prefix. Note that unambiguity is a property of products and not a property of traces. Another interpretation (that we do not use here) of unambiguous products could be to take the *subset* of $L_0 a L_1$ that consists of all traces that have an unambiguous factorization.

Our main step for these results has been a more general investigation of polynomials over commutative languages, i.e., languages where membership of traces does not depend on the order of the letters. We have given an algebraic characterization of such languages in terms of the Mal'cev product.

We have presented three characterizations with temporal logic of trace languages whose syntactic monoid is in **DA**. The first characterization is the class of languages that is expressible by formulae in local temporal logic using only the two simple temporal operators XF and YP:

- $\mathsf{XF}\varphi$ holds at all positions such that in their strict future there exist a position where φ holds.
- $\mathsf{YP}\varphi$ holds at all positions such that in their strict past there exist a position where φ holds.

Secondly, we have shown that adding the operator M to this logic does not change its expressive power.

- $\mathsf{M}\varphi$ holds at all positions if there exists a position where φ holds.

This operator enables us to change to another connected component within a trace. The third characterization of trace languages whose syntactic monoid is in **DA** is by temporal formulae using only the two types of operators X_a and Y_a for letters $a \in \Sigma$.

- $\mathsf{X}_a\varphi$ holds at all positions such that at the first position in the strict future labeled by the letter a the formula φ holds. It is false at positions with no position labeled by the letter a in the future.
- $\mathsf{Y}_a\varphi$ holds at all positions such that at the last position in the strict past labeled by the letter a the formula φ holds. It is false at positions with no position labeled by the letter a in the past.

All three fragments are local. The term *local* means that formulae are evaluated at positions. Another widely used approach for temporal logic over traces is *global* temporal logic. In global logic, formulae are evaluated at factorizations of a trace. The idea is that a global position within a trace t is a 'cut' of t into parts t_0 and t_1 such that $t = t_0t_1$. The trace t_0 represents the past of the cut and t_1 is the future of the cut.

We have shown that the following two fragments of first order logic also corresponds to the variety **DA** in the sense that these fragments can express exactly those languages whose syntactic monoid is in **DA**. The first characterization is by FO^2 first order formulae with only two different variables and where the only allowed binary relation besides equality $=$ is the dependence relation $\twoheadrightarrow$. For two positions ν and χ we have $\nu \twoheadrightarrow \chi$ if χ occurs after ν and if the labels of these two positions are dependent. The second characterization also uses only these two binary relations. A trace language is in $\Delta_2[\twoheadrightarrow]$ if there exists two first order formulae with only one quantifier alternation, both expressing this language. The first of these two formulae starts with existential quantifiers followed by a block of universal quantifiers. The second of these two formulae starts with a block of universal quantifiers followed by a block of existential quantifiers.

Both first order fragments are quite natural. To ask which languages can be expressed by formulae using only two variables is interesting, since three variables are already sufficient to express all first order properties [GM02a]. The fragment Δ_2 is the largest subclass of trace languages that can be expressed by one quantifier alternation, i.e., two quantifier blocks, starting with a block of existential quantifiers that is closed under complementation. Symmetrically, Δ_2 is also the largest subclass of trace languages that can be expressed by one quantifier alternation starting with a block of universal quantifiers and that is closed under complementation.

The algebraic characterization of polynomials led to a logical characterization in terms of formulae over dependence graphs with two quantifier blocks

starting with a block of existential quantifiers. This fragment is not closed under complementation.

Additionally, we investigated the fragments FO^2 and Δ_2 over partial orders $<$. Surprisingly, both fragments have a larger expressive power than their (equally expressive) counterparts over dependence graphs. This additional expressive power arises from the fact, that within both fragments we can express concurrency of two letters. By allowing the temporal operators XF, YP and the operator Eco to express parallelism, we obtained a characterization of FO^2 over partial orders. We have that

- $\mathsf{Eco}\varphi$ holds at all positions such that there exist a concurrent position where φ holds.

We also showed that within FO^2 one cannot express parallelism of three letters. This is possible in Δ_2. We also showed that for every $n \in \mathbb{N}$ there exists a trace monoid $\mathbb{M}(\Sigma, I)$ and a trace language $L \subseteq \mathbb{M}(\Sigma, I)$ that is expressible in FO^2 but not in Δ_n. Note that every trace language that can be expressed by a first order formula lies in some fragment Δ_n for $n \in \mathbb{N}$. Over words we have that FO^2 and Δ_2 have the same expressive power since the relations $\rightarrowtriangle$ and $<$ coincide.

Most results of the form that a specific property is not expressible within a particular logical fragment are obtained by suitable Ehrenfeucht-Fraïssé (EF) games. We presented characterizations with EF games for the fragments FO^2 and for first-order logic with a bounded number of quantifier alternations.

Since membership within the variety **DA** is decidable, membership within any of the language theoretic or logical characterizations is decidable.

The two main open problems are to find algebraic characterizations of the fragments FO^2 and Δ_2 over partial orders. For both fragments, this would be desirable in order to be able to decide the membership problem. Both fragments, FO^2 and Δ_2, do not correspond to classical varieties of monoids. Over words both of them correspond to the variety **DA**. If, over traces, they would correspond to a variety that lies strictly above **DA** then by Eilenbergs theorem we would also obtain a larger class of word languages, which is not possible. One possible approach for a characterization of FO^2 and Δ_2 could be so called $\mathcal{C}$-varieties which were introduced by Straubing [Str02]. $\mathcal{C}$-varieties are an extension of classical varieties. Since their introduction, a lot of tools for classical varieties have been extended to $\mathcal{C}$-varieties over

words [Cha06, CPS06, ÉI03, Kun03, PS05]. A first step for an algebraic characterization of FO^2 and Δ_2 over traces could be to extend the theory of $\mathcal{C}$-varieties to traces. A possible framework for this would be combining an approach by Volkov [Vol00] with $\mathcal{C}$-varieties.

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Index