# Control of Uncertain Systems with $\ell_1$ and Quadratic Performance Objectives

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Sicher ist, dass nichts sicher ist. Selbst das nicht!

The only certain thing is that nothing is certain. Not even that!

*Joachim Ringelnatz (1883–1934)*

# Contents

# Acronyms

| Acronym | Meaning |
|---|---|
| FD | finite-dimensional |
| FIR | finite impulse response |
| LFT | linear fractional transformation |
| LMI | linear matrix inequality |
| LP | linear program, linear programming |
| LPV | linear parameter-varying |
| LTI | linear time-invariant |
| LTV | linear time-varying |
| MIMO | multi-input multi-output |
| SDP | semi-definite program |
| SISO | single-input single-output |
| TI | time-invariant |
| TV | time-varying |

# Abstract

This thesis presents novel analysis and synthesis concepts for linear control systems with parametric uncertainties. Different performance objectives such as $\ell_1$, $\mathcal{H}_\infty$, $\mathcal{H}_2$, and quadratic performance are considered. In the analysis section, upper bounds on the $\ell_\infty$-gain (or the $\ell_1$-norm) of uncertain systems are developed. These bounds exhibit different degrees of computational effort and accuracy. In particular, a new direct approach for determining the robust $\ell_\infty$-gain is proposed. The synthesis sections introduce an efficient formulation of $\mathcal{H}_\infty$ and $\mathcal{H}_2$ constraints in a general linear multi-objective control framework. Moreover, a novel control structure for the design of parameter-varying controllers is developed. Using this structure, a scheme for the synthesis of linear parameter-varying output-feedback controllers in the $\ell_1$ control framework is presented for the first time. In addition, it is shown how the control structure is applicable to other norm-based frameworks like quadratic performance control and in particular $\mathcal{H}_\infty$ control. The analysis and synthesis conditions proposed in this thesis are expressed as computationally tractable optimization problems, in particular in form of linear matrix inequalities, semi-definite programs, or iterations thereof. Several detailed examples, including a flight control problem with time-varying dynamics, demonstrate the properties and the applicability of the proposed methods.

# Deutsche Zusammenfassung

Die Regelungstechnik und das Prinzip der Rückkopplung haben unseren Alltag in den letzten Jahrzehnten zunehmend beeinflusst. Regelsysteme sind heutzutage wesentliche Bestandteile einer großen Zahl von technischen Errungenschaften. Diese Errungenschaften würden ohne die enthaltenen Regelkreise entweder gar nicht oder nur eingeschränkt funktionieren. Um nur einige Beispiele zu nennen, sei auf CD-Spieler, Digitalkameras, Mobiltelefone, Waschmaschinen, Automobile, Flug- und Raumfahrzeuge, Werkzeugmaschinen, Roboter, chemische Reaktoren, das Internet, sowie Solar-, Wind-, und Kernenergiekraftwerke verwiesen. Selbst der menschliche Körper ist auf eine unüberschaubare Anzahl von Regelkreisen angewiesen, um den Blutzuckerspiegel, die Pupillenweitung, die Körpertemperatur, die Zellteilung und so weiter zu regulieren.
*Regelung* ist, allgemein gesagt, die Tätigkeit zur Beeinflussung eines *Systems* (d.h. beispielsweise eines Geräts, eines Vorgangs oder eines Lebewesens), so dass es sich in gewünschter Weise verhält. Bei der Regelung wird das Systemverhalten fortlaufend gemessen oder beobachtet und mit dem Sollverhalten verglichen. Regelsysteme stellen eine „versteckte Technologie“ dar, das heißt, wir erfahren nur ihren Nutzen und ihre Auswirkungen, ihre Wirkungsweise bleibt aber meistens unsichtbar. Trotzdem ist die Welt, so wie wir sie heute kennen, ohne Regelungstechnik nicht mehr vorstellbar.

## Grundlagen

Regelungs-Ingenieure bearbeiten in heutiger Zeit zunehmend Aufgaben mit hoher Komplexität. Dabei spielt der *modellbasierte Ansatz* eine grundlegende Rolle für das Verständnis und für die Auslegung von Regelsystemen. Im modellbasierten Ansatz beschreiben mathematische Formulierungen die wesentlichen Funktionen und Phänomene eines Systems. Diese Formulierungen werden dann benutzt, um Vorhersagen und Entscheidungen zu treffen oder Regelstrategien zu entwerfen. Es ist allgemein anerkannt, dass detaillierte Systemmodelle für Simulationszwecke unerlässlich sind – also um das Systemverhalten am Computer zu analysieren und vorherzusagen, ohne tatsächlich Experimente durchzuführen. Doch unabhängig davon wie genau ein mathematisches Modell die Wirklichkeit abbildet, wird es immer eine Abweichung zwischen dem Modellverhalten und dem tatsächlichen Systemverhalten geben. Diesen Unterschied bezeichnet man als *Unsicherheit*. Oftmals sind Unsicherheiten im Modell auch beabsichtigt: Modelle für den Reglerentwurf sollten zwar die wesentlichen Funktionen und Phänomene eines Systems widerspiegeln, aber gleichzeitig so einfach wie möglich aufgebaut sein. In diesem Fall wird vom Regelungs-Ingenieur bewusst eine noch größere Modell-Unsicherheit als für Simulationszwecke in Kauf genommen. Trotz dieser Unsicherheit muss ein sinnvolles Reglerentwurfsverfahren Stabilität und die angestrebten Regelgüteeigenschaften garantieren. Diese Überlegungen führen zum Gebiet der *robusten Regelung*. Dort wird untersucht, wie man unsichere Systeme analysieren kann und in welcher Weise Unsicherheiten beim Reglerentwurf berücksichtigt werden können.

**Robuste Störunterdrückung**

Das Fachgebiet der robusten Regelungstheorie hat für die Klasse der linearen zeitinvarianten und der linearen parameterveränderlichen (LPV) Systeme ein hohes Maß an Reife erlangt. Gute Einführungen in die robuste Regelung enthalten zum Beispiel die Texte Skogestad and Postlethwaite (2005) und Sanchez-Pena and Sznaier (1998). Die Mehrzahl der Resultate auf diesem Gebiet befasst sich mit quadratischen oder damit verwandten Regelgüte- und Stabilitätskriterien. Bekannte Beispiele sind die Methode der kleinsten Quadrate, $\mathcal{L}_2$-Signalnormen, $\mathcal{H}_2$- und $\mathcal{H}_\infty$-Systemnormen oder integral-quadratische Beschränkungen. Daraus hervorgegangene Methoden wie die $\mathcal{H}_\infty$-Regelung sind erfolgreich auf zahlreiche reale Probleme in der akademischen Forschung und in der Industrie angewandt worden. Interpretationen des zugehörigen Systemverhaltens in Form von Energie-, Dissipativitäts- oder Frequenzbereichsbetrachtungen tragen zur Attraktivität der quadratischen Kriterien bei. In der Praxis ist es aber oft wünschenswert, Eigenschaften wie den maximalen Regelfehler, das Überschwingen, die maximalen Stellgrößen oder andere Zeitbereichseigenschaften einer Systemantwort direkt zu beeinflussen. Dies ist mit quadratischen Ansätzen zwar prinzipiell möglich, jedoch oft nur indirekt und mittels zahlreicher Iterationsschritte.

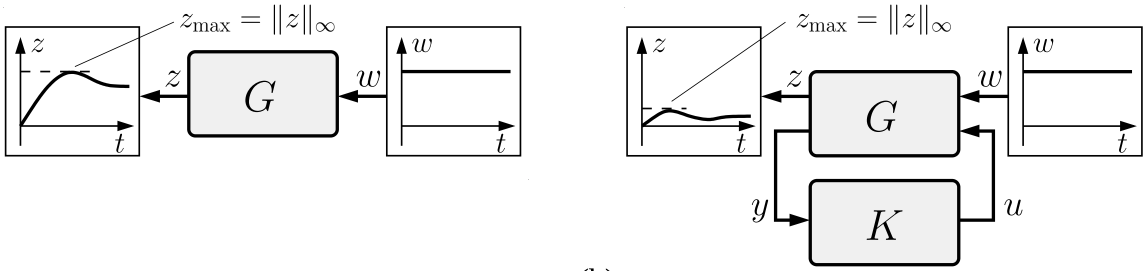
Um sich mit den genannten Zeitbereichseigenschaften einer Systemantwort direkt zu befassen, ist es naheliegend, die Regelgüte mittels der *$\mathcal{L}_\infty$-Signalnorm*

$$\|v\|_\infty := \sup_t \max_i |v_i(t)|$$

zu bestimmen. Diese Norm gibt die maximale Amplitude der Komponenten $v_i$ eines Signalvektors $v$ über der Zeit $t$ an. Um ein entsprechendes Gütemaß für ein stabiles System $G$ zu erhalten, wird oft die sogenannte *Amplitudenverstärkung* oder *$\mathcal{L}_\infty$-Verstärkung*

$$\|G\|_{\infty-\mathrm{ind}} := \sup_{0<\|w\|_\infty<\infty} \frac{\|Gw\|_\infty}{\|w\|_\infty}$$

verwendet. Diese Verstärkung beziffert den schlimmstmöglichen Amplitudenwert des Systemausgangs $z = Gw$, normalisiert mittels der maximalen Amplitude des Eingangs $w$ und unter der Annahme verschwindender Anfangsbedingungen. In anderen Worten: die $\mathcal{L}_\infty$-Verstärkung beschreibt, wie gut ein System persistente (d.h. nicht verschwindende) Störungen abschwächt. Der genannte Verstärkungsbegriff wird in Abbildung 1(a) veranschaulicht. Man spricht von $\mathcal{L}_\infty$-verstärkungsbasierter Störunterdrückung, wenn ein stabilisierender Regler gesucht wird, der die $\mathcal{L}_\infty$-




(a) (b)

*Abbildung 1:* $\mathcal{L}_\infty$-Verstärkung. (a) Falls der Eingang $w$ des Systems $G$ den schlimmstmöglichen Eingang im Sinne der $\mathcal{L}_\infty$-Verstärkung darstellt, ist die $\mathcal{L}_\infty$-Verstärkung $\|G\|_{\infty-\mathrm{ind}}$ gleich der maximalen Amplitude des entsprechenden Ausgangssignals $z$ dividiert durch die maximale Amplitude von $w$. (b) Wenn $G$ so durch einen Regler $K$ kompensiert wird, dass die $\mathcal{L}_\infty$-Verstärkung abnimmt, dann ist auch die maximale Amplitude von $z$, bezogen auf den schlimmstmöglichen Eingang $w$ (der ein anderer sein kann als zuvor), kleiner als in (a) oder gleich.

Verstärkung des geschlossenen Kreises minimiert bzw. beschränkt. Als Beispiel sei $z$ der Regelfehler, dessen Amplitude klein sein soll. Dieser Sachverhalt ist in Abbildung 1(b) dargestellt. Es kann gezeigt werden, dass die $\mathcal{L}_\infty$-Verstärkung für lineare zeitinvariante Systeme gleich der $\mathcal{L}_1$-*Norm* der Impulsantwort des Systems ist. Deshalb wird der Begriff $\mathcal{L}_1$-optimale Regelung für das Gebiet der $\mathcal{L}_\infty$-verstärkungsbasierten Störunterdrückung benutzt. Außer zur Störunterdrückung kann diese Methode auch für viele andere Regelaufgaben verwendet werden, so zum Beispiel zur Sollwertregelung, zur Solltrajektorienfolge, zur Minimierung des Ressourcenverbrauchs oder zum Filterentwurf. Die zugehörige Literatur betrachtet fast ausschließlich zeitdiskrete Entwurfsmethoden, da in diesem Fall lösbare und auf einem Rechner umsetzbare Entwurfsbedingungen formuliert werden können. Die entsprechenden Gütemaße für zeitdiskrete Signale werden $\ell_\infty$- und $\ell_1$-Normen genannt. Dieselben Ideen sind auf Regelgütebetrachtungen für unsichere Systeme übertragbar. Mit Hilfe der eingeführten Normen ist es darüberhinaus möglich, Unsicherheiten quantitativ zu beschreiben, das heißt hinsichtlich ihres Ein-/Ausgangsverhaltens und ihrer maximalen Verstärkung. Wenn Regelungsziele anhand mehrerer Normbedingungen definiert werden, oder wenn verschiedene Normen in einem Reglerentwurfsproblem benutzt werden, spricht man von *Mehrziel-Regelung*. Zusammengefasst ist das Forschungsgebiet der Störunterdrückung ein wesentlicher Bestandteil der robusten Regelung.

Während sich der Großteil der Beiträge auf dem Gebiet der robusten linearen Regelung mit $\mathcal{H}_\infty$- und $\mathcal{H}_2$-Regelung befasst, hat auch das $\ell_1$-Paradigma einige grundlegende und vielversprechende Resultate hervorgebracht. In der $\ell_1$-Literatur wird die Analyse und Synthese von Regelsystemen sowohl ohne als auch mit Unsicherheiten behandelt. Detaillierte Literaturübersichten sind in den einzelnen Kapiteln dieser Arbeit angegeben. Die bisher verfügbaren Analyseverfahren sind einfache Normberechnungen im nominellen Fall, während für Modelle mit Unsicherheiten die „Small-gain"-Theorie in Kombination mit Skalierungen verwendet wird. Die bisher vorliegenden Entwurfsverfahren behandeln den nominellen Reglerentwurf in Form von linearen Programmen.

Die Literatur diskutiert darüberhinaus den robusten Entwurf von linearen zeitinvarianten Reglern bezüglich strukturierter dynamischer Unsicherheiten mit Hilfe von Iterationen über lineare Programme.
Zusammenfassend läßt sich festhalten, dass $\ell_1$-Regelgütebetrachtungen die Festlegung von Regelgütezielen im Zeitbereich sowie die Berücksichtigung von Robustheitseigenschaften ermöglichen. Obwohl einige grundlegende Resultate vorliegen, hat die Fachliteratur der $\ell_1$-Regelung weniger Aufmerksamkeit geschenkt als der Regelung mittels quadratischen Gütekriterien.

**Forschungsrichtungen dieser Arbeit**

Motiviert durch die oben beschriebenen Fakten beschäftigt sich diese Arbeit überwiegend mit dem Problembereich der $\ell_1$-optimalen Regelung. Das Ziel dieser Dissertation ist die Entwicklung von neuen und rechentechnisch effizienten Methoden für die Analyse und Regelung unsicherer Systeme. Insbesondere betrachten wir lineare dynamische Systeme der Form

$$\begin{bmatrix} x(k+1) \\ y(k) \end{bmatrix} = \begin{bmatrix} A(\rho(k)) & B(\rho(k)) \\ C(\rho(k)) & D(\rho(k)) \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}, \qquad x(0) = x_0$$

oder entsprechende zeitkontinuierliche Formulierungen. Dabei bezeichnen $x$ die Zustände, $y$ die Ausgänge und $u$ die Eingänge des Systems. Die Modellkoeffizienten können möglicherweise nichtlinear von zeitvarianten Parametern $\rho$ abhängen. Solche Systeme spielen eine wichtige Rolle, wenn sich physikalische Systemparameter durch variable Betriebsbedingungen oder durch Drift ändern oder wenn sich diese Parameter nicht exakt bestimmen lassen. Es gibt hierfür eine Vielzahl von Beispielen in nahezu allen Anwendungsgebieten wie in der Robotik, der Flugregelung, bei CD-Spielern, bei Magnetlagern, und so weiter.
Wie im Folgenden näher erläutert wird, mangelt es auf dem Gebiet der $\ell_1$-Regelung an Ansätzen für die obige Systemklasse. Auf der einen Seite ist kein allgemeingültiges Resultat zur direkten Analyse des robusten $\ell_\infty$-Verstärkung eines LPV-Systems vorhanden. Ein derartiges Analysewerkzeug ist erforderlich, um die Güte eines Systems zu beurteilen. Andererseits ist die effiziente Synthese von $\ell_1$-optimalen Reglern größtenteils ungelöst. Insbesondere gibt es keine Vorgehensweise zum Entwurf parameterveränderlicher Regler für LPV-Systeme im Rahmen der $\ell_1$-Regelung. Diese Arbeit stellt theoretische Beiträge zu diesen offenen Fragen vor. Dabei wird besonderer Wert darauf gelegt, die vorgeschlagenen Analyse- und Synthesemethoden als rechentechnisch lösbare, möglichst konvexe Optimierungsprobleme zu formulieren.
Aus mehreren Gründen sind die vorgestellten Beiträge für die $\ell_1$-Regelungsmethodik wichtig. Zunächst werden dadurch die theoretischen Grundlagen der $\ell_1$-optimalen Regelung erweitert. Damit steht eine größere Vielfalt an Analyse- und Synthesewerkzeugen zur Verfügung, so wie dies für Formulierungen mit quadratischer Regelgüte schon der Fall ist. Desweiteren wird die Anwendbarkeit und die praktische Bedeutung der $\ell_1$-optimalen Regelung vergrößert. Außerdem bilden die Beiträge dieser Arbeit Bausteine für eine allgemeine Methodik zur Mehrziel-Regelung. Damit wird eine Regelungsmethodik mit $\mathcal{H}_\infty$-, $\mathcal{H}_2$-, sowie $\ell_1$-Normbeschränkungen sowohl für li-

neare zeitinvariante als auch für parameterabhängige Systeme ermöglicht. Aus einem globalen Blickwinkel gesehen stellen Fortschritte in diese Richtungen einen wichtigen Schritt dar, um das Störunterdrückungs-Paradigma weithin anwendbar, theoretisch fundiert und praxisrelevant zu gestalten.

## Gliederung und Forschungsbeiträge der Arbeit

Der folgende Überblick erläutert die Gliederung der Dissertation und fasst die darin vorgestellten Beiträge kurz zusammen.

**Kapitel 2 – Performance Analysis (Analyse der Regelgüte) –** stellt neue Methoden zur Analyse der $\ell_\infty$-Verstärkung von linearen Systemen mit zeitveränderlichen, parametrischen Unsicherheiten vor. Teile dieses Kapitels basieren auf Rieber *et al.* (2006*b*); Rieber *et al.* (2007).

- Lösung von unsicheren Systemen: Die Lösung eines unsicheren Systems im Zeitbereich wird in Form von linearen, fraktionalen Transformationen charakterisiert. Eine derartige Charakterisierung ermöglicht Robustheitsbetrachtungen für unsichere Systemantworten.
- Star-Norm-Analyse: Es werden neue Matrizenungleichungen zur nominellen und robusten Regelgüte-Untersuchung basierend auf der sogenannten Star-Norm hergeleitet. Diese Ungleichungen erlauben die Bestimmung von oberen Schranken für die robuste $\ell_\infty$-Verstärkung.
- Analyse der $\ell_\infty$-Verstärkung: Es werden erstmals Methoden zur Berechnung von vergleichsweise genauen oberen Schranken für die $\ell_\infty$-Verstärkung (oder die $\ell_1$-Norm) von Systemen mit zeitveränderlichen oder zeitinvarianten Parameter-Unsicherheiten vorgeschlagen. Dies schließt konzeptionell eine Lücke in der $\ell_1$-Literatur.

**Kapitel 3 – Synthesis of LTI Controllers (Entwurf von linearen zeitinvarianten Reglern) –** behandelt lineare Matrizenungleichungen zur Synthese von linearen, zeitinvarianten Reglern mit Mehrziel- oder Robustheitsgarantien. Dieses Kapitel basiert teilweise auf Rieber and Allgöwer (2005); Rieber *et al.* (2006*a*); Rieber and Allgöwer (2006*a*).

- Synthese von Mehrziel-Reglern: Eine neue und rechentechnisch effiziente konvexe Formulierung von $\mathcal{H}_\infty$- und $\mathcal{H}_2$-Normbeschränkungen für eine allgemeine Mehrziel-Reglerentwurfsmethode wird eingeführt. Die Formulierung hat rechentechnische Vorteile gegenüber bekannten Ansätzen. Die betrachtete Entwurfsmethodik beinhaltet $\mathcal{H}_\infty$-, $\mathcal{H}_2$-, und $\ell_1$-Normbeschränkungen sowie Zeitbereichsvorgaben.
- Robuster Reglerentwurf: Lineare Matrizenungleichungen für den Entwurf von robusten, linearen, zeitinvarianten Zustandsrückführungen im zeitdiskreten Bereich werden hergeleitet. Dieses Resultat ergänzt verwandte Methoden aus der Literatur.

**Kapitel 4 – Synthesis of LPV Controllers (Entwurf von linearen parameterabhängigen Reglern)** – betrachtet den Entwurf von robusten, linearen, parameterabhängigen Reglern. Dabei wird angenommen und ausgenutzt, dass die unsicheren Systemparameter gemessen werden können. Teile dieses Kapitels basieren auf Rieber and Allgöwer (2003); Rieber *et al.* (2005*a*); Rieber and Allgöwer (2006*b*).

- Regelstruktur: Eine neue Struktur für den Entwurf von LPV-Reglern wird entwickelt und diskutiert. Die Struktur vermeidet die Einführung von zusätzlichen Unsicherheiten, wie dies in der Literatur üblich ist. Zudem ist die Struktur unabhängig von den angewandten Regelgütekriterien.
- LPV-Synthese: Zum ersten Mal wird ein Verfahren zum Entwurf von $\ell_1$-optimalen LPV-Ausgangsrückführungen vorgestellt.
- LPV-Synthese: Die neue Regelstruktur wird im Zusammenhang mit $\ell_1$-, $\mathcal{H}_\infty$- und quadratischen Gütekriterien angewandt.

**Kapitel 5 – Application Examples for LPV Controller Synthesis (Anwendungsbeispiele für den Entwurf von linearen parameterabhängigen Reglern)** – präsentiert zwei Anwendungen für die vorgeschlagenen LPV-Entwurfsmethoden. Die Diskussion der Beispiele basiert teilweise auf Rieber and Allgöwer (2003); Rieber and Allgöwer (2006*b*). Zunächst wird ein einfaches Regelsystem herangezogen, um die Eigenschaften der neuen Regelstruktur im Zusammenhang mit der Minimierung der $\ell_\infty$-Verstärkung zu veranschaulichen. Ein zweites Beispiel aus dem Bereich der Flugregelung, speziell ein Nickwinkel-Regelungsproblem, zeigt eine konkrete Anwendung auf, die in der Literatur schon mehrfach untersucht wurde. Vergleiche mit anderen Entwurfsmethoden werden hinsichtlich der Regelgüte, des gleichförmigen Regelverhaltens und des rechentechnischen Aufwandes durchgeführt. Beide Beispiele liefern darüberhinaus praktische Erkenntnisse über die Wahl der Optimierungskriterien und der Gewichtungsfunktionen innerhalb der $\ell_1$-Methodik.

**Kapitel 6 – Conclusions (Schlussfolgerungen)** – enthält eine Zusammenfassung der Arbeit, diskutiert ihre Ergebnisse und gibt einen Ausblick auf mögliche zukünftige Entwicklungen der behandelten Themen.

Mehrere Anhänge stellen zusätzliche Materialien zusammen, um die Dissertation so eigenständig lesbar wie möglich zu halten.

# Chapter 1

# Introduction

Automatic control technology and the principle of feedback have pervaded our everyday life more and more during the last decades. In fact, control systems are essential components of a large variety of technological achievements. These achievements would not be working as beneficially or not at all without built-in regulators and control loops. To name just a few examples, we refer to CD players, digital cameras, mobile phones, washing machines, cars, air- and spacecrafts, robots, chemical reactors, the Internet, and solar, wind or nuclear power plants. Even our own bodies rely on a vast amount of control loops to regulate the level of blood sugar, the opening of the eye pupil, the body temperature, cell division, and so on.

*Control* is, generally speaking, the activity of influencing a *system* (i.e. for example a device, a process, or a living organism) such that it behaves in a desired way. In feedback control, the system behavior is continuously measured or monitored, and compared to the desired behavior. Control constitutes a “hidden technology”, which means that we experience its benefits and effects, but it is mostly invisible how it works. Yet the world as we know it is unimaginable without control.

## 1.1 Background

As control engineers address increasingly complex tasks, the *model-based approach* plays a fundamental role in understanding and designing control systems. In this approach, mathematical formulations describe the principal functions and phenomena of a system. These formulations are then used to come up with predictions, decisions, and control strategies for the system. It is commonly agreed upon that detailed system models are necessary for simulation purposes – that is, for analyzing and predicting system behavior on a computer without actually doing hardware experiments. However, no matter how accurate a mathematical model represents reality, there will always be a difference between the model behavior and the real-world behavior of the considered system. This difference is called *uncertainty*. Often, uncertainties are introduced by intention: models for controller design should include the essential functions and phenomena of a system while being as simply structured as possible. In this case, even bigger model uncertainty than for simulation purposes is deliberately accepted by the control engineer. Despite this uncertainty, a useful control system design has to guarantee stability and desired performance. These considerations lead to the field of *robust control*. It is investigated there how uncertain systems can be analyzed, and how controllers can be designed such that the uncertainty is taken into account.

**Robust Disturbance Attenuation**

The field of robust control theory has reached a high level of maturity for the classes of linear time-invariant (LTI) and linear parameter-varying (LPV) systems. Good introductions and overviews on robust control are given in Skogestad and Postlethwaite (2005) and in Sanchez-Pena and Sznaier (1998), for example. The majority of the results in this area considers quadratic-type performance and stability criteria. Well-known examples are least squares, $\mathcal{L}_2$ signal norms, $\mathcal{H}_2$ and $\mathcal{H}_\infty$ system norms, and integral quadratic constraints. Related performance frameworks like $\mathcal{H}_\infty$ control have been successfully applied to many real-world problems both in academia and industry. Interpretations of the corresponding system behavior in terms of energy, dissipativity, or frequency-domain properties contribute to the attractiveness of quadratic criteria. In practice, it may however be more desirable to directly influence the maximum control error, the response overshoot, the maximum values of control inputs, or other time-domain properties of a system response. Such goals can be achieved by quadratic-type approaches in principle, but often only indirectly and with numerous design iterations.

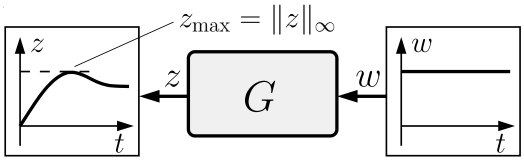
To address the mentioned time-domain properties of a system response more directly, it is natural to consider performance in terms of the $\mathcal{L}_\infty$ *signal norm*

$$\|v\|_\infty := \sup_t \max_i |v_i(t)|.$$

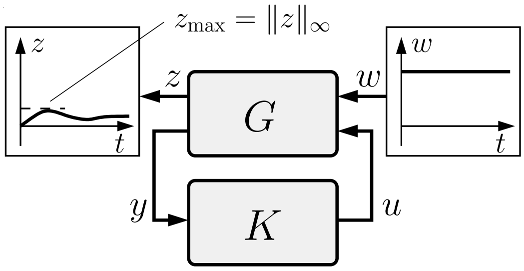
The $\mathcal{L}_\infty$-norm measures the maximum amplitude of the components $v_i$ of a signal vector $v$ over time $t$. To obtain a corresponding measure for a stable system $G$, often the so-called *amplitude-gain* or $\mathcal{L}_\infty$*-gain*

$$\|G\|_{\infty-\mathrm{ind}} := \sup_{0<\|w\|_\infty<\infty} \frac{\|Gw\|_\infty}{\|w\|_\infty}$$

is used. This gain characterizes the worst-case amplitude of the system output $z = Gw$ normalized by the maximum amplitude of the input $w$ under the assumption of zero initial conditions. In other words, the $\mathcal{L}_\infty$-gain describes how well a system attenuates persistent disturbances. The gain notion is visualized in Figure 1.1(a). One speaks of $\mathcal{L}_\infty$-gain based disturbance attenuation if a stabilizing controller is sought such that the $\mathcal{L}_\infty$-gain of the closed-loop system is minimized or bounded. As a common situation, think of $z$ being the control error, the amplitude of which is supposed to be kept small. This is depicted in Figure 1.1(b). It can be shown that, for LTI systems, the $\mathcal{L}_\infty$-gain is equal to the $\mathcal{L}_1$*-norm* of the system's impulse response. Therefore the name $\mathcal{L}_1$-optimal control is used for the field of $\mathcal{L}_\infty$-gain based disturbance attenuation. Besides the rejection of disturbances, many other control goals can be formulated in such a framework, for example




(a)




(b)

*Figure 1.1:* $\mathcal{L}_\infty$-gain. (a) Suppose that the input $w$ of the system $G$ is the worst-case input in terms of the $\mathcal{L}_\infty$-gain. Then the $\mathcal{L}_\infty$-gain $\|G\|_{\infty-\text{ind}}$ is equal to the maximum amplitude of the corresponding output signal $z$ divided by the maximum amplitude of $w$. (b) If $G$ is compensated by a controller $K$ such that the $\mathcal{L}_\infty$-gain decreases, then the maximum amplitude of $z$ normalized by the worst-case input $w$ (which may be a different one than before) is smaller than in (a) or equal.

setpoint control, following of reference commands, minimization of resource consumption, or filtering problems. The related literature treats almost only discrete-time design methods, since in this case it is possible to formulate tractable synthesis conditions that can be solved on a computer. The corresponding measures for discrete-time signals and systems are denoted $\ell_\infty$- and $\ell_1$-norms, respectively. The same ideas carry over to performance considerations for uncertain systems. With help of the above norm descriptions, it is furthermore possible to quantify uncertainties in terms of their input/output behavior and their maximum gain. If control goals are formulated in terms of several norm constraints, or when different norms are used within one control design problem, one speaks of *multi-objective control*. In summary, the disturbance attenuation framework constitutes an essential part of robust control.

While the vast amount of contributions in the field of robust linear control is concerned with $\mathcal{H}_\infty$ and $\mathcal{H}_2$ control, the $\ell_1$ paradigm has also seen a number of basic and promising results. The $\ell_1$ control literature treats analysis and synthesis for systems both with and without uncertainties. Detailed literature overviews are given in the different chapters of this work. The available analysis results are straightforward norm computations in the nominal case, whereas for models with uncertainties, small-gain theory in combination with scalings is applied. The synthesis methods proposed so far treat nominal control design in terms of linear programs (LPs). The literature moreover discusses robust design of LTI controllers with respect to structured dynamic uncertainties using iterations over LPs.

Summarizing, $\ell_1$ performance objectives allow to specify desired control goals in the time-domain and to address robustness issues. Although there have been a number of basic results, the literature has paid less attention to $\ell_1$ control than to quadratic-type performance frameworks.

### Research Directions of this Work

Motivated by the above facts, this work is primarily concerned with the problem of $\ell_1$-optimal control. Our main research direction is the development of novel and computationally tractable analysis and synthesis methods for uncertain systems. In particular, we consider LPV systems of the form

$$\begin{bmatrix} x(k+1) \\ y(k) \end{bmatrix} = \begin{bmatrix} A(\rho(k)) & B(\rho(k)) \\ C(\rho(k)) & D(\rho(k)) \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}, \qquad x(0) = x_0,$$

or corresponding continuous-time formulations. Therein, $x$ denotes the states, $y$ the outputs, and $u$ the inputs of the system. The model coefficients may depend on time-varying parameters $\rho$, possibly in a nonlinear fashion. Such systems play an important role when physical system parameters are subject to change due to varying operating conditions and drift, or when these parameters cannot be determined accurately in the first place. There is an abundance of examples in almost all application fields like robotics, aerial vehicles, CD players, magnetic levitation, and so on.

As discussed in more detail later on, there is a lack of approaches for the above system class in the $\ell_1$ framework. On the one hand, there is no general result to analyze the robust $\ell_\infty$-gain of an LPV system directly. Yet such an analysis tool is essential for judging robust system performance quantitatively. On the other hand, the efficient synthesis of $\ell_1$-optimal controllers is a largely open problem. In particular, there is no scheme for the design of parameter-varying controllers for LPV systems in the framework of $\ell_1$ control. This work provides theoretical contributions to these open questions. It is our particular focus to propose analysis and synthesis methods as tractable, possibly convex, optimization problems.

For various reasons, contributions in these directions are important within the $\ell_1$ performance framework. First, the theoretical foundation of $\ell_1$-optimal control is extended. Hence, a wider variety of tractable analysis and synthesis tools is available similarly to quadratic-type performance formulations. Second, the applicability and practical relevance of $\ell_1$-optimal control is augmented. Moreover, our contributions are building blocks for a general multi-objective analysis and synthesis framework. Thus a control methodology including $\mathcal{H}_\infty$-, $\mathcal{H}_2$-, as well as $\ell_1$-norm constraints for both LTI and LPV systems is made possible. From a global point of view, advances in these directions are important step towards rendering the disturbance attenuation paradigm widely applicable, theoretically sound, and practically relevant.

## 1.2 Outline and Contributions of this Work

The following overview displays an outline of this thesis and briefly summarizes its contributions.

**Chapter 2 – Performance Analysis –** provides new tools to analyze the $\ell_\infty$-gain of linear systems subject to time-varying parametric uncertainties. Parts of this chapter are based on Rieber *et al.* (2006*b*); Rieber *et al.* (2007).

- Solutions of uncertain systems: The time-domain response of an uncertain system is characterized in terms of linear fractional transformations. Such a characterization enables robustness analysis of uncertain system responses.

- Star-norm based performance analysis: We derive new matrix inequality conditions for the nominal and robust performance analysis of discrete-time systems based on star-norms. These inequalities allow to compute upper bounds on the robust $\ell_\infty$-gain.
- $\ell_\infty$-gain based performance analysis: We introduce methods to compute relatively tight upper bounds on the $\ell_\infty$-gain (or $\ell_1$-norm) of systems with time-varying or time-invariant parametric uncertainties. This conceptually closes a gap in the $\ell_1$ control literature.

**Chapter 3 – Synthesis of LTI Controllers –** addresses LMI conditions for the synthesis of multi-objective or robust LTI controllers. This chapter is based partly on Rieber and Allgöwer (2005); Rieber *et al.* (2006*a*); Rieber and Allgöwer (2006*a*).

- Multi-objective controller synthesis: A new and efficient convex formulation of $\mathcal{H}_\infty$ and $\mathcal{H}_2$ constraints in a general multi-objective synthesis framework is introduced. The formulation has computational advantages over existing ones. The framework includes $\mathcal{H}_\infty$-, $\mathcal{H}_2$-, and $\ell_1$-norm constraints, as well as time-domain constraints.
- Robust controller synthesis: We derive LMI conditions to synthesize robust LTI state-feedback controllers in discrete time. The result complements methods from the literature.

**Chapter 4 – Synthesis of LPV Controllers –** considers the problem of synthesizing robust LPV controllers (or gain-scheduled controllers) for LPV systems. In this context, measurability of the uncertain system parameters is assumed and exploited. Parts of this chapter are based on Rieber and Allgöwer (2003); Rieber *et al.* (2005*a*); Rieber and Allgöwer (2006*b*).

- Control structure: A novel structure for LPV controller design is developed and discussed. The structure avoids the introduction of additional uncertainties as is commonly done in the literature. Moreover, the structure is independent of the applied performance framework.
- LPV synthesis: For the first time, a scheme to synthesize $\ell_1$-optimal LPV output-feedback controllers is proposed.
- LPV synthesis: The novel control structure is applied in the context of $\ell_1$-optimal, $\mathcal{H}_\infty$-optimal, and quadratic performance synthesis.

**Chapter 5 – Application Examples for LPV Controller Synthesis –** presents two applications of the proposed LPV controller synthesis approaches. The discussion of the examples is based in part on Rieber and Allgöwer (2003); Rieber and Allgöwer (2006*b*). First, a simple control system is used to demonstrate the properties of the proposed control structure for $\ell_\infty$-gain minimization. A second example from flight control, namely a pitch axis attitude control problem, provides a concrete application that is well-studied in the literature. Comparisons in terms of performance, uniform behavior, and computational effort are carried out with respect to other synthesis methods. Both examples additionally give insight into the selection of optimization criteria and weightings in the $\ell_1$ framework.

**Chapter 6 – Conclusions –** provides a summary of the thesis, discusses its achievements, and gives an outlook on possible further developments of the considered topics.

Several appendices contain additional material to make the thesis as self-contained as possible.

## 1.3 Preliminaries and Notation

This section briefly introduces some preliminary definitions and basic notation to enable a compact and precise statement of problem formulations and results. Related definitions are found in Dahleh and Diaz-Bobillo (1995); Zhou *et al.* (1996); Skogestad and Postlethwaite (2005).

**General**

$|\cdot|$ is the absolute value of a number, and $\|v\| = \sqrt{v^T v}$ denotes the Euclidean vector norm. $\mathrm{Co}(S)$ represents the convex hull of a set $S$.

**Matrices**

The symbols $I$ and $0$ denote the identity and zero matrices of appropriate dimension, respectively. To address components of matrices, we use $M_{i:}$ to denote the $i^{\text{th}}$ row of a matrix $M$, $M_{:j}$ for the $j^{\text{th}}$ column, and $M_{ij}$ for the element with index $(i, j)$. $M^T$ and $M^*$ denote transposition and complex conjugate transposition of $M$, respectively, and $M^{-T}$ represents the transpose of the inverse of $M$. A square matrix $M$ is called symmetric if $M = M^T$, and Hermitian if $M = M^*$. $M^\dagger$ ($M^\ddagger$) is a left-inverse (right-inverse) of $M$ with the property $M^\dagger M = I$ ($MM^\ddagger = I$). A Hermitian matrix is said to be negative (semi-)definite, denoted by $M < 0$ ($M \leq 0$), if $x^* M x < 0$ ($x^* M x \leq 0$) for all nonzero $x$, which is equivalent to all eigenvalues of $M$ being less than (less than or equal to) zero. Analogous definitions for positive (semi-)definiteness hold. The notation $M < N$ stands for $M - N < 0$, and $[*]^T MN < 0$ is used to abbreviate $N^T MN < 0$. The notation $\begin{bmatrix} A & B \\ * & C \end{bmatrix}$ represents $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$, $\mathrm{diag}(M_1, M_2)$ abbreviates the block-diagonal form $\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}$, and $\mathrm{col}(M_1, M_2)$ represents $\begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$, similarly for more than two arguments. $\lambda_{\max}(M)$ ($\lambda_{\min}(M)$) is the largest (smallest) eigenvalue of a Hermitian matrix $M$. $\sigma_{\max}(M) := \sqrt{\lambda_{\max}(M^* M)}$ denotes the maximum singular value of a matrix $M$. $\rho(M) := \max_i |\lambda_i(M)|$ is the spectral radius of a matrix $M$. The kernel and the image of a matrix $M$ are denoted by $\ker(M)$ and $\mathrm{im}(M)$, respectively.

**Operators on Sequences and Functions**

The truncation operator $\mathcal{P}_N$ acting on an infinite sequence $x = \{x(0), x(1), x(2), ...\}$ is defined by $\mathcal{P}_N x := \{x(0), ..., x(N), 0, ...\}$. The right shift operator $\mathcal{S}$ is defined by $\mathcal{S}x := \{0, x(0), x(1), ...\}$. Convolution of two (possibly matrix-valued) sequences is denoted by $x * y$, given componentwise by $(x * y)_{ij}(k) = \sum_p \sum_q x_{ip}(k-q) y_{pj}(q)$. The $\mathcal{Z}$-transform of a right-sided sequence $x$ is $\hat{x}(z) = \mathcal{Z}(x)(z) := \sum_{k=0}^{\infty} x(k) z^{-k}$. The Laplace-transform of a right-sided function $x$ is $\hat{x}(s) = \mathcal{L}(x)(s) := \int_0^\infty x(t) e^{-st} dt$. The arguments $z$ and $s$ are occasionally dropped for brevity.

### Systems and Interconnections

An operator (or a map) representing a dynamic system is denoted by a capital letter such as $G$. If $G$ acts on an object $w$, the outcome is denoted by $z = Gw$. The impulse response corresponding to an operator $G$ is also denoted by $G$ (with a slight abuse of notation), whereas the corresponding transfer function (if existing) is called $\hat{G}$. An operator $G$ is said to be causal (or proper) if $\mathcal{P}_k G = \mathcal{P}_k G \mathcal{P}_k$ for all $k \geq 0$. An operator $G$ is said to be time-invariant if $\mathcal{S}G = G\mathcal{S}$. A state-space realization of a transfer function $\hat{G}$ is occasionally written as

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](z) := C(zI - A)^{-1}B + D = \hat{G}(z).$$

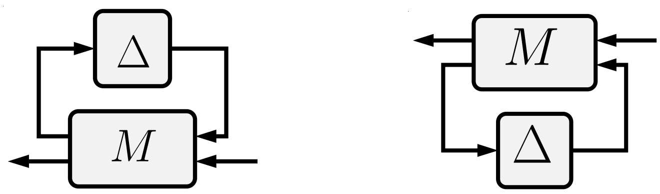
Regular letters ($G$, $A$, $B$, ...) are used for open-loop systems, whereas script letters ($\mathcal{G}$, $\mathcal{A}$, $\mathcal{B}$, ...) are used for closed-loop systems with a controller connected.
The upper linear fractional transformation (LFT) of two matrices $\Delta$ and $M$ with appropriate partitioning is defined as

$$\mathcal{F}_u(M, \Delta) = \Delta \star \underbrace{\left[\begin{array}{c:c} A & B \\ \hdashline C & D \end{array}\right]}_{M} := D + C(I - \Delta A)^{-1} \Delta B,$$

provided the inverse $(I - \Delta A)^{-1}$ exists. The lower LFT is defined similarly as

$$\mathcal{F}_l(M, \Delta) = \underbrace{\left[\begin{array}{c:c} A & B \\ \hdashline C & D \end{array}\right]}_{M} \star \Delta := A + B(I - \Delta D)^{-1} \Delta C.$$

LFTs are a way of describing feedback interconnections as depicted in Figure 1.2 and are special cases of the star product, see e.g. Zhou *et al.* (1996, Section 10.4).
Maps are also used to represent interconnections of systems. For example, $G_1 G_2$ stands for a series connection of two systems represented by the maps $G_1$ and $G_2$, $G_1 + G_2$ for a parallel connection, and $G_1 \star G_2$ for an LFT interconnection, like it is common for transfer functions.




(a) (b)

*Figure 1.2:* (a) Upper LFT $\Delta \star M$. (b) Lower LFT $M \star \Delta$.

# Chapter 2

# Performance Analysis

*This chapter introduces a collection of tools to analyze stability and $\ell_\infty$-gain performance as well as star-norm performance of linear systems subject to time-varying parametric uncertainties. These tools complement the established methods for dynamic uncertainties, and provide trade-offs between accuracy and computational complexity. The main results are*

- *matrix inequality conditions for nominal and robust star-norm performance analysis of discrete-time systems (Theorems 2.1 and 2.4),*
- *a matrix inequality condition to estimate the $\ell_1$-norm of an impulse response tail (Corollary 2.2),*
- *an LFT characterization of the response of an uncertain system (Lemmas 2.5 and 2.11), and*
- *matrix inequality conditions for upper bounds on the $\ell_\infty$-gain of linear systems with time-invariant parametric uncertainty (Theorems 2.7, 2.8, 2.9, and 2.10) or time-varying parametric uncertainty (Section 2.5.3).*

## 2.1 Overview

Robust analysis problems are of central importance in control to certify stability of an uncertain control system and to quantify its performance. A general framework for robustness analysis is given by the problem of investigating the uncertain system $\bar{\mathcal{G}}(\Delta)$ depicted in Figure 2.1(a). $\bar{\mathcal{G}}(\Delta)$ consists of an LTI part and an uncertainty, connected to each other in an LFT fashion. In the literature, the $\Delta$-block is sometimes called a perturbation. The uncertainty may be real parametric, LTI, linear time-varying (LTV), or nonlinear. During the last 25 years, numerous methods have been developed for the analysis of linear systems subject to different classes of uncertainties or perturbations.

In principle, two approaches are available to characterize robust stability and robust performance of uncertain linear systems. First, there is the small-gain based approach, where stability and performance are investigated using scaled system norms. System performance is usually immersed

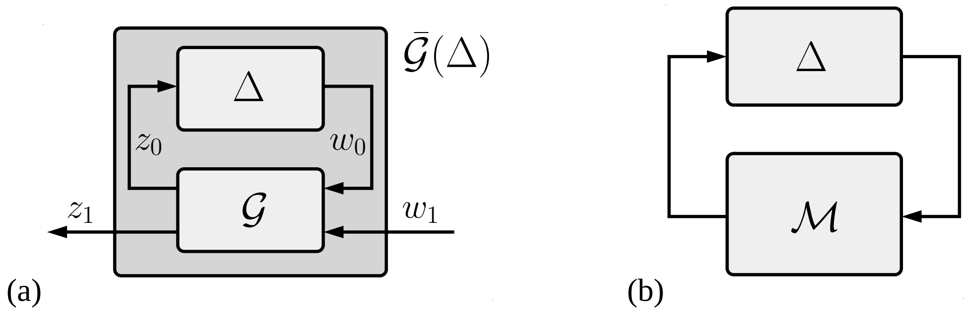



*Figure 2.1:* (a) The uncertain closed-loop system $\bar{\mathcal{G}}(\Delta)$ consists of an LTI part $\mathcal{G}$ and an uncertainty block $\Delta$. (b) Setup for the robust stability problem.

into a stability notion, such that it is enough to develop conditions just for the internal stability of the loop in Figure 2.1(b). Uncertainties with bounded $\mathcal{L}_2$- or $\ell_2$-gain or bounded $\mathcal{L}_\infty$- or $\ell_\infty$-gain are particularly well-studied. There are tractable exact conditions for LTV or nonlinear uncertainties, whereas only tractable upper bounds exist for LTI uncertainties in general. The well-known time-invariant structured singular value $\mu$ is a concept for exactly characterizing robust stability and robust performance. It can be related to scaled norms and constitutes the most prominent robust performance characterization to date. Some important contributions for $\mathcal{L}_2$- or $\ell_2$-stable uncertainties are reported in Doyle and Stein (1981); Doyle (1982); Safonov (1982); Packard and Doyle (1993); Megretski (1993); Shamma (1994). Results for $\mathcal{L}_\infty$- or $\ell_\infty$-stable perturbations have been developed analogously, see Dahleh and Ohta (1988); Khammash and Pearson (1991); Khammash and Pearson (1993); Dahleh and Khammash (1993); Dahleh and Diaz-Bobillo (1995).

The second main approach to robust analysis relies on Lyapunov stability and performance characterizations via dissipation inequalities, Riccati equations, and LMIs. There, state-space representations of uncertain systems play a major role in the development of stability and performance conditions. To address different types of uncertainties, often additional variables – called multipliers – with various complexity levels are introduced. Moreover, different types of Lyapunov function candidates are considered, for example quadratic, quadratic parameter-dependent, piecewise affine, polynomial, etc. The reader is pointed to the references Boyd *et al.* (1994); Helmersson (1995); Gahinet *et al.* (1996); Oliveira *et al.* (1999); Apkarian and Tuan (2000); Scherer (2000*b*); Iwasaki and Shibata (2001); Chesi *et al.* (2005); Biswas *et al.* (2005); Oliveira and Peres (2006) for more information. During the last decade, a key question has been the tractability of analysis conditions, in particular the convexity of the conditions. Since the development of so-called interior-point algorithms (Alizadeh, 1991; Nesterov and Nemirovski, 1994), it is in principle possible to numerically solve convex optimization problems and in particular semi-definite programs (SDPs) quite efficiently. SDP and LMI constraints play an important role in formulating control-related problems (Boyd *et al.*, 1994; Vandenberghe and Boyd, 1996). Some robust analysis results that are used in this work are collected in Appendix B.8.

Although there have been a vast number of results on uncertain system analysis, there are still many open questions. This chapter focuses on some of these and investigates conditions to compute the $\ell_\infty$-gain of systems with parametric uncertainties or upper bounds thereof. The results conceptually fill a gap in the $\ell_\infty$-gain analysis literature, since parametric uncertainties are only treated there for some special cases or using conservative upper bounds. The development starts off in Section 2.2 with a brief review on nominal $\ell_1$ performance analysis, to set the computational approaches into perspective. Moreover, new matrix inequality conditions to determine the star-norm of an LTI system are derived. After defining the problem setup in more detail in Section 2.3, the previous star-norm results are extended to the robust case in Section 2.4. Thereby, upper bounds on the robust $\ell_\infty$-gain are attained. The problem of computing the robust $\ell_\infty$-gain is approached more directly in Section 2.5, yielding new upper bounds with low conservatism. An example finally indicates the differences between known and new methods, and investigates trade-offs between accuracy and computational effort. The reader is referred to Appendix A for a brief introduction to signal and operator norms. Parts of this chapter are based on Rieber *et al.* (2006*b*); Rieber *et al.* (2007).

## 2.2 Nominal Performance Analysis

This section recalls well-known results for $\ell_1$ performance analysis of discrete-time LTI systems without uncertainties. Moreover, a novel matrix inequality condition for star-norm performance of discrete-time LTI systems is introduced, which gives an upper bound on the peak-induced norm and on the $\ell_1$-norm. Based on the star-norm condition, a new method to compute an upper bound on the $\ell_1$-norm of an impulse response tail is described. Finally, a simple example shows the application of the various norm computations.

### 2.2.1 $\ell_1$ Performance

First it is recalled how the $\ell_\infty$-gain of a discrete-time LTI system, i.e. its $\ell_1$ performance, can be determined. Suppose a system $\mathcal{G}$ of dimension $p_1 \times q_1$ without uncertainties has the impulse response $\mathcal{G} = \{\mathcal{G}(0), \mathcal{G}(1), ...\}$. From a state-space realization

$$\begin{bmatrix} \xi(k+1) \\ z_1(k) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \begin{bmatrix} \xi(k) \\ w_1(k) \end{bmatrix} \tag{2.1}$$

with $n_\xi = \dim(\xi)$, the Markov parameters of the impulse response are computed as

$$\mathcal{G}(k) = \begin{cases} \mathcal{D} & \text{for } k = 0 \\ \mathcal{C}\mathcal{A}^{k-1}\mathcal{B} & \text{for } k \geq 1. \end{cases}$$

Then the $\ell_\infty$-gain is obtained as the $\ell_1$-norm of $\mathcal{G}$, i.e.

$$\|\mathcal{G}\|_1 := \max_{1\leq i\leq p_1} \sum_{j=1}^{q_1} \sum_{k=0}^{\infty} |\mathcal{G}_{ij}(k)| \leq \underbrace{\max_{1\leq i\leq p_1} \sum_{j=1}^{q_1} \sum_{k=0}^{N} |\mathcal{G}_{ij}(k)|}_{\gamma^N} + \underbrace{\max_{1\leq i\leq p_1} \sum_{j=1}^{q_1} \sum_{k=N+1}^{\infty} |\mathcal{G}_{ij}(k)|}_{\gamma^{r,N}} \tag{2.2}$$

due to the subadditive property of the $\max$-operator, where $\mathcal{G}_{ij}(k)$ are the components of the matrix $\mathcal{G}(k)$. In this context, $N$ is not an exponent of $\gamma$, but rather an index. It is obvious from (2.2) that $\gamma^{r,N} \geq 0$ and

$$\gamma^N \leq \|\mathcal{G}\|_1 \leq \gamma^N + \gamma^{r,N}.$$

The value of $\|\mathcal{G}\|_1$ is approximated by truncation as $\|\mathcal{G}\|_1 \approx \gamma^N$ provided that $\gamma^{r,N}$, which is the $\ell_1$-norm of the truncated remainder (or tail), is sufficiently small. To judge whether this is the case, Dahleh and Diaz-Bobillo (1995, Chapter 4.4.3) describe a procedure to obtain upper bounds on $\gamma^{r,N}$ based on Hankel-norm approximations. A different (and presumably less conservative) method is proposed in the next section.

### 2.2.2 Star-Norm Performance

It is clear from (2.2) that one has to find the impulse response $\mathcal{G}$ – i.e. a solution of (2.1) – to compute the $\ell_\infty$-gain of a system $\mathcal{G}$. Since getting the solution may be computationally expensive, another possibility is to use cheaper upper bounds. To this end, a novel matrix inequality condition is introduced to characterize the so-called star-norm and hence to determine an upper bound on the peak-induced norm. Related results for continuous-time systems are given in Abedor *et al.* (1996), Scherer *et al.* (1997). A similar result for discrete-time systems is available in Bu *et al.* (1996).

**Theorem 2.1** *Consider the system $\mathcal{G}$ with realization (2.1) and $\xi(0) = 0$. Suppose there exist $\mu > 0$, $0 < \lambda < 1$, and $X = X^T$ satisfying*

$$\begin{bmatrix} \mathcal{A}^T X \mathcal{A} - \lambda X & \mathcal{A}^T X \mathcal{B} \\ \mathcal{B}^T X \mathcal{A} & \mathcal{B}^T X \mathcal{B} - \mu I \end{bmatrix} < 0, \tag{2.3}$$

$$\begin{bmatrix} (\lambda - 1)X + \mathcal{C}^T \mathcal{C} & \mathcal{C}^T \mathcal{D} \\ \mathcal{D}^T \mathcal{C} & (\mu - \gamma^2)I + \mathcal{D}^T \mathcal{D} \end{bmatrix} < 0. \tag{2.4}$$

*Then*

- $\|z_1\|_{\text{peak}} < \gamma$ *for* $\|w_1\|_{\text{peak}} \leq 1$, *and moreover* $\|\mathcal{G}\|_{\text{peak-ind}} < \gamma$,
- $\|\mathcal{G}\|_1 < \gamma\sqrt{q_1}$, *and*
- *$\mathcal{A}$ has all its eigenvalues in the open unit disk.*

Proof: See Appendix D.1. ■

The value $\gamma$ appearing in Theorem 2.1 is just an upper bound on the system's peak-induced norm. The smallest achievable $\gamma$ is called the star-norm $\|\mathcal{G}\|_\star$ of $\mathcal{G}$. Mathematically speaking,

$$\|\mathcal{G}\|_\star := \inf_{\mu,\lambda,X} \gamma \quad \text{such that} \quad \mu > 0,\ 0 < \lambda < 1,\ (2.3),\ (2.4) \quad \text{hold.} \tag{2.5}$$

Hence $\|\mathcal{G}\|_{\text{peak-ind}} \leq \|\mathcal{G}\|_\star$ and

$$\|\mathcal{G}\|_1 \leq \sqrt{q_1}\|\mathcal{G}\|_{\text{peak-ind}} \leq \sqrt{q_1}\|\mathcal{G}\|_\star.$$

Since $\gamma^2$ enters the inequalities affinely, a minimization of $\gamma^2$ subject to the inequality conditions in Theorem 2.1 would be possible if the conditions were linear in the other unknowns. However, the two conditions (2.3)–(2.4) do not constitute LMIs due to products between $\lambda$ and $X$. Still, the global minimum of $\gamma^2$ is found by combining the minimization of $\gamma(\lambda)^2$ for fixed $\lambda$ with a line search over $0 < \lambda < 1$, which promises to be computationally tractable for reasonably-sized problems.

**Remark 2.1** *The inequality (2.4) is equivalently rewritten into*

$$\begin{bmatrix} (1-\lambda)X & 0 & \mathcal{C}^T \\ 0 & (\gamma^2-\mu)I & \mathcal{D}^T \\ \mathcal{C} & \mathcal{D} & I \end{bmatrix} > 0 \tag{2.6}$$

*by using the Schur Lemma (Lemma B.5) with respect to the (3,3)-block in (2.6).* □

**Computing an Upper Bound on the $\ell_1$-Norm of an Impulse Response Tail**

In Dahleh and Diaz-Bobillo (1995, Chapter 4.4.3), a Hankel-norm based procedure is described to compute an upper bound on $\gamma^{r,N}$ in (2.2), i.e. the $\ell_1$-norm of the impulse response tail. Consider the system (2.1). Suppose that $\mathcal{P}$ and $\mathcal{Q}_i$ are the (discrete-time) reachability and observability Gramians of the state-space realization $\left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C}_{i:} & \mathcal{D}_{i:} \end{array}\right]$, respectively. In particular, $\mathcal{P}$ and $\mathcal{Q}_i$ are the solutions of

$$\mathcal{A}\mathcal{P}\mathcal{A}^T - \mathcal{P} + \mathcal{B}\mathcal{B}^T = 0, \qquad \mathcal{A}^T\mathcal{Q}_i\mathcal{A} - \mathcal{Q}_i + \mathcal{C}_{i:}^T\mathcal{C}_{i:} = 0,$$

respectively. Then an upper bound on $\gamma^{r,N}$ as given in Dahleh and Diaz-Bobillo (1995) is

$$\gamma^{r,N} \leq \max_{1\leq i\leq p_1} 2\sqrt{q_1}\sum_{l=1}^{n}\tilde{\sigma}_{i,l}, \tag{2.7}$$

where $\tilde{\sigma}_{i,l} = \sqrt{\lambda_l(\mathcal{Q}_i\mathcal{A}^N\mathcal{P}(\mathcal{A}^T)^N)}$, and $\lambda_l(M)$ is the $l^{\text{th}}$ eigenvalue of the matrix $M$.

As an alternative, one can use Theorem 2.1 to compute an upper bound on $\gamma^{r,N}$. To this end, observe that the impulse response tail $\mathcal{G}_{r,N} := \{\mathcal{G}(N+1), \mathcal{G}(N+2), ...\}$ belonging to (2.1) is the impulse response of

$$\begin{bmatrix} \xi_r(k+1) \\ z_r(k) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{A}^{N+1}\mathcal{B} \\ \mathcal{C} & \mathcal{C}\mathcal{A}^N\mathcal{B} \end{bmatrix}\begin{bmatrix} \xi_r(k) \\ w_r(k) \end{bmatrix}. \tag{2.8}$$

Hence, $\gamma^{r,N} = \|\mathcal{G}_{r,N}\|_1$ is upper-bounded by using a star-norm computation on (2.8) as follows.

**Corollary 2.2** *Consider the system (2.1) with $\xi(0) = 0$. Suppose there exist $\mu > 0$, $0 < \lambda < 1$, and $X = X^T$ satisfying*

$$\begin{bmatrix} \mathcal{A}^T X \mathcal{A} - \lambda X & \mathcal{A}^T X \mathcal{A}^{N+1}\mathcal{B} \\ (\mathcal{A}^{N+1}\mathcal{B})^T X \mathcal{A} & (\mathcal{A}^{N+1}\mathcal{B})^T X \mathcal{A}^{N+1}\mathcal{B} - \mu I \end{bmatrix} < 0,$$

$$\begin{bmatrix} (\lambda - 1)X + \mathcal{C}^T\mathcal{C} & \mathcal{C}^T\mathcal{C}\mathcal{A}^N\mathcal{B} \\ (\mathcal{C}\mathcal{A}^N\mathcal{B})^T\mathcal{C} & (\mu - (\eta^N)^2)I + (\mathcal{C}\mathcal{A}^N\mathcal{B})^T\mathcal{C}\mathcal{A}^N\mathcal{B} \end{bmatrix} < 0.$$

*Then $\gamma^{r,N} < \eta^N\sqrt{q_1}$, and $\mathcal{A}$ has all its eigenvalues in the open unit disk.*

Proof: Note that the two matrix inequalities in this lemma are nothing else than (2.3)–(2.4) with $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ replaced by the system matrices in (2.8). Hence, the statement $\gamma^{r,N} = \|\mathcal{G}_{r,N}\|_1 < \eta^N\sqrt{q_1}$ follows directly from the second item in Theorem 2.1. Stability of $\mathcal{A}$ follows in the same way as in the proof of Theorem 2.1. ■

Since the value of $(\eta^N)^2$ appears affinely in the inequalities of Corollary 2.2, it can be minimized subject to an SDP in combination with a line search over $\lambda$. In a large number of numerical examples, we found that using Corollary 2.2 has always been less conservative than using the upper bound from Dahleh and Diaz-Bobillo (1995). See Section 2.2.3 for an example.

**Remark 2.2** *There exists a simpler realization of the system having the tail as its impulse response than (2.8), see Dahleh and Diaz-Bobillo (1995, Chapter 4.4.3). However, that realization is not suited for the extension of Corollary 2.2 to the robust case. Therefore we use realization (2.8) even for the nominal case presented here.* □

### 2.2.3 Example

The norm computations discussed so far are illustrated by a simple academic example. The example is also helpful in understanding the basic ideas used in the robust $\ell_\infty$-gain computation in Section 2.5. Consider the stable discrete-time dynamic system $\mathcal{G}$ with realization (2.1), where

$$\mathcal{A} = \begin{bmatrix} 0.2 & 0.01 \\ -0.1 & -0.01 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 3 & 2 \\ 3 & 1 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}, \quad \mathcal{D} = \begin{bmatrix} 1 & -2 \\ -4 & 3 \end{bmatrix},$$

and $n_\xi = p_1 = q_1 = 2$. One can think of $\mathcal{G}$ as a closed-loop system with two disturbance inputs and two performance outputs. An $\ell_1$-norm computation using the expression for $\gamma^N$ in (2.2) with e.g. $N = 10$ or $N = 1000$ impulse response elements yields an almost perfect approximation of $\|\mathcal{G}\|_1 \approx \gamma^N = 29.59580$, which is used to compare the subsequent upper bound computations.
Table 2.1 lists upper bounds on the $\ell_1$-norm according to $\|\mathcal{G}\|_1 \leq \gamma^N + \gamma^{r,N}$. There, $\gamma^N$ is computed for different values of $N$ to investigate different accuracy levels. $\gamma^{r,N}$ is upper-bounded either by (2.7) from Dahleh and Diaz-Bobillo (1995) or by our approach from Corollary 2.2. It is evident from comparing rows 2 and 3 or rows 4 and 5 of Table 2.1 that the upper bound on $\gamma^{r,N}$

is more accurate for our approach in both the $N = 2$ and $N = 6$ cases. The absolute difference is marginal in this example, but the relative difference is significant. In fact, the star-norm based upper bounds are about $40\%$ smaller than the Hankel-norm based approximations. This means that the same accuracy is achievable for truncation with smaller $N$. The shown trend is confirmed in a large number of other example systems, where the star-norm based approximation was always considerably tighter than the Hankel-norm based one. Lower bounds are the values for $\gamma^N$. Moreover, Table 2.1 shows the value $\sqrt{q_1}\|\mathcal{G}\|_\star$, computed via Theorem 2.1, which defines a generally conservative upper bound on $\|\mathcal{G}\|_1$.

*Table 2.1:* Nominal performance computations for the example system.

| Norm computation | Result | $\gamma^N$ | Upper bound on $\gamma^{r,N}$ |
|---|---|---|---|
| $\|\mathcal{G}\|_1$ | 29.59580 | - | - |
| Upper bound (2.2), $N = 2$, using (2.7) | 30.11154 | 29.46000 | 0.65154 |
| Upper bound (2.2), $N = 2$, using Lemma 2.2 | 29.87332 | 29.46000 | 0.41332 |
| Upper bound (2.2), $N = 6$, using (2.7) | 29.59654 | 29.59560 | $9.4400{\cdot}10^{-4}$ |
| Upper bound (2.2), $N = 6$, using Lemma 2.2 | 29.59620 | 29.59560 | $5.9867{\cdot}10^{-4}$ |
| $\sqrt{q_1}\|\mathcal{G}\|_\star$ | 34.36 | - | - |

## 2.3 Robust Performance Analysis: Problem Setup

Before addressing the robust performance analysis in detail, the considered system and uncertainty classes are introduced. It is desired to analyze an uncertain discrete-time plant $\bar{\mathcal{G}}(\Delta)$, obtained from the interconnection of a discrete-time LTI part $\mathcal{G}$ and an uncertainty block $\Delta$ as depicted in Figure 2.1(a). It holds that $\bar{\mathcal{G}}(\Delta) = \Delta \star \mathcal{G}$. Assume that $\Delta$ is a function $\Delta : \{0, 1, 2, ...\} \to \mathbf{\Delta}$, where $\mathbf{\Delta} \subset \mathbb{R}^{q_0 \times p_0}$ is some compact set with $0 \in \mathbf{\Delta}$. Hence $\Delta$ can be interpreted as a time-varying structured parametric uncertainty in general. The case of time-invariant $\Delta = \mathrm{const.}$ is contained as a special case. Define moreover the sets of time-varying, rate-bounded time-varying, and time-invariant sequences respectively as

$$\begin{aligned}
\mathbf{\Delta}_{\mathrm{TV}} &:= \{\Delta | \Delta(k) \in \mathbf{\Delta}\ \forall k\}\,, \\
\mathbf{\Delta}_{\mathrm{RB}} &:= \{\Delta | \Delta(k) \in \mathbf{\Delta},\ |\Delta_{ij}(k+1) - \Delta_{ij}(k)| \leq e_{ij}\ \forall i, j, k\}\,, \\
\mathbf{\Delta}_{\mathrm{TI}} &:= \{\Delta | \Delta(k) \in \mathbf{\Delta},\ \Delta(k+1) = \Delta(k)\ \forall k\}\,.
\end{aligned}$$

To make the system formulation more concrete, we consider a discrete-time LPV system $\bar{\mathcal{G}}(\Delta)$ with a state-space realization obtained from the interconnection between $\mathcal{G}$ and $\Delta$ such that

$$\begin{bmatrix} \xi(k+1) \\ z_0(k) \\ z_1(k) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B}_0 & \mathcal{B}_1 \\ \mathcal{C}_0 & \mathcal{D}_{00} & \mathcal{D}_{01} \\ \mathcal{C}_1 & \mathcal{D}_{10} & \mathcal{D}_{11} \end{bmatrix} \begin{bmatrix} \xi(k) \\ w_0(k) \\ w_1(k) \end{bmatrix}, \tag{2.9}$$

$$w_0(k) = \Delta(k) z_0(k), \tag{2.10}$$

with the state $\xi(k) \in \mathbb{R}^{n_\xi}$, outputs $z_i(k) \in \mathbb{R}^{p_i}$, and inputs $w_j(k) \in \mathbb{R}^{q_j}$. The corresponding closed-loop system $\bar{\mathcal{G}}(\Delta)$ is

$$\begin{bmatrix} \xi(k+1) \\ z_1(k) \end{bmatrix} = \begin{bmatrix} \bar{\mathcal{A}}(\Delta(k)) & \bar{\mathcal{B}}_1(\Delta(k)) \\ \bar{\mathcal{C}}_1(\Delta(k)) & \bar{\mathcal{D}}_{11}(\Delta(k)) \end{bmatrix} \begin{bmatrix} \xi(k) \\ w_1(k) \end{bmatrix}, \tag{2.11}$$

where

$$\begin{aligned} \begin{bmatrix} \bar{\mathcal{A}}(\Delta(k)) & \bar{\mathcal{B}}_1(\Delta(k)) \\ \bar{\mathcal{C}}_1(\Delta(k)) & \bar{\mathcal{D}}_{11}(\Delta(k)) \end{bmatrix} &= \Delta(k) \star \left[\begin{array}{c:cc} \mathcal{D}_{00} & \mathcal{C}_0 & \mathcal{D}_{01} \\ \hdashline \mathcal{B}_0 & \mathcal{A} & \mathcal{B}_1 \\ \mathcal{D}_{10} & \mathcal{C}_1 & \mathcal{D}_{11} \end{array}\right] \\ &= \begin{bmatrix} \mathcal{A} + \mathcal{B}_0(I - \Delta(k)\mathcal{D}_{00})^{-1}\Delta(k)\mathcal{C}_0 & \mathcal{B}_1 + \mathcal{B}_0(I - \Delta(k)\mathcal{D}_{00})^{-1}\Delta(k)\mathcal{D}_{01} \\ \mathcal{C}_1 + \mathcal{D}_{10}(I - \Delta(k)\mathcal{D}_{00})^{-1}\Delta(k)\mathcal{C}_0 & \mathcal{D}_{11} + \mathcal{D}_{10}(I - \Delta(k)\mathcal{D}_{00})^{-1}\Delta(k)\mathcal{D}_{01} \end{bmatrix}. \end{aligned} \tag{2.12}$$

The interconnection (2.9)–(2.10) is said to be well-posed if $I - \Delta(k)\mathcal{D}_{00}$ is non-singular for all $\Delta(k) \in \mathbf{\Delta}$.

From the LFT interconnection of $\mathcal{G}$ and $\Delta$ or from (2.12) it is clear that $\bar{\mathcal{G}}(\Delta)$ may depend on the uncertain real time-varying parameters $\Delta$ in a rational manner. Note that polynomial or affine parameter dependence and polytopic uncertainty descriptions, which are extensively studied in the literature, are included in this LFT description as special cases. A commonly encountered situation is the case where the LPV system $\bar{\mathcal{G}}(\Delta)$ depends rationally on an uncertain parameter vector $\delta(k) = [\delta_1(k), ..., \delta_{n_\delta}(k)]^T$ with $\delta_i(k) \in [\delta_{\min,i}, \delta_{\max,i}]$ for all $k$ and for $i = 1, ..., n_\delta$. This situation can be transformed into the structure of Figure 2.1(a) with diagonal $\Delta(\delta(k))$. Detailed descriptions on how to perform such transformations are given for example in Zhou *et al.* (1996, Chapter 10); Hecker and Varga (2004); Scherer and Weiland (2005, Section 6.2). Some simple examples are listed in Appendix B.7. Note that in $\mathbf{\Delta}_{\mathrm{TV}}$, $\mathbf{\Delta}_{\mathrm{RB}}$, and $\mathbf{\Delta}_{\mathrm{TI}}$, no restrictions to diagonally structured $\Delta$ or to square $\Delta$ are assumed.

The problem to be considered in the remainder of this chapter is the stability and performance analysis of a given system $\bar{\mathcal{G}}(\Delta)$. To this end, the following robust stability notion is defined, cf. Scherer (2000*b*).

**Definition 2.1** *(Uniform exponential stability, discrete time)*
*Suppose that the interconnection (2.9)–(2.10) is well-posed. Then the system (2.11) is said to be uniformly exponentially stable if there exist constants $0 < \alpha < 1$, $\beta > 0$ such that $\|\xi(k)\| \leq \beta\alpha^{k-k_0}\|\xi(k_0)\|\ \forall k \geq k_0 \geq 0$ for all considered uncertainties $\Delta(\cdot)$ and for every system trajectory $\xi(\cdot)$ with $w_1 = 0$.* □

The notion of $\ell_1$ performance is extended to the robust case as defined next.

**Definition 2.2** *(Robust $\ell_1$ performance, discrete time)*
*Suppose that the interconnection (2.9)–(2.10) is well-posed. Then the system (2.11) is said to have robust $\ell_1$ performance with performance level $\gamma$ if $\|z_1\|_\infty \leq \gamma\|w_1\|_\infty$ for all considered uncertainties $\Delta(\cdot)$ and for every system trajectory $\xi(\cdot)$ with $\xi(0) = 0$.* □

Robust peak-induced performance can be defined similarly. The robust star-norm cannot be defined in such a compact manner, but is rather characterized by the solution of certain matrix inequalities, cf. (2.5).

## 2.4 Robust Star-Norm Performance Analysis

In this section, the result for star-norm based nominal performance analysis of discrete-time systems (Theorem 2.1) is generalized to systems with time-varying parametric uncertainty. The obtained robust star-norm conditions yield an upper bound on the robust $\ell_\infty$-gain of uncertain systems. To this end, a matrix inequality for analyzing the robust star-norm performance of uncertain systems is introduced. The approach is based on a constant Lyapunov function, uses static full-block multipliers to deal with the uncertainties, and applies a convex hull relaxation of the parameter set to obtain a finite number of inequalities.
The following theorem gives two equivalent sufficient conditions for robust star-norm performance.

**Theorem 2.3** *Consider the system (2.11) obtained from the interconnection (2.9)–(2.10) with* $\xi(0) = 0$, $\Delta \in \mathbf{\Delta}_{\mathrm{TV}}$. *The following two statements are equivalent:*

*(i) (2.9)–(2.10) is well-posed and there exist* $\mu > 0$, $0 < \lambda < 1$, *and* $X = X^T$ *satisfying*

$$[*]^T \left[\begin{array}{cc:c} -\lambda X & 0 & 0 \\ 0 & X & 0 \\ \hdashline 0 & 0 & -\mu I \end{array}\right] \left[\begin{array}{cc} I & 0 \\ \bar{\mathcal{A}}(\Delta) & \bar{\mathcal{B}}_1(\Delta) \\ \hdashline 0 & I \end{array}\right] < 0, \tag{2.13}$$

$$[*]^T \left[\begin{array}{c:cc} (\lambda-1)X & 0 & 0 \\ \hdashline 0 & (\mu-\gamma^2)I & 0 \\ 0 & 0 & I \end{array}\right] \left[\begin{array}{cc} I & 0 \\ \hdashline 0 & I \\ \bar{\mathcal{C}}_1(\Delta) & \bar{\mathcal{D}}_{11}(\Delta) \end{array}\right] < 0 \qquad \forall \Delta \in \mathbf{\Delta}. \tag{2.14}$$

*(ii) There exist* $\mu > 0$, $0 < \lambda < 1$, $X = X^T$, $P_1 = P_1^T$, *and* $P_2 = P_2^T$ *satisfying*

$$\begin{bmatrix} \Delta \\ I \end{bmatrix}^T \underbrace{\begin{bmatrix} Q_1 & S_1 \\ S_1^T & R_1 \end{bmatrix}}_{P_1} \begin{bmatrix} \Delta \\ I \end{bmatrix} \geq 0, \quad \begin{bmatrix} \Delta \\ I \end{bmatrix}^T \underbrace{\begin{bmatrix} Q_2 & S_2 \\ S_2^T & R_2 \end{bmatrix}}_{P_2} \begin{bmatrix} \Delta \\ I \end{bmatrix} \geq 0 \quad \forall \Delta \in \mathbf{\Delta}, \tag{2.15}$$

$$[*]^T \left[\begin{array}{cc:cc:c} -\lambda X & 0 & 0 & 0 & 0 \\ 0 & X & 0 & 0 & 0 \\ \hdashline 0 & 0 & Q_1 & S_1 & 0 \\ 0 & 0 & S_1^T & R_1 & 0 \\ \hdashline 0 & 0 & 0 & 0 & -\mu I \end{array}\right] \left[\begin{array}{ccc} I & 0 & 0 \\ \mathcal{A} & \mathcal{B}_0 & \mathcal{B}_1 \\ \hdashline 0 & I & 0 \\ \mathcal{C}_0 & \mathcal{D}_{00} & \mathcal{D}_{01} \\ \hdashline 0 & 0 & I \end{array}\right] < 0, \tag{2.16}$$

$$[*]^T \left[\begin{array}{c:cc:cc} (\lambda-1)X & 0 & 0 & 0 & 0 \\ \hdashline 0 & Q_2 & S_2 & 0 & 0 \\ 0 & S_2^T & R_2 & 0 & 0 \\ \hdashline 0 & 0 & 0 & (\mu-\gamma^2)I & 0 \\ 0 & 0 & 0 & 0 & I \end{array}\right] \left[\begin{array}{ccc} I & 0 & 0 \\ \hdashline 0 & I & 0 \\ \mathcal{C}_0 & \mathcal{D}_{00} & \mathcal{D}_{01} \\ \hdashline 0 & 0 & I \\ \mathcal{C}_1 & \mathcal{D}_{10} & \mathcal{D}_{11} \end{array}\right] < 0. \tag{2.17}$$

*Moreover, if either (i) or (ii) holds, then*

- $\|z_1\|_{\text{peak}} < \gamma$ *for* $\|w_1\|_{\text{peak}} \leq 1,$ *and moreover* $\|\bar{\mathcal{G}}(\Delta)\|_{\text{peak}-\text{ind}} < \gamma$ *for all* $\Delta \in \mathbf{\Delta}_{\text{TV}}$,
- $\|\bar{\mathcal{G}}(\Delta)\|_{\infty-\text{ind}} < \gamma\sqrt{q_1}$ *for all* $\Delta \in \mathbf{\Delta}_{\text{TV}}$, *and*
- *(2.11) is uniformly exponentially stable with respect to* $\Delta \in \mathbf{\Delta}_{\text{TV}}$.

Proof: See Appendix D.2. ■

The theorem is an extension of Theorem 2.1 to systems of class (2.11). The smallest achievable $\gamma$ is called the robust star-norm $\|\bar{\mathcal{G}}\|_\star$ of $\bar{\mathcal{G}}(\cdot)$. Some comments about the result are in order. Both conditions (i) and (ii) are sufficient for stability and performance. The advantage of using condition (ii) lies in the fact that the rational parameter dependence of the inequalities (2.13)–(2.14) is converted into conditions (2.15) with quadratic parameter dependence. The price to be paid is the introduction of new unknowns $P_i = \begin{bmatrix} Q_i & S_i \\ S_i^T & R_i \end{bmatrix}$, $i = 1, 2$, in (2.16)–(2.17).
The conditions of Theorem 2.3 contain the semi-infinite constraints (2.15), which are difficult to check. To obtain a finite set of constraints, one possibility is to apply Lemma B.12, provided that $\mathbf{\Delta}$ is defined as the convex hull of a number of generators. As a result, the semi-infinite constraints (2.15) are converted into a finite number of inequalities, at the expense of some conservatism in general. This leads to a tractable version of Theorem 2.3 presented next.

**Theorem 2.4** *Consider the system (2.11) obtained from the interconnection (2.9)–(2.10) with* $\xi(0) = 0$, $\Delta \in \mathbf{\Delta}_{\text{TV}}$, $\mathbf{\Delta} = \text{Co}(\{\Delta_1, ..., \Delta_{n_\Delta}\})$. *If there exist* $\mu > 0$, $0 < \lambda < 1$, $X = X^T$, $P_1 = P_1^T$, *and* $P_2 = P_2^T$ *satisfying*

$$Q_1 < 0, \qquad Q_2 < 0, \tag{2.18}$$

$$\begin{bmatrix} \Delta_l \\ I \end{bmatrix}^T \underbrace{\begin{bmatrix} Q_1 & S_1 \\ S_1^T & R_1 \end{bmatrix}}_{P_1} \begin{bmatrix} \Delta_l \\ I \end{bmatrix} \geq 0, \quad \begin{bmatrix} \Delta_l \\ I \end{bmatrix}^T \underbrace{\begin{bmatrix} Q_2 & S_2 \\ S_2^T & R_2 \end{bmatrix}}_{P_2} \begin{bmatrix} \Delta_l \\ I \end{bmatrix} \geq 0, \quad l = 1, ..., n_\Delta, \tag{2.19}$$

$$\left[\begin{array}{ccc} I & 0 & 0 \\ \mathcal{A} & \mathcal{B}_0 & \mathcal{B}_1 \\ \hdashline 0 & I & 0 \\ \mathcal{C}_0 & \mathcal{D}_{00} & \mathcal{D}_{01} \\ \hdashline 0 & 0 & I \end{array}\right]^T \left[\begin{array}{cc:cc:c} -\lambda X & 0 & 0 & 0 & 0 \\ 0 & X & 0 & 0 & 0 \\ \hdashline 0 & 0 & Q_1 & S_1 & 0 \\ 0 & 0 & S_1^T & R_1 & 0 \\ \hdashline 0 & 0 & 0 & 0 & -\mu I \end{array}\right] \left[\begin{array}{ccc} I & 0 & 0 \\ \mathcal{A} & \mathcal{B}_0 & \mathcal{B}_1 \\ \hdashline 0 & I & 0 \\ \mathcal{C}_0 & \mathcal{D}_{00} & \mathcal{D}_{01} \\ \hdashline 0 & 0 & I \end{array}\right] < 0, \tag{2.20}$$

$$\begin{bmatrix} I & 0 & 0 \\ \hline 0 & I & 0 \\ \mathcal{C}_0 & \mathcal{D}_{00} & \mathcal{D}_{01} \\ \hline 0 & 0 & I \\ \mathcal{C}_1 & \mathcal{D}_{10} & \mathcal{D}_{11} \end{bmatrix}^T \left[\begin{array}{c|cc|cc} (\lambda-1)X & 0 & 0 & 0 & 0 \\ \hline 0 & Q_2 & S_2 & 0 & 0 \\ 0 & S_2^T & R_2 & 0 & 0 \\ \hline 0 & 0 & 0 & (\mu-\gamma^2)I & 0 \\ 0 & 0 & 0 & 0 & I \end{array}\right] \begin{bmatrix} I & 0 & 0 \\ \hline 0 & I & 0 \\ \mathcal{C}_0 & \mathcal{D}_{00} & \mathcal{D}_{01} \\ \hline 0 & 0 & I \\ \mathcal{C}_1 & \mathcal{D}_{10} & \mathcal{D}_{11} \end{bmatrix} < 0, \quad (2.21)$$

*then*

- $\|z_1\|_{\text{peak}} < \gamma$ *for* $\|w_1\|_{\text{peak}} \leq 1$, *and moreover* $\|\bar{\mathcal{G}}(\Delta)\|_{\text{peak-ind}} < \gamma$ *for all* $\Delta \in \mathbf{\Delta}_{\text{TV}}$,
- $\|\bar{\mathcal{G}}(\Delta)\|_{\infty-\text{ind}} < \gamma\sqrt{q_1}$ *for all* $\Delta \in \mathbf{\Delta}_{\text{TV}}$, *and*
- *(2.11) is uniformly exponentially stable with respect to* $\Delta \in \mathbf{\Delta}_{\text{TV}}$.

Proof: In condition (ii) of Theorem 2.3, add the two constraints $Q_1 < 0$ and $Q_2 < 0$. According to Lemma B.12, these constraints together with (2.15) are equivalent to (2.18)–(2.19). Hence the conditions (2.18)–(2.21) are sufficient for the result, cf. Theorem 2.3. ■

This theorem thus gives an upper bound $\gamma$ on the peak-induced norm of $\bar{\mathcal{G}}(\cdot)$, and the upper bound computation is based on an SDP in combination with a line search over $\lambda$. To determine $n_\Delta$ in a particular application, suppose that there are $n_\delta$ different scalar parameters $\delta_i$, $i = 1, ..., n_\delta$, in the uncertainty structure $\Delta$, with box constraints $\delta_i \in [\delta_{\min,i}, \delta_{\max,i}]$. Then $n_\Delta = 2^{n_\delta}$. Apart from these box constraints, a large number of other descriptions for $\mathbf{\Delta}$ is conceivable, and they lead to different types of convex hull formulations.

**Remark 2.3** *Instead of using Lemma B.12 to create a finite set of constraints, other relaxation techniques with generally less conservatism may be applied to Theorem 2.3. Possible alternatives include the Polya relaxation and matrix-sum-of-squares relaxations as described in Scherer (2006), Scherer and Hol (2006). Since these techniques come with a possibly larger computational burden, they are not considered in more detail here. It is noted however that the extension of analysis results to controller synthesis conditions often requires the introduction of constraints like (2.18), see Theorem 4.3. Hence the convex hull relaxation seems to be a reasonable choice if analysis results are to be extended to synthesis conditions.* □

**Analysis Using Parameter-Dependent Lyapunov Functions**

In Theorem 2.3 as well as in Theorems B.17, B.18, and B.19, the analysis is essentially based on fixed quadratic Lyapunov function candidates of the form $V(x(k)) = x(k)^T X x(k)$. However, one can find examples of stable systems, where stability cannot be shown using such fixed Lyapunov functions. To obtain less conservative conditions, there exists the possibility of addressing the analysis problems using parameter-dependent Lyapunov function candidates

$$V(x(k), \bar{\Delta}(k)) = x(k)^T X(\bar{\Delta}(k)) x(k),$$

where the function $\bar{\Delta}(k)$ is dependent on the components of $\Delta(k)$ in some pre-defined fashion. It is however unclear in general how to best choose the function $\bar{\Delta}(\cdot)$. Stability and performance conditions using such parameter-dependent Lyapunov function candidates can be transformed into tractable analysis conditions.

It goes beyond the scope of this thesis to delve further into this topic, yet an extension of Theorem 2.3 to the case of a parameter-dependent Lyapunov function is stated in Appendix C. The interested reader is moreover referred to e.g. Gahinet *et al.* (1996); Apkarian and Tuan (2000); Iwasaki (2001); Wu and Dong (2006).

## 2.5 Robust $\ell_\infty$-Gain Analysis

As briefly discussed in Section 2.1, the literature on robust $\ell_\infty$-gain analysis has mostly treated dynamic uncertainties (Khammash and Pearson, 1991; Dahleh and Khammash, 1993). Parametric uncertainties have only been considered for the restrictive system class of so-called rank-one problems (Dahleh and Diaz-Bobillo, 1995, Section 7.8). The main obstacle in the $\ell_\infty$-gain framework is to find the impulse response of (2.11) explicitly. In contrast, "quadratic" approaches using the $\mathcal{H}_\infty$- or star-norm do not require an explicit solution to be found, which significantly simplifies the problem (as in Theorems 2.3 and B.19).

In this section, a computational approach for analyzing the robust stability and the robust $\ell_\infty$-gain (or the robust $\ell_1$ performance) of the uncertain discrete-time linear system (2.11) obtained from the interconnection (2.9)–(2.10) is addressed directly. We provide procedures tailored to the cases of time-invariant or time-varying parametric uncertainty, with or without bounds on the rate of variation.

The computation of upper bounds on the robust $\ell_1$ performance is split into two parts as in (2.2). First, the $\ell_\infty$-gain of a truncated impulse response is determined by using relaxations for uncertain matrix inequalities. In this context, compact expressions for the parameter dependence of the system's Markov parameters are derived. Second, the $\ell_\infty$-gain of the impulse response tail is bounded from above by a star-norm condition. All computations are based on robust SDP, and provide a trade-off between computational burden and accuracy. Hence, the general robust $\ell_1$ performance analysis problem for linear systems with parametric uncertainties can be addressed with relatively low conservatism for the first time.

Next, a special case of the stated problem is considered in order to present our approach without too cluttered notation. Full generality is attained subsequently in Sections 2.5.2 and 2.5.3.

### 2.5.1 SISO Plant and Time-Invariant Uncertainty

Suppose for now that (2.11) is a single-input/single-output (SISO) system, i.e. $p_1 = q_1 = 1$, and that $\Delta(k) = \Delta = \mathrm{const}$. Since $\Delta$ is time-invariant, the response of (2.11) at time-step $k$ to an input $w_1$ with $\xi(0) = 0$ is obtained by convolution as

$$z_1(k) = \sum_{l=0}^{k} \bar{\mathcal{G}}(k-l,\Delta) w_1(l) = \sum_{l=0}^{k} \bar{\mathcal{G}}(l,\Delta) w_1(k-l).$$

The dependence of the system's Markov parameters $\bar{\mathcal{G}}(k,\Delta)$ on $\Delta$ is compactly characterized by the next statement. Define $\Psi_k(\Delta) := \operatorname{diag}(\Delta, ..., \Delta)$ with $k+1$ $\Delta$-blocks.

**Lemma 2.5**

$$\bar{\mathcal{G}}(k,\Delta) = \begin{cases} \bar{\mathcal{D}}_{11}(\Delta) & \text{for} \quad k = 0 \\ \bar{\mathcal{C}}_1(\Delta)\bar{\mathcal{A}}(\Delta)^{k-1}\bar{\mathcal{B}}_1(\Delta) & \text{for} \quad k \geq 1 \end{cases} \tag{2.22}$$

$$= \Psi_k(\Delta) \star \underbrace{\left[\begin{array}{c:c} H_k^A & H_k^B \\ \hdashline H_k^C & H_k^D \end{array}\right]}_{H_k}, \tag{2.23}$$

*where*

$$\left[\begin{array}{c:c} H_0^A & H_0^B \\ \hdashline H_0^C & H_0^D \end{array}\right] := \left[\begin{array}{c:c} \mathcal{D}_{00} & \mathcal{D}_{01} \\ \hdashline \mathcal{D}_{10} & \mathcal{D}_{11} \end{array}\right], \tag{2.24}$$

$$\left[\begin{array}{c:c} H_k^A & H_k^B \\ \hdashline H_k^C & H_k^D \end{array}\right] := \left[\begin{array}{ccccc:c} \mathcal{D}_{00} & \mathcal{C}_0\mathcal{B}_0 & \mathcal{C}_0\mathcal{A}\mathcal{B}_0 & \cdots & \mathcal{C}_0\mathcal{A}^{k-1}\mathcal{B}_0 & \mathcal{C}_0\mathcal{A}^{k-1}\mathcal{B}_1 \\ 0 & \mathcal{D}_{00} & \mathcal{C}_0\mathcal{B}_0 & & \vdots & \vdots \\ \vdots & & \ddots & & \mathcal{C}_0\mathcal{B}_0 & \mathcal{C}_0\mathcal{B}_1 \\ 0 & \cdots & & & \mathcal{D}_{00} & \mathcal{D}_{01} \\ \hdashline \mathcal{D}_{10} & \mathcal{C}_1\mathcal{B}_0 & \mathcal{C}_1\mathcal{A}\mathcal{B}_0 & \cdots & \mathcal{C}_1\mathcal{A}^{k-1}\mathcal{B}_0 & \mathcal{C}_1\mathcal{A}^{k-1}\mathcal{B}_1 \end{array}\right], \quad k = 1, 2, .... \tag{2.25}$$

Proof: The proof is only sketched briefly. The expression (2.22) follows by evaluating the equations of (2.11) at different time instances, and then repeatedly inserting the equations into each other. From (2.22), the LFT representation (2.23) is obtained as follows. Just replace the parameter-dependent matrices $\bar{\mathcal{A}}(\Delta)$ etc. with the LFT (2.12). Using Lemma B.4(ii) repeatedly leads to (2.23) with the structures (2.24) and (2.25). ■

Thus the robust $\ell_1$ performance level $\gamma$ is given by

$$\gamma = \sup_{\Delta \in \boldsymbol{\Delta}_{\mathrm{TI}}} \|\bar{\mathcal{G}}(\Delta)\|_{\infty-\mathrm{ind}} = \sup_{\Delta \in \boldsymbol{\Delta}} \|\bar{\mathcal{G}}(\Delta)\|_1 = \sup_{\Delta \in \boldsymbol{\Delta}} \sum_{k=0}^{\infty} |\bar{\mathcal{G}}(k,\Delta)| = \sup_{\Delta \in \boldsymbol{\Delta}} \sum_{k=0}^{\infty} |\Psi_k(\Delta) \star H_k|,$$

which can be upper-bounded similarly to (2.2) by

$$\gamma \leq \underbrace{\sup_{\Delta \in \boldsymbol{\Delta}} \sum_{k=0}^{N} |\bar{\mathcal{G}}(k,\Delta)|}_{\gamma^N} + \underbrace{\sup_{\Delta \in \boldsymbol{\Delta}} \sum_{k=N+1}^{\infty} |\bar{\mathcal{G}}(k,\Delta)|}_{\gamma^{r,N}} \tag{2.26}$$

due to the subadditive property of the $\sup$-operator. In this context, $N$ is not an exponent of $\gamma$, but rather an index. The rationale behind this split computation is as follows. We show in the sequel that upper bounds on $\gamma^N$ are obtainable with relatively low conservatism as a convex optimization problem, although the computational demand may be large. On the other hand, conservative upper bounds on $\gamma^{r,N}$ are derived. These bounds converge to zero for $N \to \infty$ for uniformly exponentially stable systems. Hence a trade-off between computational demand and accuracy is introduced by the choice of $N$. In general the larger the value of $N$, the higher the accuracy, but also the larger the computational burden. It is detailed next how to compute upper bounds on $\gamma^N$ and $\gamma^{r,N}$ via LMI relaxations.

**Robust $\ell_1$-Norm of a Finite Impulse Response**

It is a standard procedure to resolve the absolute value in the computation of $\gamma^N$ in (2.26) as

$$\gamma^N = \inf \nu \tag{2.27}$$

$$\text{s.t.} \quad \sum_{k=0}^{N} \gamma_k < \nu, \qquad \gamma_k > 0, \tag{2.28}$$

$$-\gamma_k < \bar{\mathcal{G}}(k,\Delta) < \gamma_k \quad \forall \Delta \in \boldsymbol{\Delta}, \tag{2.29}$$

$$k = 0, 1, ..., N.$$

Since the problem (2.27)–(2.29) is in general rational in the uncertain parameters, it is difficult to use this formulation to compute $\gamma^N$. However, the two scalar inequalities in (2.29) can be treated as LMIs with a rational dependence on $\Delta$. Thus it is possible to state a new condition for computing $\gamma^N$, which is the first main result of this section.

**Theorem 2.6** *The interconnection (2.9)–(2.10) is well-posed and (2.29) holds if and only if there exist $P_k^a = (P_k^a)^T$, $P_k^b = (P_k^b)^T$ satisfying (2.32)–(2.35) below. Moreover,*

$$\gamma^N = \inf \nu \tag{2.30}$$

$$\text{s.t.} \quad \sum_{k=0}^{N} \gamma_k < \nu, \qquad \gamma_k > 0, \tag{2.31}$$

$$\begin{bmatrix} \Psi_k(\Delta) \\ I \end{bmatrix}^T \underbrace{\begin{bmatrix} Q_k^a & S_k^a \\ (S_k^a)^T & R_k^a \end{bmatrix}}_{P_k^a} \begin{bmatrix} \Psi_k(\Delta) \\ I \end{bmatrix} \geq 0 \qquad \forall \Delta \in \boldsymbol{\Delta}, \tag{2.32}$$

$$\begin{bmatrix} \Psi_k(\Delta) \\ I \end{bmatrix}^T \underbrace{\begin{bmatrix} Q_k^b & S_k^b \\ (S_k^b)^T & R_k^b \end{bmatrix}}_{P_k^b} \begin{bmatrix} \Psi_k(\Delta) \\ I \end{bmatrix} \geq 0 \qquad \forall \Delta \in \boldsymbol{\Delta}, \tag{2.33}$$

$$\begin{bmatrix} I & 0 \\ H_k^A & H_k^B \\ \hdashline 0 & 1 \\ H_k^C & H_k^D \end{bmatrix}^T \left[\begin{array}{cc:cc} Q_k^a & S_k^a & 0 & 0 \\ (S_k^a)^T & R_k^a & 0 & 0 \\ \hdashline 0 & 0 & -\gamma_k & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{array}\right] \begin{bmatrix} I & 0 \\ H_k^A & H_k^B \\ \hdashline 0 & 1 \\ H_k^C & H_k^D \end{bmatrix} < 0, \tag{2.34}$$

$$\begin{bmatrix} I & 0 \\ H_k^A & H_k^B \\ \hdashline 0 & 1 \\ H_k^C & H_k^D \end{bmatrix}^T \left[\begin{array}{cc:cc} Q_k^b & S_k^b & 0 & 0 \\ (S_k^b)^T & R_k^b & 0 & 0 \\ \hdashline 0 & 0 & -\gamma_k & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \end{array}\right] \begin{bmatrix} I & 0 \\ H_k^A & H_k^B \\ \hdashline 0 & 1 \\ H_k^C & H_k^D \end{bmatrix} < 0, \tag{2.35}$$

$$k = 0, 1, ..., N.$$

Proof: To prove the first part, rewrite the inequalities $\bar{\mathcal{G}}(k,\Delta) < \gamma_k$ and $-\gamma_k < \bar{\mathcal{G}}(k,\Delta)$ in (2.29) equivalently as

$$[*]^T \begin{bmatrix} -\gamma_k & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \Psi_k(\Delta) \star H_k \end{bmatrix} < 0, \quad [*]^T \begin{bmatrix} -\gamma_k & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \Psi_k(\Delta) \star H_k \end{bmatrix} < 0, \tag{2.36}$$

respectively. Using these LMI representations and the Full-Block S-Procedure (Lemma B.11), the condition $\bar{\mathcal{G}}(k,\Delta) < \gamma_k \ \forall \Delta \in \mathbf{\Delta}$ is equivalent to the existence of $P_k^a$ satisfying (2.32) and (2.34) (due to compactness of $\mathbf{\Delta}$). Similarly, $-\gamma_k < \bar{\mathcal{G}}(k,\Delta) \ \forall \Delta \in \mathbf{\Delta}$ is equivalent to the existence of $P_k^b$ satisfying (2.33) and (2.35). Furthermore well-posedness follows since the same type of LFT with the inverse $(I - \Delta \mathcal{D}_{00})^{-1}$ appears in (2.9)–(2.10) and (2.29). Concerning the second part, since (2.29) can be equivalently replaced by (2.32)–(2.35) for all $k$, the new condition for computing $\gamma^N$ follows directly from (2.27)–(2.29). ■

Even though Theorem 2.6 gives an equivalent new condition for computing $\gamma^N$, it contains semi-infinite constraints (2.32)–(2.33). By way of Lemma B.12, a finite number of constraints is obtained as in Theorem 2.4. The next result states an optimization problem to compute an upper bound on $\gamma^N$ with a finite number of LMIs, which is the second main result of this section.

**Theorem 2.7** *Suppose that* $\mathbf{\Delta} = \mathrm{Co}(\{\Delta_1, ..., \Delta_{n_\Delta}\})$ *and that there exist* $P_k^a = (P_k^a)^T$, $P_k^b = (P_k^b)^T$ *satisfying (2.39)–(2.42) below. Then (2.9)–(2.10) is well-posed and (2.29) holds. Moreover,*

$$\gamma^N \leq \tilde{\gamma}^N,$$

*where*

$$\tilde{\gamma}^N = \inf \nu \tag{2.37}$$

$$\text{s.t.} \quad \sum_{k=0}^{N} \gamma_k < \nu, \qquad \gamma_k > 0, \tag{2.38}$$

$$Q_k^a < 0, \qquad \begin{bmatrix} \Psi_k(\Delta_l) \\ I \end{bmatrix}^T \underbrace{\begin{bmatrix} Q_k^a & S_k^a \\ (S_k^a)^T & R_k^a \end{bmatrix}}_{P_k^a} \begin{bmatrix} \Psi_k(\Delta_l) \\ I \end{bmatrix} \geq 0, \quad l = 1, ..., n_\Delta, \tag{2.39}$$

$$Q_k^b < 0, \qquad \begin{bmatrix} \Psi_k(\Delta_l) \\ I \end{bmatrix}^T \underbrace{\begin{bmatrix} Q_k^b & S_k^b \\ (S_k^b)^T & R_k^b \end{bmatrix}}_{P_k^b} \begin{bmatrix} \Psi_k(\Delta_l) \\ I \end{bmatrix} \geq 0, \quad l = 1, ..., n_\Delta, \tag{2.40}$$

$$\begin{bmatrix} I & 0 \\ H_k^A & H_k^B \\ \hdashline 0 & 1 \\ H_k^C & H_k^D \end{bmatrix}^T \left[\begin{array}{cc:cc} Q_k^a & S_k^a & 0 & 0 \\ (S_k^a)^T & R_k^a & 0 & 0 \\ \hdashline 0 & 0 & -\gamma_k & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{array}\right] \begin{bmatrix} I & 0 \\ H_k^A & H_k^B \\ \hdashline 0 & 1 \\ H_k^C & H_k^D \end{bmatrix} < 0, \tag{2.41}$$

$$\begin{bmatrix} I & 0 \\ H_k^A & H_k^B \\ \hdashline 0 & 1 \\ H_k^C & H_k^D \end{bmatrix}^T \left[\begin{array}{cc:cc} Q_k^b & S_k^b & 0 & 0 \\ (S_k^b)^T & R_k^b & 0 & 0 \\ \hdashline 0 & 0 & -\gamma_k & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \end{array}\right] \begin{bmatrix} I & 0 \\ H_k^A & H_k^B \\ \hdashline 0 & 1 \\ H_k^C & H_k^D \end{bmatrix} < 0, \tag{2.42}$$

$$k = 0, 1, ..., N.$$

Proof: Add the constraints $Q_k^a < 0$, $Q_k^b < 0$ to the conditions in Theorem 2.6. According to Lemma B.12, $Q_k^a < 0$, $Q_k^b < 0$ together with (2.32)–(2.33) are equivalent to (2.39)–(2.40). Because of the two additional constraints, we now have a sufficient condition for well-posedness and (2.29) to hold. Likewise, $\gamma^N \leq \tilde{\gamma}^N$. ■

Theorem 2.7 thus gives an upper bound on $\gamma^N$, and the upper bound computation is based on an SDP. To determine $n_\Delta$, suppose that there are $n_\delta$ different scalar parameters $\delta_i$, $i = 1, ..., n_\delta$, in the uncertainty structure $\Delta$, with box constraints $\delta_i \in [\delta_{\min,i}, \delta_{\max,i}]$. Then $n_\Delta = 2^{n_\delta}$.

The exact number of multipliers needed in this basic setup is discussed in Section 2.5.2 for the multi-input/multi-output (MIMO) case. To reduce the computational burden, a number of restrictions on the multipliers $P_k^j$ are possible. These are described in Section 2.5.4. For comments on other relaxation techniques to create finite sets of constraints, see Remark 2.3 in Section 2.4.

**Robust $\ell_1$-Norm of an Impulse Response Tail**

After the treatment of the first part $\gamma^N$ of (2.26), it is described next how to obtain an upper bound on the second part $\gamma^{r,N}$, i.e. the robust $\ell_1$-norm of the sequence $\bar{\mathcal{G}}_{r,N}(\Delta) := \{\bar{\mathcal{G}}(N+1,\Delta), \bar{\mathcal{G}}(N+2,\Delta), ...\}$. To this end, the impulse response tail $\bar{\mathcal{G}}_{r,N}(\Delta)$ is determined, and an upper bound on its $\ell_\infty$-gain is computed using a star-norm estimate. Similarly to the discussion preceding Lemma 2.2, first one has to find a state-space realization of a system having $\bar{\mathcal{G}}_{r,N}(\Delta)$ as its impulse response. This is valid for

$$\begin{bmatrix} \xi_r(k+1) \\ z_r(k) \end{bmatrix} = \begin{bmatrix} \bar{\mathcal{A}}(\Delta) & \bar{\mathcal{A}}(\Delta)^{N+1}\bar{\mathcal{B}}_1(\Delta) \\ \bar{\mathcal{C}}_1(\Delta) & \bar{\mathcal{C}}_1(\Delta)\bar{\mathcal{A}}(\Delta)^N\bar{\mathcal{B}}_1(\Delta) \end{bmatrix} \begin{bmatrix} \xi_r(k) \\ w_r(k) \end{bmatrix}. \tag{2.43}$$

Next express this state-space realization as the LFT

$$\begin{bmatrix} \bar{\mathcal{A}}_{r,N}(\Delta) & \bar{\mathcal{B}}_{r,N}(\Delta) \\ \bar{\mathcal{C}}_{r,N}(\Delta) & \bar{\mathcal{D}}_{r,N}(\Delta) \end{bmatrix} := \begin{bmatrix} \bar{\mathcal{A}}(\Delta) & \bar{\mathcal{A}}(\Delta)^{N+1}\bar{\mathcal{B}}_1(\Delta) \\ \bar{\mathcal{C}}_1(\Delta) & \bar{\mathcal{C}}_1(\Delta)\bar{\mathcal{A}}(\Delta)^N\bar{\mathcal{B}}_1(\Delta) \end{bmatrix} = \Psi_{N+1}(\Delta) \star \underbrace{\left[\begin{array}{c:cc} L_N^{D00} & L_N^{C0} & L_N^{D01} \\ \hdashline L_N^{B0} & L_N^{A} & L_N^{B1} \\ L_N^{D10} & L_N^{C1} & L_N^{D11} \end{array}\right]}_{L_N},$$

where (see (2.24)–(2.25) for abbreviations)

$$\begin{aligned} &L_N^A := \mathcal{A}, && L_N^{D00} := H_{N+1}^A, && L_N^{D01} := H_{N+1}^B, \\ &L_N^{B1} := \mathcal{A}^{N+1}\mathcal{B}_1, && L_N^{B0} := \begin{bmatrix} \mathcal{B}_0 & \mathcal{A}\mathcal{B}_0 & \cdots & \mathcal{A}^{N+1}\mathcal{B}_0 \end{bmatrix}, && L_N^{D10} := H_{N+1}^C, \\ &L_N^{C1} := \mathcal{C}_1, && L_N^{C0} := \begin{bmatrix} \mathcal{C}_0^T & 0 & \cdots & 0 \end{bmatrix}^T, && L_N^{D11} := H_{N+1}^D. \end{aligned}$$

Now we are ready to state the following result on how to obtain a star-norm based upper bound on the $\ell_\infty$-gain of the map $w_r \mapsto z_r$, i.e. on $\gamma^{r,N}$.

**Theorem 2.8** *Consider the system (2.43) with $\xi_r(0) = 0$, $\Delta \in \boldsymbol{\Delta} = \mathrm{Co}(\{\Delta_1, ..., \Delta_{n_\Delta}\})$. If there exist $\mu > 0$, $0 < \lambda < 1$, $X = X^T$, $P_1 = P_1^T$, and $P_2 = P_2^T$ satisfying (2.45)–(2.48) below, then*

- *(2.9)–(2.10) is well-posed and (2.11) is uniformly exponentially stable with respect to $\Delta \in \boldsymbol{\Delta}_{\mathrm{TI}}$,*
- *$\|\bar{\mathcal{G}}_{r,N}(\Delta)\|_{\mathrm{peak-ind}} < \eta^N$ for all $\Delta \in \boldsymbol{\Delta}_{\mathrm{TI}}$, and*
- *$\gamma^{r,N} \le \tilde{\gamma}^{r,N} := \tilde{\eta}^N \sqrt{q_1}$, where*

$$\tilde{\eta}^N = \inf \eta^N \tag{2.44}$$

$$\text{s.t.} \quad Q_1 < 0, \qquad [*]^T \underbrace{\begin{bmatrix} Q_1 & S_1 \\ S_1^T & R_1 \end{bmatrix}}_{P_1} \begin{bmatrix} \Psi_{N+1}(\Delta_l) \\ I \end{bmatrix} \ge 0, \qquad l = 1, ..., n_\Delta, \tag{2.45}$$

$$Q_2 < 0, \qquad [*]^T \underbrace{\begin{bmatrix} Q_2 & S_2 \\ S_2^T & R_2 \end{bmatrix}}_{P_2} \begin{bmatrix} \Psi_{N+1}(\Delta_l) \\ I \end{bmatrix} \ge 0, \qquad l = 1, ..., n_\Delta, \tag{2.46}$$

$$[*]^T \left[\begin{array}{cc:cc:c} -\lambda X & 0 & 0 & 0 & 0 \\ 0 & X & 0 & 0 & 0 \\ \hdashline 0 & 0 & Q_1 & S_1 & 0 \\ 0 & 0 & S_1^T & R_1 & 0 \\ \hdashline 0 & 0 & 0 & 0 & -\mu I \end{array}\right] \left[\begin{array}{ccc} I & 0 & 0 \\ L_N^A & L_N^{B0} & L_N^{B1} \\ \hdashline 0 & I & 0 \\ L_N^{C0} & L_N^{D00} & L_N^{D01} \\ \hdashline 0 & 0 & I \end{array}\right] < 0, \tag{2.47}$$

$$[*]^T \left[\begin{array}{c:cc:cc} (\lambda - 1)X & 0 & 0 & 0 & 0 \\ \hdashline 0 & Q_2 & S_2 & 0 & 0 \\ 0 & S_2^T & R_2 & 0 & 0 \\ \hdashline 0 & 0 & 0 & (\mu - (\eta^N)^2)I & 0 \\ 0 & 0 & 0 & 0 & I \end{array}\right] \left[\begin{array}{ccc} I & 0 & 0 \\ \hdashline 0 & I & 0 \\ L_N^{C0} & L_N^{D00} & L_N^{D01} \\ \hdashline 0 & 0 & I \\ L_N^{C1} & L_N^{D10} & L_N^{D11} \end{array}\right] < 0. \tag{2.48}$$

Proof: We give only a sketch of the proof. Applying Lemmas B.12 and B.11 to the inequalities (2.45)–(2.48) implies well-posedness and

$$\begin{bmatrix} \bar{\mathcal{A}}_{r,N}(\Delta)^T X \bar{\mathcal{A}}_{r,N}(\Delta) - \lambda X & \bar{\mathcal{A}}_{r,N}(\Delta)^T X \bar{\mathcal{B}}_{r,N}(\Delta) \\ \bar{\mathcal{B}}_{r,N}(\Delta)^T X \bar{\mathcal{A}}_{r,N}(\Delta) & \bar{\mathcal{B}}_{r,N}(\Delta)^T X \bar{\mathcal{B}}_{r,N}(\Delta) - \mu I \end{bmatrix} < 0, \tag{2.49}$$

$$\begin{bmatrix} (\lambda - 1)X + \bar{\mathcal{C}}_{r,N}(\Delta)^T \bar{\mathcal{C}}_{r,N}(\Delta) & \bar{\mathcal{C}}_{r,N}(\Delta)^T \bar{\mathcal{D}}_{r,N}(\Delta) \\ \bar{\mathcal{D}}_{r,N}(\Delta)^T \bar{\mathcal{C}}_{r,N}(\Delta) & (\mu - (\eta^N)^2) I + \bar{\mathcal{D}}_{r,N}(\Delta)^T \bar{\mathcal{D}}_{r,N}(\Delta) \end{bmatrix} < 0 \tag{2.50}$$

for all $\Delta \in \boldsymbol{\Delta}$ similar to the proof of Theorem 2.3. These two inequalities imply $\|\bar{\mathcal{G}}_{r,N}(\Delta)\|_{\mathrm{peak-ind}} < \eta^N$ for all $\Delta \in \boldsymbol{\Delta}_{\mathrm{TI}}$, compare to Lemma 2.2. Uniform exponential stability follows along the same lines as in the proof of Theorem 2.3. Finally, the upper bound on $\gamma^{r,N}$ follows from

$$\gamma^{r,N} = \sup_{\Delta \in \boldsymbol{\Delta}} \|\bar{\mathcal{G}}_{r,N}(\Delta)\|_1 \leq \sqrt{q_1} \sup_{\Delta \in \boldsymbol{\Delta}} \|\bar{\mathcal{G}}_{r,N}(\Delta)\|_{\mathrm{peak-ind}} \leq \tilde{\eta}^N \cdot \sqrt{q_1} = \tilde{\gamma}^{r,N}.$$

■

The result states how the star-norm $\tilde{\eta}^N$ of an uncertain impulse response tail and an upper bound $\tilde{\gamma}^{r,N}$ on $\gamma^{r,N}$ can be computed. Moreover a robust stability result for (2.11) is included. Again, $N$ is an index rather than an exponent. The computation of $\tilde{\gamma}^{r,N}$ is based on an SDP in combination with a line search. The result is also valid for MIMO systems and/or time-varying parametric uncertainties.

**Remark 2.4** *The conditions of Theorem 2.4 are obtained as a special case of Theorem 2.8 with* $N = -1$. □

**A Procedure for Robust $\ell_1$ Performance Analysis**

Next, some more properties of the proposed upper bounds $\gamma^N$ and $\gamma^{r,N}$ are explored. Subsequently, a procedure on how to use these bounds to compute approximations of the robust $\ell_\infty$-gain of (2.11) is described. The next theorem states (a) that $\tilde{\gamma}^N$ is a non-decreasing sequence, (b) that the sequence $\tilde{\gamma}^{r,N}$ converges to zero, and (c) that $\tilde{\gamma}^N + \tilde{\gamma}^{r,N}$ is an upper bound on the $\ell_\infty$-gain $\gamma$.

**Theorem 2.9** *The following properties hold:*

*(a)* $\tilde{\gamma}^{N+1} \geq \tilde{\gamma}^N \quad \forall N$,

*(b)* $\gamma^{r,N} \to 0$ *and* $\tilde{\gamma}^{r,N} \to 0$ *for* $N \to \infty$,

*(c)* $\gamma \leq \tilde{\gamma}^N + \tilde{\gamma}^{r,N} \quad \forall N$.

Proof: See Appendix D.3. ■

It is suggested by Theorem 2.9 that an approximate computation of the $\ell_\infty$-gain $\gamma$ of (2.11) proceeds along the following lines. Choose a rather small $N$ and determine $\tilde{\gamma}^N$ and $\tilde{\gamma}^{r,N}$ using Theorems 2.7 and 2.8. If $\tilde{\gamma}^{r,N}$ is below a desired accuracy level, use $\tilde{\gamma}^N + \tilde{\gamma}^{r,N}$ as an upper bound on the $\ell_\infty$-gain $\gamma$ of (2.11), otherwise increase $N$ and redo the computations. Note that uniform exponential stability is implied by $\tilde{\gamma}^{r,N} < \infty$. This procedure and the properties stated in Theorem 2.9 are exemplified in Section 2.6.

### 2.5.2 MIMO Plant and Time-Invariant Uncertainty

After having seen the procedure for SISO plants, it is straightforward to extend the main results of the last section (Theorems 2.7 and 2.8) to the MIMO case. Suppose now that $p_1 \geq 1$, $q_1 \geq 1$, and $\Delta(k) = \Delta = \text{const}$. Let $\bar{\mathcal{G}}_{ij}(k, \Delta)$ denote the $(i, j)^{\text{th}}$ element of the matrix $\bar{\mathcal{G}}(k, \Delta)$. Hence instead of the bounding inequality (2.26), we now have

$$\begin{aligned}\gamma &= \sup_{\Delta \in \boldsymbol{\Delta}} \max_{1 \leq i \leq p_1} \sum_{j=1}^{q_1} \sum_{k=0}^{\infty} |\bar{\mathcal{G}}_{ij}(k, \Delta)| \\ &\leq \underbrace{\sup_{\Delta \in \boldsymbol{\Delta}} \max_{1 \leq i \leq p_1} \sum_{j=1}^{q_1} \sum_{k=0}^{N} |\bar{\mathcal{G}}_{ij}(k, \Delta)|}_{\gamma^N} + \underbrace{\sup_{\Delta \in \boldsymbol{\Delta}} \max_{1 \leq i \leq p_1} \sum_{j=1}^{q_1} \sum_{k=N+1}^{\infty} |\bar{\mathcal{G}}_{ij}(k, \Delta)|}_{\gamma^{r,N}},\end{aligned} \tag{2.51}$$

and the optimization (2.27)–(2.29) is replaced by

$$\gamma^N = \inf \nu \tag{2.52}$$

$$\text{s.t.} \quad \sum_{j=1}^{q_1} \sum_{k=0}^{N} \gamma_{ijk} < \nu, \qquad \gamma_{ijk} > 0, \tag{2.53}$$

$$-\gamma_{ijk} < \bar{\mathcal{G}}_{ij}(k, \Delta) < \gamma_{ijk} \quad \forall \Delta \in \boldsymbol{\Delta}, \tag{2.54}$$

$$i = 1, ..., p_1, \quad j = 1, ..., q_1, \quad k = 0, 1, ..., N.$$

The essential difference to the SISO case is the larger number of inequalities in (2.54). This just increases the number of matrix inequalities and the number of multipliers, but does not introduce structurally new elements. Hence a MIMO version of Theorem 2.7 can be stated.

**Theorem 2.10** *Suppose that $\boldsymbol{\Delta} = \mathrm{Co}(\{\Delta_1, ..., \Delta_{n_\Delta}\})$, and that there exist $P^a_{ijk} = (P^a_{ijk})^T$, $P^b_{ijk} = (P^b_{ijk})^T$ satisfying (2.57)–(2.60) below. Then (2.9)–(2.10) is well-posed and (2.54) holds. Moreover,*

$$\gamma^N \leq \tilde{\gamma}^N,$$

*where*

$$\tilde{\gamma}^N = \inf \nu \tag{2.55}$$

$$\text{s.t.} \quad \sum_{j=1}^{q_1} \sum_{k=0}^{N} \gamma_{ijk} < \nu, \qquad \gamma_{ijk} > 0, \tag{2.56}$$

$$Q^a_{ijk} < 0, \qquad [*]^T \underbrace{\begin{bmatrix} Q^a_{ijk} & S^a_{ijk} \\ (S^a_{ijk})^T & R^a_{ijk} \end{bmatrix}}_{P^a_{ijk}} \begin{bmatrix} \Psi_k(\Delta_l) \\ I \end{bmatrix} \geq 0, \quad l = 1, ..., n_\Delta, \tag{2.57}$$

$$Q^b_{ijk} < 0, \qquad [*]^T \underbrace{\begin{bmatrix} Q^b_{ijk} & S^b_{ijk} \\ (S^b_{ijk})^T & R^b_{ijk} \end{bmatrix}}_{P^b_{ijk}} \begin{bmatrix} \Psi_k(\Delta_l) \\ I \end{bmatrix} \geq 0, \quad l = 1, ..., n_\Delta, \tag{2.58}$$

$$[*]^T \left[\begin{array}{cc:cc} Q^a_{ijk} & S^a_{ijk} & 0 & 0 \\ (S^a_{ijk})^T & R^a_{ijk} & 0 & 0 \\ \hdashline 0 & 0 & -\gamma_{ijk} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{array}\right] \left[\begin{array}{cc} I & 0 \\ H_k^A & (H_k^B)_{:j} \\ \hdashline 0 & 1 \\ (H_k^C)_{i:} & (H_k^D)_{ij} \end{array}\right] < 0, \tag{2.59}$$

$$[*]^T \left[\begin{array}{cc:cc} Q^b_{ijk} & S^b_{ijk} & 0 & 0 \\ (S^b_{ijk})^T & R^b_{ijk} & 0 & 0 \\ \hdashline 0 & 0 & -\gamma_{ijk} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \end{array}\right] \left[\begin{array}{cc} I & 0 \\ H_k^A & (H_k^B)_{:j} \\ \hdashline 0 & 1 \\ (H_k^C)_{i:} & (H_k^D)_{ij} \end{array}\right] < 0, \tag{2.60}$$

$$i = 1, ..., p_1, \quad j = 1, ..., q_1, \quad k = 0, 1, ..., N.$$

Proof: We only point out the differences to the case before, since the proof follows along the same lines as the proofs of Theorems 2.6 and 2.7. In the MIMO case, the inequalities (2.54) have to be addressed by Lemmas B.11 and B.12 for all $i, j, k$ instead of just for $k$. Note in this context that

$$\bar{\mathcal{G}}_{ij}(k, \Delta) = (\Psi_k(\Delta) \star H_k)_{ij} = \left[\begin{array}{c:c} H_k^A & (H_k^B)_{:j} \\ \hdashline (H_k^C)_{i:} & (H_k^D)_{ij} \end{array}\right].$$

The derivation then follows analogously as for the conditions in Theorem 2.7. ■

Concerning the upper bound $\tilde{\gamma}^{r,N}$ of the impulse response tail described in Theorem 2.8, no special properties or notations for SISO systems were exploited. Hence the computation of $\tilde{\gamma}^{r,N}$ can be adopted without any changes also in the MIMO case. Likewise, the properties discussed in Theorem 2.9 carry over to the MIMO case.

**Some facts about computational complexity**

It is important to know estimates about the computational complexity of the analysis procedure before addressing a specific problem and before making a choice for $N$. We give the number of free decision variables and the number of inequalities involved in Theorems 2.7 and 2.8, for simplicity under the assumption $p_0 = q_0$. To compute $\tilde{\gamma}^N$, one needs

- $2p_1q_1\left(q_0^2(\frac{2}{3}N^3 + 3N^2 + \frac{13}{3}N + 2) + q_0(\frac{1}{2}N^2 + \frac{3}{2}N + 1)\right) + p_1q_1(N+1)$ free scalar decision variables $(\gamma_{ijk}, P^a_{ijk}, P^b_{ijk})$ and
- $p_1q_1(N+1)(5 + 2n_\Delta) + p_1$ inequalities.

To find $\tilde{\gamma}^{r,N}$, one needs

- $q_0^2(4N^2 + 16N + 16) + q_0(4N + 8) + \frac{1}{2}(n^2 + n) + 2$ free scalar decision variables $(\mu, \eta^N, X, P_1, P_2)$ and
- $6 + 2n_\Delta$ inequalities.

Thus the number of variables scales with $N^3$ overall. It is evident that the method requires a large value of $N$ and thus a large computational effort if the system's impulse response does not decay to zero rapidly, for example if the absolute values of some eigenvalues of $\bar{A}$ are close to one. This is a common problem of all $\ell_1$-norm based analysis and synthesis approaches, and no general work-around with low conservatism has been found so far. There are, however, certain relaxations possible to decrease the number of multiplier variables $P^a_{ijk}$, $P^b_{ijk}$, $P_1$, $P_2$, see Section 2.5.4.

### 2.5.3 MIMO Plant and Time-Varying Uncertainty

After having treated the case of time-invariant parametric uncertainty in the Sections 2.5.1 and 2.5.2, time-varying parametric uncertainties are discussed next. Recall that for the computation of the robust star-norm with constant Lyapunov function in Section 2.4, it did not matter if the uncertainty was time-varying or time-invariant. For the robust $\ell_\infty$-gain computation suggested here, differences in both the results and the computational complexity appear. Moreover the presentation is split up in two separate parts for unrestricted uncertainty and for rate-bounded uncertainty.

**Unrestricted Time-Varying Uncertainty**

Suppose that the values of $\Delta$ are allowed to change over time (i.e. $\Delta \in \mathbf{\Delta}_{\mathrm{TV}}$) and that $p_1 \geq 1$, $q_1 \geq 1$. The response of (2.11) at time step $k$ to an input $w_1$ with $\xi(0) = 0$ is

$$z_1(k) = \sum_{l=0}^{k} \bar{\mathcal{G}}(k, l, \Delta) w_1(l).$$

Similarly to Lemma 2.5, expressions for the Markov parameters $\bar{\mathcal{G}}(k, l, \Delta)$ are given next.

**Lemma 2.11** *Using the definitions of Lemma 2.5, it holds that*

$$\bar{\mathcal{G}}(k,l,\Delta) = \begin{cases} \bar{\mathcal{D}}_{11}(\Delta(k)), & l = k \\ \bar{\mathcal{C}}_1(\Delta(k))\left(\prod_{i=l+1}^{k-1} \bar{\mathcal{A}}(\Delta(i))\right) \bar{\mathcal{B}}_1(\Delta(l)), & l < k \end{cases} \tag{2.61}$$

$$= \underbrace{\mathrm{diag}(\Delta(k), ..., \Delta(l))}_{\Lambda_{k,l}(\Delta)} \star \left[\begin{array}{c:c} H^A_{k-l} & H^B_{k-l} \\ \hdashline H^C_{k-l} & H^D_{k-l} \end{array}\right]. \tag{2.62}$$

Proof: We only sketch the proof briefly. The special form (2.61) of the Markov parameters $\bar{\mathcal{G}}(k, l, \Delta)$ follows similarly to the time-invariant case by repeatedly inserting the equations of (2.11) into each other. The representation (2.62) in terms of an LFT is obtained by replacing the parameter-dependent matrices $\bar{\mathcal{A}}(\Delta(k))$ etc. with the LFT (2.12), and repeatedly applying Lemma B.4(ii). ■

An expression for the robust $\ell_\infty$-gain of (2.11) in the time-varying case is stated in the next lemma.

**Lemma 2.12** *The robust $\ell_1$ performance of (2.11) is*

$$\gamma = \sup_{\Delta \in \mathbf{\Delta}_{\mathrm{TV}}} \|\bar{\mathcal{G}}(\Delta)\|_{\infty-\mathrm{ind}} = \sup_{\Delta \in \mathbf{\Delta}_{\mathrm{TV}}} \max_{1 \leq i \leq p_1} \sum_{j=1}^{q_1} \sum_{k=0}^{\infty} |(\Lambda_{k,0}(\Delta) \star H_k)_{ij}|. \tag{2.63}$$

Proof: The expression for the $\ell_\infty$-gain of a time-varying system is well-known and can for example be found in Khammash and Pearson (1991); Dahleh and Diaz-Bobillo (1995, Section 2.3). Its extension to the robust $\ell_\infty$-gain used here is

$$\begin{aligned}\sup_{\Delta\in\boldsymbol{\Delta}_{\mathrm{TV}}} \|\bar{\mathcal{G}}(\Delta)\|_{\infty-\mathrm{ind}} &= \sup_{\Delta\in\boldsymbol{\Delta}_{\mathrm{TV}}} \sup_{k\geq 0} \max_{1\leq i\leq p_1} \sum_{j=1}^{q_1}\sum_{l=0}^{k} |\bar{\mathcal{G}}_{ij}(k,l,\Delta)| \\ &= \sup_{\Delta\in\boldsymbol{\Delta}_{\mathrm{TV}}} \sup_{k\geq 0} \max_{1\leq i\leq p_1} \sum_{j=1}^{q_1}\sum_{l=0}^{k} |(\Lambda_{k,l}(\Delta)\star H_{k-l})_{ij}|.\end{aligned}$$

The particular values of the $\Delta(i)$ are not significant, but only the fact that all components of $\Lambda_{k,l}(\Delta)$ obey $\Delta(i)\in\boldsymbol{\Delta}$ for all $i$. Hence for example $\Lambda_{0,0}(\Delta)$ and $\Lambda_{1,1}(\Delta)$ have the same meaning in the preceding formula. The evaluation of the supremum over $k$ thus leads to the simplified expression (2.63). ∎

Hence the proposed split for computing upper bounds on the robust $\ell_\infty$-gain is

$$\gamma \leq \underbrace{\sup_{\Delta\in\boldsymbol{\Delta}_{\mathrm{TV}}} \max_{1\leq i\leq p_1} \sum_{j=1}^{q_1}\sum_{k=0}^{N} |(\Lambda_{k,0}(\Delta)\star H_k)_{ij}|}_{\gamma^N} + \underbrace{\sup_{\Delta\in\boldsymbol{\Delta}_{\mathrm{TV}}} \max_{1\leq i\leq p_1} \sum_{j=1}^{q_1}\sum_{k=N+1}^{\infty} |(\Lambda_{k,0}(\Delta)\star H_k)_{ij}|}_{\gamma^{r,N}}.$$

Compared to the time-invariant case (2.51) with $\bar{\mathcal{G}}(k,\Delta) = \mathrm{diag}(\Delta,...,\Delta)\star H_k$, the time-varying case has an uncertainty structure of $\bar{\mathcal{G}}(k,0,\Delta) = \mathrm{diag}(\Delta(k),\Delta(k-1),...,\Delta(0))\star H_k$. The only difference to developments in the time-invariant case is the time-dependence of the $\Lambda_{k,0}$-structures. It follows that Theorems 2.8 and 2.10 still hold for the time-varying case, provided that the multiplier constraints therein are rewritten accordingly. The details are described in the following.

In the time-invariant case, the convex hull relaxation via Lemma B.12 means the replacement of semi-infinite constraints by a finite number of constraints according to the scheme

$$\begin{aligned}&\Psi_0(\Delta) = \Delta \quad \forall\Delta\in\boldsymbol{\Delta} && \rightsquigarrow && \Psi_0(\Delta_l) = \Delta_l, \quad l=1,...,n_\Delta, \\ &\Psi_1(\Delta) = \begin{bmatrix}\Delta & 0\\ 0 & \Delta\end{bmatrix} \quad \forall\Delta\in\boldsymbol{\Delta} && \rightsquigarrow && \Psi_1(\Delta_l) = \begin{bmatrix}\Delta_l & 0\\ 0 & \Delta_l\end{bmatrix}, \quad l=1,...,n_\Delta,\end{aligned}$$

and so on, since the $\Delta$-blocks are just repeated. In the time-varying case however, the replacement has to be done according to the pattern

$$\begin{aligned}&\Lambda_{0,0}(\Delta) = \Delta(0) \quad \forall\Delta\in\boldsymbol{\Delta}_{\mathrm{TV}} && \rightsquigarrow && \tilde{\Lambda}_{0,0}(\Delta) = \Delta_{l_0}, \quad l_0=1,...,n_\Delta, \\ &\Lambda_{1,0}(\Delta) = \begin{bmatrix}\Delta(1) & 0\\ 0 & \Delta(0)\end{bmatrix} \quad \forall\Delta\in\boldsymbol{\Delta}_{\mathrm{TV}} && \rightsquigarrow && \tilde{\Lambda}_{1,0}(\Delta) = \begin{bmatrix}\Delta_{l_1} & 0\\ 0 & \Delta_{l_0}\end{bmatrix}, \\ &&&&& l_0=1,...,n_\Delta, \quad l_1=1,...,n_\Delta,\end{aligned}$$

since the $\Delta$-blocks in the structure $\Lambda_{k,0}(\Delta)$ satisfy $\Delta(k), ..., \Delta(0) \in \mathbf{\Delta}$ independently of each other. Based on this observation, define $\tilde{\Lambda}_{k,0}(\Delta) := \mathrm{diag}(\Delta_{l_k}, ..., \Delta_{l_0})$. The above considerations are summarized as follows.

In Theorem 2.10, the conditions (2.57)–(2.58) are replaced by

$$Q^a_{ijk} < 0, \qquad [*]^T \begin{bmatrix} Q^a_{ijk} & S^a_{ijk} \\ (S^a_{ijk})^T & R^a_{ijk} \end{bmatrix} \begin{bmatrix} \tilde{\Lambda}_{k,0}(\Delta) \\ I \end{bmatrix} \geq 0, \qquad l_0 = 1, ..., n_\Delta, \; ..., \; l_k = 1, ..., n_\Delta,$$

$$Q^b_{ijk} < 0, \qquad [*]^T \begin{bmatrix} Q^b_{ijk} & S^b_{ijk} \\ (S^b_{ijk})^T & R^b_{ijk} \end{bmatrix} \begin{bmatrix} \tilde{\Lambda}_{k,0}(\Delta) \\ I \end{bmatrix} \geq 0, \qquad l_0 = 1, ..., n_\Delta, \; ..., \; l_k = 1, ..., n_\Delta.$$

In Theorem 2.8, the conditions (2.45)–(2.46) are replaced by

$$Q_1 < 0, \qquad [*]^T \begin{bmatrix} Q_1 & S_1 \\ S_1^T & R_1 \end{bmatrix} \begin{bmatrix} \tilde{\Lambda}_{N+1,0}(\Delta) \\ I \end{bmatrix} \geq 0, \qquad l_0 = 1, ..., n_\Delta, \; ..., \; l_{N+1} = 1, ..., n_\Delta,$$

$$Q_2 < 0, \qquad [*]^T \begin{bmatrix} Q_2 & S_2 \\ S_2^T & R_2 \end{bmatrix} \begin{bmatrix} \tilde{\Lambda}_{N+1,0}(\Delta) \\ I \end{bmatrix} \geq 0, \qquad l_0 = 1, ..., n_\Delta, \; ..., \; l_{N+1} = 1, ..., n_\Delta.$$

It is clear from this discussion that the time-varying nature of the parameters is taken into account by imposing additional constraints on the multipliers $P^a_{ijk}$, $P^b_{ijk}$, $P_1$, $P_2$ compared to the case with time-invariant parameters. In general, the values $\tilde{\gamma}^N$ and $\tilde{\gamma}^{r,N}$ increase as a consequence, as should be expected.

To complement the discussion on the number of parameters, suppose as before that there are $n_\delta$ different scalar parameters in the uncertainty $\Delta$, with box constraints $\delta_i \in [\delta_{\min,i}, \delta_{\max,i}]$. Then the number of constraints per multiplier $P^m_{ijk}$ is $n_\Delta(k) = 2^{(k+1)n_\delta}$, $k = 0, ..., N$ for computing $\tilde{\gamma}^N$. Likewise the number of constraints per multiplier $P_i$ is $n_\Delta(N+1) = 2^{(N+2)n_\delta}$ for computing $\tilde{\gamma}^{r,N}$.

**Rate-Bounded Time-Varying Uncertainty**

It is sketched briefly, how time-varying parameters with bounds on the rate of variation as in the set $\mathbf{\Delta}_{\mathrm{RB}}$ are treated within our method. Suppose for simplicity that the scalar parameter $\Delta$ (i.e. $n_\delta = 1$) satisfies $\Delta(k) \in [-d, d]$ for all $k$. If no further constraints on the parameter trajectory are imposed, still one has $|\Delta(k+1) - \Delta(k)| \leq 2d$ in discrete time, since the value of $\Delta$ can change at most from $-d$ to $d$ (or vice versa) from one time-step to the next one. This is indicated in Figure 2.2(a), where the parameter value box in the $\Delta(k)$-$\Delta(k+1)$-space is not intersected by the dashed lines indicating the rate bounds. Hence, $n_\Delta(0) = 2$, $n_\Delta(1) = 4$, $n_\Delta(2) = 8$, ..., $n_\Delta(k) = 2^{(k+1)n_\delta}$ etc. in this situation.

If a rate bound $|\Delta(k+1) - \Delta(k)| \leq 2e$ with $0 < e < d$ is imposed, then the situation changes to the one shown in Figure 2.2(b), where the two dashed lines obtained from the rate bound are intersecting the parameter value box. It follows that $n_\Delta(0) = 2$, $n_\Delta(1) = 6$ now. The six generators forming the convex hull for $k = 1$ are the six points indicated by the bullets in Figure 2.2(b). Going to the three-dimensional $\Delta(k)$-$\Delta(k+1)$-$\Delta(k+2)$-space, it can be seen that $n_\Delta(2) = 14$, where the 14 generators lie on the edges of a cube. These considerations may be further generalized to $k > 3$

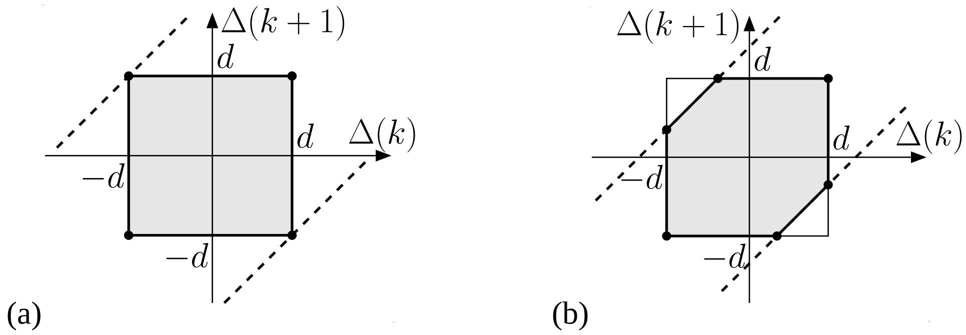



*Figure 2.2:* For a scalar time-varying parameter with $\Delta(k) \in [-d, d]$ and bounds on the rate of variation (dashed), the possible parameter values in the $\Delta(k)$-$\Delta(k+1)$-space are shown (grey box): (a) no rate bound; (b) with effective rate bound.

and to $n_\delta > 1$. Since the computational effort for time-varying parameters without rate-bounds is already significant, no more details on these generalizations are presented here. Using sum-of-squares relaxations (Scherer, 2006; Scherer and Hol, 2006), it is possible to avoid the complexity stemming from the convex hull relaxation. The price to be paid is the possibly high order of polynomials in the relaxation, however.

Even though more inequalities than in the standard time-varying parameter case have to be considered, the resulting performance is expected to be better. This is because only a part of the parameter box in the non-rate-bounded case has to be taken into account for robustness considerations. For $e = 0$, the time-invariant case is recovered. Related results in this context have been presented in Dietz and Scherer (2004) for stability analysis and in Dietz and Scherer (2005) for robust $\mathcal{L}_2$-gain analysis.

### 2.5.4 Reduction of Number of Multipliers

To reduce the computational burden of Theorems 2.7, 2.8, and 2.10, there are certain different ways of decreasing the number of involved multipliers. Such an action introduces conservatism in general. Some of these ways are briefly described next, while the effects of the relaxations are studied exemplarily in Section 2.6. The approaches are roughly ordered by conservatism (least conservative one first) and computational demand (highest demand first).

(a) Set $P^a_{ijk} = P^b_{ijk}$ for all $i, j, k$. This approach divides the number of multipliers by two. In all our numerical examples, no increased conservatism has been observed by imposing this relaxation, although no general proof has been found so far.

(b) Re-use multipliers from previous time-steps. That is,

$$P^a_{ij0} = \begin{bmatrix} Q^a_{ij0} & S^a_{ij0} \\ (S^a_{ij0})^T & R^a_{ij0} \end{bmatrix}, \qquad P^a_{ij1} = \begin{bmatrix} Q^a_{ij0} & Q^a_{ij1,1} & S^a_{ij0} & S^a_{ij1,1} \\ (Q^a_{ij1,1})^T & Q^a_{ij1,2} & S^a_{ij1,2} & S^a_{ij1,3} \\ (S^a_{ij0})^T & (S^a_{ij1,2})^T & R^a_{ij0} & R^a_{ij1,1} \\ (S^a_{ij1,1})^T & (S^a_{ij1,3})^T & (R^a_{ij1,1})^T & R^a_{ij1,2} \end{bmatrix},$$

and so on for $k > 1$, the same for index $b$. This reduces the number of decision variables from the order of $N^3$ to the order of $N^2$.

(c) Like (b), but additionally setting the “off-diagonal” blocks $Q^a_{ij1,1}$, $R^a_{ij1,1}$, $S^a_{ij1,1}$, $S^a_{ij1,2}$ (and likewise for $k > 1$ and for index $b$) to zero.

(d) Like (c), but additionally setting $Q^a_{ij1,2} = Q^a_{ij0}$, $R^a_{ij1,2} = R^a_{ij0}$, $S^a_{ij1,3} = S^a_{ij0}$ (and likewise for $k > 1$ and for index $b$), i.e. using $P^a_{ij0}$ repeatedly.

(e) Include weightings into the plant (2.9) such that $\Delta(k)^T\Delta(k) \leq I \;\forall\, k$. Then set $P^a_{ijk} = P^b_{ijk} = \mathrm{diag}(-I, I)$ for all $i, j, k$, maybe even just for certain $k$. Alternatively, set $Q^a_{ijk} = -I$, $S^a_{ijk} = 0$ for all $i, j, k$, and set $R^a_{ijk} = R$ for a fixed $R$ for all $i, j, k$, implying $\Delta(k)^T\Delta(k) \leq R \;\forall\, k$. Both choices actually allow for dynamic norm-bounded uncertainties (Scherer and Weiland, 2005, Section 6.5) and are expected to be overly conservative.

Other possibilities for multiplier restriction are conceivable, see e.g. Scherer (2000*b*); Scherer and Weiland (2005, Section 6.5); Wu and Dong (2006).

### 2.5.5 Lower Bounds

To judge the accuracy of the computed upper bounds, one way is to additionally determine lower bounds on the $\ell_\infty$-gain $\gamma$. If lower and upper bounds are close to each other, it is concluded that $\gamma$ is approximately equal to the best upper bound.

There exist several related possibilities to compute lower bounds on the robust $\ell_\infty$-gain $\gamma$ of the uncertain linear system (2.11). All of the methods considered here use a finite set of predefined uncertainties. In the time-invariant case, this set is described as $\mathbf{\Delta}_{\mathrm{low,TI}} := \{\Delta_1, ..., \Delta_r | \Delta_i \in \mathbf{\Delta}\}$, whereas in the time-varying case, the set is $\mathbf{\Delta}_{\mathrm{low,TV}} := \{\Delta_1, ..., \Delta_r | \Delta_i \in \mathbf{\Delta}_{\mathrm{TV}}\}$.

The simplest method to compute a lower bound on $\gamma$ is to take one or more specific uncertainty sequences, insert them into the system description (2.11), compute its impulse response, and determine the corresponding $\ell_1$-norms as described in Section 2.2.1. The maximum of these $\ell_1$-norms is the best approximation using such an approach. The uncertainties considered in this context could be vertices of a parameter polytope or of the convex hull, or randomly generated parameter values or sequences.

A more elaborate alternative is presented here, similar to the discussions in Calafiore and Campi (2005); Scherer (2006). Instead of solving (2.52)–(2.54) for all $\Delta \in \mathbf{\Delta}_{\mathrm{TV}}$, a lower bound on $\gamma^N$ and hence on $\gamma$ is

$$\underline{\gamma}^N = \inf \nu \tag{2.64}$$

$$\text{s.t.} \quad \sum_{j=1}^{q_1} \sum_{k=0}^{N} \gamma_{ijk} < \nu, \qquad \gamma_{ijk} > 0, \tag{2.65}$$

$$-\gamma_{ijk} < \bar{\mathcal{G}}_{ij}(k, \Delta) < \gamma_{ijk} \quad \forall \Delta \in \mathbf{\Delta}_{\mathrm{low,TI}}, \tag{2.66}$$

$$i = 1, ..., p_1, \quad j = 1, ..., q_1, \quad k = 0, 1, ..., N$$

or respectively for all $\Delta \in \mathbf{\Delta}_{\text{low,TV}}$. Hence

$$\underline{\gamma}^N \leq \gamma^N \leq \gamma \qquad \forall N \geq 0.$$

It has to be noted that the quality of lower bounds obtained in this way largely depends on the number $r$ and the spread of the sample uncertainties, i.e. the larger $r$ the better. The accuracy of these lower bounds is studied in the example of Section 2.6.

## 2.6 Example

The various tools developed in this chapter are now exemplified with a simple academic, yet very general example. The example system is an extension of the system in Section 2.2.3 to the uncertain case. Consider a MIMO system $\bar{\mathcal{G}}(\Delta)$ with two parameters $\delta_1$ and $\delta_2$. Its realization is of the form (2.9)–(2.10) with

$$\left[\begin{array}{c:c:c} \mathcal{A} & \mathcal{B}_0 & \mathcal{B}_1 \\ \hdashline \mathcal{C}_0 & \mathcal{D}_{00} & \mathcal{D}_{01} \\ \hdashline \mathcal{C}_1 & \mathcal{D}_{10} & \mathcal{D}_{11} \end{array}\right] = \left[\begin{array}{rr:rr:rr} 0.2 & 0.01 & 0.1 & 0.2 & 3 & 2 \\ -0.1 & -0.01 & 0.3 & -0.2 & 3 & 1 \\ \hdashline 0.2 & -0.3 & 0.4 & 0.3 & 3 & 1 \\ 0.8 & 0.5 & -0.6 & 0.1 & 2 & 7 \\ \hdashline 2 & 1 & 1 & 2 & 1 & -2 \\ 2 & 3 & -1 & 4 & -4 & 3 \end{array}\right], \quad \Delta(k) = \begin{bmatrix} \delta_1(k) & 0 \\ 0 & \delta_2(k) \end{bmatrix},$$

and $\delta_1(k) \in [-0.1, 0.5]$, $\delta_2(k) \in [-0.3, 0.6]$ for all $k$. Note the nonlinear dependence on $\Delta$ since $\Delta(\cdot)\mathcal{D}_{00} \not\equiv 0$. $\Delta$ is first treated as a time-invariant uncertainty. We have $n = p_0 = p_1 = q_0 = q_1 = 2$ and $n_\Delta = 2^{q_0} = 4$ since

$$\Delta \in \mathbf{\Delta} = \mathrm{Co}(\{(-0.1, -0.3), (-0.1, 0.6), (0.5, -0.3), (0.5, 0.6)\}).$$

The hard- and software used for the following computations is described in Appendix E. An upper bound on $\gamma$ is computed by means of the robust star-norm of (2.11) according to Theorem 2.4 and yields $\gamma < 94.65$. The computation time (CPU time) is $5.6$ s. Note that this upper bound also holds in the time-varying case. Another upper bound on $\gamma$ is obtained using small-gain arguments for LTV uncertainties as described in Appendix B.8, Theorem B.20. The result is $\gamma \leq 86.81$, with a computation time of $2.4$ s.

A better upper bound is computed by means of the relation (2.51). The numerical values of $\tilde{\gamma}^N$ and $\tilde{\gamma}^{r,N}$ are computed according to Theorems 2.10 and 2.8 for different $N$. From the results in Table 2.2 it can be seen that $\tilde{\gamma}^{r,N}$ decays to small values and is close to zero for $N = 5$, indicating that $\tilde{\gamma}^N$ does not change much more for large $N$. We thus have the less conservative upper bound $\gamma \leq 73.53$, computed in $62$ s for $N = 5$.

A lower bound on $\gamma$ is obtained by a standard $\ell_1$-norm computation at the "worst-case" vertex of the parameter box, $(\delta_1, \delta_2) = (0.5, 0.6)$, resulting in $\gamma \geq 66.00$. Better lower bounds for different

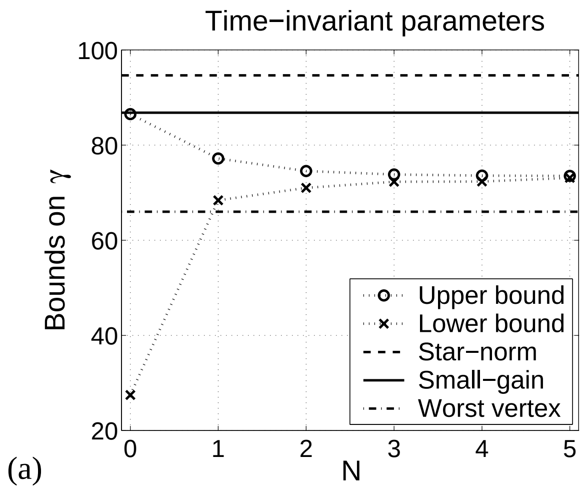
*Table 2.2:* Example. Robust $\ell_1$ performance analysis for time-invariant $\Delta$.

| $N$ | $\tilde{\gamma}^N$ | $\tilde{\gamma}^{r,N}$ | $\tilde{\gamma}^N + \tilde{\gamma}^{r,N}$ | CPU time [s] |
|---|---|---|---|---|
| 0 | 28.33 | 58.18 | 86.51 | 3.6 |
| 1 | 69.06 | 8.123 | 77.18 | 4.7 |
| 2 | 71.53 | 3.022 | 74.55 | 7.0 |
| 3 | 73.03 | 0.7857 | 73.82 | 12.3 |
| 4 | 73.36 | 0.2292 | 73.59 | 28.2 |
| 5 | 73.47 | 0.05905 | 73.53 | 62.1 |

values of $N$ are computed using (2.64)–(2.66) with $r = 500$ random points spread uniformly in the parameter space. The best lower bound is obtained as $\gamma \geq 73.10$ for $N = 5$, with a CPU time of about $48$ min. Together with the upper bounds $\tilde{\gamma}^N$, the lower bounds $\underline{\gamma}^N$ are depicted in Figure 2.3(a). The lower and upper bounds get quite close for $N = 5$, which suggests a very small or vanishing relaxation gap. It is concluded that the true robust $\ell_\infty$-gain $\gamma$ of the example system with respect to time-invariant parametric uncertainties satisfies $73.10 \leq \gamma \leq 73.53$, which is a significantly better estimate than the star-norm or small-gain based ones.

Taking time-varying $\Delta$ into account (see Section 2.5.3) is computationally more expensive due to the increased number of multiplier constraints. The computation times are in the order of minutes to some hours. The results are shown in Figure 2.3(b) together with upper and lower bounds. A range of $69.41 \leq \gamma \leq 78.10$ is established. Interestingly, the lower bounds that depend on $N$ are smaller than in the time-invariant case. This is partly due to the large variety of allowed time-varying parameter trajectories, which requires a large value of $r$ in the lower bound computation. Also, the existence of a relaxation gap is possible. All discussed trends are generally confirmed in other examples, both for time-invariant and for time-varying parametric uncertainties.

To study the effect of multiplier reduction as described in Section 2.5.4, the values $\tilde{\gamma}^N$ in the time-invariant $\Delta$ case are computed and shown for $N = 1$ and $N = 5$ in Table 2.3. It is evident



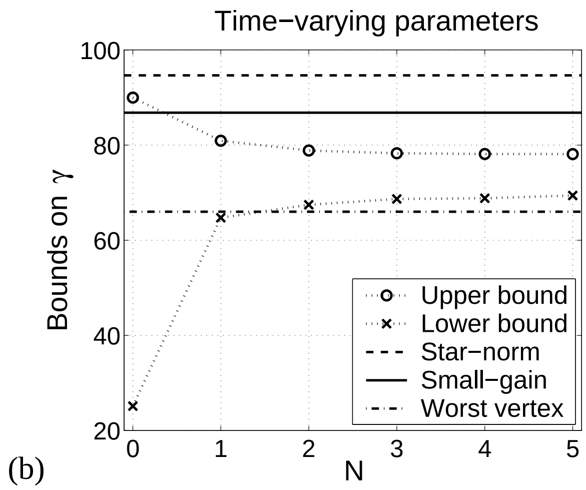




*Figure 2.3:* Example. Analysis of the robust $\ell_\infty$-gain $\gamma$ for (a) time-invariant and (b) time-varying $\Delta$. Different upper and lower bounds are shown.

that only the reduction approaches (a) and (b) yield satisfactory results. The approaches (c), (d), and (e), where parts of the multiplier are set to zero, lead to lower computation times, however at the expense of introducing considerable conservatism. Other examples generally confirm that reduction approach (a) does not introduce conservatism.

*Table 2.3:* Example. Effect of multiplier reduction on $\tilde{\gamma}^N$-values with time-invariant $\Delta$.

| Approach | $N = 1$ | $N = 5$ | CPU time for $N = 5$ [s] |
|---|---|---|---|
| No reduction | 69.06 | 73.47 | 16.9 |
| (a) | 69.06 | 73.47 | 12.7 |
| (b) | 69.06 | 75.03 | 16.2 |
| (c) | 73.21 | 105.0 | 6.2 |
| (d) | 80.35 | 183.4 | 5.1 |
| (e), only for $k = N$ | 92.85 | 115.1 | 10.6 |
| (e), for $k = 0, \dots, N$ | 106.1 | 274.2 | 1.5 |

## 2.7 Summary

This chapter introduces novel approaches to analyze stability and performance of linear systems with parametric uncertainties. The system's state-space matrices may be rationally dependent on time-invariant or time-varying parameters. First, upper bounds on the robust $\ell_\infty$-gain are computed based on the star-norm. As an auxiliary result, a novel matrix inequality condition to determine the star-norm of an LTI system is introduced. Although the star-norm based approach is conservative, it is useful for SISO systems, or to quickly get an estimate of the $\ell_\infty$-gain of MIMO systems, or to get an estimate of the peak-induced norm of a system. Second, a more direct approach using $\ell_1$-norm computations of uncertain systems is introduced. To this end, the response of an uncertain system is characterized in terms of an LFT. The resulting upper bounds on the robust $\ell_\infty$-gain are considerably less conservative than existing methods, yet with a possibly high computational price. Accuracy and computational effort can be traded off by various means. An example demonstrates the properties and the applicability of the approaches.
In connection with the well-known analysis results for systems with dynamic uncertainties, there is now a set of tools available for computing upper bounds on the robust $\ell_\infty$-gain of uncertain systems, tailored to the various uncertainty classes.

# Chapter 3

# Synthesis of LTI Controllers

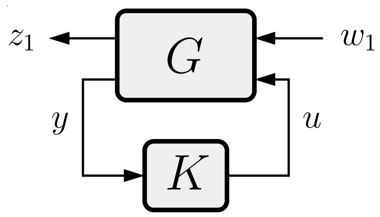
*In this chapter, SDP formulations for the synthesis of LTI controllers are established. After briefly reviewing the background of nominal $\ell_1$-optimal controller design, the synthesis of multi-objective and robust LTI controllers is treated. The main results are*

- *a novel and computationally efficient formulation for $\mathcal{H}_\infty$ and $\mathcal{H}_2$ constraints in a general multi-objective controller synthesis framework (Section 3.3.2), and*
- *an LMI condition to design robust state-feedback controllers for discrete-time linear systems with parametric uncertainties using a quadratic performance criterion (Theorem 3.5).*

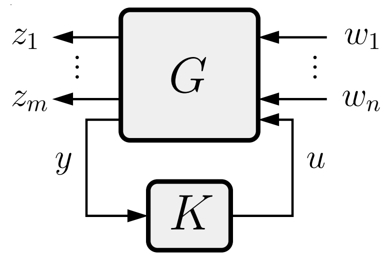
## 3.1 Overview

The design of LTI controllers that satisfy norm-based performance objectives like $\mathcal{H}_\infty$, $\mathcal{H}_2$, or $\ell_1$ performance criteria has been thoroughly studied in the literature for systems without uncertainties. If an uncertain system is addressed, methods like $\mathcal{H}_\infty$- or $\ell_1$-optimal control are also applicable to synthesize controllers with robust stability guarantees against unstructured dynamic uncertainties. In the same way, multi-objective control approaches allow to include the same type of robust stability guarantees with additional nominal or robust norm-based performance specifications. It is possible to solve these types of synthesis problems using convex conditions. Hence an efficient controller design is possible in such a norm-based framework. Introductions and overviews on the topics of $\mathcal{H}_\infty$, $\mathcal{H}_2$, $\ell_1$, and multi-objective control are given in Dahleh and Diaz-Bobillo (1995); Zhou *et al.* (1996); Sanchez-Pena and Sznaier (1998); Skogestad and Postlethwaite (2005); Rieber and Allgöwer (2006*a*), for example.

If, on the other hand, there is more information on the uncertainties available, it is less conservative to consider more involved robustness conditions. The same holds true for the inclusion of robust performance specifications. In these cases it is necessary to rely on the concept of structured and possibly mixed uncertainties. To come up with controller synthesis procedures, often the existing robust analysis conditions are extended. This leads to the well-known methods of $\mu$ synthesis using $D$-$K$-iterations (Doyle, 1982; Doyle *et al.*, 1991; Packard and Doyle, 1993), LMI-based synthesis




(a)




(b)

*Figure 3.1:* Control setups: generalized plant $G$ in loop with controller $K$. (a) $\ell_1$-optimal synthesis. (b) Multi-objective synthesis.

using static or dynamic multipliers (Helmersson, 1995; Scherer, 2000*b*; Köse and Scherer, 2006), or synthesis with respect to $\ell_\infty$-bounded uncertainties (Dahleh and Khammash, 1993; Khammash *et al.*, 2001). Convex conditions are obtained for robust state-feedback controllers, whereas it is not known how to formulate the robust output-feedback synthesis exactly in a convex way.

This chapter contributes in two ways to the synthesis of LTI controllers. A new efficient formulation of $\mathcal{H}_\infty$ and $\mathcal{H}_2$ constraints in a general multi-objective control setting is proposed in Section 3.3. The formulation includes $\mathcal{H}_\infty$-, $\mathcal{H}_2$-, and $\ell_1$-norm constraints as well as time-domain constraints. Comparisons to existing approaches show the favorable properties of the proposed method in terms of complexity and computation times. Furthermore, in Section 3.4 a convex condition for robust state-feedback controller synthesis in a discrete-time setting is derived to complement existing continuous-time results. Before going into details, the synthesis of $\ell_1$-optimal LTI controllers is reviewed briefly in Section 3.2 as a foundation for subsequent sections. This chapter is based partly on Rieber and Allgöwer (2005); Rieber *et al.* (2006*a*); Rieber and Allgöwer (2006*a*).

## 3.2 Review of $\ell_1$-Optimal Controller Synthesis

We briefly review the synthesis of $\ell_1$-optimal controllers for systems without uncertainties. This is done to facilitate the understanding of the synthesis procedures in Section 3.3 (multi-objective controller synthesis) and in Section 4.5 (LPV controller synthesis).

The $\ell_1$ control problem was formulated in Barabanov and Granichin (1984); Vidyasagar (1986). The $\ell_1$ control literature generally treats discrete-time problems, since only these lead to numerically tractable synthesis conditions. Suppose a finite-dimensional discrete-time LTI plant $G$ without uncertainties is given as in Figure 3.1(a) and has the state-space realization

$$\begin{bmatrix} x(k+1) \\ z_1(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} A & B_1 & B_u \\ C_1 & D_{11} & D_{1u} \\ C_y & D_{y1} & D_{yu} \end{bmatrix} \begin{bmatrix} x(k) \\ w_1(k) \\ u(k) \end{bmatrix}$$

with states $x(k) \in \mathbb{R}^{n_x}$. A controller takes measurements $y(k) \in \mathbb{R}^{n_y}$ and acts on the system by means of the control inputs $u(k) \in \mathbb{R}^{n_u}$. Exogenous inputs $w_1(k) \in \mathbb{R}^{q_1}$ such as disturbances or reference commands enter the system externally, whereas outputs $z_1(k) \in \mathbb{R}^{p_1}$ are used to quantify the system performance.

It is assumed that $(A, B_u)$ is stabilizable and $(A, C_y)$ is detectable, which is necessary and sufficient for the existence of a stabilizing LTI output-feedback controller $K$ with realization

$$\begin{bmatrix} x_K(k+1) \\ u(k) \end{bmatrix} = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \begin{bmatrix} x_K(k) \\ y(k) \end{bmatrix}, \tag{3.1}$$

see e.g. Zhou *et al.* (1996, Section 12.1). Denote the closed-loop map from $w_1$ to $z_1$ as $\mathcal{G}$.

The *standard* $\ell_1$ *control problem* is to find an internally stabilizing controller such that the closed-loop $\ell_1$ performance $\|\mathcal{G}\|_1$ is minimized. To convert this problem into a more tractable form, the well-known Youla parameterization of all stabilizing controllers and all stable closed-loop maps is applied (Youla *et al.*, 1976; Zhou *et al.*, 1996, Chapter 12). Lemma B.13 in the appendix describes a version of this parameterization in detail. Thus, all asymptotically stable closed-loop transfer matrices are written as

$$\hat{\mathcal{G}}(z) = \hat{H}(z) - \hat{U}(z)\hat{Q}(z)\hat{V}(z),$$

where $\hat{Q}$ is a free stable parameter, and $\hat{H}$, $\hat{U}$, $\hat{V}$ are fixed and stable transfer matrices. The standard $\ell_1$ control problem can hence be stated alternatively as finding the optimal value

$$\gamma := \inf_{Q \in \ell_1^{n_u \times n_y}} \|H - U * Q * V\|_1. \tag{3.2}$$

and the optimal argument $Q_{\mathrm{opt}}$ thereof. It turns out that this problem is infinite-dimensional with infinitely many constraints. Moreover, the solution may not even exist or may be non-unique. Rigorous characterizations of the solution are given in Dahleh and Pearson (1987*a*); Dahleh and Pearson (1987*b*); Diaz-Bobillo and Dahleh (1993). Various approaches to solve (3.2) in a tractable manner are presented in Staffans (1993); Dorea and Hennet (1997); Elia and Dahleh (1998); Khammash (2000); Casavola and Famularo (2003); Hurak *et al.* (2006). We briefly recall the scaled-$Q$ method of Khammash (2000) next.

A way to parameterize all stable $Q$ parameters is by means of an infinite sequence

$$Q = \{Q(0), Q(1), Q(2), ...\} \quad \text{corresponding to} \quad \hat{Q}(z) = Q(0) + Q(1)z^{-1} + Q(2)z^{-2} + \cdots \tag{3.3}$$

with $Q(k) \in \mathbb{R}^{n_u \times n_y}$ for all $k$. A tractable problem is obtained from (3.2) by additionally imposing $Q(k) = 0$ for $k > N$ and $\|Q\|_1 \leq \gamma_Q$ for some given $N$ and $\gamma_Q$, respectively. These restrictions on the free parameter lead to the *suboptimal problem*

$$\begin{aligned} \overline{\gamma}_N := &\inf_{Q \in \ell_1^{n_u \times n_y}} \|H - U * Q * V\|_1 \\ &\text{s.t.} \quad Q(k) = 0 \quad \text{for} \quad k > N, \qquad \|Q\|_1 \leq \gamma_Q. \end{aligned} \tag{3.4}$$

It can be shown that a solution to (3.4) exists under mild technical assumptions. Moreover, $\overline{\gamma}_N \geq \overline{\gamma}_{N+1} \geq \gamma$ for all $N \geq 0$, and $\overline{\gamma}_N \to \gamma$ as $N \to \infty$, if $\gamma$ exists (Khammash, 2000). This means that $\overline{\gamma}_N$ is an upper bound on $\gamma$ and converges monotonically to $\gamma$ from above. Note that $\|Q\|_1 \leq \gamma_Q$ does not constrain the problem if $\gamma_Q$ is chosen to be greater than or equal to the $\ell_1$-norm of the optimal argument $Q_{\mathrm{opt}}$ (where one does not necessarily know $\gamma_Q$ in advance). If no solution to (3.2) exists, then the constraint $\|Q\|_1 \leq \gamma_Q$ regularizes the problem insofar as undesired solutions with large $\ell_1$-norm of $Q$ are excluded (Khammash, 2000). It has to be noted that this approach may result in controllers with high order due to generic choices of stable basis functions in the Youla parameterization. Hence it may be necessary to do controller order reduction steps afterwards. This is a common problem for all $\ell_1$ analysis and synthesis methods.

Similarly, a relaxation of (3.2) is obtained by discarding some of the involved convolution constraints. In particular, one considers the *superoptimal problem*

$$\begin{aligned} \underline{\gamma}_N := & \inf_{Q \in \ell_1^{n_u \times n_y}} \|H - U * Q * V\|_1 \\ & \text{s.t.} \qquad (H - U * Q * V)(k) = 0 \quad \text{for} \quad k > N, \qquad \|Q\|_1 \leq \gamma_Q. \end{aligned} \tag{3.5}$$

Again, a solution to this formulation can be shown to exist with the properties $\underline{\gamma}_N \leq \underline{\gamma}_{N+1} \leq \gamma$ for all $N \geq 0$ and $\underline{\gamma}_N \to \gamma$ as $N \to \infty$ if $\gamma$ exists (Khammash, 2000). Thus $\underline{\gamma}_N$ is a lower bound on $\gamma$ and converges monotonically to $\gamma$ from below.

Both (3.4) and (3.5) can be transformed into finite-dimensional LPs and hence allow a computationally efficient controller synthesis in principle (Khammash, 2000). A controller satisfying a prescribed performance tolerance $\delta > 0$ is obtained if for some $N = N_0$ the lower and upper bounds are sufficiently close according to $|\overline{\gamma}_{N_0} - \underline{\gamma}_{N_0}| \leq \delta$. Then a suboptimal controller $K$ achieving the objective value $\overline{\gamma}_{N_0}$ is constructed from the corresponding Youla parameter $Q$ using the formula (B.4).

It is summarized next why the $\ell_1$ framework is attractive for controller synthesis. More elaborate discussions can be found in Vidyasagar (1986); Dahleh and Khammash (1993); Dahleh and Diaz-Bobillo (1995). First, the setup considers persistent disturbances, i.e. disturbances with bounded $\ell_\infty$-norm, which can be useful if for example bounded-energy disturbances ($\ell_2$ signals) are not suited to model a problem. Moreover, the $\ell_1$ framework allows to incorporate time-domain design specifications like maximum amplitudes or maximum rates of variation directly and intuitively. Finally, the controller synthesis is done via LP problems, which allows efficient computation even for larger-sized problems. Current drawbacks of the $\ell_1$ framework are the often high order of the resulting controllers, and the possibly large size of the LPs. Accounts on how the $\ell_1$ framework can be used for practical applications are found in e.g. Spillman and Ridgely (1997); Tadeo *et al.* (1998); Malaterre and Khammash (2000); Rieber *et al.* (2005*b*); Stemmer *et al.* (2005); Rieber and Allgöwer (2006*b*).

Some interesting properties of $\ell_1$-optimal synthesis are that the optimal controller may be non-unique, that the optimal controller may be dynamic even in the state-feedback case, that nonlinear static state-feedback performs as well as linear dynamic feedback, and that nonlinear controllers may result in better performance than the optimal linear controller (Diaz-Bobillo and Dahleh, 1992; Dahleh and Shamma, 1992; Shamma, 1993; Blanchini and Sznaier, 1995; Stoorvogel, 1995; Shamma, 1996). Relations to continuous-time controllers and sampled-data implementations are established in Dullerud and Francis (1992); Ohta *et al.* (1992); Blanchini and Sznaier (1994); Chen and Francis (1995).

## 3.3 Efficient Multi-Objective Controller Synthesis

Norm-based control paradigms such as $\mathcal{H}_\infty$-, $\mathcal{H}_2$-, and $\ell_1$-optimal control are suited to handle control system performance requirements in a quantitative fashion. Moreover, these paradigms are able to deal with uncertain systems by means of robustness considerations with respect to unstructured perturbations. In particular the $\mathcal{H}_\infty$ control paradigm has received widespread attention for practical applications in academia and industry lately. Yet the mentioned approaches differ in the characterization of disturbances, of performance, and of uncertainties. If in a control problem there is more than one type of disturbance, or more than one type of performance criterion, or more than one type of uncertainty description involved, then it is desirable to consider a combination of the mentioned norm-based approaches. The so-called multi-objective control framework for linear systems formalizes this idea by including different types of control system requirements into one optimization problem. These requirements are directly imposed during the controller synthesis, and hence do not have to be achieved by time-consuming iterative tuning procedures. Altogether the multi-objective framework is suited to handle quantitative control performance requirements, to trade-off conflicting design goals, and to include robustness considerations.

Theoretical developments on $\mathcal{H}_\infty$ control have been initiated in Zames (1981) and have led to different solution strategies as described in Francis (1987); Doyle *et al.* (1989); Gahinet and Apkarian (1994); Iwasaki and Skelton (1994); Scherer *et al.* (1997); Masubuchi *et al.* (1998); Oliveira *et al.* (2002). The origins of $\mathcal{H}_2$ control go back to the classical linear quadratic regulator problem treated in the 1950s and 1960s, see e.g. Bryson and Ho (1975); Kailath (1980); Zhou *et al.* (1996). Modern solution approaches are presented in Doyle *et al.* (1989); Scherer *et al.* (1997); Masubuchi *et al.* (1998); Oliveira *et al.* (2002). Interest in the mixed-norm controller synthesis started at the end of the 1980s by considering $\mathcal{H}_2/\mathcal{H}_\infty$ constraints and has been extended with star-norm and time-domain constraints (Bernstein and Haddad, 1989; Khargonekar and Rotea, 1991; Rotstein and Sideris, 1994; Scherer *et al.*, 1997; Masubuchi *et al.*, 1998; Bu and Sznaier, 2000; Ebihara and Hagiwara, 2004). True multi-objective approaches with potentially less conservatism have been addressed soon after for different combinations of the $\mathcal{H}_2$-, $\mathcal{H}_\infty$-, and $\ell_1$-norms (Voulgaris, 1994; Scherer, 1995; Chen and Wen, 1995; Salapaka *et al.*, 1997; Young and Dahleh, 1997; Sznaier and Bu, 1998; Hindi *et al.*, 1998; Salapaka *et al.*, 1999; Scherer, 2000*a*; Scherer, 2001*b*), leading to a uniform and general formulation in Qi *et al.* (2001); Qi *et al.* (2004). The meaning of constraints in multi-objective control is for example discussed in Skogestad and Postlethwaite (2005); Rieber and Allgöwer (2006*a*).

Several contributions have shown ways of formulating the multi-objective control problem in a convex manner as an SDP, which in principle can be solved efficiently with widely available numerical algorithms. Yet it is essential to restrict memory usage and computation time by using efficient formulations. In this context, critical issues are the size of optimization constraints, the number of involved variables, and the computation time. The high computational complexity is mainly due to the possibly large controller order (because of Youla parameterization), the number of involved constraints, and multivariable formulations.

This section proposes a novel formulation for $\mathcal{H}_\infty$ and $\mathcal{H}_2$ constraints in a multi-objective control setting using the Youla parameterization. Moreover, our formulation is compared to existing approaches in terms of computational complexity in a quantitative way, showing its favorable properties for a selection of typical applications examples. To this end, resulting LMI sizes and free variable sizes are analyzed, and computation times for some typically sized problems are evaluated. The study reveals from both an analytical and a practical perspective, where the advantages and drawbacks of the considered methods are.

### 3.3.1 Multi-Objective Control Formulation and Relaxations

First, a general formulation for multi-objective controller synthesis based on the Youla parameterization is reviewed. The presentation goes along the lines of Qi *et al.* (2004). Consider a finite-dimensional LTI plant $G$ with state-space realization

$$\begin{bmatrix} x(k+1) \\ z_1(k) \\ \vdots \\ z_m(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} A & B_1 & \cdots & B_n & B_u \\ C_1 & D_{11} & \cdots & D_{1n} & D_{1u} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_m & D_{m1} & \cdots & D_{mn} & D_{mu} \\ C_y & D_{y1} & \cdots & D_{yn} & D_{yu} \end{bmatrix} \begin{bmatrix} x(k) \\ w_1(k) \\ \vdots \\ w_n(k) \\ u(k) \end{bmatrix}$$

and state-vector $x(k) \in \mathbb{R}^{n_x}$, measurements $y(k) \in \mathbb{R}^{n_y}$, control inputs $u(k) \in \mathbb{R}^{n_u}$, performance outputs $z_i(k) \in \mathbb{R}^{p_i}$, and exogenous inputs $w_j(k) \in \mathbb{R}^{q_j}$. It is assumed that $(A, B_u)$ is stabilizable and $(A, C_y)$ is detectable, which is a necessary and sufficient condition for the existence of a stabilizing LTI output-feedback controller $K$ with realization (3.1). The control configuration is depicted in Figure 3.1(b).

Applying the Youla parameterization of all stabilizing controllers (see Lemma B.13), all stable closed-loop transfer functions $\hat{\mathcal{G}}_{ij}$ of the transfer channel $w_j \mapsto z_i$ are expressed as

$$\hat{\mathcal{G}}_{ij}(\hat{Q})(z) = \hat{H}_{ij}(z) - \hat{U}_i(z)\hat{Q}(z)\hat{V}_j(z) \tag{3.6}$$

in terms of a common free parameter $\hat{Q} \in \mathcal{RH}_\infty^{n_u \times n_y}$. Without loss of generality, assume that the transfer matrices $\hat{U}_i$ have full column normal rank, and that the $\hat{V}_j$ have full row normal rank.

Assume furthermore that the $\hat{U}_i$ and $\hat{V}_j$ have finite support, i.e. their components are polynomial in $z^{-1}$. The rank assumptions can be satisfied by removing redundant controls or measurements (Dahleh and Diaz-Bobillo, 1995, Section 6.2). If the transfer functions $\hat{U}_i$ and $\hat{V}_j$ are rational in $z^{-1}$, the finite support assumption is achieved by absorbing the denominators in $\hat{Q}$ (Khammash, 2000). The free parameter $\hat{Q}$ is chosen in the form (compare (3.3))

$$\hat{Q}(z) = \sum_{k=0}^{\infty} Q(k) z^{-k}. \tag{3.7}$$

With the above definitions and preliminaries in place, a general multi-objective control problem is defined next. The formulation considers minimizing a linear combination of the $\mathcal{H}_\infty$-, $\mathcal{H}_2$-, and $\ell_1$-norms of certain transfer channels, subject to norm constraints on the free parameter $Q$ and on specified transfer channels.

**Problem 3.1** *(Multi-objective control problem)*

$$\begin{aligned}
\mu := \inf_Q \sum_{l=1}^{r_1} \alpha_l \|\mathcal{G}_{i_l j_l}(Q)\|_1 + \sum_{l=s_1+1}^{r_2} \alpha_l \|\hat{\mathcal{G}}_{i_l j_l}(\hat{Q})\|_2 + \sum_{l=s_2+1}^{r_3} \alpha_l \|\hat{\mathcal{G}}_{i_l j_l}(\hat{Q})\|_\infty \\
\text{s.t.} \quad & \|\mathcal{G}_{i_l j_l}(Q)\|_1 \le \gamma_l, \quad l = r_1+1, ..., s_1, \\
& \|\hat{\mathcal{G}}_{i_l j_l}(\hat{Q})\|_2 \le \gamma_l, \quad l = r_2+1, ..., s_2, \\
& \|\hat{\mathcal{G}}_{i_l j_l}(\hat{Q})\|_\infty \le \gamma_l, \quad l = r_3+1, ..., s_3, \\
& \underline{g}_l(k) \le (\mathcal{G}_{i_l j_l}(Q) * h_l)(k) \le \overline{g}_l(k) \quad \forall k, \quad l = s_3+1, ..., s_4, \\
& \|Q\|_1 \le \gamma_Q. \qquad \square
\end{aligned}$$

Here, $1 \le r_1 \le s_1 \le r_2 \le s_2 \le r_3 \le s_3 \le s_4$ are constants to enumerate the involved constraints. $\alpha_l$, $\gamma_l$, $\gamma_Q$ are given constants, and $h_l$, $\underline{g}_l$, $\overline{g}_l$ are given sequences. The indices are $i_l \in \{1, ..., m\}$, $j_l \in \{1, ..., n\}$ to choose the considered transfer channels. It is assumed that the bounds $\gamma_l$ and the sequences $\underline{g}_l(k)$, $\overline{g}_l(k)$ are chosen such that the associated constraints are feasible. See Rieber and Allgöwer (2006*a*) for a discussion on the practical meaning of the various constraints. For comments on the additional regularizing constraint $\|Q\|_1 \le \gamma_Q$, see Section 3.2.

To make the formulation of Problem 3.1 more concrete, two special cases are given as examples. The well-known $\mathcal{H}_2$/$\mathcal{H}_\infty$ problem

$$\inf_{K \text{ stabilizing}} \|\hat{\mathcal{G}}_{11}\|_2 \quad \text{s.t.} \quad \|\hat{\mathcal{G}}_{22}\|_\infty \le \gamma$$

can be rewritten in the form of Problem 3.1. Another special case is for example

$$\inf_{K \text{ stabilizing}} \|\hat{\mathcal{G}}_{11}\|_2 + 3\|\hat{\mathcal{G}}_{22}\|_\infty \quad \text{s.t.} \quad \|\mathcal{G}_{21}\|_1 \le \gamma_1, \qquad \|\hat{\mathcal{G}}_{12}\|_\infty \le \gamma_2.$$

Problem 3.1 is not directly tractable in general, since infinitely many constraints (the convolution constraints stemming from the system dynamics) and infinitely many degrees of freedom (the parameters in $Q$, see (3.7)) are involved. However, tractable relaxations resulting in upper and lower bounds on $\mu$ are derived in the literature, see Scherer (1995); Hindi *et al.* (1998); Sznaier and Bu (1998); Scherer (1999); Khammash (2000); Salapaka and Dahleh (2000); Scherer (2000*a*); Qi *et al.* (2004); Rieber *et al.* (2006*a*) for a rather complete picture. These bounds can be made arbitrarily close to $\mu$, and thus a suitable controller can be computed. In particular the upper bound relaxation is important for controller synthesis, since the controller to be implemented is computed using this relaxation. Hence the following treatment focuses on the upper bound.

**Upper Bound Approximation**

The upper bound approximation considered in Qi *et al.* (2004) is based on the idea of truncating the infinite number of variables (degrees of freedom) in the $Q$ parameter of Problem 3.1, see also (3.7). The result is a suboptimal approximation to Problem 3.1 with guarantees for the existence of solutions and with certain convergence properties. This is described next. Suppose that instead of the infinite sequence $Q$ corresponding to (3.7), a truncated version $Q^N := \mathcal{P}_N Q$ is considered. Then a relaxation of Problem 3.1 is the following.

**Problem 3.2** *(Upper bound relaxation)*

$$
\begin{aligned}
\overline{\mu}_N := \inf_{Q^N} \sum_{l=1}^{r_1} \alpha_l \gamma_l + \sum_{l=s_1+1}^{r_2} \alpha_l \gamma_l + \sum_{l=s_2+1}^{r_3} \alpha_l \gamma_l \\
\text{s. t.} \quad & \|\mathcal{G}_{i_l j_l}(Q^N)\|_1 \le \gamma_l, \quad l = 1, ..., s_1, \\
& \|\hat{\mathcal{G}}_{i_l j_l}(\hat{Q}^N)\|_2 \le \gamma_l, \quad l = s_1 + 1, ..., s_2, \\
& \|\hat{\mathcal{G}}_{i_l j_l}(\hat{Q}^N)\|_\infty \le \gamma_l, \quad l = s_2 + 1, ..., s_3, \\
& \underline{g}_l(k) \le (\mathcal{G}_{i_l j_l}(Q^N) * h_l)(k) \le \overline{g}_l(k) \quad \forall k, \quad l = s_3 + 1, ..., s_4, \\
& \|Q^N\|_1 \le \gamma_Q. \qquad \square
\end{aligned}
$$

The following theorem states that the sequence $\overline{\mu}_N$ is non-increasing and converges to $\mu$ from above.

**Theorem 3.1** *(Qi* et al.*, 2004)*

*For Problem 3.2 it holds true that* $\overline{\mu}_N \ge \overline{\mu}_{N+1} \ge \mu$ *for all* $N$, *and* $\overline{\mu}_N \to \mu$ *for* $N \to \infty$. ■

Hence the upper bound relaxation leads to a monotonically converging upper bound approximation of the general multi-objective control problem. As in the pure $\ell_1$ case (Section 3.2), the controller order may be high.

It is possible to cast the $\ell_1$ and time-domain template constraints of Problem 3.2 into a finite-dimensional LP, see e.g. Dahleh and Diaz-Bobillo (1995); Khammash (2000). A formulation for addressing the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ constraints is proposed in Hindi *et al.* (1998), and used in Qi *et al.* (2001); Qi (2002). An equivalent formulation with considerably less computational complexity is stated in Scherer (2000*a*). Next, an alternative novel formulation regarding $\mathcal{H}_\infty$ and $\mathcal{H}_2$ constraints is introduced, and subsequently compared to the existing approaches in terms of computational complexity.

### 3.3.2 Formulation of the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ Constraints

To address the $\mathcal{H}_\infty$ constraint, the Bounded Real Lemma version presented next is applied.

**Lemma 3.2** *(Gahinet and Apkarian, 1994)*
*Consider a transfer function $\hat{\mathcal{G}}_{ij}(z) = \mathcal{C}_i(zI - \mathcal{A})^{-1}\mathcal{B}_j + \mathcal{D}_{ij}$. The following two statements are equivalent.*

*(i) There exists $X = X^T > 0$ such that*

$$\begin{bmatrix} \mathcal{A}^T X \mathcal{A} - X & \mathcal{A}^T X \mathcal{B}_j & \mathcal{C}_i^T \\ \mathcal{B}_j^T X \mathcal{A} & \mathcal{B}_j^T X \mathcal{B}_j - \gamma I & \mathcal{D}_{ij}^T \\ \mathcal{C}_i & \mathcal{D}_{ij} & -\gamma I \end{bmatrix} < 0. \tag{3.8}$$

*(ii) All eigenvalues of $\mathcal{A}$ are inside the open unit circle and $\|\hat{\mathcal{G}}_{ij}\|_\infty < \gamma$.* ■

Similarly, a matrix inequality characterization for an $\mathcal{H}_2$ constraint is as follows.

**Lemma 3.3** *(Hindi* et al.*, 1998)*
*Consider a transfer function $\hat{\mathcal{G}}_{ij}(z) = \mathcal{C}_i(zI - \mathcal{A})^{-1}\mathcal{B}_j + \mathcal{D}_{ij}$. The following two statements are equivalent.*

*(i) There exist $X = X^T > 0$ and $S$ such that*

$$\begin{bmatrix} \mathcal{A}^T X \mathcal{A} - X & \mathcal{A}^T X \mathcal{B}_j \\ \mathcal{B}_j^T X \mathcal{A} & \mathcal{B}_j^T X \mathcal{B}_j - I \end{bmatrix} < 0, \quad \begin{bmatrix} X & 0 & \mathcal{C}_i^T \\ 0 & I & \mathcal{D}_{ij}^T \\ \mathcal{C}_i & \mathcal{D}_{ij} & S \end{bmatrix} > 0, \quad \mathrm{trace}(S) - \gamma^2 < 0. \tag{3.9}$$

*(ii) All eigenvalues of $\mathcal{A}$ are inside the open unit circle and $\|\hat{\mathcal{G}}_{ij}\|_2 < \gamma$.* ■

When the closed-loop dynamic matrix $\mathcal{A}$ is known to be stable (as in the constraints associated with Problem 3.2), the condition $X > 0$ is automatically satisfied by any feasible $X$ and can be dropped (Hindi *et al.*, 1998). To use these analysis LMIs directly for controller synthesis, a special form of state-space description $\{\mathcal{A}, \mathcal{B}_j, \mathcal{C}_i, \mathcal{D}_{ij}\}$ is introduced for each closed-loop transfer function $\hat{\mathcal{G}}_{ij}$ as described in the sequel. The crucial point of this description is that only $\mathcal{C}_i$ and $\mathcal{D}_{ij}$ depend on the unknown $Q$ parameter. It follows that the conditions (3.8) and (3.9) are still LMIs in the unknowns $X$, $S$, and the Youla parameter $Q$.

The idea of the proposed formulation is to look at the description of the closed loop as a transfer matrix in the $z$-domain, where all unknowns are in the numerator. This transfer matrix is then converted into a state-space description at once. The number of states of the resulting realization is relatively low since all denominators of the transfer matrix' components are identical.

As a first step, each component of the $N$-tap Youla parameter $Q^N = \{Q(0), ..., Q(N), 0, ...\}$ in (3.6) is rewritten into the transfer function

$$\hat{Q}^N_{\nu\eta}(z) = \frac{1}{z^N} \underbrace{\left(Q_{\nu\eta}(0)z^N + \cdots + Q_{\nu\eta}(N)\right)}_{n_{Q,\nu\eta}(z)}, \qquad \nu = 1, ..., n_u, \quad \eta = 1, ..., n_y.$$

The components of $\hat{H}_{ij} \in \ell_1^{p_i \times q_j}$, $\hat{U}_i \in \ell_1^{p_i \times n_u}$, and $\hat{V}_j \in \ell_1^{n_y \times q_j}$ are written as

$$(\hat{H}_{ij})_{\xi\lambda}(z) = \frac{n_{H,ij,\xi\lambda}(z)}{d_U(z)d_V(z)}, \qquad (\hat{U}_i)_{\xi\nu}(z) = \frac{n_{U,i,\xi\nu}(z)}{d_U(z)}, \qquad (\hat{V}_j)_{\eta\lambda}(z) = \frac{n_{V,j,\eta\lambda}(z)}{d_V(z)},$$
$$\xi = 1, ..., p_i, \quad \lambda = 1, ..., q_j, \quad \nu = 1, ..., n_u, \quad \eta = 1, ..., n_y,$$

where $n_{H,ij,\xi\lambda}$ is a polynomial of degree $2n_x$, and $n_{U,i,\xi\nu}$, $n_{V,j,\eta\lambda}$, $d_U$, and $d_V$ are polynomials of degree $n_x$, obtained from the Youla parameterization in Lemma B.13. Note that the denominators do not depend on the indices, and that the denominator of $\hat{H}_{ij}(z)$ is obtained from the denominators of $\hat{U}_i(z)$ and $\hat{V}_j(z)$. Now the components of the closed-loop transfer matrix $\hat{\mathcal{G}}_{ij}$ are given by

$$\begin{aligned}(\hat{\mathcal{G}}_{ij})_{\xi\lambda}(z) &= (\hat{H}_{ij} - \hat{U}_i\hat{Q}\hat{V}_j)_{\xi\lambda}(z) = (\hat{H}_{ij})_{\xi\lambda}(z) - \sum_{\eta=1}^{n_y}\sum_{\nu=1}^{n_u} (\hat{U}_i)_{\xi\nu}(z)\hat{Q}_{\nu\eta}(z)(\hat{V}_j)_{\eta\lambda}(z) \\ &= \frac{n_{H,ij,\xi\lambda}(z)z^N - \sum_{\eta=1}^{n_y}\sum_{\nu=1}^{n_u} n_{U,i,\xi\nu}(z)n_{V,j,\eta\lambda}(z)n_{Q,\nu\eta}(z)}{d_U(z)d_V(z)z^N}. \end{aligned} \tag{3.10}$$

Note that all unknowns $Q_{\nu\eta}(k)$ are contained in the polynomials $n_{Q,\nu\eta}(z)$ and hence in the numerators of $\hat{\mathcal{G}}_{ij}$, and that the dependence on these unknowns is affine. Moreover, all components of $\hat{\mathcal{G}}_{ij}$ share the same denominator. To transform this transfer matrix into a state-space description, the following procedure is applied.

Observe that any proper rational transfer matrix $\hat{T}(z)$ can be written in the form

$$\hat{T}(z) = \frac{1}{d(z)} \left(M_r z^r + M_{r-1}z^{r-1} + \cdots + M_0\right) \tag{3.11}$$

for some $r$, where the $M_\zeta$ are matrices that do not depend on $z$, and $d(z) = z^r + d_{r-1}z^{r-1} + \cdots + d_0$ with scalar $d_\zeta$ is the least common multiple of the denominators of the entries of $\hat{T}(z)$. Based on this structure, a (not necessarily minimal) state-space realization of $\hat{T}(z)$ is derived by the following lemma, which extends a scheme for strictly proper systems given in Kailath (1980, Section 6.1).

**Lemma 3.4** *Given a proper transfer matrix $\hat{T}(z)$ with $q$ inputs, and matrices $M_\zeta$ and scalars $d_\zeta$ as in the factorization (3.11). Then*

$$\hat{T}(z) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{ccccc|c} -d_{r-1}I & -d_{r-2}I & \cdots & & -d_0 I & I \\ I & 0 & \cdots & & 0 & 0 \\ 0 & I & 0 & \cdots & \vdots & \vdots \\ \vdots & & \ddots & & & \\ 0 & \cdots & 0 & I & 0 & 0 \\ \hline W_{r-1} & W_{r-2} & \cdots & & W_0 & M_r \end{array}\right], \tag{3.12}$$

*where* $W_\zeta := M_\zeta - d_\zeta M_r$*,* $\zeta = 0, \ldots, r-1$*. The identity matrices in (3.12) are of dimension* $q \times q$*.*

Proof: From (3.11) it follows that

$$\hat{T}(z) = M_r + \frac{1}{d(z)} \left( (M_{r-1} - d_{r-1} M_r) z^{r-1} + \cdots + M_0 - d_0 M_r \right).$$

The first term $M_r$ is the direct feed-through term in (3.12), whereas the realization of the second term follows directly as in Kailath (1980, Section 6.1). The result is easily verified by comparing the difference equation corresponding to (3.12) with the one corresponding to (3.11). ■

By using Lemma 3.4 on $\hat{\mathcal{G}}_{ij}(z)$ with the components (3.10), a state-space realization

$$\hat{\mathcal{G}}_{ij}(z) = \left[ \begin{array}{c|c} \mathcal{A} & \mathcal{B}_j \\ \hline \mathcal{C}_i & \mathcal{D}_{ij} \end{array} \right] \tag{3.13}$$

is obtained. The crucial fact in this representation is that both $\mathcal{A}$ and $\mathcal{B}_j$ do not depend on the unknowns $Q(0), ..., Q(N)$, and that $\mathcal{C}_i$ and $\mathcal{D}_{ij}$ depend affinely on these unknowns. Hence, by inserting (3.13) into the inequalities of Lemmas 3.2 and 3.3, tractable LMI conditions for controller synthesis are obtained. Note that $d(z) = d_U(z) d_V(z) z^N$ is a polynomial of order $2n_x + N$. Thus the number of states in (3.13) is $(2n_x + N)q_j$. The process of obtaining (3.13) from the Youla parameterization of $\hat{\mathcal{G}}_{ij}$ can be automated.

### 3.3.3 Complexity Analysis

This section analyzes the three approaches of Hindi *et al.* (1998), of Scherer (2000*a*), and of Section 3.3.2 with respect to resulting LMI sizes and number of free variables. Furthermore a comparison of computation times is made for some typical problem sizes. Hereafter, the methods are abbreviated as HHB98, S00, and R06 for Hindi *et al.* (1998), Scherer (2000*a*), and our approach, respectively. A summary of HHB98 and S00 is provided in Rieber *et al.* (2006*a*).

**Analysis of Problem Sizes**

To estimate the computational burden, the number of unknowns (in $X$) and the number of constraints are most important (Helmersson (1995, Section 3.2); Boyd and Vandenberghe (2004, Chapter 11)), but also the LMI size plays a significant role. The sparsity of LMIs (as e.g. in S00) is favorable, but not easily quantifiable. Likewise, we do not consider complexity of LMI solvers here. To analyze the problem sizes resulting from the three described approaches, we use the $\mathcal{H}_\infty$ condition (3.8) and the respective state-space realizations (3.12), as well as the corresponding conditions for HHB98 and S00. The $\mathcal{H}_2$ case follows similarly using (3.9) and is thus omitted.

Recall that the closed-loop transfer function $\hat{\mathcal{G}}_{ij}$ of one $\mathcal{H}_\infty$ constraint has $p_i$ outputs and $q_j$ inputs, $K$ has $n_u$ outputs (control variables) and $n_y$ inputs (measurements), $G$ has $n_x$ states, and $Q(k) = 0$ for $k > N$. Note that in the case of "wide" $\hat{U}_i$ $(p_i < n_u)$ and/or "tall" $\hat{V}_j$ $(n_y > q_j)$, inner-outer factorizations of $\hat{U}_i$ and $\hat{V}_j$ can be performed for each constraint such that $n_u = p_i$ and $n_y = q_j$.

From this data, the number of states of the three state-space realizations are

- HHB98: $2n_x + (2n_x + Np_i) \cdot \min\{p_i, n_u\} \cdot \min\{n_y, q_j\}$,
- S00: $2n_x + N \cdot \min\{p_i, q_j, n_y, n_u\}$,
- R06: $(2n_x + N) \cdot q_j$.

Let this number of states be called $n_{\mathcal{G}}$, then the number of unknowns in the $X$ variable of the LMI condition is $n_{\mathcal{G}}(n_{\mathcal{G}} + 1)/2$ for all three methods respectively ($X = X^T$). The LMI size, i.e. the number of rows (or columns) of the LMI condition, can be derived as

- HHB98: $2n_x + (2n_x + Np_i) \cdot \min\{p_i, n_u\} \cdot \min\{n_y, q_j\} + p_i + q_j$,
- S00: $4n_x + 2N \cdot \min\{p_i, q_j, n_y, n_u\} + p_i + q_j$,
- R06: $(2n_x + N) \cdot q_j + p_i + q_j$.

To visualize and interpret the above-stated formulas, a comparison of the number of unknowns is shown in Figure 3.2 for different problem sizes. Especially for MIMO problems, S00 uses the least amount of unknowns, and the number of unknowns of R06 is in the same order. In contrast, the $X$ variable in HHB98 becomes very large. A comparison of LMI sizes is depicted in Figure 3.3. It is

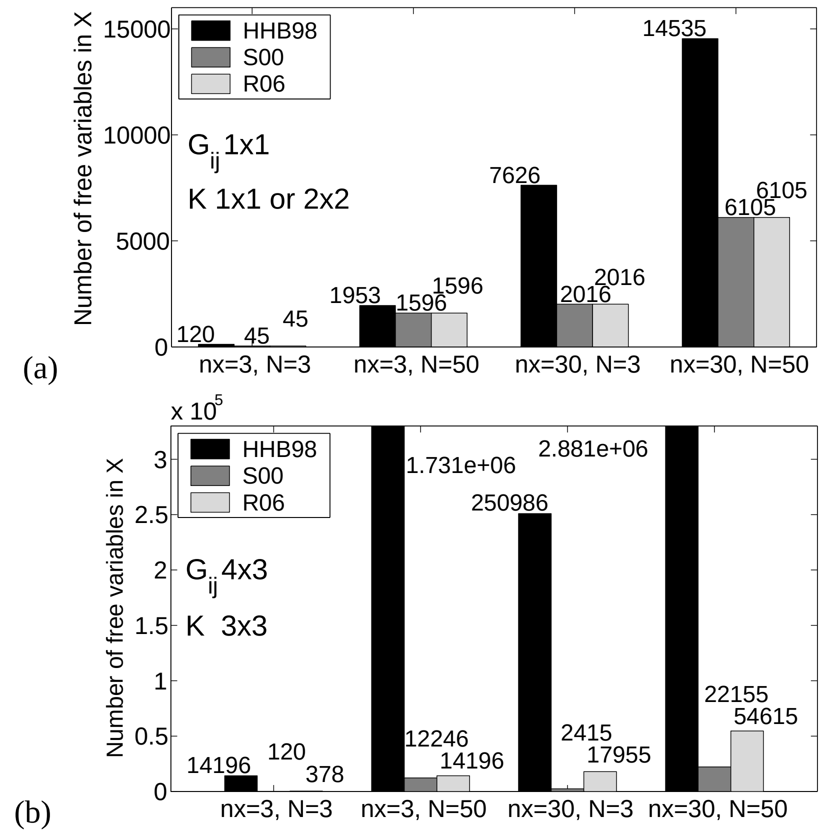



*Figure 3.2:* Comparison of the number of free scalar variables in $X$ for one $\mathcal{H}_\infty$ constraint. (a) SISO case; (b) MIMO case. Different problem sizes with respect to $\mathcal{H}_\infty$ channel dimension ($\dim \mathcal{G}_{ij}$), controller dimension ($\dim K$), number of plant states $n_x$, and number of taps $N$ in the Youla parameter are considered.

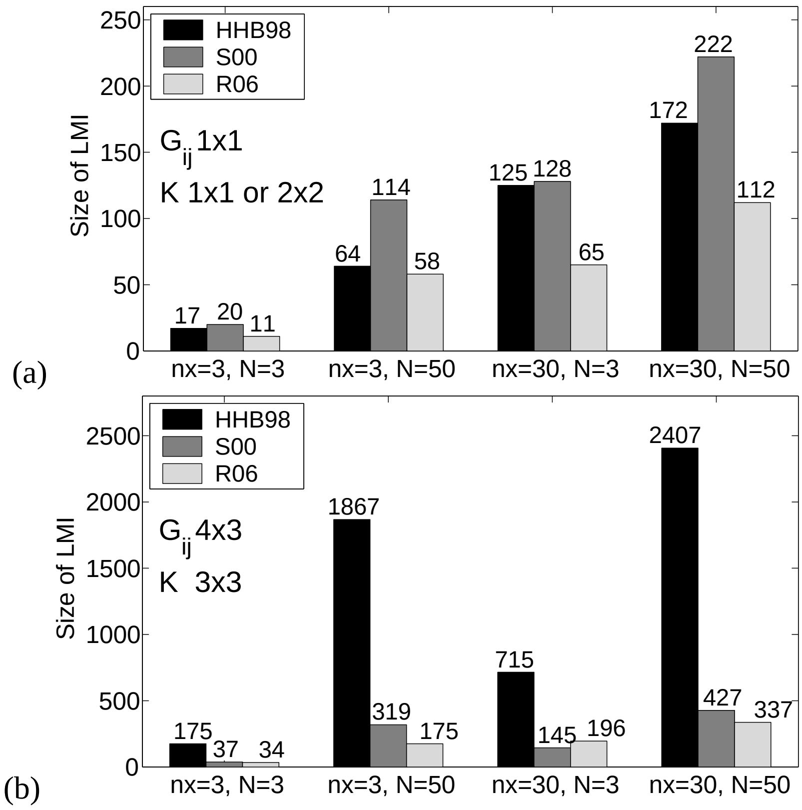



*Figure 3.3:* Comparison of LMI size of one $\mathcal{H}_\infty$ constraint, cf. Figure 3.2.

evident that the LMI size of HHB98 grows significantly for MIMO problems, since a large number of states in the closed-loop realization is built up. In contrast, the LMI size stays manageable for larger problems for both S00 and R06, with advantages for R06.

In the case of multiple performance constraints, the above-mentioned inner-outer factorization may not be or may only partly be possible. Still, R06 is unaffected by that and the numbers for each constraint just add up. For HHB98 and S00 however, $p_i$ and $q_j$ have to be replaced by $\sum_{i=1}^m p_i$ and $\sum_{j=1}^m q_j$, respectively, in the $\min$-operations, which degrades the numbers relative to R06. One might imagine other special situations, in which the complexity numbers of the three methods change more or less relative to each other. Yet the study in this section gives an impression of general tendencies.

Based on this analysis, it is clear that HHB98 has to be considered the least efficient formulation. S00 and R06 are to be favored for all kinds of problem sizes. Depending on the application, either S00 or R06 is better suited. For SISO or low-dimensional MIMO constraints, which often appear in practical applications, R06 appears to be more efficient. S00 has to be preferred for larger MIMO constraints due to the relatively small number of unknowns in $X$.

**Comparison of Computation Time**

CPU times are compared for the solution of simple SISO and MIMO multi-objective $\ell_1/\mathcal{H}_\infty$ problems. All results are obtained using the MATLAB LMI Toolbox (Gahinet *et al.*, 1995) with hardware as described in Appendix E. With more efficient SDP solvers, the computation times presented here may be improved. However, it is the point of this study to investigate the three methods under the same conditions with widely available general purpose software and evaluate their performance relative to each other.

To be more precise, the same Youla parameterizations are applied in all three methods. Exactly the same formulation is used for the $\ell_1$ parts of the considered problems. Moreover, exactly the same algorithm is used to set up the optimization problems for the HHB98 and the R06 methods. The only difference (which makes a huge difference in the results) is the state-space realization of $\hat{\mathcal{G}}_{ij}$. We stress that, apart from slight numerical differences, the same controller orders and the same performance are to be expected from all three methods theoretically.

First, for the unstable plant

$$\hat{G}(z) = \frac{2z}{z^2 - 2}$$

with two states, the optimization

$$\inf_{K \text{ stabilizing}} \|GK(I+GK)^{-1}\|_1 \quad \text{s.t.} \quad \|(I+\hat{G}\hat{K})^{-1}\|_\infty \leq 2.4$$

is carried out with $N = 3$ and $N = 20$. The model is taken from Chen and Wen (1995). The $\mathcal{H}_\infty$ constraint transfer function and $\hat{K}$ are both $1 \times 1$. All three methods achieve the upper bound 2.227 for $N = 3$ and for $N = 20$. Computation times in Figure 3.4(a) show the superiority of R06, which is expected from the upper graph in Figure 3.3. S00 performs better than HHB98 since its number of optimization variables is considerably smaller.

As a second example, the unstable plant

$$\hat{G}(z) = \frac{1}{z^4 + 2z^2 + 0.5z + 0.25} \cdot \begin{bmatrix} 2z^2, & -4 \end{bmatrix}$$

with four states and two inputs is considered. Based on Figure 3.5, the optimization problem

$$\inf_K \|w_2 \mapsto z_1\|_1 \quad \text{s.t.} \quad \|[w_1, w_2] \mapsto [z_2, z_3]\|_\infty \leq 2.8$$

is solved with $N = 3$ and $N = 10$. Hence, the $\mathcal{H}_\infty$ constraint transfer function and $\hat{K}$ are both $2\times2$. All three methods achieve the upper bound 2.216 for $N = 3$, and with S00/R06 the upper bound 2.181 is obtained for $N = 10$. It is evident from the computation times shown in Figure 3.4(b) that the HHB98 method is inferior to S00 and R06, which perform orders of magnitude better in the case of $N = 3$. For a Youla parameter of higher order ($N = 10$), S00 and R06 still solve this problem in acceptable time.

These examples confirm the trends from the analysis in the preceding paragraph. In summary, R06 appears to be best suited for SISO constraints and moderately sized MIMO constraints. Using the proposed method, further examples related to atomic force microscopy (AFM) and to fuel injection are discussed in Keßler (2005).

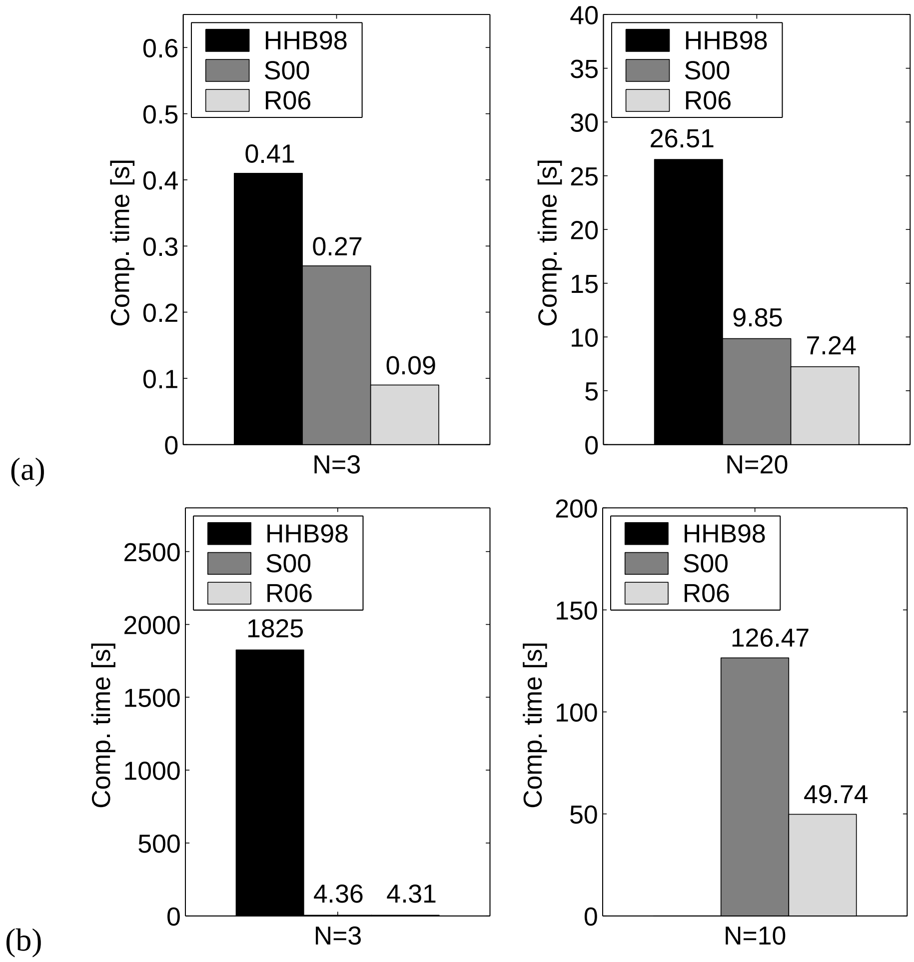



*Figure 3.4:* Comparison of CPU times for multi-objective $\ell_1/\mathcal{H}_\infty$ controller synthesis. (a) SISO example. (b) MIMO example. In the case $N = 10$, the available memory was exhausted for the HHB98 method before obtaining a solution.

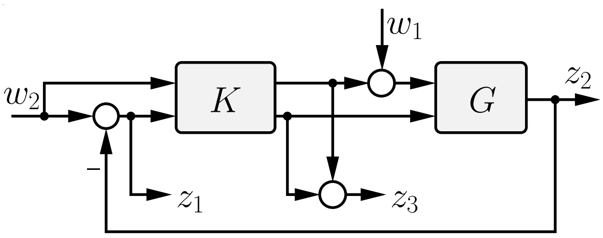



*Figure 3.5:* Control setup for the multi-objective $\ell_1/\mathcal{H}_\infty$ MIMO example.

## 3.4 Robust State-Feedback in Discrete Time

This section looks at the question of finding a fixed state-feedback gain providing robust stabilization or robust quadratic performance in face of parametric uncertainties. Such a result is useful to do state-feedback control, to derive a robust Youla parameterization, to design observer gains, or whenever state-feedback gains are needed as intermediate steps of a controller synthesis procedure. Although it is not known how to obtain convex conditions for robust output-feedback, there are convex robust state-feedback procedures available in the case of continuous-time systems, see for example Scherer (2000*b*); Scherer and Weiland (2005, Section 8.1). The derivation of the discrete-time result presented here is slightly more involved than the continuous-time counterpart. Moreover, a discrete-time version has not been published so far to the best of the author's knowledge. First, the problem of robust quadratic performance is considered, before the result is specialized to robust stabilization.

**Robust Quadratic Performance**

Consider an uncertain system obtained from the interconnection

$$\begin{bmatrix} x(k+1) \\ z_0(k) \\ z_1(k) \end{bmatrix} = \begin{bmatrix} A & B_0 & B_1 & B_2 \\ C_0 & D_{00} & D_{01} & D_{02} \\ C_1 & D_{10} & D_{11} & D_{12} \end{bmatrix} \begin{bmatrix} x(k) \\ w_0(k) \\ w_1(k) \\ u(k) \end{bmatrix}, \tag{3.14}$$

$$w_0(k) = \Delta(k) z_0(k). \tag{3.15}$$

The meaning of the symbols is explained in Section 3.3.1. The structure of the state-feedback controller $K$ is taken to be

$$u(k) = Fx(k)$$

with $F \in \mathbb{R}^{n_u \times n_x}$ Hence the closed-loop system is of the form (2.11) with

$$\begin{bmatrix} \bar{\mathcal{A}}(\Delta(k)) & \bar{\mathcal{B}}_1(\Delta(k)) \\ \bar{\mathcal{C}}_1(\Delta(k)) & \bar{\mathcal{D}}_{11}(\Delta(k)) \end{bmatrix} = \Delta(k) \star \left[\begin{array}{c:cc} D_{00} & C_0 + D_{02}F & D_{01} \\ \hdashline B_0 & A + B_2 F & B_1 \\ D_{10} & C_1 + D_{12}F & D_{11} \end{array}\right]. \tag{3.16}$$

The interconnection is depicted in Figure 3.6. The goal of this section is to find a constant state-feedback gain $F$ achieving uniform exponential stability and robust quadratic performance with respect to $\Delta \in \boldsymbol{\Delta}_{\mathrm{TV}}$. The used performance notion is defined precisely next.

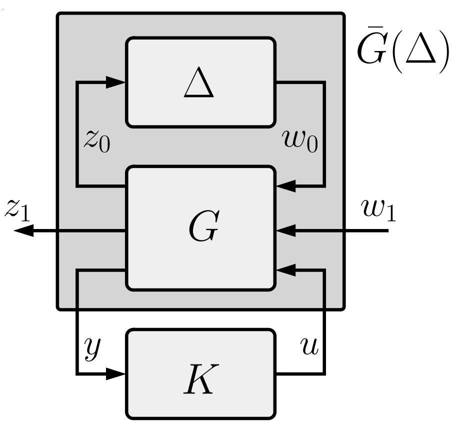



*Figure 3.6:* The robust control problem is to find a controller $K$ for the uncertain plant $\bar{G}(\Delta) = \Delta \star G$. In the state-feedback case it is assumed that $y = x$.

**Definition 3.1** *(Robust quadratic performance, discrete time)*
*Suppose that the interconnection (2.9)–(2.10) is well-posed. Then the system (2.11) is said to have robust quadratic performance with performance index $P_p = P_p^T$ if there exists an $\varepsilon > 0$ such that*

$$\sum_{k=0}^{\infty} \begin{bmatrix} w_1(k) \\ z_1(k) \end{bmatrix}^T P_p \begin{bmatrix} w_1(k) \\ z_1(k) \end{bmatrix} \leq -\varepsilon \sum_{k=0}^{\infty} w_1(k)^T w_1(k)$$

*for all considered uncertainties $\Delta(\cdot)$ and for every system trajectory $\xi(\cdot)$ with $\xi(0) = 0$.* □

**Remark 3.1** *$\mathcal{H}_\infty$ performance is a special case of quadratic performance by means of the choice $P_p = \begin{bmatrix} -\gamma^2 I & 0 \\ 0 & I \end{bmatrix}$ (or equivalently $P_p = \begin{bmatrix} -\gamma I & 0 \\ 0 & \frac{1}{\gamma} I \end{bmatrix}$), which implies that the $\ell_2$-gain of the map $w_1 \mapsto z_1$ is smaller than $\gamma$.* □

The next theorem gives sufficient conditions on the existence of such a state-feedback gain $F$, and provides a way for obtaining $F$ from the solution of a set of LMIs.

**Theorem 3.5** *The following two statements are equivalent.*

*(i) The interconnection (3.14)–(3.15) is well-posed and there exists $X = X^T > 0$ satisfying*

$$\begin{bmatrix} I & 0 \\ \bar{\mathcal{A}}(\Delta) & \bar{\mathcal{B}}_1(\Delta) \\ \hline 0 & I \\ \bar{\mathcal{C}}_1(\Delta) & \bar{\mathcal{D}}_{11}(\Delta) \end{bmatrix}^T \left[\begin{array}{cc|cc} -X & 0 & 0 & 0 \\ 0 & X & 0 & 0 \\ \hline 0 & 0 & Q_p & S_p \\ 0 & 0 & S_p^T & R_p \end{array}\right] \begin{bmatrix} I & 0 \\ \bar{\mathcal{A}}(\Delta) & \bar{\mathcal{B}}_1(\Delta) \\ \hline 0 & I \\ \bar{\mathcal{C}}_1(\Delta) & \bar{\mathcal{D}}_{11}(\Delta) \end{bmatrix} < 0 \qquad \forall \Delta \in \mathbf{\Delta}. \tag{3.17}$$

*(ii) There exist $Y = Y^T$, $M$, $\tilde{Q} = \tilde{Q}^T$, $\tilde{R} = \tilde{R}^T \geq 0$, $\tilde{S}$ satisfying*

$$\begin{bmatrix} -I \\ \Delta^T \end{bmatrix}^T \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{bmatrix} \begin{bmatrix} -I \\ \Delta^T \end{bmatrix} \leq 0 \qquad \forall \Delta \in \mathbf{\Delta}, \tag{3.18}$$

$$\begin{bmatrix} B_0\tilde{Q}B_0^T + Y + B_1\tilde{Q}_pB_1^T & B_0\tilde{Q}D_{00}^T - B_0\tilde{S} + B_1\tilde{Q}_pD_{01}^T & \cdots \\ * & D_{00}\tilde{Q}D_{00}^T - D_{00}\tilde{S} - \tilde{S}^TD_{00}^T + \tilde{R} + D_{01}\tilde{Q}_pD_{01}^T & \cdots \\ * & * & \\ * & * & \end{bmatrix}$$

$$\begin{bmatrix} \cdots & B_0\tilde{Q}D_{10}^T + B_1\tilde{Q}_pD_{11}^T - B_1\tilde{S}_p & AY + B_2M \\ \cdots & D_{00}\tilde{Q}D_{10}^T - \tilde{S}^TD_{10}^T + D_{01}\tilde{Q}_pD_{11}^T - D_{01}\tilde{S}_p & C_0Y + D_{02}M \\ & D_{10}\tilde{Q}D_{10}^T + D_{11}\tilde{Q}_pD_{11}^T - D_{11}\tilde{S}_p - \tilde{S}_p^TD_{11}^T + \tilde{R}_p & C_1Y + D_{12}M \\ & * & Y \end{bmatrix} > 0, \tag{3.19}$$

*where $\begin{bmatrix} \tilde{Q}_p & \tilde{S}_p \\ \tilde{S}_p^T & \tilde{R}_p \end{bmatrix} := \begin{bmatrix} Q_p & S_p \\ S_p^T & R_p \end{bmatrix}^{-1}$ with an appropriate partitioning.*

*Moreover, if either (i) or (ii) holds, then the state-feedback gain* $F = MY^{-1}$ *provides uniform exponential stability and robust quadratic performance with index* $P_p = \begin{bmatrix} Q_p & S_p \\ S_p^T & R_p \end{bmatrix}$, $R_p \geq 0$, $Q_p \leq 0$, *for (2.11) with respect to* $\Delta \in \mathbf{\Delta}_{\mathrm{TV}}$.

Proof: See Appendix D.4 ■

If $P_p$ is not invertible, then a slight perturbation can be used to make it non-singular without essentially changing the performance index. The semi-infinite constraint (3.18) can be converted into a finite number of conditions with help of Lemma B.12. See the conversion of Theorem 2.3 into Theorem 2.4 on how to proceed in this case. Altogether, the discrete-time robust state-feedback problem is thus formulated as a finite-dimensional SDP, which is efficiently solvable using widely available algorithms.

**Robust Stabilization**

Theorem 3.5 is now specialized to the case of robust stabilization. To this end, the performance channels in (3.14) are discarded. The precise problem statement is to find a fixed state-feedback gain $F$ such that the system

$$\xi(k+1) = \bar{\mathcal{A}}(\Delta(k))\xi(k) \tag{3.20}$$

with

$$\bar{\mathcal{A}}(\Delta(k)) := \bar{A}(\Delta(k)) + \bar{B}_2(\Delta(k))F = \Delta(k) \star \left[\begin{array}{c:c} D_{00} & C_0 + D_{02}F \\ \hdashline B_0 & A + B_2 F \end{array}\right] \tag{3.21}$$

is uniformly exponentially stable with respect to $\Delta \in \mathbf{\Delta}_{\mathrm{TV}}$. In other words, find a state-feedback gain for the stabilization of a system, where the plant matrices are rationally dependent on uncertain parameters. The following theorem provides a solution to the stated problem, and gives sufficient convex conditions for the existence and the computation of the desired state-feedback gain $F$.

**Corollary 3.6** *The following two statements are equivalent.*

*(i) The LFT in (3.21) is well-posed and there exists* $X = X^T > 0$ *satisfying*

$$\bar{\mathcal{A}}(\Delta)^T X \bar{\mathcal{A}}(\Delta) - X < 0 \qquad \forall \Delta \in \mathbf{\Delta}. \tag{3.22}$$

*(ii) There exist* $Y = Y^T$, $M$, $\tilde{Q} = \tilde{Q}^T$, $\tilde{R} = \tilde{R}^T \geq 0$, $\tilde{S}$ *satisfying*

$$\begin{bmatrix} -I \\ \Delta^T \end{bmatrix}^T \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{bmatrix} \begin{bmatrix} -I \\ \Delta^T \end{bmatrix} \leq 0 \qquad \forall \Delta \in \mathbf{\Delta}, \tag{3.23}$$

$$\begin{bmatrix} B_0\tilde{Q}B_0^T + Y & B_0\tilde{Q}D_{00}^T - B_0\tilde{S} & AY + B_2M \\ * & D_{00}\tilde{Q}D_{00}^T - D_{00}\tilde{S} - \tilde{S}^T D_{00}^T + \tilde{R} & C_0Y + D_{02}M \\ * & * & Y \end{bmatrix} > 0. \tag{3.24}$$

*Moreover, if either (i) or (ii) holds, then the state-feedback gain $F = MY^{-1}$ results in a matrix $\bar{\mathcal{A}}(\Delta)$ according to (3.21) such that (3.20) is uniformly exponentially stable with respect to $\Delta \in \mathbf{\Delta}_{\mathrm{TV}}$.*

Proof: The equivalence between (i) and (ii) and the computation of the state-feedback gain $F$ are special cases of the robust quadratic performance result in Theorem 3.5, see the proof there. In all the inequalities, the rows and columns with system matrices containing 1 in the index, or rows and columns with elements of $\tilde{P}_p$ have to be canceled. Item (i) implies uniform exponential stability as in the proof of Theorem 2.3. ■

As before, the semi-infinite constraint (3.23) can be converted into a finite number of conditions using the convex hull relaxation of Lemma B.12, see the transition from Theorem 2.3 to Theorem 2.4.

## 3.5 Summary

This chapter considers the design of LTI controllers for finite-dimensional dynamic systems. A novel and efficient formulation of $\mathcal{H}_\infty$ and $\mathcal{H}_2$ constraints in a general multi-objective control setting is proposed. The formulation's efficiency is analyzed with respect to LMI size, number of unknowns, and computation time for typical examples in comparison to existing approaches. The results show that the proposed method is superior to Hindi *et al.* (1998), comparable to Scherer (2000*a*), and particularly well-suited for reasonably sized problems. Moreover, a convex formulation of discrete-time robust state-feedback synthesis with a quadratic performance criterion complements the existing LMI-based design methods. This state-feedback result is beneficial e.g. for the design of auxiliary state-feedback gains and observer gains, or for deriving a robust Youla parameterization.

# Chapter 4

# Synthesis of LPV Controllers

*In this chapter, the problem of synthesizing robust LPV controllers (or gain-scheduled controllers) for LPV systems is treated. The main difference to robust fixed controller design is that measurability of uncertain parameters is assumed and favorably exploited. The main results are*

- *a novel control structure for LPV controller design, which is independent of the applied performance criteria (Section 4.3),*
- *conditions for the realizability of LPV controllers obtained with the proposed control structure (Definitions 4.1 to 4.5),*
- *a scheme for $\ell_1$-optimal LPV output-feedback controller synthesis (Section 4.5) with robust stability and robust performance results (Theorems 4.1 and 4.2), and*
- *matrix inequality conditions for synthesizing LPV controllers using the proposed control structure with a quadratic performance criterion (Theorem 4.3, Corollaries 4.4 and 4.5).*

## 4.1 Overview

After considering the synthesis of LTI controllers in the previous chapter, the attention is now turned towards the synthesis of LPV controllers. The idea for this kind of compensator is based on the assumption that the system to be controlled has some time-varying parameters (e.g. model coefficients) that are measurable in real-time. Hence it is natural to use this parameter information in the controller, such that larger operating regions are covered or better performance is achieved. To be more concrete, consider an LPV system

$$\begin{bmatrix} x(k+1) \\ z_1(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} \bar{A}(\Delta(k)) & \bar{B}_1(\Delta(k)) & \bar{B}_2(\Delta(k)) \\ \bar{C}_1(\Delta(k)) & \bar{D}_{11}(\Delta(k)) & \bar{D}_{12}(\Delta(k)) \\ \bar{C}_2(\Delta(k)) & \bar{D}_{21}(\Delta(k)) & \bar{D}_{22}(\Delta(k)) \end{bmatrix} \begin{bmatrix} x(k) \\ w_1(k) \\ u(k) \end{bmatrix}. \tag{4.1}$$

On the one hand, this system class encompasses linear difference equations with parameter-depending coefficients, which can be obtained from differential equation descriptions. Common examples are flight dynamics depending on velocity and altitude (Packard and Balas, 1992), robotic systems with varying mass or mass distribution (Rieber, 2001; Rieber and Taylor, 2004), or compact disc players (Dettori and Scherer, 2001). On the other hand, nonlinear systems linearized around specific trajectories can be cast into the LPV framework (Rugh and Shamma, 2000; Khalil, 2002). Even general input-affine nonlinear systems

$$x(k+1) = f(x(k)) + g(x(k))u, \qquad f : \mathbb{R}^n \to \mathbb{R}^n, \quad g : \mathbb{R}^n \to \mathbb{R}^n,$$

can be seen as LPV systems. To this end, they are rewritten as $x(k+1) = A(x(k))x(k)+B(x(k))u$ (which is possible if $f(0) = 0$) and treated as the LPV system

$$x(k+1) = A(\delta(k))x(k) + B(\delta(k))u,$$

introducing the "parameter vector" $\delta(k) := x(k)$. Finally, LPV systems arise in several control frameworks, whenever the design includes tuning knobs as parameters. An example is an $\mathcal{H}_\infty$ design with a weighting function used to influence the closed-loop bandwidth $\omega_\mathrm{b}$. Suppose that $\omega_\mathrm{b}$ is not fixed, but rather left as an unknown time-invariant parameter that can be manipulated by an engineer during operation of the controller. Then an LPV design problem arises.

Early gain-scheduling approaches as described in Rugh and Shamma (2000) use a collection of controllers designed for frozen values of the parameter. The overall control scheme switches between these locally valid controllers. However, in this case usually no guarantees for closed-loop stability can be given, or the methods only work for constant or slowly varying parameters. On the other hand, fixed robust controllers developed from small-gain criteria may be applied to LPV systems, guaranteeing stability over the whole range of parameter variation. Yet with such an approach, there is considerable conservatism involved in general, since the measurement information about current parameter values is discarded. This leads to the desire for a controller taking into account the parameter measurements in real-time, in order to change its dynamic behavior along with the plant. Such controllers are called gain-scheduled or LPV controllers.

Systematic gain-scheduled control of LPV systems with measurable parameters has received wide attention during the 1990s, especially in the framework of $\mathcal{H}_\infty$ control. Overviews are given in Rugh and Shamma (2000); Khalil (2002). Some exemplary references are Packard (1994); Apkarian and Gahinet (1995); Gahinet *et al.* (1995); Apkarian *et al.* (1995); Feron *et al.* (1996); Apkarian and Adams (1998); Bennani *et al.* (1998); Scorletti and El Ghaoui (1998); Shamma and Xiong (1999); Scherer (2000*b*); Scherer (2001*a*); Wu (2001); Wu and Dong (2006). The beauty of these approaches is that they provide stability and performance guarantees of the overall gain-scheduling scheme, even for large parameter variation rates, while improving performance over robust controllers. Moreover, the parameters do not need to be assumed as slowly varying. This is in contrast to ad-hoc techniques like switching between frozen-parameter controllers.

This section discusses a novel control structure for LPV controller design and its application in $\ell_\infty$- and $\ell_2$-gain based control. After Section 4.2 describes the problem setup, Section 4.3 introduces the

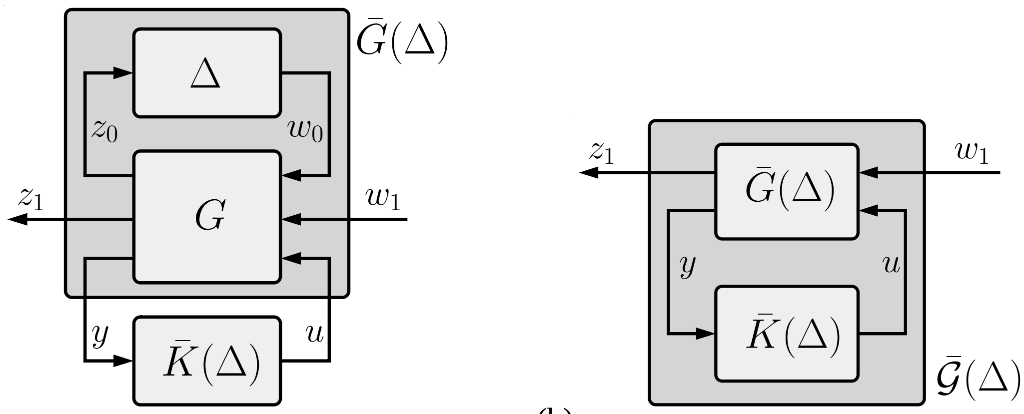



*Figure 4.1:* (a) The LPV control problem is to find a parameter-dependent controller $\bar{K}(\Delta)$ for the LPV plant $\bar{G}(\Delta) = \Delta \star G$. (b) The uncertain closed-loop system $\bar{\mathcal{G}}(\Delta)$ is the LFT of $\bar{G}(\Delta)$ and $\bar{K}(\Delta)$ and denoted by $\bar{\mathcal{G}}(\Delta) = \bar{G}(\Delta) \star \bar{K}(\Delta) = \Delta \star G \star \bar{K}(\Delta)$.

novel control structure. The control structure is independent of the particular performance framework. The idea associated with this structure is to transform the LPV gain-scheduling problem into a classical robust performance problem. A robust LTI controller solving this robust performance problem is converted into an LPV controller for the original LPV gain-scheduling problem. Conditions for the realizability of this LPV controller as well as state-space formulas for the controller are stated in Section 4.4. Subsequently we describe how the proposed gain-scheduling approach is used in the $\ell_1$ performance framework (Section 4.5) and in the quadratic performance framework (Section 4.6). The proposed method constitutes the first scheme for $\ell_1$-optimal LPV output-feedback synthesis. In Chapter 5, it is described how to apply the proposed method to an academic and to a more practical example. In doing so, the advantages of the proposed control structure are discussed, and the applicability of $\ell_1$-optimal control is tested. Parts of this chapter are based on Rieber and Allgöwer (2003); Rieber *et al.* (2005*a*); Rieber and Allgöwer (2006*b*).

## 4.2 Problem Setup

Consider the interconnection of an LTI system $G$ and an uncertainty block $\Delta$ as depicted in Figure 4.1(a). The symbols denote measurements $y$, control inputs $u$, exogenous inputs $w_1$, and performance outputs $z_1$. Specifications for $\Delta$ are described in Section 2.3. The LPV system obtained from this interconnection is denoted $\bar{G}(\Delta)$ and given by the LFT $\bar{G}(\Delta) = \Delta \star G$. Assume now that a measurement or a reasonably good estimate of $\Delta(k)$ is available in real-time for each time-step $k$. To use the parameter measurement favorably for control, an LPV output-feedback controller

$$u = \bar{K}(\Delta) y$$

with a certain dependence on the measured parameter information $\Delta$ is desired. The situation is depicted in Figure 4.1. The controlled uncertain system is denoted by $\bar{\mathcal{G}}(\Delta)$, see Figure 4.1(b). Particular realizations of the involved systems and of the interconnection are detailed later on.

Let the vector-valued signals have the dimensions $p_i = \dim(z_i)$ and $q_j = \dim(w_j)$, $i, j = 0, 1$, $n_y = \dim(y)$, and $n_u = \dim(u)$. The described framework is embedded into a robust performance problem as follows.

**Problem 4.1** *For the LPV plant $\bar{G}(\Delta) = \Delta \star G$, find an LPV output-feedback controller $\bar{K}(\Delta)$ such that*

*(i) the closed loop $\bar{\mathcal{G}}(\Delta) = \Delta \star G \star \bar{K}(\Delta)$ is robustly stable with respect to $\Delta \in \mathbf{\Delta}_{\mathrm{TV}}$ (in some sense to be specified more precisely later), and*

*(ii) the robust performance criterion $\sup\limits_{\Delta\in\mathbf{\Delta}_{\mathrm{TV}}} \sup\limits_{0<\|w_1\|_p<\infty} \frac{\|z_1\|_p}{\|w_1\|_p}$ is minimized.* $\square$

In this work, the two cases $p = \infty$ and $p = 2$, i.e. the $\ell_1$ and the $\mathcal{H}_\infty$ problem, are treated in more detail. To make the system formulation more concrete, state-space realizations of the involved systems are given next. We consider a realization of the discrete-time LPV plant $\bar{G}(\Delta)$ as in (4.1), obtained from the interconnection of the LTI system $G$ and the matrix-valued function $\Delta$ such that

$$\begin{bmatrix} x(k+1) \\ z_0(k) \\ z_1(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} A & B_0 & B_1 & B_2 \\ C_0 & D_{00} & D_{01} & D_{02} \\ C_1 & D_{10} & D_{11} & D_{12} \\ C_2 & D_{20} & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} x(k) \\ w_0(k) \\ w_1(k) \\ u(k) \end{bmatrix}, \tag{4.2}$$

$$w_0(k) = \Delta(k) z_0(k) \tag{4.3}$$

with the state $x(k) \in \mathbb{R}^{n_x}$. Then the system matrices in (4.1) are given by

$$\begin{bmatrix} \bar{A}(\Delta(k)) & \bar{B}_j(\Delta(k)) \\ \bar{C}_i(\Delta(k)) & \bar{D}_{ij}(\Delta(k)) \end{bmatrix} = \Delta(k) \star \left[\begin{array}{c:cc} D_{00} & C_0 & D_{0j} \\ \hdashline B_0 & A & B_j \\ D_{i0} & C_i & D_{ij} \end{array}\right], \quad i = 1,2, \quad j = 1,2 \tag{4.4}$$

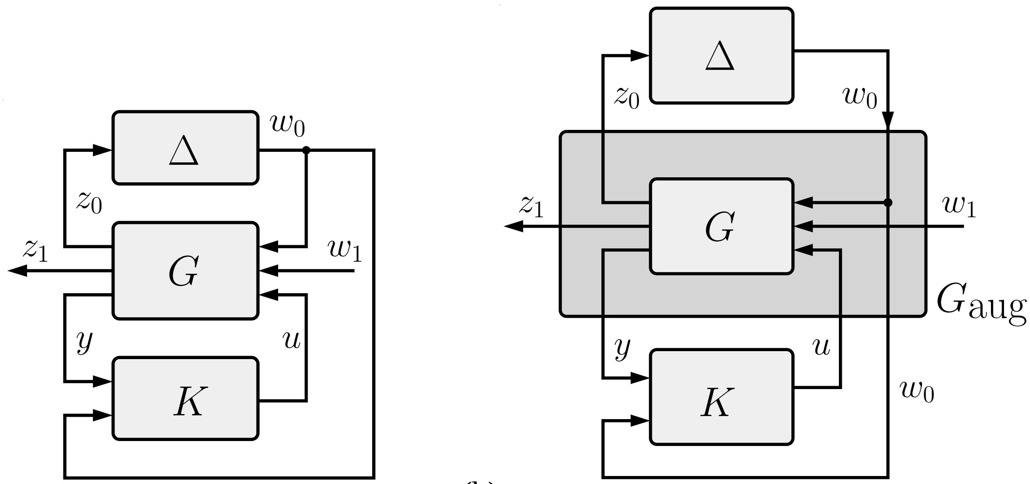
$$= \begin{bmatrix} A + B_0(I - \Delta D_{00})^{-1}\Delta C_0 & B_j + B_0(I - \Delta D_{00})^{-1}\Delta D_{0j} \\ C_i + D_{i0}(I - \Delta D_{00})^{-1}\Delta C_0 & D_{ij} + D_{i0}(I - \Delta D_{00})^{-1}\Delta D_{0j} \end{bmatrix}.$$

The corresponding closed-loop system is $\bar{\mathcal{G}}(\Delta)$ with realization (2.11).

## 4.3 A Novel Gain-Scheduling Structure

This section introduces the concepts and structures, on which our new LPV synthesis method is based. The structural setup is independent of the specific norm-based optimization approach used, i.e. it does not matter if $\mathcal{H}_\infty$, $\ell_1$ or any other performance measures are applied. However, the structure has been particularly developed for LPV control in the $\ell_1$ framework, since it possesses some specific advantages there as discussed in Section 4.5.

Consider a parameter-dependent plant $\bar{G}(\Delta) = \Delta \star G$, where the specifications of Section 4.2 hold. Assume for the moment that the signal $w_0 = \Delta z_0$ is measurable. This assumption is not valid in




(a) (b)

*Figure 4.2:* (a) The controller's parameter-dependence is realized through an LTI controller $K$ with access to the signal $w_0$. (b) The new LTI design plant is called $G_{\text{aug}}$.

general and is relaxed in the next section. A special type of parameter dependence is achieved if one feeds $w_0$ as an additional measurement to an LTI controller

$$u = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} y \\ w_0 \end{bmatrix}. \tag{4.5}$$

The controller has access to the parameter information by means of the signal $w_0$ as depicted in Figure 4.2(a). The structure in Figure 4.2(a) is redrawn as in Figure 4.2(b) to combine all elements except for the uncertainty block and the controller block into the augmented LTI plant $G_{\text{aug}}$. It is easy to see that $G_{\text{aug}}$ is

$$\begin{bmatrix} z_0 \\ z_1 \\ y \\ w_0 \end{bmatrix} = \begin{bmatrix} G_{00} & G_{01} & G_{02} \\ G_{10} & G_{11} & G_{12} \\ G_{20} & G_{21} & G_{22} \\ I & 0 & 0 \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ u \end{bmatrix} \quad \text{if} \quad G = \begin{bmatrix} G_{00} & G_{01} & G_{02} \\ G_{10} & G_{11} & G_{12} \\ G_{20} & G_{21} & G_{22} \end{bmatrix}. \tag{4.6}$$

Hence the original LPV design problem of Figure 4.1 has been transformed into a robust control problem, where a fixed robust controller $K$ is sought for the plant $G_{\text{aug}}$ in face of the uncertainty $\Delta$. Using this structure, only the original uncertainty block $\Delta$ appears in the problem setting. This is in contrast to $\mathcal{H}_\infty$ and quadratic performance approaches like in Packard (1994); Apkarian and Gahinet (1995); Scorletti and El Ghaoui (1998); Scherer (2000*b*); Scherer (2001*a*); Wu and Dong (2006), where either the $\Delta$-block is duplicated or a new $\Delta_K$-block is introduced, see Figure 4.3(a). For this additional $\Delta_K$-block, a dependence on the parameter information $\Delta$ would have to be chosen or computed by some procedure, which often increases the number and size of the inequalities involved in the problem's solution compared to robust controller synthesis. In the method of Scherer (2000*b*), which is related to quadratic and other LMI-based performance criteria, the additional $\Delta_K$-block does not introduce additional complexity in the synthesis equations.

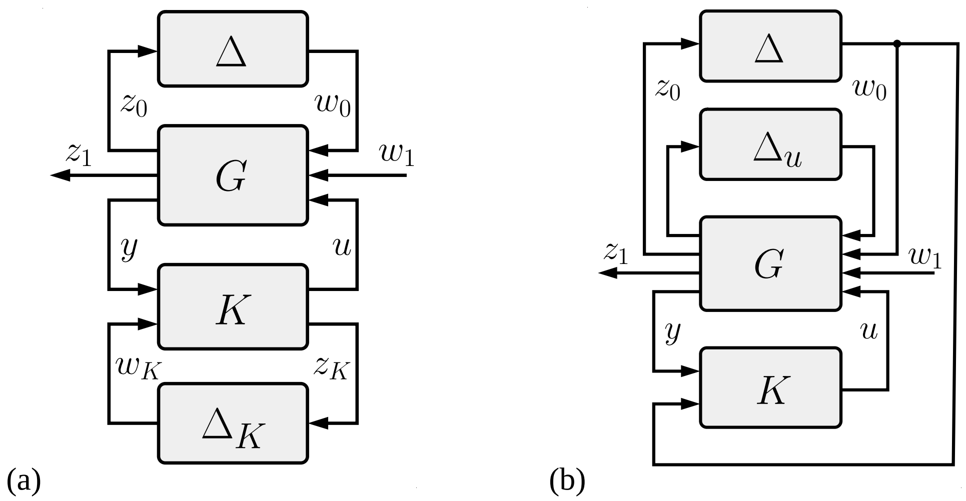



*Figure 4.3:* (a) Alternative gain-scheduling structure often used in the literature. (b) Proposed gain-scheduling structure with additional (dynamic) uncertainty block $\Delta_u$.

**Remark 4.1** *The control structure in Figure 4.2(a) has first been introduced in Rieber and Allgöwer (2003) and further been investigated in Rieber* et al. *(2005*a*). The same control structure has been discussed independently in Wu and Lu (2004), under the restrictive assumption that $w_0$ is measurable in general. There, the structure is used mainly to investigate convex synthesis conditions for robust controller design, but not in a gain-scheduling scheme as proposed here. Moreover, it is shown in Wu and Lu (2004) that introducing an additional $\Delta_K$-block for gain-scheduling does not improve performance, if $w_0$ is fed to the controller $K$. The controller construction described in Section 4.4 below relaxes the somewhat artificial assumption on $w_0$.* □

**Remark 4.2** *The commonly applied LPV control structure of Figure 4.3(a) uses the more complex augmented plant and uncertainty structures*

$$\begin{bmatrix} z_K \\ z_0 \\ z_1 \\ y \\ w_K \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & I \\ 0 & G_{00} & G_{01} & G_{02} & 0 \\ 0 & G_{10} & G_{11} & G_{12} & 0 \\ 0 & G_{20} & G_{21} & G_{22} & 0 \\ I & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_K \\ w_0 \\ w_1 \\ u \\ z_K \end{bmatrix}, \qquad \Delta_{\text{aug}} = \begin{bmatrix} \Delta_K & 0 \\ 0 & \Delta \end{bmatrix} \tag{4.7}$$

*instead of (4.6) and $\Delta$.* □

Using the structure of Figure 4.2(b), Problem 4.1 can be rewritten in a more concrete way as follows.

**Problem 4.2** *For the LPV plant $\Delta \star G_{\text{aug}}$, find an LTI output-feedback controller $K$ such that*

*(i) the closed loop $\bar{\mathcal{G}}(\Delta) = \Delta \star G_{\text{aug}} \star K$ is robustly stable with respect to $\Delta \in \mathbf{\Delta}_{\text{TV}}$ (in some sense to be specified more precisely later), and*

*(ii) the robust performance criterion $\sup\limits_{\Delta \in \mathbf{\Delta}_{\text{TV}}} \sup\limits_{0<\|w_1\|_p<\infty} \frac{\|z_1\|_p}{\|w_1\|_p}$ is minimized.* □

The optimal performance is denoted $\gamma$. It is easy to see that a controller obtained with the structure in Figure 4.2(b) has the same or better performance than a robust controller that does not have access to the parameter information. To this end, suppose that there exists a fixed robust controller $u = K_{\mathrm{robust}} y$ for the plant $\bar{G}(\Delta)$ without access to the parameter information $w_0$, and suppose that this controller achieves robust stability and a certain performance $\gamma = \gamma_{\mathrm{robust}}$. A controller $u = K\,\mathrm{col}(y, w_0)$ with $K = [K_1 \quad K_2]$ can always be chosen such that $K_1 = K_{\mathrm{robust}}$ and $K_2 \equiv 0$. Then $K$ achieves robust stability and a performance level $\gamma = \gamma_{\mathrm{robust}}$. However, $K_2 \not\equiv 0$ introduces additional degrees of freedom. Hence $K$ achieves $\gamma = \gamma_{\mathrm{LPV}} \leq \gamma_{\mathrm{robust}}$ in general.

The proposed setting of Figure 4.2(a) can be extended in a straightforward manner to also include unknown bounded structured or unstructured dynamic uncertainties $\Delta_u$ that are not subject to measurement of any kind, allowing for robust gain-scheduling. The corresponding structure is displayed in Figure 4.3(b).

## 4.4 Realization of the LPV Controller

So far the LTI controller $K$ with inputs $y$ and $w_0$ is considered as a robust controller for the plant $G_{\mathrm{aug}}$. It has been said before that the signal $w_0$ is not available as a measurement in general. Moreover, the original goal was to find an LPV controller $\bar{K}(\Delta)$ for the plant $\bar{G}(\Delta)$ as depicted in Figure 4.1. To this end, it is shown next how and under which conditions an LPV controller $\bar{K}(\Delta)$ with input $y$ can be constructed from the auxiliary LTI controller $K$. Assume in this section that a controller $K$ is given, achieving robust stability and a certain robust performance $\gamma$ of the closed loop in Figure 4.2(b).

It turns out that the realization of $\bar{K}(\Delta)$ becomes particularly simple if the channel $w_1 \mapsto z_0$ of $G$ has a vanishing transfer map $G_{01} \equiv 0$. The general case $G_{01} \not\equiv 0$ is dealt with next, whereas the reduction to the frequently arising case $G_{01} \equiv 0$ is presented afterwards. Furthermore it is discussed, under which conditions $G_{01} \equiv 0$ is valid and how the realization conditions for $\bar{K}(\Delta)$ can be achieved or relaxed. Preliminary discussions on the topic are contained in Fritsch (2004).

### 4.4.1 General Case

The following definition states conditions for the existence of an LPV controller $\bar{K}(\Delta)$ constructed from $K$ and $G_{\mathrm{aug}}$, and gives a formula for such a controller.

**Definition 4.1** *Consider Figure 4.2(b) with the augmented plant $G_{\mathrm{aug}}$ (4.6), the uncertainty block $\Delta$, and a controller $K$ (4.5). Suppose that*

1. *$G_{21}$ is a left-invertible map, and*
2. *$I - \Delta(\tilde{G}_{00} + \tilde{G}_{02} K_2)$ is an invertible map for all $\Delta \in \mathbf{\Delta}_{\mathrm{TV}}$ (for the abbreviations see below).*

*Then an LPV controller $\bar{K}(\Delta)$ can be constructed according to*

$$\bar{K}(\Delta) = K_1 + K_2\breve{G}(\Delta)\breve{G}_{02} = \left[\begin{array}{c:c} K_1 & K_2 \\ \hdashline G_{01}G_{21}^{\dagger} + \tilde{G}_{02}K_1 & \tilde{G}_{00} + \tilde{G}_{02}K_2 \end{array}\right] \star \Delta \tag{4.8}$$

*with the abbreviations*

$$\begin{aligned} \tilde{G}_{00} &:= G_{00} - G_{01}G_{21}^{\dagger}G_{20}, & \breve{G}_{02} &:= G_{01}G_{21}^{\dagger} + \tilde{G}_{02}K_1, \\ \tilde{G}_{02} &:= G_{02} - G_{01}G_{21}^{\dagger}G_{22}, & \breve{G}(\Delta) &:= \Big(I - \Delta(\tilde{G}_{00} + \tilde{G}_{02}K_2)\Big)^{-1}\Delta. \end{aligned} \qquad \square$$

See Appendix D.5 for a derivation of (4.8). If the conditions in Definition 4.1 are fulfilled, then an LPV controller $\bar{K}(\Delta)$ can be obtained from the LTI controller $K$. Some comments about the definition are in order. First, it is visible that the controller's dependence on $\Delta$ is rational in general. This means that a certain nonlinear parameter dependence is taken into account, and hence a better performance compared to, say, just affine dependence may be achieved. On the other hand it has to be noted that the left-invertibility of $G_{21}$ is a somewhat restrictive condition. This condition essentially amounts to demanding the reconstruction of the external disturbance $w_1$ from $y$ and $u$. Moreover, it requires $n_y \geq q_1$. By looking into the derivation of the controller realization above from a transfer-function perspective (Rieber *et al.*, 2005*a*), it becomes clear that it is not advisable to use the given formulas in the presence of unstable zeros of $\hat{G}_{21}(z)$. In this case, $\hat{G}_{21}(z)^{\dagger}$ is an unstable system, and the slightest numerical or modeling errors would introduce unstable modes into the closed-loop system. Furthermore, the invertibility condition (2.) is nothing more than an extended well-posedness assumption on the closed-loop. Section 4.4.3 discusses relaxations to circumvent or enforce the given conditions.

A state-space version of Definition 4.1 is presented next to make the involved conditions and expressions more concrete, and to provide simple formulas to actually implement the LPV controller. To this end, consider the following state-space realizations of the LPV controller $\bar{K}(\Delta)$

$$\begin{bmatrix} \xi_K(k+1) \\ u(k) \end{bmatrix} = \begin{bmatrix} \bar{A}_K(\Delta(k)) & \bar{B}_K(\Delta(k)) \\ \bar{C}_K(\Delta(k)) & \bar{D}_K(\Delta(k)) \end{bmatrix} \begin{bmatrix} \xi_K(k) \\ y(k) \end{bmatrix}, \tag{4.9}$$

of the auxiliary LTI controller $K$

$$\begin{bmatrix} x_K(k+1) \\ u(k) \end{bmatrix} = \begin{bmatrix} A_K & B_{K1} & B_{K2} \\ C_K & D_{K1} & D_{K2} \end{bmatrix} \begin{bmatrix} x_K(k) \\ y(k) \\ w_0(k) \end{bmatrix}, \tag{4.10}$$

and of the augmented design plant $G_{\text{aug}}$

$$\begin{bmatrix} x(k+1) \\ z_0(k) \\ z_1(k) \\ y(k) \\ w_0(k) \end{bmatrix} = \begin{bmatrix} A & B_0 & B_1 & B_2 \\ C_0 & D_{00} & D_{01} & D_{02} \\ C_1 & D_{10} & D_{11} & D_{12} \\ C_2 & D_{20} & D_{21} & D_{22} \\ 0 & I & 0 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ w_0(k) \\ w_1(k) \\ u(k) \end{bmatrix}. \tag{4.11}$$

**Definition 4.2** *Consider the augmented plant $G_{\text{aug}}$ (4.11) interconnected with an uncertainty block $\Delta$ as in (4.3), and a controller $K$ (4.10). Suppose that*

1. *$D_{21}$ has full column rank, and*
2. *$I - \Delta(k)(\tilde{D}_{00} + \tilde{D}_{02}D_{K2})$ is non-singular for all $\Delta(k) \in \mathbf{\Delta}$ (for the abbreviations see below).*

*Then an LPV controller $\bar{K}(\Delta)$ can be constructed according to (4.9), where $\xi_K = \mathrm{col}(x, x_K)$ and*

$$\left[\begin{array}{c:c} \bar{A}_K(\Delta(k)) & \bar{B}_K(\Delta(k)) \\ \hdashline \bar{C}_K(\Delta(k)) & \bar{D}_K(\Delta(k)) \end{array}\right]$$
$$= \left[\begin{array}{cc:c} \tilde{A} + \breve{B}\breve{D}(\Delta(k))\tilde{C}_0 & \left(\tilde{B}_2 + \breve{B}\breve{D}(\Delta(k))\tilde{D}_{02}\right) C_K & \breve{B}_2 + \breve{B}\breve{D}(\Delta(k))\breve{D}_{02} \\ B_{K2}\breve{D}(\Delta(k))\tilde{C}_0 & A_K + B_{K2}\breve{D}(\Delta(k))\tilde{D}_{02}C_K & B_{K1} + B_{K2}\breve{D}(\Delta(k))\breve{D}_{02} \\ \hdashline D_{K2}\breve{D}(\Delta(k))\tilde{C}_0 & \left(I + D_{K2}\breve{D}(\Delta(k))\tilde{D}_{02}\right) C_K & D_{K1} + D_{K2}\breve{D}(\Delta(k))\breve{D}_{02} \end{array}\right]$$

*with the abbreviations*

$$\begin{aligned} \tilde{A} &:= A - B_1 D_{21}^{\dagger} C_2, & \breve{B} &:= \tilde{B}_0 + \tilde{B}_2 D_{K2}, \\ \tilde{B}_0 &:= B_0 - B_1 D_{21}^{\dagger} D_{20}, & \breve{B}_2 &:= B_1 D_{21}^{\dagger} + \tilde{B}_2 D_{K1}, \\ \tilde{B}_2 &:= B_2 - B_1 D_{21}^{\dagger} D_{22}, & \breve{D}_{02} &:= D_{01} D_{21}^{\dagger} + \tilde{D}_{02} D_{K1}, \\ \tilde{C}_0 &:= C_0 - D_{01} D_{21}^{\dagger} C_2, & \breve{D}(\Delta(k)) &:= \left(I - \Delta(k)(\tilde{D}_{00} + \tilde{D}_{02} D_{K2})\right)^{-1} \Delta(k), \\ \tilde{D}_{00} &:= D_{00} - D_{01} D_{21}^{\dagger} D_{20}, & \tilde{D}_{02} &:= D_{02} - D_{01} D_{21}^{\dagger} D_{22}. \end{aligned}$$ □

See Appendix D.5 for a derivation of the given formulas. With Definitions 4.1 and 4.2, realization conditions and easily implementable state-space formulas of the LPV controller $\bar{K}(\Delta)$ are available. The state-space realization given in Definitions 4.2 exhibits a state-dimension that is equal to the number of states of $A$ plus the number of states of $A_K$. The realization is hence minimal in the sense that in general no further state reduction is possible in an analytic way, provided that the realizations of $G_{\text{aug}}$ and $K$ are minimal. Further state reduction of the resulting LPV controller may be achieved in particular cases. Moreover, model reduction techniques that preserve closed-loop stability and (approximately) also performance could be applied on the resulting LPV controller. However, due to the controller's parameter dependence, this is a largely open problem and not further pursued here.

### 4.4.2 Simple Case

If the transfer operator $G_{01}$ of the channel $w_1 \mapsto z_0$ in (4.6) is identically zero, the realization of $\bar{K}(\Delta)$ becomes much simpler than in the general case. Moreover, the conditions for existence of $\bar{K}(\Delta)$ become considerably less restrictive. This special case is of practical importance as discussed at the end of the section. Analogous to Definitions 4.1 and 4.2, the following statements give simpler conditions and realizations for the LPV controller $\bar{K}(\Delta)$. The first two definitions treat the case $G_{01} \equiv 0$, the third one treats the case if additionally $C_0 = 0$ in the state-space version.

**Definition 4.3** *Consider Figure 4.2(b) with the augmented plant $G_{\text{aug}}$ (4.6), the uncertainty block $\Delta$, and a controller $K$ (4.5). Suppose that*

1. $G_{01} \equiv 0$*, and*
2. $I - \Delta(G_{00} + G_{02}K_2)$ *is an invertible map for all* $\Delta \in \boldsymbol{\Delta}_{\text{TV}}$.

*Then an LPV controller $\bar{K}(\Delta)$ can be constructed according to*

$$\bar{K}(\Delta) = \left(I + K_2\tilde{G}(\Delta)G_{02}\right) K_1 = \left[\begin{array}{c:c} K_1 & K_2 \\ \hdashline G_{02}K_1 & G_{00} + G_{02}K_2 \end{array}\right] \star \Delta \tag{4.12}$$

*with the abbreviation*

$$\tilde{G}(\Delta) := \left(I - \Delta(G_{00} + G_{02}K_2)\right)^{-1} \Delta. \qquad \square$$

See Appendix D.5 for a derivation of (4.12). A state-space version of Definition 4.3 is stated next.

**Definition 4.4** *Consider the augmented plant $G_{\text{aug}}$ (4.11) interconnected with an uncertainty block $\Delta$ as in (4.3), and a controller $K$ (4.10). Suppose that*

1. $G_{01} \equiv 0$*, i.e.* $D_{01} = 0$, $x(0) = 0$*, and all modes of the system* $x(k+1) = Ax(k) + B_1 w_1(k)$, $z_0(k) = C_0 x(k)$ *are unreachable or unobservable (or both), and*
2. $I - \Delta(k)(D_{00} + D_{02}D_{K2})$ *is non-singular for all* $\Delta(k) \in \boldsymbol{\Delta}$.

*Then an LPV controller $\bar{K}(\Delta)$ can be constructed according to (4.9), where $\xi_K = \operatorname{col}(x, x_K)$ and*

$$\left[\begin{array}{c:c} \bar{A}_K(\Delta(k)) & \bar{B}_K(\Delta(k)) \\ \hdashline \bar{C}_K(\Delta(k)) & \bar{D}_K(\Delta(k)) \end{array}\right]$$
$$= \left[\begin{array}{cc:c} A + \tilde{B}\tilde{D}(\Delta(k))C_0 & \left(B_2 + \tilde{B}\tilde{D}(\Delta(k))D_{02}\right) C_K & \left(B_2 + \tilde{B}\tilde{D}(\Delta(k))D_{02}\right) D_{K1} \\ B_{K2}\tilde{D}(\Delta(k))C_0 & A_K + B_{K2}\tilde{D}(\Delta(k))D_{02}C_K & B_{K1} + B_{K2}\tilde{D}(\Delta(k))D_{02}D_{K1} \\ \hdashline D_{K2}\tilde{D}(\Delta(k))C_0 & \left(I + D_{K2}\tilde{D}(\Delta(k))D_{02}\right) C_K & \left(I + D_{K2}\tilde{D}(\Delta(k))D_{02}\right) D_{K1} \end{array}\right]$$

*with the abbreviations*

$$\tilde{B} := B_0 + B_2 D_{K2}, \qquad \tilde{D}(\Delta(k)) := \left(I - \Delta(k)(D_{00} + D_{02}D_{K2})\right)^{-1} \Delta(k). \qquad \square$$

See Appendix D.5 for a derivation of the given formulas. If additionally $C_0 = 0$ holds in the state-space version, Definition 4.4 can be even further simplified as shown next.

**Definition 4.5** *Consider the augmented plant $G_{\text{aug}}$ (4.11) interconnected with an uncertainty block $\Delta$ as in (4.3), and a controller $K$ (4.10). Suppose that*

1. $C_0 = 0$ *and* $D_{01} = 0$,
2. $I - \Delta(k)(D_{00} + D_{02}D_{K2})$ *is non-singular for all* $\Delta(k) \in \mathbf{\Delta}$.

*Then an LPV controller $\bar{K}(\Delta)$ can be constructed according to (4.9), where $\xi_K = x_K$ and*

$$\left[\begin{array}{c:c} \bar{A}_K(\Delta(k)) & \bar{B}_K(\Delta(k)) \\ \hdashline \bar{C}_K(\Delta(k)) & \bar{D}_K(\Delta(k)) \end{array}\right] = \left[\begin{array}{c:c} A_K + B_{K2}\tilde{D}(\Delta(k))D_{02}C_K & B_{K1} + B_{K2}\tilde{D}(\Delta(k))D_{02}D_{K1} \\ \hdashline \left(I + D_{K2}\tilde{D}(\Delta(k))D_{02}\right)C_K & \left(I + D_{K2}\tilde{D}(\Delta(k))D_{02}\right)D_{K1} \end{array}\right]$$

*with the abbreviations as in Definition 4.4.* □

See Appendix D.5 for a derivation of the given formulas. Definitions 4.3, 4.4, and 4.5 provide realization conditions and easily implementable state-space formulas of the LPV controller $\bar{K}(\Delta)$ also for the simple case $G_{01} \equiv 0$. It is argued next, why this special case is of practical importance.

**When is the Simple Case Applicable?**

This section discusses some situations that lead to $G_{01} \equiv 0$, i.e. the transfer map $w_1 \mapsto z_0$ in (4.6) or in (4.2) vanishes. It turns out that one encounters this situation quite often, hence it is of practical importance.

As a first observation, consider a standard control design problem for a parameter-dependent system. Usually one starts out with a model with control inputs $u$ and measurement outputs $y$ such as in

$$\begin{bmatrix} x(k+1) \\ y(k) \end{bmatrix} = \begin{bmatrix} \bar{A}(\Delta(k)) & \bar{B}(\Delta(k)) \\ \bar{C}(\Delta(k)) & \bar{D}(\Delta(k)) \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}.$$

Then the parameter-dependent model is factored into

$$\begin{bmatrix} x(k+1) \\ z_0(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} A & B_0 & B_2 \\ C_0 & D_{00} & D_{02} \\ C_2 & D_{20} & D_{22} \end{bmatrix} \begin{bmatrix} x(k) \\ w_0(k) \\ u(k) \end{bmatrix}, \qquad (4.13)$$
$$w_0(k) = \Delta(k)z_0(k).$$

Note that $B_1$ and $D_{01}$ are not present, hence $G_{01} \equiv 0$ here. This does not change if exogenous inputs $w_1$ and performance outputs $z_1$ are introduced, as long as there is no direct influence from $w_1$ on $x$ (such as disturbances acting on the states or input disturbances). This last assumption is often valid in practice, see also the examples in Chapter 5.

The property $G_{01} \equiv 0$ is preserved if additionally the feedback error signal is filtered by an integrator. Suppose that there is a reference command $w_1$, and that the controller now obtains the signal $\tilde{y} = G_{\text{int}}(w_1 - y)$ instead of $y$, where $\hat{G}_{\text{int}}(z) = \tau I/(z-1)$ is a discrete-time integrator with sampling interval $\tau$. Thus the state-space representation (4.13) is augmented to

$$\left[\begin{array}{c} x_{\text{int}}(k+1) \\ x(k+1) \\ \hdashline z_0(k) \\ \tilde{y}(k) \end{array}\right] = \left[\begin{array}{cc:ccc} I & -\tau C_2 & -\tau D_{20} & \tau I & -\tau D_{22} \\ 0 & A & B_0 & 0 & B_2 \\ \hdashline 0 & C_0 & D_{00} & 0 & D_{02} \\ I & 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} x_{\text{int}}(k) \\ x(k) \\ \hdashline w_0(k) \\ w_1(k) \\ u(k) \end{array}\right]. \tag{4.14}$$

Observe that the realization (4.14) still implies $G_{01} \equiv 0$.

Secondly, one has $B_1 = 0$ whenever there are no disturbances acting on the states directly, i.e. if $\bar{B}_1(\Delta) \equiv 0$ in (4.1). If moreover $\bar{D}_{11}(\Delta(k))$ and $\bar{D}_{21}(\Delta(k))$ are not parameter-dependent, i.e. no parameter-dependence is introduced by the definition of performance channels, then one can find a realization (4.2)–(4.3) such that $D_{01} = 0$ and thus $G_{01} \equiv 0$.

As a third instance, one has $C_0 = 0$ and $D_{01} = 0$ and hence $G_{01} \equiv 0$ on the admittedly rare occasion that only $\bar{B}_2(\Delta(k))$, $\bar{D}_{12}(\Delta(k))$, and $\bar{D}_{22}(\Delta(k))$ are (possibly) parameter-dependent. In all these cases, the less restrictive realization conditions and simpler formulas of Section 4.4.2 can be used.

### 4.4.3 Relaxation of the Realization Conditions

The realization conditions of Definitions 4.1 and 4.2 are restrictive in general. If one cannot take advantage of the case $G_{01} \equiv 0$, there are yet a number of situations where these conditions can be influenced or circumvented as described next.

First, there is the possibility to impose conditions on the chosen plant description and on the controller. Appropriate modeling can often lead to $D_{00} = 0$ and $D_{01} = 0$. If one additionally imposes $D_{K2} = 0$ (which may result in some performance loss or in shrinking of the robust stability margins), condition 2 is trivially satisfied. On the other hand, slight perturbations of controller and/or plant matrices lead to non-singularity of the matrix in condition 2 without considerable performance loss.

To achieve the full-column rank of $D_{21}$, some external disturbances may have to be left out of the design model. Apart from that, one can also think of estimating or observing the signal $w_1$ instead of reconstructing it completely. If $\hat{G}_{21}(z)$ has unstable zeros, a remedy may be found using methods for stable inversion of non-minimum phase systems (Chen and Paden, 1996; George *et al.*, 1999, for example).

As a final special case, suppose that the signal $w_1$ is available just like a measurement $y$ in real-time. This strong assumption holds when $w_1$ is estimated, or when no unknown external disturbances are modeled in $w_1$, but only measurable disturbances, reference commands and the like. Then the signal $w_1$ can be fed to the controller directly in addition to $y$. Hence, even in the case $G_{01} \not\equiv 0$, a realization of $\bar{K}(\Delta)$ is given by Definition 4.4, where the input $y$ is replaced by $\text{col}(y, w_1)$, and

$\bar{B}_K(\Delta(k))$, $\bar{D}_K(\Delta(k))$ are replaced by

$$\bar{B}_K(\Delta(k)) = \begin{bmatrix} \left(B_2 + \tilde{B}\tilde{D}(\Delta(k))D_{02}\right) D_{K1} & B_1 + \tilde{B}\tilde{D}(\Delta(k))D_{01} \\ B_{K1} + B_{K2}\tilde{D}(\Delta(k))D_{02}D_{K1} & B_{K2}\tilde{D}(\Delta(k))D_{01} \end{bmatrix},$$
$$\bar{D}_K(\Delta(k)) = \begin{bmatrix} \left(I + D_{K2}\tilde{D}(\Delta(k))D_{02}\right) D_{K1} & D_{K2}\tilde{D}(\Delta(k))D_{01} \end{bmatrix}.$$

This discussion concludes the section about realization of the LPV controller $\bar{K}(\Delta)$. Next, more concrete versions of Problem 4.2 using the control structure of Figure 4.2(b) are investigated.

## 4.5 Gain-Scheduling in the $\ell_1$ Performance Framework

Whereas there are powerful methods for LPV controller design in the $\mathcal{H}_\infty$ framework, there has been little progress for LPV controller design with respect to the $\ell_1$ framework. This is mainly due to a lack of LMI conditions to tightly characterize $\ell_\infty$-stable systems and the corresponding $\ell_1$-norm. There is however a state-feedback approach to parameter-varying control based on set-valued analysis and viability theory in Shamma and Xiong (1999).

The control structure developed in the previous sections is now applied to the problem of $\ell_1$-optimal LPV output-feedback control. A robust stability proof for the proposed LPV controllers is given. The method presented here constitutes the first instance of gain-scheduled output-feedback controller synthesis in the $\ell_1$ framework.

Suppose that the parameter dependence of $\bar{K}(\Delta)$ is chosen as in Figure 4.2. Hence the problem is reduced to finding an LTI controller $K$ instead of $\bar{K}(\Delta)$. In doing so, only the original uncertainty block $\Delta$ appears in the problem setting. The proposed approach in connection with the special control structure is particularly suited for $\ell_\infty$-gain based control, since in this framework only robust performance methods using small-gain theory exist; there is yet no LMI approach similar to the $\mathcal{H}_\infty$ framework. Additional complexity and increased computation times would be the result by considering a structure with additional $\Delta_K$-block as in Figure 4.3(a). Moreover, conservatism would be introduced due to repeated components in the $\Delta$ and $\Delta_K$ blocks. By applying the control structure introduced in Figure 4.2, this increased complexity and conservatism are avoided.

For simplicity of the presentation, assume $\dim z_1 = \dim w_1 = p_1$. This can be achieved by introducing dummy inputs/outputs that have essentially no effect on performance, whereas the case $\dim z_1 \neq \dim w_1$ requires slightly more involved formulations. For the same reason it is assumed that $\dim z_0 = \dim w_0 = p_0$. $\ell_\infty$ stability and robust $\ell_\infty$ stability of an input/output map are defined next. For related definitions and more details, consult for example Desoer and Vidyasagar (1975); Khalil (2002); Dahleh and Diaz-Bobillo (1995); Khammash and Pearson (1991).

**Definition 4.6** *($\ell_\infty$ stability)*
*A map $\mathcal{G}$ is said to be $\ell_\infty$-stable if it is causal, maps $\ell_\infty^n$ to $\ell_\infty^m$, and satisfies $\|\mathcal{G}\|_{\infty-\mathrm{ind}} < \infty$.* □

The corresponding continuous-time property is called $\mathcal{L}_\infty$ stability. A definition for robust stability of the interconnection in Figure 2.1(b) is recalled next, applying the norm-bounded uncertainty set

$$\mathbf{\Delta}_{\mathrm{NB}} := \{\Delta = \mathrm{diag}(\Delta_1, ..., \Delta_n) | \Delta_i : \ell_\infty^{m_i} \to \ell_\infty^{m_i} \text{ is causal LTV and } \|\Delta_i\|_{\infty-\mathrm{ind}} \leq 1\} \tag{4.15}$$

with $p_0 = \sum_{i=1}^n m_i$.

**Definition 4.7** *(Robust $\ell_\infty$ stability)*
*Suppose that $\mathcal{M}$ and $\Delta$ in Figure 2.1(b) are $\ell_\infty$-stable maps, and that $\Delta \in \mathbf{\Delta}_{\mathrm{NB}}$. Then the interconnection in Figure 2.1(b) is said to be robustly $\ell_\infty$-stable if $(I - \mathcal{M}\Delta)^{-1}$ is $\ell_\infty$-stable for all $\Delta \in \mathbf{\Delta}_{\mathrm{NB}}$.* □

Now suppose that the uncertainty block $\Delta \in \mathbf{\Delta}_{\mathrm{TV}}$ is viewed as an LTV operator from $\ell_\infty^{p_0}$ to $\ell_\infty^{p_0}$ belonging to the norm-bounded uncertainty set (4.15). For block-diagonal $\Delta$-blocks it holds that $\mathbf{\Delta}_{\mathrm{TV}} \subset \mathbf{\Delta}_{\mathrm{NB}}$, if corresponding scalings are incorporated into the plant $G$. This viewpoint on the uncertainty leads to the following reformulation of Problem 4.2.

**Problem 4.3** *For the LPV plant $\Delta \star G_{\mathrm{aug}}$, find an LTI output-feedback controller $K$ such that*

*(i) the closed loop $\bar{\mathcal{G}}(\Delta) = \Delta \star G_{\mathrm{aug}} \star K$ is robustly $\ell_\infty$-stable with respect to $\Delta \in \mathbf{\Delta}_{\mathrm{NB}}$, and*

*(ii) the $\ell_\infty$-gain $\sup\limits_{\Delta \in \mathbf{\Delta}_{\mathrm{NB}}} \|\bar{\mathcal{G}}(\Delta)\|_{\infty-\mathrm{ind}}$ is minimized.* □

Note that some conservatism is introduced by our method due to the view on the uncertainty $\Delta$ as being composed of LTV dynamic uncertainty blocks in the set $\mathbf{\Delta}_{\mathrm{NB}}$, even though the original uncertainty is real-parametric. The main reason for doing so is that so far, robust synthesis within the $\ell_1$ framework is only tractable for time-varying dynamic uncertainties with a reasonable computational effort.

### 4.5.1 Controller Synthesis via $E$-$Q$-Iterations

Approaches to solve such a robust performance problem are described in Dahleh and Khammash (1993); Dahleh and Diaz-Bobillo (1995); Khammash *et al.* (2001). These approaches are based on the robust stability results of Theorems B.20 and B.21. We briefly describe the main steps of the synthesis procedure. First, suppose that a virtual LTV uncertainty $\Delta_v : \ell_\infty^{p_1} \to \ell_\infty^{p_1}$, $\Delta_v$ causal with $\|\Delta_v\|_{\infty-\mathrm{ind}} \leq 1$, is connected to the performance channel between $z_1$ and $w_1$ to convert the robust performance problem into a robust stability problem (Dahleh and Khammash, 1993; Dahleh and Diaz-Bobillo, 1995). This results in the augmented uncertainty structure $\tilde{\Delta} = \mathrm{diag}(\Delta_v, \Delta)$. The virtual uncertainty $\Delta_v$ added to convert the robust performance problem into a robust stability problem has to be LTV even in the case where the plant uncertainty is real-parametric (Dahleh and Diaz-Bobillo, 1995). Since no methods for considering mixed uncertainties are available in the $\ell_1$ framework, this fact alleviates the conservatism associated with $\mathbf{\Delta}_{\mathrm{NB}}$ somewhat. Second, determine a Youla parameterization

$$\Phi(Q) = H - UQV$$

of the closed-loop map $\Phi = G_{\text{aug}} \star K = \text{col}(w_0, w_1) \mapsto \text{col}(z_0, z_1)$, see Lemma B.13. Third, find (approximate) solutions $Q$, $E$, $\mu$, $\gamma$ of the optimization problem

$$\gamma := \inf_{Q \in \ell_1^{n_u \times (n_y + p_0)}} \inf_{0 < \mu \in \mathbb{R}} \inf_{E \in \boldsymbol{E}} \left\| E^{-1} \begin{bmatrix} I & 0 \\ 0 & \frac{1}{\mu} I \end{bmatrix} \Phi(Q) E \right\|_1 , \tag{4.16}$$

where $\boldsymbol{E} = \{E = \text{const.} \,|\, E\tilde{\Delta} = \tilde{\Delta}E \;\forall \tilde{\Delta} \in \boldsymbol{\Delta}_{\text{NB}}\}$ is the set of appropriately structured constant matrices commuting with $\tilde{\Delta}$ (Dahleh and Diaz-Bobillo, 1995, Chapter 7), and the identity matrices are dimensioned according to the partitioning of $\text{col}(z_0, z_1)$. If an optimal value of $\gamma < 1$ is obtained, then robust stability and robust performance with an $\ell_\infty$-gain corresponding to the minimal $\mu$ is guaranteed. If no value $\gamma < 1$ can be found, then robust stability with respect to the considered uncertainty set $\boldsymbol{\Delta}_{\text{NB}}$ is not attainable. In contrast to the usual formulations in the literature (Dahleh and Khammash, 1993; Dahleh and Diaz-Bobillo, 1995; Khammash *et al.*, 2001), the additional infimization over $\mu$ is introduced to allow for a robust $\ell_\infty$-gain $\mu$ larger than 1 while using the virtual uncertainty approach. Alternatively, $\mu$ can be fixed to a value smaller than the optimal $\ell_\infty$-gain (which one does not know beforehand in general), and then find the true $\ell_\infty$-gain after the synthesis in an analysis step.

(Sub)optimal solutions to the infinite-dimensional non-convex optimization problem (4.16) can be obtained by three approaches in principal.

- Use $E$-$Q$-iterations (Dahleh and Khammash, 1993) similar to the $D$-$K$-iterations in $\mu$ synthesis. To this end, start with an initial guess for the matrix $E$ and for the real scalar $\mu$ (like e.g. $E = I$, $\mu = 1$). Then repeat the following two steps until convergence is achieved or the performance is satisfactory.

  (a) Compute an optimal $Q$ from (4.16) for fixed $E$ and $\mu$, which is a standard $\ell_1$-optimal synthesis problem, solvable by an LP as described in Section 3.2.

  (b) Choose some value for $\mu$. Insert the $Q$ from step (a) into (4.16) and compute an optimal scaling $E$. This can be done by applying for example the Perron-Frobenius theorem as in Dahleh and Khammash (1993, Section 7.2); Dahleh and Diaz-Bobillo (1995, Section 7.3.1), see also Appendix B.8, or by using a convex parameterization as in Dahleh and Diaz-Bobillo (1995, Exercise 7.4). Using bisection on $\mu$, repeat this step until $E$ and $\mu$ are found such that the $\ell_1$-norm in (4.16) is minimal or smaller than 1.

  As for $D$-$K$-iterations, this approach may result in local solutions only (if any).

- A globally optimal solution with pre-defined accuracy can be attained using LP relaxations on (4.16) in combination with a branch-and-bound procedure (Khammash *et al.*, 2001). The approach may demand a large computational effort however. Experience shows that the algorithm often produces similar results as $E$-$Q$-iterations in similar time.

- A third method by Sokolov (2002) provides lower computational complexity than Khammash *et al.* (2001), but is only suited for certain coprime factor representations of the uncertainty.

As described in Section 4.4, an LPV controller $\bar{K}(\Delta)$ can be reconstructed from $Q$ and $K$.

### 4.5.2 Robust Stability and Robust Performance

Next, a result on robust stability and robust performance of the closed loop in Figure 4.1(a) with the controller (4.8) is given. It states that, under certain conditions, $\bar{K}(\Delta)$ guarantees robust stability of the closed loop if $K$ does so, and that the closed-loop maps $w_1 \mapsto z_1$ of the structures in Figures 4.1(a) and 4.2(b) are equivalent.

**Theorem 4.1** *Consider the two control structures of Figures 4.1(a) and 4.2(b), with the controllers (4.8) and (4.5), respectively. Suppose that* $(I - \Delta\tilde{G}_{00})^{-1}$ *is* $\ell_\infty$*-stable for all* $\Delta \in \mathbf{\Delta}_{\mathrm{NB}}$*, that* $G_{01}G_{21}^{\dagger}G_{20}$ *is* $\ell_\infty$*-stable, that* $G_{21}$ *is left-invertible, and that* $\big(I - \Delta(\tilde{G}_{00} + \tilde{G}_{02}K_2)\big)^{-1}$ *is* $\ell_\infty$*-stable for all* $\Delta \in \mathbf{\Delta}_{\mathrm{NB}}$*. Suppose moreover that* $G_{\mathrm{aug}} \star K$ *is stable and that* $K$ *robustly stabilizes* $\Delta \star G_{\mathrm{aug}}$ *in the* $\ell_\infty$ *sense with respect to* $\Delta \in \mathbf{\Delta}_{\mathrm{NB}}$*. Then* $\bar{K}(\Delta)$ *robustly stabilizes the original plant* $\Delta \star G$ *in the* $\ell_\infty$ *sense with respect to* $\Delta \in \mathbf{\Delta}_{\mathrm{NB}}$*, and*

$$\Delta \star G \star \bar{K}(\Delta) = \Delta \star G_{\mathrm{aug}} \star K.$$

Proof: See Appendix D.6. ■

It is concluded that robust stability is achieved by $\bar{K}(\Delta)$ constructed from a stabilizing auxiliary controller $K$, and that the performance properties achieved by $K$ and $\bar{K}(\Delta)$ are the same. The condition on $I - \Delta\tilde{G}_{00}$ is interpreted as a well-posedness assumption of our special control structure. Yet the conditions of Theorem 4.1 can be restrictive in practice. For the simple case $G_{01} \equiv 0$ discussed next, the corresponding conditions are much less restrictive. Hence the simple case will be more relevant in actual applications of the proposed control structure. Thus a robust stability and performance result corresponding to Theorem 4.1 is established for the simple case next.

**Theorem 4.2** *Consider the two control structures of Figures 4.1(a) and 4.2(b), with the controllers (4.8) and (4.5), respectively. Suppose that the interconnection between* $G$ *and* $\Delta$ *is well-posed, that* $G_{01} \equiv 0$*, and that* $\big(I - \Delta(G_{00} + G_{02}K_2)\big)^{-1}$ *is* $\ell_\infty$*-stable for all* $\Delta \in \mathbf{\Delta}_{\mathrm{NB}}$*. Suppose moreover that* $G_{\mathrm{aug}} \star K$ *is stable and that* $K$ *robustly stabilizes* $\Delta \star G_{\mathrm{aug}}$ *in the* $\ell_\infty$ *sense with respect to* $\Delta \in \mathbf{\Delta}_{\mathrm{NB}}$*. Then* $\bar{K}(\Delta)$ *robustly stabilizes the original plant* $\Delta \star G$ *in the* $\ell_\infty$ *sense with respect to* $\Delta \in \mathbf{\Delta}_{\mathrm{NB}}$*, and*

$$\Delta \star G \star \bar{K}(\Delta) = \Delta \star G_{\mathrm{aug}} \star K.$$

Proof: See Appendix D.6. ■

To summarize this section, it is noted again that our approach converts the problem of synthesizing an $\ell_1$-optimal LPV controller into the problem of finding a robust $\ell_1$-optimal LTI controller. To this end, a robust performance problem has to be solved. The desired robust LTI controller can be computed with well-known robust synthesis methods for structured dynamic uncertainties. These methods are the only tractable methods for robust synthesis in $\ell_1$ to this date. Other robustness approaches that avoid the small-gain theory altogether are currently not available for $\ell_1$ control. Hence it is concluded that the presented gain-scheduling approach is not only the first one in the $\ell_1$ framework, but also the only feasible approach from a computational point of view at the moment.

## 4.6 Gain-Scheduling in the Quadratic Performance Framework

The control structure proposed in Section 4.3 is now used in the well-known quadratic performance framework, which includes $\mathcal{L}_2$-gain based control (i.e. $\mathcal{H}_\infty$ control) as a special case. The presentation is based on state-space realizations of the involved systems, and follows an LMI approach in the spirit of Scherer (2000*b*). In particular, continuous-time systems are considered for reasons of comparability to published results.

The continuous-time formulations corresponding to the discrete-time systems considered before are introduced next, focusing only on the main differences to discrete time. The LPV system $\bar{G}(\Delta)$ is described by

$$\begin{bmatrix} \dot{x}(t) \\ z_1(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \bar{A}(\Delta(t)) & \bar{B}_1(\Delta(t)) & \bar{B}_2(\Delta(t)) \\ \bar{C}_1(\Delta(t)) & \bar{D}_{11}(\Delta(t)) & \bar{D}_{12}(\Delta(t)) \\ \bar{C}_2(\Delta(t)) & \bar{D}_{21}(\Delta(t)) & \bar{D}_{22}(\Delta(t)) \end{bmatrix} \begin{bmatrix} x(t) \\ w_1(t) \\ u(t) \end{bmatrix}, \tag{4.17}$$

obtained from the interconnection of $G$ and $\Delta$ as given by

$$\begin{bmatrix} \dot{x}(t) \\ z_0(t) \\ z_1(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A & B_0 & B_1 & B_2 \\ C_0 & D_{00} & D_{01} & D_{02} \\ C_1 & D_{10} & D_{11} & D_{12} \\ C_2 & D_{20} & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ w_0(t) \\ w_1(t) \\ u(t) \end{bmatrix}, \tag{4.18}$$

$$w_0(t) = \Delta(t) z_0(t). \tag{4.19}$$

It is assumed without loss of generality that $D_{22} = 0$ (see Appendix B.6). $\Delta$ is assumed to be a function $\Delta : [0, \infty) \to \mathbf{\Delta}$, and $\mathbf{\Delta}_{\mathrm{TV}} := \{\Delta | \Delta(t) \in \mathbf{\Delta}\ \forall t\}$ in this section. The interconnection of the controlled system $\mathcal{G} = G \star K$ with the uncertainty $\Delta$ is described by

$$\begin{bmatrix} \dot{\xi}(t) \\ z_0(t) \\ z_1(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B}_0 & \mathcal{B}_1 \\ \mathcal{C}_0 & \mathcal{D}_{00} & \mathcal{D}_{01} \\ \mathcal{C}_1 & \mathcal{D}_{10} & \mathcal{D}_{11} \end{bmatrix} \begin{bmatrix} \xi(t) \\ w_0(t) \\ w_1(t) \end{bmatrix}, \tag{4.20}$$

$$w_0(t) = \Delta(t) z_0(t). \tag{4.21}$$

The corresponding uncertain closed-loop system $\bar{\mathcal{G}}(\Delta)$ is

$$\begin{bmatrix} \dot{\xi}(t) \\ z_1(t) \end{bmatrix} = \begin{bmatrix} \bar{\mathcal{A}}(\Delta(t)) & \bar{\mathcal{B}}_1(\Delta(t)) \\ \bar{\mathcal{C}}_1(\Delta(t)) & \bar{\mathcal{D}}_{11}(\Delta(t)) \end{bmatrix} \begin{bmatrix} \xi(t) \\ w_1(t) \end{bmatrix}. \tag{4.22}$$

As stability notion, the following definition is used.

**Definition 4.8** *(Uniform exponential stability, continuous time)*
*Suppose that the interconnection (4.20)–(4.21) is well-posed. Then the system (4.22) is said to be uniformly exponentially stable if there exist constants $\alpha > 0$, $\beta > 0$ such that $\|\xi(t)\| \leq \beta e^{-\alpha(t-t_0)}\|\xi(t_0)\|$ $\forall t \geq t_0 \geq 0$ for all considered uncertainties $\Delta(\cdot)$, and for every system trajectory $\xi(\cdot)$ with $w_1 = 0$.* □

A continuous-time definition of robust quadratic performance is stated additionally.

**Definition 4.9** *(Robust quadratic performance, continuous time)*
*Suppose that the interconnection (4.20)–(4.21) is well-posed. Then the system (4.22) is said to have robust quadratic performance with performance index $P_p = P_p^T$ if there exists an $\varepsilon > 0$ such that*

$$\int_0^\infty \begin{bmatrix} w_1(t) \\ z_1(t) \end{bmatrix}^T P_p \begin{bmatrix} w_1(t) \\ z_1(t) \end{bmatrix} dt \leq -\varepsilon \int_0^\infty w_1(t)^T w_1(t)\, dt$$

*for all considered uncertainties $\Delta(\cdot)$ and for every system trajectory $\xi(\cdot)$ with $\xi(0) = 0$.* □

Now we are ready to describe the problem to be studied. Like in the previous section, the LPV system $\bar{G}(\Delta)$ with an LFT factorization into $G$ and $\Delta$ as in Figure 4.1(a) is considered, with the state-space realizations (4.17)–(4.19). Apply the control structure of Figure 4.2(b) to the problem. This leads to the interconnection

$$\begin{bmatrix} \dot{x}(t) \\ z_0(t) \\ z_1(t) \\ y(t) \\ w_0(t) \end{bmatrix} = \begin{bmatrix} A & B_0 & B_1 & B_2 \\ C_0 & D_{00} & D_{01} & D_{02} \\ C_1 & D_{10} & D_{11} & D_{12} \\ C_2 & D_{20} & D_{21} & 0 \\ 0 & I & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w_0(t) \\ w_1(t) \\ u(t) \end{bmatrix}, \tag{4.23}$$

$$w_0(t) = \Delta(t) z_0(t) \tag{4.24}$$

of $G_{\text{aug}}$ and $\Delta$. Now an LTI output-feedback controller $K$ with realization

$$\begin{bmatrix} \dot{x}_K(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} A_K & B_{K1} & B_{K2} \\ C_K & D_{K1} & D_{K2} \end{bmatrix} \begin{bmatrix} x_K(t) \\ y(t) \\ w_0(t) \end{bmatrix} \tag{4.25}$$

is sought according to the following problem.

**Problem 4.4** *For the LPV plant $\Delta \star G_{\mathrm{aug}}$, find an LTI output-feedback controller $K$ such that*

*(i) the closed loop $\bar{\mathcal{G}}(\Delta) = \Delta \star G_{\mathrm{aug}} \star K$ is uniformly exponentially stable with respect to $\Delta \in \mathbf{\Delta}_{\mathrm{TV}}$, and*

*(ii) the closed loop $\bar{\mathcal{G}}(\Delta)$ achieves robust quadratic performance with index $P_p$.* □

A special case of this setup is the robust $\mathcal{H}_\infty$ problem, see Remark 3.1. In the following sections, the matrix inequality conditions arising from the special control structure in Figure 4.2(a) are derived and analyzed. We stress that other performance frameworks like $\mathcal{H}_2$-, generalized $\mathcal{H}_2$-, or star-norm performance can be treated analogously. See Scherer (2000*b*) for an overview.
Moreover we want to mention that robust stability and robust performance results as in Section 4.5.2 can be derived for the case of robust $\ell_2$-stability. Corresponding results are omitted for brevity.

### 4.6.1 Controller Synthesis via Matrix Inequality Conditions

This section derives matrix inequality conditions for synthesizing LPV controllers using the control structure of Figure 4.2(a). First, conditions for the existence of such controllers are given. In Section 4.6.2, a procedure on how to compute an actual controller from the solution of the inequalities is described. The setup under consideration uses static multipliers for characterizing the involved parametric uncertainty as in the Full-Block S-Procedure (Lemma B.11). The general case is dealt with next, whereas restricted multipliers are discussed afterwards. A corollary with convex conditions for robust stability without robust performance concludes the section.

**The General Case**

The following theorem states matrix inequality conditions for the existence of a controller as desired in Problem 4.4. Comments on how to solve these conditions are given subsequently.

**Theorem 4.3** *The following two statements are equivalent.*

*(i) There exist $Q = Q^T$, $R = R^T$, $S$, and $v := [X, Y, K, L_1, L_2, M, N_1, N_2]$ with $X = X^T$, $Y = Y^T$ satisfying*

$$\begin{bmatrix} Y & I \\ I & X \end{bmatrix} > 0, \tag{4.26}$$

$$[*]^T \left[\begin{array}{cc:cc:cc} 0 & I & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & Q & S & 0 & 0 \\ 0 & 0 & S^T & R & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & Q_p & S_p \\ 0 & 0 & 0 & 0 & S_p^T & R_p \end{array}\right] \left[\begin{array}{ccc} I & 0 & 0 \\ \boldsymbol{A}(v) & \boldsymbol{B}_0(v) & \boldsymbol{B}_1(v) \\ \hdashline 0 & I & 0 \\ \boldsymbol{C}_0(v) & \boldsymbol{D}_{00}(v) & \boldsymbol{D}_{01}(v) \\ \hdashline 0 & 0 & I \\ \boldsymbol{C}_1(v) & \boldsymbol{D}_{10}(v) & \boldsymbol{D}_{11}(v) \end{array}\right] < 0, \tag{4.27}$$

$$\begin{bmatrix} \Delta \\ I \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} \Delta \\ I \end{bmatrix} \geq 0 \qquad \forall \Delta \in \boldsymbol{\Delta}, \tag{4.28}$$

$$Q \leq 0, \tag{4.29}$$

*where*

$$\left[\begin{array}{c:c:c} \boldsymbol{A}(v) & \boldsymbol{B}_0(v) & \boldsymbol{B}_1(v) \\ \hdashline \boldsymbol{C}_0(v) & \boldsymbol{D}_{00}(v) & \boldsymbol{D}_{01}(v) \\ \hdashline \boldsymbol{C}_1(v) & \boldsymbol{D}_{10}(v) & \boldsymbol{D}_{11}(v) \end{array}\right] := \tag{4.30}$$

$$\left[\begin{array}{cc:c:c} AY & A & B_0 & B_1 \\ 0 & XA & XB_0 & XB_1 \\ \hdashline C_0Y & C_0 & D_{00} & D_{01} \\ \hdashline C_1Y & C_1 & D_{10} & D_{11} \end{array}\right] + \left[\begin{array}{cc} 0 & B_2 \\ I & 0 \\ \hdashline 0 & D_{02} \\ \hdashline 0 & D_{12} \end{array}\right] \begin{bmatrix} K & L_1 & L_2 \\ M & N_1 & N_2 \end{bmatrix} \left[\begin{array}{cc:c:c} I & 0 & 0 & 0 \\ 0 & C_2 & D_{20} & D_{21} \\ 0 & 0 & I & 0 \end{array}\right].$$

*(ii) There exist $X = X^T$, $Y = Y^T$, $\tilde{Q} = \tilde{Q}^T$, $\tilde{R} = \tilde{R}^T$, $\tilde{S}$, and $R = R^T$ satisfying*

$$\begin{bmatrix} Y & I \\ I & X \end{bmatrix} > 0, \tag{4.31}$$

$$\Psi^T \left[\begin{array}{cc} I & 0 \\ A & B_1 \\ \hdashline C_0 & D_{01} \\ \hdashline 0 & I \\ C_1 & D_{11} \end{array}\right]^T \left[\begin{array}{cc:c:cc} 0 & X & 0 & 0 & 0 \\ X & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & R & 0 & 0 \\ \hdashline 0 & 0 & 0 & Q_p & S_p \\ 0 & 0 & 0 & S_p^T & R_p \end{array}\right] \left[\begin{array}{cc} I & 0 \\ A & B_1 \\ \hdashline C_0 & D_{01} \\ \hdashline 0 & I \\ C_1 & D_{11} \end{array}\right] \Psi < 0, \tag{4.32}$$

$$\Theta^T \left[\begin{array}{ccc} A^T & C_0^T & C_1^T \\ -I & 0 & 0 \\ \hdashline B_0^T & D_{00}^T & D_{10}^T \\ 0 & -I & 0 \\ \hdashline B_1^T & D_{01}^T & D_{11}^T \\ 0 & 0 & -I \end{array}\right]^T \left[\begin{array}{cc:cc:cc} 0 & Y & 0 & 0 & 0 & 0 \\ Y & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & \tilde{Q} & \tilde{S} & 0 & 0 \\ 0 & 0 & \tilde{S}^T & \tilde{R} & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & \tilde{Q}_p & \tilde{S}_p \\ 0 & 0 & 0 & 0 & \tilde{S}_p^T & \tilde{R}_p \end{array}\right] \left[\begin{array}{ccc} A^T & C_0^T & C_1^T \\ -I & 0 & 0 \\ \hdashline B_0^T & D_{00}^T & D_{10}^T \\ 0 & -I & 0 \\ \hdashline B_1^T & D_{01}^T & D_{11}^T \\ 0 & 0 & -I \end{array}\right] \Theta > 0, \tag{4.33}$$

$$\begin{bmatrix} -I \\ \Delta^T \end{bmatrix}^T \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{bmatrix} \begin{bmatrix} -I \\ \Delta^T \end{bmatrix} \leq 0 \qquad \forall \Delta \in \boldsymbol{\Delta}, \tag{4.34}$$

$$\tilde{R} \geq 0, \tag{4.35}$$

$$R = (\tilde{R} - \tilde{S}^T \tilde{Q}^{-1} \tilde{S})^{-1}, \tag{4.36}$$

*where $\Theta$, $\Psi$ are basis matrices of* $\ker\left([B_2^T \quad D_{02}^T \quad D_{12}^T]\right)$, $\ker\left([C_2 \quad D_{21}]\right)$*, respectively, and* $\begin{bmatrix} \tilde{Q}_p & \tilde{S}_p \\ \tilde{S}_p^T & \tilde{R}_p \end{bmatrix} = \begin{bmatrix} Q_p & S_p \\ S_p^T & R_p \end{bmatrix}^{-1}$ *is partitioned accordingly.*

*Furthermore, if either (i) or (ii) holds, then there exists a controller (4.25) for the plant (4.23) achieving uniform exponential stability and robust quadratic performance with index* $P_p = P_p^T = \begin{bmatrix} Q_p & S_p \\ S_p^T & R_p \end{bmatrix}$, $R_p \geq 0$, *for the closed-loop system (4.22) with respect to* $\Delta \in \mathbf{\Delta}_{\mathrm{TV}}$ .

Proof: See Appendix D.7. ■

A finite set of conditions is obtained using for example the convex hull relaxation (Lemma B.12) as seen in the transition from Theorem 2.3 to Theorem 2.4.

**Remark 4.3** *For the special case of* $\mathcal{L}_2$*-gain based analysis, i.e. with*

$$P_p = \begin{bmatrix} -\gamma^2 I & 0 \\ 0 & I \end{bmatrix}, \qquad \tilde{P}_p = P_p^{-1} = \begin{bmatrix} -\frac{1}{\gamma^2} I & 0 \\ 0 & I \end{bmatrix},$$

*a minimization of* $\gamma^2 > 0$ *can be carried out using the Schur complement of (4.33) with respect to* $\gamma^2$. *In this case, (4.33) is equivalent to*

$$\begin{bmatrix} [*]^T \left[\begin{array}{cc:cc:c} 0 & Y & 0 & 0 & 0 \\ Y & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & \tilde{Q} & \tilde{S} & 0 \\ 0 & 0 & \tilde{S}^T & \tilde{R} & 0 \\ \hdashline 0 & 0 & 0 & 0 & I \end{array}\right] \left[\begin{array}{ccc} A^T & C_0^T & C_1^T \\ -I & 0 & 0 \\ \hdashline B_0^T & D_{00}^T & D_{10}^T \\ 0 & -I & 0 \\ \hdashline 0 & 0 & -I \end{array}\right] \Theta & \Theta^T \begin{bmatrix} B_1 \\ D_{01} \\ D_{11} \end{bmatrix} \\ \begin{bmatrix} B_1^T & D_{01}^T & D_{11}^T \end{bmatrix} \Theta & \gamma^2 I \end{bmatrix} > 0. \qquad \square$$

Note that (4.28) and (4.34) imply $R \geq 0$ and $\tilde{Q} \leq 0$, respectively, since $0 \in \mathbf{\Delta}$ by assumption. The conditions (4.31)–(4.36) are not convex in the unknowns due to the constraint (4.36). Yet one can try to obtain solutions using one of the following procedures.

- Find $Y$, $\tilde{Q}$, $\tilde{R}$, $\tilde{S}$ satisfying (4.33)–(4.35). This is a convex problem. Then compute $R$ using (4.36). Finally find $X$ satisfying (4.31)–(4.32). If this last step is not feasible, relax the performance index $P_p$ and redo the procedure.
- A procedure similar to $D$-$K$-iterations as in $\mu$ synthesis is conceivable. Details are described in Scherer (2000*b*).

There are, however, relaxations to convert the conditions (4.31)–(4.36) into convex ones as described next. How to construct a controller (4.25) from the solutions of (4.31)–(4.36) or of (4.26)–(4.29) is detailed in Section 4.6.2.

### A Convex Relaxation Using Constrained Multipliers

In Theorem 4.3, the multipliers $P = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}$ and $\tilde{P} = \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{bmatrix}$ can be constrained by setting $S = \tilde{S} = 0$, possibly leading to worse achievable performance. Yet in this case, the following result with convex conditions can be stated.

**Corollary 4.4** *Suppose there exist $X = X^T$, $Y = Y^T$, $\tilde{Q} = \tilde{Q}^T$, and $\tilde{R} = \tilde{R}^T$ satisfying (4.31), (4.33), (4.34) with $\tilde{S} = 0$, and*

$$\begin{bmatrix} \Gamma_1 & \Gamma_2^T \\ \Gamma_2 & -\tilde{R} \end{bmatrix} < 0, \tag{4.37}$$

*where*

$$\Gamma_1 = \Psi^T \left( \begin{bmatrix} A^T X + XA & XB_1 \\ B_1^T X & 0 \end{bmatrix} + [*]^T \begin{bmatrix} Q_p & S_p \\ S_p^T & R_p \end{bmatrix} \begin{bmatrix} 0 & I \\ C_1 & D_{11} \end{bmatrix} \right) \Psi,$$
$$\Gamma_2 = \begin{bmatrix} C_0 & D_{01} \end{bmatrix} \Psi,$$

*$\Theta$, $\Psi$ are basis matrices of* $\ker\left([B_2^T\ D_{02}^T\ D_{12}^T]\right)$, $\ker\left([C_2\ D_{21}]\right)$*, respectively, and* $\begin{bmatrix} \tilde{Q}_p & \tilde{S}_p \\ \tilde{S}_p^T & \tilde{R}_p \end{bmatrix} = \begin{bmatrix} Q_p & S_p \\ S_p^T & R_p \end{bmatrix}^{-1}$ *is partitioned accordingly. Then there exists a controller (4.25) for the plant (4.23) achieving uniform exponential stability and robust quadratic performance with index $P_p = P_p^T = \begin{bmatrix} Q_p & S_p \\ S_p^T & R_p \end{bmatrix}$, $R_p \geq 0$, for the closed-loop system (4.22) with respect to $\Delta \in \mathbf{\Delta}_{\mathrm{TV}}$ .*

Proof: It just needs to be shown that the existence of a solution satisfying (4.31), (4.33), (4.34), (4.37) with $\tilde{S} = 0$ implies the existence of a solution satisfying (4.31)–(4.36). To this end, it is shown that (4.37) is equivalent to (4.32), (4.36), $\tilde{R} > 0$, $\tilde{S} = 0$. With $\tilde{S} = 0$, (4.36) reduces to $R^{-1} = \tilde{R}$. Rewriting (4.32) as

$$[*]^T \left[\begin{array}{cc:cc} 0 & X & 0 & 0 \\ X & 0 & 0 & 0 \\ \hdashline 0 & 0 & Q_p & S_p \\ 0 & 0 & S_p^T & R_p \end{array}\right] \left[\begin{array}{cc} I & 0 \\ A & B_1 \\ \hdashline 0 & I \\ C_1 & D_{11} \end{array}\right] \Psi + \Psi^T \begin{bmatrix} C_0 & D_{01} \end{bmatrix}^T R \begin{bmatrix} C_0 & D_{01} \end{bmatrix} \Psi < 0$$

and using the Schur Lemma (Lemma B.5) with respect to $R^{-1} = \tilde{R} > 0$ shows that (4.32) is equivalent to (4.37) then. Hence a solution of the conditions in this result has the desired property.
■

**Remark 4.4** *Corollary 4.4 is a more general version of Theorem 1 in Wu and Lu (2004), with an alternative proof. The result in Wu and Lu (2004) states the following: It is an LMI problem (and hence convex) to verify the existence of a robust $\mathcal{H}_\infty$-optimal controller that not only has access to the measurement $y$, but also to the signal $w_0$ as in Figure 4.2(a), if the multipliers are restricted to be block-diagonal. Our result extends the theorem in Wu and Lu (2004) insofar, as the more general quadratic performance is considered instead of $\mathcal{H}_\infty$ performance. Moreover, from our result a structural relation to other performance criteria such as robust $\mathcal{H}_2$- or robust star-norm or to discrete-time conditions is visible. Hence, the result can be generalized in the spirit of Scherer (2000*b*).* □

**A Convex Relaxation For Systems with $C_0 = 0,\ D_{01} = 0$**

For completeness, a specialization of Theorem 4.3 to systems with $C_0 = 0$, $D_{01} = 0$ is made. As in Corollary 4.4, convex synthesis conditions result in this particular case. See the end of Section 4.4.2 for comments on this restriction of the system class.

**Corollary 4.5** *Suppose there exist $X = X^T$, $Y = Y^T$, $\tilde{Q} = \tilde{Q}^T$, $\tilde{R} = \tilde{R}^T$, and $\tilde{S}$ satisfying (4.31), (4.33), (4.34), (4.35), and*

$$\Psi^T \left[\begin{array}{cc} I & 0 \\ A & B_1 \\ \hline 0 & I \\ C_1 & D_{11} \end{array}\right]^T \left[\begin{array}{cc:cc} 0 & X & 0 & 0 \\ X & 0 & 0 & 0 \\ \hdashline 0 & 0 & Q_p & S_p \\ 0 & 0 & S_p^T & R_p \end{array}\right] \left[\begin{array}{cc} I & 0 \\ A & B_1 \\ \hline 0 & I \\ C_1 & D_{11} \end{array}\right] \Psi < 0, \tag{4.38}$$

*where $\Theta$ and $\Psi$ are basis matrices of $\ker\left(\begin{bmatrix} B_2^T & D_{02}^T & D_{12}^T \end{bmatrix}\right)$ and $\ker\left(\begin{bmatrix} C_2 & D_{21} \end{bmatrix}\right)$, respectively, and $\begin{bmatrix} \tilde{Q}_p & \tilde{S}_p \\ \tilde{S}_p^T & \tilde{R}_p \end{bmatrix} = \begin{bmatrix} Q_p & S_p \\ S_p^T & R_p \end{bmatrix}^{-1}$ is partitioned accordingly. Then there exists a controller (4.25) for the plant (4.23) with $C_0 = 0$, $D_{01} = 0$ achieving uniform exponential stability and robust quadratic performance with index $P_p = P_p^T = \begin{bmatrix} Q_p & S_p \\ S_p^T & R_p \end{bmatrix}$, $R_p \geq 0$, for the closed-loop system (4.22) with respect to $\Delta \in \boldsymbol{\Delta}_{\mathrm{TV}}$ .*

Proof: It just has to be shown that (4.38) is equivalent to (4.32), (4.36), $C_0 = 0$, $D_{01} = 0$. If $C_0 = 0$, $D_{01} = 0$ in (4.32), then all terms related to $R$ vanish, leading to (4.38). The condition (4.36) is dropped since $R$ does not appear anymore. This leads to the conditions as stated. ■

**Convex Conditions for the Robust Stability Case**

Interestingly, from Theorem 4.3 one can obtain inequality conditions that result in a convex problem if just uniform exponential stability is considered without robust quadratic performance. The next corollary states these conditions. The proof contains a way of finding solutions to the conditions using a convex problem. The result is helpful to easily get a stabilizing LPV output-feedback controller.

**Corollary 4.6** *Suppose there exist $X = X^T$, $Y = Y^T$, $\tilde{Q} = \tilde{Q}^T$, $\tilde{R} = \tilde{R}^T$, $\tilde{S}$, and $R = R^T$ satisfying*

$$\begin{bmatrix} Y & I \\ I & X \end{bmatrix} > 0, \tag{4.39}$$

$$\Psi^T \left[\begin{array}{c} I \\ A \\ \hline C_0 \end{array}\right]^T \left[\begin{array}{cc:c} 0 & X & 0 \\ X & 0 & 0 \\ \hdashline 0 & 0 & R \end{array}\right] \left[\begin{array}{c} I \\ A \\ \hline C_0 \end{array}\right] \Psi < 0, \tag{4.40}$$

$$\Theta^T \left[\begin{array}{cc} A^T & C_0^T \\ -I & 0 \\ \hline B_0^T & D_{00}^T \\ 0 & -I \end{array}\right]^T \left[\begin{array}{cc|cc} 0 & Y & 0 & 0 \\ Y & 0 & 0 & 0 \\ \hline 0 & 0 & \tilde{Q} & \tilde{S} \\ 0 & 0 & \tilde{S}^T & \tilde{R} \end{array}\right] \left[\begin{array}{cc} A^T & C_0^T \\ -I & 0 \\ \hline B_0^T & D_{00}^T \\ 0 & -I \end{array}\right] \Theta > 0, \tag{4.41}$$

$$\begin{bmatrix} -I \\ \Delta^T \end{bmatrix}^T \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{bmatrix} \begin{bmatrix} -I \\ \Delta^T \end{bmatrix} \leq 0 \qquad \forall \Delta \in \boldsymbol{\Delta}, \tag{4.42}$$

$$\tilde{R} \geq 0, \tag{4.43}$$

$$R = (\tilde{R} - \tilde{S}^T \tilde{Q}^{-1} \tilde{S})^{-1}, \tag{4.44}$$

*where* $\Theta$, $\Psi$ *are basis matrices of* $\ker\left([B_2^T \quad D_{02}^T]\right)$, $\ker(C_2)$, *respectively. Then there exists a controller (4.25) for the plant (4.23) achieving uniform exponential stability of the closed-loop system (4.22) with respect to* $\Delta \in \boldsymbol{\Delta}_{\mathrm{TV}}$ *. Moreover, finding solutions to the given conditions is a convex problem.*

Proof: The conditions in this corollary are just the same as the ones in Theorem 4.3(ii) after discarding all rows and columns associated with the performance channel $w_1 \mapsto z_1$. Hence a sufficient condition for the existence of a controller guaranteeing uniform exponential stability of (4.22) is established. How to obtain the solution in form of a convex problem is described next. First, find $Y$, $\tilde{Q}$, $\tilde{R}$, $\tilde{S}$ satisfying (4.41)–(4.43). Then compute $R$ from (4.44). Now observe that (4.40) is equivalent to

$$\Psi^T \left(A^T X + XA + C_0^T R C_0\right) \Psi < 0.$$

By the Projection Lemma (Lemma B.9), the last inequality holds if and only if there exists $\tilde{L}$ such that

$$A^T X + XA + C_0^T R C_0 + \tilde{L} C_2 + (\tilde{L} C_2)^T < 0$$

(set $U = I$ and $V = C_2$ in the Projection Lemma). By setting $\tilde{L} = XL$ for some $X > 0$, this condition holds if and only if there exists $L$ such that

$$(A + LC_2)^T X + X(A + LC_2) + C_0^T R C_0 < 0. \tag{4.45}$$

Now fix $L$ such that $A + LC_2$ is stable (i.e. has all its eigenvalues in the left half plane), which is always possible since the nominal system has to satisfy that $(A, C_2)$ is detectable. Finally find an $X$ satisfying (4.45) and (4.39). This is always possible since $A + LC_2$ was rendered stable, and (4.45) is a Lyapunov inequality. Hence $X$ just has to be large enough in terms of its smallest eigenvalue. Then all conditions of the theorem are satisfied by using only convex conditions. ■

As pointed out in the proof, it is possible to solve the seemingly non-convex conditions (4.39)–(4.44) using a procedure that only involves LMIs. Unfortunately, so far no way has been found to similarly convexify the conditions in Theorem 4.3, which includes robust performance specifications.

## 4.6.2 Controller Construction

In Theorems 4.3 to 4.6, only the question of the existence of a controller with desirable properties has been addressed. To complement these results, a procedure to construct a controller (4.25) from the solution variables $\{X,\, Y,\, \tilde{Q},\, \tilde{R},\, \tilde{S},\, R\}$ or from $v$ is reviewed and made explicit. The procedure is an extension of schemes presented in Gahinet and Apkarian (1994); Gahinet (1996) and basically follows from arguments in Scherer and Weiland (2005, Section 4.5.3). The procedure can be obtained by tracing the steps in the proof of Theorem 4.3 backwards.

After obtaining solutions $X$, $Y$, $\tilde{Q}$, $\tilde{R}$, $\tilde{S}$, $R$ satisfying the inequalities (4.31)–(4.36), first compute $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{bmatrix}^{-1}$. Second, find $K$, $L_i$, $M$, $N_i$ satisfying (4.27). To this end, one has to find a solution $Z$ satisfying a matrix inequality of the type

$$\begin{bmatrix} I \\ W_1^T Z W_2 + W_3 \end{bmatrix}^T \underbrace{\begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_2^T & \Pi_3 \end{bmatrix}}_{\Pi} \begin{bmatrix} I \\ W_1^T Z W_2 + W_3 \end{bmatrix} < 0.$$

A possibility to rewrite such a quadratic matrix inequality into an LMI in $Z$ is provided by the next lemma.

**Lemma 4.7** *Suppose $\tilde{\Pi}_3 > 0$, $\Pi_1$, $\Pi_2$, $\Sigma$, $W_1$, $W_2$, $W_3$ are constant matrices. Then there exists $Z$ such that*

$$\tilde{\Pi}_3 > 0 \quad \textit{and} \quad \begin{bmatrix} I \\ W_1^T Z W_2 + W_3 \end{bmatrix}^T \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_2^T & \Sigma\tilde{\Pi}_3\Sigma^T \end{bmatrix} \begin{bmatrix} I \\ W_1^T Z W_2 + W_3 \end{bmatrix} < 0 \tag{4.46}$$

*if and only if there exists $Z$ such that*

$$\begin{bmatrix} \Pi_1 + \Pi_2(W_1^T Z W_2 + W_3) + (W_1^T Z W_2 + W_3)^T \Pi_2^T & (W_1^T Z W_2 + W_3)^T \Sigma \\ \Sigma^T (W_1^T Z W_2 + W_3) & -\tilde{\Pi}_3^{-1} \end{bmatrix} < 0. \tag{4.47}$$

Proof: The result is a direct consequence of applying Lemma B.6 to (4.46) with $v = Z$. ■

Using this lemma, (4.27) is hence equivalent to (4.47) with

$$\Pi_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & Q_p \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} I & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & S_p \end{bmatrix}, \quad \tilde{\Pi}_3 = \begin{bmatrix} R & 0 \\ 0 & R_p \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix},$$

$$Z = \begin{bmatrix} K & L_1 & L_2 \\ M & N_1 & N_2 \end{bmatrix}, \quad W_1^T Z W_2 + W_3 = \begin{bmatrix} \boldsymbol{A}(v) & \boldsymbol{B}_0(v) & \boldsymbol{B}_1(v) \\ \boldsymbol{C}_0(v) & \boldsymbol{D}_{00}(v) & \boldsymbol{D}_{01}(v) \\ \boldsymbol{C}_1(v) & \boldsymbol{D}_{10}(v) & \boldsymbol{D}_{11}(v) \end{bmatrix}.$$

With these relations, $K$, $L_i$, $M$, $N_i$ can be computed from the LMI (4.47). In a third step, find non-singular $U$, $V$ satisfying

$$XY + UV^T = I.$$

The choices $V = I$, $U = I - XY$, or $V = -Y$, $U = X - Y^{-1}$, or similar constructions are possible, among others. Fourth, compute $A_K$, $B_{Kj}$, $C_K$, $D_{Kj}$ from

$$\begin{bmatrix} A_K & B_{K1} & B_{K2} \\ C_K & D_{K1} & D_{K2} \end{bmatrix} = \begin{bmatrix} U & XB_2 \\ 0 & I \end{bmatrix}^{-1} \left( \begin{bmatrix} K & L_1 & L_2 \\ M & N_1 & N_2 \end{bmatrix} - \begin{bmatrix} XAY & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} V^T & 0 & 0 \\ C_2Y & I & 0 \\ 0 & 0 & I \end{bmatrix}^{-1},$$

which is obtained by solving (D.34) for $A_K$ etc. This is always possible since the inverses exist due to the conditions on $X$, $Y$, $U$, $V$. Finally, from (4.25), a continuous-time LPV controller $\bar{K}(\Delta)$ is realized as described in Section 4.4.

The procedures described in Theorem 4.3, Corollary 4.4, and this section are exemplified in Chapter 5.

## 4.7 Summary

This chapter treats the synthesis of LPV output-feedback controllers for LPV systems, where uncertain parameters are assumed to be measurable. The parameter dependence of the plant may be affine, polynomial or rational, with possibly unbounded parameter variation rates. A novel control structure for realizing the dependence of the controller on the measured parameters is introduced and investigated. The structure reduces the LPV synthesis problem to a standard LTI robust performance problem. The difference to existing approaches is however that no additional uncertainty blocks are introduced. Moreover, no heuristic choices for gridding of the parameter space are necessary. The control structure is applied to LPV output-feedback synthesis in the $\ell_1$ and quadratic performance settings. The novel structure has particular advantages for $\ell_1$-optimal LPV control design, since only the uncertainty block of the plant is involved.

# Chapter 5

# Application Examples for LPV Controller Synthesis

*This application chapter presents several synthesis examples, where LPV controllers are designed for LPV plants. First, a simple control system is used to demonstrate the properties of the control structure proposed in Chapter 4 for $\ell_\infty$-gain minimization. The second example is related to a flight control problem studied in a large number of LPV synthesis publications. It is therefore well-suited to test our $\ell_\infty$-gain and quadratic performance synthesis methods and to compare the results to alternative approaches. In both examples, simulation studies are conducted. Comparisons in terms of performance, uniform behavior, and computational effort are carried out with respect to existing approaches. The examples of this chapter are based in part on Rieber and Allgöwer (2003); Rieber and Allgöwer (2006*b*).*

## 5.1 A Simple Example

An academic example is presented first. It is kept simple and used to verify the properties and the applicability of the proposed LPV control structure in connection with the $\ell_1$ optimality framework.

**Plant and Problem Description**

Consider the discrete-time LPV system resulting from the interconnection

$$\begin{bmatrix} x(k+1) \\ z_1(k) \\ y(k) \end{bmatrix} = \left[\begin{array}{cc:c:c} \rho(k) & -0.5 & 0 & 0 \\ 0.3 & 0.1 & 0 & 1 \\ \hdashline 1 & 0 & 0.1 & 0 \\ \hdashline 1 & 0 & 0.1 & 0 \end{array}\right] \begin{bmatrix} x(k) \\ w_1(k) \\ u(k) \end{bmatrix},$$

$$\rho(k) = \rho_0 + \Delta(k)\rho_1$$

with $\rho_0 = 1.2$, $\rho_1 = 0.12$, and $\Delta(k) \in [-1, 1]$ for all $k$. It can be verified that this plant is unstable for certain constant or varying parameter values, such as e.g. $\rho(k) = 1.32$ for all $k$. Suppose that a measurement of $\rho$ (or of $\Delta$ respectively) is available at each time-step.

Now a stabilizing controller is sought such that persistent output disturbances $w_1$ are rejected and small control error amplitudes are achieved. More specifically, the $\ell_\infty$-gain of the map $w_1 \mapsto z_1$ is to be minimized by means of an appropriate controller.

**Controller Synthesis**

First, an integrator $\hat{G}_{\text{int}}(z) = 1/(z-1)$ is attached to the measurement output $y$ to achieve asymptotic rejection of steps or step-like disturbances. Second, the plant $G$ (including $G_{\text{int}}$) is obtained as

$$\begin{bmatrix} x(k+1) \\ \hline z_0(k) \\ \hline z_1(k) \\ \hline y(k) \end{bmatrix} = \left[\begin{array}{ccc:c:c:c} 1 & 1 & 0 & 0 & 0.1 & 0 \\ 0 & \rho_0 & -0.5 & 1 & 0 & 0 \\ 0 & 0.3 & 0.1 & 0 & 0 & 1 \\ \hdashline 0 & \rho_1 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 1 & 0 & 0 & 0.1 & 0 \\ \hdashline 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x(k) \\ \hline w_0(k) \\ \hline w_1(k) \\ \hline u(k) \end{bmatrix}$$

with $w_0(k) = \Delta(k) z_0(k)$. It can be shown that $G_{01} \equiv 0$, thus the simple LPV controller realization of Section 4.4.2 is applicable. For this plant, three controllers are designed. All synthesis schemes are implemented in MATLAB, using toolboxes and hardware as described in Appendix E.

On the one hand, a robust LTI output-feedback controller $u = K_{\text{robust}} y$ is designed to guarantee $\ell_\infty$ stability of the closed loop and to achieve a small robust $\ell_\infty$-gain. The Youla parameterization is chosen such that two closed-loop poles each are located at $0.01$, $0.02$, $0.03$. The Youla parameter $Q$ is taken to be a finite-impulse response (FIR) filter with $N = 15$ taps, see (3.3). Using $E$-$Q$-iterations (Section 4.5), which consist of the scaled-$Q$ approach and Perron eigenvector computations, a robust controller is computed with a robust $\ell_\infty$-gain of $\mu_{\text{opt}} = 1.16$ and optimal scales $E_{\text{opt}} = \begin{bmatrix} 0.13 & 0 \\ 0 & 0.99 \end{bmatrix}$. The computation time is $3.4$ s for two iterations.

On the other hand, a robust LPV output-feedback controller $u = \bar{K}(\Delta) y$ is designed using the novel control structure of Figure 4.2. To obtain the auxiliary controller $u = K\,\text{col}(y, w_0)$, the same synthesis steps and design specifications as for the robust controller are applied, i.e. no different tuning is carried out for the two controllers. $K$ achieves a robust $\ell_\infty$-gain of $\mu_{\text{opt}} = 0.306$, which is about $74\%$ better than for the robust controller. The optimal scales are $E_{\text{opt}} = \begin{bmatrix} 0.025 & 0 \\ 0 & 1.0 \end{bmatrix}$. The computation time is $2.8$ s for two iterations. From the auxiliary controller $K$, the LPV controller $\bar{K}(\Delta)$ is obtained as described in Section 4.4.2. Note that the only difference between the controllers $K_{\text{robust}}$ and $K$ is the additional input $w_0$ for $K$.

Finally, for comparison purposes, a robust LPV output-feedback controller $u = \bar{K}_{\text{trad}}(\Delta) y$ is designed using the traditional LPV structure in Figure 4.3(a) with $\Delta_K = \Delta$. The same synthesis steps and design specifications as before are carried out. A robust $\ell_\infty$-gain of $\mu_{\text{opt}} = 0.46$ is achieved, with optimal scales $E_{\text{opt}} = \begin{bmatrix} 0.040 & 0.081 & 0 \\ 0 & 0.11 & 0 \\ 0 & 0 & 2.718 \end{bmatrix}$. The computation time is 288 s for three iterations. This relatively large computation time stems from the fact that the computation of $E$-scales has to be done via an optimization problem rather than via Perron eigenvector calculations due to the repeated uncertainty block $\Delta$. Even if the other two controllers are computed via the same optimization methods, their computation time is smaller since the involved overall uncertainty block is smaller.

While the synthesis of LPV controller $\bar{K}_{\text{trad}}(\Delta)$ using the traditional LPV structure requires the most computational effort, there is essentially no difference between computation times for $K_{\text{robust}}$ and $\bar{K}(\Delta)$. The order of the auxiliary LTI controllers is $n_x + N = 18$ for all three methods, and for the final controllers it is 19 for $K_{\text{robust}}$ and $\bar{K}_{\text{trad}}(\Delta)$, and 22 for $\bar{K}(\Delta)$. In all three cases, the dynamics matrix is sparse in the sense that more than 85% of the entries are zero. No model reduction steps are invoked here, since the comparison between the three control approaches is the main point of this study. Note however that the order of $\ell_1$ controllers can often be reduced considerably with little or no performance degradation as discussed in Khammash (2000); Rieber *et al.* (2005*b*); Rieber and Allgöwer (2006*a*).

**Results**

The robust controller and the two LPV controllers are combined with the integrator $G_{\text{int}}$ and connected to the original plant. Table 5.1 displays the robust $\ell_\infty$-gain $\mu_{\text{opt}}$ of the three resulting closed-loop systems as computed in the synthesis step. Additionally, the $\ell_\infty$-gain for three frozen parameter values are shown. The $\ell_\infty$-gains for frozen parameter values are lower bounds of the system's robust $\ell_\infty$-gain. All these numbers indicate that the two LPV controllers have a significantly better overall robust performance than the robust controller, and that the controller $\bar{K}(\Delta)$ resulting from our novel control structure clearly has the best performance (i.e. the lowest amplitudes). Moreover, for our proposed controller $\bar{K}(\Delta)$, the performance is quite uniform over the range of different parameter values, whereas the performance changes significantly more for the other two controllers.

*Table 5.1:* Example 1. Closed-loop performance for the three considered controllers. The numbers indicate the robust $\ell_\infty$-gain $\mu_{\text{opt}}$, and the $\ell_\infty$-gain for the nominal and the two extreme parameter values.

| Controller | $\mu_{\text{opt}}$ | $\|\|\bar{\mathcal{G}}(\Delta = -1)\|\|_1$ | $\|\|\bar{\mathcal{G}}(\Delta = 0)\|\|_1$ | $\|\|\bar{\mathcal{G}}(\Delta = 1)\|\|_1$ |
|---|---|---|---|---|
| Robust controller $K_{\text{robust}}$ | 1.16 | 0.37 | 0.27 | 0.62 |
| Traditional LPV cont. $\bar{K}_{\text{trad}}(\Delta)$ | 0.459 | 0.31 | 0.26 | 0.41 |
| Novel LPV controller $\bar{K}(\Delta)$ | 0.306 | 0.29 | 0.26 | 0.30 |

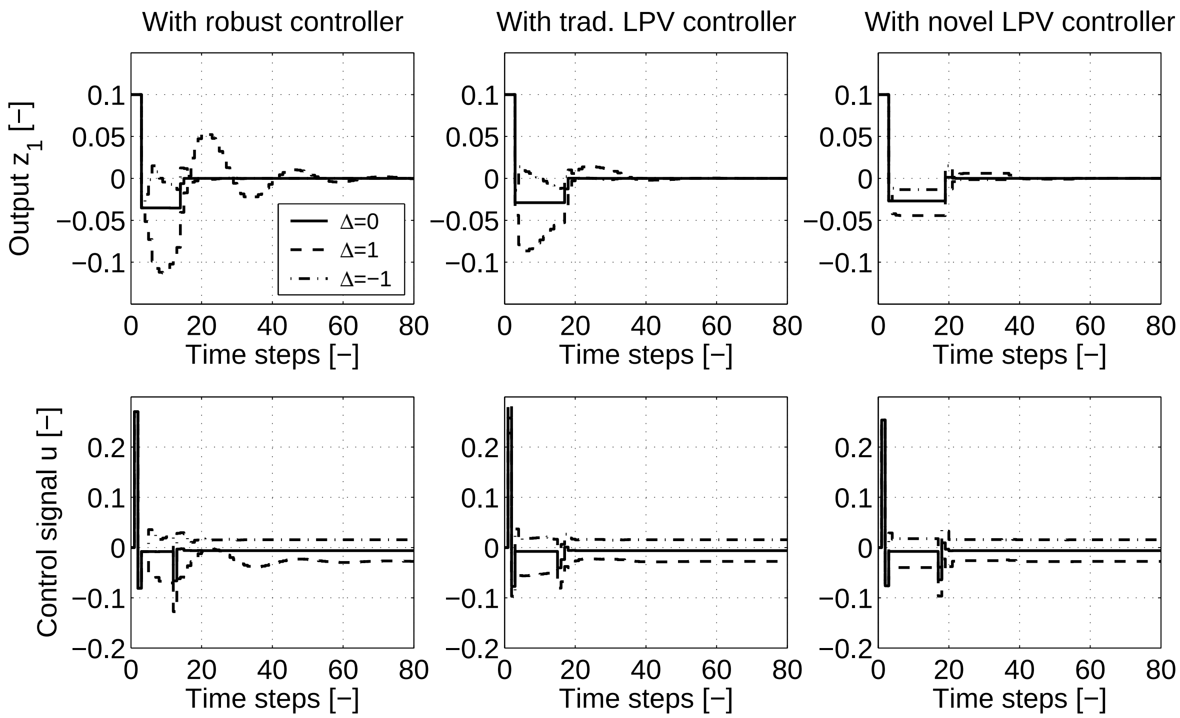



*Figure 5.1:* Example 1. Closed-loop responses $z_1$ and $u$ with the three controllers for three fixed parameter values. The input $w_1$ is a unit step disturbance.

Next, discrete-time simulations are carried out using MATLAB/Simulink. The LPV systems $\bar{G}(\Delta)$, $\bar{K}(\Delta)$, and $\bar{K}_{\mathrm{trad}}(\Delta)$ are implemented as S-functions, scheduled on the parameter $\Delta$ respectively $\rho$. Simulation results in response to a unit step disturbance $w_1$ for different fixed values of the parameter $\rho$ are shown in Figure 5.1. The trends from the $\ell_\infty$-gain computations of Table 5.1 are confirmed: the LPV controller resulting from our new approach has superior performance and produces a much more uniform behavior than the robust controller and than the traditional LPV controller.

A second set of simulations is depicted in Figure 5.2. The closed loop is simulated in response to a sine disturbance with $w_1(k) = 0.9\sin(0.3k)$ for $0 \leq k \leq 30$. This time, a discontinuously varying parameter as in Figure 5.3 is applied. All three controllers are stabilizing the system, but with a considerably better performance for the novel LPV controller. Moreover, the control action $u$ of the novel LPV controller has a lower amplitude compared to the other controllers, and is thus applied with more efficiency.

Interestingly, the three controllers also have different robust stability margins for frozen parameter values. The designs guarantee stability for $\Delta \in [-1, 1]$. The robust controller $K_{\mathrm{robust}}$ produces a stable response for constant or slowly-varying $\Delta \in [-4.7, 1.6]$, the traditional LPV controller $\bar{K}_{\mathrm{trad}}(\Delta)$ for $\Delta \in [-8.2, 4.0]$, and the novel LPV controller $\bar{K}(\Delta)$ for $\Delta \in [-6.2, 4.2]$. This indicates that the two LPV controllers have a considerably larger stability margin. The traditional LPV controller has the largest stability margin. Yet in the "difficult" region of positive $\Delta$ (i.e. frozen $\Delta$-values, for which the system is unstable), the novel LPV controller stabilizes for slightly larger values.

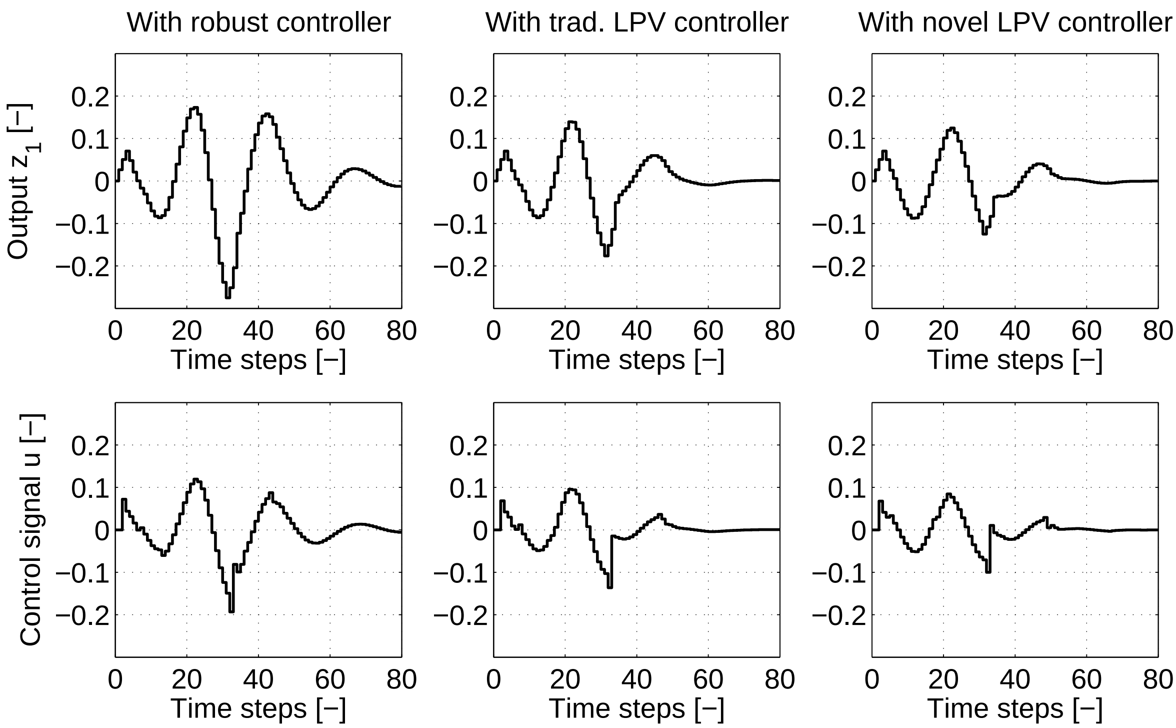



*Figure 5.2:* Example 1. Closed-loop responses $z_1$ and $u$ with the three controllers for a varying parameter as depicted in Figure 5.3. The input $w_1$ is a sine disturbance.

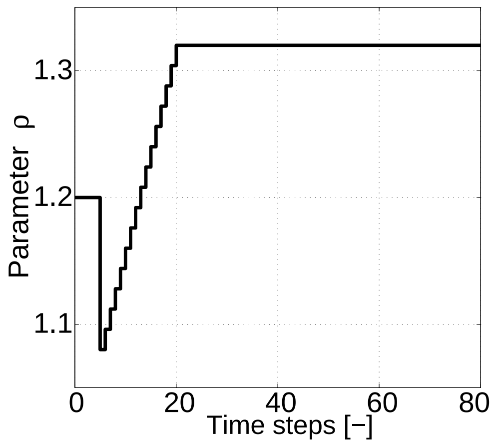



*Figure 5.3:* Example 1. Parameter $\rho$ applied in the simulations of Figure 5.2.

## 5.2 A Flight Control Problem

The second example describes the application of the proposed LPV synthesis methods to a flight control problem. We consider the reduced model of the pitch-axis dynamics of a missile with two measurable aerodynamical parameters. The model or variants thereof are used as a standard example in the gain-scheduling literature. In contrast to the first example, the emphasis is put more on approaches to the control problem and qualitative results, rather than numerical performance comparisons.

First, different LFT models are derived from the original continuous-time LPV model. It is discussed how models with different complexity can be obtained, and how the dimensions of the uncertainty block are kept small. Then different controller design approaches using the novel LPV

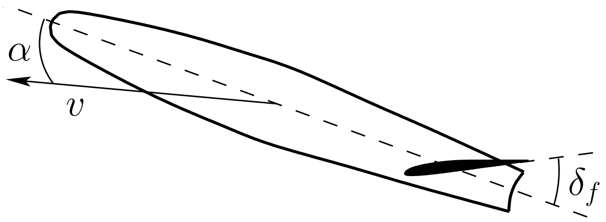



*Figure 5.4:* Example 2. The missile pitch dynamics include the fin deflection $\delta_f$ and the angle of attack $\alpha$, which is the angle between the longitudinal body axis and the air speed vector $v$.

control structure are tested on the pitch-axis model. A discrete-time $\ell_1$ design discusses several time-domain performance criteria and gives insight into control structure selection. A continuous-time $\mathcal{H}_\infty$ design allows comparisons with established methods and with the previously treated $\ell_1$ setup. Simulation studies of the LPV controllers in loop with the original continuous-time LPV plant demonstrate the controller performance and verify the applicability and the properties of the gain-scheduling scheme. Preliminaries to the study in this section are discussed in Fritsch (2004); Kotman (2006).

### 5.2.1 Parameter-Varying Models

**Continuous-Time LPV Model**

The considered example is borrowed from Gahinet *et al.* (1995, Chapter 7) and concerns controller design for the pitch-axis behavior of a missile. The model is a simplified version of a standard model in the gain-scheduling literature, see for example Packard and Balas (1992); Apkarian *et al.* (1995); Bennani *et al.* (1998). We restrict ourselves to the simplified model to concentrate on the key issues and properties in LPV controller synthesis. The original pitch-axis dynamics are nonlinear and highly dependent on the angle of attack, the speed, and the altitude. These dependencies are captured in the parameters injected into the system. The simplified LPV model is given in continuous time as

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} -Z_\alpha(t) & 1 \\ -M_\alpha(t) & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta_f, \tag{5.1}$$

$$\begin{bmatrix} a_v \\ \beta \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \tag{5.2}$$

where the symbols denote the angle of attack $\alpha$, the pitch rate $\beta$, the fin deflection $\delta_f$, and the vertical acceleration $a_v$. Figure 5.4 visualizes these symbols. All variables are normalized. The time-varying aerodynamical coefficients are restricted by $Z_\alpha(t) \in [0.5, 4]$ and $M_\alpha(t) \in [0, 106]$ for all times $t$ and are measurable in real-time. The time unit is seconds, all other variables are taken dimensionless.

**Continuous-Time LFT Model**

From (5.1)–(5.2), an LFT factorization is performed as described in Appendix B.7. It is easy to see that the model

$$\dot{x}(t) = \begin{bmatrix} -2.25 & 1 \\ -53 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} w_0(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta_f(t), \tag{5.3}$$

$$z_0(t) = \begin{bmatrix} 1.75 & 0 \\ 53 & 0 \end{bmatrix} x(t), \tag{5.4}$$

$$w_0(t) = \underbrace{\mathrm{diag}(\bar{Z}_\alpha(t), \bar{M}_\alpha(t))}_{\Delta(t)} z_0(t) \tag{5.5}$$

with $\bar{Z}_\alpha(t) \in [-1, 1]$ and $\bar{M}_\alpha(t) \in [-1, 1]$ is equivalent to (5.1). The model (5.2)–(5.5) is used in the following for continuous-time LPV controller design.

**Discrete-Time LFT Model, Method I**

Since the $\ell_1$ gain-scheduling approach of Section 4.5 is a discrete-time synthesis procedure, discrete-time equivalents of (5.1)–(5.2) are considered additionally. Let $(A_d(\rho), B_d(\rho), C_d, D_d)$ denote a corresponding zero-order hold (ZOH) equivalent (Apkarian, 1997; Franklin *et al.*, 1998, Section 4.3) of (5.1)–(5.2). Both $A_d(\rho)$ and $B_d(\rho)$ depend nonlinearly on the parameter vector $\rho = [Z_\alpha, M_\alpha]^T$. The sampling time $\tau = 0.01$ s is chosen as a compromise between incorporation of essential dynamical effects and restriction of the size of the controller synthesis optimization problem in the sequel. Two approximate procedures on how to obtain the factorization (4.2)–(4.3) from $(A_d(\rho), B_d(\rho), C_d, D_d)$ without performance channels are described next.

In method I, the ZOH equivalent's nonlinear parameter dependence is approximated by an affine one, and then the factorization into (4.2)–(4.3) is performed using standard methods as in Zhou *et al.* (1996, Chapter 10); Hecker and Varga (2004); Scherer and Weiland (2005, Section 6.2) or in Appendix B.7.

To this end, let $\bar{Z}_\alpha$ and $\bar{M}_\alpha$ represent $Z_\alpha$ and $M_\alpha$, with their ranges projected onto the range $[-1, 1]$ each. Approximations $\bar{A}_{dI}$ and $\bar{B}_{dI}$ of $A_d$ and $B_d$ are found such that the elements of $\bar{A}_{dI}$ and $\bar{B}_{dI}$ are affine functions of the two parameters $\bar{Z}_\alpha$ and $\bar{M}_\alpha$, that is

$$x(k+1) = \underbrace{\begin{bmatrix} \bar{A}_{dI,11} & \bar{A}_{dI,12} \\ \bar{A}_{dI,21} & \bar{A}_{dI,22} \end{bmatrix}}_{\bar{A}_{dI}} x(k) + \underbrace{\begin{bmatrix} \bar{B}_{dI,1} \\ \bar{B}_{dI,2} \end{bmatrix}}_{\bar{B}_{dI}} \delta_f(k) \tag{5.6}$$

with

$$\bar{A}_{dI,11}(\bar{Z}_\alpha, \bar{M}_\alpha) = a_{11,0} + a_{11,1}\bar{Z}_\alpha + a_{11,2}\bar{M}_\alpha,$$

and analogously for the other elements in $\bar{A}_{dI}$ and $\bar{B}_{dI}$. The coefficients $a_{ij,l}$ are chosen such that the absolute error sum over the parameter ranges is minimal for each matrix element. Mathematically speaking,

$$\{a_{ij,0}, a_{ij,1}, a_{ij,2}\} = \arg\inf_{a_{ij,l}} \sum_{m,n} |\bar{A}_{dI,ij}(\bar{Z}_{\alpha,m}, \bar{M}_{\alpha,n}) - A_{d,ij}(Z_{\alpha,m}, M_{\alpha,n})|, \qquad i = 1,2,\ j = 1,2,$$

where the continuous ranges of $Z_\alpha$ and $\bar{Z}_\alpha$ are gridded into 10 equi-distantly spaced intervals each, that is $Z_{\alpha,m} = 0.5 + (m-1)(4-0.5)/10$ and $\bar{Z}_{\alpha,m} = -1 + (m-1)2/10$, $m = 1, \dots, 11$, analogously for $M_{\alpha,n}$ and $\bar{M}_{\alpha,n}$. The elements of $\bar{B}_{dI,i}$ are chosen correspondingly using coefficients $b_{i,l}$. Numerical values of $a_{ij,l}$, $b_{i,l}$ are

$$\begin{aligned}
&a_{11,0} = 0.97532, && a_{11,1} = -1.7082 \cdot 10^{-2}, && a_{11,2} = -2.6086 \cdot 10^{-3},\\
&a_{12,0} = 9.8797 \cdot 10^{-3}, && a_{12,1} = -8.6124 \cdot 10^{-5}, && a_{12,2} = -8.7210 \cdot 10^{-6},\\
&a_{21,0} = -0.52315, && a_{21,1} = 0, && a_{21,2} = -0.52315,\\
&a_{22,0} = 0.99737, && a_{22,1} = 1.47768 \cdot 10^{-5}, && a_{22,2} = -2.6285 \cdot 10^{-3}.\\
&b_{1,0} = 4.9606 \cdot 10^{-5}, && b_{1,1} = -2.9236 \cdot 10^{-7}, && b_{1,2} = -2.6056 \cdot 10^{-8},\\
&b_{2,0} = 9.9913 \cdot 10^{-3}, && b_{2,1} = -2.1741 \cdot 10^{-7}, && b_{2,2} = -9.1438 \cdot 10^{-6}.
\end{aligned}$$

The resulting absolute error sum is below $0.008$ for each element, confirming that an affine approximation is of good accuracy. After obtaining the described affine approximation, a linear fractional representation of (5.6) is performed using

$$w_0(k) = \underbrace{\operatorname{diag}\left(\bar{Z}_\alpha(k), \bar{Z}_\alpha(k), \bar{M}_\alpha(k), \bar{M}_\alpha(k)\right)}_{\Delta(k)} z_0(k), \tag{5.7}$$

leading to

$$x(k+1) = \begin{bmatrix} a_{11,0} & a_{12,0} \\ a_{21,0} & a_{22,0} \end{bmatrix} x(k) + \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} w_0(k) + \begin{bmatrix} b_{1,0} \\ b_{2,0} \end{bmatrix} \delta_f(k) \tag{5.8}$$

$$z_0(k) = \begin{bmatrix} a_{11,1} & a_{12,1} \\ a_{21,1} & a_{22,1} \\ a_{11,2} & a_{12,2} \\ a_{21,2} & a_{22,2} \end{bmatrix} x(k) + \begin{bmatrix} b_{1,1} \\ b_{2,1} \\ b_{1,2} \\ b_{2,2} \end{bmatrix} \delta_f(k). \tag{5.9}$$

Note that a $4 \times 4$ $\Delta$-block is necessary to obtain the factorization, where $|\Delta_{ii}(k)| \leq 1$ for all $k$.

**Discrete-Time LFT Model, Method II**

As it is visible from method I, the factorization (5.7)–(5.9) is found after assuming a specific (affine) structure of the matrix elements, and then performing the factorization, which determines the size of the $\Delta$-block. In method II, the issue is addressed the other way round. Such a procedure has not been reported in the literature to the best of our knowledge. In particular, the $\Delta$-block is chosen to have a $2 \times 2$ diagonal structure with $|\Delta_{ii}(k)| \leq 1$ for all $k$, such that

$$w_0(k) = \underbrace{\operatorname{diag}\left(\bar{Z}_\alpha(k), \bar{M}_\alpha(k)\right)}_{\Delta(k)} z_0(k). \tag{5.10}$$

Second, all degrees of freedom in (4.2) are allowed to be present, that is

$$x(k+1) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} x(k) + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} w_0(k) + \begin{bmatrix} b_{13} \\ b_{23} \end{bmatrix} \delta_f(k), \tag{5.11}$$

$$z_0(k) = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} x(k) + \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} w_0(k) + \begin{bmatrix} d_{13} \\ d_{23} \end{bmatrix} \delta_f(k). \tag{5.12}$$

This is equivalent to allowing a rational parameter dependence of the system matrices, which is well suited to represent the original nonlinear dependence, even though just a $2 \times 2$ $\Delta$-block is used. Now (5.10)–(5.12) is combined analytically into

$$x(k+1) = \begin{bmatrix} \bar{A}_{dII,11} & \bar{A}_{dII,12} \\ \bar{A}_{dII,21} & \bar{A}_{dII,22} \end{bmatrix} x(k) + \begin{bmatrix} \bar{B}_{dII,1} \\ \bar{B}_{dII,2} \end{bmatrix} \delta_f(k),$$

with $\bar{A}_{dII,ij}$ and $\bar{B}_{dII,i}$ being rationally dependent on $\bar{Z}_\alpha$, $\bar{M}_\alpha$, and $a_{pq}$, etc. Since most free coefficients $a_{pq}$, ... appear in all system matrix elements, the coefficient optimization cannot be carried out for each matrix element separately as in method I. Instead, the optimization of method I is extended to

$$\begin{aligned}\{a_{pq}, b_{pq}, c_{pq}, d_{pq}\} = \arg \inf_{a_{pq},b_{pq},c_{pq},d_{pq}} & \sum_{i,j,m,n} |\bar{A}_{dII,ij}(\bar{Z}_{\alpha,m}, \bar{M}_{\alpha,n}) - A_{d,ij}(Z_{\alpha,m}, M_{\alpha,n})| \\ & + \sum_{i,m,n} |\bar{B}_{dII,i}(\bar{Z}_{\alpha,m}, \bar{M}_{\alpha,n}) - B_{d,i}(Z_{\alpha,m}, M_{\alpha,n})|,\end{aligned}$$

and done in one shot, again with the same gridding as in method I. Numerical values of $a_{pq}$, $b_{pq}$, $c_{pq}$, $d_{pq}$ are

$$\begin{array}{llll} a_{11} = 0.97537, & a_{12} = 0.0098705, & a_{21} = -0.52374, & a_{22} = 0.99737, \\ b_{11} = -0.035139, & b_{12} = 0.0035962, & b_{21} = 0.0094172, & b_{22} = 0.80450, \\ b_{13} = 4.9600 \cdot 10^{-5}, & b_{23} = 0.0099913, & c_{11} = 0.47831, & c_{12} = 0.0024022, \\ c_{21} = -0.65097, & c_{22} = -0.0031942, & d_{11} = -0.011354, & d_{12} = -0.011571 \\ d_{21} = 0.011908, & d_{22} = 0.0023395, & d_{13} = 8.0977 \cdot 10^{-6}, & d_{23} = -1.0789 \cdot 10^{-5}. \end{array}$$

The resulting absolute error sum is $0.018$ and slightly larger than the combined error sum in method I. Thus, by proceeding as in method II, a similarly accurate model as in method I is obtained, but with a much smaller $\Delta$-block. Either model from method I or II can now be used as an approximate discrete-time representation of (5.1)–(5.2).

### 5.2.2 Controller Design and Simulation Results

Two types of controllers are designed for the missile control problem. First, $\ell_1$-optimal LPV controllers based on Section 4.5 are obtained using the discrete-time LFT model II (5.10)–(5.12), whereas $\mathcal{H}_\infty$-optimal LPV controllers are synthesized by means of Section 4.6 using the continuous-time LFT model (5.1)–(5.5). It is the overall design goal that the closed-loop step response of $a_v$ reaches a value of $1$ in less than $0.5$ s with a smooth settling behavior.

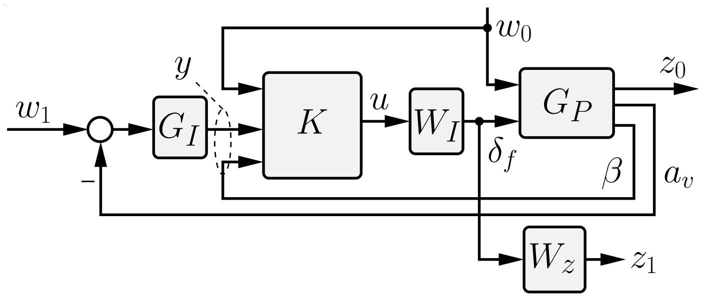



*Figure 5.5:* Example 2. Setup for $\ell_1$ synthesis.

**$\ell_1$-Optimal Controller Synthesis**

The $\ell_1$-optimal controller design for the discrete-time plant resulting from method II is described next. The design for method I works analogously and is thus omitted.

Consider the setup sketched in Figure 5.5, which is the same as in Gahinet *et al.* (1995) except for the weighting functions. This is due to the fact that objectives in $\ell_1$ design are fundamentally different from the ones in $\mathcal{H}_\infty$ design. $G_P$ represents the system equations (5.11)–(5.12) augmented by the measurement equation (5.2). The signal $w_1$ is a reference command for the vertical acceleration $a_v$. An integrator with transfer function $G_I(z) = \tau/(z-1)$ is included to provide zero steady-state error. Moreover a low-pass input filter $W_I(z) = 0.7/(z-0.3)$ is included. It turns out that a slightly better suppression of parameter variation effects is achieved with help of this filter.

When choosing the control error $w_1 - a_v$ as performance output $z_1$, a very fast and uniform command tracking is obtained, however at the price of unacceptably large control amplitudes. Instead, we take $z_1 = W_z\delta_f$, that is the control input weighted by the constant gain $W_z$. Note that $y$ combines the feedback signals fed to $K$, and $u$ is the output of $K$. Based on Figure 5.5, the augmented plant (4.6) is derived in a straightforward manner as the map $\mathrm{col}(w_0, w_1, u) \mapsto \mathrm{col}(z_0, z_1, y, w_0)$. It turns out that $G_{01} \equiv 0$, hence the simple case for reconstructing the LPV controller $\bar{K}(\Delta)$ is applicable. The optimization problem is of the form

$$\mu_{\text{opt}} := \inf_{E\in\boldsymbol{E}} \inf_{Q\in\ell_1^{1\times 4}} \|E^{-1}\Phi(Q)E\|_1 \tag{5.13}$$

with an additional regularizing constraint $\|Q\|_1 \leq \eta$, where the closed-loop mapping is $\mathrm{col}(z_0, z_1) = \Phi\,\mathrm{col}(w_0, w_1)$, and $\boldsymbol{E} := \{\mathrm{diag}(E_1, E_2, 1) | E_1, E_2 \in \mathbb{R}\}$. For this problem, a suboptimal Youla parameter $Q$ is computed using $E$-$Q$-iterations as described in Section 4.5. The $Q$ parameter is taken to be an FIR filter with $N = 20$ taps. Furthermore, $W_z = 0.012$ and $\eta = 10^7$. From $Q$, an LTI controller $K$ and an LPV controller $\bar{K}(\Delta)$ are constructed as described in Lemma B.13 and Section 4.4.2. $\bar{K}(\Delta)$ is of order $30$, with a sparse matrix structure. Order reduction usually leads to $\ell_1$ controllers with considerably lower order (Khammash, 2000), but is not further considered here. The controller achieves $\mu = 7.67$, and robust stability is implied by $\|w_0 \mapsto z_0\|_1 = 0.26 < 1$. The controller performance is illustrated by means of simulation results. Therein, the discrete-time LPV controller running at a sampling time of $\tau = 0.01$ s is connected to the original continuous-time plant. The parameters are measured at the beginning of each sampling interval, just as the feedback measurements, and assumed constant until the next sampling. Hence, a sampled-data

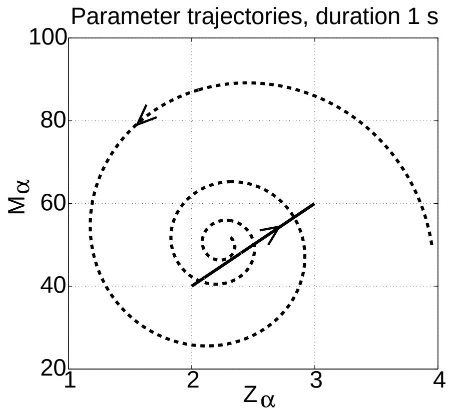



*Figure 5.6:* Example 2. Two exemplary time-varying parameter trajectories.

implementation is simulated, providing a test for the practical applicability of the controller. Simulations are carried out for the fixed parameters $(Z_\alpha, M_\alpha) \in \{(2.25, 53), (0.5, 0), (4, 0), (0.5, 106), (4, 106)\}$, i.e. the nominal case and the four extreme cases. Two time-varying parameter trajectories as displayed in Figure 5.6 are also considered. The simulation results in Figure 5.7 depict these seven different cases. The step responses of $a_v$ (top row) can hardly be distinguished, indicating the uniform behavior resulting from the gain-scheduled controller. Yet it is seen that the control action (bottom row) appears to change its values quite often with large gradients. To alleviate this "hectic" behavior, the optimization (5.13) is augmented by a bound on the control action variation rate, that is (with a slight abuse of notation)

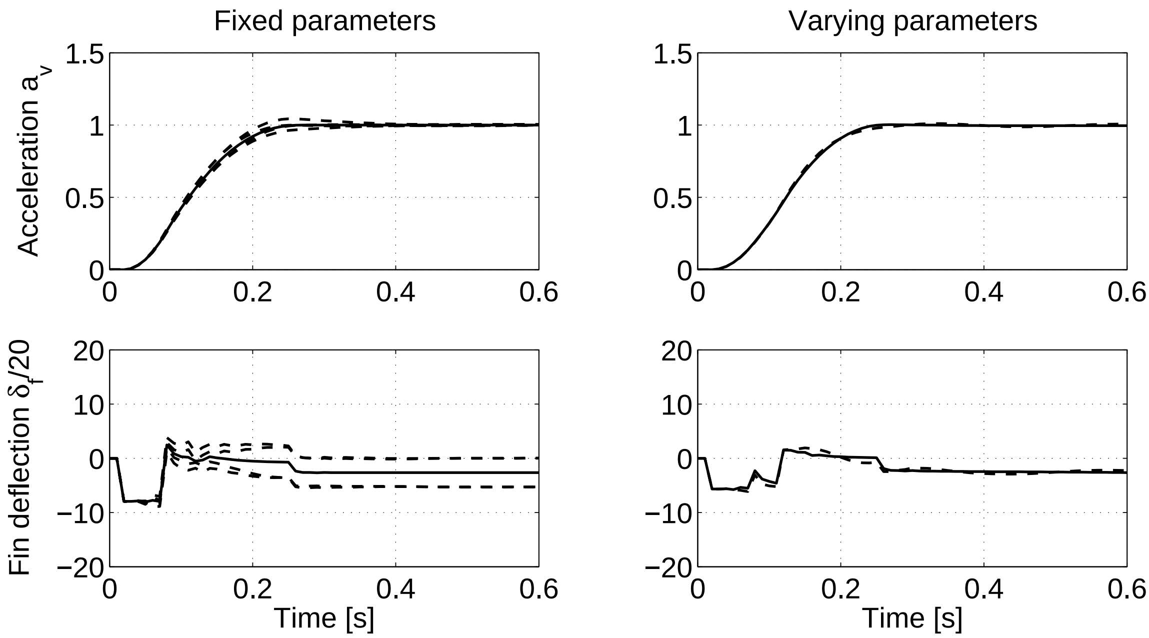



*Figure 5.7:* Example 2. Simulation results of the LPV $\ell_1$ controller without condition (5.14). Step responses of $a_v$ (top row) and corresponding control action $\delta_f$ (bottom row) are shown for five fixed parameters (left) and two time-varying parameters (right). The fin deflection is divided by 20 to conform with the images of the LMI Toolbox manual.

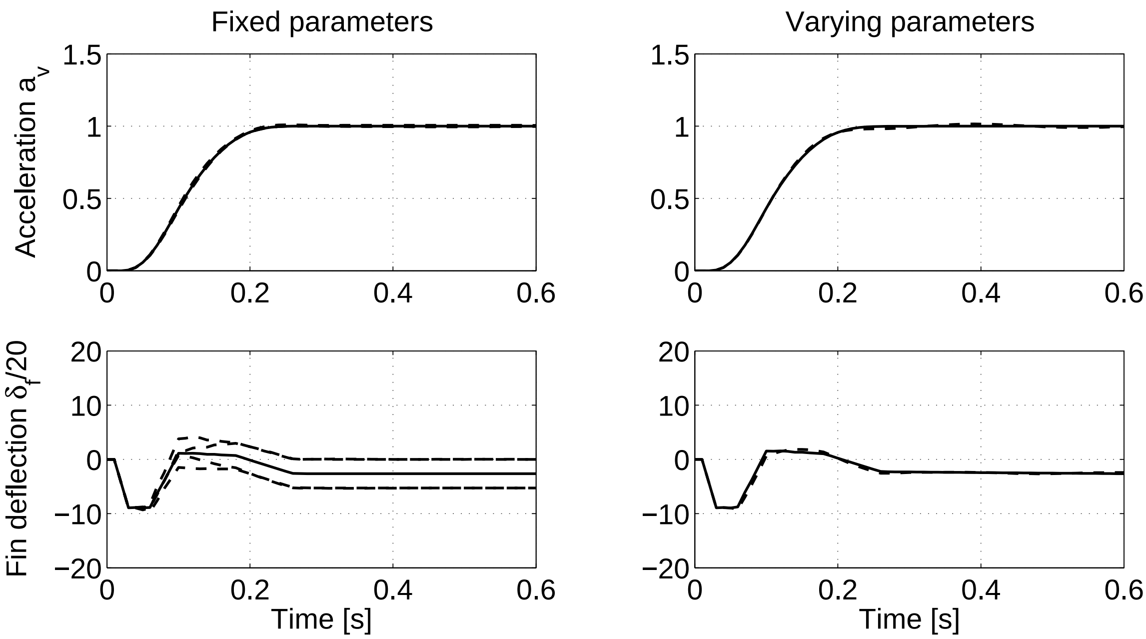



*Figure 5.8:* Example 2. Simulation results of the LPV $\ell_1$ controller with condition (5.14), cf. Figure 5.7.

$$\left\| \frac{z-1}{\tau z} \hat{\Phi}_{11}(z) \right\|_1 \leq u_d, \tag{5.14}$$

where $z_1 = \Phi_{11} w_1$. Here $u_d = 150$ is chosen, furthermore the performance weight is changed to $W_z = 0.005$. Now $\mu = 4.30$ is achieved, and robust stability is implied by $\|w_0 \mapsto z_0\|_1 = 0.17 < 1$. Interestingly, even though the bound (5.14) is imposed only for the nominal parameter case, it favorably affects the control action for all parameters, as seen in Figure 5.8. The maximum control signal amplitude has increased, but the control action changes more gradually and smoothly, which is better suited in a real application. The uniform behavior and the almost complete absence of overshoot sets these results apart from virtually all other designs in the literature (Apkarian *et al.*, 1995; Gahinet *et al.*, 1995; Bennani *et al.*, 1998), see also the $\mathcal{H}_\infty$ design below.

Finally note that the $\ell_1$ results are obtained using almost no tuning or design iterations (apart from playing around a little with one constant gain), which is in strong contrast to classical design methods or to $\mathcal{H}_\infty$ methods. The results for a $4 \times 4$ $\Delta$-block model from method I are similar. However, robust stability is not as easily obtained (due to a larger number of uncertainty blocks and their repeated structure), and computation times are higher.

### $\mathcal{H}_\infty$-Optimal Controller Synthesis

To set the obtained results into perspective, also $\mathcal{H}_\infty$-optimal LPV controllers are designed for this example. This is done by applying the LMI framework of Section 4.6. Now the continuous-time LFT model (5.2)–(5.5) is used, and a design according to the example in Chapter 7 of the MATLAB LMI Toolbox manual (Gahinet *et al.*, 1995) is carried out.

The control objectives are represented by a weighted mixed-sensitivity minimization according to

$$\gamma := \inf_{K \text{ stabilizing}} \left\| \begin{matrix} W_1(I+G_PK)^{-1} \\ W_2K(I+G_PK)^{-1} \end{matrix} \right\|_\infty ,$$

where the weights

$$W_1(s) = \frac{2.01}{s+0.201}, \qquad W_2(s) = \frac{9.678s^3 + 0.0269s^2}{s^3 + 1.206 \cdot 10^4 s^2 + 1.136 \cdot 10^7 s + 1.066 \cdot 10^{10}}$$

from Gahinet *et al.* (1995) are applied. Figure 5.9 shows the corresponding control scheme, which is straightforwardly transformed into (4.18).

Based on the described setup, several existence conditions for LPV $\mathcal{H}_\infty$ controllers are tested. Note that $\mathcal{H}_\infty$ conditions are a special case of quadratic performance, see Remark 3.1. The results in Table 5.2 indicate that our approaches achieve better performance than the method implemented in the LMI Toolbox, and the same performance as the LPV approach in (Scherer, 2000*b*), which uses a structure with additional $\Delta_K$-block according to Figure 4.3(a). Due to numerical ill-conditioning of the optimal controller, a suboptimal controller achieving $\gamma = 0.169$ obtained from the convex conditions of Corollary 4.4 is used for the simulations in the sequel. The corresponding LPV controller $\bar{K}(\Delta)$ is obtained as described in Section 4.6.2.

The simulation results for the LPV $\mathcal{H}_\infty$ controller in Figure 5.10 show a behavior which is comparable to the results obtained in the literature. This tendency is expected from the achieved $\mathcal{H}_\infty$-norm. The results do not change significantly when uniform noise up to $\pm 5\%$ is added to the parameter measurement fed to the controller, as depicted in Figure 5.11. Such a test is important for practical purposes. To complement the simulations, the controller obtained in Gahinet *et al.* (1995) is simulated in Figure 5.12, which confirms the above statements about existing control methods. The control action has smaller amplitudes, but the convergence behavior is worse than

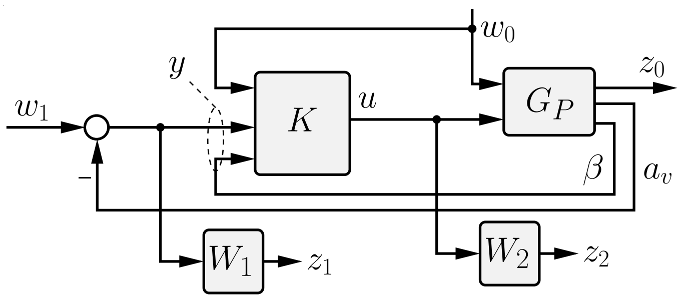



*Figure 5.9:* Example 2. Setup for $\mathcal{H}_\infty$ synthesis.

*Table 5.2:* Example 2. Optimal $\mathcal{H}_\infty$ performance $\gamma$ for LPV controller synthesis.

| Approach | $\gamma$ |
|---|---|
| LMI Toolbox, Chapter 7 | $0.205$ |
| Scherer (2000*b*) | $0.157$ |
| Theorem 4.3 | $0.157$ |
| Corollary 4.4 ($\tilde{S} = 0$) | $0.157$ |

in Figure 5.10. Specific comparisons to the $\ell_1$ results are not in order, since $\ell_1$ and $\mathcal{H}_\infty$ designs pursue different objectives. But as a general impression note that the $\ell_1$ controller shows favorable properties in terms of overshoot, maximum control signal amplitude, and uniformity with respect to parameter changes.

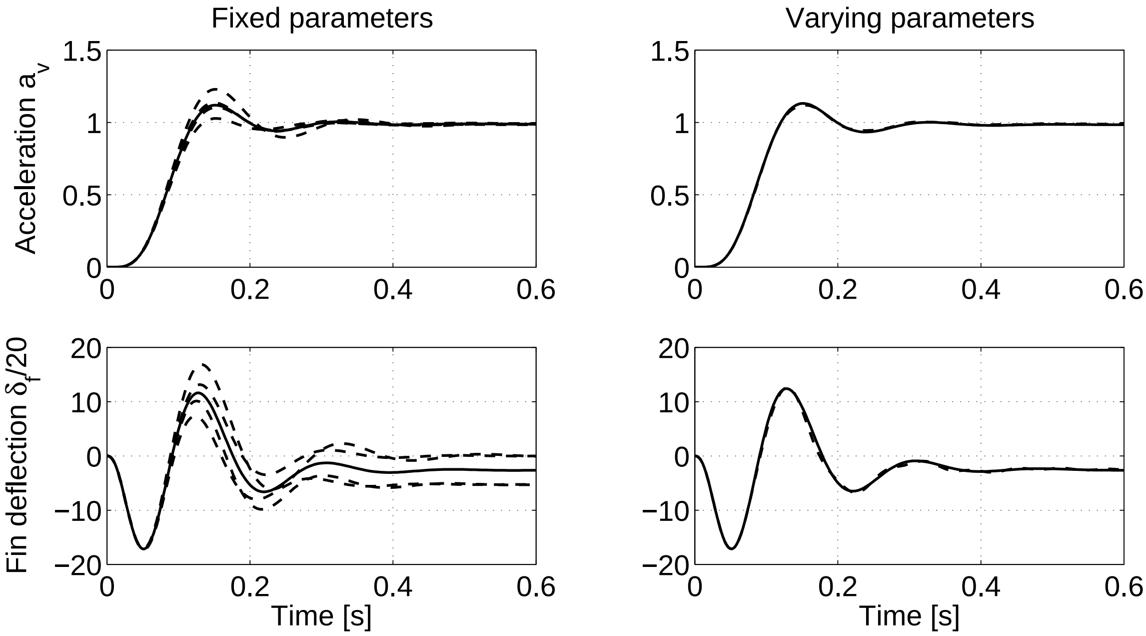



*Figure 5.10:* Example 2. Simulation results of the LPV $\mathcal{H}_\infty$ controller using the novel control structure, cf. Figure 5.7.

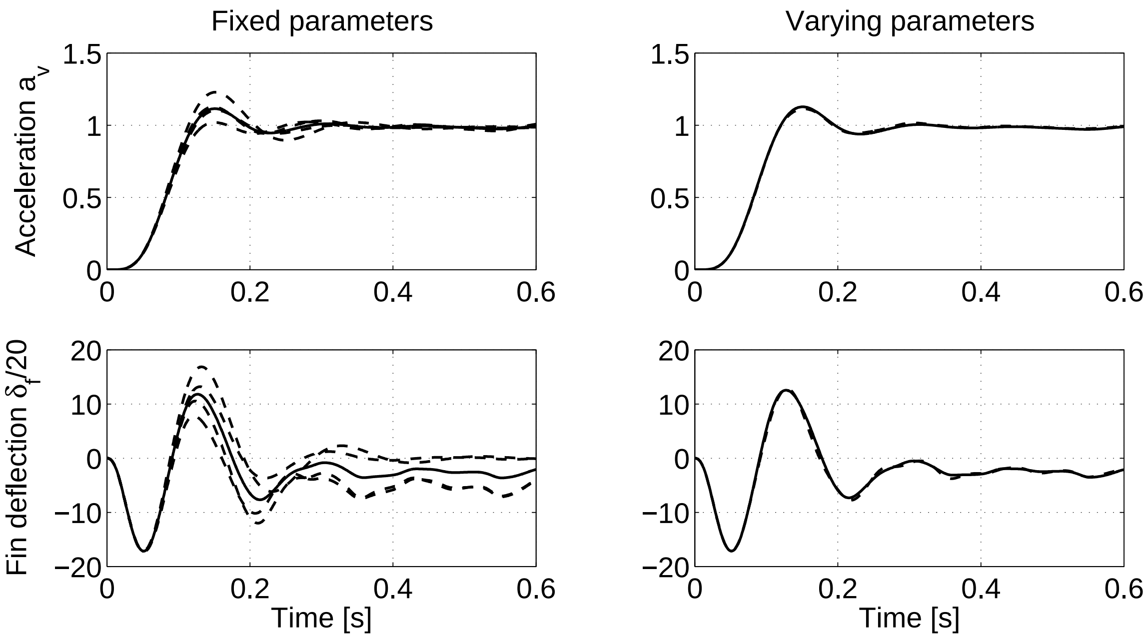



*Figure 5.11:* Example 2. Simulation results of the LPV $\mathcal{H}_\infty$ controller using the novel control structure, with additional parameter noise, cf. Figure 5.7.

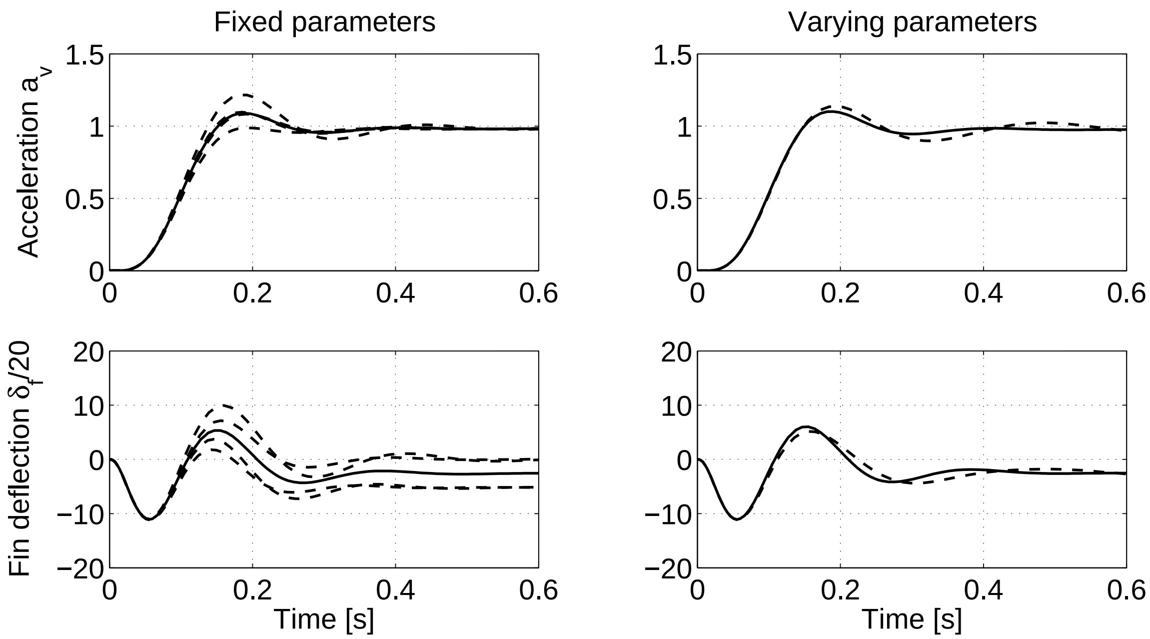



*Figure 5.12:* Example 2. Simulation results of the LPV $\mathcal{H}_\infty$ controller as computed in the LMI Toolbox demo, cf. Figure 5.7.

## 5.3 Summary

This chapter presents two examples for LPV controller design. The first example shows the favorable properties of our novel control structure in an $\ell_1$ setting. The second example applies the control structure to a standard flight control example both in the $\ell_1$ and the $\mathcal{H}_\infty$ framework. It is visible that the proposed control structure tends to produce uniform behavior for different parameter values, i.e. the different step responses are very close to each other. Moreover, the settling behavior of the step response indicates good performance. Especially in $\ell_1$-optimal LPV controller design, the avoidance of introducing additional uncertainty blocks provides low conservatism and small computation times. This means that – compared to the traditional LPV controller structure – a better performance is achieved while the complexity of the design process is smaller.

Furthermore, it is discussed in this chapter how discrete-time models with low complexity can be obtained, how control setup and weights may be selected in $\ell_1$ control, and how favorable time-domain properties can be achieved. Based on the simulation results, it is concluded that the LPV $\ell_1$ controller may serve as a benchmark of what could be achieved in the flight control example.

Further experience with $\ell_1$-optimal control has been gained in another application, as reported in Rieber *et al.* (2005*b*); Stemmer *et al.* (2005). There, an $\ell_1$-optimal controller has been designed for the position and vibration control of piezo scanners in atomic force microscopy (AFM). The feedforward/feedback controller has been experimentally tested on an industrial microscope. The imaging accuracy and speed are comparable to results obtained with $\mathcal{H}_\infty$ controllers (Schitter *et al.*, 2004). However, the $\ell_1$ controller design featured simple performance weightings (constant gains) and an almost complete absence of tuning. Practical insights obtained from such simulation and experimental studies are very important, since only little experience with applications of $\ell_1$-optimal control exists at this point in time.

# Chapter 6

# Conclusions

**Summary**

Norm-based disturbance attenuation is one of the most promising control paradigms at present. Methods like $\mathcal{H}_\infty$ and $\ell_1$ control are applicable to a wide range of system classes and control problems. Most importantly, these methods allow to include quantitative design goals and robustness considerations into the controller synthesis. While $\mathcal{H}_\infty$-optimal control and other quadratic-type frameworks are well-developed and still a focus of further considerations, $\ell_1$-optimal control had reached an intermediate level of maturity. Indeed, available results include nominal analysis and design, as well as robustness considerations for systems with dynamic uncertainties. To extend the theoretic foundation and the applicability of $\ell_1$-optimal control, this thesis presents novel and computationally tractable conditions for the control of uncertain systems.

On the analysis side, an LFT characterization of uncertain system responses is derived, which makes the response generally accessible to robustness analysis. Matrix inequality conditions for star-norm based performance analysis of discrete-time systems are developed, both for systems with and without parametric uncertainties. Although these conditions establish upper bounds on the robust $\ell_\infty$-gain, a direct approach for determining this gain is proposed moreover. The direct approach allows to compute relatively tight upper bounds, and trades off accuracy and computational effort. It is tailored to the cases of time-invariant, time-varying, or rate-bounded parametric uncertainties.

Synthesis of LTI and LPV controllers is addressed in various ways. A novel convex formulation of $\mathcal{H}_\infty$ and $\mathcal{H}_2$ constraints for linear multi-objective controller synthesis is proposed. The formulation shows its efficiency compared to existing methods. The state-feedback literature is complemented with a discrete-time version of LMI conditions for robust LTI controller synthesis with quadratic performance guarantees in face of parametric uncertainties. To get a new perspective on gain-scheduling problems, a novel structure is created that converts the LPV controller synthesis into an LTI robust synthesis problem without introducing additional uncertainty blocks. The new structure is exploited for $\ell_1$ and quadratic performance synthesis, and leads to the first solution of LPV

output-feedback controller design in the $\ell_1$ framework. Finally, several examples discuss the properties and the applicability of the proposed approaches, also in comparison to existing methods.
Together with well-known results from $\mathcal{H}_\infty$- and $\mathcal{H}_2$-optimal control, our developments are building blocks for a general multi-objective framework to address LTI and LPV systems. Hence our work broadens the scope of the disturbance attenuation framework, and is an important step towards making it a widely applicable, theoretically sound, and practically relevant control paradigm.

**Outlook**

To conclude this thesis, some ideas about open questions and future challenges are suggested. We start with particular ideas related to our contributions, go on with suggestions on the topic of $\ell_1$ control, and finish with a more global perspective.
First, the presented $\ell_\infty$-gain analysis approach of Section 2.5 can be beneficial in at least two ways for further developments. Using the proposed LFT time-domain response characterization, the method may be extended in a straightforward manner to other performance objectives, in which the impulse response or the solution of the system is involved directly. In this context, one can think of new approaches to robust model predictive or receding horizon control, where usually a solution of the system is computed and optimized over a finite horizon. Moreover, new direct $\ell_\infty$-gain based procedures for LPV controller synthesis or robust filter design are conceivable, however at a considerable computational price.
Second, further conditions may be included into the multi-objective framework of Section 3.3, although it already covers a great variety of possible design goals. One can think of pole placement constraints, star-norm constraints, etc. Another possibility is to investigate general formulations without using the Youla parameterization along the lines of Scherer (2001*b*).
Third, the control structure proposed in Section 4.3 may be applied not only to the $\ell_1$ and quadratic performance frameworks, but also to robust $\mathcal{H}_2$-, generalized $\mathcal{H}_2$-, and star-norm performance setups. See Scherer (2000*b*) for a general discussion. Extending these ideas, LPV controller synthesis in a multi-objective framework is conceivable, either using common (constant or parameter-depending) Lyapunov functions, or scaled-norm approaches based on small-gain conditions.
More generally speaking, the $\ell_1$ framework still has potential for improvement in terms of other system classes. One can think of extending the available results to differential-algebraic systems. Preliminary steps have been taken by the author in connection with Huck (2005). A framework for the design of $\ell_\infty$-gain-optimal nonlinear controllers for linear systems is addressed in e.g. Blanchini and Sznaier (1995); Shamma (1996), but warrants more general investigations towards output feedback. $\ell_\infty$-gain based control of nonlinear systems is discussed in Figueiredo and Guarong (1989); Poolla and Shamma (1995); Lu (1998); Fialho and Georgiou (1999); Huang and James (2003), for example. Good bases for a general treatment of nonlinear systems in this direction are input-to-state stability (ISS) (Sontag, 2006) and viability theory (Aubin, 1991). Yet further steps need to be taken, in particular towards computational tractability. On a different level are methods to synthesize optimal controllers with a given maximum order, or with a pre-defined

structure. Such approaches can be interesting for practical applications, see e.g. Blanchini and Sznaier (2000); Qi *et al.* (2004); Tantaris *et al.* (2006) and references therein.

An important topic related to $\ell_1$ control is the field of model reduction. To reduce the order of a given $\ell_1$-optimal controller while (approximately) keeping the closed-loop performance is essential for practical applications with little computational resources. This reduction can take place at different points in the design process. So the Youla parameter could be reduced, or the final controller obtained from the Youla parameter. Also a largely open problem is the field of stability- and performance-preserving order-reduction of LPV controllers.

Furthermore, the application of $\ell_1$ control to practical examples is encouraged. Simulation studies and experiments using $\ell_1$ controllers should indicate how to find meaningful control setups and performance weightings for control design.

From a more global point of view, there is still considerable work to be done in the field of numerical algorithms for optimal control. Although there are a number of general purpose solvers available, algorithms specialized to a certain control problem at hand may help in finding solutions reliably and reducing computation times significantly. An important topic in this context is the conditioning of numerical solutions, which influences the sensitivity of achieved properties on slight perturbations of the solution, see e.g. Keel and Bhattacharyya (1997).

Together with the results in this thesis, solutions to the discussed problems will further open up new application fields to the norm-based disturbance attenuation paradigm.

# Appendix A

# Signal and Operator Norms

The problems treated in this thesis are not only concerned with the stability of control systems, but also with their performance. To quantify the achieved or desired performance, norms for signals and system operators are used. These norms are defined and explained in this appendix. See for example Zhou *et al.* (1996); Skogestad and Postlethwaite (2005); Khalil (2002); Dahleh and Diaz-Bobillo (1995); Scherer and Weiland (2005) for more information.

**Signal Norms**

To measure the "size" of a signal, three norms are defined here. The $\ell_\infty$-norm

$$\|v\|_\infty := \sup_k \max_{1\leq i\leq n} |v_i(k)|$$

measures the maximum amplitude of a sequence $v = \{v(k)\}_{k=0}^{\infty}$ with $v(k) = [v_1(k), ..., v_n(k)]^T \in \mathbb{R}^n$. The $\ell_\infty$-norm is defined on the space $\ell_\infty^n$, the Banach space of right-sided bounded real vector sequences of dimension $n$. An alternative norm on $\ell_\infty^n$ is the peak-norm

$$\|v\|_{\text{peak}} := \sup_k \sqrt{v(k)^T v(k)}.$$

For $\dim(v) = n$ it holds that $\|v\|_\infty \leq \|v\|_{\text{peak}} \leq \sqrt{n}\|v\|_\infty$ (Khalil, 2002, Appendix A). On the other hand, the $\ell_2$-norm

$$\|v\|_2 := \sqrt{\sum_{k=0}^{\infty} v(k)^T v(k)}$$

measures the energy of a signal. It is defined on the Hilbert space $\ell_2^n$ of right-sided square summable real vector sequences of dimension $n$. Continuous-time versions of the given definitions follow analogously as

$$\|v\|_\infty := \operatorname{ess\,sup}_t \max_{1\leq i\leq n} |v_i(t)|, \qquad \|v\|_{\text{peak}} := \operatorname{ess\,sup}_t \sqrt{v(t)^T v(t)}, \qquad \|v\|_2 := \sqrt{\int_0^\infty v(t)^T v(t) dt}.$$

**Operator Norms**

As for signals, there are notions to measure the "size" of a stable operator. To this end, we consider the $\ell_p$-induced norm (or $\ell_p$-gain)

$$\|G\|_{p-\mathrm{ind}} := \sup_{0<\|w\|_p<\infty} \frac{\|Gw\|_p}{\|w\|_p}$$

of a map $G : \ell_p^n \to \ell_p^m$. $G$ is said to be $\ell_p$-stable (or just stable) if it is causal and $\|G\|_{p-\mathrm{ind}} < \infty$. The $\ell_p$-gain indicates the worst-case amplification of the input $w$ in terms of the $\ell_p$-norm.

It can be shown that for $p = \infty$, the $\ell_\infty$-gain $\|G\|_{\infty-\mathrm{ind}}$ of stable LTI systems is equal to the $\ell_1$-norm

$$\|G\|_1 := \max_{1\le i\le m} \sum_{j=1}^{n} \sum_{k=0}^{\infty} |G_{ij}(k)|$$

of the operator's impulse response $G \in \ell_1^{m\times n}$, see e.g. Zhou *et al.* (1996, Section 4.5). The space $\ell_1^{m\times n}$ is the Banach space of right-sided absolutely summable real matrix sequences of dimension $m \times n$. For the peak-induced norm $\|G\|_{\mathrm{peak-ind}}$, an equality relation to another norm is not available. However, $\|G\|_{\mathrm{peak-ind}}$ can be bounded from above by the so-called star-norm, see Section 2.2.2. Moreover it holds that $\frac{1}{\sqrt{m}}\|G\|_{\mathrm{peak-ind}} \le \|G\|_{\infty-\mathrm{ind}} \le \sqrt{n}\|G\|_{\mathrm{peak-ind}}$ for $\dim(G) = m \times n$ (Khalil, 2002, Appendix A). Hence the peak-induced norm is suited to serve as an upper bound on the $\ell_\infty$-induced norm.

In the case of $p = 2$, it can be shown that the $\ell_2$-gain $\|G\|_{2-\mathrm{ind}}$ of stable LTI systems is equal to the $\mathcal{H}_\infty$-norm

$$\|\hat{G}\|_\infty := \sup_{0\le\theta\le 2\pi} \sigma_{\max}\left(\hat{G}(e^{j\theta})\right)$$

defined on $\mathcal{H}_\infty^{m\times n}$, see e.g. Zhou *et al.* (1996, Section 4.3). The space $\mathcal{H}_\infty^{m\times n}$ is the Hardy space of complex matrix functions $\hat{G}(z)$ essentially bounded on the unit circle and bounded and analytic outside the unit disc. In the case of finite-dimensional LTI systems, it is enough to consider $\mathcal{RH}_\infty^{m\times n}$, which is the space of proper and real-rational stable $m \times n$ transfer matrices.

Another system measure (not directly related to $\ell_p$-gains) is the $\mathcal{H}_2$-norm

$$\|\hat{G}\|_2 := \sqrt{\frac{1}{2\pi}\int_0^{2\pi} \text{Trace}\left(\hat{G}(e^{j\theta})^*\hat{G}(e^{j\theta})\right) d\theta}$$

defined on $\mathcal{RH}_\infty^{m\times n}$.

Continuous-time versions of the considered operator norms are

$$\|G\|_1 := \max_{1\le i\le m} \sum_{j=1}^{n} \int_{t=0}^{\infty} |G_{ij}(t)|dt,$$

$$\|\hat{G}\|_\infty := \sup_{\omega\in\mathbb{R}} \sigma_{\max}\left(\hat{G}(j\omega)\right),$$

$$\|\hat{G}\|_2 := \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}\left(\hat{G}(j\omega)^*\hat{G}(j\omega)\right) d\omega}.$$

It is common in the literature to denote control methods related to $\ell_2$- or $\mathcal{L}_2$-gain performance as $\mathcal{H}_\infty$ control methods, even for LPV or for nonlinear systems. It follows from the definition of the $\ell_2$-gain that $\mathcal{H}_\infty$ control methods may be interpreted as being concerned with the worst-case energy-gain of a system. Methods related to $\ell_\infty$-gain performance are likewise often denoted $\ell_1$ control methods in the literature. These methods deal with the worst-case amplitude-gain of a system, or more generally with time-domain properties of system outputs. The reader is referred to a number of references that discuss properties and usefulness of $\mathcal{H}_\infty$, $\mathcal{H}_2$, and $\ell_1$ control approaches, for example to Desoer and Vidyasagar (1975); Zames (1981); Doyle and Stein (1981); Dahleh and Khammash (1993); Dahleh and Diaz-Bobillo (1995); Sanchez-Pena and Sznaier (1998); Skogestad and Postlethwaite (2005); Rieber and Allgöwer (2006*a*).

# Appendix B

# Auxiliary Results

This appendix collects results from the literature concerning matrices, LFTs, matrix inequalities, and parameterizations of stabilizing controllers that are used throughout this thesis. Moreover, some results on robust stability and performance analysis are recalled as a foundation for developments in this thesis. All matrices are supposed to be appropriately dimensioned, and all LFTs are supposed to be well-defined. Sometimes the given references are not the original references, but rather recent publications with nice presentations or proofs.

## B.1 Matrices

The following three lemmas state well-known results for matrices.

**Lemma B.1** *(Matrix inverses, see Zhou* et al. *(1996, Section 2.3); Horn and Johnson (1985, Section 0.7.3); Skogestad and Postlethwaite (2005, Section 3.2))*
*The following statements hold:*

*(i) Suppose $A$ and $D$ are both non-singular. Then*

$$(A - BD^{-1}C)^{-1} = A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1}.$$

*(ii) Suppose $M$ and $A$ (or $D$ respectively) are both non-singular, and let $V := D - CA^{-1}B$ (or $W := A - BD^{-1}C$). Then*

$$\underbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}_{M}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}BV^{-1}CA^{-1} & -A^{-1}BV^{-1} \\ -V^{-1}CA^{-1} & V^{-1} \end{bmatrix}$$

$$\left( = \begin{bmatrix} W^{-1} & -W^{-1}BD^{-1} \\ -D^{-1}CW^{-1} & D^{-1} + D^{-1}CW^{-1}BD^{-1} \end{bmatrix} \right).$$

*(iii) Suppose that $I - AB$ is non-singular. Then*

$$A(I - BA)^{-1} = (I - AB)^{-1}A.$$

■

**Lemma B.2** *(Schur Complement, see Zhou* et al. *(1996, Section 2.3))*
*Suppose $A$ (or $D$ respectively) is non-singular. Then*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix}$$
$$\left( = \begin{bmatrix} I & BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I & 0 \\ D^{-1}C & I \end{bmatrix} \right). \quad \blacksquare$$

**Lemma B.3** *(Left/right inverse, see Zhou* et al. *(1996, Section 2.9))*
*Suppose $M$ is a complex matrix with full column-rank. Then a left inverse of $M$ is given by $M^\dagger = (M^*M)^{-1}M^*$. Likewise a right inverse of a full row-rank matrix $M$ is given by $M^\ddagger = M^*(MM^*)^{-1}$.* $\blacksquare$

Moreover, the obvious identities

$$V^TXV + W^TYW = \begin{bmatrix} V \\ W \end{bmatrix}^T \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \begin{bmatrix} V \\ W \end{bmatrix} = \begin{bmatrix} W \\ V \end{bmatrix}^T \begin{bmatrix} Y & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} W \\ V \end{bmatrix}$$

are often used.

## B.2 Linear Fractional Transformations

Some rules for operations on LFTs are summarized in the following lemma.

**Lemma B.4** *(LFT rules, see Zhou* et al. *(1996, Section 10.1))*
*The following statements hold:*

*(i) The sum of two LFTs is again an LFT, in particular*

$$\Delta_1 \star \left[\begin{array}{c:c} A_1 & B_1 \\ \hdashline C_1 & D_1 \end{array}\right] + \Delta_2 \star \left[\begin{array}{c:c} A_2 & B_2 \\ \hdashline C_2 & D_2 \end{array}\right] = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \star \left[\begin{array}{cc:c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hdashline C_1 & C_2 & D_1 + D_2 \end{array}\right].$$

*(ii) The product of two LFTs is again an LFT, in particular*

$$\left(\Delta_1 \star \left[\begin{array}{c:c} A_1 & B_1 \\ \hdashline C_1 & D_1 \end{array}\right]\right) \cdot \left(\Delta_2 \star \left[\begin{array}{c:c} A_2 & B_2 \\ \hdashline C_2 & D_2 \end{array}\right]\right) = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \star \left[\begin{array}{cc:c} A_1 & B_1C_2 & B_1D_2 \\ 0 & A_2 & B_2 \\ \hdashline C_1 & D_1C_2 & D_1D_2 \end{array}\right].$$

*(iii) Suppose $D$ is invertible. Then the inverse of an LFT is again an LFT, in particular*

$$\left(\Delta \star \left[\begin{array}{c:c} A & B \\ \hdashline C & D \end{array}\right]\right)^{-1} = \Delta \star \left[\begin{array}{c:c} A - BD^{-1}C & -BD^{-1} \\ \hdashline D^{-1}C & D^{-1} \end{array}\right].$$

*Analogous formulas hold for left and right inverses of an LFT.*

*(iv) An upper LFT can be expressed as a lower LFT and vice versa, in particular*

$$\Delta \star \left[\begin{array}{c:c} A & B \\ \hdashline C & D \end{array}\right] = \left[\begin{array}{c:c} D & C \\ \hdashline B & A \end{array}\right] \star \Delta. \qquad \blacksquare$$

## B.3 Matrix Inequalities

The well-known *Schur Lemma* stated next is applied to rewrite or linearize matrix inequalities. Details on how to handle non-strict inequalities are given in e.g. Boyd *et al.* (1994).

**Lemma B.5** *(Schur Lemma, see Horn and Johnson (1985, Section 7.7); Boyd* et al. *(1994, Section 2.1))*
*Let $Q = Q^T$, $R = R^T$. Then*

$$\begin{aligned} & \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} < 0 \\ \Leftrightarrow \quad & R < 0 \quad \textit{and} \quad Q - SR^{-1}S^T < 0 \\ \Leftrightarrow \quad & Q < 0 \quad \textit{and} \quad R - S^TQ^{-1}S < 0. \end{aligned} \qquad \blacksquare$$

The following result is useful for linearizing matrix inequalities with a certain structure, and is a direct consequence of the Schur Lemma.

**Lemma B.6** *(Scherer and Weiland, 2005, Section 4.2)*
*Suppose $S$, $U$, $V$ are constant, and $Q(v) = Q(v)^T$, $R(v) = R(v)^T$, $W(v)$ are affine in $v$. Then there exists $v$ such that*

$$R(v) > 0 \quad \textit{and} \quad \underbrace{\begin{bmatrix} V \\ W(v) \end{bmatrix}^T \begin{bmatrix} Q(v) & S \\ S^T & UR(v)^{-1}U^T \end{bmatrix} \begin{bmatrix} V \\ W(v) \end{bmatrix}}_{V^TQV+V^TSW+W^TS^TV+W^TUR^{-1}U^TW} < 0$$

*if and only if there exists $v$ such that*

$$\begin{bmatrix} V^TQ(v)V + V^TSW(v) + W(v)^TS^TV & W(v)^TU \\ U^TW(v) & -R(v) \end{bmatrix} < 0. \qquad \blacksquare$$

The *Dualization Lemma* presented next gives another possibility of rewriting matrix inequalities. It can be used to transform bilinear inequalities into LMIs.

**Lemma B.7** *(Dualization Lemma, see Scherer and Weiland (2005, Section 4.5))*
*Suppose $P = P^T = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}$ is non-singular with $R \geq 0$, and let $\tilde{P} = P^{-1} = \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{bmatrix}$ with $\tilde{Q} \leq 0$. Then*

$$\begin{bmatrix} I \\ W \end{bmatrix}^T P \begin{bmatrix} I \\ W \end{bmatrix} < 0 \qquad \Leftrightarrow \qquad \begin{bmatrix} W^T \\ -I \end{bmatrix}^T \tilde{P} \begin{bmatrix} W^T \\ -I \end{bmatrix} > 0. \qquad \blacksquare$$

The following three lemmas provide the possibility to eliminate variables from matrix inequalities. The first one is the well-known *Finsler Lemma*, whereas the second and third result – the *Projection Lemma* and the *Elimination Lemma* – provide generalizations thereof.

**Lemma B.8** *(Finsler Lemma, see Oliveira and Skelton (2001))*
*Let* $x \in \mathbb{R}^n$, $P = P^T \in \mathbb{R}^{n\times n}$, $V \in \mathbb{R}^{m\times n}$ *such that* $\operatorname{rank}(V) < n$, *and* $V^\perp$ *be a basis matrix of* $\ker(V)$, *i.e.* $\operatorname{im}(V^\perp) = \ker(V)$. *Then the following statements are equivalent:*

*(i)* $x^T P x < 0 \quad \forall\, Vx = 0, x \neq 0.$

*(ii)* $(V^\perp)^T P V^\perp < 0.$

*(iii) There exists* $\mu \in \mathbb{R}$ *such that* $\quad P - \mu V^T V < 0.$

*(iv) There exists* $Z \in \mathbb{R}^{n\times m}$ *such that* $\quad P + ZV + V^T Z^T < 0.$ ■

**Lemma B.9** *(Projection Lemma, see Gahinet and Apkarian (1994))*
*Suppose* $P = P^T$, *and let* $U^\perp$, $V^\perp$ *be basis matrices of* $\ker(U)$, $\ker(V)$, *i.e.* $\operatorname{im}(U^\perp) = \ker(U)$, $\operatorname{im}(V^\perp) = \ker(V)$. *Then the following statements are equivalent:*

*(i) There exists* $Z$ *such that* $\quad P + U^T Z V + V^T Z^T U < 0.$

*(ii)* $Ux = 0$ *or* $Vx = 0$ *imply* $x^T P x < 0$ *or* $x = 0.$

*(iii)* $(U^\perp)^T P U^\perp < 0$ *and* $(V^\perp)^T P V^\perp < 0.$ ■

**Lemma B.10** *(Elimination Lemma, see Scherer (2001*a*, Appendix A.2); Scherer and Weiland (2005, Section 4.5))*
*Suppose* $P = P^T = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}$ *is non-singular with* $R \geq 0$, *and let* $\tilde{P} = P^{-1} = \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{bmatrix}$ *with* $\tilde{Q} \leq 0$. *Let* $U^\perp$, $V^\perp$ *be basis matrices of* $\ker(U)$, $\ker(V)$, *i.e.* $\operatorname{im}(U^\perp) = \ker(U)$, $\operatorname{im}(V^\perp) = \ker(V)$. *Then there exists* $Z$ *such that*

$$\begin{bmatrix} I \\ U^T Z V + W \end{bmatrix}^T P \begin{bmatrix} I \\ U^T Z V + W \end{bmatrix} < 0$$

*if and only if*

$$(V^\perp)^T \begin{bmatrix} I \\ W \end{bmatrix}^T P \begin{bmatrix} I \\ W \end{bmatrix} V^\perp < 0 \quad \textit{and} \quad (U^\perp)^T \begin{bmatrix} W^T \\ -I \end{bmatrix}^T \tilde{P} \begin{bmatrix} W^T \\ -I \end{bmatrix} U^\perp > 0. \qquad ■$$

## B.4 Uncertain Matrix Inequalities

The following version of the *Full-Block S-Procedure* is useful for rewriting matrix inequalities with a rational dependence on uncertain parameters by introducing multipliers.

**Lemma B.11** *(Full-Block S-Procedure, see Scherer (2000*b*); Iwasaki and Shibata (2001); Scherer (2006))*
*Suppose* $R_p = R_p^* \geq 0$, $Q_p = Q_p^*$, $S_p$, $H = \left[\begin{array}{c:c} A & B \\ \hdashline C & D \end{array}\right]$ *are given, and* $\boldsymbol{\Delta}$ *is some set of complex matrices. Then* $\Delta \star H$ *is well-posed and*

$$\begin{bmatrix} I \\ \Delta \star H \end{bmatrix}^* \begin{bmatrix} Q_p & S_p \\ S_p^* & R_p \end{bmatrix} \begin{bmatrix} I \\ \Delta \star H \end{bmatrix} < 0 \quad \forall \Delta \in \boldsymbol{\Delta} \tag{B.1}$$

*if there exists* $Q = Q^*$, $R = R^*$, $S$ *satisfying*

$$\begin{bmatrix} \Delta \\ I \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} \Delta \\ I \end{bmatrix} \geq 0 \quad \forall \Delta \in \boldsymbol{\Delta} \tag{B.2}$$

*and*

$$\left[\begin{array}{cc} I & 0 \\ A & B \\ \hdashline 0 & I \\ C & D \end{array}\right]^* \left[\begin{array}{cc:cc} Q & S & 0 & 0 \\ S^* & R & 0 & 0 \\ \hdashline 0 & 0 & Q_p & S_p \\ 0 & 0 & S_p^* & R_p \end{array}\right] \left[\begin{array}{cc} I & 0 \\ A & B \\ \hdashline 0 & I \\ C & D \end{array}\right] < 0. \tag{B.3}$$

*If moreover* $\boldsymbol{\Delta}$ *is compact, then equivalence holds.* ■

**Remark B.1** *In the preceding lemma, (B.1) and (B.3) may be replaced by*

$$(\Delta \star H)^* P_p (\Delta \star H) < 0 \quad \forall \Delta \in \boldsymbol{\Delta}$$

*and*

$$\left[\begin{array}{cc} I & 0 \\ A & B \\ \hdashline C & D \end{array}\right]^* \left[\begin{array}{cc:c} Q & S & 0 \\ S^* & R & 0 \\ \hdashline 0 & 0 & P_p \end{array}\right] \left[\begin{array}{cc} I & 0 \\ A & B \\ \hdashline C & D \end{array}\right] < 0,$$

*respectively, where* $P_p = P_p^*$ *(Wu and Dong, 2006).* □

Finally we give a result for converting a semi-infinite constraint like (B.2) into a finite set of constraints by introducing a possibly conservative relaxation. The lemma states that it is enough to check the considered uncertain matrix inequality for a finite number of $\Delta$ values instead of for all $\Delta \in \boldsymbol{\Delta}$, if the set of multipliers is restricted by imposing $Q < 0$.

**Lemma B.12** *(Scherer, 2000b)*
*Suppose that* $\mathbf{\Delta} = \mathrm{Co}(\{\Delta_1, ..., \Delta_{n_\Delta}\})$. *Then*

$$\begin{aligned} & Q < 0 \quad \textit{and} \quad \begin{bmatrix} \Delta \\ I \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} \Delta \\ I \end{bmatrix} \geq 0 \quad \forall \Delta \in \mathbf{\Delta} \\ \Leftrightarrow \quad & Q < 0 \quad \textit{and} \quad \begin{bmatrix} \Delta_l \\ I \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} \Delta_l \\ I \end{bmatrix} \geq 0, \quad l = 1, ..., n_\Delta. \end{aligned}$$

■

## B.5 Youla Parameterization

This section describes the Youla parameterization of all stabilizing finite-dimensional LTI output-feedback controllers and all stable closed-loop maps, following Zhou *et al.* (1996, Chapter 12); Sanchez-Pena and Sznaier (1998, Chapter 3). This parameterization has been developed in Kucera (1972) and Youla *et al.* (1976), and is also known as the YBJK parameterization or the $Q$-parameterization. We give a discrete-time description, but the same formulas hold in continuous time as well.

**Lemma B.13** *Let a system $G$ with state-space realization*

$$\begin{bmatrix} x(k+1) \\ z_1(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} x(k) \\ w_1(k) \\ u(k) \end{bmatrix}$$

*be given, and let $F$ and $L$ be such that $A + B_2F$ and $A + LC_2$ are asymptotically stable. Then all internally stabilizing finite-dimensional LTI output-feedback controllers $u = Ky$ are given by*

$$\hat{K}(z) = \hat{J}(z) \star \hat{Q}(z), \tag{B.4}$$

*with $\hat{Q} \in \mathcal{RH}_\infty$, $\det(I + D_{22}\hat{Q}(\infty)) \neq 0$, and*

$$\hat{J}(z) = \left[\begin{array}{c|cc} A + B_2F + LC_2 + LD_{22}F & -L & B_2 + LD_{22} \\ \hline F & 0 & I \\ -(C_2 + D_{22}F) & I & -D_{22} \end{array}\right].$$

*Furthermore, all internally stable closed-loop maps $z_1 = \mathcal{G}w_1$ are given by*

$$\hat{\mathcal{G}}(z) = \hat{H}(z) - \hat{U}(z)\hat{Q}(z)\hat{V}(z),$$

*where*

$$\hat{H}(z) = \left[\begin{array}{cc|c} A + B_2F & -B_2F & B_1 \\ 0 & A + LC_2 & B_1 + LD_{21} \\ \hline C_1 + D_{12}F & -D_{12}F & D_{11} \end{array}\right],$$

$$\hat{U}(z) = \left[\begin{array}{c|c} A + B_2F & -B_2 \\ \hline C_1 + D_{12}F & -D_{12} \end{array}\right], \qquad \hat{V}(z) = \left[\begin{array}{c|c} A + LC_2 & B_1 + LD_{21} \\ \hline C_2 & D_{21} \end{array}\right].$$

■

## B.6 Loop-Shifting

The procedure of loop-shifting allows to eliminate the direct-feedthrough term of a plant model for controller design. See for example Sanchez-Pena and Sznaier (1998, Section 3.4.2); Zhou *et al.* (1996, Section 12.3.4, Section 17.2) for details. The same formulas as for the discrete-time case hold in continuous time as well. Consider a plant $G$ with realization

$$\begin{bmatrix} x(k+1) \\ z_1(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} x(k) \\ w_1(k) \\ u(k) \end{bmatrix}.$$

Suppose now that one has designed a controller $\tilde{K}$

$$\begin{bmatrix} x_K(k+1) \\ u(k) \end{bmatrix} = \begin{bmatrix} \tilde{A}_K & \tilde{B}_K \\ \tilde{C}_K & \tilde{D}_K \end{bmatrix} \begin{bmatrix} x(k) \\ \tilde{y}(k) \end{bmatrix}$$

for the plant $\tilde{G}$

$$\begin{bmatrix} x(k+1) \\ z_1(k) \\ \tilde{y}(k) \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ w_1(k) \\ u(k) \end{bmatrix},$$

which is sometimes more convenient than designing a controller for $G$. It can be shown that the controller $K$ that yields the same closed-loop map $w_1 \mapsto z_1$ for $G$ as $\tilde{K}$ does for $\tilde{G}$ is given by the transfer matrix $\hat{K}(z) = (I + \hat{\tilde{K}}(z) D_{22})^{-1} \hat{\tilde{K}}(z)$ or in state-space form as

$$\begin{bmatrix} x_K(k+1) \\ u(k) \end{bmatrix} = \begin{bmatrix} \tilde{A}_K - \tilde{B}_K D_{22} V \tilde{C}_K & \tilde{B}_K - \tilde{B}_K D_{22} V \tilde{D}_K \\ V\tilde{C}_K & V\tilde{D}_K \end{bmatrix} \begin{bmatrix} x(k) \\ y(k) \end{bmatrix},$$

where $V := (I + \tilde{D}_K D_{22})^{-1}$. To understand the construction, note that $\tilde{G}$ is obtained from $G$ by the transformation $\tilde{y} = y - D_{22}u$. The result is obtained by plugging in this transformation into a realization of $u = \tilde{K}\tilde{y}$ and solving for $u$.

## B.7 Examples of Linear Fractional Representations

This section exemplifies possible representations of LPV systems in linear fractional form. More complicated LPV systems can be treated similarly as these simple examples. Further details can be found in Zhou *et al.* (1996, Chapter 10); Scherer and Weiland (2005, Chapter 6); Hecker and Varga (2004). Consider the LPV system

$$\left[\begin{array}{c} x(k+1) \\ \hline z_1(k) \\ \hline y(k) \end{array}\right] = \left[\begin{array}{cc|c|c} -0.1 & 1 & 0 & 0 \\ \rho(k) & 0 & 0 & 1 \\ \hline -1 & 0 & 1 & 0 \\ \hline -1 & 0 & 1 & 0 \end{array}\right] \left[\begin{array}{c} x(k) \\ \hline w_1(k) \\ \hline u(k) \end{array}\right]$$

with state $x(k) \in \mathbb{R}^2$. In the following, all coefficients $\rho_i \in \mathbb{R}$. For the affine parameter dependence $\rho(k) = \rho_0 + \rho_1 \delta(k)$, an LFT representation is given by

$$\begin{bmatrix} x(k+1) \\ \hdashline z_0(k) \\ \hdashline z_1(k) \\ \hdashline y(k) \end{bmatrix} = \left[\begin{array}{cc:c:c:c} -0.1 & 1 & 0 & 0 & 0 \\ \rho_0 & 0 & 1 & 0 & 1 \\ \hdashline \rho_1 & 0 & 0 & 0 & 0 \\ \hdashline -1 & 0 & 0 & 1 & 0 \\ \hdashline -1 & 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} x(k) \\ \hdashline w_0(k) \\ \hdashline w_1(k) \\ \hdashline u(k) \end{bmatrix}, \qquad w_0(k) = \underbrace{\delta(k)}_{\Delta(k)} z_0(k).$$

For the polynomial parameter dependence $\rho(k) = \rho_0 + \rho_1 \delta(k) + \rho_2 \delta(k)^2$, an LFT representation is

$$\begin{bmatrix} x(k+1) \\ \hdashline z_0(k) \\ \hdashline z_1(k) \\ \hdashline y(k) \end{bmatrix} = \left[\begin{array}{cc:cc:c:c} -0.1 & 1 & 0 & 0 & 0 & 0 \\ \rho_0 & 0 & 1 & 0 & 0 & 1 \\ \hdashline \rho_1 & 0 & 0 & 1 & 0 & 0 \\ \rho_2 & 0 & 0 & 0 & 0 & 0 \\ \hdashline -1 & 0 & 0 & 0 & 1 & 0 \\ \hdashline -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} x(k) \\ \hdashline w_0(k) \\ \hdashline w_1(k) \\ \hdashline u(k) \end{bmatrix}, \qquad w_0(k) = \underbrace{\begin{bmatrix} \delta(k) & 0 \\ 0 & \delta(k) \end{bmatrix}}_{\Delta(k)} z_0(k).$$

For the rational parameter dependence $\rho(k) = \frac{3+6\delta(k)}{2-8\delta(k)} = 1.5 + \frac{9\delta(k)}{1-4\delta(k)}$, $\delta(k) \in (-\frac{1}{4}, \frac{1}{4})$, a possible representation is

$$\begin{bmatrix} x(k+1) \\ \hdashline z_0(k) \\ \hdashline z_1(k) \\ \hdashline y(k) \end{bmatrix} = \left[\begin{array}{cc:c:c:c} -0.1 & 1 & 0 & 0 & 0 \\ 1.5 & 0 & 1 & 0 & 1 \\ \hdashline 9 & 0 & 4 & 0 & 0 \\ \hdashline -1 & 0 & 0 & 1 & 0 \\ \hdashline -1 & 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} x(k) \\ \hdashline w_0(k) \\ \hdashline w_1(k) \\ \hdashline u(k) \end{bmatrix}, \qquad w_0(k) = \underbrace{\delta(k)}_{\Delta(k)} z_0(k).$$

## B.8 Stability and Performance Analysis

Some selected results for stability and performance analysis are briefly reviewed next.

**Nominal Star-Norm Performance in Continuous Time**

A sufficient matrix inequality condition for star-norm performance of a continuous-time LTI system is given by the following result.

**Theorem B.14** *(Scherer* et al. *(1997); Scherer and Weiland (2005, Section 3.3))*
*Consider the system $G$ with realization*

$$\begin{bmatrix} \dot{x}(t) \\ z_1(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(t) \\ w_1(t) \end{bmatrix} \tag{B.5}$$

*and $x(0) = 0$. Suppose that there exist $\mu > 0$, $\lambda > 0$, and $X$ satisfying*

$$\begin{bmatrix} A^T X + XA + \lambda X & XB \\ B^T X & -\mu I \end{bmatrix} < 0, \qquad \begin{bmatrix} \lambda X & 0 & C^T \\ 0 & (\gamma - \mu) I & D^T \\ C & D & \gamma I \end{bmatrix} > 0.$$

*Then* $\|z_1\|_{\text{peak}} < \gamma$ *for* $\|w_1\|_{\text{peak}} \leq 1$*, and* $\|G\|_{\text{peak}-\text{ind}} < \gamma$*. Moreover,* $A$ *has all its eigenvalues in the open left half-plane.* ■

The smallest achievable value $\gamma$ satisfying the conditions of Theorem B.14 is called the star-norm of the system $G$.

### Nominal Quadratic Performance in Continuous Time

Quadratic performance is described in Definition 3.1 and 4.9 for the robust case. The corresponding nominal notion is recalled next for completeness.

**Definition B.1** *(Quadratic performance, continuous time)*
*The system (B.5) is said to have quadratic performance with performance index* $P_p = P_p^T$ *if there exists an* $\varepsilon > 0$ *such that*

$$\int_{t=0}^{\infty} \begin{bmatrix} w_1(t) \\ z_1(t) \end{bmatrix}^T P_p \begin{bmatrix} w_1(t) \\ z_1(t) \end{bmatrix} dt \leq -\varepsilon \int_{t=0}^{\infty} w_1(t)^T w_1(t) dt$$

*for every system trajectory* $x(\cdot)$ *with* $x(0) = 0$*.* □

The following result states a necessary and sufficient LMI condition for quadratic performance of a continuous-time LTI system.

**Theorem B.15** *(Scherer and Weiland, 2005)*
*The system (B.5) has quadratic performance with respect to a performance index* $P_p$ *with* $R_p \geq 0$*, and all eigenvalues of* $A$ *are inside the open left half-plane, if and only if there exists* $X > 0$ *satisfying*

$$\left[\begin{array}{cc} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{array}\right]^T \left[\begin{array}{cc|cc} 0 & X & 0 & 0 \\ X & 0 & 0 & 0 \\ \hline 0 & 0 & Q_p & S_p \\ 0 & 0 & S_p^T & R_p \end{array}\right] \left[\begin{array}{cc} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{array}\right] < 0.$$ ■

### Nominal Quadratic Performance in Discrete Time

The discrete-time notion corresponding to Definition B.1 is recalled next.

**Definition B.2** *(Quadratic performance, discrete time)*
*The system (2.1) is said to have quadratic performance with performance index* $P_p = P_p^T$ *if there exists an* $\varepsilon > 0$ *such that*

$$\sum_{k=0}^{\infty} \begin{bmatrix} w_1(k) \\ z_1(k) \end{bmatrix}^T P_p \begin{bmatrix} w_1(k) \\ z_1(k) \end{bmatrix} \leq -\varepsilon \sum_{k=0}^{\infty} w_1(k)^T w_1(k)$$

*for every system trajectory* $x(\cdot)$ *with* $x(0) = 0$*.* □

The following result states a necessary and sufficient LMI condition for quadratic performance of a discrete-time LTI system.

**Theorem B.16** *The system (2.1) has quadratic performance with respect to a performance index $P_p$ with $R_p \geq 0$, and all eigenvalues of $A$ are inside the open unit disk, if and only if there exists $X > 0$ satisfying*

$$\left[\begin{array}{cc} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{array}\right]^T \left[\begin{array}{cc|cc} -X & 0 & 0 & 0 \\ 0 & X & 0 & 0 \\ \hline 0 & 0 & Q_p & S_p \\ 0 & 0 & S_p^T & R_p \end{array}\right] \left[\begin{array}{cc} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{array}\right] < 0. \qquad \blacksquare$$

Note that this is a straightforward analogon to the corresponding continuous-time condition, insofar as the $2 \times 2$ block matrix $\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix}$ of the continuous-time condition is replaced by the $2 \times 2$ block matrix $\begin{bmatrix} -X & 0 \\ 0 & X \end{bmatrix}$. This is a standard replacement to adapt certain continuous-time conditions to the discrete-time setting (Scherer and Weiland, 2005, Section 4.7).

### Uniform Exponential Stability in Continuous Time

The next theorem gives sufficient conditions for robust stability in terms of uniform exponential stability of continuous-time systems, see Definition 4.8.

**Theorem B.17** *(Scherer, 2000*b*)*
*The uncertain system (4.22), obtained from the interconnection (4.20)–(4.21), is uniformly exponentially stable with respect to $\Delta \in \mathbf{\Delta}_{\mathrm{TV}}$ if any one of the following equivalent statements holds.*

*(i) (4.20)–(4.21) is well-posed and there exists an $X > 0$ satisfying*

$$\bar{\mathcal{A}}(\Delta)^T X + X\bar{\mathcal{A}}(\Delta) = \begin{bmatrix} I \\ \bar{\mathcal{A}}(\Delta) \end{bmatrix}^T \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \begin{bmatrix} I \\ \bar{\mathcal{A}}(\Delta) \end{bmatrix} < 0 \qquad \forall \Delta \in \mathbf{\Delta}.$$

*(ii) There exist $X > 0$ and $Q = Q^T$, $R = R^T$, $S$ satisfying*

$$\begin{bmatrix} \Delta \\ I \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} \Delta \\ I \end{bmatrix} \geq 0 \qquad \forall \Delta \in \mathbf{\Delta},$$

$$\left[\begin{array}{cc} I & 0 \\ \mathcal{A} & \mathcal{B}_0 \\ \hline 0 & I \\ \mathcal{C}_0 & \mathcal{D}_{00} \end{array}\right]^T \left[\begin{array}{cc|cc} 0 & X & 0 & 0 \\ X & 0 & 0 & 0 \\ \hline 0 & 0 & Q & S \\ 0 & 0 & S^T & R \end{array}\right] \left[\begin{array}{cc} I & 0 \\ \mathcal{A} & \mathcal{B}_0 \\ \hline 0 & I \\ \mathcal{C}_0 & \mathcal{D}_{00} \end{array}\right] < 0. \qquad \blacksquare$$

**Uniform Exponential Stability in Discrete Time**

Similarly, a robust stability condition of discrete-time systems is stated.

**Theorem B.18** *The uncertain system (2.11), obtained from the interconnection (2.9)–(2.10), is uniformly exponentially stable with respect to $\Delta \in \mathbf{\Delta}_{\mathrm{TV}}$ if any one of the following equivalent statements holds.*

*(i) (2.9)–(2.10) is well-posed and there exists an $X > 0$ satisfying*

$$\bar{\mathcal{A}}(\Delta)^T X \bar{\mathcal{A}}(\Delta) - X = \begin{bmatrix} I \\ \bar{\mathcal{A}}(\Delta) \end{bmatrix}^T \begin{bmatrix} -X & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} I \\ \bar{\mathcal{A}}(\Delta) \end{bmatrix} < 0 \qquad \forall \Delta \in \mathbf{\Delta}.$$

*(ii) There exist $X > 0$ and $Q = Q^T$, $R = R^T$, $S$ satisfying*

$$\begin{bmatrix} \Delta \\ I \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} \Delta \\ I \end{bmatrix} \geq 0 \qquad \forall \Delta \in \mathbf{\Delta},$$

$$\begin{bmatrix} I & 0 \\ \mathcal{A} & \mathcal{B}_0 \\ \hline 0 & I \\ \mathcal{C}_0 & \mathcal{D}_{00} \end{bmatrix}^T \left[\begin{array}{cc|cc} -X & 0 & 0 & 0 \\ 0 & X & 0 & 0 \\ \hline 0 & 0 & Q & S \\ 0 & 0 & S^T & R \end{array}\right] \begin{bmatrix} I & 0 \\ \mathcal{A} & \mathcal{B}_0 \\ \hline 0 & I \\ \mathcal{C}_0 & \mathcal{D}_{00} \end{bmatrix} < 0.$$

Proof: We only give a sketch of the proof. The equivalence of (i) and (ii) follows by the Full-Block S-Procedure (Lemma B.11) similarly to the proof of Theorem 2.3. Uniform exponential stability is obtained using an "integrating" factor, also as in the proof of Theorem 2.3. ■

**Robust Quadratic Performance in Continuous Time**

An equivalent set of sufficient conditions for robust quadratic performance of continuous-time systems is described next.

**Theorem B.19** *(Scherer, 2000*b*)*
*The following two statements are equivalent:*

*(i) (4.20)–(4.21) is well posed and there exists an $X > 0$ satisfying*

$$\begin{bmatrix} I & 0 \\ \bar{\mathcal{A}}(\Delta) & \bar{\mathcal{B}}_1(\Delta) \\ \hline 0 & I \\ \bar{\mathcal{C}}_1(\Delta) & \bar{\mathcal{D}}_{11}(\Delta) \end{bmatrix}^T \left[\begin{array}{cc|cc} 0 & X & 0 & 0 \\ X & 0 & 0 & 0 \\ \hline 0 & 0 & Q_p & S_p \\ 0 & 0 & S_p^T & R_p \end{array}\right] \begin{bmatrix} I & 0 \\ \bar{\mathcal{A}}(\Delta) & \bar{\mathcal{B}}_1(\Delta) \\ \hline 0 & I \\ \bar{\mathcal{C}}_1(\Delta) & \bar{\mathcal{D}}_{11}(\Delta) \end{bmatrix} < 0 \qquad \forall \Delta \in \mathbf{\Delta}. \tag{B.6}$$

*(ii) There exist $X > 0$ and $Q = Q^T$, $R = R^T$, $S$ satisfying*

$$\begin{bmatrix} \Delta \\ I \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} \Delta \\ I \end{bmatrix} \geq 0 \qquad \forall \Delta \in \mathbf{\Delta}, \tag{B.7}$$

$$\left[\begin{array}{ccc} I & 0 & 0 \\ \mathcal{A} & \mathcal{B}_0 & \mathcal{B}_1 \\ \hdashline 0 & I & 0 \\ \mathcal{C}_0 & \mathcal{D}_{00} & \mathcal{D}_{01} \\ \hdashline 0 & 0 & I \\ \mathcal{C}_1 & \mathcal{D}_{10} & \mathcal{D}_{11} \end{array}\right]^T \left[\begin{array}{cc:cc:cc} 0 & X & 0 & 0 & 0 & 0 \\ X & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & Q & S & 0 & 0 \\ 0 & 0 & S^T & R & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & Q_p & S_p \\ 0 & 0 & 0 & 0 & S_p^T & R_p \end{array}\right] \left[\begin{array}{ccc} I & 0 & 0 \\ \mathcal{A} & \mathcal{B}_0 & \mathcal{B}_1 \\ \hdashline 0 & I & 0 \\ \mathcal{C}_0 & \mathcal{D}_{00} & \mathcal{D}_{01} \\ \hdashline 0 & 0 & I \\ \mathcal{C}_1 & \mathcal{D}_{10} & \mathcal{D}_{11} \end{array}\right] < 0. \tag{B.8}$$

*Moreover, if either (i) or (ii) holds, then (4.22), obtained from the interconnection (4.20)–(4.21), is uniformly exponentially stable and has robust quadratic performance with performance index* $P_p = \begin{bmatrix} Q_p & S_p \\ S_p^T & R_p \end{bmatrix}$ *with respect to* $\Delta \in \mathbf{\Delta}_{\mathrm{TV}}$. ■

In Scherer (2000*b*), a collection of similar continuous-time formulations for robust $\mathcal{H}_2$ performance, robust generalized $\mathcal{H}_2$ performance, and robust star-norm performance in continuous time is given. A corresponding discrete-time condition of robust quadratic performance is given by substituting the $2 \times 2$ block matrix $\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix}$ with $\begin{bmatrix} -X & 0 \\ 0 & X \end{bmatrix}$.

**Robust Stability and $\ell_\infty$-Gain Performance with respect to $\ell_\infty$-Bounded Uncertainties**

Necessary and sufficient conditions for robust stability of discrete-time systems with respect to structured nonlinear or LTV dynamic uncertainties with bounded $\ell_\infty$-gain are derived in Khammash and Pearson (1991); Khammash and Pearson (1993); Dahleh and Khammash (1993), applying Definition 4.7. The setup is depicted in Figure 2.1(b). The next theorem summarizes these results for the particular case of SISO uncertainty blocks in $\Delta$. Define the uncertainty class as

$$\mathbf{\Delta}_{\mathrm{NB1}} := \left\{ \Delta = \mathrm{diag}(\Delta_1, ..., \Delta_n) \mid \Delta_i : \ell_\infty^1 \to \ell_\infty^1 \text{ causal nonlinear or LTV and } \|\Delta_i\|_{\infty-\mathrm{ind}} \leq 1 \right\}.$$

For the map $\mathcal{M} \in \ell_1^{n \times n}$, define $\tilde{\mathcal{M}} := \begin{bmatrix} \|\mathcal{M}_{11}\|_1 & \cdots & \|\mathcal{M}_{1n}\|_1 \\ \vdots & & \vdots \\ \|\mathcal{M}_{n1}\|_1 & \cdots & \|\mathcal{M}_{nn}\|_1 \end{bmatrix}$.

**Theorem B.20** *(Dahleh and Khammash (1993); Dahleh and Diaz-Bobillo (1995, Proposition 7.3.1))*
*Consider a map $\mathcal{M} \in \ell_1^{n \times n}$ and uncertainties $\Delta \in \mathbf{\Delta}_{\mathrm{NB1}}$. The system in Figure 2.1(b) possesses robust $\ell_\infty$ stability if and only if any one of the following equivalent conditions holds.*

*(i)* $\rho(\tilde{\mathcal{M}}) < 1$.

*(ii)* $x \leq \tilde{\mathcal{M}}x$ *and* $x \geq 0$ *imply* $x = 0$*, where the inequalities are to be interpreted component-wise.*

*(iii)* $\inf_{E \in \boldsymbol{E}} \|E^{-1}\mathcal{M}E\|_1 < 1$*, where* $\boldsymbol{E} := \{\mathrm{diag}(E_1, ..., E_n) \,|\, 0 < E_i \in \mathbb{R}\}$. ■

Note that the above conditions include both nonlinear and LTV dynamic uncertainties. The result can be generalized to MIMO uncertainty blocks $\Delta_i$ as follows. Define the uncertainty class as

$$\boldsymbol{\Delta}_{\mathrm{NB2}} := \{\Delta = \mathrm{diag}(\Delta_1, ..., \Delta_n) \,|\, \Delta_i : \ell_\infty^{m_i} \to \ell_\infty^{m_i} \text{ causal nonlinear or LTV and } \|\Delta_i\|_{\infty-\mathrm{ind}} \leq 1\}$$

with $m = \sum_{i=1}^n m_i$. Partition the map $\mathcal{M} \in \ell_1^{m \times m}$ as $\mathcal{M} = \begin{bmatrix} \mathcal{M}_{11} & \cdots & \mathcal{M}_{1n} \\ \vdots & & \vdots \\ \mathcal{M}_{n1} & \cdots & \mathcal{M}_{nn} \end{bmatrix}$, where $\mathcal{M}_{ij} \in \ell_1^{m_i \times m_j}$. Let the set $\Lambda$ be an index set for all possible collections of rows from the row blocks $[\mathcal{M}_{i1} \,...\, \mathcal{M}_{in}]$, and for each $\lambda = [\lambda_1, ..., \lambda_n] \in \Lambda$, define

$$\tilde{\mathcal{M}}_\lambda := \begin{bmatrix} \|(\mathcal{M}_{11})_{\lambda_1:}\|_1 & \cdots & \|(\mathcal{M}_{1n})_{\lambda_1:}\|_1 \\ \vdots & & \vdots \\ \|(\mathcal{M}_{n1})_{\lambda_n:}\|_1 & \cdots & \|(\mathcal{M}_{nn})_{\lambda_n:}\|_1 \end{bmatrix},$$

where $(\mathcal{M}_{ij})_{\lambda_l:}$ is the $\lambda_l^{\mathrm{th}}$ row of the block matrix $\mathcal{M}_{ij}$.

**Theorem B.21** *(Dahleh and Diaz-Bobillo, 1995, Theorem 7.3.3)*
*Consider a map* $\mathcal{M} \in \ell_1^{m \times m}$*, uncertainties* $\Delta \in \boldsymbol{\Delta}_{\mathrm{NB2}}$*, and an index set* $\Lambda$ *as described above. The system in Figure 2.1(b) possesses robust* $\ell_\infty$ *stability if and only if any one of the following equivalent conditions holds.*

*(i)* $\rho(\tilde{\mathcal{M}}_\lambda) < 1$ *for all* $\lambda \in \Lambda$.

*(ii)* $\|[E_1\mathcal{M}_{i1} \,...\, E_n\mathcal{M}_{in}]\|_1 < E_i$ *for some* $0 < E_i \in \mathbb{R}$*, for all* $i = 1, ..., n$.

*(iii)* $\inf_{E \in \boldsymbol{E}} \|E^{-1}\mathcal{M}E\|_1 < 1$*, where* $\boldsymbol{E} := \{\mathrm{diag}(E_1 I_{m_1}, ..., E_n I_{m_n}) \,|\, 0 < E_i \in \mathbb{R}\}$*, and* $I_{m_i}$ *is the identity matrix of dimension* $m_i$. ■

For the case of LTI uncertainties, the conditions are only sufficient. Yet if one restricts the uncertainty class to SISO LTI blocks, the well-known structured singular value condition from $\mu$ analysis is exact, see for example Packard and Doyle (1993); Dahleh and Diaz-Bobillo (1995, Chapter 7); Zhou *et al.* (1996, Chapter 11). For MIMO LTI blocks, just an upper bound can be obtained from the $\mu$ condition. Parametric uncertainties are just addressed for the restrictive class of rank-one problems, that is the case where $\mathcal{M}$ can be reduced to the dimension $n \times 1$ or $1 \times n$ (Dahleh and Diaz-Bobillo, 1995).

Finally, the robust $\ell_1$ performance analysis problem can be converted into a robust stability problem by connecting a virtual uncertainty block to the performance channel as shown in Khammash and Pearson (1991); Dahleh and Khammash (1993). A concise overview of conditions related to different classes of $\ell_\infty$- and $\ell_2$-bounded uncertainties is given in Dahleh and Khammash (1993); Dahleh and Diaz-Bobillo (1995, Chapter 7) together with remarks on computational complexity.

# Appendix C

# Performance Analysis Using Parameter-Dependent Lyapunov Functions

This appendix states an extension of Theorem 2.3 to the case of a parameter-dependent Lyapunov function. A certain form of parameter-dependent Lyapunov function candidates is described by

$$V(x(k), \bar{\Delta}(k)) = x(k)^T X(\bar{\Delta}(k)) x(k), \tag{C.1}$$

where

$$X(\bar{\Delta}(k)) = (\bar{\Delta}(k) \star Y)^T Y_0 (\bar{\Delta}(k) \star Y) > 0, \quad Y = \left[\begin{array}{c:c} Y_1 & Y_2 \\ \hdashline Y_3 & Y_4 \end{array}\right], \tag{C.2}$$

and the function $\bar{\Delta}(k)$ is chosen to be dependent on the entries of $\Delta(k)$ in some pre-defined fashion. The dimensions of $\bar{\Delta}(k)$ and of $\Delta(k)$ may be different. See below for some examples. Denote the uncertainty set for $\bar{\Delta}$ corresponding to $\boldsymbol{\Delta}$ by $\bar{\boldsymbol{\Delta}}$.

The following theorem states two equivalent sufficient conditions for robust star-norm performance of discrete-time linear systems with time-varying parametric uncertainties. The proof is omitted for brevity.

**Theorem C.1** *Consider the system (2.11), obtained from the interconnection (2.9)–(2.10), with $\xi(0) = 0$, $\Delta \in \boldsymbol{\Delta}_{\mathrm{TV}}$. The following two statements are equivalent:*

*(i) (2.9)–(2.10) is well-posed and there exist $\mu > 0$, $0 < \lambda < 1$, and $Y_0 = Y_0^T$ satisfying for all $\Delta \in \boldsymbol{\Delta}$, $\bar{\Delta}$, $\bar{\Delta}^+ \in \bar{\boldsymbol{\Delta}}$*

$$[*]^T \left[\begin{array}{cc:c} -\lambda X(\bar{\Delta}) & 0 & 0 \\ 0 & X(\bar{\Delta}^+) & 0 \\ \hdashline 0 & 0 & -\mu I \end{array}\right] \left[\begin{array}{cc} I & 0 \\ \bar{\mathcal{A}}(\Delta) & \bar{\mathcal{B}}_1(\Delta) \\ \hdashline 0 & I \end{array}\right] < 0,$$

$$[*]^T \left[\begin{array}{c:cc} (\lambda-1) X(\bar{\Delta}) & 0 & 0 \\ \hdashline 0 & (\mu-\gamma^2) I & 0 \\ 0 & 0 & I \end{array}\right] \left[\begin{array}{cc} I & 0 \\ \hdashline 0 & I \\ \bar{\mathcal{C}}_1(\Delta) & \bar{\mathcal{D}}_{11}(\Delta) \end{array}\right] < 0.$$

*(ii) There exist* $\mu > 0$, $0 < \lambda < 1$, $Y_0 = Y_0^T$, $Y$, *and* $P_i = P_i^T$ *satisfying for all* $\Delta \in \mathbf{\Delta}$, $\bar{\Delta}$, $\bar{\Delta}^+ \in \bar{\mathbf{\Delta}}$

$$[*]^T \underbrace{\left[\begin{array}{c:c} Q_1 & S_1 \\ \hdashline S_1^T & R_1 \end{array}\right]}_{P_1} \left[\begin{array}{ccc} \bar{\Delta}^+ & 0 & 0 \\ 0 & \bar{\Delta} & 0 \\ 0 & 0 & \Delta \\ \hdashline I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{array}\right] \geq 0, \qquad [*]^T \underbrace{\left[\begin{array}{c:c} Q_2 & S_2 \\ \hdashline S_2^T & R_2 \end{array}\right]}_{P_2} \left[\begin{array}{cc} \bar{\Delta} & 0 \\ 0 & \Delta \\ \hdashline I & 0 \\ 0 & I \end{array}\right] \geq 0,$$

$$[*]^T \left[\begin{array}{cc:c:c:c} -\lambda Y_0 & 0 & 0 & 0 & 0 \\ 0 & Y_0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & Q_1 & S_1 & 0 \\ \hdashline 0 & 0 & S_1^T & R_1 & 0 \\ \hdashline 0 & 0 & 0 & 0 & -\mu I \end{array}\right] \left[\begin{array}{ccccc} 0 & Y_3 & Y_4 & 0 & 0 \\ Y_3 & 0 & Y_4\mathcal{A} & Y_4\mathcal{B}_0 & Y_4\mathcal{B}_1 \\ \hdashline I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ \hdashline Y_1 & 0 & Y_2\mathcal{A} & Y_2\mathcal{B}_0 & Y_2\mathcal{B}_1 \\ 0 & Y_1 & Y_2 & 0 & 0 \\ 0 & 0 & \mathcal{C}_0 & \mathcal{D}_{00} & \mathcal{D}_{01} \\ \hdashline 0 & 0 & 0 & 0 & I \end{array}\right] < 0,$$

$$[*]^T \left[\begin{array}{c:c:c:cc} (\lambda-1)Y_0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & Q_1 & S_1 & 0 & 0 \\ \hdashline 0 & S_1^T & R_1 & 0 & 0 \\ \hdashline 0 & 0 & 0 & (\mu-\gamma^2)I & 0 \\ 0 & 0 & 0 & 0 & I \end{array}\right] \left[\begin{array}{cccc} Y_3 & Y_4 & 0 & 0 \\ \hdashline I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ \hdashline Y_1 & Y_2 & 0 & 0 \\ 0 & \mathcal{C}_0 & \mathcal{D}_{00} & \mathcal{D}_{01} \\ \hdashline 0 & 0 & 0 & I \\ 0 & \mathcal{C}_1 & \mathcal{D}_{10} & \mathcal{D}_{11} \end{array}\right] < 0.$$

*Moreover, if either (i) or (ii) holds, then*

- $\|z_1\|_{\text{peak}} < \gamma$ *for* $\|w_1\|_{\text{peak}} \leq 1$, *and moreover* $\|\bar{\mathcal{G}}(\Delta)\|_{\text{peak-ind}} < \gamma$ *for all* $\Delta \in \mathbf{\Delta}_{\text{TV}}$,
- *(2.11) is uniformly exponentially stable with respect to* $\Delta \in \mathbf{\Delta}_{\text{TV}}$. ∎

Rate bounds can be included in this result by adding the restriction $(\bar{\Delta}^+ - \bar{\Delta})_{ij} \leq d_{ij}$ according to the definition of the set $\mathbf{\Delta}_{\text{RB}}$. This is in contrast to Theorem 2.3, where no difference between the sets $\mathbf{\Delta}_{\text{TV}}$ and $\mathbf{\Delta}_{\text{RB}}$ can be made. See Section 2.5.3 and Figure 2.2 for more details on rate bounds. A pure stability condition can be obtained by discarding rows and columns related to the performance channel $w_1 \mapsto z_1$. Conditions for checking robust quadratic performance are obtained by analogously extending Theorem B.19 or its discrete-time counter-part.

**Examples of Parameter-Dependent Lyapunov Functions**

For the special form (C.1)–(C.2) of parameter-dependent Lyapunov function candidates, we give some exemplary choices for the parameter dependence of $X(\bar{\Delta}(k))$, see also Wu and Dong (2006). The case of constant $X$ is contained as a special case in all examples. More efficient formulations with smaller $\bar{\Delta}$-blocks may be possible in a particular application. The dependence on $k$ is dropped. An example for affine parameter dependence in the case of two scalar parameters $\Delta = \mathrm{diag}(\delta_1, \delta_2)$ is

$$X(\bar{\Delta}) = M_0 + \delta_1(M_1 + M_1^T) + \delta_2(M_2 + M_2^T)$$

for

$$\bar{\Delta} = \begin{bmatrix} \delta_1 I & 0 \\ 0 & \delta_2 I \end{bmatrix}, \qquad Y_0 = \begin{bmatrix} 0 & 0 & M_1 \\ 0 & 0 & M_2 \\ M_1^T & M_2^T & M_0 \end{bmatrix} = Y_0^T, \qquad Y = \left[\begin{array}{cc:c} 0 & 0 & I \\ 0 & 0 & I \\ \hdashline I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{array}\right],$$

where the $M_i$ are free variables. The size of the $M_i$ is the same as the size of $X(\bar{\Delta})$. Similar to the affine parameter dependence above, a quadratic dependence is obtained as

$$X(\bar{\Delta}) = M_0 + \delta_1(M_1 + M_1^T) + \delta_2(M_2 + M_2^T) + \delta_1\delta_2(M_{12} + M_{12}^T) + \delta_1^2 M_{11} + \delta_2^2 M_{22}$$

for

$$\bar{\Delta} = \begin{bmatrix} \delta_1 I & 0 \\ 0 & \delta_2 I \end{bmatrix}, \qquad Y_0 = \begin{bmatrix} M_{11} & M_{12} & M_1 \\ M_{12}^T & M_{22} & M_2 \\ M_1^T & M_2^T & M_0 \end{bmatrix} = Y_0^T, \qquad Y = \left[\begin{array}{cc:c} 0 & 0 & I \\ 0 & 0 & I \\ \hdashline I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{array}\right].$$

The 3rd-order polynomial dependence

$$X(\bar{\Delta}) = M_0 + \delta_1(M_1 + M_1^T) + \delta_1^2(M_2 + M_2^T) + \delta_1^3(M_3 + M_3^T)$$

for one scalar parameter $\Delta = \delta_1$ is achieved by using

$$\bar{\Delta} = \begin{bmatrix} \delta_1 I & 0 & 0 \\ 0 & \delta_1 I & 0 \\ 0 & 0 & \delta_1 I \end{bmatrix}, \quad Y_0 = \begin{bmatrix} 0 & 0 & 0 & M_1 \\ 0 & 0 & 0 & M_2 \\ 0 & 0 & 0 & M_3 \\ M_1^T & M_2^T & M_3^T & M_0 \end{bmatrix} = Y_0^T, \quad Y = \left[\begin{array}{ccc:c} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ \hdashline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{array}\right].$$

A certain rational dependence as given in

$$X(\Delta) = M_0 + M_1^T\delta_1(I - \delta_1 Y_1)^{-1} + \delta_1(I - \delta_1 Y_1)^{-T}M_1 + \delta_1(I - \delta_1 Y_1)^{-T}M_2\delta_1(I - \delta_1 Y_1)^{-1}$$

for one scalar parameter $\Delta = \delta_1$ is achieved by using

$$\bar{\Delta} = \delta_1 I, \qquad Y_0 = \begin{bmatrix} M_2 & M_1 \\ M_1^T & M_0 \end{bmatrix}, \qquad Y = \left[\begin{array}{c:c} Y_1 & I \\ \hdashline I & 0 \\ 0 & I \end{array}\right].$$

for some fixed $Y_1$.

# Appendix D

# Proofs

## D.1 Proof of Theorem 2.1

Multiplying (2.3) by $\mathrm{col}(\xi(k), w_1(k))^T$ and $\mathrm{col}(\xi(k), w_1(k))$ from left and right, respectively, and using the system equations (2.1) leads to

$$\xi(k+1)^T X \xi(k+1) - \lambda \xi(k)^T X \xi(k) < \mu w_1(k)^T w_1(k) \quad \forall k, \tag{D.1}$$

which implies

$$\xi(k)^T X \xi(k) \le \frac{\mu}{1-\lambda} \quad \forall k. \tag{D.2}$$

This can be seen as follows. Define $V(k) := \xi(k)^T X \xi(k)$ and $W(k) := \lambda^{-k+1} V(k)$. After multiplying (D.1) with $\lambda^{-k} > 0$, we have $W(k+1) - W(k) < \lambda^{-k} \mu w_1(k)^T w_1(k)$ $\forall k$. Using this relation together with $\xi(0) = 0$, $W(0) = 0$, and $w_1(k)^T w_1(k) \le 1$ $\forall k$, we have from $W(k) - W(0) = \sum_{l=0}^{k-1} (W(l+1) - W(l))$ that $W(k) < \mu \sum_{l=0}^{k-1} \lambda^{-l}$ $\forall k$. Hence, using $0 < \lambda < 1$, it holds that $V(k) = \lambda^{k-1} W(k) < \mu \sum_{l=0}^{k-1} \lambda^{k-1-l} = \mu \sum_{l=0}^{k-1} \lambda^l \le \frac{\mu}{1-\lambda}$ $\forall k$, which is (D.2).
Furthermore, by continuity there exists an $\epsilon > 0$ such that from (2.4) we have

$$\begin{bmatrix} (\lambda-1)X + \mathcal{C}^T\mathcal{C} & \mathcal{C}^T\mathcal{D} \\ \mathcal{D}^T\mathcal{C} & (\mu-\gamma^2)I + \mathcal{D}^T\mathcal{D} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \epsilon I \end{bmatrix} \le 0.$$

Multiplying this inequality by $\mathrm{col}(\xi(k), w_1(k))^T$ and $\mathrm{col}(\xi(k), w_1(k))$ from left and right, respectively, and applying the system equations (2.1) as well as $w_1(k)^T w_1(k) \le 1$ $\forall k$ leads to

$$z_1(k)^T z_1(k) \le (1-\lambda)\xi(k)^T X \xi(k) + (\gamma^2 - \mu) - \epsilon \quad \forall k. \tag{D.3}$$

(D.3) together with (D.2) implies $z_1(k)^T z_1(k) \le \gamma^2 - \epsilon < \gamma^2$ for some $\epsilon > 0$ for all $k$. Hence, $\|\mathcal{G}\|_{\mathrm{peak-ind}} \le \sqrt{\gamma^2 - \epsilon} < \gamma$, which proves the first part of the result. The upper bound $\|\mathcal{G}\|_1 < \gamma\sqrt{q_1}$ follows right away from $\|\mathcal{G}\|_1 \le \sqrt{q_1}\|\mathcal{G}\|_{\mathrm{peak-ind}}$.
Stability of $\mathcal{A}$ can be seen as follows. The (1,1)-element of (2.4) together with $\lambda - 1 < 0$ implies $(\lambda-1)X \le (\lambda-1)X + \mathcal{C}^T\mathcal{C} < 0$ and thus $X > 0$. From (2.3) it hence follows that $\mathcal{A}^T X \mathcal{A} - X <$

$\mathcal{A}^T X \mathcal{A} - X + (1-\lambda)X = \mathcal{A}^T X \mathcal{A} - \lambda X < 0$. By virtue of the quadratic Lyapunov function $U(\xi(k)) = \xi(k)^T X \xi(k) > 0$ with variation $U(\xi(k+1)) - U(\xi(k)) = \xi(k)^T(\mathcal{A}^T X \mathcal{A} - X)\xi(k) < 0$, we have that $\mathcal{A}$ is Schur stable, i.e. all eigenvalues of $\mathcal{A}$ are in the open unit disk. This concludes the proof.

## D.2 Proof of Theorem 2.3

The proof demonstrates the application of the Full-Block S-Procedure (Lemma B.11) and follows along similar lines as the arguments in Scherer (2000*b*); Scherer and Weiland (2005, Section 6.6). First, we show the equivalence between (i) and (ii). The inequality (2.13) can be rewritten as

$$[*]^T \left[\begin{array}{cc:c} -\lambda X & 0 & 0 \\ 0 & -\mu I & 0 \\ \hdashline 0 & 0 & X \end{array}\right] \left[\begin{array}{cc} I & 0 \\ 0 & I \\ \hdashline \multicolumn{2}{c}{\Delta \star \left[\begin{array}{c:cc} \mathcal{D}_{00} & \mathcal{C}_0 & \mathcal{D}_{01} \\ \hdashline \mathcal{B}_0 & \mathcal{A} & \mathcal{B}_1 \end{array}\right]} \end{array}\right] < 0 \qquad \forall \Delta \in \mathbf{\Delta}$$

by using (2.12) and some row and column permutations. According to Lemma B.11, this inequality and well-posedness of the involved LFT are equivalent to the existence of a multiplier $P_1$ satisfying (2.16) and the first condition in (2.15). To this end recall that $\mathbf{\Delta}$ is compact. The exact form of (2.16) is obtained after some row and column permutations again. Analogously, (2.14) is equivalent to (2.17) and the second part of (2.15). Thus, (i) and (ii) are equivalent.

It remains to show that (i) implies the three statements at the end of the theorem. First, observe that (2.13) and (2.14) are equivalent to

$$\begin{bmatrix} \bar{\mathcal{A}}(\Delta)^T X \bar{\mathcal{A}}(\Delta) - \lambda X & \bar{\mathcal{A}}(\Delta)^T X \bar{\mathcal{B}}_1(\Delta) \\ \bar{\mathcal{B}}_1(\Delta)^T X \bar{\mathcal{A}}(\Delta) & \bar{\mathcal{B}}_1(\Delta)^T X \bar{\mathcal{B}}_1(\Delta) - \mu I \end{bmatrix} < 0, \tag{D.4}$$

$$\begin{bmatrix} (\lambda-1)X + \bar{\mathcal{C}}_1(\Delta)^T \bar{\mathcal{C}}_1(\Delta) & \bar{\mathcal{C}}_1(\Delta)^T \bar{\mathcal{D}}_{11}(\Delta) \\ \bar{\mathcal{D}}_{11}(\Delta)^T \bar{\mathcal{C}}_1(\Delta) & (\mu - \gamma^2)I + \bar{\mathcal{D}}_{11}(\Delta)^T \bar{\mathcal{D}}_{11}(\Delta) \end{bmatrix} < 0 \qquad \forall \Delta \in \mathbf{\Delta}. \tag{D.5}$$

By using basically the same arguments as in the proof of Theorem 2.1, it follows that $z_1(k)^T z_1(k) < \gamma^2$ for all $w_1(k)^T w_1(k) \leq 1$, for all $\Delta(k) \in \mathbf{\Delta}$, and for all $k$. Also $\|\bar{\mathcal{G}}(\Delta)\|_{\text{peak-ind}} < \gamma$. Thus the first item of the theorem holds. The upper bound $\|\bar{\mathcal{G}}(\Delta)\|_{\infty-\text{ind}} < \gamma\sqrt{q_1}$ follows right away from $\|\bar{\mathcal{G}}(\Delta)\|_{\infty-\text{ind}} \leq \sqrt{q_1}\|\bar{\mathcal{G}}(\Delta)\|_{\text{peak-ind}}$.

Uniform exponential stability is shown next. Similar proofs with slightly different arguments are available e.g. in Scherer (2000*b*, Theorem 10.1) and Wu and Dong (2006, Theorem 5). Define $V(k) = \xi(k)^T X \xi(k)$ and $W(k) = \delta^{-k+1}V(k)$. From the (1,1)-element of (D.5) we have that $X > 0$ since $\lambda - 1 < 0$. From the (1,1)-element of (D.4) it follows that $\bar{\mathcal{A}}(\Delta)^T X \bar{\mathcal{A}}(\Delta) - X < \bar{\mathcal{A}}(\Delta)^T X \bar{\mathcal{A}}(\Delta) - X + (1-\lambda)X = \bar{\mathcal{A}}(\Delta)^T X \bar{\mathcal{A}}(\Delta) - \lambda X < 0$ for all $\Delta \in \mathbf{\Delta}$. By continuity and compactness of $\mathbf{\Delta}$ it follows that there exists $0 < \epsilon < 1$ such that $\bar{\mathcal{A}}(\Delta)^T X \bar{\mathcal{A}}(\Delta) - X + \epsilon X \leq 0$ for all $\Delta \in \mathbf{\Delta}$. Multiplying with $\xi(k)^T$ and $\xi(k)$ from left and right leads to $V(k+1) - V(k) + \epsilon V(k) =$

$V(k+1) - \delta V(k) \le 0$ for all $k \ge 0$ with $0 < \delta := 1 - \epsilon < 1$, where $\Delta(\cdot)$ is any admissible uncertainty sequence and $\xi(\cdot)$ is an arbitrary unforced system trajectory. Multiplying with the "integrating factor" $\delta^{-k}$ results in $W(k+1) - W(k) \le 0$ for all $k \ge 0$. With the identity $W(k) - W(k_0) = \sum_{l=k_0}^{k-1}(W(k+1) - W(k))$ for some $k_0$ such that $k \ge k_0 \ge 0$, we have that $W(k) \le W(k_0)$ and thus $V(k) \le V(k_0)\delta^{k-k_0}$ for all $k \ge k_0 \ge 0$. Due to the relation $\lambda_{\min}(X)\xi(k)^T\xi(k) \le V(k) \le \lambda_{\max}(X)\xi(k)^T\xi(k)$ (Khalil, 2002, Chapter 4), it follows with the norm $\|v\| = \sqrt{v^T v}$ that

$$\|\xi(k)\|^2 \le \frac{V(k)}{\lambda_{\min}(X)} \le \frac{V(k_0)\delta^{k-k_0}}{\lambda_{\min}(X)} \le \frac{\lambda_{\max}(X)}{\lambda_{\min}(X)}\delta^{k-k_0}\|\xi(k_0)\|^2 \quad \forall k \ge k_0 \ge 0$$

and hence

$$\|\xi(k)\| \le \sqrt{\frac{\lambda_{\max}(X)}{\lambda_{\min}(X)}}\left(\sqrt{\delta}\right)^{k-k_0}\|\xi(k_0)\| \quad \forall k \ge k_0 \ge 0,$$

which proves uniform exponential stability of (2.11) according to Definition 2.1 with explicit expressions for $\alpha$ and $\beta$. ■

## D.3 Proof of Theorem 2.9

(a) All constraints used in the computation of $\tilde{\gamma}^N$ are also present in the computation of $\tilde{\gamma}^{N+1}$, thus $\tilde{\gamma}^{N+1}$ cannot be smaller than $\tilde{\gamma}^N$.

(b) If the matrix inequalities for computing $\tilde{\eta}^N$ are feasible, it is inferred that there exists $X > 0$ such that $\bar{\mathcal{A}}(\Delta)^T X \bar{\mathcal{A}}(\Delta) - X < 0 \ \forall \Delta \in \mathbf{\Delta}$ (as in the proof of Theorem 2.1). Factorizing $X = Y^T Y$ with some square and non-singular $Y$, it follows from the previous inequality that $\|Y\bar{\mathcal{A}}(\Delta)Y^{-1}\| < 1 \ \forall \Delta \in \mathbf{\Delta}$ ($\|\cdot\|$ is the spectral matrix norm). Since the LFT is well-posed, it follows by continuity and compactness that there exists some $\nu < 1$ with $\|Y\bar{\mathcal{A}}(\Delta)Y^{-1}\| \le \nu$ $\forall \Delta \in \mathbf{\Delta}$. Therefore

$$\|\bar{\mathcal{A}}(\Delta)^k\| = \|Y^{-1}(Y\bar{\mathcal{A}}(\Delta)Y^{-1})^k Y\| \le \|Y\|\|Y^{-1}\|\nu^k \qquad \forall \Delta \in \mathbf{\Delta}$$

and, using the geometric series,

$$\sum_{k=N}^{\infty}\|\bar{\mathcal{A}}(\Delta)^k\| \le \sum_{k=N}^{\infty}\|Y\|\|Y^{-1}\|\nu^k = \|Y\|\|Y^{-1}\|\frac{\nu^N}{1-\nu} \quad \forall \Delta \in \mathbf{\Delta}.$$

Thus we have $\sum_{k=N}^{\infty}\|\bar{\mathcal{A}}(\Delta)^k\| \to 0$ for $N \to \infty$ uniformly for $\Delta \in \mathbf{\Delta}$, and

$$\gamma^{r,N} = \sum_{k=N+1}^{\infty}|\bar{\mathcal{C}}_1(\Delta)\bar{\mathcal{A}}(\Delta)^{k-1}\bar{\mathcal{B}}_1(\Delta)| \le \|\bar{\mathcal{C}}_1(\Delta)\|\|\bar{\mathcal{B}}_1(\Delta)\|\sum_{k=N}^{\infty}\|\bar{\mathcal{A}}(\Delta)^k\| \to 0$$

for $N \to \infty$.

It is sketched now how to show $\tilde{\gamma}^{r,N} \to 0$. Suppose again that the LMIs for computing $\tilde{\eta}^N$ are feasible for some $X$. Now consider (2.49)–(2.50) for some $M \geq N$ instead of $N$, some $\eta$ instead of $\eta^N$, fixed $\lambda$, and fixed $X > 0$. Taking Schur complements (Lemma B.5) leads to

$$\bar{\mathcal{B}}_{r,M}^T(X - X\bar{\mathcal{A}}_{r,M}(\bar{\mathcal{A}}_{r,M}^T X\bar{\mathcal{A}}_{r,M} - \lambda X)^{-1}\bar{\mathcal{A}}_{r,M}^T X)\bar{\mathcal{B}}_{r,M} - \mu I < 0, \tag{D.6}$$

$$(\eta^2 - \mu)I - \bar{\mathcal{D}}_{r,M}^T\left(\begin{bmatrix} 0 & I \end{bmatrix}\begin{bmatrix} (1-\lambda)X & \bar{\mathcal{C}}_{r,M}^T \\ \bar{\mathcal{C}}_{r,M} & I \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ I \end{bmatrix}\right)\bar{\mathcal{D}}_{r,M} > 0, \tag{D.7}$$

where the dependence on $\Delta$ has been dropped for brevity. Observe that $\bar{\mathcal{A}}_{r,M}$ and $\bar{\mathcal{C}}_{r,M}$ do not depend on $M$. Moreover $\bar{\mathcal{B}}_{r,M}$ and $\bar{\mathcal{D}}_{r,M}$ converge to zero for $M \to \infty$ (as $\bar{\mathcal{A}}(\Delta)^M$ does). For arbitrarily small $\eta > 0$, one can choose $M_0$ and a small $\mu$ such that (D.6)–(D.7) are satisfied for all $M \geq M_0$. This shows $\tilde{\gamma}^{r,N} = \sqrt{q_1}\tilde{\eta}^N \to 0$ for $N \to \infty$.

(c) Because of $\gamma^N \leq \tilde{\gamma}^N$ and $\gamma^{r,N} \leq \tilde{\gamma}^{r,N}$, we have that $\gamma \leq \gamma^N + \gamma^{r,N} \leq \tilde{\gamma}^N + \tilde{\gamma}^{r,N}$. ■

## D.4 Proof of Theorem 3.5

We first show the equivalence between (i) and (ii). To this end, observe that the chain of equivalences

$$X > 0, \qquad \left[\begin{array}{cc} I & 0 \\ \bar{\mathcal{A}}(\Delta) & \bar{\mathcal{B}}_1(\Delta) \\ \hline 0 & I \\ \bar{\mathcal{C}}_1(\Delta) & \bar{\mathcal{D}}_{11}(\Delta) \end{array}\right]^T \left[\begin{array}{cc|cc} -X & 0 & 0 & 0 \\ 0 & X & 0 & 0 \\ \hline 0 & 0 & Q_p & S_p \\ 0 & 0 & S_p^T & R_p \end{array}\right] \left[\begin{array}{cc} I & 0 \\ \bar{\mathcal{A}}(\Delta) & \bar{\mathcal{B}}_1(\Delta) \\ \hline 0 & I \\ \bar{\mathcal{C}}_1(\Delta) & \bar{\mathcal{D}}_{11}(\Delta) \end{array}\right] < 0$$

if and only if

$$X > 0,$$
$$\begin{bmatrix} -X & 0 \\ 0 & 0 \end{bmatrix} + [*]^T X^{-1}\begin{bmatrix} X\bar{\mathcal{A}}(\Delta) & X\bar{\mathcal{B}}_1(\Delta) \end{bmatrix} + [*]^T\begin{bmatrix} Q_p & S_p \\ S_p^T & R_p \end{bmatrix}\begin{bmatrix} 0 & I \\ \bar{\mathcal{C}}_1(\Delta) & \bar{\mathcal{D}}_{11}(\Delta) \end{bmatrix} < 0$$

if and only if (by Schur Lemma, Lemma B.5)

$$\begin{bmatrix} -X & 0 & \bar{\mathcal{A}}(\Delta)^T X \\ 0 & 0 & \bar{\mathcal{B}}_1(\Delta)^T X \\ X\bar{\mathcal{A}}(\Delta) & X\bar{\mathcal{B}}_1(\Delta) & -X \end{bmatrix} + [*]^T\begin{bmatrix} Q_p & S_p \\ S_p^T & R_p \end{bmatrix}\begin{bmatrix} 0 & I & 0 \\ \bar{\mathcal{C}}_1(\Delta) & \bar{\mathcal{D}}_{11}(\Delta) & 0 \end{bmatrix} < 0$$

holds. Together with the identity

$$\begin{bmatrix} -X & 0 & \bar{\mathcal{A}}(\Delta)^T X \\ 0 & 0 & \bar{\mathcal{B}}_1(\Delta)^T X \\ X\bar{\mathcal{A}}(\Delta) & X\bar{\mathcal{B}}_1(\Delta) & -X \end{bmatrix} = [*]^T\begin{bmatrix} -X & 0 & 0 \\ 0 & -X & X \\ 0 & X & 0 \end{bmatrix}\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ \bar{\mathcal{A}}(\Delta) & \bar{\mathcal{B}}_1(\Delta) & 0 \end{bmatrix},$$

we have that (3.17) is equivalent to

$$[*]^T \left[\begin{array}{ccc|cc} -X & 0 & 0 & 0 & 0 \\ 0 & -X & X & 0 & 0 \\ 0 & X & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & Q_p & S_p \\ 0 & 0 & 0 & S_p^T & R_p \end{array}\right] \left[\begin{array}{ccc} I & 0 & 0 \\ 0 & 0 & I \\ \bar{\mathcal{A}}(\Delta) & \bar{\mathcal{B}}_1(\Delta) & 0 \\ \hline 0 & I & 0 \\ \bar{\mathcal{C}}_1(\Delta) & \bar{\mathcal{D}}_{11}(\Delta) & 0 \end{array}\right] < 0 \quad \forall \Delta \in \mathbf{\Delta}.$$

Using (3.16) and the Full-Block S-Procedure (Lemma B.11), the last condition is equivalent to

$$\begin{bmatrix} \Delta \\ I \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} \Delta \\ I \end{bmatrix} \geq 0 \qquad \forall \Delta \in \mathbf{\Delta}, \tag{D.8}$$

$$[*]^T \left[\begin{array}{ccc|cc|cc} -X & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -X & X & 0 & 0 & 0 & 0 \\ 0 & X & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & Q & S & 0 & 0 \\ 0 & 0 & 0 & S^T & R & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & Q_p & S_p \\ 0 & 0 & 0 & 0 & 0 & S_p^T & R_p \end{array}\right] \left[\begin{array}{cccc} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ A + B_2 F & B_1 & 0 & B_0 \\ \hline 0 & 0 & 0 & I \\ C_0 + D_{02} F & D_{01} & 0 & D_{00} \\ \hline 0 & I & 0 & 0 \\ C_1 + D_{12} F & D_{11} & 0 & D_{10} \end{array}\right] < 0. \tag{D.9}$$

(D.9) is equivalent to

$$\begin{bmatrix} -X & 0 & (A + B_2 F)^T X & 0 \\ * & 0 & B_1^T X & 0 \\ * & * & -X & X B_0 \\ * & * & * & 0 \end{bmatrix} + [*]^T \left[\begin{array}{cc|cc} Q & S & 0 & 0 \\ S^T & R & 0 & 0 \\ \hline 0 & 0 & Q_p & S_p \\ 0 & 0 & S_p^T & R_p \end{array}\right] \left[\begin{array}{cccc} 0 & 0 & 0 & I \\ C_0 + D_{02} F & D_{01} & 0 & D_{00} \\ \hline 0 & I & 0 & 0 \\ C_1 + D_{12} F & D_{11} & 0 & D_{10} \end{array}\right] < 0.$$

Define $Y := X^{-1}$ and $M := F X^{-1}$. Multiplying the last inequality with the symmetric matrix $\mathrm{diag}(X^{-1}, I, X^{-1}, I)$ from left and right leads to

$$\begin{bmatrix} -Y & 0 & (AY + B_2 M)^T & 0 \\ * & 0 & B_1^T & 0 \\ * & * & -Y & B_0 \\ * & * & * & 0 \end{bmatrix} + [*]^T \left[\begin{array}{cc|cc} Q & S & 0 & 0 \\ S^T & R & 0 & 0 \\ \hline 0 & 0 & Q_p & S_p \\ 0 & 0 & S_p^T & R_p \end{array}\right] \left[\begin{array}{cccc} 0 & 0 & 0 & I \\ C_0 Y + D_{02} M & D_{01} & 0 & D_{00} \\ \hline 0 & I & 0 & 0 \\ C_1 Y + D_{12} M & D_{11} & 0 & D_{10} \end{array}\right] < 0,$$

which, after some row/column permutations, is equivalent to

$$[*]^T \left[\begin{array}{cccc|ccc} -Y & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -Y & 0 & I & 0 & 0 & 0 \\ 0 & 0 & Q & 0 & 0 & S & 0 \\ 0 & 0 & 0 & Q_p & 0 & 0 & S_p \\ \hline 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & S^T & 0 & 0 & R & 0 \\ 0 & 0 & 0 & S_p^T & 0 & 0 & R_p \end{array}\right] \left[\begin{array}{cccc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ \hline AY + B_2 M & 0 & B_0 & B_1 \\ C_0 Y + D_{02} M & 0 & D_{00} & D_{01} \\ C_1 Y + D_{12} M & 0 & D_{10} & D_{11} \end{array}\right] < 0. \tag{D.10}$$

Applying the Dualization Lemma (Lemma B.7), (D.8), (D.10), and $R \geq 0$ are equivalent to (3.18), $\tilde{Q} \leq 0$, and

$$[*]^T \begin{bmatrix} -Y^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & \tilde{Q} & 0 & 0 & \tilde{S} & 0 \\ 0 & 0 & 0 & \tilde{Q}_p & 0 & 0 & \tilde{S}_p \\ 0 & I & 0 & 0 & Y & 0 & 0 \\ 0 & 0 & \tilde{S}^T & 0 & 0 & \tilde{R} & 0 \\ 0 & 0 & 0 & \tilde{S}_p^T & 0 & 0 & \tilde{R}_p \end{bmatrix} \begin{bmatrix} (AY + B_2M)^T & (C_0Y + D_{02}M)^T & (C_1Y + D_{12}M)^T \\ 0 & 0 & 0 \\ B_0^T & D_{00}^T & D_{10}^T \\ B_1^T & D_{01}^T & D_{11}^T \\ -I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix} > 0,$$

where

$$\begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{bmatrix} := \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}^{-1}, \qquad \begin{bmatrix} \tilde{Q}_p & \tilde{S}_p \\ \tilde{S}_p^T & \tilde{R}_p \end{bmatrix} := \begin{bmatrix} Q_p & S_p \\ S_p^T & R_p \end{bmatrix}^{-1}.$$

Using the zero row in the outer factor, the second row and column of the inner factor are eliminated. Applying the Schur Lemma (Lemma B.5) with respect to the element $-Y^{-1}$ of the inner factor and carrying out the matrix multiplications finally leads to (3.19). Hence the equivalence between (i) and (ii) is shown. Since all steps can be traced backwards, the gain $F = MY^{-1}$ satisfies (3.17). Uniform exponential stability follows from (i) using the same arguments as in the proof of Theorem 2.3, whereas robust quadratic performance follows from Theorem B.19. This concludes the proof.

## D.5 Derivation of Formulas in Defs. 4.1, 4.2, 4.3, 4.4, and 4.5

The controller formulas of the mentioned definitions are derived in this section.

**Definition 4.1**

The equations associated with plant $G_{\text{aug}}$, uncertainty $\Delta$, and auxiliary controller $K$, which are involved in reconstructing the LPV controller $\bar{K}(\Delta)$ from $K$, are stated here explicitly for reference in the derivation:

$$z_0 = G_{00}w_0 + G_{01}w_1 + G_{02}u, \tag{D.11}$$

$$z_1 = G_{10}w_0 + G_{11}w_1 + G_{12}u, \tag{D.12}$$

$$y = G_{20}w_0 + G_{21}w_1 + G_{22}u, \tag{D.13}$$

$$w_0 = \Delta z_0, \tag{D.14}$$

$$u = K_1y + K_2w_0. \tag{D.15}$$

Suppose that $G_{21}$ is a left-invertible map. From (D.13) it hence follows that

$$w_1 = G_{21}^{\dagger}(y - G_{20}w_0 - G_{22}u). \tag{D.16}$$

Next use (D.14) and successively insert (D.11), (D.16), (D.15) to obtain

$$\begin{aligned}
w_0 &= \Delta z_0 \\
&= \Delta(G_{00}w_0 + G_{01}w_1 + G_{02}u) \\
&= \Delta\Big((\underbrace{G_{00} - G_{01}G_{21}^{\dagger}G_{20}}_{\tilde{G}_{00}})w_0 + G_{01}G_{21}^{\dagger}y + (\underbrace{G_{02} - G_{01}G_{21}^{\dagger}G_{22}}_{\tilde{G}_{02}})u\Big) \\
&= \Delta\Big((\underbrace{G_{01}G_{21}^{\dagger} + \tilde{G}_{02}K_1}_{\breve{G}_{02}})y + (\tilde{G}_{00} + \tilde{G}_{02}K_2)w_0\Big).
\end{aligned}$$

According to assumption 2, $I - \Delta(\tilde{G}_{00} + \tilde{G}_{02}K_2)$ is an invertible map. Then the last equation can be solved for $w_0$ to yield

$$w_0 = \underbrace{\Big(I - \Delta(\tilde{G}_{00} + \tilde{G}_{02}K_2)\Big)^{-1}\Delta}_{\breve{G}(\Delta)}\breve{G}_{02}y. \tag{D.17}$$

Finally insert (D.17) into (D.15) to obtain

$$u = \Big(K_1 + K_2\breve{G}(\Delta)\breve{G}_{02}\Big)\,y,$$

which is an expression for the controller $\bar{K}(\Delta)$. The LFT factorization follows by inspection. This completes the derivation of the given formula.

**Definition 4.2**

The state-space version follows along similar lines, but is slightly more involved. The equations (4.11), (4.10), (4.3) are stated here explicitly for reference in the derivation:

$$x(k+1) = Ax(k) + B_0w_0(k) + B_1w_1(k) + B_2u(k), \tag{D.18}$$
$$z_0(k) = C_0x(k) + D_{00}w_0(k) + D_{01}w_1(k) + D_{02}u(k), \tag{D.19}$$
$$z_1(k) = C_1x(k) + D_{10}w_0(k) + D_{11}w_1(k) + D_{12}u(k), \tag{D.20}$$
$$y(k) = C_2x(k) + D_{20}w_0(k) + D_{21}w_1(k) + D_{22}u(k), \tag{D.21}$$
$$w_0(k) = \Delta(k)z_0(k), \tag{D.22}$$
$$x_K(k+1) = A_Kx_K(k) + B_{K1}y(k) + B_{K2}w_0(k), \tag{D.23}$$
$$u(k) = C_Kx_K(k) + D_{K1}y(k) + D_{K2}w_0(k). \tag{D.24}$$

In the following, the argument $k$ is dropped for brevity. Suppose that $n_y \geq q_1$ and that $D_{21}$ has full column rank. This implies that $D_{21}$ is left-invertible (see Lemma B.3). From (D.21) it hence follows that

$$w_1 = D_{21}^{\dagger}(y - C_2x - D_{20}w_0 - D_{22}u). \tag{D.25}$$

Next use (D.22) and successively insert (D.19), (D.25), (D.24) to obtain

$$\begin{aligned}
w_0 &= \Delta z_0 \\
&= \Delta(C_0 x + D_{00} w_0 + D_{01} w_1 + D_{02} u) \\
&= \Delta\Big( (\underbrace{C_0 - D_{01} D_{21}^{\dagger} C_2}_{\tilde{C}_0}) x + (\underbrace{D_{00} - D_{01} D_{21}^{\dagger} D_{20}}_{\tilde{D}_{00}}) w_0 + D_{01} D_{21}^{\dagger} y + (\underbrace{D_{02} - D_{01} D_{21}^{\dagger} D_{22}}_{\tilde{D}_{02}}) u \Big) \\
&= \Delta\Big( \tilde{C}_0 x + \tilde{D}_{02} C_K x_K + (\underbrace{D_{01} D_{21}^{\dagger} + \tilde{D}_{02} D_{K1}}_{\breve{D}_{02}}) y + (\tilde{D}_{00} + \tilde{D}_{02} D_{K2}) w_0 \Big).
\end{aligned}$$

According to assumption 2, $I - \Delta(\tilde{D}_{00} + \tilde{D}_{02} D_{K2})$ is non-singular for all $\Delta(k) \in \boldsymbol{\Delta}$. Then the last equation can be solved for $w_0$ to yield

$$w_0 = \underbrace{\Big( I - \Delta(\tilde{D}_{00} + \tilde{D}_{02} D_{K2}) \Big)^{-1} \Delta}_{\breve{D}(\Delta)} \Big( \tilde{C}_0 x + \tilde{D}_{02} C_K x_K + \breve{D}_{02} y \Big) . \tag{D.26}$$

Next use (D.18) and successively insert (D.25), (D.24), (D.26) to obtain

$$\begin{aligned}
x(k+1) &= (\underbrace{A - B_1 D_{21}^{\dagger} C_2}_{\tilde{A}}) x + (\underbrace{B_0 - B_1 D_{21}^{\dagger} D_{20}}_{\tilde{B}_0}) w_0 + B_1 D_{21}^{\dagger} y + (\underbrace{B_2 - B_1 D_{21}^{\dagger} D_{22}}_{\tilde{B}_2}) u \\
&= \tilde{A} x + \tilde{B}_2 C_K x_K + (\underbrace{B_1 D_{21}^{\dagger} + \tilde{B}_2 D_{K1}}_{\breve{B}_2}) y + (\underbrace{\tilde{B}_0 + \tilde{B}_2 D_{K2}}_{\breve{B}}) w_0 \\
&= \Big( \tilde{A} + \breve{B} \breve{D}(\Delta) \tilde{C}_0 \Big) x + \Big( \tilde{B}_2 + \breve{B} \breve{D}(\Delta) \tilde{D}_{02} \Big) C_K x_K + \Big( \breve{B}_2 + \breve{B} \breve{D}(\Delta) \breve{D}_{02} \Big) y.
\end{aligned}$$

Insert (D.26) into (D.23) to get

$$x_K(k+1) = B_{K2} \breve{D}(\Delta) \tilde{C}_0 x + \Big( A_K + B_{K2} \breve{D}(\Delta) \tilde{D}_{02} C_K \Big) x_K + \Big( B_{K1} + B_{K2} \breve{D}(\Delta) \breve{D}_{02} \Big) y.$$

Finally insert (D.26) into (D.24) to obtain

$$u = D_{K2} \breve{D}(\Delta) \tilde{C}_0 x + \Big( I + D_{K2} \breve{D}(\Delta) \tilde{D}_{02} \Big) C_K x_K + \Big( D_{K1} + D_{K2} \breve{D}(\Delta) \breve{D}_{02} \Big) y.$$

These expressions are the ones in Definition 4.2.

**Definition 4.3**

Using $G_{01} \equiv 0$, (D.17) reduces to

$$w_0 = \underbrace{(I - \Delta(G_{00} + G_{02} K_2))^{-1} \Delta}_{\tilde{G}(\Delta)} G_{02} K_1 y, \tag{D.27}$$

provided that the inverse exists. Inserting (D.27) into (D.15) yields

$$u = \Big( I + K_2 \tilde{G}(\Delta) G_{02} \Big) K_1 y.$$

The LFT factorization follows by inspection. This completes the derivation of the given formula.

**Definition 4.4**

The state-space version is shown next. $G_{01} \equiv 0$ implies $D_{01} = 0$, hence (D.26) reduces to

$$w_0 = \underbrace{(I - \Delta(D_{00} + D_{02}D_{K2}))^{-1}\Delta}_{\tilde{D}(\Delta)}(C_0 x + D_{02}C_K x_K + D_{02}D_{K1}y), \tag{D.28}$$

provided that the inverse exists. Inserting (D.28) into (D.24) yields

$$u = D_{K2}\tilde{D}(\Delta)C_0 x + \Big(I + D_{K2}\tilde{D}(\Delta)D_{02}\Big) C_K x_K + \Big(I + D_{K2}\tilde{D}(\Delta)D_{02}\Big) D_{K1}y.$$

Note that this equation is dependent on $x$ only via the matrix $C_0$. It follows that we can drop the term $B_1 w_1$ in (D.18), since $G_{01} \equiv 0$ (together with $x(0) = 0$) means that all modes of $G_{01}$ are unreachable or unobservable or both. Hence from (D.18), where (D.24), (D.28) are inserted subsequently, one has

$$\begin{aligned} x(k+1) &= Ax + B_0 w_0 + B_2 u \\ &= Ax + B_2 C_K x_k + B_2 D_{K1} y + (\underbrace{B_0 + B_2 D_{K2}}_{\tilde{B}})w_0 \\ &= \Big(A + \tilde{B}\tilde{D}(\Delta)C_0\Big) x + \Big(B_2 + \tilde{B}\tilde{D}(\Delta)D_{02}\Big) C_K x_K + \Big(B_2 + \tilde{B}\tilde{D}(\Delta)D_{02}\Big) D_{K1}y. \end{aligned}$$

Finally insert (D.28) into (D.23) to obtain

$$x_K(k+1) = B_{K2}\tilde{D}(\Delta)C_0 x + \Big(A_K + B_{K2}\tilde{D}(\Delta)D_{02}C_K\Big) x_K + \Big(B_{K1} + B_{K2}\tilde{D}(\Delta)D_{02}D_{K1}\Big) y.$$

This completes the derivation.

**Definition 4.5**

If additionally $C_0 = 0$, the dependence on $x$ vanishes and one has the state-space realization in Definition 4.5.

## D.6 Proof of Theorems 4.1 and 4.2

First, the proof of Theorem 4.1 is given. The proof of Theorem 4.2 follows by simplifications.

**Proof of Theorem 4.1**

For the closed-loop system of Figure 4.2(b) (with controller $K$) it holds that

$$G_{\text{aug}} \star K = \begin{bmatrix} G_{00} + G_{02}\Pi(K_1 G_{20} + K_2) & G_{01} + G_{02}\Pi K_1 G_{21} \\ G_{10} + G_{12}\Pi(K_1 G_{20} + K_2) & G_{11} + G_{12}\Pi K_1 G_{21} \end{bmatrix} \tag{D.29}$$

and

$$\begin{aligned} \Delta \star G_{\text{aug}} \star K = {} & G_{11} + G_{12}\Pi K_1 G_{21} + \big(G_{10} + G_{12}\Pi(K_1 G_{20} + K_2)\big)\cdot \\ & \Big(I - \Delta\left(G_{00} + G_{02}\Pi(K_1 G_{20} + K_2)\right)\Big)^{-1}\Delta\big(G_{01} + G_{02}\Pi K_1 G_{21}\big), \end{aligned}$$

where $\Pi = (I - K_1 G_{22})^{-1}$. Since the interconnection of Figure 4.2(b) is assumed to be robustly $\ell_\infty$-stable, it follows that $G_{\text{aug}} \star K$ is causal and has a bounded $\ell_\infty$-norm, see Definition 4.7. All involved mappings and uncertainties are causal and linear. Moreover it follows that $(I - \Delta M_1)^{-1}$ with

$$M_1 := G_{00} + G_{02}\Pi(K_1 G_{20} + K_2)$$

from the upper left block in (D.29) is $\ell_\infty$-stable for all $\Delta \in \mathbf{\Delta}_{\text{NB}}$ and thus also for all $\Delta \in \mathbf{\Delta}_{\text{TV}}$. On the other hand, for the closed-loop system of Figure 4.1(a) (with controller $\bar{K}(\Delta)$) it holds that

$$\Delta \star G \star \bar{K}(\Delta) = \underbrace{\begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}}_{\tilde{\Delta}} \star \underbrace{\left[\begin{array}{cc:c:cc} 0 & 0 & 0 & 0 & I \\ 0 & G_{00} & G_{01} & G_{02} & 0 \\ \hdashline 0 & G_{10} & G_{11} & G_{12} & 0 \\ \hdashline 0 & G_{20} & G_{21} & G_{22} & 0 \\ I & 0 & 0 & 0 & 0 \end{array}\right]}_{\tilde{G}} \star \underbrace{\begin{bmatrix} K_1 & K_2 \\ G_{01}G_{21}^\dagger + \tilde{G}_{02}K_1 & \tilde{G}_{00} + \tilde{G}_{02}K_2 \end{bmatrix}}_{\tilde{K}}$$

and

$$\begin{aligned} \tilde{G} \star \tilde{K} &= \begin{bmatrix} K_3 G_{22}\Pi K_2 + K_4 & K_3(I + G_{22}\Pi K_1)G_{20} & K_3(I + G_{22}\Pi K_1)G_{21} \\ G_{02}\Pi K_2 & G_{00} + G_{02}\Pi K_1 G_{20} & G_{01} + G_{02}\Pi K_1 G_{21} \\ G_{12}\Pi K_2 & G_{10} + G_{12}\Pi K_1 G_{20} & G_{11} + G_{12}\Pi K_1 G_{21} \end{bmatrix} \\ &= \begin{bmatrix} G_{00} + G_{02}\Pi K_2 - \alpha & G_{02}\Pi K_1 G_{20} + \alpha & G_{01} + G_{02}\Pi K_1 G_{21} \\ G_{02}\Pi K_2 & G_{00} + G_{02}\Pi K_1 G_{20} & G_{01} + G_{02}\Pi K_1 G_{21} \\ G_{12}\Pi K_2 & G_{10} + G_{12}\Pi K_1 G_{20} & G_{11} + G_{12}\Pi K_1 G_{21} \end{bmatrix}, \end{aligned} \tag{D.30}$$

where

$$\begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix} := \tilde{K}, \qquad \alpha := G_{01}G_{21}^\dagger G_{20}.$$

Since $\alpha$ and $G_{\text{aug}} \star K$ are causal and have bounded $\ell_\infty$-norm, the same holds true for $\tilde{G} \star \tilde{K}$. The extended uncertainty $\tilde{\Delta}$ is $\ell_\infty$-stable just as $\Delta$. To prove robust $\ell_\infty$-stability of the interconnection of Figure 4.1(a), it remains to show that $(I - \tilde{\Delta} M_2)^{-1}$ is $\ell_\infty$-stable for all $\Delta \in \mathbf{\Delta}_{\text{NB}}$, where

$$M_2 := \begin{bmatrix} G_{00} + G_{02}\Pi K_2 - \alpha & G_{02}\Pi K_1 G_{20} + \alpha \\ G_{02}\Pi K_2 & G_{00} + G_{02}\Pi K_1 G_{20} \end{bmatrix}$$

from the upper left $2 \times 2$ block in (D.30). To this end, note that

$$\begin{aligned} &\left( \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} G_{00} + G_{02}\Pi K_2 - \alpha & G_{02}\Pi K_1 G_{20} + \alpha \\ G_{02}\Pi K_2 & G_{00} + G_{02}\Pi K_1 G_{20} \end{bmatrix} \right) \underbrace{\begin{bmatrix} I & I \\ I & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ I & -I \end{bmatrix}}_{I} \\ &= \begin{bmatrix} I - \Delta\big(G_{00} + G_{02}\Pi(K_1 G_{20} + K_2)\big) & I - \Delta\big(G_{00} + G_{02}\Pi K_2 - \alpha\big) \\ I - \Delta\big(G_{00} + G_{02}\Pi(K_1 G_{20} + K_2)\big) & -\Delta G_{02}\Pi K_2 \end{bmatrix} \begin{bmatrix} 0 & I \\ I & -I \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} I & 0 \\ I & I \end{bmatrix} \begin{bmatrix} I - \Delta\big(G_{00} + G_{02}\Pi(K_1G_{20} + K_2)\big) & 0 \\ 0 & -\big(I - \Delta\tilde{G}_{00}\big) \end{bmatrix} \cdot$$

$$\begin{bmatrix} I & \Big(I - \Delta\big(G_{00} + G_{02}\Pi(K_1G_{20} + K_2)\big)\Big)^{-1}\big(I - \Delta(G_{00} + G_{02}\Pi K_2 - \alpha)\big) \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & I \\ I & -I \end{bmatrix},$$

using Lemma B.2 for the last equality. Finally,

$$\begin{aligned} &\big(I - \tilde{\Delta}M_2\big)^{-1} \qquad\qquad\qquad\qquad \text{(D.31)} \\ &= \begin{bmatrix} I & I \\ I & 0 \end{bmatrix} \begin{bmatrix} I & -\Big(I - \Delta\big(G_{00} + G_{02}\Pi(K_1G_{20} + K_2)\big)\Big)^{-1}\big(I - \Delta(G_{00} + G_{02}\Pi K_2 - \alpha)\big) \\ 0 & I \end{bmatrix} \cdot \\ &\quad \begin{bmatrix} \Big(I - \Delta\big(G_{00} + G_{02}\Pi(K_1G_{20} + K_2)\big)\Big)^{-1} & 0 \\ 0 & -\big(I - \Delta\tilde{G}_{00}\big)^{-1} \end{bmatrix} \begin{bmatrix} 0 & I \\ I & -I \end{bmatrix} \end{aligned}$$

is a causal mapping with bounded $\ell_\infty$-norm, since $(I - \Delta M_1)^{-1}$ and $(I - \Delta\tilde{G}_{00})^{-1}$ are $\ell_\infty$-stable for all $\Delta \in \mathbf{\Delta}_{\text{NB}}$. Hence, robust $\ell_\infty$-stability of $(I - \tilde{\Delta}M_2)^{-1}$ is concluded.

Equality of the closed-loop maps $\Delta \star G_{\text{aug}} \star K$ and $\Delta \star G \star \bar{K}(\Delta)$ is shown next. The construction of $\bar{K}(\Delta)$ as in Definition 4.1 amounts to just re-arranging the structure of Figure 4.2(b) into the one in Figure 4.1(a). An algebraic proof follows. From (D.30), we have

$$\begin{aligned} \Delta \star G \star \bar{K}(\Delta) &= \tilde{\Delta} \star \tilde{G} \star \tilde{K} \\ &= G_{11} + G_{12}\Pi K_1 G_{21} + \begin{bmatrix} G_{12}\Pi K_2 & G_{10} + G_{12}\Pi K_1 G_{20} \end{bmatrix} \cdot \qquad \text{(D.32)} \\ &\quad \underbrace{\begin{bmatrix} I - \Delta(G_{00} + G_{02}\Pi K_2 - \alpha) & -\Delta(G_{02}\Pi K_1 G_{20} + \alpha) \\ -\Delta G_{02}\Pi K_2 & I - \Delta(G_{00} + G_{02}\Pi K_1 G_{20}) \end{bmatrix}}_{\Gamma}^{-1} \cdot \\ &\quad \begin{bmatrix} I \\ I \end{bmatrix} \Delta(G_{01} + G_{02}\Pi K_1 G_{21}). \end{aligned}$$

The inverse of $\Gamma$ exists, see condition (D.31) above. Using $\begin{bmatrix} A & B \\ A & D \end{bmatrix}^{-1} \begin{bmatrix} I \\ I \end{bmatrix} = \begin{bmatrix} A^{-1} \\ 0 \end{bmatrix}$, the equalities

$$\begin{aligned} \Gamma^{-1} \begin{bmatrix} I \\ I \end{bmatrix} &= \Bigg(\Gamma \underbrace{\begin{bmatrix} I & I \\ I & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ I & -I \end{bmatrix}}_{I}\Bigg)^{-1} \begin{bmatrix} I \\ I \end{bmatrix} \\ &= \begin{bmatrix} 0 & I \\ I & -I \end{bmatrix}^{-1} \begin{bmatrix} I - \Delta\big(G_{00} + G_{02}\Pi(K_1G_{20} + K_2)\big) & I - \Delta(G_{00} + G_{02}\Pi K_2 - \alpha) \\ I - \Delta\big(G_{00} + G_{02}\Pi(K_1G_{20} + K_2)\big) & -\Delta G_{02}\Pi K_2 \end{bmatrix}^{-1} \begin{bmatrix} I \\ I \end{bmatrix} \\ &= \begin{bmatrix} I & I \\ I & 0 \end{bmatrix} \begin{bmatrix} \Big(I - \Delta\big(G_{00} + G_{02}\Pi(K_1G_{20} + K_2)\big)\Big)^{-1} \\ 0 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} I \\ I \end{bmatrix} \Big(I - \Delta\big(G_{00} + G_{02}\Pi(K_1 G_{20} + K_2)\big)\Big)^{-1}$$

hold. Substituting the last expression into (D.32), it follows that

$$\Delta \star G \star \bar{K}(\Delta) = \Delta \star G_{\mathrm{aug}} \star K,$$

which proves the equivalence of the closed-loop maps. This concludes the proof of Theorem 4.1.

**Proof of Theorem 4.2**

In difference to Theorem 4.1, with the assumptions of Theorem 4.2 we now have

$$M_2 := \begin{bmatrix} G_{00} + G_{02}\Pi K_2 & G_{02}\Pi K_1 G_{20} \\ G_{02}\Pi K_2 & G_{00} + G_{02}\Pi K_1 G_{20} \end{bmatrix}$$

and

$$\begin{aligned} &\big(I - \tilde{\Delta} M_2\big)^{-1} \\ &= \begin{bmatrix} I & I \\ I & 0 \end{bmatrix} \begin{bmatrix} I & -\Big(I - \Delta\big(G_{00} + G_{02}\Pi(K_1 G_{20} + K_2)\big)\Big)^{-1}\big(I - \Delta(G_{00} + G_{02}\Pi K_2)\big) \\ 0 & I \end{bmatrix} \cdot \\ &\quad \begin{bmatrix} \Big(I - \Delta\big(G_{00} + G_{02}\Pi(K_1 G_{20} + K_2)\big)\Big)^{-1} & 0 \\ 0 & -\big(I - \Delta G_{00}\big)^{-1} \end{bmatrix} \begin{bmatrix} 0 & I \\ I & -I \end{bmatrix}. \end{aligned}$$

Thus $(I - \tilde{\Delta} M_2)^{-1}$ is a causal mapping with bounded $\ell_\infty$-norm, and robust $\ell_\infty$-stability is concluded. Equality of the closed-loop maps follows in the same way as before, where simplifications due to $G_{01} \equiv 0$ are applicable. This concludes the proof of Theorem 4.2.

## D.7 Proof of Theorem 4.3

The proof is divided into two parts. First, it is shown that (i) implies the existence of a controller guaranteeing uniform exponential stability and robust quadratic performance. Second, the equivalence between (i) and (ii) is shown. The proof uses ideas from Scherer *et al.* (1997); Scherer (2000*b*); Scherer and Weiland (2005). The derivation of the robust synthesis inequalities is based on the analysis inequalities for robust quadratic performance, applying a congruence transformation of the involved inequalities, a nonlinear transformation of the controller parameters, and the Elimination Lemma. The special control structure of the problem is exploited to this end.

To address the first part of the proof, observe that the interconnection of the augmented plant (4.23) and the controller (4.25) is given by (4.20), where

$$\left[\begin{array}{c:c:c} \mathcal{A} & \mathcal{B}_0 & \mathcal{B}_1 \\ \hdashline \mathcal{C}_0 & \mathcal{D}_{00} & \mathcal{D}_{01} \\ \hdashline \mathcal{C}_1 & \mathcal{D}_{10} & D_{11} \end{array}\right] = \left[\begin{array}{cc:c:c} A & 0 & B_0 & B_1 \\ 0 & 0 & 0 & 0 \\ \hdashline C_0 & 0 & D_{00} & D_{01} \\ \hdashline C_1 & 0 & D_{10} & D_{11} \end{array}\right] + \left[\begin{array}{cc} 0 & B_2 \\ I & 0 \\ \hdashline 0 & D_{02} \\ \hdashline 0 & D_{12} \end{array}\right] \begin{bmatrix} A_K & B_{K1} & B_{K2} \\ C_K & D_{K1} & D_{K2} \end{bmatrix} \left[\begin{array}{cc:c:c} 0 & I & 0 & 0 \\ C_2 & 0 & D_{20} & D_{21} \\ 0 & 0 & I & 0 \end{array}\right].$$

Then the closed-loop system is the interconnection of (4.20) and (4.21), or (4.22). As stated in Theorem B.19, sufficient conditions for uniform exponential stability and robust quadratic performance of (4.22) are the existence of $\mathcal{X} > 0$ and $Q$, $R$, $S$ satisfying (B.7) and (B.8). (B.7) is included as condition (4.28). (B.8) can be rewritten as

$$\begin{bmatrix} I & 0 & 0 \\ \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B}_0 & \mathcal{X}\mathcal{B}_1 \\ \hline 0 & I & 0 \\ \mathcal{C}_0 & \mathcal{D}_{00} & \mathcal{D}_{01} \\ \hline 0 & 0 & I \\ \mathcal{C}_1 & \mathcal{D}_{10} & \mathcal{D}_{11} \end{bmatrix}^T \left[\begin{array}{cc|cc|cc} 0 & I & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & Q & S & 0 & 0 \\ 0 & 0 & S^T & R & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & Q_p & S_p \\ 0 & 0 & 0 & 0 & S_p^T & R_p \end{array}\right] \begin{bmatrix} I & 0 & 0 \\ \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B}_0 & \mathcal{X}\mathcal{B}_1 \\ \hline 0 & I & 0 \\ \mathcal{C}_0 & \mathcal{D}_{00} & \mathcal{D}_{01} \\ \hline 0 & 0 & I \\ \mathcal{C}_1 & \mathcal{D}_{10} & \mathcal{D}_{11} \end{bmatrix} < 0, \tag{D.33}$$

which contains products of unknowns $\mathcal{X}$, $Q$, $S$, $R$, $A_K$, $B_{Kj}$, $C_K$, $D_{Kj}$. To partly resolve these bilinear terms, a nonlinear transformation is invoked as follows, similarly to Scherer *et al.* (1997); Masubuchi *et al.* (1998); Scherer and Weiland (2005, Section 4.2). Use the parameterizations (partitioned accordingly to $\mathcal{A}$)

$$\mathcal{X} = \begin{bmatrix} X & U \\ U^T & X_2 \end{bmatrix}, \quad \mathcal{X}^{-1} = \begin{bmatrix} Y & V \\ V^T & Y_2 \end{bmatrix}, \quad \mathcal{Y} = \begin{bmatrix} Y & I \\ V^T & 0 \end{bmatrix}, \quad \mathcal{Z} = \begin{bmatrix} I & 0 \\ X & U \end{bmatrix},$$

$$\begin{bmatrix} K & L_1 & L_2 \\ M & N_1 & N_2 \end{bmatrix} = \begin{bmatrix} XAY & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} U & XB_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_K & B_{K1} & B_{K2} \\ C_K & D_{K1} & D_{K2} \end{bmatrix} \begin{bmatrix} V^T & 0 & 0 \\ C_2Y & I & 0 \\ 0 & 0 & I \end{bmatrix}, \tag{D.34}$$

which imply

$$XY + UV^T = I, \qquad \mathcal{Y}^T\mathcal{X} = \mathcal{Z}, \qquad \boldsymbol{X}(v) := \mathcal{Y}^T\mathcal{X}\mathcal{Y} = \begin{bmatrix} Y & I \\ I & X \end{bmatrix},$$

$$\left[\begin{array}{c|c|c} \boldsymbol{A}(v) & \boldsymbol{B}_0(v) & \boldsymbol{B}_1(v) \\ \hline \boldsymbol{C}_0(v) & \boldsymbol{D}_{00}(v) & \boldsymbol{D}_{01}(v) \\ \hline \boldsymbol{C}_1(v) & \boldsymbol{D}_{10}(v) & \boldsymbol{D}_{11}(v) \end{array}\right] = \left[\begin{array}{c|c|c} \mathcal{Y}^T\mathcal{X}\mathcal{A}\mathcal{Y} & \mathcal{Y}^T\mathcal{X}\mathcal{B}_0 & \mathcal{Y}^T\mathcal{X}\mathcal{B}_1 \\ \hline \mathcal{C}_0\mathcal{Y} & \mathcal{D}_{00} & \mathcal{D}_{01} \\ \hline \mathcal{C}_1\mathcal{Y} & \mathcal{D}_{10} & \mathcal{D}_{11} \end{array}\right]$$

with (4.30). Note that the expressions for $\boldsymbol{A}(v)$ etc. are affine in the new variables $v$.

Two congruence transformations are carried out next. Multiply $\mathcal{X} > 0$ by $\mathcal{Y}$ from the right and by $\mathcal{Y}^T$ from the left. Likewise, multiply (D.33) by $\mathrm{diag}(\mathcal{Y}, I, I)$ from the right and by its transpose from the left. The transformed inequalities are (4.26) and (4.27). Since all steps can be traced backwards (all transformations are invertible), the existence of a solution for conditions (4.26)–(4.28) is sufficient for the existence of a controller (4.25) achieving uniform exponential stability and robust quadratic performance for the corresponding closed-loop system (4.22). The extra condition (4.29) is used in the second part of the proof. This concludes the first part of the proof.

In the second part of the proof, we show equivalence between (i) and (ii). The inequality (4.27) still contains products of the unknowns $v$, $Q$, $R$, $S$. After permuting rows and columns of (4.27), the expression (4.30) is directly inserted there. Now the Elimination Lemma (Lemma B.10) is applied as follows. Let $\Theta = \mathrm{col}(\Theta_1, \Theta_2, \Theta_3)$, $\Psi = \mathrm{col}(\Psi_1, \Psi_2)$ be basis matrices of $\ker\left(\begin{bmatrix} B_2^T & D_{02}^T & D_{12}^T \end{bmatrix}\right)$, $\ker\left(\begin{bmatrix} C_2 & D_{21} \end{bmatrix}\right)$, respectively. Then $\mathrm{col}(\Theta_1, 0, \Theta_2, \Theta_3)$ is a basis of $\ker\left(\begin{bmatrix} 0 & I & 0 & 0 \\ B_2^T & 0 & D_{02}^T & D_{12}^T \end{bmatrix}\right)$, and $\mathrm{col}(0, \Psi_1, 0, \Psi_2)$ is a basis of $\ker\left(\left[\begin{array}{cc:c:c} I & 0 & 0 & 0 \\ 0 & C_2 & D_{20} & D_{21} \\ 0 & 0 & I & 0 \end{array}\right]\right)$. By using the Elimination Lemma, (4.27) is rewritten into two new conditions. These are simplified by eliminating zero rows and columns, which leads to (4.32)–(4.33). Then the existence of $K$, $L_1$, $L_2$, $M$, $N_1$, $N_2$ satisfying (4.27) is equivalent to (4.32)–(4.33) plus the coupling condition (4.36). Note that $Q$ and $S$ do not appear any more due to the special structure of the above matrices, following from the structure of the plant $G_{\mathrm{aug}}$. Finally, by the Dualization Lemma (Lemma B.7), (4.28)–(4.29) and (4.34)–(4.35) are equivalent. This shows equivalence between (i) and (ii) and concludes the second part of the proof.

# Appendix E

# MATLAB Function Collection

This appendix lists a collection of MATLAB functions, which were developed by the author in connection with this thesis. These functions treat analysis and synthesis problems for multi-objective and robust control system design. The functions use the standard MATLAB toolboxes Control System Toolbox, Robust Control Toolbox, LMI Toolbox, and Optimization Toolbox (see the World Wide Web at `www.mathworks.com`), as well as the freely available toolboxes YALMIP (Löfberg, 2004) and SeDuMi (Sturm, 1999).
The numerical results in this thesis were obtained using MATLAB 6.5.1 (R13) on a Personal Computer with a 3 GHz Intel Xeon processor and 2 GB RAM, running in a server network under the operating system Linux 2.4.

**Analysis**

- $\ell_1$-norm computation for a FD LTI discrete-time system (upper and lower bounds with arbitrary accuracy). See Section 2.2.1 for details.
- Star-norm computation for a FD LTI discrete-time system (upper bound with arbitrary accuracy). See Section 2.2.2.
- $\ell_1$-norm computation for the tail of a discrete-time impulse response (upper bound). See Section 2.2.2.
- $\ell_1$-norm computation for a FD discrete-time linear system with LTV dynamic uncertainties (exact up to numerical accuracy for non-repeated uncertainties). See Section B.8.
- Star-norm computation for a FD discrete-time linear system with parametric TV uncertainties (upper bound). See Section 2.4.
- $\ell_1$-norm computation for a FD discrete-time linear system with parametric TI or TV uncertainties (upper and lower bounds). See Section 2.5.
- $\ell_1$-norm computation for a FD LTI discrete-time descriptor system (upper bound with arbitrary accuracy).

**Synthesis**

- Youla parameterization for stabilizable and detectable FD LTI systems (continuous and discrete time). See Section B.5.
- Youla parameterization for stabilizable and detectable FD systems with TI parametric uncertainties (continuous and discrete time). See Sections B.5 and 3.4.
- $\ell_1$-optimal LTI output-feedback controller synthesis for FD LTI discrete-time systems based on the scaled-$Q$ method (Khammash, 2000). Lower and upper bounds on the performance are computed. See Section 3.2.
- Multi-objective LTI output-feedback controller synthesis for FD LTI discrete-time systems, including $\mathcal{H}_\infty$-, $\mathcal{H}_2$-, $\ell_1$-norm constraints and time-domain template constraints. Lower and upper bounds on the performance are computed. See Section 3.3.
- Robust $\ell_1$-optimal LTI output-feedback controller synthesis for FD linear discrete-time systems with structured LTV or nonlinear uncertainties based on a branch-and-bound method (Khammash *et al.*, 2001). See Sections B.8 and 4.5.
- Robust $\ell_1$-optimal LTI output-feedback controller synthesis for FD linear discrete-time systems with structured LTV or nonlinear uncertainties based on $E$-$Q$-iterations (Dahleh and Khammash, 1993; Dahleh and Diaz-Bobillo, 1995). See Sections B.8 and 4.5.
- Robust $\ell_1$-optimal LPV output-feedback controller synthesis for FD linear discrete-time systems with measurable structured TV parameters. See Section 4.5.
- Robust LPV output-feedback controller synthesis for FD linear continuous-time systems with measurable structured TV parameters with respect to a quadratic performance criterion. See Section 4.6.
- Simple $\ell_1$-optimal controller synthesis for FD LTI discrete-time descriptor systems.

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