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Generalizations of the Finite Element Method

Review Article

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Abstract: This paper is concerned with the generalization of the finite element method via the use of non-polynomial enrichment functions. Several methods employ this general approach, e.g. the extended finite element method and the generalized finite element method. We review these approaches and interpret them in the more general framework of the partition of unity method. Here we focus on fundamental construction principles, approximation properties and stability of the respective numerical method. To this end, we consider meshbased and meshfree generalizations of the finite element method and the use of smooth, discontinuous, singular and numerical enrichment functions.

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1. Introduction

The classical finite element method (FEM) is a well-established tool in scientific computing and widely used in many areas of application, see e.g. [91, 92]. The success of the FEM can be attributed in part to the following key properties: The FEM employs a variational approach with piecewise polynomial shape functions and shows an improved geometry handling compared e.g. with the finite difference method.

These features render the FEM a very versatile numerical approach and it can be regarded as a general purpose solver e.g. for the discretization of partial differential equations (PDE). However, this flexibility comes at a price — mesh-generation. The construction of good quality meshes is not an easy task and accounts for a large percentage of the total (computational and economical) cost of an FE simulation. Moreover, we must acknowledge the fact that (piecewise) polynomials are very much appropriate for the approximation of smooth functions but they are not tailored

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for the approximation of non-smooth functions. Here, local *geometric* mesh refinement must be employed. This may yield an optimal asymptotic convergence behavior but can involve a large number of refinement steps and degrees of freedom to reach the required accuracy.

This effect can only be avoided by abandoning the restriction to piecewise polynomial shape functions in the FEM; i.e., by the generalization of the FEM. Then, we can employ an *algebraic* refinement of the approximation space $V^{\mathrm{FE}} = \operatorname{span}\langle \phi_i^{\mathrm{FE}} \rangle_{i=1}^{N}$ which can provide a much more efficient approximation than geometric mesh refinement with respect to the number of unknowns. However, the incorporation of non-polynomial shape functions in V^{FE} must respect the global regularity constraints, i.e. the inter-element continuity conditions. To this end, the partition of unity (PU) property

$$\sum_{i=1}^{N} \phi_i^{\mathrm{FE}} \equiv 1$$

of e.g. the piecewise linear FE shape functions ϕ_i^{FE} is utilized. With the PU approach we attain an *enriched* approximation space by

$$V_E^{\mathrm{FE}} = V^{\mathrm{FE}} + \eta \sum_{\lambda \in \Lambda} \phi_\lambda^{\mathrm{FE}} \tag{1}$$

where η denotes a specific (solution- or problem-dependent) non-smooth enrichment function and $\Lambda \subset \{1, \dots, N\}$ defines the subset of algebraically refined FE shape functions.[1] This approach led to the introduction of the special finite elements of [4], the extended finite element method (XFEM) [8, 12, 57, 58], and the generalized finite element method [1–3, 27, 28, 32, 64, 79, 81], see also [35] for an extensive collection of references on XFEM and GFEM. Note that these methods are essentially identical and can all be casted in the more general framework of the partition of unity method (PUM) [6, 55] where the approximation space is defined via

$$V^{\mathrm{PU}} = \sum_{i=1}^{N} \varphi_i V_i, \qquad \text{with} \quad V_i = \mathcal{P}^{p_i} + \mathcal{E}_i \tag{2}$$

with $\mathcal{P}^{p_i} = \operatorname{span}\langle \psi_i^s \rangle$ denoting the space of polynomials of degree $p \leq p_i$ and $\mathcal{E}_i = \operatorname{span}\langle \eta_i^t \rangle$ a problem-dependent (local) enrichment space. Obviously, the choice of $\varphi_i = \phi_i^{\mathrm{FE}}$, $\mathcal{P}^{p_i} = \operatorname{span}\langle 1 \rangle$ and $\mathcal{E}_\lambda = \operatorname{span}\langle \eta \rangle$ for $\lambda \in \Lambda$ yields $V^{\mathrm{PU}} = V_E^{\mathrm{FE}}$ of (1).

Thus, in this overview we treat a certain generalization of the FEM as a specific PUM and present the existing techniques in this context. Therefore, we focus on the two main components of the construction of a PUM approximation space (2), the choice of the partition of unity functions φ_i and the selection of local approximation spaces $V_i = \mathcal{P}^{p_i} + \mathcal{E}_i$, and their impact on the approximation and stability properties of the resulting scheme.

2. Generalizations of the Finite Element Method

It was already observed in the 1970s that the approximation properties of classical FEM are not tailored for the treatment of problems with singular solutions. To overcome this issue, certain globally supported non-polynomial functions were added explicitly to the FE approximation space, e.g. in [13, 18, 34]. This approach is essentially based on a global singular function representation of the solution which is also the foundation of the methods proposed, e.g. in [15–17, 19, 20].

The use of compactly supported non-polynomial shape functions to resolve local details of the solution was presented in [4] for the treatment of second order PDE with rough coefficients. In particular, the use of products $\phi_i^{\mathrm{FE}} \eta_i$ of piecewise linear FE shape functions ϕ_i^{FE} with problem-dependent approximation functions η_i was proposed. From this approach originated the work on the partition of unity method (PUM) in [6, 55] which will be the framework employed in this paper to generalize the FEM.

[1] *This approach is in the XFEM context often referred to as extrinsic (local) enrichment.*

2.1. Partition of unity approach

The abstract ingredients of a PUM are

- a partition of unity (PU) $\{\varphi_i : i = 1, \dots, N\}$ with

$$\varphi_i \in \mathcal{C}^r(\mathbb{R}^D) \qquad \text{and patches} \qquad \omega_i = \operatorname{supp}^\circ \varphi_i,$$

- a collection of local approximation spaces

$$V_i = V_i(\omega_i) = \operatorname{span} \langle \vartheta_i^n \rangle$$

  defined on the patches ω_i for $i = 1, \dots, N$.

With these two ingredients we define the PUM space

$$V^{\mathrm{PU}} = \sum_{i=1}^{N} \varphi_i V_i = \operatorname{span} \langle \varphi_i \vartheta_i^n \rangle; \tag{3}$$

i.e., the shape functions of a PUM space are simply defined as the products of the PU functions φ_i and the local approximation functions ϑ_i^n. The PU functions provide the locality and global regularity of the product functions whereas the functions ϑ_i^n equip V^{PU} with its approximation power. Thus, we refer to the PU functions φ_i also as h-components and denote the local approximation functions ϑ_i^n also as (generalized) p-components of the PUM space V^{PU}.

2.2. Fundamental approximation results

To study the approximation properties of the PUM space V^{PU} we need to introduce some notation and specific assumptions on the PU and the local approximation spaces due to [6].

Definition 2.1 (partition of unity).
Let $\Omega \subset \mathbb{R}^D$ be an open set. Let $\{\varphi_i : i = 1, \dots, N\}$ be a collection of Lipschitz functions $\varphi_i \colon \mathbb{R}^D \to \mathbb{R}$ with

$$\sum_{i=1}^{N} \varphi_i \equiv 1 \ \text{ on } \ \overline{\Omega}, \qquad \|\varphi_i\|_{L^\infty(\mathbb{R}^d)} \le C_\infty, \qquad \omega_i = \operatorname{supp}(\varphi_i)^\circ, \qquad \|\nabla \varphi_i\|_{L^\infty(\mathbb{R}^d)} \le \frac{C_\nabla}{\operatorname{diam} \omega_i} \tag{4}$$

where each support ω_i is a Lipschitz domain, C_∞ and C_∇ are two positive constants. The collection of functions $\{\varphi_i : i = 1, \dots, N\}$ is referred to as a partition of unity (PU) and the PU is said to be of degree $k \in \mathbb{N}_0$ if $\varphi_i \in \mathcal{C}^k(\mathbb{R}^D)$ for all $i = 1, \dots, N$. The sets ω_i are called patches and their collection is referred to as a cover $C_\Omega = \{\omega_i : i = 1, \dots, N\}$ of the domain Ω. Let us further introduce the covering index $\lambda_{C_\Omega} \colon \Omega \to \mathbb{N}$ such that

$$\lambda_{C_\Omega}(x) = \operatorname{card} \{\omega_i \in C_\Omega : x \in \omega_i\}.$$

A family of PUs is said to satisfy a finite covering property if

$$\lambda_{C_\Omega}(x) \le M \in \mathbb{N} \qquad \text{holds for all} \quad x \in \Omega \tag{5}$$

with M independent of $N = \operatorname{card} C_\Omega$.

For PUM spaces (3) which employ a PU $\{\varphi_i\}$ satisfying Definition 2.1 there hold the following error estimates due to [6].

Theorem 2.2.
Let $\Omega \subset \mathbb{R}^D$ be a Lipschitz domain. Let $\{\varphi_i\}$ be a partition of unity according to Definition 2.1. Let a collection of local approximation spaces $V_i = \operatorname{span}\langle \vartheta_i^n \rangle \subset H^1(\omega_i)$ be given. Let $u \in H^1(\Omega)$ be the function to be approximated. Assume that the local approximation spaces V_i have the following approximation properties: On each patch $\Omega \cap \omega_i$, the function u can be approximated by a function $u_i \in V_i$ such that

$$\|u - u_i\|_{L^2(\Omega\cap\omega_i)} \leq \hat{\epsilon}_i, \qquad \|\nabla(u - u_i)\|_{L^2(\Omega\cap\omega_i)} \leq \tilde{\epsilon}_i \tag{6}$$

hold for all $i = 1, \ldots, N$. Then the function

$$u^{\mathrm{PU}} = \sum_{i=1}^{N} \varphi_i u_i \in V^{\mathrm{PU}} \subset H^1(\Omega)$$

satisfies the global estimates

$$\begin{aligned} \left\|u - u^{\mathrm{PU}}\right\|_{L^2(\Omega)} &\leq \sqrt{M} C_\infty \left(\sum_{i=1}^{N} \hat{\epsilon}_i^2 \right)^{1/2}, \\ \left\|\nabla\left(u - u^{\mathrm{PU}}\right)\right\|_{L^2(\Omega)} &\leq \sqrt{2M} \left(\sum_{i=1}^{N} \left(\frac{C_\nabla}{\operatorname{diam} \omega_i} \right)^2 \hat{\epsilon}_i^2 + C_\infty^2 \tilde{\epsilon}_i^2 \right)^{1/2}. \end{aligned} \tag{7}$$

Note, however, that Theorem 2.2 is an abstract approximation result only and involves a specific choice of the approximation u^{PU}. It does not state that this approximation is the best-approximation in V^{PU} nor the uniqueness of the representation $\sum_{i=1}^{N} \varphi_i u_i$. In fact, the above assumptions are *not* sufficient to ensure the uniqueness of the representation $u^{\mathrm{PU}} = \sum_{i=1}^{N} \varphi_i u_i$; i.e., the shape functions $\varphi_i \vartheta_i^n$ may be linearly dependent in the PUM with the assumptions of Theorem 2.2. In the construction above all approximation power comes from the local approximation spaces V_i and the PU provides a smooth transition between the local approximations $u_i \in V_i$ only. The shape functions $\varphi_i \vartheta_i^n$, however, may satisfy a higher order reproducing condition than just the local approximation functions ϑ_i^n, e.g. if the PU functions φ_i are capable of resolving a polynomial of order $p > 0$ which is of course the case for $\varphi_i = \phi_i^{\mathrm{FE}}$, and the results of Theorem 2.2 are not optimal in such cases.

3. Variants of the PUM

In this section we review the most prominent variants of the PUM, i.e. we present the most common choices of components employed in the various methods. In particular, we discuss the use of meshbased and meshfree PU functions φ_i as well as the use of local enrichment spaces $\mathcal{E}_i$ spanned by analytical or numerical functions in the definition of a PUM space

$$V^{\mathrm{PU}} = \sum_{i=1}^{N} \varphi_i V_i, \qquad \text{with} \quad V_i = \mathcal{P}^{p_i} + \mathcal{E}_i. \tag{8}$$

First, we focus on the local enrichment spaces $\mathcal{E}_i = \operatorname{span}\langle \eta_i^t \rangle$ and thus the approximation properties of the resulting method. To this end, we rewrite (8) as

$$V^{\mathrm{PU}} = \sum_{i=1}^{N} \varphi_i \mathcal{P}^{p_i} + \sum_{i=1}^{N} \varphi_i \mathcal{E}_i = V^{\mathrm{ORG}} + V^{\mathrm{ENR}}. \tag{9}$$

Observe that $\mathcal{E}_i = 0$ is a valid definition and thus enrichment may be employed everywhere in the domain or just locally where $\mathcal{E}_i \neq 0$. In subsection 3.2 we are then concerned with the choice and construction of the PU functions φ_i and thus the stability properties of the resulting shape functions $\varphi_i \vartheta_i^n$.

3.1. Enrichment functions

The abstract motivation for the introduction of local enrichment spaces $\mathcal{E}_i$ in (9) is to bring additional approximation capabilities (collected in V^{ENR}) to an existing method (or implementation) which originally employs the approximation space V^{ORG}.

3.1.1. Smooth enrichment functions

Let us consider the case

$$V^{\mathrm{ORG}} = \sum_{i=1}^{N} \varphi_i \mathcal{P}^{p_i} = \sum_{i=1}^{N} \phi_i^{\mathrm{FE}} \operatorname{span}\langle 1 \rangle$$

with piecewise linear Lagrange FE functions ϕ_i^{FE}; i.e., we want to improve the approximation power of a typical h-version FE code by enrichment. Here, a first very natural choice (in one space dimension) is to use a polynomial enrichment space

$$V^{\mathrm{ENR}} = \sum_{i=1}^{N} \varphi_i \mathcal{E}_i = \sum_{i=1}^{N} \phi_i^{\mathrm{FE}} \operatorname{span}\langle (x - x_i)^p \rangle_{p=1,\dots,p_i^{\max}}$$

to equip the code with higher order approximation properties without inflicting significant changes to its internal structure. Note also that due to the use of local enrichment spaces $\mathcal{E}_i = \operatorname{span}\langle (x - x_i)^p \rangle_{p=1,\dots,p_i^{\max}}$ this procedure automatically adds the capability for p-adaptive computations to the code; i.e., $p_i^{\max}$ can be chosen for each node x_i (each patch ω_i) individually and independently. Thus, the PUM enrichment concept can be used as a simple strategy to extend an existing h-adaptive FE implementation to obtain an hp-adaptive (generalized) FE code. Moreover, not only are the $p_i^{\max}$ chosen on the patches ω_i completely independent of each other, but we can also use different local basis functions e.g. by using different local coordinate systems on the patches ω_i.

For the approximation of problems with strongly anisotropic solutions, e.g. internal or boundary layers which are often treated by classical discretization techniques on graded or Shishkin meshes, this freedom can be exploited to improve the approximation properties of the resulting scheme in an a-priori solution adapted fashion [26]. If the direction of the maximal variation of the solution is essentially known we can use this information on each patch locally to define an appropriate local coordinate system, e.g. $(\xi_{(i)}, \zeta_{(i)})$ on patch ω_i where $\xi_{(i)}$ denotes the axis of maximal variation of the solution on this patch and then define the local enrichment space e.g.

$$\mathcal{E}_i = \operatorname{span}\left\langle \xi_{(i)}^p \right\rangle.$$

If more characteristics of the behavior of the solution are known, e.g. exponential growth, this information can easily be incorporated into the local approximation spaces e.g. via the selection of appropriate basis functions. Note that we can essentially use a known mesh grading function to select appropriate local approximation spaces.

The two enrichment approaches introduced above are essentially aimed at problems (solutions with sharp gradients or weak singularities) which are usually treated by locally refined meshes and they can provide a similar quality in the approximation based on a simple, e.g. uniform, mesh with less degrees of freedom than a classical method on graded or locally refined meshes.

This decoupling of approximation power and number of degrees of freedom is a core property of the PU approach and renders a PUM especially suitable for problems with highly oscillatory solutions which usually require a very fine global mesh resolution to resolve the characteristics of the solution and avoid so-called pollution. In these applications the local enrichment spaces $\mathcal{E}_i$ are necessary everywhere in the domain (e.g. for all patches ω_i) to obtain an acceptable approximation at all. For the high wave number Helmholtz equation, for instance, the use of generalized harmonic polynomials or local spaces based on plane waves have been employed successfully [55, 80, 82].

In summary, we can say that the use of smooth, e.g. polynomial enrichment functions in a PUM, is a valid and interesting choice which can serve at least one of the following three purposes.

- Extend an existing h-version code and equip it with the capability for (anisotropic) p-adaptive discretizations.
- Avoid mesh grading yet obtain a similar quality of approximation on a uniform mesh.
- Remove global refinement constraints (e.g. h depending on the wave number k).

Here the common effect is that the total number of degrees of freedom necessary to obtain a certain accuracy is reduced substantially.

3.1.2. Discontinuous enrichment functions

This effect is becoming even more pronounced when we consider the approximation of problems with internal interfaces due to jumping coefficients of the PDE (e.g. material properties). In a standard FE discretization of an interface or transmission problem it is assumed that the interface is aligned with the mesh lines; i.e., the coefficients are locally smooth in each element (often assumed to be constant in each element). Obviously, this assumption yields a minimal mesh refinement to resolve the geometry of the interface and thus a minimal number of degrees of freedom in a classical FE approximation. Moreover, if we consider an instationary process with a moving interface (e.g. free surface flow) mesh refinement and coarsening must be employed in every time-step which increases the computational work further. This issue was one of the central motives in the development of meshless or meshfree methods [10, 31, 55] and later on for the XFEM.

For the treatment of interface problems with a PUM we introduce the local enrichment spaces $\mathcal{E}_i$ which encode jumps across interfaces Γ that intersect the patches ω_i in an arbitrary fashion.

Kinks

Let us first consider the treatment of so-called weak discontinuities or solutions with kinks; i.e., we consider the approximation of functions $u\colon \Omega \to \mathbb{R}$ with discontinuous first order derivatives ∇u. The prototypical situation is

$$u = \operatorname{abs} f_1, \qquad u = \max(0, f_2), \qquad \text{or} \qquad u = \min(0, f_3)$$

with a smooth function $f_i\colon \Omega \to \mathbb{R}$ which changes its sign in Ω. The roots $f(x) = 0$ describe the interface $\Gamma = \{x \in \Omega : f(x) = 0\}$. Assume that the interface Γ is closed and induces a disjoint decomposition of our domain Ω into Ω_+ and Ω_-. Moreover, let us assume that the smooth part V^{ORG} of our approximation space (9) can resolve linear functions. Then it is very natural to look for an enrichment space V^{ENR} such that the enriched approximation space V^{PU} (9) can resolve functions that are piecewise linear in Ω and linear in Ω_+ and Ω_-. To this end, consider a simple situation in two space dimensions where the interface is given by $\Gamma = \{(x, y) : y = 0\}$. Thus, we are looking for an approximation space V^{PU} such that $y_+, y_- \in V^{\mathrm{PU}}$, where

$$y_+ = \max(0, y) \qquad \text{and} \qquad y_- = \min(0, y).$$

This can easily be attained with the choice of

$$\mathcal{E}_i = \operatorname{span}\langle y_+, y_- \rangle \tag{10}$$

for all ω_i with $i = 1, \ldots, n$. However, on patches ω_i with $\Gamma \cap \omega_i = \emptyset$ we have either $y = y_+$ or $y = y_-$ which can already be represented in V^{ORG} without enrichment. Therefore it is sufficient to use (10) on patches ω_i with $\Gamma \cap \omega_i \neq \emptyset$. Now observe that $y_+ + y_- = y \in \mathcal{E}_i$ which can again be resolved by V^{ORG}. This obvious linear dependence can be circumvented easily via the choice

$$\mathcal{E}_i = \operatorname{span}\langle |y| \rangle$$

which seems to correspond to the choice of

$$\mathcal{E}_i = \operatorname{span}\langle |\operatorname{sdist}((x, y), \Gamma)| \rangle$$

where $\operatorname{sdist}((x,y),\Gamma)$ denotes the *signed distance* to Γ for an arbitrary interface.[2] Note, however, that this is true for interfaces Γ where the intersections $\Gamma \cap \omega_i \neq \emptyset$ are line segments only.[3] In general, when the intersections $\Gamma \cap \omega_i \neq \emptyset$ are curved (or piecewise linear), the signed distance $\eta_\Gamma(x,y) = \operatorname{sdist}((x,y),\Gamma)$ cannot be represented in V^{ORG} and we must use the two-dimensional enrichment spaces

$$\mathcal{E}_i = \operatorname{span}\left\langle \max(0,\eta_\Gamma), \min(0,\eta_\Gamma)\right\rangle = \operatorname{span}\langle \eta_\Gamma, |\eta_\Gamma|\rangle$$

to resolve the kink at the interface.

Several extensions of this approach to higher order approximations are possible: If the PU functions φ_i already recover polynomials of higher order, e.g. when the ϕ_i are higher order Lagrange FE functions, then the resulting products $\varphi_i\mathcal{E}_i$ have an implicit higher order reproducing property.[4] On the other hand, we can also construct enrichment spaces $\mathcal{E}_i$ which explicitly reproduce higher order (piecewise) polynomials. To this end, we can simply define the enrichment functions $\eta_i \in \mathcal{E}_i$ via

$$\eta_i(x,y) = \eta_\Gamma(x,y)\,p(x,y) + |\eta_\Gamma(x,y)|\,q(x,y)$$

where p and q denote higher order polynomials. Following the presentation above, the most natural choice, however, seems to view the signed distance to the interface as a generalized coordinate and to consider higher order powers of the distance for the enrichment, i.e. with $l = 1,\dots,p_i$ use

$$\mathcal{E}_i = \operatorname{span}\left\langle \max(0,\eta_\Gamma)^l, \min(0,\eta_\Gamma)^l\right\rangle$$

as local enrichment spaces.[5]

Jumps

Let us now consider the approximation of functions u with strong discontinuities across an interface Γ. Here, we allow for $u_+ \neq u_-$ with

$$u_+(x) = \lim_{\epsilon\to 0} u(x+\epsilon n(x)), \qquad u_-(x) = \lim_{\epsilon\to 0} u(x-\epsilon n(x)),$$

where $n(x)$ denotes the oriented normal to Γ at $x \in \Gamma$. A very natural choice of appropriate enrichment functions is the definition

$$\eta_i(x,y) = \operatorname{signum}\eta_\Gamma(x,y)\cdot p(x,y)$$

with $p \in \mathcal{P}^{p_i}$ which essentially models the piecewise constant gradients of the distance function $|\eta_\Gamma(x,y)|$ via the Haar function signum. Note that these enrichment functions are already higher order since the definition above corresponds to the enriched local approximation spaces

$$V_i = \mathcal{P}^{p_i} + \mathcal{E}_i = \mathcal{P}^{p_i} + \operatorname{signum}\eta_\Gamma \cdot \mathcal{P}^{p_i}.$$

[2] *This is often referred to as* abs*-enrichment in the XFEM where the signed distance to Γ is usually given by a level-set function Φ.*

[3] *When finite elements are employed as PU, the intersection $\Gamma \cap E \neq \emptyset$ with a single element E must be a line segment. This is, for instance, the case when the signed distance is determined from a discrete level-set function $\Phi \in V^{\mathrm{ORG}}$ on the same mesh.*

[4] *This technique seems to be the most common approach employed in the XFEM to achieve higher order approximation with discontinuous enrichment.*

[5] *An optimal construction for the local approximation spaces $V_i = \mathcal{P}^{p_i} + \mathcal{E}_i$ would be based on a coordinate system that employs the normal to the interface (via the distance) to capture the kink and the (locally) tangential directions to describe smooth components.*

In summary, the enrichment spaces $\mathcal{E}_i$ for the resolution of (weakly and strongly) discontinuous solutions are essentially *multiplicative* extensions of the smooth polynomial part $\mathcal{P}^{p_i}$ of the local approximation spaces V_i on a particular patch ω_i. Moreover, the enrichment is truly *local* since it must be employed only on patches ω_i that are intersected (completely) by the interface Γ to attain the optimal global convergence rate.

Finally, let us remark that the presentation above focused on conforming treatments of kinks and jumps. For the approximation of kinks, i.e. of transmission problems, there are also non-conforming PU-based methods [46, 53] which also provide optimal convergence. These schemes are similar in spirit to the non-conforming treatment of essential boundary conditions discussed below.

3.1.3. Singular enrichment functions

For the approximation of singular solutions an *additive* enrichment is more appropriate since polynomials cannot account for the strong local variation in the solution or its gradient; i.e., the local approximation spaces are defined as

$$V_i = \mathcal{P}^{p_i} + \mathcal{E}_i$$

with enrichment spaces $\mathcal{E}_i$ which are very different from (piecewise) polynomials. Let us consider the prototypical singular function

$$s(x) = \|x - c\|_2^\alpha \qquad \text{with} \quad \alpha < 1$$

and its efficient numerical approximation by a PU method. Obviously, the function s cannot be approximated accurately by polynomials at the point $x = c \in \bar{\Omega}$ which we assume to be known.

In a classical FEM geometric refinement of the patches ω_i with $c \in \omega_i$ must be employed to recover the optimal convergence rate.[6] In a PU method we replace the geometric refinement of such a patch with the appropriate enrichment of its local approximation space; i.e., we employ

$$V_i = \mathcal{P}^{p_i} + \mathcal{E}_i(s) \tag{11}$$

on patches ω_i with $c \in \omega_i$. For the selection of optimal enrichment spaces $\mathcal{E}_i(s)$ the characteristic singular behavior of s must also be known; i.e., α must be known. This essentially corresponds to a priori regularity results for the considered PDE problem like the following lemma from [54] for the Poisson problem, see also [43, 44].

Lemma 3.1.
Let $\Omega \in \mathbb{R}^2$ be a polygon with vertices c_j and internal angles θ_j for $j = 1, \dots, J$. Define for each vertex c_j the singularity functions

$$s_{j,l}(r_j, \beta_j) = \begin{cases} r^{l\pi/\beta_j} \sin l\frac{\pi}{\beta_j} & \text{if } l\frac{\pi}{\beta_j} \notin \mathbb{N}, \\ r^{l\pi/\beta_j} \left(\ln r_j \cdot \sin l\frac{\pi}{\beta_j} + \beta_j \cos l\frac{\pi}{\beta_j} \right) & \text{if } l\frac{\pi}{\beta_j} \in \mathbb{N}, \end{cases} \tag{12}$$

where (r_j, β_j) denote polar coordinates with respect to the origin c_j such that the two linear segments of the boundary $\partial\Omega$ meeting at the vertex c_j fall on the lines $\beta_j = 0$ and $\beta_j = \theta_j$. Let $f \in H^{-1+k}(\Omega)$ with $k \notin \mathbb{N}$ and $k > 0$, then the solution u of

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

can be written as

$$u = \sum_{j=1}^{J} \sum_{l \in N_k} a_{j,l} s_{j,l} + u_0 \tag{13}$$

with $N_k = \{l \in \mathbb{N} : l\pi < k\beta_j\}$ for some coefficients $a_{j,l} \in \mathbb{R}$ and $u_0 \in H^{1+k}(\Omega)$.

[6] *Actually adaptive refinement must be employed in the vicinity of the singular point c.*

The key to the selection of appropriate enrichment spaces $\mathcal{E}_i$ is the representation (13) of the solution u.[7] If we restrict (13) to a particular patch ω_i and assume that the smooth part u_0 of the solution u can be approximated sufficiently well by the polynomial part $\mathcal{P}^{p_i}$ of (11), we can define the enrichment space

$$\mathcal{E}_i = \operatorname{span}\langle s_{j,l}\rangle$$

to obtain a locally optimal approximation in V_i; i.e., the rate of convergence is given by $\min(p_i, k)$. This, however, is not sufficient to obtain optimal convergence on Ω since the singularity functions $s_{j,l}$ are global functions and may dominate the behavior of the solution in an area larger than ω_i. Thus, the enrichment for singular solutions must be *non-local* to recover the optimal convergence rate globally; i.e., we must enrich patches ω_i with a finite distance to the singular points c_j not only those patches with $c_j \in \omega_i$. This is sometimes referred to as *geometric* enrichment whereas the enrichment of patches ω_i with $c_j \in \omega_i$ only is denoted as *topological* enrichment [7, 49, 70, 74, 75]. In practice, only the singularity function $s_{j,1}$, i.e. the dominant singularity, is used as an enrichment function.

Expansions of the solution u similar to (13) exist e.g. for mixed Dirichlet–Neumann boundary conditions or fracture mechanics problems in two space dimensions. In crack simulations with PU type methods a very common choice [8, 11, 12, 27, 28, 32, 57, 62] of the respective singular enrichment space is

$$\mathcal{E}_{\text{tip}} = \operatorname{span}\left\langle \sqrt{r}\cos\frac{\beta}{2}, \sqrt{r}\sin\frac{\beta}{2}, \sqrt{r}\sin\beta\sin\frac{\beta}{2}, \sqrt{r}\sin\beta\cos\frac{\beta}{2}\right\rangle.$$

3.1.4. Numerical enrichment functions

Even if an expansion such as (13) with explicitly given singularity functions (12) is not known, e.g. if the singularity exponents of $s_{j,l}$ are not known analytically, we can still pursue our enrichment approach. However, we now have to precompute appropriate numerical enrichment spaces $\mathcal{E}_i$ on patches ω_i via auxiliary problems defined on subsets $\Omega_i \subset \Omega$ which satisfy $\omega_i \subseteq \Omega_i$.

Observe that the use of numerical enrichment functions may yield better results than the use of analytical enrichment functions. Recall that the main ingredient in the design of enrichment functions for the approximation of discontinuous solutions across a closed interface Γ was the signed distance function $\eta_\Gamma(x) = \operatorname{sdist}(x, \Gamma)$. If the interface Γ is sufficiently smooth, η_Γ captures the relevant solution behavior rather well. However, if the interface Γ is not smooth, e.g. if Γ is given by a polygon, this is no longer the case. Then, we essentially encounter a similar situation as in Lemma 3.1 and the singular behavior of the solution at the vertices of the polygon is not resolved by the enrichment function $\eta_\Gamma(x) = \operatorname{sdist}(x, \Gamma)$. One approach to deal with this issue is to use the enrichment by η_Γ only on patches ω_i where the intersections $\gamma_i = \Gamma \cap \omega_i$ are smooth and to use the respective singularity functions (12) on patches ω_i which contain a vertex of Γ. This approach, however, may lead to a rather large number of enrichment functions and also requires that the patches resolve the interface in the following sense: A patch ω_i may contain at most one of the vertices of Γ. This essentially means that we need to employ local geometric refinement of the patches which intersect the interface to obtain a valid enrichment scheme. This obviously yields a substantial conflict with the main idea of replacing geometric refinement by the selection of appropriate local approximation spaces and may furthermore be challenging to realize in a global approximation on Ω.

To overcome these issues numerical enrichment functions or handbook functions can be employed. These numerical enrichment functions are simply solutions of certain local problems which are approximated e.g. by the approach above or adaptive FEM. These local problems can either be defined a priori [83, 84] or a posteriori [29, 48] which is then referred to as global-local enrichment. In the a priori approach the constructed local problems are usually not as much tailored to the concrete global problem at hand, e.g. the local problems may employ (several) reference boundary conditions. Thus, the computed local solutions may be useful as enrichment functions in other simulations and are therefore stored permanently in a handbook. The global-local enrichment functions on the other hand may be more adapted to the considered problem and thus can provide a better approximation.

[7] *This is also the foundation of the global enrichment approaches and methods proposed in [15–17, 19, 20].*

3.1.5. Summary

The approximation properties of polynomials are well-suited for the numerical treatment of smooth solutions. For the approximation of discontinuous or singular functions piecewise polynomials with adaptive geometric refinement can be used. The geometric refinement, however, leads to an increase in the number of unknowns and storage demands which may be prohibitive. These issues can be avoided to a large extent via the use of problem-dependent non-polynomial local enrichment functions which can resolve those components of the solution u which cannot be approximated well by polynomials, i.e. the local enrichment functions should provide good approximation for the function $u - \Pi_{\mathcal{P}^{p_i}} u$ where $\Pi_{\mathcal{P}^{p_i}}$ denotes a projector onto the space of polynomials of degree $\leq p_i$ on a particular patch ω_i.

Besides the enrichment with smooth or polynomial functions to obtain a higher order method from a first order implementation, there are essentially two types of enrichment spaces: multiplicative and additive enrichments. Multiplicative enrichments essentially change the local *polynomial* approximation spaces $V_i = \mathcal{P}^{p_i}$ to a *piecewise polynomial* approximation space $V_i = \mathcal{P}^{p_i} + \eta_M \mathcal{P}^{p_i}$ where η_M denotes a (weakly or strongly) discontinuous function. Thus, multiplicative enrichments are *higher order* and *local* by design, i.e. are defined on patches ω_i which intersect the respective interface. Additive enrichments on the other hand are truly *non-polynomial* and usually deal with *singular* solution behavior. Therefore, additive enrichments must be employed in a *non-local* fashion to account for the domain of influence of the specific singularity function used for the enrichment.

Areas of application for the discussed enrichment spaces are for instance: transmission and interface problems, shear bands, cracks [8, 57], cohesive cracks [52], dislocations [36, 89], free surface flow [21, 22, 45], polycrystals and grain boundaries [76, 78], thin fiber composites [65, 66]. See [9, 35, 47] for further references.

Recall that the main goal of an enriched PU approximation is that we obtain a better approximation with less degrees of freedom. Thus, the PUM offers a convenient approach to construct coarse approximation spaces (small number of unknowns) which can, however, resolve fine details (large variations or oscillations with the support of a single basis function, i.e. a patch ω_i). This renders the approach very promising for the construction of multiscale methods [5, 33] and the design of robust multilevel solvers [40, 73]. In some sense we can view the algebraic refinement of an approximation space by problem-dependent enrichment functions as a continuum version of operator-dependent multigrid or algebraic multigrid methods [85]. This interpretation, however, assumes that the linear systems arising from a PUM discretization are well-conditioned, an issue that is related to the particular choice of basis functions. So far we were only concerned with the abstract *approximation properties* of the respective PUM space and ignored the potential impact the enrichments may have on the *stability* of the resulting PUM and its shape functions. However it is obvious that in the non-local additive enrichment case we will encounter stability problems.

3.2. Partition of unity functions

The selection of local approximation spaces V_i in a PUM is on an abstract level straightforward: Among all spaces V_i with fixed dimension $\dim V_i$, choose V_i such that the local error bounds $\hat{\epsilon}_i$ and $\tilde{\epsilon}_i$ of (6) in Theorem 2.2 are minimal. The selection criteria for the PU functions φ_i with $\operatorname{supp} \varphi_i = \omega_i$ are more vague. Theorem 2.2 allows for the use of any PU whose first derivatives are bounded by $\operatorname{diam} \omega_i^{-1}$ to obtain the global bounds (7). Thus, the assumptions and results of Theorem 2.2 give much freedom in the particular choice of the PU and the selection of the PU functions φ_i can for instance be guided by the following criteria.

1. The PU should provide higher order reproduction to further improve the global approximation properties.
2. The functions φ_i should be easy to implement or there should be an efficient implementation available.
3. The construction of the PU functions φ_i should be independent of a mesh.
4. The implementation of a PU with arbitrary regularity should be simple.
5. The functions φ_i should be chosen so that the product functions $\varphi_i \vartheta_i^n$ which span the global PUM space V^{PU} (3) provide a stable basis of V^{PU}.

Criteria 1 and 2 suggest the use of classical Lagrange FE shape functions as the PU functions φ_i which is the choice made in XFEM and GFEM approaches. Criteria 3 and 4 imply the use of meshfree methods for the construction of the PU [24, 30, 38, 39, 51]. And criterion 5 seems to favor the use of a flat-top PU [40, 63, 67, 68].

3.2.1. Finite element functions as PU

Considering the fact that we are interested in a generalization of the FEM, the most natural choice of PU functions φ_i probably is the use of classical FE shape functions ϕ_i^{FE}. However, the theoretical results of subsection 2.2 do not yield the expected error bounds for arbitrary local approximation spaces V_i and can yield linearly dependent product functions $\varphi_i \vartheta_i^n$. Moreover, the point of view taken in subsection 2.2 is completely different than in the classical FE. The analysis of the PUM is based on patches $\omega_i = \operatorname{supp} \varphi_i$ whereas the classical FE theory considers elements $E \subset \Omega$ such that for all $i = 1, \dots, N$ the restrictions $\phi_i^{\mathrm{FE}}|_E$ are polynomials.

Let us consider the following simple one-dimensional situation: Let $\Omega = (0,1)$ be the unit interval and define

$$\varphi_0(x) = 1 - x \qquad \text{and} \qquad \varphi_1(x) = x$$

as the respective linear FE shape functions. Furthermore, we define the local approximation spaces

$$V_0 = \operatorname{span}\langle 1, x\rangle, \qquad V_1 = \operatorname{span}\langle 1, x\rangle.$$

Obviously, the choice of components satisfies the assumptions above, however, the resulting PUM shape functions are linearly dependent. Observe that the PU functions φ_0 and φ_1 are linear polynomials as are all local approximation functions ϑ_i^n. Hence, the four product functions $\varphi_i \vartheta_i^n$ are the quadratic polynomials

$$(1-x), \quad x(1-x), \quad x, \quad x^2.$$

Yet, there exist only three linearly independent quadratic polynomials on the interval Ω so that the four shape functions of our PUM space $V^{\mathrm{PU}} = \varphi_0 V_0 + \varphi_1 V_1$ must be linearly dependent. On the other hand, the space $V^{\mathrm{PU}} = \mathcal{P}^2$ obviously has better reproduction properties than the local spaces $V_i = \mathcal{P}^1$ so that better global error bounds than those of Theorem 2.2 can be attained. Thus, there is a benefit of using FE shape functions as the PU: higher order convergence. This, however, comes at the price that the product functions may be linearly dependent if we use polynomial local approximation spaces or ill-conditioned in general.

Another issue that is often referred to in the XFEM literature is the notion of blending elements which we shortly discuss in the following. Consider the PUM approximation space

$$V^{\mathrm{PU}} = \sum_{i=1}^{N} \varphi_i \mathcal{P}^{p_i} + \sum_{i=1}^{N} \varphi_i \mathcal{E}_i = V^{\mathrm{ORG}} + V^{\mathrm{ENR}}$$

and let us assume that $\mathcal{P}^{p_i} = \operatorname{span}\langle 1\rangle$ and $\varphi_i = \phi_i^{\mathrm{FE}}$. Then we have

$$V^{\mathrm{PU}} = V^{\mathrm{FE}} + \sum_{i=1}^{N} \phi_i^{\mathrm{FE}} \mathcal{E}_i$$

where $\mathcal{E}_i$ is an arbitrary local enrichment space and may satisfy $\mathcal{E}_i \neq 0$ for some nodes only, i.e.

$$V^{\mathrm{PU}} = V^{\mathrm{FE}} + \sum_{i \in \Lambda} \phi_i^{\mathrm{FE}} \mathcal{E}_i$$

with $\Lambda \subset \{1, \dots, N\}$. For simplicity let these enrichment spaces be one-dimensional and be generated by a single global enrichment function η, i.e. any function $v^{\mathrm{PU}} \in V^{\mathrm{PU}}$ can be written as

$$v^{\mathrm{PU}}(x) = \sum_{i=1}^{N} \tilde{v}_i \phi_i^{\mathrm{FE}}(x) + \sum_{i \in \Lambda} \tilde{e}_i \phi_i^{\mathrm{FE}}(x)\, \eta|_{\omega_i}(x)$$

with coefficients $\tilde{v}_i$ and $\tilde{e}_i$. Then it is obvious that the function η cannot be exactly represented globally on Ω by a function $v^{\mathrm{PU}} \in V^{\mathrm{PU}}$. Moreover, η cannot be recovered in V^{PU} on the enriched subdomain

$$\Omega_E = \bigcup_{i \in \Lambda} \omega_i, \qquad \text{since} \qquad 1 \not\equiv \sum_{i \in \Lambda} \phi_i^{\mathrm{FE}} \quad \text{on} \quad \Omega_E.$$

Only on the smaller subdomain

$$\Omega_F = \left\{ x \in \Omega_E : 1 \equiv \sum_{i \in \Lambda} \phi_i^{\mathrm{FE}} \right\}$$

can η be represented exactly in V^{PU}. All elements in $\Omega_E \setminus \Omega_F$ are called blending elements. The question is if this issue has an adverse effect on the global approximation properties. To answer this question, let us consider two cases: First a singular enrichment function η as in 3.1.3 and then a piecewise polynomial enrichment function as in 3.1.2.

Recall that a singular enrichment function is employed for the approximation of a singular solution u. Thus, here we assume that an expansion of the solution as in (13) is known a priori. Due to the rather local character of a singularity, however, we can generalize such a decomposition in such a fashion that

$$u = \sum_{j=1}^{J} \sum_{l \in N_k} a_{j,l} \chi_j s_{j,l} + \tilde{u}_0$$

with a cut-off function χ_i that satisfies $\chi_i \equiv 1$ in a vicinity of the vertex c_i and $\chi_i \equiv 0$ in a vicinity of all other vertices c_j and a regular function $\tilde{u}_0$. Thus, the enriched approximation space V^{PU} must not be able to represent $s_{j,l}$ exactly but only $\chi_j s_{j,l}$ in its support which is obviously attainable by the enrichment space $\sum_{i \in \Lambda} \phi_i^{\mathrm{FE}} \mathcal{E}_i$ only. Thus, the overall convergence rate is determined by the approximation properties of V^{FE} with respect to the regular part $\tilde{u}_0$. Therefore, blending elements have no impact on the approximation properties in the case of singular enrichment functions.

When we use the kink enrichment we are essentially interested in the construction of an approximation space V^{PU} that can represent piecewise polynomials that are globally continuous and may have a jumping derivative across a prescribed interface Γ that decomposes the domain Ω into two disjoint subdomains Ω_- and Ω_+. For the sake of simplicity let us consider linear polynomials only. The line of argument in the PUM given above is that the *local* approximation spaces V_i determine the reproduction order and the PU functions merely merge the local approximations together. Thus, if we assume that the local approximation spaces V_i on patches $\omega_i = \operatorname{supp} \phi_i^{\mathrm{FE}}$ which do not intersect the interface Γ can reproduce linear polynomials, i.e. $V_i = \mathcal{P}^1$, and that the V_i on patches $\omega_i \cap \Gamma \neq \emptyset$ can reproduce piecewise linear functions, for instance, by the choice of

$$V_i = \mathcal{P}^1 + \operatorname{span} \left\langle \max(0, \eta_\Gamma), \min(0, \eta_\Gamma) \right\rangle,$$

we have exact reproduction in V^{PU}. With a finite element PU, however, reproducing conditions are not inferred on the patch level but on the element level. Thus enrichment functions must also be viewed as element shape functions via their restriction to elements. Consequently we would like to enrich those elements which intersect the interface Γ only. The PUM framework, however, requires that we enrich patches. To this end, we simply extend the element enrichment functions by zero to the neighboring elements (the blending elements) to obtain a patch-based enrichment function. This, however, conflicts with the global continuity constraints. To fix this issue, we change the element enrichment

functions in such a fashion that they vanish at element boundaries to obtain a valid global approximation space.[8] Yet, the required characteristics of the enrichment functions, i.e. piecewise linearity with respect to the interface Γ, must be preserved. Thus, blending elements are really just a technical difficulty arising due to the fact that we mix FEM approximation properties with PUM approximation properties. However, the approximation properties of FE shape functions are derived from element properties under the constraint of global continuity whereas the global PUM space inherits its approximation properties from independent local spaces defined on patches.

In summary, the use of a PU build from FE shape functions is in principle possible and can provide a better convergence than indicated by Theorem 2.2 but leads to several technical challenges. The most essential drawback of a finite element PU is that it somewhat destroys the local character of the PUM. Recall that it is not sufficient to have linear independence of the local basis functions ϑ_i^n of V_i to infer that the product functions $\phi_i^{\mathrm{FE}}\vartheta_i^n$ are globally linearly independent. Thus the selection of a well-conditioned basis for the enriched global approximation space cannot be based on local properties. This issue becomes even more pronounced when non-local enrichment must be employed.

Recall that for the approximation of a singular solution a simple topological enrichment is not sufficient to attain the optimal convergence rate. Thus geometric enrichment is usually employed, i.e. all patches ω_i in a certain neighborhood of the singular point c of the solution are enriched. Obviously this procedure yields a rather large number of enrichment functions η_i^t which are usually given as restrictions of global functions η^t on the computational domain Ω, compare with (12). Therefore, the restrictions $\eta_i^t = \eta^t\restriction_{\omega_i}$ of the enrichment functions η^t to a particular patch ω_i are not scaled with respect to the local supports ω_i and may be ill-conditioned or even linearly dependent on ω_i, even if the enrichment functions η^t are well-conditioned on a global scale. Furthermore, the coupling between the spaces $\mathcal{P}^{p_i}$ and $\mathcal{E}_i$ on the patch ω_i must be considered since

$$V_i(\omega_i) = \mathcal{P}^{p_i}(\omega_i) + \mathcal{E}_i(\omega_i)$$

is not a direct splitting. The system of functions $\langle\vartheta_i^m\rangle = \langle\psi_i^s, \eta_i^t\rangle$ degenerates from a basis of V_i to a generating system if the restricted enrichment functions $\eta_i^t = \eta^t\restriction_{\omega_i}$ can be represented (well-approximated) by polynomials $\psi_i^s \in \mathcal{P}^{p_i}$ on the patch ω_i.

The elimination of these linear dependencies (the ill-conditioning) and the selection of a well-conditioned basis for the space V^{PU} is of great practical relevance and one of the main challenges in an enriched PUM computation. Probably the most general approach to solve this issue is via the imposition of an additional condition on the PU functions, the flat-top property.[9]

3.2.2. Flat-top PU

Definition 3.2 (flat-top property).
Let $\{\varphi_i\}$ be a partition of unity satisfying (4). Let us define the sub-patches $\omega_{\mathrm{FT},i} \subset \omega_i$ such that $\varphi_i\restriction_{\omega_{\mathrm{FT},i}} \equiv 1$. Then, the PU is said to have the flat-top property, if there exists a constant C_{FT} such that for all patches ω_i

$$\mu(\omega_i) \leq C_{\mathrm{FT}}\,\mu(\omega_{\mathrm{FT},i} \cap \Omega) \tag{14}$$

where $\mu(A)$ denotes the Lebesgue measure of $A \subset \mathbb{R}^d$. Note that $C_\infty = 1$ holds for a PU with the flat-top property.

[8] *Note that also a non-vanishing constant extension of the element enrichment functions is sufficient.*

[9] *A rather simple approach to this issue in the case of singular enrichment functions can be summarized as follows. For simplicity let us assume that the local enrichment spaces $\mathcal{E}_i$ are one-dimensional and given by a single global enrichment function η, i.e. any local approximation function $v_i \in V_i$ can be written as*

$$v_i(x) = \sum_{s=1}^{d_i} \tilde{v}_i^s \psi_i^s(x) + \tilde{e}_i \eta\restriction_{\omega_i}(x)$$

with coefficients $\tilde{v}_i$ and $\tilde{e}_i$ and $d_i = \dim \mathcal{P}^{p_i}$. Recall that the enrichment function essentially serves the purpose to capture the leading term in an expansion like (13). Thus if the employed enrichment function η satisfies $\eta = Cs_{j,1}$ for some constant C it is sufficient to set all local coefficients $\tilde{e}_i$ to the same value and determine this single coefficient on a patch ω_i with $c_j \in \omega_i$ [49, 86, 88]. This approach can either be interpreted as imposing constraints on the local approximation spaces or as a clustering of the associated PU functions and is also referred to as flat-top enrichment [35].

This property is sufficient to ensure that the product functions $\varphi_i \vartheta_i^m$ are linearly independent, provided that the employed local approximation functions ϑ_i^m are linearly independent with respect to $\omega_{\mathrm{FT},i} = \{x \in \omega_i : \varphi_i(x) = 1\}$.[10] Hence, we obtain global stability of the product functions $\varphi_i \vartheta_i^m$ from the local stability of the approximation functions ϑ_i^m on ω_i according to the following lemma.

Lemma 3.3 (stability).
Let the local spaces V_i for $i = 1, \dots, N$ and their respective basis functions ϑ_i^m for $m = 1, \dots, d_i = \dim V_i$ satisfy

$$\kappa_{1,i}^{-1}\, \mu(\omega_{\mathrm{FT},i} \cap \Omega) \sum_{m=1}^{d_i} (u_i^m)^2 \leq \|u_i\|^2_{L^2(\omega_{\mathrm{FT},i} \cap \Omega)}, \qquad \|u_i\|^2_{L^2(\omega_i \cap \Omega)} \leq \kappa_{2,i}\, \mu(\omega_i) \sum_{m=1}^{d_i} (u_i^m)^2 \tag{15}$$

for any $u_i = \sum_{m=1}^{d_i} u_i^m \vartheta_i^m \in V_i$. Moreover, let $\mu(\omega_i) \asymp h^d$ and assume that the PU has the flat-top property. Then the product functions $\varphi_i \vartheta_i^m$ are a basis for the global PUM space V^{PU} and there holds

$$K_1^{-1} \left(\sum_{i=1}^{N} \sum_{m=1}^{d_i} (u_i^m)^2 \right)^{1/2} \leq h^{-d/2} \|u^{\mathrm{PU}}\|_{L^2(\Omega)} \leq K_2 \left(\sum_{i=1}^{N} \sum_{m=1}^{d_i} (u_i^m)^2 \right)^{1/2}$$

where $u^{\mathrm{PU}} = \sum_{i=1}^{N} \varphi_i u_i \in V^{\mathrm{PU}}$ with

$$K_1 = \min_{i=1,\dots,N} \kappa_{1,i} \qquad \text{and} \qquad K_2 = \sqrt{M} \max_{i=1,\dots,N} \kappa_{2,i}.$$

Proof. With the help of (4) and (5) we easily obtain the upper bound

$$\left\|u^{\mathrm{PU}}\right\|^2_{L^2(\Omega)} = \int_\Omega \left(\sum_{i=1}^{N} \varphi_i(x)\, u_i(x) \right)^2 dx \leq M C_\infty^2 \sum_{i=1}^{N} \|u_i\|^2_{L^2(\omega_i \cap \Omega)}.$$

For the respective lower bound we utilize the flat-top property (14). Obviously, there holds

$$\left\|u^{\mathrm{PU}}\right\|^2_{L^2(\Omega)} \geq \sum_{j=1}^{N} \int_{\omega_{\mathrm{FT},j} \cap \Omega} \left(\sum_{i=1}^{N} \varphi_i(x)\, u_i(x) \right)^2 dx.$$

Observe that a flat-top PU has a particular Kronecker property

$$\varphi_i \restriction_{\omega_{\mathrm{FT},j}} = \delta_{ij}$$

which yields the estimate

$$\left\|u^{\mathrm{PU}}\right\|^2_{L^2} \geq \sum_{j=1}^{N} \int_{\omega_{\mathrm{FT},j}} u_j^2(x)\, dx = \sum_{j=1}^{N} \|u_j\|^2_{L^2(\omega_{\mathrm{FT},i} \cap \Omega)}.$$

The assertion then follows from (15). □

[10] *A more general condition that ensures the linear independence is given in [23]. This condition, however, does not give rise to a simple constructition process which is essential for an easy and efficient implementation.*

Thus, we must be concerned with the construction of appropriate local basis functions ϑ_i^m only. The global basis functions $\varphi_i\vartheta_i^n$ are then automatically L^2-stable. In the case $V_i = \mathcal{P}^{p_i}$ the conditions (15) follow from the stability of the polynomials $\pi \in V_i$ on $\omega_{\mathrm{FT},i}$ since

$$\|\pi\|_{L^2(\omega_i)} \leq C\|\pi\|_{L^2(\omega_{\mathrm{FT},i})}$$

holds with an absolute constant depending on the polynomial degree p_i and C_{FT} of (14) due to the Hölder inequality and the estimates [25]

$$\|\pi\|_{L^\infty(\omega_i)} \leq C\left(\frac{\operatorname{diam}\omega_i}{\operatorname{diam}\omega_{\mathrm{FT},i}}\right)^{p_i}\|\pi\|_{L^\infty(\omega_{\mathrm{FT},i})}, \qquad \|\pi\|_{L^\infty(\omega_{\mathrm{FT},i})} \leq \mu(\omega_{\mathrm{FT},i})^{-1/2}\|\pi\|_{L^2(\omega_{\mathrm{FT},i})},$$

so that K_1 and K_2 are dependent on the maximal polynomial degree only. In the general case, when $\mathcal{E}_i \neq 0$, the conditions (15) are explicitly enforced via a local preconditioning/projection approach [70, 74], i.e. this construction yields a direct splitting of the local approximation spaces V_i with respective stable local basis functions

$$V_i(\omega_i) = \operatorname{span}\langle\vartheta_i^m\rangle = \mathcal{P}^{p_i}(\omega_i) + \mathcal{E}_i(\omega_i) = \mathcal{P}^{p_i}(\omega_i) \oplus \mathcal{D}_i(\omega_i).$$

Thus, a stable basis for the global PUM space

$$V^{\mathrm{PU}} = \sum_{i=1}^{N}\varphi_i\mathcal{P}^{p_i} \oplus \sum_{i=1}^{N}\varphi_i\mathcal{D}_i$$

can readily be obtained for arbitrary enrichment spaces with a flat-top PU.

To attain this highly desirable property in practice, we must construct such a flat-top PU. In one-dimension space this is trivial and many constructions, e.g. by piecewise polynomials, are available. In multiple dimensions the one-dimensional techniques can of course be tensorized, e.g. [63], but this severely limits the applicable geometric refinement schemes for the patches ω_i. A more general approach is the use of a mesh and the respective dual-mesh to define the patches ω_i and flat-top sub-patches $\omega_{\mathrm{FT},i}$ on which the PU functions are then defined as piecewise polynomials [67]. Besides these meshbased approaches there is also a meshfree construction of a flat-top PU [38, 39] which allows for rather arbitrary geometric refinement of the patches and avoids the issue of mesh generation.

Meshfree PU

The fundamental construction principle employed in [39, 40, 68] for the construction of the PU $\{\varphi_i\}$ is a d-binary tree. Based on the given point data $P = \{x_i : i = 1, \dots, \hat{N}\}$, we sub-divide a bounding-box $\mathcal{C}_\Omega \supset \Omega$ of the domain Ω until each cell

$$\mathcal{C}_i = \prod_{l=1}^{d}(c_i^l - h_i^l, c_i^l + h_i^l)$$

associated with a leaf of the tree contains at most a single point $x_i \in P$. We obtain an overlapping cover $C_\Omega = \{\omega_i\}$ from this tree by defining the cover patches ω_i as

$$\omega_i = \prod_{l=1}^{d}(c_i^l - \alpha h_i^l, c_i^l + \alpha h_i^l), \qquad \alpha > 1.$$

Note that we define a cover patch ω_i for leaf-cells $\mathcal{C}_i$ that contain a point $x_i \in P$ as well as for *empty* cells that do not contain any point from P.

To obtain a PU on a cover C_Ω with $N = \operatorname{card} C_\Omega$ we define a weight function $W_i\colon \Omega \to \mathbb{R}$ with $\operatorname{supp} W_i = \omega_i$ for each cover patch ω_i by

$$W_i(x) = \begin{cases} \mathcal{W} \circ T_i(x) & \text{if } x \in \omega_i, \\ 0 & \text{else,} \end{cases}$$

with the affine transforms $T_i\colon \overline{\omega}_i \to [-1,1]^d$ and $\mathcal{W}\colon [-1,1]^d \to \mathbb{R}$ the reference d-linear B-spline. By simple averaging of these weight functions we obtain the Shepard functions [77]

$$\varphi_i(x) = \frac{W_i(x)}{S_i(x)} \qquad \text{with} \qquad S_i(x) = \sum_{l=1}^{N} W_l(x). \tag{16}$$

The functions (16) define an admissible PU and with the choice of $\alpha < 1 + 2^{-L-1}$, where L denotes the irregularity of the employed d-binary tree, this PU has the flat-top property [71].

3.3. Implementation

The main difference in the efficient implementation of the PUM and classical FEM is in the assembly of the stiffness matrix. The enrichment functions $\eta_i^t \in \mathcal{E}_i$ on a particular patch ω_i are discontinuous or singular functions and the respective entries of the stiffness matrix can usually not be computed analytically. Thus, numerical integration must, in general, be employed in the PUM. Unfortunately, a detailed numerical analysis about the effect of numerical integration in the PUM is missing, i.e. an a priori error bound on the consistency error introduced by numerical integration in the spirit of Strang's Lemmata for the FEM is yet not available for the PUM. Thus, the PUM literature concerned with the implementation of the method, e.g. [14, 87] for FE-based PUM and [68, 69] for meshfree PUM, focuses on the construction of integration schemes that yield sufficient accuracy in practice with an acceptable operation count. Recall also that the elementary geometric entity in PUM is a patch $\omega_i = \operatorname{supp}\varphi_i$ and that these patches overlap (each other and the computational domain) in a rather arbitrary fashion. Thus, the decomposition of the computational domain into disjoint elements which we have in FEM is, in general, not (directly) available in PUM.

3.3.1. Enrichment functions

Let us shortly review the approaches [14, 35, 87] employed for the numerical integration of discontinuous and singular enrichment functions. For the sake of simplicity let us assume that the PU is given by FE shape functions and thus a mesh which decomposes the computational domain into disjoint elements is available. Then, we must be concerned with the integration of a discontinuous or singular function on a single element only where the location of the discontinuity or singularity is known a priori. If the integrand is discontinuous at the considered element, a straightforward approach is to employ e.g. a constrained mesh generation scheme locally on this element to resolve the discontinuity explicitly and to use a standard numerical integration rule on the cells of this local mesh. If the integrand is singular we can employ a local mesh generator that uses e.g. a gradient based adaptive refinement near the singularity together with standard numerical integration rules or singularity adapted rules, see also [59, 60]. These procedures can of course be coupled with an adaptive (integration) error control. Yet, the crucial point is that there is, in general, no reliable a priori bound on the required integration accuracy which would yield a valid stopping criterion for this adaptation process.

3.3.2. Partition of unity

If the PU is not constructed on a mesh but e.g. via the meshfree approach (16) we only have the overlapping patches ω_i available. Numerical integration of the weak form based on these patches is in essence comparable to a node oriented assembly of FE stiffness matrix and thus leads to an overhead in the number of evaluations of the shape functions $\varphi_i\vartheta_i^n$. Moreover, the PU functions φ_i are, in general, not smooth, e.g. $\nabla\varphi_i$ is discontinuous within the patch ω_i if a linear spline is used as a weight function in (16). Thus, the patches ω_i must be decomposed into disjoint cells which respect the discontinuities of the PU functions φ_i [68, 69]. Note that this construction and the selection of integration rules on the resulting cells must at least preserve the fundamental analytical properties of the PU, i.e.

$$\sum_{i=1}^{N} \varphi_i = 1 \quad \Longrightarrow \quad \sum_{i=1}^{N} \nabla\varphi_i = 0,$$

in the assembly of the stiffness matrix.

3.3.3. Essential boundary conditions

The implementation of essential boundary conditions in the PUM is in general somewhat more involved than in the FEM. First of all the conditions imposed on the PU functions (4) are essentially independent of the computational domain. That is the patches ω_i do not have to align with the domain boundaries. Moreover, the local approximation spaces V_i must only provide the approximation properties (6), they do not need to satisfy essential boundary conditions explicitly.

If the PU is composed of FE shape functions, we may assume that the underlying mesh is a discrete realization of the computational domain such that the implementation of essential boundary conditions is very much similar to the classical FEM. Only in the case when enrichment functions are employed near the boundary of the domain some special treatment is required. This, however, is usually just a minor technical issue that can be dealt with easily.[11]

The conforming treatment of essential boundary conditions in a general PUM requires the use of local approximation spaces V_i that satisfy the boundary conditions explicitly. In general, this requirement may be hard to realize a priori since the PUM makes no assumption on how the computational domain and its boundary are given. However, in [72] an a posteriori construction is presented for flat-top PUs. This general algebraic approach constructs a direct splitting $V_i = V_{i,I} \oplus V_{i,B}$ of the local approximation spaces V_i on patches $\omega_i \cap \partial\Omega \neq \emptyset$ into a space $V_{i,I}$ that satisfies vanishing boundary conditions and $V_{i,B}$ which must be used for the approximation of non-homogeneous boundary conditions only.

There are a number of non-conforming approaches to the implementation of essential boundary conditions which have been suggested in the context of the PUM and meshfree methods in general [3], e.g. penalty methods, Lagrange multipliers, collocation techniques, or Nitsche's method. The most natural approach seems to be Nitsche's method [61] which was originally proposed for unfitted FEM and is often employed, e.g. in Mortar methods and for the treatment of interface problems.

To quickly review this approach, let us consider the linear elastic model problem

$$\begin{aligned} -\operatorname{\mathbf{div}} \boldsymbol{\sigma}(u) &= f && \text{in} \quad \Omega \subset \mathbb{R}^d, \\ \boldsymbol{\sigma}(u) \cdot n &= g_N && \text{on} \quad \Gamma_N \subset \partial\Omega, \\ (\boldsymbol{\sigma}(u) \cdot n) \cdot t &= 0 && \text{on} \quad \Gamma_D = \partial\Omega \setminus \Gamma_N, \\ u \cdot n &= g_{D,n} && \text{on} \quad \Gamma_D = \partial\Omega \setminus \Gamma_N. \end{aligned} \tag{17}$$

Testing the PDE with an arbitrary function $v \in H^1(\Omega)$, we obtain

$$-\int_\Omega \operatorname{\mathbf{div}} \boldsymbol{\sigma}(u) \cdot v \, dx = \int_\Omega \boldsymbol{\sigma}(u) : \boldsymbol{\epsilon}(v) \, dx - \int_{\partial\Omega} \boldsymbol{\sigma}(u) n \cdot v \, ds = \int_\Omega f v \, dx$$

via integration by parts due to the symmetry of $\boldsymbol{\sigma}(u)$. Decomposing the boundary $\partial\Omega$ into the Neumann Γ_N and Dirichlet parts $\Gamma_D = \partial\Omega \setminus \Gamma_N$, we can bring the Neumann boundary conditions of (17) to the right-hand side. On the left-hand side only the *consistency* term on the Dirichlet boundary Γ_D remains.

$$\int_\Omega \boldsymbol{\sigma}(u) : \boldsymbol{\epsilon}(v) \, dx - \int_{\Gamma_D} \boldsymbol{\sigma}(u) n \cdot v \, ds = \int_\Omega f v \, dx + \int_{\Gamma_N} g_N v \, ds.$$

On Γ_D we have essential boundary conditions in normal direction but vanishing natural boundary conditions in tangential direction. Therefore, we split the integrand of the consistency term on the left-hand side into its normal and tangential parts:

$$\int_{\Gamma_D} \boldsymbol{\sigma}(u) n \cdot v \, ds = \int_{\Gamma_D} \big((n \cdot \boldsymbol{\sigma}(u) n) n + (t \cdot \boldsymbol{\sigma}(u) n) t \big) \cdot v \, ds$$

[11] *If the PU, however, is given by FE shape functions that arise from an unfitted FEM non-conforming approaches to the implementation of essential boundary conditions must be employed.*

which yields

$$\int_\Omega \boldsymbol{\sigma}(u) : \boldsymbol{\epsilon}(v)\ dx - \int_{\Gamma_D} (n \cdot \boldsymbol{\sigma}(u) n) n \cdot v\ ds = \int_\Omega f v\, dx + \int_{\Gamma_N} g_N v\, ds$$

due to $(\boldsymbol{\sigma}(u) \cdot n) \cdot t = 0$ on Γ_D. Now we are in a position to symmetrize the bilinear form consistently using the available Dirichlet boundary values $u \cdot n = g_{D,n}$ only and obtain

$$\int_\Omega \boldsymbol{\sigma}(u) : \boldsymbol{\epsilon}(v)\ dx - \int_{\Gamma_D} \left((n \cdot \boldsymbol{\sigma}(u) n) n \cdot v + (n \cdot \boldsymbol{\sigma}(v) n) n \cdot u \right)\ ds = \int_\Omega f v\, dx + \int_{\Gamma_N} g_N v\, ds - \int_{\Gamma_D} g_{D,n} (n \cdot \boldsymbol{\sigma}(v) n)\ ds.$$

Finally, we introduce a symmetric regularization term which again may only involve available boundary information to attain the weak formulation

$$a_\beta(u, v) = \langle l_\beta, v \rangle \qquad \text{for all} \quad v \in V^{\text{PU}} \tag{18}$$

with the symmetric bilinear form

$$a_\beta(u, v) = \int_\Omega \boldsymbol{\sigma}(u) : \boldsymbol{\epsilon}(v)\ dx - \int_{\Gamma_D} (n \cdot \boldsymbol{\sigma}(u) n) n \cdot v\ ds - \int_{\Gamma_D} (n \cdot \boldsymbol{\sigma}(v) n) n \cdot u\ ds + \beta \int_{\Gamma_D} (u \cdot n)(v \cdot n)\ ds$$

and the corresponding linear form

$$\langle l_\beta, v \rangle = \int_\Omega f v\, dx + \int_{\Gamma_N} g_N v\, ds - \int_{\Gamma_D} g_{D,n} (n \cdot \boldsymbol{\sigma}(v) n)\ ds + \beta \int_{\Gamma_D} g_{D,n} v \cdot n\ ds.$$

There is a unique solution $u^{\text{PU}} \in V^{\text{PU}}$ of (18) if the regularization parameter β is chosen large enough; i.e., the regularization parameter $\beta = \beta_{V^{\text{PU}}}$ is dependent on the discretization space V^{PU}. This solution u^{PU} satisfies optimal error bounds if the space V^{PU} admits the inverse estimate

$$\|(n \cdot \boldsymbol{\sigma}(v) n)\|^2_{L^2(\Gamma_D)} \leq C^2_{V^{\text{PU}}} \|v\|^2_E = C^2_{V^{\text{PU}}} \int_\Omega \boldsymbol{\sigma}(v) : \boldsymbol{\epsilon}(v)\ dx \tag{19}$$

for all $v \in V^{\text{PU}}$ with a constant $C_{V^{\text{PU}}}$ depending on the cover C_Ω and the employed local bases $\langle \vartheta_i^m \rangle$ only. If (19) holds and $C_{V^{\text{PU}}}$ is known explicitly, the regularization parameter $\beta_{V^{\text{PU}}}$ can be chosen as $\beta_{V^{\text{PU}}} > 2C^2_{V^{\text{PU}}}$ to obtain a symmetric positive definite linear system [61]. Hence, the main task associated with the use of Nitsche's approach in the PUM context is the efficient and automatic computation of the constant $C_{V^{\text{PU}}}$ [41, 42, 68].

3.3.4. Conditioning and solution techniques

As was discussed in subsection 3.2, the selection of an appropriate stable basis for general PUM approximation spaces is not straightforward and thus the arising linear systems can be severely ill-conditioned. This ill-conditioning is observed when the PU is given by FE shape functions, e.g. linear hat functions. However, only a few papers deal with the question of stability and efficient solution techniques for the PUM with a FE PU, e.g. [2, 50, 56, 90]. Thus, there is no optimal linear solver such as classical multigrid for the FEM available in this context. Recently the stable GFEM was proposed in [2] which aims at removing the ill-conditioning by changing the local enrichment spaces in a similar spirit as was done in [74]. Also the flat-top enrichment strategy [49, 86, 88] essentially changes the enrichment space, yet in a more global sense, to reduce the ill-conditioning.

For a PUM with a flat-top PU, however, we can construct stable basis functions for arbitrary enrichments and there is a multilevel solver [37, 40, 73, 74] that shows optimal convergence independent of the enrichment functions in practice. The optimality of this approach was shown in [23] for polynomial enrichment spaces.

4. Concluding remarks

The PUM is a general framework that allows for an easy integration of application-specific (a priori) information in the approximation space. Its mathematical foundation is solid and elegant yet its realization poses some technical challenges. If a classical *h*-version FE implementation serves as the basis of this framework (GFEM and XFEM), special attention must be paid to stability, ill-conditioning and the enrichment functions should be built in an element-wise fashion. If the PUM is based on a flat-top PU, the implementation must essentially be built from scratch. The elegance and extendability of the framework renders the approach very interesting from a mathematical as well as an application point of view.

References

[1] Aragón A.M., Duarte C.A., Geubelle P.H., Generalized finite element enrichment functions for discontinuous gradient fields, Internat. J. Numer. Methods Engrg., 2010, 82(2), 242–268

[2] Babuška I., Banerjee U., Stable generalized finite element method, Comput. Methods Appl. Mech. Engrg. (in press), DOI: 10.1016/j.cma.2011.09.012

[3] Babuška I., Banerjee U., Osborn J.E., Meshless and generalized finite element methods: a survey of some major results, In: Meshfree Methods for Partial Differential Equations, Bonn, 2001, Lect. Notes Comput. Sci. Eng., 26, Springer, Berlin, 2003, 1–20

[4] Babuška I., Caloz G., Osborn J.E., Special finite element methods for a class of second order elliptic problems with rough coefficients, SIAM J. Numer. Anal., 1994, 31(4), 945–981

[5] Babuška I., Lipton R., Optimal local approximation spaces for generalized finite element methods with application to multiscale problems, Multiscale Model. Simul., 2011, 9(1), 373–406

[6] Babuška I., Melenk J.M., The partition of unity method, Internat. J. Numer. Methods Engrg., 1997, 40(4), 727–758

[7] Béchet E., Minnebo H., Moës N., Burgardt B., Improved implementation and robustness study of the X-FEM for stress analysis around cracks, Internat. J. Numer. Methods Engrg., 2005, 64(8), 1033–1056

[8] Belytschko T., Black T., Elastic crack growth in finite elements with minimal remeshing, Internat. J. Numer. Methods Engrg., 1999, 45(5), 601–620

[9] Belytschko T., Gracie R., Ventura G., A review of extended/generalized finite element methods for material modelling, Modelling and Simulation in Materials Science and Engineering, 2009, 17(4), #043001

[10] Belytschko T., Krongauz Y., Organ D., Fleming M., Krysl P., Meshless methods: an overview and recent developments, Comput. Methods Appl. Mech. Engrg., 1996, 139(1-4), 3–47

[11] Belytschko T., Lu Y.Y., Gu L., Crack Propagation by Element-free Galerkin methods, Engrg. Fracture Mech., 1995, 51(2), 295–315

[12] Belytschko T., Moës N., Usui S., Parimi C., Arbitrary discontinuities in finite elements, Internat. J. Numer. Methods Engrg., 2001, 50(4), 993–1013

[13] Benzley S.E., Representation of singularities with isoparametric finite elements, Internat. J. Numer. Methods Engrg., 1974, 8(3), 537–545

[14] Bordas S., Nguyen P.V., Dunant C., Guidom A., Nguyen-Dang H., An extended finite element library, Internat. J. Numer. Methods Engrg., 2006, 71(6), 703–732

[15] Brenner S.C., Multigrid methods for the computation of singular solutions and stress intensity factors I: Corner singularities, Math. Comp., 1999, 68(266), 559–583

[16] Brenner S.C., Sung L.-Y., Multigrid method for the computation of singular solutions and stress intensity factors II: Crack singularities, BIT, 1997, 37(3), 623–643

[17] Brenner S.C., Sung L., Multigrid methods for the computation of singular solutions and stress intensity factors III: Interface singularities, Comput. Methods Appl. Mech. Engrg., 2003, 192(41-42), 4687–4702

[18] Byskov E., The calculation of stress intensity factors using the finite element method with cracked elements, Internat. J. Fracture, 1970, 6(2), 159–167

[19] Cai Z., Kim S., A finite element method using singular functions for the poisson equation: corner singularities, SIAM J. Numer. Anal., 2001, 39(1), 286–299
[20] Cai Z., Kim S., Shin B.-C., Solution methods for the Poisson equation with corner singularities: numerical results, SIAM J. Sci. Comput., 2001, 23(2), 672–682
[21] Chessa J., Belytschko T., An extended finite element method for two-phase fluids, Trans. ASME J. Appl. Mech., 2003, 70(1), 10–17
[22] Chessa J., Belytschko T., An enriched finite element method and level sets for axisymmetric two-phase flow with surface tension, Internat. J. Numer. Methods Engrg., 2003, 58(13), 2041–2064
[23] Dahmen W., Dekel S., Petrushev P., Multilevel preconditioning for partition of unity methods: some analytic concepts, Numer. Math., 2007, 107(3), 503–532
[24] De S., Bathe K.J., The method of finite spheres, Comput. Mech., 2000, 25(4), 329–345
[25] DeVore R.A., Lorentz G.G., Constructive Approximation, Grundlehren Math. Wiss., 303, Springer, Berlin, 1993
[26] Duarte C.A., Babuška I., Mesh-independent *p*-orthotropic enrichment using the generalized finite element method, Internat. J. Numer. Methods Engrg., 2002, 55(12), 1477–1492
[27] Duarte C.A., Babuška I., Oden J.T., Generalized finite element methods for three-dimensional structural mechanics problems, Comput. & Structures, 2000, 77(2), 215–232
[28] Duarte C.A., Hamzeh O.N., Liszka T.J., Tworzydlo W.W., A generalized finite element method for the simulation of three-dimensional dynamic crack propagation, Internat. J. Numer. Methods Engrg., 2001, 190(15-17), 2227–2262
[29] Duarte C.A., Kim D.-J., Analysis and applications of a generalized finite element method with global-local enrichment functions, Comput. Methods Appl. Mech. Engrg., 2007, 197(6-8), 487–504
[30] Duarte C.A., Oden J.T., An *h-p* adaptive method using clouds, Comput. Methods Appl. Mech. Engrg., 1996, 139(1-4), 237–262
[31] Duarte C.A., Oden J.T., *H-p* clouds — an *h-p* meshless method, Numer. Methods Partial Differential Equations, 1996, 12(6), 673–705
[32] Duarte C.A., Reno L.G., Simone A., A high-order generalized FEM for through-the-thickness branched cracks, Internat. J. Numer. Methods Engrg., 2007, 72(3), 325–351
[33] Fish J., Yuan Z., Multiscale enrichment based on partition of unity, Internat. J. Numer. Methods Engrg., 2005, 62(10), 1341–1359
[34] Fix G.J., Gulati S., Wakoff G.I., On the use of singular functions with finite element approximations, J. Comput. Phys., 1973, 13(2), 209–228
[35] Fries T.-P., Belytschko T., The extended/generalized finite element method: an overview of the method and its applications, Internat. J. Numer. Methods Engrg., 2010, 84(3), 253–304
[36] Gracie R., Ventura G., Belytschko T., A new fast finite element method for dislocations based on interior dicontinuities, Internat. J. Numer. Methods Engrg., 2007, 69(2), 423–441
[37] Griebel M., Oswald P., Schweitzer M.A., A particle-partition of unity method. VI. A *p*-robust multilevel solver, In: Meshfree Methods for Partial Differential Equations II, Lect. Notes Comput. Sci. Eng., 43, Springer, Berlin, 2005, 71–92
[38] Griebel M., Schweitzer M.A., A particle-partition of unity method for the solution of elliptic, parabolic, and hyperbolic PDEs, SIAM J. Sci. Comput., 2000, 22(3), 853–890
[39] Griebel M., Schweitzer M.A., A particle-partition of unity method. II. Efficient cover construction and reliable integration, SIAM J. Sci. Comput., 2002, 23(5), 1655–1682
[40] Griebel M., Schweitzer M.A., A particle-partition of unity method. III. A multilevel solver, SIAM J. Sci. Comput., 2002, 24(2), 377–409
[41] Griebel M., Schweitzer M.A., A particle-partition of unity method. V. Boundary conditions, In: Geometric Analysis and Nonlinear Partial Differential Equations, Springer, Berlin, 2003, 519–542
[42] Griebel M., Schweitzer M.A., A particle-partition of unity method. VII. Adaptivity, In: Meshfree Methods for Partial Differential Equations III, Lect. Notes Comput. Sci. Eng., 57, Springer, Berlin, 2007, 121–147
[43] Grisvard P., Elliptic Problems in Nonsmooth Domains, Monogr. Stud. Math., 24, Pitman, Boston, 1985
[44] Grisvard P., Singularities in Boundary Value Problems, Rech. Math. Appl., 22, Springer, Berlin, 1992
[45] Groß S., Reusken A., An extended pressure finite element space for two-phase incompressible flows with surface tension, J. Comput. Phys., 2007, 224(1), 40–58

[46] Hansbo A., Hansbo P., An unfitted finite element method, based on Nitsche's method, for elliptic interface problems, Comput. Methods Appl. Mech. Engrg., 2002, 191(47-48), 5537–5552
[47] Huerta A., Belytschko T., Fernández-Méndez S., Rabczuk T., Meshfree methods, In: Encyclopedia of Computational Mechanics, 1, John Wiley & Sons, Chichester, 2004, chapter 10, 279–309
[48] Kim D.-J., Duarte C.A., Proença S.P., Generalized Finite Element Method with global-local enrichments for nonlinear fracture analysis, In: Mechanics of Solids in Brazil 2009, Rio de Janeiro, April 28–30, 2009, ABCM Symposium Series in Solid Mechanics, 2, Brazilian Society of Mechanical Sciences and Engineering, 2009, 317–330
[49] Laborde P., Pommier J., Renard Y., Salaün M., Higher order extended finite element method for cracked domains, Internat. J. Numer. Methods Engrg., 2005, 64(3), 354–381
[50] Li H., A note on the conditioning of a class of generalized finite element methods, Appl. Numer. Math. (in press), DOI: 10.1016/j.apnum.2011.05.004
[51] Macri M., De S., Shepard M.S., Hierarchical tree-based discretization for the method of finite spheres, Comput. & Structures, 2003, 81(8-11), 789–803
[52] Mariani S., Perego U., Extended finite element method for quasi-brittle fracture, Internat. J. Numer. Methods Engrg., 2003, 58(1), 103–126
[53] Mazzucato A.L., Nistor V., Qu Q., A non-conforming generalized finite element method for transmission problems, preprint available at http://www.math.psu.edu/mazzucat/preprint/GFEM.pdf
[54] Melenk J.M., On approximation in meshless methods, In: Frontiers of Numerical Analysis, Durham, July 4–9, 2004, Universitext, Springer, Berlin, 2005, 65–141
[55] Melenk J.M., Babuška I., The partition of unity finite element method: basic theory and applications, Comput. Methods Appl. Mech. Engrg., 1996, 139(1-4), 289–314
[56] Menk A., Bordas S.P.A., A robust preconditioning technique for the extended finite element method, Internat. J. Numer. Methods Engrg., 2011, 85(13), 1609–1632
[57] Moës N., Dolbow J., Belytschko T., A finite element method for crack growth without remeshing, Internat. J. Numer. Methods Engrg., 1999, 46(1), 131–150
[58] Mohammadi S., Extended Finite Element Method, Blackwell, Oxford, 2008
[59] Mousavi S.E., Sukumar N., Generalized Duffy transformation for integrating vertex singularities, Comput. Mech., 2010, 45(2-3), 127–140
[60] Mousavi S.E., Sukumar N., Generalized Gaussian quadrature rules for discontinuities and crack singularities in the extended finite element method, Comput. Methods Appl. Mech. Engrg., 2010, 199(49-52), 3237–3249
[61] Nitsche J., Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind, Abh. Math. Sem. Univ. Hamburg, 1971, 36, 9–15
[62] Oden J.T., Duarte C.A., Clouds, Cracks and FEM's, In: Recent Developments in Computational and Applied Mechanics, International Center for Numerical Methods in Engineering, CIMNE, Barcelona, 1997, 302–321
[63] Oh H.-S., Jae J.W., Hong W.T., The generalized product partition of unity for the meshless methods, J. Comput. Phys., 2010, 229(5), 1600–1620
[64] Pereira J.P., Duarte C.A., Guoy D., Jiao X., *hp*-generalized FEM and crack surface representation for non-planar 3-D cracks, Internat. J. Numer. Methods Engrg., 2009, 77(5), 601–633
[65] Radtke F.K.F., Simone A., Sluys L.J., A partition of unity finite element method for obtaining elastic properties of continua with embedded thin fibres, Internat. J. Numer. Methods Engrg., 2010, 84(6), 708–732
[66] Radtke F.K.F., Simone A., Sluys L.J., A partition of unity finite element method for simulating non-linear debonding and matrix failure in thin fibre composites, Internat. J. Numer. Methods Engrg., 2011, 86(4-5), 453–476
[67] Riker C., Holzer S.M., The mixed-cell-complex partition-of-unity method, Comput. Methods Appl. Mech. Engrg., 2009, 198(13-14), 1235–1248
[68] Schweitzer M.A., A Parallel Multilevel Partition of Unity Method for Elliptic Partial Differential Equations, Lect. Notes Comput. Sci. Eng., 29, Springer, Berlin, 2003
[69] Schweitzer M.A., Efficient implementation and parallelization of meshfree and particle methods — the parallel multilevel partition of unity method, In: Frontiers of Numerical Analysis, Durham, July 4–9, 2004, Universitext, Springer, Berlin, 2005, 195–262
[70] Schweitzer M.A., A particle-partition of unity method. VIII. Hierarchical enrichment, In: Meshfree Methods for Partial Differential Equations IV, Lect. Notes Comput. Sci. Eng., 65, Springer, Berlin, 2008, 279–302
[71] Schweitzer M.A., Generalized Finite Element and Meshfree Methods, Habilitation thesis, Univeristät Bonn, 2008

[72] Schweitzer M.A., An algebraic treatment of essential boundary conditions in the particle-partition of unity method, SIAM J. Sci. Comput., 2008/09, 31(2), 1581–1602
[73] Schweitzer M.A., Robust multilevel partition of unity method for problems with jumping coefficients, Technical report, Universität Stuttgart, 2011
[74] Schweitzer M.A., Stable enrichment and local preconditioning in the particle-partition of unity method, Numer. Math., 2011, 118(1), 137–170
[75] Schweitzer M.A., Multilevel particle-partition of unity method, Numer. Math., 2011, 118(2), 307–328
[76] Shabir Z., Van der Giessen E., Duarte C.A., Simone A., The role of cohesive properties on intergranular crack propagation in brittle polycrystals, Modelling and Simulation in Materials Science and Engineering, 2011, 19(3), #035006
[77] Shepard D., A two-dimensional interpolation function for irregularly-spaced data, In: Proceedings of the 1968 23rd ACM National Conference, Association for Computing Machinery, New York, 1968, 517–524
[78] Simone A., Duarte C.A., Van der Giessen E., A generalized finite element method for polycrystals with discontinuous grain boundaries, Internat. J. Numer. Methods Engrg., 2006, 67(8), 1122–1145
[79] Strouboulis T., Babuška I., Copps K., The design and analysis of the generalized finite element method, Comput. Methods Appl. Mech. Engrg., 2000, 181(1-3), 43–69
[80] Strouboulis T., Babuška I., Hidajat R., The generalized finite element method for Helmholtz equation: theory, computation, and open problems, Comput. Methods Appl. Mech. Engrg., 2006, 195(37-40), 4711–4731
[81] Strouboulis T., Copps K., Babuška I., The generalized finite element method, Comput. Methods Appl. Mech. Engrg., 2001, 190(32-33), 4081–4193
[82] Strouboulis T., Hidajat R., Babuška I., The generalized finite element method for Helmholtz equation. II. Effect of choice of handbook functions, error due to absorbing boundary conditions and its assessment, Comput. Meth. Appl. Mech. Engrg., 2008, 197(5), 364–380
[83] Strouboulis T., Zhang L., Babuška I., Generalized finite element method using mesh-based handbooks: application to problems in domains with many voids, Comput. Methods Appl. Mech. Engrg., 2003, 192(28-30), 3109–3161
[84] Strouboulis T., Zhang L., Babuška I., *p*-version of the generalized FEM using mesh-based handbooks with applications to multiscale problems, Internat. J. Numer. Methods Engrg., 2004, 60(10), 1639–1672
[85] Stüben K., An introduction to algebraic multigrid, An appendix to: Trottenberg U., Oosterlee C.W., Schüller A., Multigrid, Academic Press, San Diego, 2001, 413–532
[86] Sukumar N., Pask J.E., Classical and enriched finite element formulations for Bloch-periodic boundary conditions, Internat. J. Numer. Methods Engrg., 2009, 77(8), 1121–1138
[87] Sukumar N., Prévost J.-H., Modeling quasi-static crack growth with the extended finite element method. I. Computer implementation, Internat. J. Solids Structures, 2003, 40(26), 7513–7537
[88] Ventura G., Gracie R., Belytschko T., Fast integration and weight function blending in the extended finite element method, Internat. J. Numer. Methods Engrg., 2009, 77(1), 1–29
[89] Ventura G., Moran B., Belytschko T., Dislocations by partition of unity, Internat. J. Numer. Methods Engrg., 2005, 62(11), 1463–1487
[90] Xu J., Zikatanov L.T., On multigrid methods for generalized finite element methods, In: Meshfree Methods for Partial Differential Equations, Bonn, 2001, Lect. Notes Comput. Sci. Eng., 26, Springer, Berlin, 2003, 401–418
[91] Catalogue of Finite Element Books, http://www.solid.ikp.liu.se/fe/index.html
[92] National Agency for Finite Element Methods and Standards, http://www.nafems.org