

Balance Laws: Non Local Mixed Systems and IBVPs

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Abstract

Scalar hyperbolic balance laws in several space dimensions play a central role in this thesis. First, we deal with a new class of mixed parabolic–hyperbolic systems on all $\mathbb{R}^n$: we obtain the basic well-posedness theorems, devise an *ad hoc* numerical algorithm, prove its convergence and investigate the qualitative properties of the solutions. The extension of these results to bounded domains requires a deep understanding of the initial boundary value problem (IBVP) for hyperbolic balance laws. The last part of the thesis provides rigorous estimates on the solution to this IBVP, under precise regularity assumptions.

In Chapter 1 we introduce a predator–prey model. A non local and non linear balance law is coupled with a parabolic equation: the former describes the evolution of the predator density, the latter that of prey. The two equations are coupled both through the convective part of the balance law and the source terms. The drift term is a non local function of the prey density. This allows the movement of predators to be directed towards the regions where the concentration of prey is higher. We prove the well-posedness of the system, hence the existence and uniqueness of solution, the continuous dependence on the initial data and various stability estimates.

In Chapter 2 we devise an algorithm to compute approximate solutions to the mixed system introduced above. The balance law is solved numerically by a Lax–Friedrichs type method via dimensional splitting, while the parabolic equation is approximated through explicit finite-differences. Both source terms are integrated by means of a second order Runge–Kutta scheme. The key result in Chapter 2 is the convergence of this algorithm. The proof relies on a careful tuning between the parabolic and the hyperbolic methods and exploits the non local nature of the convective part in the balance law.

This algorithm has been implemented in a series of `Python` scripts. Using them, we obtain information about the possible order of convergence and we investigate the qualitative properties of the solutions. Moreover, we observe the formation of a striking pattern: while prey diffuse, predators accumulate on the vertices of a regular lattice.

The analytic study of the system above is on all $\mathbb{R}^n$. However, both possible biological applications and numerical integrations suggest that the boundary plays a relevant role. With the aim of studying the mixed hyperbolic–parabolic system in a bounded domain, we noticed that for balance laws known results lack some of the estimates necessary to deal with the coupling. In Chapter 3 we then focus on the IBVP for a general balance law in a bounded domain. We prove the well-posedness of this problem, first with homogeneous boundary condition, exploiting the vanishing viscosity technique and the doubling of variables method, then for the non homogeneous case, mainly thanks to elliptic techniques. We pay particular attention to the regularity assumptions and provide rigorous estimates on the solution.

Riassunto

Questa tesi riguarda leggi di bilancio scalari iperboliche in più dimensioni spaziali. Nella prima parte viene introdotta una nuova classe di sistemi parabolico–iperbolici su $\mathbb{R}^n$: abbiamo dimostrato la buona posizione, sviluppato un algoritmo *ad hoc* per calcolare le soluzioni numeriche, verificato la sua convergenza e studiato le proprietà qualitative delle soluzioni. Per estendere tali risultati a domini limitati, è necessaria una dettagliata conoscenza del problema ai valori iniziali e al bordo (IBVP) per una generale legge di bilancio. L’ultima parte della tesi fornisce stime rigorose sulla soluzione di questo IBVP, sotto precise ipotesi di regolarità.

Nel Capitolo 1 abbiamo introdotto un modello preda–predatore. Una legge di bilancio non locale e non lineare è accoppiata ad un’equazione parabolica: la prima descrive l’evoluzione dei predatori, la seconda delle prede. Il termine convettivo è una funzione non locale della densità delle prede. Il movimento dei predatori è quindi diretto verso le regioni in cui la concentrazione delle prede è maggiore. Abbiamo dimostrato l’esistenza e l’unicità della soluzione del sistema, la dipendenza continua dai dati iniziali e varie stime di stabilità.

Nel Capitolo 2 abbiamo sviluppato un algoritmo per calcolare soluzioni approssimate del sistema misto di cui al Capitolo 1. Per la legge di bilancio abbiamo utilizzato uno schema di tipo Lax–Friedrichs con *dimensional splitting*, mentre l’equazione parabolica è stata approssimata attraverso differenze finite. I termini di sorgente sono stati integrati con un metodo tipo Runge–Kutta al secondo ordine. Il risultato principale del Capitolo 2 è la convergenza dell’algoritmo. Tale dimostrazione si basa su un’attenta sincronizzazione dei metodi utilizzati per le due equazioni e sfrutta la natura non locale del termine convettivo della legge di bilancio.

Questo algoritmo è stato implementato in una serie di procedure **Python**. Queste ci hanno permesso di stimare l’ordine di convergenza dell’algoritmo e di indagare le proprietà qualitative delle soluzioni. Abbiamo inoltre osservato la formazione di un particolare *pattern*: mentre le prede diffondono, i predatori si accumulano sui vertici di un reticolo regolare.

Lo studio analitico del sistema è in tutto $\mathbb{R}^n$. Tuttavia, sia le possibili applicazioni biologiche sia le integrazioni numeriche suggeriscono che il bordo giochi un ruolo chiave. Ci siamo accorti che fra i risultati noti per le leggi di bilancio mancano alcune stime necessarie per studiare tale sistema misto in un dominio limitato. Nel Capitolo 3 della tesi ci siamo quindi concentrati sull’IBVP per una generale legge di bilancio in un dominio limitato. Ne abbiamo dimostrato la buona posizione, utilizzando la tecnica della *vanishing viscosity* e il metodo del *doubling of variables*, dapprima per condizioni al bordo omogenee e in seguito nel caso non omogeneo, principalmente grazie a tecniche ellittiche. Abbiamo prestato particolare attenzione alle condizioni di regolarità e fornito stime rigorose sulla soluzione.

Zusammenfassung

Skalare hyperbolische Erhaltungsgleichungen in mehreren Raumdimensionen spielen eine zentrale Rolle in dieser Arbeit. Zuerst behandeln wir eine neue Klasse von gemischten parabolisch–hyperbolischen Systemen in $\mathbb{R}^n$: Wir zeigen, dass die grundlegenden Wohlgestelltheits–Theoreme für diese Systeme gelten, entwickeln einen *ad hoc* numerischen Algorithmus, beweisen seine Konvergenz und untersuchen die qualitative Eigenschaften der Lösungen. Die Erweiterung dieser Ergebnisse auf beschränkte Gebiete erfordert ein tiefes Verständnis der Anfangsrandwertprobleme (IBVP) für hyperbolische Erhaltungsgleichungen. Im letzten Teil der Arbeit werden rigorose Abschätzungen für die Lösung dieses IBVP, die spezielle Regularitätsannahmen erfüllen, hergeleitet.

In Kapitel 1 stellen wir ein Räuber–Beute-Modell vor. Eine nichtlokale und nichtlineare Erhaltungsgleichung wird mit einer parabolischen Gleichung gekoppelt: Die hyperbolische Gleichung beschreibt dabei die Dichteevolution der Räuber, die parabolische beschreibt die der Beute. Die zwei Gleichungen sind sowohl durch den konvektiven Teil der Erhaltungsgleichung als auch durch die Quellterme gekoppelt. Der Drift Term ist eine nichtlokale Funktion, die von der Beutedichte abhängt. Das ermöglicht die Bewegung der Raubtiere zu den Regionen, in denen die Konzentration der Beute höher ist. Wir beweisen die Wohlgestelltheit des Systems und damit sowohl die Existenz und Eindeutigkeit der Lösung, als auch die stetige Abhängigkeit der Lösung von den Anfangsdaten und verschiedene Stabilitätsabschätzungen.

In Kapitel 2 entwickeln wir einen Algorithmus, um Näherungslösungen für das oben eingeführte hyperbolisch-parabolische System zu berechnen. Die Erhaltungsgleichung wird numerisch durch eine Lax–Friedrichs–Methode mit Dimensionssplitting gelöst, während die Lösung der parabolischen Gleichung durch explizite Finite–Differenzen berechnet wird. Beide Quellterme werden durch ein Runge–Kutta Schema zweiter Ordnung integriert. Das Hauptresultat in Kapitel 2 ist die Konvergenz des Algorithmus. Der Beweis beruht auf einer sorgfältigen Abstimmung zwischen der parabolischen und hyperbolischen Methode und nutzt den nicht lokalen Charakter des konvektiven Teils in der Erhaltungsgleichung.

Dieser Algorithmus wurde in **Python** implementiert um verschiedene numerische Studien durchzuführen. Durch diese Studien haben wir Informationen über die mögliche Konvergenzordnung und die qualitativen Eigenschaften der Lösungen erhalten. Darüber hinaus haben wir die Bildung eines markanten Musters für die Räuber-Beute-Verteilung beobachtet: während sich die Beute durch Diffusion auf dem ganzen Rechengebiet verteilt, sammeln sich die Räuber auf den Ecken eines regelmäßigen Rechtecksgitters.

Die analytische Untersuchung des obigen Systems wurde auf ganz $\mathbb{R}^n$ durchgeführt. Jedoch legen sowohl mögliche biologische Anwendungen als auch numerische Integrationen nahe, dass die Grenzen eine relevante Rolle spielen. Allerdings fehlen für die Erhaltungsgleichungen einige der Abschätzungen, die notwendig sind um die bekannten Resultate für gekoppelte hyperbolisch–parabolische Systeme auf beschränkten Gebieten anwenden zu können. In Kapitel 3 konzentrieren wir uns auf das IBVP für eine allgemeine Erhaltungsgleichung auf einem beschränkten Gebiet. Wir beweisen die Wohlgestelltheit dieses Problems zunächst mit homogenen Randbedingungen, indem wir vanishing–viscosity Technik und das Variablenverdopplungsverfahren anwenden. Die Erweiterung auf den nichthomogenen Fall nutzt vor allem elliptische Techniken. Wir achten dabei besonders auf die Regularitätsannahmen und leiten rigorose Abschätzungen für die Lösung her.

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Introduction

Scalar hyperbolic balance laws in several space dimensions play a central role in this thesis. First, we deal with a new class of mixed parabolic–hyperbolic systems on all $\mathbb{R}^n$. In this regard, we obtain the basic well–posedness theorems, we devise an *ad hoc* numerical algorithm, we prove its convergence and we investigate the qualitative properties of the solutions. The natural extension of these results to bounded domain requires a deeper understanding of the initial boundary value problem for hyperbolic balance laws. Under precise regularity assumptions, the last part of this thesis provides rigorous estimates on the solution to this IBVP.

In Chapter 1 (based on [20, 24, 25]), we introduce a predator–prey model describing two competing populations. A non local and non linear balance law is coupled with a parabolic equation: the former describes the evolution of predator density, the latter that of prey. The two equations are connected both through the convective part of the balance law and through the source terms. More precisely, the drift term in the balance law is a non local function of the prey density. This allows to model the fact that predators move towards regions where there is a higher concentration of prey. At the same time, prey diffuse. The source terms describe the feeding, the birth rate of the prey and the mortality of the predators, similarly to what happens in Lotka–Volterra equations.

We establish the analytic theory for this class of systems, proving the basic well–posedness results, which consist in a theorem ensuring existence, uniqueness, continuous dependence of solutions on the initial data and various stability estimates. These results are obtained by studying, separately but symmetrically, the parabolic and the hyperbolic equations. Differently from the standard parabolic literature, here we are forced to obtain estimates on the solution to the prey equation in the $\mathbf{L}^1$, $\mathbf{L}^\infty$ and $\mathbf{BV}$ norms. Indeed, it is only in these norms that stability estimates on the solutions to hyperbolic balance laws are currently available. These results allow us to deal with the full mixed system, taking advantage of the non local nature of the convective term in the predators’ equation.

A qualitative study of the solution to this new system requires reliable numerical integrations. For this purpose, in Chapter 2 (based on [61]) we devise an algorithm to efficiently compute approximate solutions to the coupled system introduced in Chapter 1. The predators’ equation is solved numerically by a Lax–Friedrichs type method via dimensional splitting, while the prey equation is approximated by an explicit finite–difference method. Both source terms are integrated by means of a second order Runge–Kutta method. The operator splitting technique *glues* the algorithm together.

The key result in Chapter 2 is the convergence of this algorithm. In particular, we show the strong convergence of the hyperbolic variable in $\mathbf{L}^1$ and the weak* convergence in $\mathbf{L}^\infty$ of the parabolic part. The proof relies on the one hand on a careful tuning between the hyperbolic and the parabolic methods. On the other hand, it exploits strongly the non local

nature of the convective part in the balance law. As usual, (discretised) **BV** type estimates allow to use a compactness criterion.

This algorithm has been implemented in a series of `Python` scripts. Using them, we document the theoretically proved convergence and obtain information about the possible order of convergence. Further qualitative properties of the solutions are then investigated. First, we show how some features of the solutions depend on the choice of various parameters. Secondly, a striking pattern formation phenomenon leads to a peculiar asymptotic state, see for instance Figure 1.3 and Figure 2.5: while prey diffuse, predators accumulate on the vertices of a regular lattice.

The analytic study of the system above is on all $\mathbb{R}^n$. However, both possible biological applications and numerical integrations suggest that the boundary plays a relevant role. We aim to study this mixed hyperbolic–parabolic system in a bounded domain. In the literature, many results on parabolic equations in bounded domains are available. Concerning balance laws, known results lack the estimates necessary to deal with the coupled system. In order to fix these issues, in Chapter 3 (based on [26]) we focus on the initial–boundary value problem for a general balance law in a bounded domain. We prove the well–posedness of this problem, showing the existence of an entropy solution, its uniqueness and its $\mathbf{L}^1$ Lipschitz continuity as a function of time, of the initial datum and of the boundary datum. First, we exploit the vanishing viscosity technique and the doubling of variables method to solve the initial boundary value problem with homogeneous boundary condition. The non homogeneous case is then recovered by means mainly of elliptic techniques. We pay particular attention to the regularity requirements and provide rigorous estimates on the solution.

Where we go from here

The mixed parabolic–hyperbolic system introduced in Chapter 1 suggests several further research directions. First, results on more general source terms are apparently easily available, through suitable modifications of the techniques developed in the present thesis. Such an extension would, in particular, comprehend the case of hyperbolic prey and parabolic predators. A rigorous analytic study of the qualitative properties of solutions discovered through numerical integrations is very appealing. A first step in this direction consists in providing bounds on the dependence of the solution on the source terms and on the convective term, i.e. on all the parameters entering the system. Far more difficult is the characterisation of the pattern appearing as asymptotic state and of the initial data leading to this pattern. We remark that the parameters characterising the model, e.g. prey birth rate or predators death rate, can be used as *controls* to pursue specific goals, for instance the maximisation of predators’ growth or the partial confinement of prey.

From the numerical point of view, a rigorous estimate on the convergence speed of the numerical scheme introduced in Chapter 2 is still unavailable. Moreover, we stress that, due to the non locality of the model, large convolution products need to be estimated at each time step. This causes the overall computing time of the present numerical scheme to be very high. Substituting Lax–Friedrichs algorithm for the balance law and the forward explicit method for the parabolic equation with different integration procedures might lead to a significant reduction in the computational cost. Indeed the sample integrations displayed in Chapters 1 and 2 were possible thanks to a careful parallelisation of the convolution products.

Concerning the basic theory of the IBVP for a general scalar balance law in several space

dimensions studied in Chapter 3, we remark that the dependence of the solution on the flux is still an open question, despite the fact that almost 40 years passed since the publication of the classical work by Bardos, Leroux and Nédélec [10]. A more accessible open problem is the extension of the present results to less regular domains, such as (hyper)rectangles, and to less regular boundary data.

Chapter 1

Hyperbolic Predators vs. Parabolic Prey

The content of this Chapter appeared on *Communications in Mathematical Sciences* as [25].

1.1 Introduction

Consider the following predator–prey model

$$\begin{cases} \partial_t u + \nabla \cdot (u v(w)) = (\alpha w - \beta) u, \\ \partial_t w - \mu \Delta w = (\gamma - \delta u) w, \end{cases} \quad (1.1.1)$$

where $u = u(t, x)$, respectively $w = w(t, x)$, is the predator, respectively prey, density at time $t \in \mathbb{R}^+$ and position $x \in \mathbb{R}^n$. Prey diffuse according to a parabolic equation, since $\mu > 0$. Here, γ is the prey birth rate and δ the prey mortality due to the predators. The predator density evolves according to a hyperbolic balance law, where the coefficient α in the source term accounts for the increase in the predator density due to feeding on prey, while β is the predator mortality rate. The flow $u v(w)$ accounts for the preferred predators' direction. The velocity v is in general a *non local* and nonlinear function of the prey density. A typical choice can be

$$v(w) = \kappa \frac{\nabla(w * \eta)}{\sqrt{1 + \|\nabla(w * \eta)\|^2}}, \quad (1.1.2)$$

meaning that predators move towards regions of higher concentrations of prey. Indeed, when η is a positive smooth mollifier with $\int_{\mathbb{R}^n} \eta dx = 1$, the space convolution $(w(t) * \eta)(x)$ has the meaning of an average of the prey density at time t around position x . The denominator $\sqrt{1 + \|\nabla(w * \eta)\|^2}$ is merely a smooth normalisation factor, so that the positive parameter κ is the maximal predator speed.

Two key features of the model (1.1.1) are the following. First, while prey diffuse in all directions due to the Laplacian in the w equation, predators in (1.1.1) have a directed movement, for instance drifting towards regions where the prey density is higher. This allows, for instance, to describe predators chasing prey. Second, predators have a well defined horizon. Indeed, the radius of the support of η in (1.1.2) defines how far predators can “*feel*” the presence of prey and, hence, the direction in which they move.

The aim of this Chapter is to study the class of models (1.1.1) under suitable assumptions on v . We prove below existence, uniqueness, continuous dependence on the initial datum, and various stability estimates for the solutions to (1.1.1). Here, solutions are found in the space $\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV}$ for the predators and in $\mathbf{L}^1 \cap \mathbf{L}^\infty$ for the prey. Thus, solutions are here understood in the distributional sense, see definitions 1.1, 1.3, and 1.6. Moreover, all analytic results hold in any space dimension, the explicit dependence of the constants entering the estimates being duly reported in the proofs below. Besides, qualitative properties of solutions are shown by means of numerical integrations.

With reference to possible biological applications, the words *prey* and *predator* should be here understood in their widest sense. The diffusion in the second equation may well describe the evolution of a chemical substance or also of temperature. Indeed, setting for instance $\delta = 0$, the second equation decouples from the first, and the first one fits into [37, Formula (0.1)], see also [18, 19]. In this connection, we recall that the interest in non local hyperbolic models is increasing in several fields.

Various multi-D models devoted to crowd dynamics are considered in [19, Section 4] and in [18] in the case of a single population, in [21] for several populations. In these works, solutions are understood in the weak sense of Kruřkov, see [47], and well posedness is proven in any space dimension.

Non local models for aggregation and swarming are presented in [33, 34], where the existence of smooth or Lipschitz continuous solutions is proved in 1D and in 2D, the n dimensional case being considered in [35]. Due to the biological motivation, in these papers only one population is considered.

In structured population biology, the use of non local models based on conservation laws is very common, also in a measure valued setting, see for example [1, 15, 29] and the references therein.

On the other hand, the use of purely parabolic equations in predator–prey models with spatial distributions is rather classical, see for instance [57, Section 1.2]. With respect to these models, the use of a first order differential operator in the predator density allows to describe the directed movement of predators and ensures that they have a finite propagation speed. Indeed, if the initial distribution of predators has compact support, then the region they occupy grows with finite speed and remains compact for all times, as proved below.

As analytical tools, in this Chapter we consider separately the equations

$$\partial_t u + \nabla \cdot (c(t, x) u) = b(t, x) u \quad \text{and} \quad \partial_t w - \mu \Delta w = a(t, x) w. \quad (1.1.3)$$

For the former, we exploit the classical results by Kruřkov [47] and the more recent stability estimates proved in [22, 52]. The literature on the latter equation in (1.1.3) is vast; however, our considering it in $\mathbf{L}^1 \cap \mathbf{L}^\infty$ on all $\mathbb{R}^n$ seems to be somewhat unconventional, hence we provide detailed proofs of the necessary estimates. The two equations (1.1.3) are here studied following exactly the same template and analogous results are obtained. Once the necessary estimates for the solutions to (1.1.3) are proven, a fixed point argument allows us to prove the well posedness of (1.1.1) and several stability estimates.

The next section presents the analytic results: first the main theorem and then the propositions at its basis. Section 1.3 is devoted to sample numerical integrations of (1.1.1). All technical details are deferred to the final Section 1.4.

1.2 Analytic Results

This section is devoted to the well posedness theorem that constitutes the main result of this Chapter. All proofs are deferred to Section 1.4.

Our first step is the rigorous definition of solution to (1.1.1).

Definition 1.1. *Let $T > 0$ be fixed. A solution to the system (1.1.1) on $[0, T]$ is a pair $(u, w) \in \mathbf{C}^0([0, T]; \mathbf{L}^1(\mathbb{R}^n; \mathbb{R}^2))$ such that*

- *setting $a(t, x) = \gamma - \delta u(t, x)$, w is a weak solution to $\partial_t w - \mu \Delta w = a w$;*
- *setting $b(t, x) = \alpha w(t, x) - \beta$ and $c(t, x) = \left(v(w(t)) \right)(x)$, u is a weak solution to $\partial_t u + \nabla \cdot (u c) = b u$.*

The extension to the case of the Cauchy problem is immediate. Below, in Definition 1.3, respectively in Definition 1.6, we state and use different definitions of solutions to the parabolic equation $\partial_t w - \mu \Delta w = a w$, respectively to the hyperbolic equation $\partial_t u + \nabla \cdot (u c) = b u$, and prove their equivalence in Lemma 1.4, respectively in Lemma 1.7.

Throughout, we work in the spaces

$$\begin{aligned} \mathcal{X} &= (\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n; \mathbb{R}) \times (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^n; \mathbb{R}) \quad \text{and} \\ \mathcal{X}^+ &= (\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n; \mathbb{R}^+) \times (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^n; \mathbb{R}^+) \end{aligned} \quad (1.2.1)$$

with the norm

$$\|(u, w)\|_{\mathcal{X}} = \|u\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} + \|w\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})}. \quad (1.2.2)$$

System (1.1.1) is defined by a few real parameters and by the map v , which is assumed to satisfy the following condition:

(v) $v: (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^n; \mathbb{R}) \rightarrow (\mathbf{C}^2 \cap \mathbf{W}^{1,\infty})(\mathbb{R}^n; \mathbb{R}^n)$ admits a constant K and an increasing map $C \in \mathbf{L}_{loc}^\infty(\mathbb{R}^+; \mathbb{R}^+)$ such that, for all $w, w_1, w_2 \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^n; \mathbb{R})$,

$$\begin{aligned} \|v(w)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R}^n)} &\leq K \|w\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})}, \\ \|\nabla v(w)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R}^{n \times n})} &\leq K \|w\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})}, \\ \|v(w_1) - v(w_2)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R}^n)} &\leq K \|w_1 - w_2\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})}, \\ \|\nabla(\nabla \cdot v(w))\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R}^n)} &\leq C \left(\|w\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \right) \|w\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})}, \\ \|\nabla \cdot (v(w_1) - v(w_2))\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} &\leq C \left(\|w_2\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \right) \|w_1 - w_2\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})}. \end{aligned}$$

Above, the bound on the $\mathbf{L}^\infty$ norm of $v(w)$ by means of the $\mathbf{L}^1$ norm of w is typical of a non local, e.g. convolution, operator. Indeed, Lemma 1.9 below ensures that under reasonable regularity conditions on the kernel η , the operator v in (1.1.2) satisfies (v).

Relying solely on (v), we state the main result of this Chapter.

Theorem 1.2. *Fix $\alpha, \beta, \gamma, \delta \geq 0$ and $\mu > 0$. Assume that v satisfies (v). Then, there exists a map*

$$\mathcal{R}: \mathbb{R}^+ \times \mathcal{X}^+ \rightarrow \mathcal{X}^+$$

with the following properties:

1. **$\mathcal{R}$ is a semigroup:** $\mathcal{R}_0 = \text{Id}$ and $\mathcal{R}_{t_2} \circ \mathcal{R}_{t_1} = \mathcal{R}_{t_1+t_2}$ for all $t_1, t_2 \in \mathbb{R}^+$.
2. **$\mathcal{R}$ solves (1.1.1):** for all $(u_o, w_o) \in \mathcal{X}^+$, the map $t \rightarrow \mathcal{R}_t(u_o, w_o)$ solves the Cauchy Problem

$$\begin{cases} \partial_t u + \nabla \cdot (u v(w)) = (\alpha w - \beta) u \\ \partial_t w - \mu \Delta w = (\gamma - \delta u) w \\ u(0, x) = u_o(x) \\ w(0, x) = w_o(x) \end{cases}$$

in the sense of Definition 1.1. In particular, for all $(u_o, w_o) \in \mathcal{X}^+$ the map $t \rightarrow \mathcal{R}_t(u_o, w_o)$ is continuous in time.

3. **Local Lipschitz continuity in the initial datum:** for all $r > 0$ and for all $t \in \mathbb{R}^+$, there exists a positive $\mathcal{L}(t, r)$ such that for all $(u_1, w_1), (u_2, w_2) \in \mathcal{X}^+$ with

$$\|u_i\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \leq r, \quad \text{TV}(u_i) \leq r, \quad \|w_i\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \leq r, \quad \|w_i\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \leq r$$

for $i = 1, 2$, the following estimate holds:

$$\|\mathcal{R}_t(u_1, w_1) - \mathcal{R}_t(u_2, w_2)\|_{\mathcal{X}} \leq \mathcal{L}(t, r) \|(u_1, w_1) - (u_2, w_2)\|_{\mathcal{X}}.$$

4. **Growth estimates:** for all $(u_o, w_o) \in \mathcal{X}^+$ and for all $t \in \mathbb{R}^+$, denote $(u, w)(t) = \mathcal{R}_t(u_o, w_o)$. Then,

$$\begin{aligned} \|u(t)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} &\leq \|u_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \exp\left(\alpha \frac{e^{\gamma t} - 1}{\gamma} \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})}\right), \\ \|u(t)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} &\leq \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \exp\left((\alpha + K) \frac{e^{\gamma t} - 1}{\gamma} \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})}\right), \\ \|w(t)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} &\leq \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} e^{\gamma t}, \\ \|w(t)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} &\leq \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} e^{\gamma t}. \end{aligned}$$

5. **Propagation speed:** if $(u_o, w_o) \in \mathcal{X}^+$ is such that $\text{spt}(u_o) \subseteq B(0, \rho_o)$, then, for all $t \in \mathbb{R}^+$,

$$\text{spt}(u(t)) \subseteq B(0, \rho(t)) \quad \text{where} \quad \rho(t) = \rho_o + K t e^{\gamma t} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})}.$$

An explicit estimate of the Lipschitz constant $\mathcal{L}(t, r)$ is provided at (1.4.37).

Theorem 1.2 is proved through careful estimates on the parabolic problem

$$\begin{cases} \partial_t w - \mu \Delta w = a(t, x) w \\ w(t_o, x) = w_o(x) \end{cases} \quad (1.2.3)$$

and, separately, on the balance law

$$\begin{cases} \partial_t u + \nabla \cdot (c(t, x) u) = b(t, x) u \\ u(t_o, x) = u_o(x). \end{cases} \quad (1.2.4)$$

Our approaches to both the evolution equations (1.2.3) and (1.2.4) are identical. We recall below the key definitions, prove the basic well posedness results, and provide rigorous stability estimates, always referring to the spaces in (1.2.1) and with reference to the $\mathbf{L}^1$ norm.

To improve the readability of the statements below, we denote by $\mathcal{O}(t)$ an increasing smooth function of time t , depending on the space dimension n and on various norms of the coefficients a, μ in (1.2.3) and b, c in (1.2.4). All proofs are deferred to Section 1.4, where explicit estimates for all constants are provided.

Throughout, we fix $t_o, T \in \mathbb{R}^+$, with $T > t_o$, and denote

$$I = [t_o, T] \quad \text{and} \quad J = \left\{ (t_1, t_2) \in I^2 : t_1 < t_2 \right\}. \quad (1.2.5)$$

For completeness, we recall the following notions from the theory of parabolic equations. They are similar to various results in the wide literature on parabolic problems, see for instance [3, 55, 60], but here we are dealing with $\mathbf{L}^1$ solutions on the whole space.

Inspired by [60, Section 48.3], we give the following definition, where we use the notation (1.2.5).

Definition 1.3. *Let $a \in \mathbf{L}^\infty(I \times \mathbb{R}^n; \mathbb{R})$ and $w_o \in \mathbf{L}^1(\mathbb{R}^n; \mathbb{R})$. A weak solution to (1.2.3) is a function $w \in \mathbf{C}^0(I; \mathbf{L}^1(\mathbb{R}^n; \mathbb{R}))$ such that, for all test functions $\varphi \in \mathbf{C}^1(I; \mathbf{C}_c^2(\mathbb{R}^n; \mathbb{R}))$*

$$\int_{t_o}^T \int_{\mathbb{R}^n} (w \partial_t \varphi + \mu w \Delta \varphi + a w \varphi) \, dx \, dt = 0 \quad (1.2.6)$$

and $w(t_o, x) = w_o(x)$.

The following lemma is similar to various results in the literature, see for instance [60, Section 48.3], and is here recalled for completeness. The heat kernel is denoted by $H_\mu(t, x) = (4\pi\mu t)^{-n/2} \exp\left(-\|x\|^2/(4\mu t)\right)$, where $t > 0$, $x \in \mathbb{R}^n$, and $\mu > 0$ is fixed.

Lemma 1.4. *Let $a \in \mathbf{L}^\infty(I \times \mathbb{R}^n; \mathbb{R})$. Assume that $w_o \in \mathbf{L}^1(\mathbb{R}^n; \mathbb{R})$. Then,*

1. *any function w satisfying*

$$w(t, x) = (H_\mu(t - t_o) * w_o)(x) + \int_{t_o}^t \left(H_\mu(t - \tau) * (a(\tau) w(\tau)) \right)(x) \, d\tau \quad (1.2.7)$$

solves (1.2.3) in the sense of Definition 1.3;

2. *any solution to (1.2.3) in the sense of Definition 1.3 satisfies (1.2.7).*

The well posedness of (1.2.3) is now proved.

Proposition 1.5. *Let $a \in \mathbf{L}^\infty(I \times \mathbb{R}^n; \mathbb{R})$. Then, the Cauchy problem (1.2.3) generates a map $\mathcal{P}: J \times \mathbf{L}^1(\mathbb{R}^n; \mathbb{R}) \rightarrow \mathbf{L}^1(\mathbb{R}^n; \mathbb{R})$ with the following properties:*

1. **$\mathcal{P}$ is a Process:** $\mathcal{P}_{t,t} = \text{Id}$ for all $t \in I$ and $\mathcal{P}_{t_2, t_3} \circ \mathcal{P}_{t_1, t_2} = \mathcal{P}_{t_1, t_3}$ for all $t_1, t_2, t_3 \in I$, with $t_1 \leq t_2 \leq t_3$.
2. **$\mathcal{P}$ solves (1.2.3):** for all $w_o \in \mathbf{L}^1(\mathbb{R}^n; \mathbb{R})$, the function $t \rightarrow \mathcal{P}_{t_o, t} w_o$ solves the Cauchy problem (1.2.3) in the sense of Definition 1.3.

3. **Regularity in w_o :** for all $(t_o, t) \in J$, the map $\mathcal{P}_{t_o, t}: \mathbf{L}^1(\mathbb{R}^n; \mathbb{R}) \rightarrow \mathbf{L}^1(\mathbb{R}^n; \mathbb{R})$ is linear and continuous, with

$$\|\mathcal{P}_{t_o, t} w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \leq \mathcal{O}(t) \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})}.$$

4. **$\mathbf{L}^\infty$ estimate:** for all $w_o \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^n; \mathbb{R})$, for all $(t_o, t) \in J$,

$$\|\mathcal{P}_{t_o, t} w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \leq \mathcal{O}(t) \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})}.$$

5. **Stability with respect to $\mathbf{a}$:** let $a_1, a_2 \in \mathbf{L}^\infty(I \times \mathbb{R}^n; \mathbb{R})$ with $a_1 - a_2 \in \mathbf{L}^1(I \times \mathbb{R}^n; \mathbb{R})$ and call $\mathcal{P}^1, \mathcal{P}^2$ the corresponding processes. Then, for all $(t_o, t) \in J$ and for all $w_o \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^n; \mathbb{R})$,

$$\|\mathcal{P}_{t_o, t}^1 w_o - \mathcal{P}_{t_o, t}^2 w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \leq \mathcal{O}(t) \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \|a_1 - a_2\|_{\mathbf{L}^1([t_o, t] \times \mathbb{R}^n; \mathbb{R})}.$$

6. **Positivity:** if $w_o \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^n; \mathbb{R})$ and $w_o \geq 0$, then $\mathcal{P}_{t_o, t} w_o \geq 0$ for all $(t_o, t) \in J$.

7. **Regularity in (t, x) :** if $w_o \in (\mathbf{L}^1 \cap \mathbf{C}^1)(\mathbb{R}^n; \mathbb{R})$, then $(t, x) \rightarrow (\mathcal{P}_{t_o, t} w_o)(x) \in \mathbf{C}^1(I \times \mathbb{R}^n; \mathbb{R})$.

8. **Regularity in time:** for all $w_o \in \mathbf{L}^1(\mathbb{R}^n; \mathbb{R})$, the map $t \rightarrow \mathcal{P}_{t_o, t} w_o$ is in $\mathbf{C}^0(I; \mathbf{L}^1(\mathbb{R}^n; \mathbb{R}))$, and, moreover, for every $\vartheta \in]0, 1[$ and for all $\tau, t_1, t_2 \in I$ with $t_2 \geq t_1 \geq \tau > t_o$

$$\|\mathcal{P}_{t_o, t_2} w_o - \mathcal{P}_{t_o, t_1} w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \leq \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \left[\frac{n}{\tau - t_o} + \mathcal{O}(t_2) \right] |t_2 - t_1|^\vartheta.$$

9. **$\mathbf{W}^{1,1}$ estimate:** for all $w_o \in \mathbf{L}^1(\mathbb{R}^n; \mathbb{R})$, for all $(t_o, t) \in J$,

$$\|\nabla(\mathcal{P}_{t_o, t} w_o)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R}^n)} \leq \frac{\mathcal{O}(t)}{\sqrt{t - t_o}} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})}.$$

We now follow the same template used in the preceding proposition and lemma, but referring to the hyperbolic problem (1.2.4). Similarly to [28, Section 4.3] and [62, Section 3.5], we give the following definition, where we used the notation (1.2.5).

Definition 1.6. Let $b \in \mathbf{L}^\infty(I \times \mathbb{R}^n; \mathbb{R})$, $c \in \mathbf{L}^\infty(I \times \mathbb{R}^n; \mathbb{R}^n)$ and $u_o \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^n; \mathbb{R})$. A weak solution to (1.2.4) is a function $u \in \mathbf{C}^0(I; \mathbf{L}^1(\mathbb{R}^n; \mathbb{R}))$ such that for all test functions $\varphi \in \mathbf{C}_c^1(\overset{\circ}{I} \times \mathbb{R}^n; \mathbb{R})$

$$\int_{t_o}^T \int_{\mathbb{R}^n} (u \partial_t \varphi + u c \cdot \nabla \varphi + b u \varphi) \, dx \, dt = 0 \quad (1.2.8)$$

and $u(t_o, x) = u_o(x)$.

The following Lemma is analogous to Lemma 1.4, with the usual integral formula (1.2.7) replaced by integration along characteristics, see (1.2.9).

Lemma 1.7. Let c be such that $c \in (\mathbf{C}^0 \cap \mathbf{L}^\infty)(I \times \mathbb{R}^n; \mathbb{R}^n)$, $c(t) \in \mathbf{C}^1(\mathbb{R}^n; \mathbb{R}^n) \quad \forall t \in I$, $\nabla c \in \mathbf{L}^\infty(I \times \mathbb{R}^n; \mathbb{R}^{n \times n})$. Assume that $b \in \mathbf{L}^\infty(I \times \mathbb{R}^n; \mathbb{R})$ and $u_o \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^n; \mathbb{R})$. Then

1. the function u defined by

$$u(t, x) = u_o(X(t_o; t, x)) \exp \left(\int_{t_o}^t \left(b(\tau, X(\tau; t, x)) - \operatorname{div} c(\tau, X(\tau; t, x)) \right) d\tau \right), \quad (1.2.9)$$

where the map $t \mapsto X(t; t_o, x_o)$ solves the Cauchy Problem

$$\begin{cases} \dot{X} = c(t, X) \\ X(t_o) = x_o, \end{cases} \quad (1.2.10)$$

is a Kruřkov solution to (1.2.4), i.e. for all $k \in \mathbb{R}$ and for all $\varphi \in \mathbf{C}_c^1(\mathring{I} \times \mathbb{R}^n; \mathbb{R}^+)$,

$$\int_{t_o}^T \int_{\mathbb{R}^n} [(u - k)(\partial_t \varphi + c \cdot \nabla \varphi) + (b u - k \operatorname{div} c) \varphi] \operatorname{sgn}(u - k) dx dt \geq 0 \quad (1.2.11)$$

and $u(t_o, x) = u_o(x)$, hence, u solves (1.2.4) in the sense of Definition 1.6;

2. any solution to (1.2.4) in the sense of Definition 1.6 coincides with u as defined in (1.2.9).

Above, our choice of the Kruřkov definition is motivated by our using the $\mathbf{L}^1$ bounds in [22, 47, 52] in the proofs of **3.** and **5.** below.

Proposition 1.8. *We pose the assumptions:*

(b) $b \in (\mathbf{C}^1 \cap \mathbf{L}^\infty)(I \times \mathbb{R}^n; \mathbb{R}); \nabla b \in \mathbf{L}^1(I \times \mathbb{R}^n; \mathbb{R}^n);$

(c) $c \in (\mathbf{C}^2 \cap \mathbf{L}^\infty)(I \times \mathbb{R}^n; \mathbb{R}^n); \nabla c \in \mathbf{L}^\infty(I \times \mathbb{R}^n; \mathbb{R}^{n \times n}); \nabla(\operatorname{div} c) \in \mathbf{L}^1(I \times \mathbb{R}^n; \mathbb{R}^n).$

Then, the Cauchy Problem (1.2.4) generates a map $\mathcal{H}: J \times (\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n; \mathbb{R}) \rightarrow (\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n; \mathbb{R})$ with the following properties:

1. **$\mathcal{H}$ is a process:** $\mathcal{H}_{t,t} = \operatorname{Id}$ for all $t \in I$ and $\mathcal{H}_{t_2, t_3} \circ \mathcal{H}_{t_1, t_2} = \mathcal{H}_{t_1, t_3}$ for all $t_1, t_2, t_3 \in I$, with $t_1 \leq t_2 \leq t_3$.
2. **$\mathcal{H}$ solves (1.2.4):** for all $u_o \in (\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n; \mathbb{R})$, the function $t \rightarrow \mathcal{H}_{t_o, t} u_o$ solves the Cauchy problem (1.2.4) in the sense of Definition 1.6.
3. **Regularity in u_o :** for all $(t_o, t) \in J$ the map $\mathcal{H}_{t_o, t}: (\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n; \mathbb{R}) \rightarrow (\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n; \mathbb{R})$ is linear and continuous, with

$$\|\mathcal{H}_{t_o, t} u_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \leq \mathcal{O}(t) \|u_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})}.$$

4. **$\mathbf{L}^\infty$ estimate:** for all $u_o \in (\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n; \mathbb{R})$, for all $(t_o, t) \in J$,

$$\|\mathcal{H}_{t_o, t} u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \leq \mathcal{O}(t) \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})}.$$

5. **Stability with respect to b, c :** if b_1, b_2 satisfy (b) with $b_1 - b_2 \in \mathbf{L}^1(I \times \mathbb{R}^n; \mathbb{R})$ and c_1, c_2 satisfy (c) with $\operatorname{div}(c_1 - c_2) \in \mathbf{L}^1(I \times \mathbb{R}^n; \mathbb{R})$, call $\mathcal{H}^1, \mathcal{H}^2$ the corresponding processes. Then, for all $(t_o, t) \in J$ and for all $u_o \in (\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n; \mathbb{R})$,

$$\begin{aligned} & \left\| \mathcal{H}_{t_o, t}^1 u_o - \mathcal{H}_{t_o, t}^2 u_o \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \\ & \leq \mathcal{O}(t) \left(\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} + \operatorname{TV}(u_o) \right) \|c_1 - c_2\|_{\mathbf{L}^1([t_o, t]; \mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R}^n))} \\ & \quad + \mathcal{O}(t) \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \left(\|b_1 - b_2\|_{\mathbf{L}^1([t_o, t] \times \mathbb{R}^n; \mathbb{R})} + \|\operatorname{div}(c_1 - c_2)\|_{\mathbf{L}^1([t_o, t] \times \mathbb{R}^n; \mathbb{R})} \right). \end{aligned}$$

6. **Positivity:** if $u_o \in (\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n; \mathbb{R})$ and $u_o \geq 0$, then $\mathcal{H}_{t_o, t} u_o \geq 0$ for all $(t_o, t) \in J$.
7. **Total variation bound:** if $u_o \in (\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n; \mathbb{R})$, then, for all $(t_o, t) \in J$,

$$\mathrm{TV}(\mathcal{H}_{t_o, t} u_o) \leq \mathcal{O}(t) \left(\|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} + \mathrm{TV}(u_o) \right).$$

8. **Regularity in time:** for all $u_o \in (\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n; \mathbb{R})$, the map $t \rightarrow \mathcal{H}_{t_o, t} u_o$ is in $\mathbf{C}^{0,1}(I; \mathbf{L}^1(\mathbb{R}^n; \mathbb{R}))$, moreover for all $t_1, t_2 \in J$,

$$\begin{aligned} & \|\mathcal{H}_{t_o, t_2} u_o - \mathcal{H}_{t_o, t_1} u_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \\ & \leq \mathcal{O}(t_2) \left(\|u_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} + \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} + \mathrm{TV}(u_o) \right) |t_2 - t_1|. \end{aligned}$$

9. **Finite propagation speed:** let $(t_o, t) \in J$ and $u_o \in (\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n; \mathbb{R})$ have compact support $\mathrm{spt} u_o$. Then, also, $\mathrm{spt} \mathcal{H}_{t_o, t} u_o$ is compact.

1.3 Numerical Integrations

To illustrate some qualitative properties of the solutions to (1.1.1), we present the result of a few numerical integrations.

To integrate both equations we use the operator splitting algorithm to combine the differential operators and the source terms. The balance law is integrated by means of the Lax–Friedrichs scheme with dimensional splitting [53, Section 19.5]. For the parabolic equation, we use the forward finite differences algorithm. Both the source terms are solved using a second order Runge–Kutta method. For more details on the algorithm and the proof of its convergence see Chapter 2, as well as [61].

Notice that the numerical integration of (1.1.1) requires a convolution integral to be computed at each time step. This puts a constraint on the space mesh, which should be sufficiently small with respect to the radius of the support of the convolution kernel to allow a good approximation of this integral.

Here, we focus on the two-dimensional case, that is $n = 2$, and use the vector field v in (1.1.2) with the compactly supported mollifier

$$\eta(x) = \hat{\eta} \left(\ell^2 - \|x\|^2 \right)^3 \chi_{B(0, \ell)}(x) \quad \text{with } \hat{\eta} \in \mathbb{R}^+ \text{ such that } \int_{\mathbb{R}^2} \eta(x) \, dx = 1. \quad (1.3.1)$$

The analytic theory developed above is referred to as the Cauchy problem on the whole space $\mathbb{R}^2$. In both examples below, the numerical domain of integration is the rectangle $[0, 1] \times [0, 1]$. The necessary boundary conditions are different in the two cases and are specified below. The time step $(\Delta t)_P$ for the parabolic equation and the one $(\Delta t)_H$ for the hyperbolic part are chosen so that $(\Delta t)_P$ is of the order of $((\Delta t)_H)^2$. The time step for the hyperbolic equation complies with the usual CFL condition.

Below, we constrain both unknown functions u and w to remain equal to the initial datum all along the boundary, which is acceptable in the first equation since no wave in the solution to the balance law ever hits the numerical boundary. Concerning the second equation, the choice of these boundary conditions amounts to assume that the displayed solution is part of a solution defined on all $\mathbb{R}^2$ that gives a constant inflow into the computational domain.

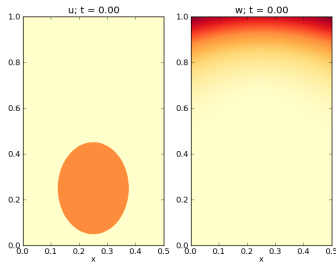
1.3.1 Predators Chasing Prey

We present a situation in which the effect of the first order transport term in the predator equation is clearly visible, as well as the well known Lotka–Volterra type effect in which a species apparently almost disappears and then its density rises again.

We set v as in (1.1.2), η as in (1.3.1) and

$$\begin{array}{llll} \alpha & = & 2 & \beta & = & 1 & \kappa & = & 1 & \ell & = & 0.0375 \\ \gamma & = & 1 & \delta & = & 2 & \mu & = & 0.5 \end{array} \quad (1.3.2)$$

with initial datum (see figure below)



$$u_o(x, y) = 4 \chi_A(x, y)$$

$$w_o(x, y) = 3(2y - 1) \max\{0, h(x, y)\} \chi_B(x, y)$$

where (1.3.3)

$$h(x, y) = (4x - 1)^2 + (4y - 2)^2 - 0.25$$

$$A = \{(x, y) \in \mathbb{R}^2 : (8x - 2)^2 + (1.25(4y - 1))^2 \leq 1\}$$

$$B = \{(x, y) \in \mathbb{R}^2 : y \geq 0.5\}.$$

The result of the numerical integration is in Figure 1.1. At first, the prey are outside the horizon of predators. Hence the latter decrease. Thanks to diffusion, some of the prey enter the region where predators feel their presence. This causes predators to move towards the highest prey density. Therefore, predators immediately increase and their effect on the prey population is clearly seen, as shown also by the graph of the integrals of u and w in Figure 1.2.

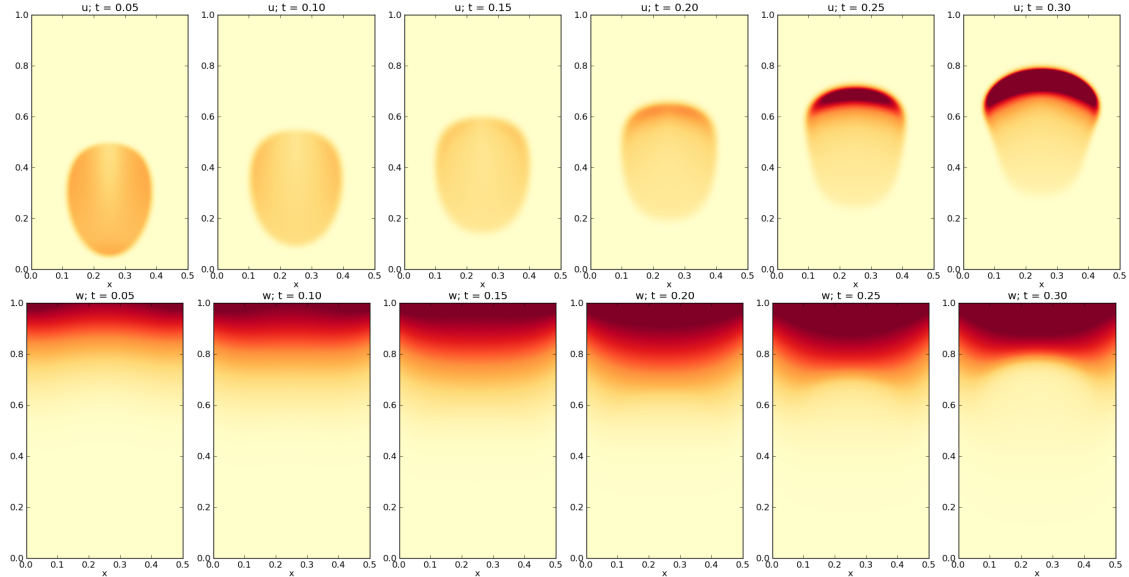


Figure 1.1: Numerical integration of (1.1.1)–(1.3.1)–(1.3.2) with initial datum (1.3.3) at different times. First row: predator density u ; second row: prey density w . The darker is the colour, the higher is the density. First, predators decrease due to lack of nutrients. Thanks to diffusion, prey reach the zone where they are “seen” by predators. Then, clearly, predators are attracted towards prey and their density starts to increase. This solution was obtained with space mesh $\Delta x = \Delta y = 0.00125$.

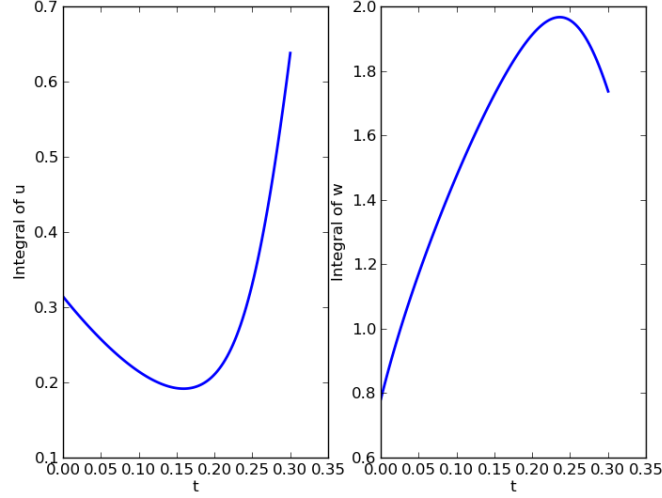


Figure 1.2: The integrals of u , left, and w , right, over the computational domain versus time; u and w are the numerical solutions to (1.1.1)–(1.3.1)–(1.3.2).

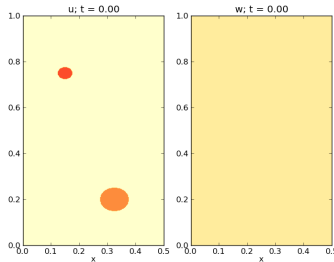
We remark that, in the present setting, as time grows, undesired effects due to the presence of the boundary become relevant.

1.3.2 A Dynamic Equilibrium

In this case, the numerical solution displays an interesting asymptotic state in which the diffusion caused by the Laplacian in the prey equation counterbalances the first order non local differential operator in the predator equation. The outcome is the onset of a discrete, quite regular, structure, see Figure 1.3. We set v as in (1.1.2), η as in (1.3.1) and

$$\begin{array}{llll} \alpha & = & 1 & \beta & = & 0.2 & \kappa & = & 1 & \ell & = & 0.0625 \\ \gamma & = & 0.4 & \delta & = & 24 & \mu & = & 0.5 & & & \end{array} \quad (1.3.4)$$

with initial datum (see figure below)



$$\begin{aligned} u_0(x, y) &= 0.25 \chi_C(x, y) + 0.2 \chi_D(x, y) \\ w_0(x, y) &= 0.2 \\ \text{where} & \\ C &= \{(x, y) \in \mathbb{R}^2 : (4x - 0.6)^2 + (4y - 3)^2 < 0.01\} \\ D &= \{(x, y) \in \mathbb{R}^2 : (4x - 1.3)^2 + (4y - 0.8)^2 < 0.04\}. \end{aligned} \quad (1.3.5)$$

In this integration, predators first almost disappear, move towards the central part of the numerical domain and then start to increase. Slowly, a regular pattern arises. Predators focus in small regions regularly distributed. These regions display a fairly stable behaviour while passing from being arranged along 5 to along 6 columns, see Figure 1.3, second line.

From the analytical point of view, this pattern can be explained as a dynamic equilibrium between the first order non local operator present in the predator equation and the Laplacian in the prey equation. Where predators accumulate, their feeding on prey causes a “hole” in

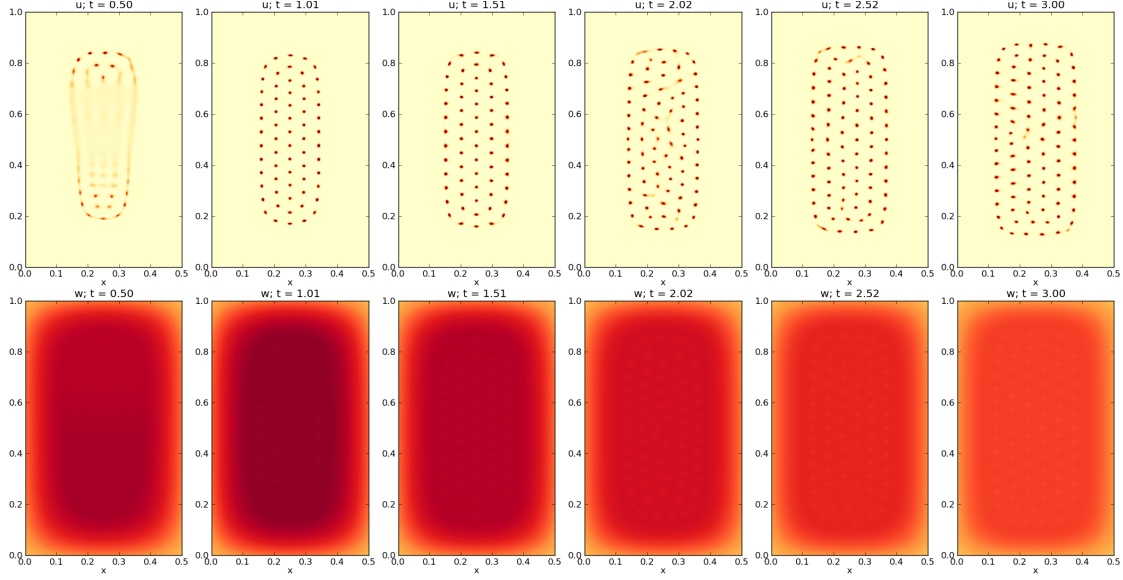


Figure 1.3: Numerical integration of (1.1.1)–(1.3.1)–(1.3.4) with initial datum (1.3.5) at different times. First row: predator density u ; second row: prey density w . The lighter is the colour, the lower is the density. First, predators decrease due to lack of nutrients and move towards the central region. Then, a discrete periodic pattern arises with predators focused in small regions regularly distributed along 5 columns and, at a later time, along 6 columns. This solution was obtained with space mesh $\Delta x = \Delta y = 0.0025$.

the prey density, see figures 1.3. As a consequence, the average gradient of the prey density, which directs the movement of predators, almost vanishes by symmetry considerations. Hence, predators almost do not move. At the same time, the diffusion of the prey keeps filling the “holes”, thus providing a persistent amount of nutrient to predators.

Coherently with this explanation, numerical integrations confirm that the above asymptotic state essentially depends on the size of the support of η . Indeed, we integrate the same initial datum (1.3.5) with the same parameters (1.3.4), except for ℓ . More precisely, we choose

$$\ell = 0.03125, \quad \ell = 0.0625, \quad \ell = 0.125, \quad \ell = 0.25.$$

We can see that as ℓ decreases, also the distance between pairwise nearest peaks in the predator density u decreases, being slightly smaller than ℓ , and more peaks are possible, see figures 1.4 and 1.5.

An other interesting feature is to see what happens when one of the parameters in (1.3.4) varies, keeping the initial datum (1.3.5) fixed. For instance, let γ assume the following values:

$$\gamma = 0.1, \quad \gamma = 0.2, \quad \gamma = 0.4, \quad \gamma = 0.6, \quad \gamma = 0.8,$$

while all the other parameters are the same. The parameter γ represents the birth rate of prey: as it gets smaller, less prey are born. The same pattern as before appears for every values of γ , with difference in number, distribution and heights of the peaks, see Figure 1.6, where we focus only on the solution for predators u at a particular time. In Figure 1.7 we can see the evolution in time of the $\mathbf{L}^1$ -norm of the densities u and w . For larger γ , the

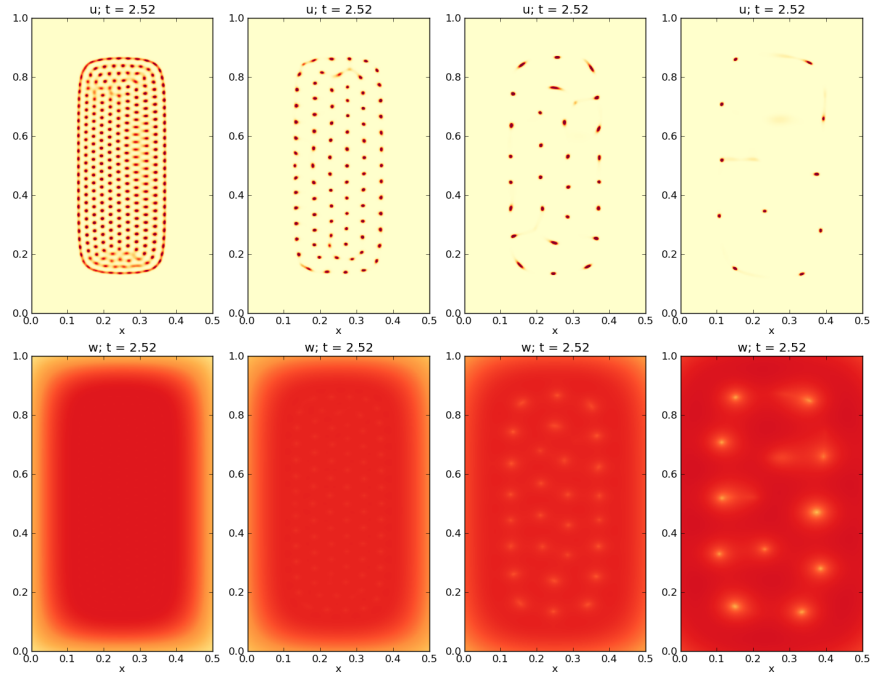


Figure 1.4: Solutions to (1.1.1)–(1.3.1)–(1.3.4) with initial datum (1.3.5) computed at time $t = 2.52$ with different values of ℓ , i.e. from the left $\ell = 0.03125$, $\ell = 0.0625$, $\ell = 0.125$ and $\ell = 0.25$. First row: predator density u ; second row: prey density w . As ℓ decreases, also the distance among peaks in the u density decreases and more peaks are possible. The darker is the colour, the higher is the density. These solutions were obtained with space mesh $\Delta x = \Delta y = 0.0025$.

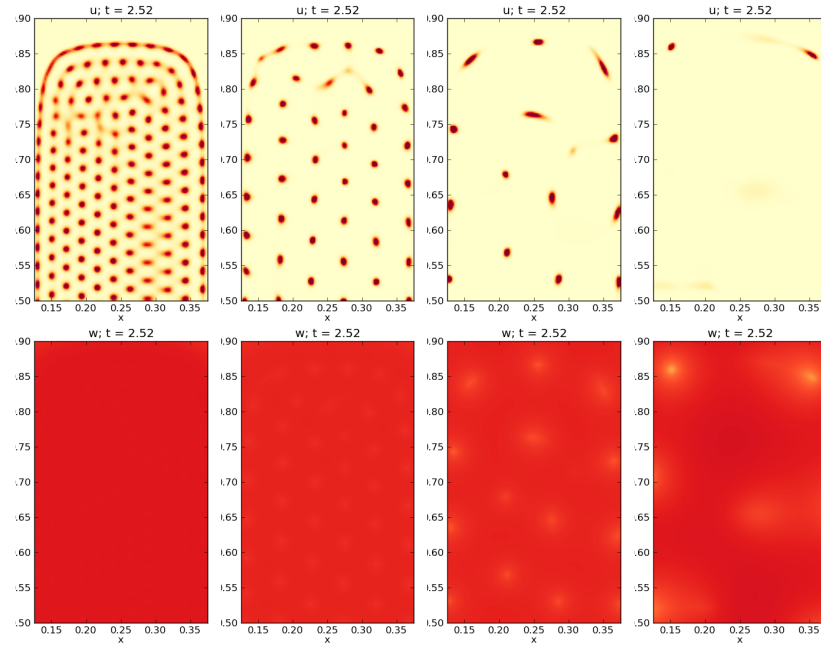


Figure 1.5: Zoom of Figure 1.4, restricted to the intervals $x \in [0.125, 0.3105]$ and $y \in [0.5, 0.9]$. The darker is the colour, the higher is the density.

total mass of predators u grows more, see the graphs on the left of Figure 1.7. The same seems to happen to the total mass of prey. However, as time passes, the great amounts of predators present for larger values of γ permits to control the prey, leading to a reduction of their mass.

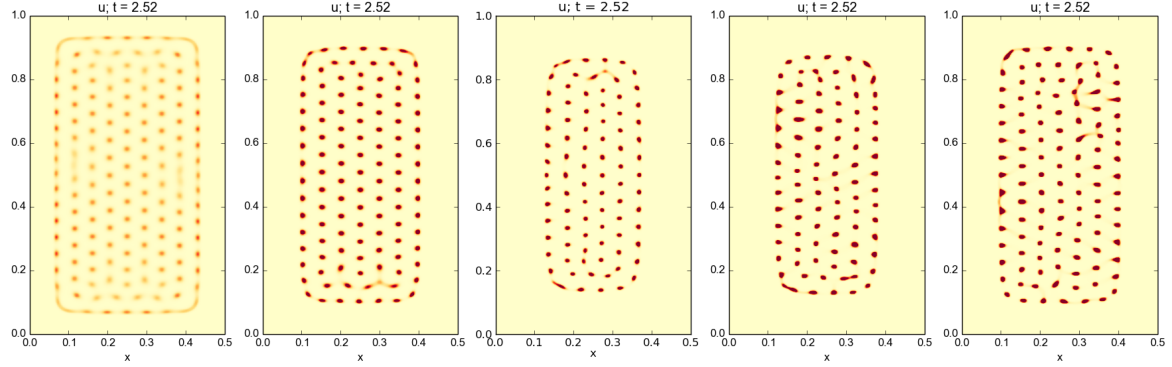


Figure 1.6: Numerical solution to (1.1.1)–(1.3.1)–(1.3.4) with initial datum (1.3.5) computed at time $t = 2.52$ for different values of γ , i.e. from the left $\gamma = 0.1$, $\gamma = 0.2$, $\gamma = 0.4$, $\gamma = 0.6$ and $\gamma = 0.8$. Only the solution for predators u is displayed, since the structure is more visible. Darker colours indicate higher densities. As γ changes, the number of peaks in u varies. The space mesh size is $\Delta x = \Delta y = 0.0025$.

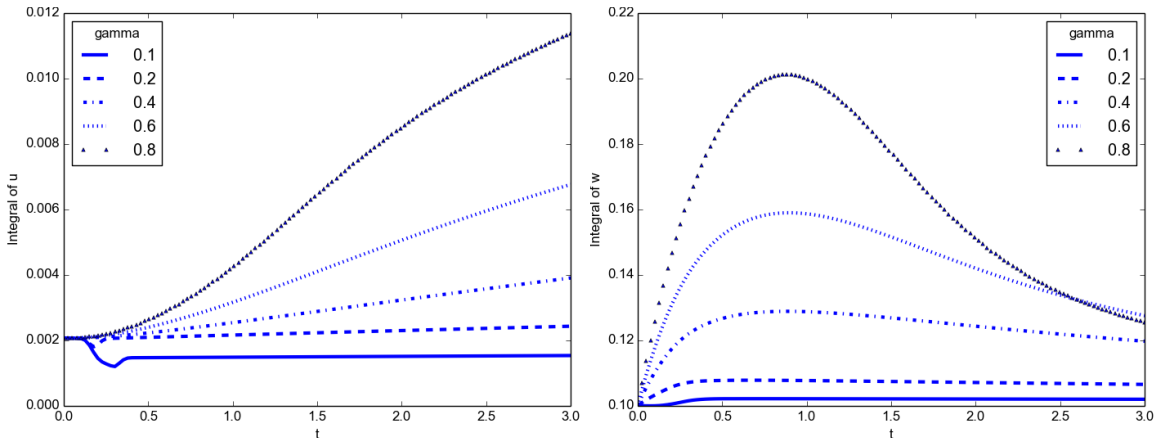


Figure 1.7: Evolution in time of the integral of u , on the left, and w , on the right, numerical solution of (1.1.1)–(1.3.1)–(1.3.4) with initial datum (1.3.5) for different values of γ , i.e. $\gamma = 0.1$, $\gamma = 0.2$, $\gamma = 0.4$, $\gamma = 0.6$ and $\gamma = 0.8$. These integrals represent the total mass of predators and prey respectively.

1.4 Technical Details

1.4.1 Proofs Related to $\partial_t w - \mu \Delta u = a(t, x) w$

We use below the constant:

$$J_n = \frac{\Gamma((n+1)/2)}{\Gamma(n/2)} \quad (1.4.1)$$

where Γ is the Gamma function. Moreover, recall the classical estimates

$$\|H_\mu(t)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} = 1 \quad (1.4.2)$$

$$\|\nabla H_\mu(t)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} = \frac{1}{\sqrt{\mu t}} \frac{\Gamma((n+1)/2)}{\Gamma(n/2)} = \frac{J_n}{\sqrt{\mu t}} \quad (1.4.3)$$

$$\begin{aligned} \partial_t H_\mu(t, x) &= \frac{1}{(4\pi\mu t)^{n/2}} \frac{\|x\|^2 - 2n\mu t}{4\mu t^2} \exp\left(-\frac{\|x\|^2}{4\mu t}\right) \\ \|\partial_t H_\mu(t)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} &\leq \frac{n}{t}. \end{aligned} \quad (1.4.4)$$

Proof of Lemma 1.4. 1. Since $H_\mu(t) \in \mathbf{C}^\infty(\mathbb{R}^n; \mathbb{R})$ and w satisfies (1.2.7), clearly $w(t) \in \mathbf{C}^\infty(\mathbb{R}^n; \mathbb{R})$ for all $t > t_o$. If $\varphi \in \mathbf{C}^1(I; \mathbf{C}_c^2(\mathbb{R}^n; \mathbb{R}))$ then $\int_{\mathbb{R}^n} \mu w \Delta \varphi \, dx = \int_{\mathbb{R}^n} \mu \Delta w \varphi \, dx$. Note first that $\mu \Delta H_\mu(t) = \partial_t H_\mu(t)$, and compute preliminarily

$$\begin{aligned} \mu \Delta w(t, x) &= \mu \Delta \left((H_\mu(t - t_o) * w_o)(x) \right) + \int_{t_o}^t \mu \Delta \left((H_\mu(t - \tau) * (a(\tau) w(\tau)))(x) \right) d\tau \\ &= (\mu \Delta H_\mu(t - t_o) * w_o)(x) + \int_{t_o}^t (\mu \Delta H_\mu(t - \tau) * (a(\tau) w(\tau)))(x) d\tau \\ &= (\partial_t H_\mu(t - t_o) * w_o)(x) + \int_{t_o}^t (\partial_t H_\mu(t - \tau) * (a(\tau) w(\tau)))(x) d\tau \\ &= \partial_t (H_\mu(t - t_o) * w_o)(x) + \int_{t_o}^t \partial_t (H_\mu(t - \tau) * (a(\tau) w(\tau)))(x) d\tau. \end{aligned}$$

Setting $\mathcal{H}(t, x) = \int_{t_o}^t (H_\mu(t - \tau) * (a(\tau) w(\tau)))(x) d\tau$, the line above becomes

$$\mu \Delta w(t, x) = \partial_t (H_\mu(t - t_o) * w_o)(x) + \frac{d}{dt} \mathcal{H}(t, x) - a(t, x) w(t, x).$$

We are now able to prove (1.2.6):

$$\begin{aligned} &\int_{t_o}^T \int_{\mathbb{R}^n} (w \partial_t \varphi + \mu \Delta w \varphi + a w \varphi) \, dx \, dt \\ &= \int_{t_o}^T \int_{\mathbb{R}^n} \left[(H_\mu(t - t_o) * w_o)(x) \partial_t \varphi(t, x) + \mathcal{H}(t, x) \partial_t \varphi(t, x) \right. \\ &\quad \left. + \partial_t \left((H_\mu(t - t_o) * w_o)(x) \right) \varphi(t, x) + \left(\frac{d}{dt} \mathcal{H}(t, x) \right) \varphi(t, x) - a(t, x) w(t, x) \varphi(t, x) \right. \\ &\quad \left. + a(t, x) w(t, x) \varphi(t, x) \right] \, dx \, dt \\ &= \int_{t_o}^T \int_{\mathbb{R}^n} \frac{d}{dt} \left[\left((H_\mu(t - t_o) * w_o)(x) + \mathcal{H}(t, x) \right) \varphi(t, x) \right] \, dx \, dt \\ &= 0. \end{aligned}$$

It is immediate to verify that if w satisfies (1.2.7), the initial condition holds.

2. Let w satisfy (1.2.7) and w_* be a weak solution to (1.2.3). Then, by the step above, the function $W = w - w_*$ is a weak solution to the linear Equation (1.2.3) with zero initial datum. By [30, Theorem 2.24], it follows that $W = 0$.

□

Proof of Proposition 1.5. For all $t \in I$ denote $A(t) = \exp \left(\int_{t_o}^t \|a(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} d\tau \right)$. The proofs of **1.**–**2.** are well known in the parabolic literature, see [3, 55]. By Lemma 1.4, recall that the solution $w(t, x) = (\mathcal{P}_{t_o, t} w_o)(x)$ to (1.2.3) satisfies (1.2.7).

3. Standard computations, using also (1.4.2), lead to:

$$\begin{aligned} \|w(t)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} &\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} H_\mu(t - t_o, x - \xi) |w_o(\xi)| d\xi dx \\ &\quad + \int_{\mathbb{R}^n} \int_{t_o}^t \int_{\mathbb{R}^n} H_\mu(t - \tau, x - \xi) |a(\tau, \xi) w(\tau, \xi)| d\xi d\tau dx \\ &\leq \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} + \int_{t_o}^t \|a(\tau) w(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} d\tau \\ &\leq \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} + \int_{t_o}^t \|a(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \|w(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} d\tau. \end{aligned}$$

An application of Gronwall's lemma yields the thesis:

$$\|w(t)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \leq A(t) \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})}.$$

4. By (1.2.7) and (1.4.2),

$$\begin{aligned} \|w(t)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} &\leq \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} + \int_{t_o}^t \|a(\tau) w(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} d\tau \\ &\leq \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} + \int_{t_o}^t \|a(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \|w(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} d\tau. \end{aligned}$$

An application of Gronwall Lemma gives the desired result:

$$\|w(t)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \leq A(t) \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})}.$$

5. Denote $w_i(t) = \mathcal{P}_{t_o, t}^i w_o$ and $A_i(t) = \exp \left(\int_{t_o}^t \|a_i(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} d\tau \right)$, for $i = 1, 2$. Use (1.2.7), (1.4.2) and 4. above:

$$\begin{aligned} &\|w_1(t) - w_2(t)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \\ &\leq \int_{\mathbb{R}^n} \int_{t_o}^t \int_{\mathbb{R}^n} H_\mu(t - \tau, x - \xi) |a_1(\tau, \xi) w_1(\tau, \xi) - a_2(\tau, \xi) w_2(\tau, \xi)| d\xi d\tau dx \\ &\leq \int_{t_o}^t \|H_\mu(t - \tau)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \|a_1(\tau) w_1(\tau) - a_2(\tau) w_2(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} d\tau \\ &= \int_{t_o}^t \|a_1(\tau) w_1(\tau) - a_2(\tau) w_2(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} d\tau \\ &\leq \int_{t_o}^t \|a_1(\tau) w_1(\tau) - a_2(\tau) w_1(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} d\tau + \int_{t_o}^t \|a_2(\tau) w_1(\tau) - a_2(\tau) w_2(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} d\tau \\ &\leq \int_{t_o}^t \|a_1(\tau) - a_2(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \|w_1(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} d\tau \\ &\quad + \int_{t_o}^t \|a_2(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \|w_1(\tau) - w_2(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} d\tau \end{aligned}$$

$$\begin{aligned} &\leq \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \int_{t_o}^t A_1(\tau) \|a_1(\tau) - a_2(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} d\tau \\ &\quad + \int_{t_o}^t \|a_2(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \|w_2(\tau) - w_1(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} d\tau. \end{aligned}$$

An application of Gronwall's lemma yields the estimate:

$$\|w_1(t) - w_2(t)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \leq A_1(t) A_2(t) \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \|a_1 - a_2\|_{\mathbf{L}^1([t_o, t] \times \mathbb{R}^n; \mathbb{R})}.$$

6. Thanks to the $\mathbf{L}^\infty$ estimate at 4., we can apply [36, Chapter 2, Section 4, Theorem 9], regularising the coefficient by means of point 5. above.

7. As is well known, note that (1.2.7) immediately ensures that w is of class $\mathbf{C}^1$.

8. By (1.2.7),

$$\begin{aligned} \|w(t_2) - w(t_1)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} &\leq \left\| (H_\mu(t_2 - t_o) - H_\mu(t_1 - t_o)) * w_o \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \\ &\quad + \int_{t_o}^{t_1} \left\| (H_\mu(t_2 - s) - H_\mu(t_1 - s)) * (a(s)w(s)) \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} ds \\ &\quad + \int_{t_1}^{t_2} \left\| H_\mu(t_2 - s) * (a(s)w(s)) \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} ds \end{aligned}$$

and we compute the three terms separately. The first one is the usual term of the heat equation, so that using (1.4.4),

$$\begin{aligned} \left\| (H_\mu(t_2 - t_o) - H_\mu(t_1 - t_o)) * w_o \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} &\leq \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \int_{t_1 - t_o}^{t_2 - t_o} \|\partial_t H_\mu(s)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} ds \\ &\leq \frac{n}{\tau - t_o} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} |t_2 - t_1|. \end{aligned}$$

Concerning the second term, use (1.4.4), point 3. above and follow the proof of [55, Proposition 4.2.4]: for every $\vartheta \in]0, 1[$ we have

$$\begin{aligned} &\int_{t_o}^{t_1} \left\| (H_\mu(t_2 - s) - H_\mu(t_1 - s)) * (a(s)w(s)) \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} ds \\ &\leq \int_{t_o}^{t_1} \left\| \left(\int_{t_1 - s}^{t_2 - s} \partial_t H_\mu(\sigma) d\sigma \right) * (a(s)w(s)) \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} ds \\ &\leq \int_{t_o}^{t_1} \left\| \int_{t_1 - s}^{t_2 - s} \partial_t H_\mu(\sigma) d\sigma \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \|a(s)w(s)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} ds \\ &\leq \int_{t_o}^{t_1} \left(\int_{t_1 - s}^{t_2 - s} \|\partial_t H_\mu(\sigma)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} d\sigma \right) \|a(s)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \|w(s)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} ds \\ &\leq \int_{t_o}^{t_1} \left(\int_{t_1 - s}^{t_2 - s} \frac{n}{\sigma} d\sigma \right) \|a(s)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} A(s) ds \\ &\leq \|a\|_{\mathbf{L}^\infty([t_o, t_1] \times \mathbb{R}^n; \mathbb{R})} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} A(t_1) \int_{t_o}^{t_1} \frac{n}{(t_1 - s)^\vartheta} \int_{t_1 - s}^{t_2 - s} \frac{1}{\sigma^{1-\vartheta}} d\sigma ds \end{aligned}$$

$$\begin{aligned}
&\leq \|a\|_{\mathbf{L}^\infty([t_o, t_1] \times \mathbb{R}^n; \mathbb{R})} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} A(t_1) \int_{t_o}^{t_1} \frac{n}{\vartheta (t_1 - s)^\vartheta} \left[(t_2 - s)^\vartheta - (t_1 - s)^\vartheta \right] ds \\
&\leq \|a\|_{\mathbf{L}^\infty([t_o, t_1] \times \mathbb{R}^n; \mathbb{R})} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} A(t_1) \int_{t_o}^{t_1} \frac{n}{\vartheta (t_1 - s)^\vartheta} |t_2 - t_1|^\vartheta ds \\
&\leq \|a\|_{\mathbf{L}^\infty([t_o, t_1] \times \mathbb{R}^n; \mathbb{R})} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} A(t_1) \frac{n (t_1 - t_o)^{1-\vartheta}}{\vartheta (1 - \vartheta)} |t_2 - t_1|^\vartheta,
\end{aligned}$$

where we used the inequality $(t_2 - s)^\vartheta - (t_1 - s)^\vartheta \leq (t_2 - t_1)^\vartheta$. Concerning the third term, use (1.4.2) and point 3. above:

$$\begin{aligned}
&\int_{t_1}^{t_2} \left\| H_\mu(t_2 - s) * (a(s)w(s)) \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} ds \\
&\leq \int_{t_1}^{t_2} \|H_\mu(t_2 - s)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \|a(s)w(s)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} ds \\
&\leq \|a\|_{\mathbf{L}^\infty([t_1, t_2] \times \mathbb{R}^n; \mathbb{R})} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} A(t_2) |t_2 - t_1|.
\end{aligned}$$

Summing up the above expressions, we obtain

$$\begin{aligned}
&\|w(t_2) - w(t_1)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \\
&\leq \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \left[\frac{n}{\tau - t_o} + \left(\frac{n (t_1 - t_o)^{1-\vartheta}}{\vartheta (1 - \vartheta)} + 1 \right) \|a\|_{\mathbf{L}^\infty([t_o, t_2] \times \mathbb{R}^n; \mathbb{R})} A(t_2) \right] |t_2 - t_1|^\vartheta,
\end{aligned}$$

and the Hölder estimate at point 8. is proved. To prove the continuity in time, we are left to check the right continuity in t_o . To this aim, use (1.4.2) and point 3. above, introduce the variable $\zeta = \xi/(\mu t)^{n/2}$, and compute

$$\begin{aligned}
\|w(t) - w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} &\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} H_\mu(t - t_o, \xi) |w_o(x - \xi) - w_o(x)| d\xi dx \\
&\quad + \int_{t_o}^t \left\| H_\mu(t - s) * (a(s)w(s)) \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} ds \\
&\leq \int_{\mathbb{R}^n} H_1(1, \zeta) \int_{\mathbb{R}^n} \left| w_o \left(x - (\mu t)^{\frac{n}{2}} \zeta \right) - w_o(x) \right| d\zeta dx \\
&\quad + A(t) \|a\|_{\mathbf{L}^\infty([t_o, t] \times \mathbb{R}^n; \mathbb{R})} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} |t - t_o|.
\end{aligned}$$

Both terms above vanish as $t \rightarrow t_o$, (in the first use the Dominated Convergence Theorem), completing the proof of continuity in time.

9. Using (1.4.3), (1.2.7), the properties of the convolution and point 3. above,

$$\begin{aligned}
&\|\nabla w(t)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \\
&\leq \|\nabla H_\mu(t - t_o)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \\
&\quad + \int_{t_o}^t \|\nabla H_\mu(t - \tau)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \|a(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \|w(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} d\tau \\
&\leq \frac{J_n}{\sqrt{\mu(t - t_o)}} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} + \|a\|_{\mathbf{L}^\infty(I \times \mathbb{R}^n; \mathbb{R})} \int_{t_o}^t \frac{J_n A(\tau)}{\sqrt{\mu(t - \tau)}} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} d\tau \\
&\leq \frac{J_n}{\sqrt{\mu(t - t_o)}} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \left(1 + 2(t - t_o) A(t) \|a\|_{\mathbf{L}^\infty(I \times \mathbb{R}^n; \mathbb{R})} \right).
\end{aligned}$$

□

1.4.2 Proofs Related to $\partial_t u + \nabla \cdot (c(t, x) u) = b(t, x) u$

We use below the constant

$$I_n = n \int_0^{\pi/2} (\cos \vartheta)^n d\vartheta = n \frac{\Gamma((n+1)/2) \Gamma(1/2)}{2 \Gamma((n+2)/2)}. \quad (1.4.5)$$

Proof of Lemma 1.7. We recall [4, Section 3] and follow the proof of [19, Lemma 5.1].

1. Let $u_{o,n} \in \mathbf{C}^1(\mathbb{R}^n; \mathbb{R})$ approximate u_o in the sense that $\|u_{o,n} - u_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \rightarrow 0$ as $n \rightarrow +\infty$. Call u_n the corresponding quantity as given by (1.2.9). Then, $\|u_n - u\|_{\mathbf{L}^\infty(I; \mathbf{L}^1(\mathbb{R}^n; \mathbb{R}))} \rightarrow 0$ as $n \rightarrow +\infty$, so that $u \in \mathbf{L}^\infty(I; \mathbf{L}^1(\mathbb{R}^n; \mathbb{R}))$. Concerning the continuity in time, by (1.2.9) $u_n \in \mathbf{C}^0(I; \mathbf{L}^1(\mathbb{R}^n; \mathbb{R}))$ and u is the uniform limit of the sequence u_n , hence we have $u \in \mathbf{C}^0(I; \mathbf{L}^1(\mathbb{R}^n; \mathbb{R}))$.

Using the flow generated by (1.2.10), introduce the change of variable $y = X(t_o; t, x)$, so that $x = X(t; t_o, y)$. Denote its Jacobian by $J(t, y) = \det(\nabla_y X(t; t_o, y))$. Then, J solves

$$\frac{dJ(t, y)}{dt} = \operatorname{div} c(t, X(t; t_o, y)) J(t, y) \quad \text{with} \quad J(t_o, y) = 1.$$

Hence

$$J(t, y) = \exp \left(\int_{t_o}^t \operatorname{div} c(\tau, X(\tau; t_o, y)) d\tau \right),$$

so that (1.2.9) can be written as

$$u(t, x) = \frac{1}{J(t, y)} u_o(y) \mathcal{B}(t, y) \quad \text{where} \quad \begin{aligned} \mathcal{B}(t, y) &= \exp \left(\int_{t_o}^t b(\tau, X(\tau; t_o, y)) d\tau \right) \\ x &= X(t; t_o, y) \end{aligned} \quad (1.4.6)$$

Let $k \in \mathbb{R}$ and $\varphi \in \mathbf{C}_c^1(I \times \mathbb{R}^n; \mathbb{R}^+)$. We prove (1.2.11) for u given as in (1.2.9):

$$\begin{aligned} & \int_{t_o}^T \int_{\mathbb{R}^n} [(u - k)(\partial_t \varphi + c \cdot \nabla \varphi) + (b u - k \operatorname{div} c) \varphi] \operatorname{sgn}(u - k) dx dt \\ &= \int_{t_o}^T \int_{\mathbb{R}^n} \left[\left(\frac{u_o(y) \mathcal{B}(t, y)}{J(t, y)} - k \right) \left(\partial_t \varphi(t, X(t; t_o, y)) + c(t, X(t; t_o, y)) \cdot \nabla \varphi(t, X(t; t_o, y)) \right) \right. \\ & \quad \left. + \left(b(t, X(t; t_o, y)) \frac{u_o(y) \mathcal{B}(t, y)}{J(t, y)} - k \operatorname{div} c(t, X(t; t_o, y)) \right) \varphi(t, X(t; t_o, y)) \right] \\ & \quad \times \operatorname{sgn} \left(\frac{u_o(y) \mathcal{B}(t, y)}{J(t, y)} - k \right) J(t, y) dy dt \\ &= \int_{t_o}^T \int_{\mathbb{R}^n} \left[u_o(y) \mathcal{B}(t, y) \frac{d}{dt} \varphi(t, X(t; t_o, y)) - k J(t, y) \frac{d}{dt} \varphi(t, X(t; t_o, y)) \right. \\ & \quad \left. + \varphi(t, X(t; t_o, y)) \frac{d}{dt} (u_o(y) \mathcal{B}(t, y)) - k \varphi(t, X(t; t_o, y)) \frac{d}{dt} J(t, y) \right] \\ & \quad \times \operatorname{sgn} (u_o(y) \mathcal{B}(t, y) - k J(t, y)) dy dt \\ &= \int_{t_o}^T \int_{\mathbb{R}^n} \frac{d}{dt} \left[(u_o(y) \mathcal{B}(t, y) - k J(t, y)) \varphi(t, X(t; t_o, y)) \right] \\ & \quad \times \operatorname{sgn} (u_o(y) \mathcal{B}(t, y) - k J(t, y)) dy dt \end{aligned}$$

$$\begin{aligned}
&= \int_{t_o}^T \int_{\mathbb{R}^n} \frac{d}{dt} \left(|u_o(y) \mathcal{B}(t, y) - k J(t, y)| \varphi(t, X(t; t_o, y)) \right) dy dt \\
&= 0.
\end{aligned}$$

It is immediate to verify that for u as in (1.2.9) the initial condition holds. By [47, Section 2], u is also a weak solution.

2. Let u be defined as in (1.2.9) and u_* be a weak solution to (1.2.4). Then, by the step above, the function $U = u - u_*$ is a weak solution to (1.2.4) with zero initial datum.

Fix $\tau \in]t_o, T]$, choose any $\varphi \in \mathbf{C}_c^1(I \times \mathbb{R}^n; \mathbb{R})$ and let $\beta_\varepsilon \in \mathbf{C}^1(I; \mathbb{R})$ such that $\beta_\varepsilon(t) = 1$ for $t \in [t_o + \varepsilon, \tau - \varepsilon]$, $\beta'_\varepsilon(t) \in [0, 2/\varepsilon]$ for $t \in [t_o, t_o + \varepsilon]$, $\beta'_\varepsilon(t) \in [-2/\varepsilon, 0]$ for $t \in [\tau - \varepsilon, \tau]$, and $\beta_\varepsilon(t) = 0$ for $t \in [\tau, T]$. Using the definition of weak solution,

$$\begin{aligned}
0 &= \int_I \int_{\mathbb{R}^n} (U \partial_t(\varphi \beta_\varepsilon) + U c \cdot \nabla(\varphi \beta_\varepsilon) + b U \varphi \beta_\varepsilon) dx dt \\
&= \int_{t_o}^\tau \int_{\mathbb{R}^n} U (\partial_t \varphi + c \cdot \nabla \varphi + b \varphi) \beta_\varepsilon dx dt + \int_{t_o}^{t_o + \varepsilon} \beta'_\varepsilon \int_{\mathbb{R}^n} U \varphi dx dt + \int_{\tau - \varepsilon}^\tau \beta'_\varepsilon \int_{\mathbb{R}^n} U \varphi dx dt \\
&= \int_{t_o}^\tau \beta_\varepsilon \int_{\mathbb{R}^n} U (\partial_t \varphi + c \cdot \nabla \varphi + b \varphi) dx dt \\
&\quad + \int_0^1 \varepsilon \beta'_\varepsilon(\varepsilon t + t_o) \int_{\mathbb{R}^n} U(\varepsilon t + t_o, x) \varphi(\varepsilon t + t_o, x) dx dt \\
&\quad + \int_{\tau - \varepsilon}^\tau \beta'_\varepsilon(t) \int_{\mathbb{R}^n} U(t, x) \varphi(t, x) dx dt.
\end{aligned}$$

As $\varepsilon \rightarrow 0$, the first term converges to $\int_{t_o}^\tau \int_{\mathbb{R}^n} U (\partial_t \varphi + c \cdot \nabla \varphi + b \varphi) dx dt$. By the Dominated Convergence Theorem, the second term tends to 0, since $U(t_o) = 0$. Concerning the third term, note that

$$\begin{aligned}
&\left| \int_{\tau - \varepsilon}^\tau \beta'_\varepsilon(t) \int_{\mathbb{R}^n} U(t, x) \varphi(t, x) dx dt + \int_{\mathbb{R}^n} U(\tau, x) \varphi(\tau, x) dx \right| \\
&= \left| \int_{\tau - \varepsilon}^\tau \beta'_\varepsilon(t) \int_{\mathbb{R}^n} (U(t, x) \varphi(t, x) - U(\tau, x) \varphi(\tau, x)) dx dt \right| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0
\end{aligned}$$

by the continuity of U in time and the smoothness of φ .

Choose $\eta \in \mathbf{C}_c^1(\mathbb{R}^n; \mathbb{R})$ and compute φ integrating backward along characteristics in

$$\begin{cases} \partial_t \varphi + c \cdot \nabla \varphi = -b \varphi \\ \varphi(\tau) = \eta \end{cases}$$

as in 3. in the proof of [19, Lemma 5.1]. Then,

$$0 = \int_{t_o}^\tau \int_{\mathbb{R}^n} U (\partial_t \varphi + c \cdot \nabla \varphi + b \varphi) dx dt - \int_{\mathbb{R}^n} U(\tau, x) \varphi(\tau, x) dx = - \int_{\mathbb{R}^n} U(\tau, x) \eta(x) dx$$

proving that $U(\tau)$ vanishes identically. By the arbitrariness of τ , $u = u_*$. $\square$

Proof of Proposition 1.8. Define $g(t, x) = b(t, x) - \operatorname{div} c(t, x)$. The equation in (1.2.4) fits into the general form

$$\partial_t u + \nabla \cdot f(t, x, u) = F(t, x, u) \quad \text{with} \quad \begin{aligned} f(t, x, u) &= c(t, x) u \\ F(t, x, u) &= b(t, x) u. \end{aligned}$$

Points **1.** and **2.** follow, for instance, from [47, Theorem 1 and Theorem 2].

3. Start from [47, Formula (3.1)], since 0 solves the equation in (1.2.4), we get

$$\|\mathcal{H}_{t_o, t} u_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \leq \|u_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \exp \left(\|b\|_{\mathbf{L}^\infty([t_o, t] \times \mathbb{R}^n; \mathbb{R})} (t - t_o) \right).$$

4. From (1.2.9) it easily follows that

$$\|\mathcal{H}_{t_o, t} u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \leq \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \exp \left(\|g\|_{\mathbf{L}^\infty([t_o, t] \times \mathbb{R}^n; \mathbb{R})} (t - t_o) \right).$$

5. We refer to [22, Theorem 2.6], see also [23], as refined in [51, Proposition 2.10] and [52, Proposition 2.9]. Indeed, compute preliminarily

$$\begin{aligned} \partial_u f(t, x, u) &= c(t, x) & \operatorname{div} f(t, x, u) &= (\operatorname{div} c(t, x)) u \\ \partial_u \nabla f(t, x, u) &= \nabla c(t, x) & \nabla^2 f(t, x, u) &= (\nabla^2 c(t, x)) u \\ \partial_u F(t, x, u) &= b(t, x) & \nabla F(t, x, u) &= (\nabla b(t, x)) u \\ F(t, x, u) - \operatorname{div} f(t, x, u) &= (b(t, x) - \operatorname{div} c(t, x)) u & &= g(t, x) u \\ \partial_u (F(t, x, u) - \operatorname{div} f(t, x, u)) &= b(t, x) - \operatorname{div} c(t, x) & &= g(t, x) \\ \nabla (F(t, x, u) - \operatorname{div} f(t, x, u)) &= (\nabla b(t, x) - \nabla (\operatorname{div} c(t, x))) u & &= \nabla g(t, x) u \end{aligned} \tag{1.4.7}$$

and introduce the quantities

$$\begin{aligned} \kappa_o^* &= (2n + 1) \|\nabla c_1\|_{\mathbf{L}^\infty([t_o, t] \times \mathbb{R}^n; \mathbb{R}^{n \times n})} + \|b_1\|_{\mathbf{L}^\infty([t_o, t] \times \mathbb{R}^n; \mathbb{R})} \\ \kappa^* &= \|b_1\|_{\mathbf{L}^\infty([t_o, t] \times \mathbb{R}^n; \mathbb{R})} + \|\operatorname{div} (c_1 - c_2)\|_{\mathbf{L}^\infty([t_o, t] \times \mathbb{R}^n; \mathbb{R})} \\ \kappa_1^* &= \|b_1\|_{\mathbf{L}^\infty([t_o, t] \times \mathbb{R}^n; \mathbb{R})} + \|\operatorname{div} (c_1 - c_2)\|_{\mathbf{L}^\infty([t_o, t] \times \mathbb{R}^n; \mathbb{R})} + (2n + 1) \|\nabla c_1\|_{\mathbf{L}^\infty([t_o, t] \times \mathbb{R}^n; \mathbb{R}^{n \times n})}. \end{aligned}$$

For $i = 1, 2$, let b_i satisfy **(b)** with $b_1 - b_2 \in \mathbf{L}^1(I \times \mathbb{R}^n; \mathbb{R})$ and c_i satisfy **(c)** with $\operatorname{div} (c_1 - c_2) \in \mathbf{L}^1(I \times \mathbb{R}^n; \mathbb{R})$. Then, it is immediate to check that the requirements in [51, Section 2] and in [52, Section 2] hold. Indeed, with obvious notation,

$$\begin{aligned} \left. \begin{aligned} c_i &\in \mathbf{L}^\infty(I \times \mathbb{R}^n; \mathbb{R}^n); \quad c_i \in \mathbf{C}^2(I \times \mathbb{R}^n; \mathbb{R}^n); \quad b_i \in \mathbf{C}^1(I \times \mathbb{R}^n; \mathbb{R}) \\ (b_i - \operatorname{div} c_i) &\in \mathbf{L}^\infty(I \times \mathbb{R}^n; \mathbb{R}) \end{aligned} \right\} &\Rightarrow \quad \mathbf{(H1^*)} \\ \left. \begin{aligned} \nabla c_1 &\in \mathbf{L}^\infty(I \times \mathbb{R}^n; \mathbb{R}^{n \times n}); \quad b_1 \in \mathbf{L}^\infty(I \times \mathbb{R}^n; \mathbb{R}); \\ \nabla b_1 - \nabla (\operatorname{div} c_1) &\in \mathbf{L}^1(I \times \mathbb{R}^n; \mathbb{R}^n) \end{aligned} \right\} &\Rightarrow \quad \mathbf{(H2^*)} \\ \left. \begin{aligned} c_1 - c_2 &\in \mathbf{L}^\infty(I \times \mathbb{R}^n; \mathbb{R}^n) \\ (b_1 - b_2) &\in \mathbf{L}^\infty(I \times \mathbb{R}; \mathbb{R}); \quad ((b_1 - b_2) - \operatorname{div} (c_1 - c_2)) \in \mathbf{L}^1(I \times \mathbb{R}^n; \mathbb{R}) \end{aligned} \right\} &\Rightarrow \quad \mathbf{(H3^*)} \end{aligned}$$

Note also that

$$\frac{\kappa_o^* e^{\kappa_o^* t} - \kappa^* e^{\kappa^* t}}{\kappa_o^* - \kappa^*} = e^{\kappa_o^* t} + \frac{\kappa^* (e^{\kappa_o^* t} - e^{\kappa^* t})}{\kappa_o^* - \kappa^*} \leq (1 + \kappa^* t) e^{\kappa_1^* t} \leq e^{\kappa^* t} e^{\kappa_1^* t}.$$

Applying [51, Proposition 2.10] and [52, Proposition 2.9] we now obtain

$$\begin{aligned} & \left\| \mathcal{H}_{t_o, t}^1 u_o - \mathcal{H}_{t_o, t}^2 u_o \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \\ & \leq e^{(\kappa^* + \kappa_1^*)(t-t_o)} \|c_1 - c_2\|_{\mathbf{L}^1([t_o, t]; \mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R}^n))} \\ & \quad \times \left[\text{TV}(u_o) + I_n \int_{t_o}^t e^{-\kappa_o^*(\tau-t_o)} \left\| \mathcal{H}_{t_o, \tau}^1 u_o \right\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \left\| \nabla g_1(\tau) \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R}^n)} d\tau \right] \\ & \quad + e^{\kappa^*(t-t_o)} \int_{t_o}^t \|g_1(\tau) - g_2(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \max_{i=1,2} \left\| \mathcal{H}_{t_o, \tau}^i u_o \right\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} d\tau \end{aligned}$$

where $\left\| \mathcal{H}_{t_o, t}^i(u_o) \right\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})}$ is estimated in 4., $g_i = b_i - \text{div } c_i$, I_n is as in (1.4.5).

6. Directly follows from (1.2.9).

7. By assumptions, b satisfies **(b)** and c satisfies **(c)**, hence, by (1.4.7), both **(H1*)** and **(H2*)** hold. From [22, Theorem 2.5] or [52, Theorem 2.2], it directly follows that

$$\text{TV}(\mathcal{H}_{t_o, t} u_o) \leq \text{TV}(u_o) e^{\kappa_o^*(t-t_o)} + I_n \int_{t_o}^t e^{\kappa_o^*(t-\tau)} \left\| \nabla g(\tau) \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R}^n)} \left\| \mathcal{H}_{t_o, \tau} u_o \right\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} d\tau \quad (1.4.8)$$

where $\left\| \mathcal{H}_{t_o, t}(u_o) \right\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})}$ is estimated in 4., I_n is as in (1.4.5) and

$$\kappa_o^* = (2n+1) \left\| \nabla c \right\|_{\mathbf{L}^\infty([t_o, t] \times \mathbb{R}^n; \mathbb{R}^{n \times n})} + \|b\|_{\mathbf{L}^\infty([t_o, t] \times \mathbb{R}^n; \mathbb{R})}.$$

8. Assume that $t_1 < t_2$. From Definition 1.6, we have that

$$\int_{t_i}^T \int_{\mathbb{R}^n} (u \partial_t \varphi + u c \cdot \nabla \varphi + b u \varphi) dx dt + \int_{\mathbb{R}^n} \varphi(t_i, x) u(t_i, x) dx = 0. \quad (1.4.9)$$

Following the proof of [28, Theorem 4.3.1], let $\varphi(t, x) = \chi(t) \psi(x)$ with $\chi \in \mathbf{C}_c^1(\mathring{I}; \mathbb{R})$, $\chi(t) = 1$ for $t \in [t_1, t_2]$, $\psi \in \mathbf{C}_c^1(\mathbb{R}^n; \mathbb{R})$ with $|\psi(x)| \leq 1$ for $x \in \mathbb{R}^n$. Subtract (1.4.9) for $i = 1$ from (1.4.9) for $i = 2$, use [5, Proposition 3.2] and the estimates at 3., 4., and 7. to obtain:

$$\begin{aligned} & \left\| u(t_2) - u(t_1) \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \\ & = \sup_{\psi \in \mathbf{C}_c^1, |\psi(x)| \leq 1} \int_{\mathbb{R}^n} \psi(x) (u(t_2, x) - u(t_1, x)) dx \\ & = \sup_{\psi \in \mathbf{C}_c^1, |\psi(x)| \leq 1} \int_{t_1}^{t_2} \int_{\mathbb{R}^n} (u(t, x) c(t, x) \cdot \nabla \psi(x) + b(t, x) u(t, x) \psi(x)) dx dt \\ & = \sup_{\psi \in \mathbf{C}_c^1, |\psi(x)| \leq 1} \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \left[-\nabla(u(t, x) c(t, x)) \psi(x) + b(t, x) u(t, x) \psi(x) \right] dx dt \\ & \leq \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \left[\left\| \nabla(u(t, x) c(t, x)) \right\| + |b(t, x) u(t, x)| \right] dt dx \\ & \leq \int_{t_1}^{t_2} \left[\int_{\mathbb{R}^n} \|u(t, x) \nabla c(t, x)\| dx + \int_{\mathbb{R}^n} \|c(t, x)\| d(\nabla u(t))(x) + \int_{\mathbb{R}^n} |b(t, x) u(t, x)| dx \right] dt \\ & \leq \int_{t_1}^{t_2} \|u(t)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \left[\|b(t)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} + \|\nabla c(t)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R}^{n \times n})} \right] dt \end{aligned}$$

$$\begin{aligned}
& + \int_{t_1}^{t_2} \|c(t)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R}^n)} \text{TV}(u(t)) \, dt \\
& \leq \|u_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} e^{\|b\|_{\mathbf{L}^\infty([t_o, t_2] \times \mathbb{R}^n; \mathbb{R})} (t_2 - t_o)} \\
& \quad \times \left[\|b\|_{\mathbf{L}^\infty([t_1, t_2] \times \mathbb{R}^n; \mathbb{R})} + \|\nabla c\|_{\mathbf{L}^\infty([t_1, t_2] \times \mathbb{R}^n; \mathbb{R}^{n \times n})} \right] |t_2 - t_1| \\
& \quad + \|c\|_{\mathbf{L}^\infty([t_1, t_2] \times \mathbb{R}^n; \mathbb{R}^n)} \int_{t_1}^{t_2} \text{TV}(u_o) e^{\kappa_o^*(t - t_o)} \, dt \\
& \quad + I_n \|c\|_{\mathbf{L}^\infty([t_1, t_2] \times \mathbb{R}^n; \mathbb{R}^n)} \int_{t_1}^{t_2} \int_{t_o}^t e^{\kappa_o^*(t - \tau)} \|\nabla g(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R}^n)} \|u(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \, d\tau \, dt \\
& \leq |t_2 - t_1| \left[\|c\|_{\mathbf{L}^\infty([t_1, t_2] \times \mathbb{R}^n; \mathbb{R}^n)} \text{TV}(u_o) e^{\kappa_o^*(t_2 - t_o)} \right. \\
& \quad + \|u_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} e^{\|b\|_{\mathbf{L}^\infty([t_o, t_2] \times \mathbb{R}^n; \mathbb{R})} (t_2 - t_o)} \left[\|b\|_{\mathbf{L}^\infty([t_1, t_2] \times \mathbb{R}^n; \mathbb{R})} + \|\nabla c\|_{\mathbf{L}^\infty([t_1, t_2] \times \mathbb{R}^n; \mathbb{R}^{n \times n})} \right] \\
& \quad + I_n \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \|c\|_{\mathbf{L}^\infty([t_1, t_2] \times \mathbb{R}^n; \mathbb{R}^n)} \\
& \quad \left. \times \int_{t_o}^{t_2} \|\nabla g(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R}^n)} e^{\kappa_o^*(t_2 - \tau) + \|g\|_{\mathbf{L}^\infty([t_o, \tau] \times \mathbb{R}^n; \mathbb{R})} (\tau - t_o)} \, d\tau \right]
\end{aligned}$$

proving point 8.

9. Directly follows from (1.2.9), since, by **(c)**, the speed of characteristics is bounded. $\square$

1.4.3 Proof of Theorem 1.2

Proof of Theorem 1.2. We are going to construct a solution to (1.1.1) as limit of a Cauchy sequence of approximate solutions. For $r > 0$, we introduce the domain

$$\mathcal{X}_r = \left\{ (u, w) \in \mathcal{X}^+ : \begin{array}{ll} \|u\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \leq r, & \text{TV}(u) \leq r \\ \|w\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \leq r, & \|w\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \leq r \end{array} \right\}.$$

Choose an initial datum

$$(u_o, w_o) \in \mathcal{X}_r \quad \text{with moreover} \quad w_o \in (\mathbf{C}^1 \cap \mathbf{W}^{1,1})(\mathbb{R}^n; \mathbb{R}^+), \quad (1.4.10)$$

and set, for $t \in [0, T]$, $(u_0(t), w_0(t)) = (u_o, w_o)$. For $i \in \mathbb{N}$, define for $(t, x) \in [0, T] \times \mathbb{R}^n$,

$$a_{i+1}(t, x) = \gamma - \delta u_i(t, x), \quad b_{i+1}(t, x) = \alpha w_i(t, x) - \beta, \quad \text{and} \quad c_{i+1}(t, x) = v(w_i(t))(x),$$

and let (u_{i+1}, w_{i+1}) be such that

$$\begin{cases} \partial_t u_{i+1} + \nabla \cdot (c_{i+1} u_{i+1}) = b_{i+1} u_{i+1} \\ \partial_t w_{i+1} - \mu \Delta w_{i+1} = a_{i+1} w_{i+1} \\ u_{i+1}(0) = u_o \\ w_{i+1}(0) = w_o. \end{cases}$$

Claim 0: For all $i \in \mathbb{N}$,

C0.1 (u_i, w_i) is well defined and in $\mathbf{L}^1([0, T]; \mathcal{X}^+)$;

C0.2 $w_i \in \mathbf{C}^1([0, T] \times \mathbb{R}^n; \mathbb{R})$ and $\nabla w_i \in \mathbf{L}^1([0, T] \times \mathbb{R}^n; \mathbb{R}^n)$;

C0.3 $a_{i+1} \in \mathbf{L}^\infty([0, T] \times \mathbb{R}^n; \mathbb{R})$;

C0.4 b_{i+1} satisfies **(b)** with $I = [0, T]$;

C0.5 c_{i+1} satisfies **(c)** with $I = [0, T]$.

Proof of Claim 0. We prove it by induction.

Case $i = 0$: is immediate by (1.4.10) and by the above definition of a_1, b_1, c_1 .

From $i - 1$ to i : Assume now that C0.1, $\dots$, C0.5 are all satisfied up to the i -th iteration. Then, Proposition 1.5 and Proposition 1.8 can now be applied, proving C0.1. Moreover, by 7. and 9. in Proposition 1.5, also C0.2 holds. Furthermore, the estimate at 3. and 4. in Proposition 1.8 ensure that C0.3 holds. Moreover, C0.1 and C0.2 directly imply C0.4 and, together with **(v)**, also C0.5, completing the proof of the present claim.

Hence, thanks to **(v)**, it clearly follows that:

$$a_{i+1} - a_i, b_{i+1} - b_i, \operatorname{div}(c_{i+1} - c_i) \in \mathbf{L}^1(I \times \mathbb{R}^n; \mathbb{R}). \quad (1.4.11)$$

In the next two claims we particularise the $\mathbf{L}^1$ and $\mathbf{L}^\infty$ estimates in 3. and 4. of Propositions 1.5 and 1.8, thanks to the explicit expressions of a , b and c .

Claim 1: For all $i \in \mathbb{N}$, if w_i is defined up to time $\hat{T}$, then for all $t \in [0, \hat{T}]$,

$$\|w_i(t)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \leq \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} e^{\gamma t} \quad \text{and} \quad \|w_i(t)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \leq \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} e^{\gamma t}.$$

Proof of Claim 1. Assume $i > 0$, the case $i = 0$ being obvious. By (1.2.7),

$$\begin{aligned} w_i(t, x) &= \int_{\mathbb{R}^n} H_\mu(t, x - \xi) w_o(\xi) \, d\xi + \int_0^t \int_{\mathbb{R}^n} H_\mu(t - \tau, x - \xi) (\gamma - \delta u_{i-1}(\tau, \xi)) w_i(\tau, \xi) \, d\xi \, d\tau \\ &\leq \int_{\mathbb{R}^n} H_\mu(t, x - \xi) w_o(\xi) \, d\xi + \int_0^t \int_{\mathbb{R}^n} \gamma H_\mu(t - \tau, x - \xi) w_i(\tau, \xi) \, d\xi \, d\tau. \end{aligned}$$

By Gronwall's lemma and (1.4.2):

$$\begin{aligned} w_i(t, x) &\leq \left(\int_{\mathbb{R}^n} H_\mu(t, x - \xi) w_o(\xi) \, d\xi \right) \exp \left(\int_0^t \int_{\mathbb{R}^n} \gamma H_\mu(t - \tau, x - \xi) \, d\xi \, d\tau \right) \\ &= e^{\gamma t} \int_{\mathbb{R}^n} H_\mu(t, x - \xi) w_o(\xi) \, d\xi. \end{aligned}$$

The proof of the claim follows.

Claim 2: For all $i \in \mathbb{N}$, if u_i is defined up to time $\hat{T}$, then for all $t \in [0, \hat{T}]$,

$$\begin{aligned} \|u_i(t)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} &\leq \|u_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \exp \left(\alpha \frac{e^{\gamma t} - 1}{\gamma} \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \right) \\ &\leq \|u_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \exp \left(\alpha t e^{\gamma t} \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \right) \\ \|u_i(t)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} &\leq \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \exp \left((\alpha + K) \frac{e^{\gamma t} - 1}{\gamma} \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \right) \end{aligned}$$

$$\leq \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \exp\left((\alpha + K)t e^{\gamma t} \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\right).$$

Proof of Claim 2. Assume $i > 0$, the case $i = 0$ being obvious. By (v) and Claim 0, we can apply Lemma 1.7, and by (1.2.9) we obtain

$$u_i(t, x) \leq u_o(X(0; t, x)) \exp \int_0^t \left(\alpha w_{i-1}(\tau, X(\tau; t, x)) - \operatorname{div} c_i(\tau, X(\tau; t, x)) \right) d\tau. \quad (1.4.12)$$

To obtain the $\mathbf{L}^1$ estimate, we adopt the notation in (1.4.6) with $t_o = 0$, $b = \alpha w_{i-1}$ and $c = c_i$, so that, using Claim 1 above, we have

$$\begin{aligned} \|u_i(t)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} &\leq \int_0^t \frac{1}{J(\tau, y)} u_o(y) \mathcal{B}(\tau, y) J(\tau, y) d\tau \\ &\leq \|u_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \exp \left(\int_0^t \alpha \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} e^{\gamma \tau} d\tau \right) \\ &= \|u_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \exp \left(\alpha \frac{e^{\gamma t} - 1}{\gamma} \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \right). \end{aligned}$$

The $\mathbf{L}^\infty$ estimate is obtained from (1.4.12) using (v) and Claim 1:

$$\begin{aligned} \|u_i(t)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} &\leq \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \exp \int_0^t \left(\alpha \|w_{i-1}(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} + K \|w_{i-1}(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \right) d\tau \\ &\leq \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \exp \int_0^t (\alpha + K) \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} e^{\gamma \tau} d\tau \\ &= \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \exp \left((\alpha + K) \frac{e^{\gamma t} - 1}{\gamma} \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \right), \end{aligned}$$

completing the proof of Claim 2.

Claim 3: For all $i \in \mathbb{N}$, if u_i is defined up to time $\hat{T}$, then for all $t \in [0, \hat{T}]$ we have $\operatorname{TV}(u_i(t)) \leq \mathcal{F}(t)$, where $\mathcal{F} \in \mathbf{C}^0([0, \hat{T}]; \mathbb{R}^+)$ depends only on the hypotheses and on the initial data, being in particular independent of i .

Proof of Claim 3. Apply 7. in 1.8 in the explicit form (1.4.8) to obtain

$$\begin{aligned} \operatorname{TV}(u_i(t)) &\leq \left[\operatorname{TV}(u_o) + I_n \int_0^t \|\nabla(b_i - \operatorname{div} c_i)(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)} \|u_i(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} d\tau \right] e^{\kappa_o^* t} \\ &\leq \operatorname{TV}(u_o) e^{\kappa_o^* t} \\ &\quad + I_n \left[\|\nabla b_i\|_{\mathbf{L}^1([0,t] \times \mathbb{R}^n;\mathbb{R}^n)} + \|\nabla \operatorname{div} c_i\|_{\mathbf{L}^1([0,t] \times \mathbb{R}^n;\mathbb{R}^n)} \right] \|u_i\|_{\mathbf{L}^\infty([0,t] \times \mathbb{R}^n;\mathbb{R})} e^{\kappa_o^* t}. \end{aligned}$$

All terms in the right hand side above are estimated by means of quantities independent of i . More precisely, using (v) and Claim 1:

$$\begin{aligned} \kappa_o^* &= (2n + 1) \|\nabla c_i\|_{\mathbf{L}^\infty([0,t] \times \mathbb{R}^n;\mathbb{R}^{n \times n})} + \|b_i\|_{\mathbf{L}^\infty([0,t] \times \mathbb{R}^n;\mathbb{R})} \\ &\leq ((2n + 1)K + \alpha) \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} e^{\gamma t} + \beta. \end{aligned}$$

Observe that $\|\nabla b_i\|_{\mathbf{L}^1([0,t] \times \mathbb{R}^n; \mathbb{R}^n)} = \alpha \|\nabla w_{i-1}\|_{\mathbf{L}^1([0,t] \times \mathbb{R}^n; \mathbb{R}^n)}$. Recall the proof of 9. in Proposition 1.5. Note preliminarily that, since $a_i(\tau, x) = \gamma - \delta u_{i-1}(\tau, x)$, by Claim 2

$$\begin{aligned} \|a_i\|_{\mathbf{L}^\infty([0,t] \times \mathbb{R}^n; \mathbb{R})} &\leq \gamma + \delta \|u_{i-1}\|_{\mathbf{L}^\infty([0,t] \times \mathbb{R}^n; \mathbb{R})} \\ &\leq \gamma + \delta \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \exp\left((\alpha + K) \frac{e^{\gamma t} - 1}{\gamma} \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})}\right) \\ &\leq \gamma + \delta \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \exp\left((\alpha + K) \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} t e^{\gamma t}\right). \end{aligned} \quad (1.4.13)$$

Use Claim 1, (1.4.13), and the expression (1.4.1) of J_n in the proof of 9. in Proposition 1.5:

$$\begin{aligned} &\|\nabla w_{i-1}(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R}^n)} \\ &\leq \|\nabla H_\mu(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \\ &\quad + \int_0^\tau \|\nabla H_\mu(\tau - s)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \|a_{i-1}(s)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \|w_{i-1}(s)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \, ds \\ &\leq \|\nabla H_\mu(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \\ &\quad + \|a_{i-1}\|_{\mathbf{L}^\infty([0,\tau] \times \mathbb{R}^n; \mathbb{R})} \int_0^\tau \|\nabla H_\mu(\tau - s)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} e^{\gamma s} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \, ds \\ &\leq \|\nabla H_\mu(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} + \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} e^{\gamma \tau} \int_0^\tau \|\nabla H_\mu(\tau - s)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \, ds \\ &\quad \times \left(\gamma + \delta \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} e^{(\alpha+K) \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \tau e^{\gamma \tau}} \right) \\ &\leq \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \left[\frac{J_n}{\sqrt{\mu \tau}} + e^{\gamma \tau} \left(\gamma + \delta \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} e^{(\alpha+K) \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \tau e^{\gamma \tau}} \right) \int_0^\tau \frac{J_n}{\sqrt{\mu(\tau - s)}} \, ds \right] \\ &\leq \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \frac{J_n}{\sqrt{\mu \tau}} \left[1 + 2\tau e^{\gamma \tau} \left(\gamma + \delta \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} e^{(\alpha+K) \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \tau e^{\gamma \tau}} \right) \right]. \end{aligned}$$

Therefore

$$\begin{aligned} &\|\nabla b_i\|_{\mathbf{L}^1([0,t] \times \mathbb{R}^n; \mathbb{R}^n)} \\ &\leq \frac{\alpha J_n \sqrt{t}}{\sqrt{\mu}} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \left[1 + 2te^{\gamma t} \left(\gamma + \delta \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} e^{(\alpha+K) \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} te^{\gamma t}} \right) \right]. \end{aligned} \quad (1.4.14)$$

Recall that $\nabla(\operatorname{div} c_i) = \nabla(\operatorname{div} v(w_{i-1}))$. By (v) and Claim 1,

$$\begin{aligned} \|\nabla(\operatorname{div} c_i)\|_{\mathbf{L}^1([0,t] \times \mathbb{R}^n; \mathbb{R}^n)} &\leq \left\| C \left(\|w_{i-1}\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \right) \|w_{i-1}\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \right\|_{\mathbf{L}^1([0,t]; \mathbb{R})} \\ &\leq \left\| C \left(\|w_{i-1}\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \right) \right\|_{\mathbf{L}^\infty([0,t]; \mathbb{R})} \|w_{i-1}\|_{\mathbf{L}^1([0,t] \times \mathbb{R}^n; \mathbb{R})} \\ &\leq C \left(\|w_{i-1}\|_{\mathbf{L}^\infty([0,t]; \mathbf{L}^1(\mathbb{R}^n; \mathbb{R}))} \right) \|w_{i-1}\|_{\mathbf{L}^1([0,t] \times \mathbb{R}^n; \mathbb{R})} \\ &\leq C \left(\|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} e^{\gamma t} \right) \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} t e^{\gamma t}. \end{aligned} \quad (1.4.15)$$

Concerning $\|u_i\|_{\mathbf{L}^\infty([0,t] \times \mathbb{R}^n; \mathbb{R})}$, it is bounded by Claim 2, completing the proof of Claim 3.

The sequence of approximate solutions we construct belongs to the set

$$X = \left\{ (u, w) \in \mathbf{L}^1([0, T]; \mathcal{X}^+) : \text{TV}(u(t)) \leq \mathcal{F}(t) \text{ for all } t \in [0, T] \right\}, \quad (1.4.16)$$

which is a complete metric space with the distance

$$\begin{aligned} d((u_1, w_1), (u_2, w_2)) &= \|u_2 - u_1\|_{\mathbf{L}^1([0, T]; \mathbf{L}^1(\mathbb{R}^n; \mathbb{R}))} + \|w_2 - w_1\|_{\mathbf{L}^1([0, T]; \mathbf{L}^1(\mathbb{R}^n; \mathbb{R}))} \\ &= \int_0^T \int_{\mathbb{R}^n} (|u_2(t, x) - u_1(t, x)| + |w_2(t, x) - w_1(t, x)|) dx dt. \end{aligned}$$

We now prove that there exist positive T and $\mathcal{K}(T, r)$ such that for all $i \in \mathbb{N}$,

$$d((u_{i+1}, w_{i+1}), (u_i, w_i)) \leq T \mathcal{K}(T, r) d((u_i, w_i), (u_{i-1}, w_{i-1})). \quad (1.4.17)$$

By (1.4.11), recall the proof of 5. in Proposition 1.5 and apply the $\mathbf{L}^\infty$ estimate in Claim 1:

$$\begin{aligned} \|w_{i+1}(t) - w_i(t)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} &\leq \int_0^t \|a_{i+1}(\tau) - a_i(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \|w_{i+1}(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} d\tau \\ &\quad + \int_0^t \|a_i(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \|w_{i+1}(\tau) - w_i(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} d\tau \\ &\leq \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \int_0^t e^{\gamma \tau} \|a_{i+1}(\tau) - a_i(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} d\tau \\ &\quad + \int_0^t \|a_i(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \|w_{i+1}(\tau) - w_i(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} d\tau. \end{aligned}$$

Apply Gronwall Lemma:

$$\begin{aligned} &\|w_{i+1}(t) - w_i(t)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \\ &\leq \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} e^{\gamma t} \|a_{i+1} - a_i\|_{\mathbf{L}^1([0, t] \times \mathbb{R}^n; \mathbb{R})} \exp\left(\|a_i\|_{\mathbf{L}^1([0, t]; \mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R}))}\right) \\ &\leq \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} e^{\gamma t} \|a_{i+1} - a_i\|_{\mathbf{L}^1([0, t] \times \mathbb{R}^n; \mathbb{R})} \exp\left(t \|a_i\|_{\mathbf{L}^\infty([0, t] \times \mathbb{R}^n; \mathbb{R})}\right). \end{aligned}$$

Hence, recalling (1.4.13), we obtain

$$\begin{aligned} &\|w_{i+1}(t) - w_i(t)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \\ &\leq \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \delta \|u_i - u_{i-1}\|_{\mathbf{L}^1([0, t]; \mathbf{L}^1(\mathbb{R}^n; \mathbb{R}))} \\ &\quad \times \exp\left(2\gamma t + \delta t \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \exp\left((\alpha + K) \frac{e^{\gamma t} - 1}{\gamma} \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})}\right)\right); \\ &\|w_{i+1} - w_i\|_{\mathbf{L}^1([0, T]; \mathbf{L}^1(\mathbb{R}^n; \mathbb{R}))} \\ &\leq \delta T \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \|u_i - u_{i-1}\|_{\mathbf{L}^1([0, T]; \mathbf{L}^1(\mathbb{R}^n; \mathbb{R}))} \\ &\quad \times \exp\left(2\gamma T + \delta T \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \exp\left((\alpha + K) \frac{e^{\gamma T} - 1}{\gamma} \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})}\right)\right) \\ &\leq T \mathcal{K}_w(T, r) \|u_i - u_{i-1}\|_{\mathbf{L}^1([0, T]; \mathbf{L}^1(\mathbb{R}^n; \mathbb{R}))}, \end{aligned} \quad (1.4.18)$$

since $(u_o, w_o) \in \mathcal{X}_r$, where

$$\mathcal{K}_w(T, r) = \delta r \exp \left[2 \gamma T + \delta T r \exp \left((\alpha + K) \frac{e^{\gamma T} - 1}{\gamma} r \right) \right]. \quad (1.4.19)$$

We now pass to estimate $\|u_{i+1} - u_i\|_{\mathbf{L}^1([0,T];\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))}$. To this aim, by (1.4.11), we start from 5. in Proposition 1.8 and use Claim 2 above:

$$\begin{aligned} & \|u_{i+1}(t) - u_i(t)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \\ & \leq e^{\kappa^* t} e^{\kappa_1^* t} \|c_{i+1} - c_i\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n))} \\ & \quad \times \left[\text{TV}(u_0) + I_n \int_0^t e^{-\kappa_o^* \tau} \|u_{i+1}(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \left\| \nabla b_{i+1}(\tau) - \nabla (\text{div } c_{i+1}(\tau)) \right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)} d\tau \right] \\ & \quad + e^{\kappa^* t} \int_0^t \left\| b_{i+1}(\tau) - b_i(\tau) - \text{div}(c_{i+1}(\tau) - c_i(\tau)) \right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \\ & \quad \times \max \left\{ \|u_{i+1}(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}, \|u_i(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \right\} d\tau \\ & \leq e^{\kappa^* t} e^{\kappa_1^* t} \|c_{i+1} - c_i\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n))} \left[\text{TV}(u_o) + I_n \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \right. \\ & \quad \times \left. \int_0^t e^{(\alpha+K)\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \tau} e^{\gamma \tau} \left\| \nabla b_{i+1}(\tau) - \nabla (\text{div } c_{i+1}(\tau)) \right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)} d\tau \right] \\ & \quad + e^{\kappa^* t} \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \\ & \quad \times \int_0^t e^{(\alpha+K)\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \tau} e^{\gamma \tau} \left\| b_{i+1}(\tau) - b_i(\tau) - \text{div}(c_{i+1}(\tau) - c_i(\tau)) \right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} d\tau. \end{aligned}$$

We proceed estimating all terms appearing in the inequality above. Begin by 5. in Proposition 1.8, (v) and Claim 1:

$$\begin{aligned} \kappa^* &= \|b_{i+1}\|_{\mathbf{L}^\infty([0,t]\times\mathbb{R}^n;\mathbb{R})} + \|\text{div}(c_{i+1} - c_i)\|_{\mathbf{L}^\infty([0,t]\times\mathbb{R}^n;\mathbb{R})} \\ &\leq \alpha \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} e^{\gamma t} + \beta + \|\text{div } c_{i+1}\|_{\mathbf{L}^\infty([0,t]\times\mathbb{R}^n;\mathbb{R})} + \|\text{div } c_i\|_{\mathbf{L}^\infty([0,t]\times\mathbb{R}^n;\mathbb{R})} \\ &\leq \alpha \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} e^{\gamma t} + \beta + 2K \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} e^{\gamma t} \\ &= \lambda \quad \text{where} \quad \lambda = (\alpha + 2K) \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} e^{\gamma t} + \beta; \end{aligned} \quad (1.4.20)$$

$$\begin{aligned} \kappa_1^* &= \|b_{i+1}\|_{\mathbf{L}^\infty([0,t]\times\mathbb{R}^n;\mathbb{R})} + \|\text{div}(c_{i+1} - c_i)\|_{\mathbf{L}^\infty([0,t]\times\mathbb{R}^n;\mathbb{R})} + (2n+1) \|\nabla c_{i+1}\|_{\mathbf{L}^\infty([0,t]\times\mathbb{R}^n;\mathbb{R})} \\ &\leq (\alpha + 2K) \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} e^{\gamma t} + \beta + (2n+1) K \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} e^{\gamma t} \\ &= \lambda_1 \quad \text{where} \quad \lambda_1 = (\alpha + (2n+3)K) \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} e^{\gamma t} + \beta. \end{aligned} \quad (1.4.21)$$

By (v) and Claim 1,

$$\begin{aligned} \|c_{i+1} - c_i\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n))} &= \|v(w_i) - v(w_{i-1})\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n))} \\ &\leq K \|w_i - w_{i-1}\|_{\mathbf{L}^1([0,t];\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))}. \end{aligned} \quad (1.4.22)$$

By (1.4.14) and (1.4.15), we obtain

$$\int_0^t e^{(\alpha+K)\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \tau} e^{\gamma \tau} \left\| \nabla b_{i+1}(\tau) - \nabla (\text{div } c_{i+1}(\tau)) \right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)} d\tau$$

$$\begin{aligned}
&\leq e^{(\alpha+K)\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}te^{\gamma t}} \|\nabla b_{i+1} - \nabla(\operatorname{div} c_{i+1})\|_{\mathbf{L}^1([0,t]\times\mathbb{R}^n;\mathbb{R}^n)} \\
&\leq e^{(\alpha+K)\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}te^{\gamma t}} \left[\|\nabla b_{i+1}\|_{\mathbf{L}^1([0,t]\times\mathbb{R}^n;\mathbb{R}^n)} + \|\nabla(\operatorname{div} c_{i+1})\|_{\mathbf{L}^1([0,t]\times\mathbb{R}^n;\mathbb{R}^n)} \right] \\
&\leq e^{(\alpha+K)\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}te^{\gamma t}} \left[C \left(\|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} e^{\gamma t} \right) \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} te^{\gamma t} \right. \\
&\quad \left. + \alpha \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} J_n \frac{\sqrt{t}}{\sqrt{\mu}} \left(1 + 2te^{\gamma t} \left(\gamma + \delta \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} e^{(\alpha+K)\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}te^{\gamma t}} \right) \right) \right] \\
&= \mathbf{M}_t(u_o, w_o), \tag{1.4.23}
\end{aligned}$$

where, for brevity, we set

$$\begin{aligned}
&\mathbf{M}_t(u_o, w_o) \\
&= \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} e^{(\alpha+K)\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}te^{\gamma t}} \sqrt{t} \left[C \left(\|w_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} e^{\gamma t} \right) \sqrt{t} e^{\gamma t} \right. \\
&\quad \left. + J_n \frac{\alpha}{\sqrt{\mu}} \left(1 + 2te^{\gamma t} \left(\gamma + \delta \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} e^{(\alpha+K)\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}te^{\gamma t}} \right) \right) \right] \tag{1.4.24}
\end{aligned}$$

$$\leq r e^{(\alpha+K)re^{\gamma t}} \sqrt{t} \left[C \left(r e^{\gamma t} \right) \sqrt{t} e^{\gamma t} + J_n \frac{\alpha}{\sqrt{\mu}} \left(1 + 2te^{\gamma t} \left(\gamma + \delta r e^{(\alpha+K)re^{\gamma t}} \right) \right) \right], \tag{1.4.25}$$

since $(u_o, w_o) \in \mathcal{X}_r$. Pass to

$$\begin{aligned}
&\int_0^t e^{(\alpha+K)\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\tau e^{\gamma \tau}} \|b_{i+1}(\tau) - b_i(\tau) - \operatorname{div}(c_{i+1}(\tau) - c_i(\tau))\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} d\tau \\
&\leq e^{(\alpha+K)\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}te^{\gamma t}} \int_0^t \|b_{i+1}(\tau) - b_i(\tau) - \operatorname{div}(c_{i+1}(\tau) - c_i(\tau))\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} d\tau. \tag{1.4.26}
\end{aligned}$$

In particular, by the definition of b_i , $(\mathbf{v})$ and Claim 1, we have

$$\begin{aligned}
&\int_0^t \|b_{i+1}(\tau) - b_i(\tau) - \operatorname{div}(c_{i+1}(\tau) - c_i(\tau))\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} d\tau \\
&\leq \int_0^t \|b_{i+1}(\tau) - b_i(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} d\tau + \int_0^t \|\operatorname{div}(c_{i+1}(\tau) - c_i(\tau))\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} d\tau \\
&\leq \alpha \int_0^t \|w_i(\tau) - w_{i-1}(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} d\tau \\
&\quad + \int_0^t C \left(\|w_{i-1}(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \right) \|w_i(\tau) - w_{i-1}(\tau)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} d\tau \\
&\leq \alpha \|w_i - w_{i-1}\|_{\mathbf{L}^1([0,t];\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))} \\
&\quad + \left\| C \left(\|w_{i-1}(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \right) \right\|_{\mathbf{L}^\infty([0,t];\mathbb{R})} \|w_i - w_{i-1}\|_{\mathbf{L}^1([0,t];\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))} \\
&\leq \left(\alpha + C \left(\|w_{i-1}\|_{\mathbf{L}^\infty([0,t]\times\mathbb{R}^n;\mathbb{R})} \right) \right) \|w_i - w_{i-1}\|_{\mathbf{L}^1([0,t];\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))} \\
&\leq \left(\alpha + C \left(\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} e^{\gamma t} \right) \right) \|w_i - w_{i-1}\|_{\mathbf{L}^1([0,t];\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}))}. \tag{1.4.27}
\end{aligned}$$

Use (1.4.20), (1.4.21), (1.4.22), (1.4.23), (1.4.26) and (1.4.27) to obtain the following bound

$$\begin{aligned} & \|u_{i+1}(t) - u_i(t)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \\ & \leq e^{\lambda t} \|w_i - w_{i-1}\|_{\mathbf{L}^1([0, t]; \mathbf{L}^1(\mathbb{R}^n; \mathbb{R}))} \left[K e^{\lambda_1 t} \left[\text{TV}(u_o) + I_n \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \mathbf{M}_t(u_o, w_o) \right] \right. \\ & \quad \left. + \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \left(\alpha + C \left(\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} e^{\gamma t} \right) \right) e^{(\alpha+K) \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} t} e^{\gamma t} \right] \end{aligned}$$

and then, recalling (1.4.20), (1.4.21) and (1.4.24),

$$\begin{aligned} & \|u_{i+1} - u_i\|_{\mathbf{L}^1([0, T]; \mathbf{L}^1(\mathbb{R}^n; \mathbb{R}))} \\ & \leq T \|w_i - w_{i-1}\|_{\mathbf{L}^1([0, T]; \mathbf{L}^1(\mathbb{R}^n; \mathbb{R}))} \exp \left((\alpha + 2K) \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} T e^{\gamma T} + \beta T \right) \\ & \quad \times \left\{ K \exp \left((\alpha + (2n+3)K) \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} T e^{\gamma T} + \beta T \right) \left\{ \text{TV}(u_o) \right. \right. \\ & \quad \left. \left. + I_n \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \sqrt{T} e^{(\alpha+K) \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} T} e^{\gamma T} \right. \right. \\ & \quad \times \left[J_n \frac{\alpha}{\sqrt{\mu}} \left(1 + 2T e^{\gamma T} (\gamma + \delta \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} e^{(\alpha+K) \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} T} e^{\gamma T}) \right) \right. \\ & \quad \left. \left. + C \left(\|w_o\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} e^{\gamma T} \right) \sqrt{T} e^{\gamma T} \right] \right\} \\ & \quad \left. + \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \left(\alpha + C \left(\|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} e^{\gamma T} \right) \right) e^{(\alpha+K) \|w_o\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} T} e^{\gamma T} \right\} \\ & \leq T \mathcal{K}_u(T, r) \|w_i - w_{i-1}\|_{\mathbf{L}^1([0, T]; \mathbf{L}^1(\mathbb{R}^n; \mathbb{R}))}, \end{aligned} \tag{1.4.28}$$

where, by (1.4.25),

$$\begin{aligned} \mathcal{K}_u(T, r) &= r e^{((2\alpha+3K)r e^{\gamma T} + \beta)T} \left\{ K e^{(2(n+1)K r e^{\gamma T} + \beta)T} \right. \\ & \quad \times \left\{ 1 + I_n r \sqrt{T} e^{(\alpha+K)r T} e^{\gamma T} \left[\frac{J_n \alpha}{\sqrt{\mu}} \left(1 + 2T e^{\gamma T} (\gamma + \delta r e^{(\alpha+K)r T} e^{\gamma T}) \right) \right. \right. \\ & \quad \left. \left. + C \left(r e^{\gamma T} \right) \sqrt{T} e^{\gamma T} \right] \right\} + \alpha + C \left(r e^{\gamma T} \right) \right\}. \end{aligned} \tag{1.4.29}$$

Therefore, the above definition of $\mathcal{K}_u(T, r)$, together with (1.4.18), (1.4.19) and (1.4.28), yields

$$\begin{aligned} & d((u_{i+1}, w_{i+1}), (u_i, w_i)) \\ &= \|u_{i+1} - u_i\|_{\mathbf{L}^1([0, T]; \mathbf{L}^1(\mathbb{R}^n; \mathbb{R}))} + \|w_{i+1} - w_i\|_{\mathbf{L}^1([0, T]; \mathbf{L}^1(\mathbb{R}^n; \mathbb{R}))} \\ & \leq T \max \{ \mathcal{K}_u(T, r), \mathcal{K}_w(T, r) \} \left[\|u_i - u_{i-1}\|_{\mathbf{L}^1([0, T]; \mathbf{L}^1(\mathbb{R}^n; \mathbb{R}))} + \|w_i - w_{i-1}\|_{\mathbf{L}^1([0, T]; \mathbf{L}^1(\mathbb{R}^n; \mathbb{R}))} \right] \end{aligned}$$

$$= T \mathcal{K}(T, r) d((u_i, w_i), (u_{i-1}, w_{i-1}))$$

where

$$\mathcal{K}(T, r) = \max \{ \mathcal{K}_u(T, r), \mathcal{K}_w(T, r) \}, \quad (1.4.30)$$

proving (1.4.17).

For any positive r , we can now choose T_r so that $T_r \mathcal{K}(T_r, r) < 1$. The sequence (u_i, w_i) converges in the space X defined in (1.4.16) to a limit, say, (u_*, w_*) . By construction, see Claim 0, both u_* and w_* attain non negative values. We now check that (u_*, w_*) solves (1.1.1) in the sense of Definition 1.1.

Clearly, $(u_*, w_*)(0) = (u_o, w_o)$. Moreover, by the above construction, we have that for any $\varphi \in \mathbf{C}_c^\infty([0, T_r] \times \mathbb{R}^n; \mathbb{R})$

$$\begin{aligned} \int_0^{T_r} \int_{\mathbb{R}^n} (w_i \partial_t \varphi + \mu w_i \Delta \varphi + (\gamma - \delta u_{i-1}) w_i \varphi) dx dt &= 0 \\ \int_0^{T_r} \int_{\mathbb{R}^n} (u_i \partial_t \varphi + u_i v(w_{i-1}) \cdot \nabla \varphi + (\alpha w_{i-1} - \beta) u_i \varphi) dx dt &= 0. \end{aligned}$$

Thanks to the $\mathbf{L}^\infty$ bounds proved in claims 1 and 2, we can apply the Dominated Convergence Theorem, ensuring that (u_*, w_*) is a weak solution to (1.1.1) for $t \in [0, T_r]$.

For all $t \in [0, T_r]$, we define $\mathcal{R}_{0,t}(u_o, w_o) = (u_*, w_*)(t)$.

Consider now a couple of initial data $(u_{1,o}, w_{1,o}), (u_{2,o}, w_{2,o})$ satisfying (1.4.10). For $t \in [0, T_r]$, we know that $(u_i, w_i)(t) = \mathcal{R}_{0,t}(u_{i,o}, w_{i,o})$ solves (1.1.1) with initial datum $(u_{i,o}, w_{i,o})$ in distributional sense. For $(t, x) \in [0, T_r] \times \mathbb{R}^n$ we define for $i = 1, 2$

$$a_i(t, x) = \gamma - \delta u_i(t, x), \quad b_i(t, x) = \alpha w_i(t, x) - \beta, \quad \text{and} \quad c_i(t, x) = v(w_i(t))(x).$$

Using the operators $\mathcal{P}$ of Proposition 1.5 and $\mathcal{H}$ of Proposition 1.8, observe that $w_i(t) = \mathcal{P}_{0,t}^i w_{i,o}$ and $u_i(t) = \mathcal{H}_{0,t}^i u_{i,o}$, for $i = 1, 2$. Moreover, note that $\mathcal{P}_{0,t}^1 w_{2,o}$ is the solution to (1.2.3) with a_1 in the source term and initial datum $w_{2,o}$, while $\mathcal{H}_{0,t}^1 u_{2,o}$ is the solution to (1.2.4) with coefficients b_1, c_1 and initial datum $u_{2,o}$.

We compute $\|\mathcal{R}_{0,t}(u_{1,o}, w_{1,o}) - \mathcal{R}_{0,t}(u_{2,o}, w_{2,o})\|_{\mathcal{X}}$ as defined in (1.2.2):

$$\begin{aligned} & \|\mathcal{R}_{0,t}(u_{1,o}, w_{1,o}) - \mathcal{R}_{0,t}(u_{2,o}, w_{2,o})\|_{\mathcal{X}} \\ &= \left\| \mathcal{H}_{0,t}^1 u_{1,o} - \mathcal{H}_{0,t}^2 u_{2,o} \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} + \left\| \mathcal{P}_{0,t}^1 w_{1,o} - \mathcal{P}_{0,t}^2 w_{2,o} \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \\ &\leq \left\| \mathcal{H}_{0,t}^1 u_{1,o} - \mathcal{H}_{0,t}^1 u_{2,o} \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} + \left\| \mathcal{H}_{0,t}^1 u_{2,o} - \mathcal{H}_{0,t}^2 u_{2,o} \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \end{aligned} \quad (1.4.31)$$

$$+ \left\| \mathcal{P}_{0,t}^1 w_{1,o} - \mathcal{P}_{0,t}^1 w_{2,o} \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} + \left\| \mathcal{P}_{0,t}^1 w_{2,o} - \mathcal{P}_{0,t}^2 w_{2,o} \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})}. \quad (1.4.32)$$

Compute each term of (1.4.31) separately. Since $\mathcal{H}_{0,t}^1$ is linear, by 3. in Proposition 1.8 and its particularisation in Claim 2, the first term in (1.4.31) is estimated by

$$\begin{aligned} \left\| \mathcal{H}_{0,t}^1 u_{1,o} - \mathcal{H}_{0,t}^1 u_{2,o} \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} &\leq \|u_{1,o} - u_{2,o}\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \exp \left(\alpha \|w_{1,o}\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} t e^{\gamma t} \right) \\ &\leq \|u_{1,o} - u_{2,o}\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \exp \left(\alpha r t e^{\gamma t} \right). \end{aligned} \quad (1.4.33)$$

Concerning the second term in (1.4.31), recall 5. in Proposition 1.8 and adapt the estimates above for $\|u_{i+1}(t) - u_i(t)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})}$, using $\mathbf{M}_t$ as defined in (1.4.24)–(1.4.25) and

$$\begin{aligned}\Theta &= \left((\alpha + K) \|w_{1,o}\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} + K \|w_{2,o}\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \right) e^{\gamma t} + \beta, \\ \Theta_1 &= \left((\alpha + 2(n+1)K) \|w_{1,o}\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} + K \|w_{2,o}\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \right) e^{\gamma t} + \beta.\end{aligned}$$

obtaining

$$\begin{aligned}& \left\| \mathcal{H}_{0,t}^1 u_{2,o} - \mathcal{H}_{0,t}^2 u_{2,o} \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \\ & \leq e^{\Theta t} \int_0^t \left\| \mathcal{P}_{0,\tau}^1 w_{1,o} - \mathcal{P}_{0,\tau}^2 w_{2,o} \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} d\tau \\ & \quad \times \left[K e^{\Theta_1 t} \left[\text{TV}(u_{2,o}) + I_n \|u_{2,o}\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \mathbf{M}_t(u_{1,o}, w_{1,o}) \right] \right. \\ & \quad \left. + \|u_{2,o}\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \left(\alpha + C \left(\|w_{2,o}\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} e^{\gamma t} \right) \right) e^{(\alpha+K)t} e^{\gamma t \max_{i=1,2} \|w_{i,o}\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})}} \right] \\ & \leq \mathcal{K}_u(t, r) \int_0^t \left\| \mathcal{P}_{0,\tau}^1 w_{1,o} - \mathcal{P}_{0,\tau}^2 w_{2,o} \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} d\tau, \tag{1.4.34}\end{aligned}$$

where $\mathcal{K}_u(t, r)$ is defined in (1.4.29).

Pass to (1.4.32). Since the map $\mathcal{P}_{0,t}^1$ is linear, by 3. in Proposition 1.5 and its particularisation in Claim 1, we have the following estimate for the first term in (1.4.32):

$$\left\| \mathcal{P}_{0,t}^1 w_{1,o} - \mathcal{P}_{0,t}^1 w_{2,o} \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \leq \|w_{1,o} - w_{2,o}\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} e^{\gamma t}. \tag{1.4.35}$$

Concerning the second term in (1.4.32), recall 5. in Proposition 1.5 and adapt the estimates above for $\|w_{i+1}(t) - w_i(t)\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})}$ to obtain

$$\begin{aligned}& \left\| \mathcal{P}_{0,t}^1 w_{2,o} - \mathcal{P}_{0,t}^2 w_{2,o} \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \\ & = \delta \|w_{2,o}\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \exp \left(2\gamma t + \delta t \|u_{2,o}\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} e^{(\alpha+K)\|w_{2,o}\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \frac{e^{\gamma t} - 1}{\gamma}} \right) \\ & \quad \times \int_0^t \left\| \mathcal{H}_{0,\tau}^1 u_{1,o} - \mathcal{H}_{0,\tau}^2 u_{2,o} \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} d\tau \\ & \leq \mathcal{K}_w(t, r) \int_0^t \left\| \mathcal{H}_{0,\tau}^1 u_{1,o} - \mathcal{H}_{0,\tau}^2 u_{2,o} \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} d\tau \tag{1.4.36}\end{aligned}$$

with $\mathcal{K}_w(t, r)$ as in (1.4.19). We rewrite (1.4.31)–(1.4.32) using (1.4.33)–(1.4.34)–(1.4.35)–(1.4.36):

$$\begin{aligned}& \left\| \mathcal{R}_{0,t}(u_{1,o}, w_{1,o}) - \mathcal{R}_{0,t}(u_{2,o}, w_{2,o}) \right\|_{\mathcal{X}} \\ & = \exp \left(\alpha r t e^{\gamma t} \right) \|u_{1,o} - u_{2,o}\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} + \mathcal{K}_u(t, r) \int_0^t \left\| \mathcal{P}_{0,\tau}^1 w_{1,o} - \mathcal{P}_{0,\tau}^2 w_{2,o} \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} d\tau \\ & \quad + e^{\gamma t} \|w_{1,o} - w_{2,o}\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} + \mathcal{K}_w(t, r) \int_0^t \left\| \mathcal{H}_{0,\tau}^1 u_{1,o} - \mathcal{H}_{0,\tau}^2 u_{2,o} \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} d\tau\end{aligned}$$

$$\leq e^{(\gamma + \alpha r e^{\gamma t})t} \left(\|u_{1,o} - u_{2,o}\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} + \|w_{1,o} - w_{2,o}\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \right) \\ + \mathcal{K}(t, r) \int_0^t \|\mathcal{R}_{0,\tau}(u_{1,o}, w_{1,o}) - \mathcal{R}_{0,\tau}(u_{2,o}, w_{2,o})\|_{\mathcal{X}} d\tau$$

and $\mathcal{K}(t, r)$ is as in (1.4.30). An application of Gronwall Lemma yields: for all $t \in [0, T_r]$

$$\|\mathcal{R}_{0,t}(u_{1,o}, w_{1,o}) - \mathcal{R}_{0,t}(u_{2,o}, w_{2,o})\|_{\mathcal{X}} \leq \mathcal{L}(t, r) \|(u_{1,o}, w_{1,o}) - (u_{2,o}, w_{2,o})\|_{\mathcal{X}}, \\ \text{where } \mathcal{L}(t, r) = \exp \left[\left(\gamma + \alpha r e^{\gamma t} + \mathcal{K}(t, r) \right) t \right] \quad \text{with } \mathcal{K}(t, r) \text{ as in (1.4.30),} \quad (1.4.37)$$

proving point 3. under condition (1.4.10).

Denote $(u(t), w(t)) = \mathcal{R}_{0,t}(u_o, w_o)$ and define

$$a(t, x) = \gamma - \delta u(t, x), \quad b(t, x) = \alpha w(t, x) - \beta, \quad \text{and} \quad c(t, x) = v(w(t))(x). \quad (1.4.38)$$

Then, with the notation in Proposition 1.5 and Proposition 1.8, by construction

$$u(t) = \mathcal{H}_{0,t}u_o \text{ and } w(t) = \mathcal{P}_{0,t}w_o, \text{ so that } \mathcal{R}_{0,t}(u_o, w_o) = (\mathcal{P}_{0,t}w_o, \mathcal{H}_{0,t}u_o). \quad (1.4.39)$$

We now extend the map $t \rightarrow \mathcal{R}_{0,t}(u_o, w_o)$ to the whole time axis. Indeed, define

$$T_* = \sup \left\{ T \in \mathbb{R}^+ : \mathcal{R}_{0,t}(u_o, w_o) \text{ is defined for all } t \in [0, T] \right\}.$$

The previous construction ensures that $\mathcal{R}_{0,t}(u_o, w_o)$ is defined at least for all $t \in [0, T_r]$, hence the supremum above is well defined. Assume now that $T_* < +\infty$. By 8. in Proposition 1.8, the first component of the map $t \rightarrow \mathcal{R}_{0,t}(u_o, w_o)$ is Lipschitz continuous with a constant uniformly bounded by a function of r on any bounded time interval. By 8. in Proposition 1.5 the second component of the map $t \rightarrow \mathcal{R}_{0,t}(u_o, w_o)$ is uniformly continuous on, say $[0, T_r/2]$ and Hölder continuous with exponent $\vartheta = 1/2$ for $t \in [T_r/2, T_*[$, the Hölder constant being bounded by a function of r . Hence, the map $t \rightarrow \mathcal{R}_{0,t}(u_o, w_o)$ is uniformly continuous for $t \in [0, T_*[$ and the pair $(U, W) = \lim_{t \rightarrow T_*^-} (u(t), w(t))$ is well defined and in $\mathcal{X}_{r_*}$ for a suitable r_* . Moreover, since $W = w(T_*) = \mathcal{P}_{0,T_*}w_o$, we have that (U, W) also satisfies (1.4.10) by 7. and 9. in Proposition 1.5. Repeating the construction above, we show that the Cauchy problem consisting of (1.1.1) with initial datum $(u, w)(T_*) = (U, W)$ admits a solution defined on $[T_*, T_* + T_{r_*}]$, which contradicts the choice of T_* , unless $T_* = +\infty$. Define a, b , and c as in (1.4.38). Then, by 8. in Proposition 1.5 and 8. in Proposition 1.8, (1.4.39) directly ensures the continuity in time of $\mathcal{R}$, for all $t \in \mathbb{R}^+$. This completes the proof of point 2. under condition (1.4.10).

Choose now a general initial datum $(u_o, w_o) \in \mathcal{X}^+$, so that $(u_o, w_o) \in \mathcal{X}_r$ for a suitable $r > 0$. Let ρ_n be a sequence of mollifiers with $\rho_n \in \mathbf{C}_c^\infty(\mathbb{R}^n; \mathbb{R}^+)$ and $\int_{\mathbb{R}^n} \rho_n(x) dx = 1$. Then, the sequence of initial data $(u_o, w_o * \rho_n)$ is in $\mathcal{X}_r$, satisfies (1.4.10), and converges to (u_o, w_o) in $\mathcal{X}$. By (1.4.37), we can uniquely extend $\mathcal{R}$ through the limit $\mathcal{R}_{0,t}(u_o, w_o) = \lim_{n \rightarrow +\infty} \mathcal{R}_{0,t}(u_o, w_o * \rho_n)$ to all $\mathcal{X}_r$ and for all $t \in \mathbb{R}^+$, completing the proof of point 2. and of point 3. for all $t \in \mathbb{R}^+$. Note that the positivity of the solution follows from Claim 0. The $\mathbf{L}^1$ and $\mathbf{L}^\infty$ estimates at point 4. now follow from Claims 1 and 2. Again, the continuity in time of the map $\mathcal{R}$ so extended directly follows from 8. in Proposition 1.5 and 8. in Proposition 1.8.

We now prove that $\mathcal{R}$ is a process. For any $(u_o, w_o) \in \mathcal{X}^+$, use the notation (1.4.38) and observe that 1. in Proposition 1.5, 1. in Proposition 1.8 and (1.4.39) ensure that the map $\mathcal{R}$ is a process. Note however that (1.1.1) is autonomous, hence definitions 1.3, 1.6, and 1.1 ensure that $\mathcal{R}$ is a semigroup, proving point 1.

Point 5. follows from 9. in Proposition 1.8, (v) , and the $\mathbf{L}^1$ estimate in 4. above. $\square$

Lemma 1.9. *Let η be such that $\nabla\eta \in (\mathbf{C}^2 \cap \mathbf{W}^{2,1} \cap \mathbf{W}^{1,\infty})(\mathbb{R}^n; \mathbb{R}^n)$. Then, the map v defined in (1.1.2) satisfies (v) with*

$$K = \max \left\{ 2\kappa \|\nabla\eta\|_{\mathbf{W}^{2,1}}, 2\kappa \|\nabla\eta\|_{\mathbf{W}^{1,\infty}}, 3\|\nabla\eta\|_{\mathbf{W}^{1,\infty}}, \frac{48}{25\sqrt{5}} \|\nabla\eta\|_{\mathbf{W}^{2,1}} \right\}, \quad (1.4.40)$$

$$C(\xi) = K(1 + K\xi) \quad \text{for all } \xi \in \mathbb{R}^+.$$

In the proof below we use the Euclidean norm $\|v\|_{\mathbb{R}^n} = \sqrt{\sum_{i=1}^n (v_i)^2}$ on vectors in $\mathbb{R}^n$ and the operator norm $\|A\|_{\mathbb{R}^n \times n} = \sup_{v: \|v\|_{\mathbb{R}^n}=1} \|Av\|_{\mathbb{R}^n}$ on $n \times n$ matrices.

Proof. The bound on $\|v(w)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})}$ is ensured by

$$\|v(w)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \leq \kappa \|\nabla\eta\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R}^n)} \|w\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})}, \quad (1.4.41)$$

see also [18, Lemma 3.1]. To estimate $\nabla v(w)$, use the identity $\nabla(fv) = f\nabla v + v \otimes \nabla f$:

$$\begin{aligned} \nabla v(w) &= \kappa \frac{1}{\left(1 + \|w * \nabla\eta\|^2\right)^{1/2}} \nabla(w * \nabla\eta) + \kappa (w * \nabla\eta) \otimes \nabla \frac{1}{\left(1 + \|w * \nabla\eta\|^2\right)^{1/2}} \\ &= \kappa \frac{w * \nabla^2\eta}{\left(1 + \|w * \nabla\eta\|^2\right)^{1/2}} - \kappa (w * \nabla\eta) \otimes \frac{(w * \nabla^2\eta)(w * \nabla\eta)}{\left(1 + \|w * \nabla\eta\|^2\right)^{3/2}}. \end{aligned}$$

Recall that $\|v_1 \otimes v_2\|_{\mathbb{R}^n \times n} \leq \|v_1\|_{\mathbb{R}^n} \|v_2\|_{\mathbb{R}^n}$ and $\|Av\|_{\mathbb{R}^n} \leq \|A\|_{\mathbb{R}^n \times n} \|v\|_{\mathbb{R}^n}$, hence

$$\begin{aligned} &\|\nabla v(w)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R}^{n \times n})} \\ &\leq \kappa \frac{\|w * \nabla^2\eta\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R}^{n \times n})}}{\sqrt{1 + \|w * \nabla\eta\|^2}} \\ &\quad + \kappa \left\| \frac{w * \nabla\eta}{\sqrt{1 + \|w * \nabla\eta\|^2}} \otimes \frac{w * \nabla^2\eta}{\sqrt{1 + \|w * \nabla\eta\|^2}} \frac{w * \nabla\eta}{\sqrt{1 + \|w * \nabla\eta\|^2}} \right\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R}^{n \times n})} \\ &\leq \kappa \|w * \nabla^2\eta\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R}^{n \times n})} \\ &\quad + \kappa \left\| \frac{w * \nabla\eta}{\sqrt{1 + \|w * \nabla\eta\|^2}} \right\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R}^n)} \|w * \nabla^2\eta\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R}^{n \times n})} \left\| \frac{w * \nabla\eta}{\sqrt{1 + \|w * \nabla\eta\|^2}} \right\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R}^n)} \\ &\leq 2\kappa \|w * \nabla^2\eta\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R}^{n \times n})} \\ &\leq 2\kappa \|\nabla^2\eta\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R}^{n \times n})} \|w\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})}, \end{aligned} \quad (1.4.42)$$

proving also that $v(w) \in \mathbf{W}^{1,\infty}(\mathbb{R}^n; \mathbb{R}^n)$. Pass now to

$$\begin{aligned} &v(w_1) - v(w_2) \\ &= \kappa \frac{(w_1 - w_2) * \nabla\eta}{\sqrt{1 + \|w_1 * \nabla\eta\|^2}} + \kappa (w_2 * \nabla\eta) \left(\frac{1}{\sqrt{1 + \|w_1 * \nabla\eta\|^2}} - \frac{1}{\sqrt{1 + \|w_2 * \nabla\eta\|^2}} \right). \end{aligned}$$

Using the elementary inequality $\left| (1+x^2)^{1/2} - (1+y^2)^{1/2} \right| \leq |x-y|$ we obtain:

$$\begin{aligned}
& v(w_1) - v(w_2) \\
&= \kappa \frac{(w_1 - w_2) * \nabla \eta}{\sqrt{1 + \|w_1 * \nabla \eta\|^2}} + \kappa \frac{w_2 * \nabla \eta}{\sqrt{1 + \|w_2 * \nabla \eta\|^2}} \frac{\sqrt{1 + \|w_2 * \nabla \eta\|^2} - \sqrt{1 + \|w_1 * \nabla \eta\|^2}}{\sqrt{1 + \|w_1 * \nabla \eta\|^2}}; \\
& \quad \left\| v(w_1) - v(w_2) \right\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R}^n)} \\
& \leq \kappa \left\| (w_1 - w_2) * \nabla \eta \right\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R}^n)} + \kappa \left| \left\| w_1 * \nabla \eta \right\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R}^n)} - \left\| w_2 * \nabla \eta \right\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R}^n)} \right| \\
& \leq 2\kappa \left\| (w_1 - w_2) * \nabla \eta \right\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R}^n)} \\
& \leq 2\kappa \|w_1 - w_2\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \|\nabla \eta\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R}^n)}. \tag{1.4.43}
\end{aligned}$$

Compute the divergence of $v(w)$ as follows:

$$\nabla \cdot v(w) = \kappa \frac{w * (\Delta \eta)}{\sqrt{1 + \|w * \nabla \eta\|^2}} \left(1 - \frac{(w * \nabla \eta) (w * \nabla \eta)}{1 + \|w * \nabla \eta\|^2} \right) = \kappa \frac{w * (\Delta \eta)}{(1 + \|w * \nabla \eta\|^2)^{3/2}}. \tag{1.4.44}$$

We now compute the gradient of (1.4.44):

$$\nabla (\nabla \cdot v(w)) = \kappa \frac{w * \nabla \Delta \eta}{(1 + \|w * \nabla \eta\|^2)^{3/2}} - 3\kappa (w * \Delta \eta) \frac{w * \nabla^2 \eta}{(1 + \|w * \nabla \eta\|^2)^2} \frac{w * \nabla \eta}{\sqrt{1 + \|w * \nabla \eta\|^2}}$$

so that

$$\begin{aligned}
& \left\| \nabla (\nabla \cdot v(w)) \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R}^n)} \\
& \leq \kappa \|w * \nabla \Delta \eta\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R}^n)} + 3\kappa \|w * \Delta \eta\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \left\| w * \nabla^2 \eta \right\|_{\mathbf{L}^\infty(\mathbb{R}; \mathbb{R}^{n \times n})} \\
& \leq \kappa \|w\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \left[\left\| \nabla \Delta \eta \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R}^n)} + 3 \left\| \Delta \eta \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \left\| \nabla^2 \eta \right\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R}^{n \times n})} \|w\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \right]. \tag{1.4.45}
\end{aligned}$$

Consider now

$$\begin{aligned}
& \nabla \cdot (v(w_1) - v(w_2)) \\
&= \kappa \frac{(w_1 - w_2) * \Delta \eta}{(1 + \|w_1 * \nabla \eta\|^2)^{3/2}} + \kappa (w_2 * \Delta \eta) \left(\frac{1}{(1 + \|w_1 * \nabla \eta\|^2)^{3/2}} - \frac{1}{(1 + \|w_2 * \nabla \eta\|^2)^{3/2}} \right).
\end{aligned}$$

Using the inequality $\left| (1+x^2)^{-3/2} - (1+y^2)^{-3/2} \right| \leq \frac{48}{25\sqrt{5}} |x-y|$,

$$\begin{aligned}
& \left\| \nabla \cdot (v(w_1) - v(w_2)) \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \\
& \leq \kappa \left\| (w_1 - w_2) * \Delta \eta \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} + \frac{48}{25\sqrt{5}} \kappa \|w_2 * \Delta \eta\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \left\| (w_1 - w_2) * \nabla \eta \right\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R}^n)} \\
& \leq \kappa \|w_1 - w_2\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \|\Delta \eta\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})}
\end{aligned}$$

$$\begin{aligned}
& + \frac{48}{25\sqrt{5}} \kappa \|w_2\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \|\Delta\eta\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \|w_1 - w_2\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \|\nabla\eta\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R}^n)} \\
& \leq \kappa \|w_1 - w_2\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \|\Delta\eta\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \left(1 + \frac{48}{25\sqrt{5}} \|\nabla\eta\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R}^n)} \|w_2\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \right). \quad (1.4.46)
\end{aligned}$$

Setting K as in (1.4.40), the inequalities above become:

$$\begin{aligned}
\|v(w)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R}^n)} & \leq K \|w\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} && \text{from (1.4.41)} \\
\|\nabla v(w)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} & \leq K \|w\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} && \text{from (1.4.42)} \\
\|v(w_1) - v(w_2)\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R}^n)} & \leq K \|w_1 - w_2\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} && \text{from (1.4.43)} \\
\|\nabla(\nabla \cdot v(w))\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R}^n)} & \leq K \left(1 + K \|w\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} \right) \|w\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} && \text{from (1.4.45)} \\
\|\nabla \cdot (v(w_1) - v(w_2))\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} & \leq K \left(1 + K \|w_2\|_{\mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})} \right) \|w_1 - w_2\|_{\mathbf{L}^1(\mathbb{R}^n; \mathbb{R})} && \text{from (1.4.46)}
\end{aligned}$$

completing the proof. $\square$

Chapter 2

Numerical Scheme for a Mixed Hyperbolic–Parabolic System

The content of this Chapter is going to appear on *ESAIM:Mathematical Modelling and Numerical Analysis* as [61].

2.1 Introduction

We consider the following Cauchy problem in two space dimensions:

$$\partial_t u + \nabla \cdot (f(u) \mathbf{v}(w)) = (\alpha w - \beta)u \quad (2.1.1a)$$

$$\partial_t w - \mu \Delta w = (\gamma - \delta u)w \quad (2.1.1b)$$

$$u(0, x, y) = u_o(x, y) \quad (2.1.1c)$$

$$w(0, x, y) = w_o(x, y) \quad (2.1.1d)$$

This is a generalisation of the predator–prey model presented in Chapter 1, see also [25]. In particular, $u = u(t, x, y)$ and $w = w(t, x, y)$ represent respectively the predator and the prey densities at time $t \in \mathbb{R}^+$ and position $(x, y) \in \mathbb{R}^2$. The parameters $\alpha, \beta, \gamma, \delta$ appearing in system (2.1.1) are all positive, μ is strictly positive. More precisely, α is the predator birth rate due to feeding on prey, β is the predator mortality rate, γ is the prey birth rate, δ is the prey mortality rate due to predators and μ represents the diffusion speed of prey.

Predator–prey models are widely studied in the literature since a long time, starting with the pioneering works of Lotka [54] and Volterra [64] in the 1920s. The model proposed therein is based on ordinary differential equations modelling the interactions of two species' populations u (predators) and w (prey), where birth and death rates depend on the interactions, see equation (2.1.2).

$$\frac{d}{dt}u = (\alpha w - \beta)u, \quad \frac{d}{dt}w = (\gamma - \delta u)w \quad (2.1.2)$$

This basic model was extended subsequently to obtain more refined predictions of population sizes [43, 11, 8] or to model the immune system response to infectious diseases [12, 59]. Further applications of the Lotka–Volterra model (2.1.2) and variations of it can be found in economy, see e.g. [40] for a pioneering work.

All these models are based on ordinary differential equations, thus implicitly assuming a homogeneous distribution of the species in space. Model (2.1.1) overcomes this deficiency and allows for spacial variations of predators and prey. More precisely, we assume that prey

diffuse in the whole space without preferred direction of motion, while predators are attracted by prey in a certain radius around them. To model this finite-range non local behaviour, the velocity vector $\mathbf{v}$ of the predators depends on a convolution of the prey density with a kernel function measuring e.g. the ability of the predators to feel the presence of prey at a certain distance. This non local term in the flux of (2.1.1a) causes the predators to move in direction of the highest prey density, thus chasing the prey. Note that the prey does not try to escape the predators.

In Chapter 1 and [25], the well-posedness of (2.1.1) was shown for $f(u) = u$ and initial data in $\mathbf{L}^1 \cap \mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R})$ with the additional constraint that u_o is of bounded total variation. The topic of the present work is to study the convergence of a finite difference scheme for the mixed hyperbolic-parabolic system (2.1.1). We choose a Lax-Friedrichs type method for the hyperbolic part, including a modification to deal with the non local term $\mathbf{v}$, and a standard five-point stencil for the discretisation of the parabolic part, see also (2.2.4) in Section 2.2. In particular, we consider a different diffusion constant in the Lax-Friedrichs flux, see (2.2.4d) and Remark 2.5. This is done in order to balance the contributions of the space-time dependent velocity field $\mathbf{v}$ to obtain positivity and an $\mathbf{L}^\infty$ bound for u .

Since the velocity function $\mathbf{v}$ depends only on w , we could view equation (2.1.1a) as a standard hyperbolic equation with space and time dependent flux function $\tilde{\mathbf{f}}(t, x, u) := f(u) \mathbf{v}(t, x)$. Equations of such type have been widely studied in the literature and especially the convergence of finite volume schemes is established in [32, 16, 17, 45] under different assumptions on $\mathbf{v}(t, x)$. Recently, a Lax-Friedrichs type method for a non local hyperbolic conservation law was studied in [6, 2]. Due to the coupling of (2.1.1a) and (2.1.1b) through the velocity function $\mathbf{v}$ and the source terms, the above results do not apply to the present case. The same holds true for the well known standard convergence results for finite difference discretisations of (quasi)linear parabolic equations, since these results are usually based on estimates in the discrete l^2 -norm. Here, the coupling of the parabolic part to a hyperbolic equation forces us to study the finite difference scheme in an l^1 setting.

To prove the convergence of the numerical scheme below, we make the following assumptions on the functions f and $\mathbf{v}$ appearing in (2.1.1):

- (f) $f \in \mathbf{C}^2(\mathbb{R}; \mathbb{R})$ and $f(0) = 0$;
- (v) $\mathbf{v}: (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^2; \mathbb{R}) \rightarrow (\mathbf{C}^2 \cap \mathbf{W}^{2,\infty})(\mathbb{R}^2; \mathbb{R}^2)$ depends on w through a convolution in space, i.e. $\mathbf{v}(w) := \mathbf{v}(\eta * w)$ for a space dependent convolution kernel $\eta \in \mathbf{L}^1(\mathbb{R}^2; \mathbb{R})$. Furthermore, there exist a constant K and an increasing map $C \in \mathbf{L}_{loc}^\infty(\mathbb{R}^+; \mathbb{R}^+)$ such that for all $w \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^2; \mathbb{R})$

$$\begin{aligned} \|\nabla \mathbf{v}(w)\|_{\mathbf{L}^\infty(\mathbb{R}^2; \mathbb{R}^{2 \times 2})} &\leq K \|w\|_{\mathbf{L}^\infty(\mathbb{R}^2; \mathbb{R})} \\ \|\nabla \mathbf{v}(w)\|_{\mathbf{L}^1(\mathbb{R}^2; \mathbb{R}^{2 \times 2})} &\leq K \|w\|_{\mathbf{L}^1(\mathbb{R}^2; \mathbb{R})} \\ \left\| \nabla (\nabla \cdot \mathbf{v}(w)) \right\|_{\mathbf{L}^\infty(\mathbb{R}^2; \mathbb{R}^2)} &\leq C \left(\|w\|_{\mathbf{L}^\infty(\mathbb{R}^2; \mathbb{R})} \right). \end{aligned}$$

With slight abuse of notation, we will also write $\mathbf{v}(t, x, y)$ instead of $\mathbf{v}(w)(t, x, y)$, to improve readability. Note that the case $f(u) = u$ is the one considered in Chapter 1 and [25]. An example of a function $\mathbf{v}$ that fulfils all requirements of assumption (v) above can be found in Section 2.5. The initial data (u_o, w_o) are chosen to fulfil the assumption

- (0) $(u_o, w_o) \in (\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^2; \mathbb{R}^+) \times (\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^2; \mathbb{R}^+)$ are positive-valued functions, i.e. $u_o \geq 0$ and $w_o \geq 0$ for a. e. $(x, y) \in \mathbb{R}^2$.

Remark 2.1. *Under the assumption $(\mathbf{f})$, existence and uniqueness of the solution to (2.1.1) in the space $(\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n; \mathbb{R}^+)$ follow by a straightforward extension of the results of Chapter 1 and [25].*

The Chapter is organised as follows. In Section 2.2, we introduce basic notations and describe the algorithm in details. To prove convergence of the given scheme, we derive bounds on the variables u and w in various norms in Section 2.3 that are needed to conclude the convergence of the scheme in Section 2.4. Finally, Section 2.5 is devoted to numerical examples including experimental convergence studies.

2.2 The Algorithm

We introduce a uniform mesh of width h along both x and y axes, given by the cartesian grid whose points are of the form

$$\{(x_i, y_j) \mid x_i = ih, y_j = jh, i, j \in \mathbb{Z}\}.$$

With slight abuse of notation, we will also write $x_{i,j} := (x_i, y_j)$, $x_{i+1/2,j} := (x_{i+1/2}, y_j) = ((i+1/2)h, jh)$ and $x_{i,j+1/2}$ defined analogously. Furthermore, we define the *parabolic* time step $\tau_p = \frac{h^2}{4\mu}$ and let the time step τ be such that

$$\tau = \tau_p \max \left\{ n \in \mathbb{N} : \frac{n\tau_p}{h} \|\partial_u f\|_{\mathbf{L}^\infty} \|\mathbf{v}\|_{\mathbf{L}^\infty} < \frac{1}{4} \right\} =: m \tau_p.$$

In other words, τ is a multiple of τ_p that satisfies the following CFL condition:

$$\frac{\tau}{h} \|\partial_u f\|_{\mathbf{L}^\infty} \|\mathbf{v}\|_{\mathbf{L}^\infty} < \frac{1}{4}. \quad (2.2.1)$$

We also define $\lambda = \frac{\tau}{h}$. We remark that $\|\mathbf{v}\|_{\mathbf{L}^\infty}$ is assumed to be *the global* supremum of $\mathbf{v}$ throughout the Chapter to simplify notations and estimates. This assumption can however be relaxed to a time–step–wise supremum, thus leading to an adaptive time stepping strategy. The same holds for $\|\partial_u f\|_{\mathbf{L}^\infty}$. Note further that τ is fixed by the choice of τ_p , contrarily to the standard case for hyperbolic equations, where (2.2.1) provides some freedom in the choice of τ . Finally, let us remark that the time step τ_p is chosen such that the numerical approximation of w^n has a particularly simple form (see (2.2.4b) below). This will improve the readability of the estimates in the following sections. However, any time step τ_p that yields a stable discretisation of w^n will also lead to a convergent scheme.

Let $(u(t, x, y), w(t, x, y))$ be the unique solution to (2.1.1) (see Chapter 1 and [25] for an existence and uniqueness result in the case of $f(u) = u$ and Remark 2.1 for the general case, under assumption $(\mathbf{f})$). To compute the solution numerically we set

$$u_{i,j}^o = \frac{1}{h^2} \int_{I_{i,j}} u_o(x, y) \, dx \, dy, \quad w_{i,j}^o = \frac{1}{h^2} \int_{I_{i,j}} w_o(x, y) \, dx \, dy, \quad (2.2.2)$$

where

$$I_{i,j} = \left[\left(i - \frac{1}{2} \right) h, \left(i + \frac{1}{2} \right) h \right] \times \left[\left(j - \frac{1}{2} \right) h, \left(j + \frac{1}{2} \right) h \right], \quad (2.2.3)$$

so that $u_{i,j}^o$ and $w_{i,j}^o$ are the cell averages of $u_o(x, y)$ and $w_o(x, y)$ respectively over the (i, j) -th cell. By (2.2.2) it follows easily that

$$\begin{aligned} \|u^o\|_{\mathbf{L}^\infty} &\leq \|u_o\|_{\mathbf{L}^\infty} & \|w^o\|_{\mathbf{L}^\infty} &\leq \|w_o\|_{\mathbf{L}^\infty} \\ \|u^o\|_{\mathbf{L}^1} &\leq \|u_o\|_{\mathbf{L}^1} & \|w^o\|_{\mathbf{L}^1} &\leq \|w_o\|_{\mathbf{L}^1} \\ \mathrm{TV}(u^o) &\leq \mathrm{TV}(u_o) & \mathrm{TV}(w^o) &\leq \mathrm{TV}(w_o), \end{aligned}$$

see also [27]. For simplicity, we denote $v_{i+1/2,j}^n = \mathbf{v}(n\tau, x_{i+1/2}, y_j) \cdot \mathbf{n}_{i+1/2,j}$, where $\mathbf{n}_{i+1/2,j}$ is the normal vector of the cell boundary at $x_{i+1/2,j}$, pointing from the cell with value $u_{i,j}$ to the cell with value $u_{i+1,j}$. The definition of $v_{i,j+1/2}^n$ follows analogously. Note that this definition, together with the rectangular grid, implies that $v_{i+1/2,j}^n$ is the x -component of the vector $\mathbf{v}^n$ at $x_{i+1/2,j}$. Analogously, $v_{i,j+1/2}^n$ denotes the y -component.

To approximate (2.1.1), we use a finite-difference scheme for the parabolic part and a Lax–Friedrichs type finite volume method for the hyperbolic part. In both equations, the source terms are treated by operator splitting, using a second order Runge–Kutta method. Further information on the use of splitting methods in system of possibly viscous conservation laws can be found in [42]. The nonlinear coupling of (2.1.1) is numerically resolved by a sequential coupling of the parabolic and the hyperbolic equation. In other words, we start computing an approximation of w^n , solving the parabolic equation by an explicit scheme with smaller time step τ_p , thus having to perform m small time steps to reach the hyperbolic time step τ defined in (2.2.1). This step involves the values of u at the previous time step $t^{n-1} := (n-1)\tau$ for the discretisation of the source term. Once, w^n is computed, we can use it to compute the velocity field $\mathbf{v}(w^n)$ and the source term in the hyperbolic equation. This balance law is now integrated by means of a Lax–Friedrichs type scheme with dimensional splitting, while its source term is included using a second order Runge–Kutta method, analogously to the parabolic equation. Note that the second order discretisation of the source terms is necessary to guarantee the positivity of the approximate solution, as shown in Paragraph 2.3.1.

Note that, in order to be able to give a valid meaning to $\mathbf{v}(w^n)$, involving a convolution, we set $w^n := \sum_{i,j} w_{i,j}^n \chi_{I_{i,j}}$. The algorithm is now defined as follows:

Algorithm 2.2 (Mixed Scheme).

$$\begin{aligned} &\text{for } n = 0, \dots, N-1 \\ &\quad W^{n,0} = w^n \end{aligned} \tag{2.2.4a}$$

$$\begin{aligned} &\text{for } l = 0, \dots, m-1 \\ &\quad W_{i,j}^{n,l+1} = \frac{1}{4} \left(W_{i+1,j}^{n,l} + W_{i-1,j}^{n,l} + W_{i,j+1}^{n,l} + W_{i,j-1}^{n,l} \right) \\ &\quad \quad \times \left[1 + \tau_p \left(\gamma - \delta u_{i,j}^n \right) \left(1 + \frac{\tau_p}{2} \left(\gamma - \delta u_{i,j}^n \right) \right) \right] \end{aligned} \tag{2.2.4b}$$

$$\begin{aligned} &\text{end} \\ &\quad w^{n+1} = W^{n,m} \end{aligned} \tag{2.2.4c}$$

$$F(u_1, u_2, t, x, y) = \frac{1}{2} (f(u_1) + f(u_2)) v(t, x, y) - \frac{1}{8\lambda} (u_1 - u_2) \tag{2.2.4d}$$

$$\begin{aligned} U_{i,j}^{n+1/2} = u_{i,j}^n - \lambda \Big[&F(u_{i+1,j}^n, u_{i,j}^n, (n+1)\tau, x_{i+1/2}, y_j) \\ &- F(u_{i,j}^n, u_{i-1,j}^n, (n+1)\tau, x_{i-1/2}, y_j) \Big] \end{aligned} \tag{2.2.4e}$$

$$U_{i,j}^{n+1} = U_{i,j}^{n+1/2} - \lambda \left[F(U_{i,j+1}^{n+1/2}, U_{i,j}^{n+1/2}, (n+1)\tau, x_i, y_{j+1/2}) - F(U_{i,j}^{n+1/2}, U_{i,j-1}^{n+1/2}, (n+1)\tau, x_i, y_{j-1/2}) \right] \quad (2.2.4f)$$

$$u_{i,j}^{n+1} = U_{i,j}^{n+1} \left[1 + \tau \left(\alpha w_{i,j}^{n+1} - \beta \right) \left(1 + \frac{\tau}{2} \left(\alpha w_{i,j}^{n+1} - \beta \right) \right) \right] \quad (2.2.4g)$$

end

The flux numerical function $F(u_1, u_2, t, x, y)$, defined in (2.2.4d) involves the evaluation of the velocity function $\mathbf{v}$ at points $x_{i+1/2,j}$ and $x_{i,j+1/2}$. Due to the definition of $\mathbf{v}$ through a convolution of w and a kernel function η , this necessitates the numerical computation of $w^n * \eta$. Since the algorithm (2.2.4) is (at most) first order, we propose to use a standard quadrature formula on the same space mesh and compute

$$(w^n * \eta)(x_i, y_j) = \sum_{k,\ell} h^2 w_{k,\ell}^n \eta_{i-k,j-\ell}, \quad (2.2.5)$$

where $\eta_{i-k,j-\ell} = \eta(x_{i-k}, y_{j-\ell})$. Notice that algorithm (2.2.4) needs values of $\mathbf{v}$ on the edges of each mesh element, such that we proceed as follows: whenever we are dealing with cells with the same x -coordinate, respectively y -coordinate, we average the corresponding component of $\mathbf{v}$ in the x -coordinate, respectively y -coordinate, thus obtaining

$$\begin{aligned} v^n(x_{i+1/2}, y_j) &= \frac{1}{2} (v^n(x_i, y_j) + v^n(x_{i+1}, y_j)) = \frac{1}{2} \left[v((w^n * \eta)(x_i, y_j)) + v((w^n * \eta)(x_{i+1}, y_j)) \right] \\ v^n(x_i, y_{j+1/2}) &= \frac{1}{2} (v^n(x_i, y_j) + v^n(x_i, y_{j+1})) = \frac{1}{2} \left[v((w^n * \eta)(x_i, y_j)) + v((w^n * \eta)(x_i, y_{j+1})) \right] \end{aligned}$$

Remark 2.3.

- The values of $\eta_{i,j}$ can be computed in a preprocessing step and stored in a matrix. This reduces the computational cost for the evaluation of $\mathbf{v}$ to a matrix-vector multiplication at each point $x_{i,j}$.
- In principle, the above quadrature formula can also be used to evaluate $(w^n * \eta)(x_{i+1/2}, y_j)$ directly, using pointwise evaluations of η at $(x_{i+1/2-k}, y_{j-l})$. However, the double storage capacity for $\eta_{i+1/2,j}$ and $\eta_{i,j+1/2}$ will be needed for this strategy.
- If the kernel function η admits an explicit form of the antiderivative, it is possible to compute the convolution $w^n * \eta$ exactly for piecewise constant w^n , since we have

$$(w^n * \eta)(x_i, y_j) := \sum_{k,l} \int_{I_{k,l}} w_{k,l}^n \eta(x_i - x, y_j - y) dx dy = \sum_{k,l} w_{k,l}^n \int_{I_{k,l}} \eta(x_i - x, y_j - y) dx dy.$$

Thus, storing the values of $\int_{I_{k,l}} \eta(x_i - x, y_j - y) dx dy$ in a preprocessing step yields an exact evaluation of the convolution.

- Observe that the differentiation property of the convolution product still holds in the discrete case. Hence, if the function $\mathbf{v}$ satisfies $(\mathbf{v})$ in the continuous case, this assumption holds also for its discrete approximation, obtained using the discrete convolution product, computed through formula (2.2.5), substituting η by $\nabla \eta$.

Remark 2.4. All estimates of Section 2.3 as well as the convergence result of Section 2.4 can be shown analogously when (2.2.4e)–(2.2.4f) are replaced by

$$U_{i,j}^{n+1} = u_{i,j}^n - \lambda \left[F(u_{i+1,j}^n, u_{i,j}^n, (n+1)\tau, x_{i+1/2}, y_j) - F(u_{i,j}^n, u_{i-1,j}^n, (n+1)\tau, x_{i-1/2}, y_j) \right. \\ \left. F(u_{i,j+1}^n, u_{i,j}^n, (n+1)\tau, x_i, y_{j+1/2}) - F(u_{i,j}^n, u_{i,j-1}^n, (n+1)\tau, x_i, y_{j-1/2}) \right].$$

Remark 2.5. The main reason for the choice of a non standard CFL condition as (2.2.1) is the space–time dependent velocity field $\mathbf{v}$. To prove the positivity of u in Lemma 2.8 we observe that the space–time dependence of $\mathbf{v}$ introduces an additional constraint on λ . The choice of $1/4$ in the CFL condition and of $1/8$ in the definition of the Lax–Friedrichs flux are optimal in the sense that they provide the largest possible CFL condition that guarantees positivity of u . More details on this can be found in the proof of Lemma 2.8 in the next section.

2.3 Bounds on w and u

The proof of convergence of the above algorithm to the unique solution of (2.1.1) is based on an extension of Helly’s theorem (see [28, Theorem 1.7.3]). To apply this theorem, we have to prove the uniform boundedness of u and w in $\mathbf{L}^1$ as well as a uniform bound on the time–space total variation. The necessary estimates are collected in this section, starting with positivity estimates in Paragraph 2.3.1 and bounds on the $\mathbf{L}^1$ and $\mathbf{L}^\infty$ norms in Paragraph 2.3.2. Once these bounds are available, we are able to prove a bound on the total variation in space in Paragraph 2.3.3. Finally, the Lipschitz continuity in time of u , proven in Paragraph 2.3.4, guarantees enough regularity of the approximate solutions to pass to the convergence proof in Section 2.4.

In the sequel, we will make use of the following Lemma in several estimates.

Lemma 2.6. Let $a, b \in \mathbb{R}$, with $a, b > 0$. The following inequality holds

$$a \sum_{k=1}^n e^{k a b} \leq \frac{1}{b} e^{(n+1) a b}.$$

Proof. Recall that $a b + 1 \leq e^{a b}$. Hence,

$$a \sum_{k=1}^n e^{k a b} = a \frac{e^{a b} (e^{n a b} - 1)}{e^{a b} - 1} \leq \frac{1}{b} e^{(n+1) a b},$$

concluding the proof. □

Remark 2.7. In what follow we restrict ourselves to the case $\gamma > 0$, in order to apply Lemma 2.6 for $b = \gamma$. However, a modification of the estimate in Lemma 2.6 to cover also the case $\gamma = 0$ is possible, at the price of more complicated formulas in the estimates below.

2.3.1 Positivity of w and u

Lemma 2.8. Let assumptions **(f)**, **(v)** and **(0)** hold. Then the approximate solution constructed by Algorithm 2.2 is such that $w_{i,j}^n \geq 0$ and $u_{i,j}^n \geq 0$ for all i, j and n .

Proof. Consider w first, in particular, focus on the sequence $(W^{n,\ell})$. Suppose $W_{i,j}^{n,\ell} \geq 0$ for all i, j and define $S = S_{i,j}^n = \tau_p \left(\gamma - \delta u_{i,j}^n \right)$. By (2.2.4b) we have

$$W_{i,j}^{n,\ell+1} = \frac{1}{4} \left(W_{i+1,j}^{n,\ell} + W_{i-1,j}^{n,\ell} + W_{i,j+1}^{n,\ell} + W_{i,j-1}^{n,\ell} \right) \left(1 + S + \frac{S^2}{2} \right).$$

The parabola $(1 + S + S^2/2)$ assumes only positive values and, by the inductive hypothesis, we deduce that $W_{i,j}^{n,\ell+1} \geq 0$. By induction, we can thus conclude that $w_{i,j}^n \geq 0$ for all i, j and n .

Consider now u and recall $v_{i+1/2,j}^{n+1} := \mathbf{v} \left((n+1)\tau, x_{i+1/2}, y_j \right) \cdot \mathbf{n}_{i+1/2,j}$. By (2.2.4d) and (2.2.4e) we have

$$\begin{aligned} U_{i,j}^{n+1/2} &= \frac{1}{8} (u_{i+1,j}^n + 6u_{i,j}^n + u_{i-1,j}^n) - \lambda f(u_{i,j}^n) \left(v_{i+1/2,j}^{n+1} - v_{i-1/2,j}^{n+1} \right) \\ &\quad - \lambda \left[\frac{f(u_{i+1,j}^n) - f(u_{i,j}^n)}{2} v_{i+1/2,j}^{n+1} - \frac{f(u_{i-1,j}^n) - f(u_{i,j}^n)}{2} v_{i-1/2,j}^{n+1} \right] \\ &= u_{i+1,j}^n \left[\frac{1}{8} - \frac{\lambda}{2} \frac{f(u_{i+1,j}^n) - f(u_{i,j}^n)}{u_{i+1,j}^n - u_{i,j}^n} v_{i+1/2,j}^{n+1} \right] \\ &\quad + u_{i-1,j}^n \left[\frac{1}{8} + \frac{\lambda}{2} \frac{f(u_{i-1,j}^n) - f(u_{i,j}^n)}{u_{i-1,j}^n - u_{i,j}^n} v_{i-1/2,j}^{n+1} \right] \\ &\quad + u_{i,j}^n \left[\frac{3}{4} + \frac{\lambda}{2} v_{i+1/2,j}^{n+1} \left(\frac{f(u_{i+1,j}^n) - f(u_{i,j}^n)}{u_{i+1,j}^n - u_{i,j}^n} - 2 \frac{f(u_{i,j}^n)}{u_{i,j}^n} \right) \right. \\ &\quad \left. - \frac{\lambda}{2} v_{i-1/2,j}^{n+1} \left(\frac{f(u_{i-1,j}^n) - f(u_{i,j}^n)}{u_{i-1,j}^n - u_{i,j}^n} - 2 \frac{f(u_{i,j}^n)}{u_{i,j}^n} \right) \right]. \end{aligned}$$

Observe that the CFL condition (2.2.1) yields

$$\frac{1}{8} \pm \frac{\lambda}{2} \frac{f(u_{i+1,j}^n) - f(u_{i,j}^n)}{u_{i+1,j}^n - u_{i,j}^n} v_{i+1/2,j}^{n+1} \geq \frac{1}{8} - \frac{\lambda}{2} \|\partial_u f\|_{\mathbf{L}^\infty} \|v\|_{\mathbf{L}^\infty} > 0,$$

and for the remaining term

$$\begin{aligned} &\frac{3}{4} + \frac{\lambda}{2} v_{i+1/2,j}^{n+1} \left[\frac{f(u_{i+1,j}^n) - f(u_{i,j}^n)}{u_{i+1,j}^n - u_{i,j}^n} - 2 \frac{f(u_{i,j}^n)}{u_{i,j}^n} \right] - \frac{\lambda}{2} v_{i-1/2,j}^{n+1} \left[\frac{f(u_{i-1,j}^n) - f(u_{i,j}^n)}{u_{i-1,j}^n - u_{i,j}^n} - 2 \frac{f(u_{i,j}^n)}{u_{i,j}^n} \right] \\ &\geq \frac{3}{4} - 3\lambda \|v\|_{\mathbf{L}^\infty} \|\partial_u f\|_{\mathbf{L}^\infty} > 0. \end{aligned}$$

Hence, using also the inductive hypothesis, we have that $U_{i,j}^{n+1/2} \geq 0$ for all i and j .

Using (2.2.4f), we can repeat the same steps as above considering $U^{n+1/2}$ instead of u^n to conclude that $U_{i,j}^{n+1} \geq 0$ for all i and j .

Finally, defining $R = R_{i,j}^{n+1} = \tau \left(\alpha w_{i,j}^{n+1} - \beta \right)$ and using (2.2.4g), we obtain

$$u_{i,j}^{n+1} = U_{i,j}^{n+1} \left(1 + R + \frac{R^2}{2} \right).$$

Analogously to w , we can conclude that $u_{i,j}^{n+1} \geq 0$ for all i, j and n . $\square$

2.3.2 L^∞ and L^1 Bounds on w and u

Lemma 2.9. *Let assumptions $(\mathbf{f})$, $(\mathbf{v})$ and $(\mathbf{0})$ hold. Then for all n the approximate solution (u^n, w^n) constructed by Algorithm 2.2 satisfies*

$$\|w^n\|_{\mathbf{L}^\infty} \leq e^{n\tau\gamma} \|w^o\|_{\mathbf{L}^\infty} \quad (2.3.1)$$

$$\|u^n\|_{\mathbf{L}^\infty} \leq \|u^o\|_{\mathbf{L}^\infty} \exp\left((2\mathcal{K}_1 + \mathcal{K}_2) \frac{1}{\gamma} e^{(n+1)\tau\gamma}\right), \quad (2.3.2)$$

where $\mathcal{K}_1, \mathcal{K}_2$ are constants depending on $\alpha, K, \|w^o\|_{\mathbf{L}^\infty}, \|\partial_u f\|_{\mathbf{L}^\infty}$.

Proof. Consider w first. By Lemma 2.8, $u_{i,j}^n \geq 0$ for all i, j and n . By (2.2.4b) we have for $0 \leq l < m$

$$\begin{aligned} W_{i,j}^{n,l+1} &\leq e^{\tau_p(\gamma - \delta u_{i,j}^n)} \frac{1}{4} \left(W_{i+1,j}^{n,l} + W_{i-1,j}^{n,l} + W_{i,j+1}^{n,l} + W_{i,j-1}^{n,l} \right) \\ &\leq e^{\tau_p\gamma} \|W^{n,l}\|_{\mathbf{L}^\infty}. \end{aligned}$$

By induction over l in the sequence $W^{n,l}$ we obtain therefore

$$\|w^{n+1}\|_{\mathbf{L}^\infty} = \|W^{n,m}\|_{\mathbf{L}^\infty} \leq e^{m\tau_p\gamma} \|w^n\|_{\mathbf{L}^\infty} = e^{\tau\gamma} \|w^n\|_{\mathbf{L}^\infty}.$$

Finally, induction over n yields

$$\|w^n\|_{\mathbf{L}^\infty} \leq e^{n\tau\gamma} \|w^o\|_{\mathbf{L}^\infty}.$$

Pass now to u and recall that by $(\mathbf{v})$, we have

$$\|\nabla v^n\|_{\mathbf{L}^\infty} \leq K \|w^n\|_{\mathbf{L}^\infty}.$$

As in Lemma 2.8, by (2.2.4d) and (2.2.4e), simple computations lead to

$$\begin{aligned} |U_{i,j}^{n+1/2}| &= U_{i,j}^{n+1/2} \\ &= u_{i+1,j}^n \left[\frac{1}{8} - \frac{\lambda}{2} \frac{f(u_{i+1,j}^n) - f(u_{i,j}^n)}{u_{i+1,j}^n - u_{i,j}^n} v_{i+1/2,j}^{n+1} \right] + u_{i-1,j}^n \left[\frac{1}{8} + \frac{\lambda}{2} \frac{f(u_{i-1,j}^n) - f(u_{i,j}^n)}{u_{i-1,j}^n - u_{i,j}^n} v_{i-1/2,j}^{n+1} \right] \\ &\quad + u_{i,j}^n \left[\frac{3}{4} + \frac{\lambda}{2} \frac{f(u_{i+1,j}^n) - f(u_{i,j}^n)}{u_{i+1,j}^n - u_{i,j}^n} v_{i+1/2,j}^{n+1} - \frac{\lambda}{2} \frac{f(u_{i-1,j}^n) - f(u_{i,j}^n)}{u_{i-1,j}^n - u_{i,j}^n} v_{i-1/2,j}^{n+1} \right. \\ &\quad \left. - \lambda \frac{f(u_{i,j}^n)}{u_{i,j}^n} (v_{i+1/2,j}^{n+1} - v_{i-1/2,j}^{n+1}) \right] \\ &\leq \|u^n\|_{\mathbf{L}^\infty} \left[\frac{1}{8} - \frac{\lambda}{2} \frac{f(u_{i+1,j}^n) - f(u_{i,j}^n)}{u_{i+1,j}^n - u_{i,j}^n} v_{i+1/2,j}^{n+1} + \frac{1}{8} + \frac{\lambda}{2} \frac{f(u_{i-1,j}^n) - f(u_{i,j}^n)}{u_{i-1,j}^n - u_{i,j}^n} v_{i-1/2,j}^{n+1} \right. \\ &\quad + \frac{3}{4} + \frac{\lambda}{2} \frac{f(u_{i+1,j}^n) - f(u_{i,j}^n)}{u_{i+1,j}^n - u_{i,j}^n} v_{i+1/2,j}^{n+1} - \frac{\lambda}{2} \frac{f(u_{i-1,j}^n) - f(u_{i,j}^n)}{u_{i-1,j}^n - u_{i,j}^n} v_{i-1/2,j}^{n+1} \\ &\quad \left. - \lambda \frac{f(u_{i,j}^n)}{u_{i,j}^n} (v_{i+1/2,j}^{n+1} - v_{i-1/2,j}^{n+1}) \right] \end{aligned}$$

$$\begin{aligned}
&\leq \|u^n\|_{\mathbf{L}^\infty} \left(1 + \tau \|\partial_u f\|_{\mathbf{L}^\infty} \left\| \partial_x v^{n+1} \right\|_{\mathbf{L}^\infty} \right) \\
&\leq \|u^n\|_{\mathbf{L}^\infty} \exp \left(\tau K \|\partial_u f\|_{\mathbf{L}^\infty} \left\| w^{n+1} \right\|_{\mathbf{L}^\infty} \right).
\end{aligned}$$

The estimate on U^{n+1} can be obtained analogously using (2.2.4f) to get

$$\left\| U^{n+1} \right\|_{\mathbf{L}^\infty} \leq \left\| U^{n+1/2} \right\|_{\mathbf{L}^\infty} \exp \left(\tau K \|\partial_u f\|_{\mathbf{L}^\infty} \left\| w^{n+1} \right\|_{\mathbf{L}^\infty} \right).$$

Concerning the source term, integrated by (2.2.4g), one can easily see that

$$\left| u_{i,j}^{n+1} \right| \leq U_{i,j}^{n+1} \exp \left[\tau \left(\alpha w_{i,j}^{n+1} - \beta \right) \right] \leq \left\| U^{n+1} \right\|_{\mathbf{L}^\infty} \exp \left(\tau \alpha \left\| w^{n+1} \right\|_{\mathbf{L}^\infty} \right).$$

Collecting the above estimates and using (2.3.1), we conclude that

$$\left\| u^{n+1} \right\|_{\mathbf{L}^\infty} \leq \|u^n\|_{\mathbf{L}^\infty} \exp \left[\tau e^{(n+1)\tau\gamma} \|w^o\|_{\mathbf{L}^\infty} (2K\|\partial_u f\|_{\mathbf{L}^\infty} + \alpha) \right].$$

Iterating over n and applying Lemma 2.6 yields

$$\begin{aligned}
\|u^n\|_{\mathbf{L}^\infty} &\leq \|u^o\|_{\mathbf{L}^\infty} \exp \left[\tau \|w^o\|_{\mathbf{L}^\infty} (2K\|\partial_u f\|_{\mathbf{L}^\infty} + \alpha) \sum_{k=1}^n e^{k\tau\gamma} \right] \\
&\leq \|u^o\|_{\mathbf{L}^\infty} \exp \left[e^{(n+1)\tau\gamma} \frac{1}{\gamma} \|w^o\|_{\mathbf{L}^\infty} (2K\|\partial_u f\|_{\mathbf{L}^\infty} + \alpha) \right].
\end{aligned}$$

Denoting

$$\mathcal{K}_1 = K \|w^o\|_{\mathbf{L}^\infty} \|\partial_u f\|_{\mathbf{L}^\infty} \quad \mathcal{K}_2 = \alpha \|w^o\|_{\mathbf{L}^\infty} \quad (2.3.3)$$

completes the proof. $\square$

Positivity and uniform boundedness of the approximate solution allow now to prove the $\mathbf{L}^1$ bounds necessary for the application of [28, Theorem 1.7.3] in the convergence proof later on.

Lemma 2.10. *Let assumptions (f), (v) and (0) hold. Then for all n the approximate solution (u^n, w^n) constructed by Algorithm 2.2 satisfies*

$$\|w^n\|_{\mathbf{L}^1} \leq e^{n\tau\gamma} \|w^o\|_{\mathbf{L}^1} \quad (2.3.4)$$

$$\|u^n\|_{\mathbf{L}^1} \leq \|u^o\|_{\mathbf{L}^1} \exp \left(\mathcal{K}_2 \frac{1}{\gamma} e^{(n+1)\tau\gamma} \right), \quad (2.3.5)$$

where $\mathcal{K}_2$ is the constant defined in Lemma 2.9, depending on $\alpha, \|w^o\|_{\mathbf{L}^\infty}$.

Proof. Consider w first. By Lemma 2.8, $u_{i,j}^n \geq 0$ and $w_{i,j}^n \geq 0$ for all i, j and n . Let $W^{n,0} = w^n$ and $0 \leq l < m$. By (2.2.4b),

$$\left\| W^{n,l+1} \right\|_{\mathbf{L}^1} = \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} h^2 W_{i,j}^{n,l+1} \leq e^{\tau_p(\gamma - \delta u_{i,j}^n)} \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} h^2 W_{i,j}^{n,l} \leq e^{\tau_p \gamma} \left\| W^{n,l} \right\|_{\mathbf{L}^1}.$$

Induction over l yields

$$\left\| w^{n+1} \right\|_{\mathbf{L}^1} = \left\| W^{n,m} \right\|_{\mathbf{L}^1} \leq e^{m\tau_p\gamma} \|w^n\|_{\mathbf{L}^1} \leq e^{\tau\gamma} \|w^n\|_{\mathbf{L}^1}.$$

Induction over n now yields

$$\|w^n\|_{\mathbf{L}^1} \leq e^{n\tau\gamma} \|w^o\|_{\mathbf{L}^1}.$$

Pass now to u . By the conservation property of the Lax–Friedrichs scheme (2.2.4d)–(2.2.4f) we have

$$\left\| U^{n+1} \right\|_{\mathbf{L}^1} = \left\| U^{n+1/2} \right\|_{\mathbf{L}^1} = \|u^n\|_{\mathbf{L}^1}.$$

To include the source term in the $\mathbf{L}^1$ -estimate, we consider (2.2.4g) and obtain

$$\begin{aligned} \left\| u^{n+1} \right\|_{\mathbf{L}^1} &= \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} h^2 u_{i,j}^{n+1} \leq \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} h^2 e^{\tau(\alpha w_{i,j}^{n+1} - \beta)} U_{i,j}^{n+1} \\ &\leq e^{\tau\alpha} \|w^{n+1}\|_{\mathbf{L}^\infty} \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} h^2 U_{i,j}^{n+1} \\ &= e^{\tau\alpha} \|w^{n+1}\|_{\mathbf{L}^\infty} \|u^n\|_{\mathbf{L}^1}. \end{aligned}$$

Using (2.3.1) and (2.3.3), this yields

$$\begin{aligned} \left\| u^{n+1} \right\|_{\mathbf{L}^1} &\leq \|u^n\|_{\mathbf{L}^1} \exp \left(\tau\alpha \|w^o\|_{\mathbf{L}^\infty} e^{(n+1)\tau\gamma} \right) \\ &\leq \|u^o\|_{\mathbf{L}^1} \exp \left(\tau\mathcal{K}_2 \sum_{k=1}^{n+1} e^{k\tau\gamma} \right) \\ &\leq \|u^o\|_{\mathbf{L}^1} \exp \left(\mathcal{K}_2 \frac{1}{\gamma} e^{(n+2)\tau\gamma} \right), \end{aligned}$$

where we applied Lemma 2.6. This completes the proof. $\square$

2.3.3 TV Estimate

Lemma 2.11. *Let assumptions $(\mathbf{f})$, $(\mathbf{v})$ and $(\mathbf{0})$ hold and fix $0 < T < \infty$. Then, for all n such that $n\tau < T$, the approximate solution (u^n, w^n) constructed by Algorithm 2.2 satisfies*

$$\text{TV}(u^n) + \text{TV}(w^n) \leq e^{n\tau\mathcal{K}_3} \left(\text{TV}(u^o) + \text{TV}(w^o) + \frac{\mathcal{K}_4}{\mathcal{K}_3} e^{\tau\mathcal{K}_5} \right)$$

where the functions $\mathcal{K}_i$, $i = 3, \dots, 5$ depend on T , various norms of u^n , w^n and $\partial_u f$ as well as on all constants $\alpha, \beta, \gamma, \delta$ and K and the function C defined in $(\mathbf{v})$.

Proof. Consider w first. In particular, focus on $W^{n,l}$ defined in (2.2.4b). To obtain a bound on the total variation of w^{n+1} , we have to estimate

$$\text{TV}(w^{n+1}) = \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} h \left[\left| w_{i+1,j}^{n+1} - w_{i,j}^{n+1} \right| + \left| w_{i,j+1}^{n+1} - w_{i,j}^{n+1} \right| \right]. \quad (2.3.6)$$

Similarly as before, we define $S_{i,j}^n = \gamma - \delta u_{i,j}^n$ for the sake of simplicity. To obtain a bound for (2.3.6), we consider

$$\begin{aligned}
& \sum_{i,j} h \left| W_{i+1,j}^{n,l+1} - W_{i,j}^{n,l+1} \right| \\
& \leq \frac{1}{4} e^{\tau_p \gamma} \sum_{i,j} h \left(\left| W_{i+2,j}^{n,l} - W_{i+1,j}^{n,l} \right| + \left| W_{i,j}^{n,l} - W_{i-1,j}^{n,l} \right| + \left| W_{i+1,j+1}^{n,l} - W_{i,j+1}^{n,l} \right| + \left| W_{i+1,j-1}^{n,l} - W_{i,j-1}^{n,l} \right| \right) \\
& \quad + \sum_{i,j} h \left| W_{i,j}^{n,l} \right| \cdot \tau_p \left| S_{i+1,j}^n \left(1 + \frac{\tau_p}{2} S_{i+1,j}^n \right) - S_{i,j}^n \left(1 + \frac{\tau_p}{2} S_{i,j}^n \right) \right| \\
& \leq e^{\tau_p \gamma} \sum_{i,j} h \left| W_{i,j}^{n,l} - W_{i-1,j}^{n,l} \right| + \left\| W_{i,j}^{n,l} \right\|_{\mathbf{L}^\infty} \sum_{i,j} h \tau_p \left| S_{i+1,j}^n - S_{i,j}^n \right| \cdot \left| 1 + \tau_p \left(\gamma - \frac{\delta}{2} (u_{i+1,j}^n + u_{i,j}^n) \right) \right| \\
& \leq e^{\tau_p \gamma} \sum_{i,j} h \left| W_{i,j}^{n,l} - W_{i-1,j}^{n,l} \right| + \tau_p (1 + \tau_p (\gamma + \delta \|u^n\|_{\mathbf{L}^\infty})) \delta \left\| W_{i,j}^{n,l} \right\|_{\mathbf{L}^\infty} \cdot \sum_{i,j} h \left| u_{i+1,j}^n - u_{i,j}^n \right| \\
& \leq e^{\tau_p \gamma} \left(\sum_{i,j} h \left| W_{i,j}^{n,l} - W_{i-1,j}^{n,l} \right| + \tau_p e^{\delta \|u^n\|_{\mathbf{L}^\infty} \tau_p} \delta \left\| W_{i,j}^{n,l} \right\|_{\mathbf{L}^\infty} \sum_{i,j} h \left| u_{i+1,j}^n - u_{i,j}^n \right| \right).
\end{aligned}$$

An analogous estimate can be derived for $\sum_{i,j} h \left| W_{i,j+1}^{n,l+1} - W_{i,j}^{n,l+1} \right|$. Induction over l yields now

$$\text{TV}(w^{n+1}) \leq e^{\tau \gamma} \left(\text{TV}(w^n) + \tau e^{\tau_p \delta \|u^n\|_{\mathbf{L}^\infty}} \delta \|w^n\|_{\mathbf{L}^\infty} \text{TV}(u^n) \right). \quad (2.3.7)$$

Pass now to u . We need to estimate the following quantity:

$$\text{TV}(u^{n+1}) = h \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \left[\left| u_{i+1,j}^{n+1} - u_{i,j}^{n+1} \right| + \left| u_{i,j+1}^{n+1} - u_{i,j}^{n+1} \right| \right]. \quad (2.3.8)$$

Denoting $R_{i,j}^{n+1} = \alpha w_{i,j}^{n+1} - \beta$ for the sake of simplicity, we obtain

$$\begin{aligned}
\text{TV}(u^{n+1}) &= \sum_{i,j} h \left[\left| u_{i+1,j}^{n+1} - u_{i,j}^{n+1} \right| + \left| u_{i,j+1}^{n+1} - u_{i,j}^{n+1} \right| \right] \\
&\leq e^{\tau \alpha \|w^{n+1}\|_{\mathbf{L}^\infty}} \sum_{i,j} h \left[\left| U_{i+1,j}^{n+1} - U_{i,j}^{n+1} \right| + \left| U_{i,j+1}^{n+1} - U_{i,j}^{n+1} \right| \right] \\
&\quad + \left\| U^{n+1} \right\|_{\mathbf{L}^\infty} \tau \sum_{i,j} h \left| R_{i+1,j}^{n+1} \left(1 + \frac{\tau}{2} R_{i+1,j}^{n+1} \right) - R_{i,j}^{n+1} \left(1 + \frac{\tau}{2} R_{i,j}^{n+1} \right) \right| \\
&\quad + \left\| U^{n+1} \right\|_{\mathbf{L}^\infty} \tau \sum_{i,j} h \left| R_{i,j+1}^{n+1} \left(1 + \frac{\tau}{2} R_{i,j+1}^{n+1} \right) - R_{i,j}^{n+1} \left(1 + \frac{\tau}{2} R_{i,j}^{n+1} \right) \right| \\
&\leq e^{\tau \alpha \|w^{n+1}\|_{\mathbf{L}^\infty}} \left(\text{TV}(U^{n+1}) + \tau \alpha e^{\tau \beta} \left\| U^{n+1} \right\|_{\mathbf{L}^\infty} \text{TV}(w^{n+1}) \right).
\end{aligned}$$

To approximate $\text{TV}(U^{n+1})$, we have to estimate

$$\sum_{i,j} h \left(\left| U_{i+1,j}^{n+1} - U_{i,j}^{n+1} \right| + \left| U_{i,j+1}^{n+1} - U_{i,j}^{n+1} \right| \right).$$

It is a well-known fact that the standard Lax–Friedrichs scheme is TVD and thus it holds $\text{TV}(U^{n+1}) \leq \text{TV}(U^{n+1/2}) \leq \text{TV}(u^n)$. The situation here however is different, since the flux does not only depend on u , but also on t and x through the component $v(w)$. The conservation law itself does therefore not satisfy the TVD–property (see Chapter 1 and [25]) and we cannot expect the numerical scheme to be TVD. To estimate the increase in total variation due to the space–time dependent velocity field, we consider the term $\sum_{i,j} h |U_{i,j+1}^{n+1} - U_{i,j}^{n+1}|$. By (2.2.4f), we have

$$\begin{aligned} U_{i,j+1}^{n+1} - U_{i,j}^{n+1} &= U_{i,j+1}^{n+1/2} - U_{i,j}^{n+1/2} \\ &\quad - \lambda \left[F \left(U_{i,j+2}^{n+1/2}, U_{i,j+1}^{n+1/2}, (n+1)\tau, x_{i,j+3/2} \right) - F \left(U_{i,j+1}^{n+1/2}, U_{i,j}^{n+1/2}, (n+1)\tau, x_{i,j+1/2} \right) \right. \\ &\quad \left. - F \left(U_{i,j+1}^{n+1/2}, U_{i,j}^{n+1/2}, (n+1)\tau, x_{i,j+1/2} \right) + F \left(U_{i,j}^{n+1/2}, U_{i,j-1}^{n+1/2}, (n+1)\tau, x_{i,j-1/2} \right) \right]. \end{aligned}$$

Add and subtract $\lambda F(U_{i,j+1}^{n+1/2}, U_{i,j}^{n+1/2}, (n+1)\tau, x_{i,j+3/2}) + \lambda F(U_{i,j}^{n+1/2}, U_{i,j-1}^{n+1/2}, (n+1)\tau, x_{i,j+1/2})$, then rearrange to obtain:

$$U_{i,j+1}^{n+1} - U_{i,j}^{n+1} = \mathcal{A}_{i,j}^n - \mathcal{B}_{i,j}^n,$$

where

$$\begin{aligned} \mathcal{A}_{i,j}^n &= U_{i,j+1}^{n+1/2} - U_{i,j}^{n+1/2} \\ &\quad - \lambda \left[F \left(U_{i,j+2}^{n+1/2}, U_{i,j+1}^{n+1/2}, (n+1)\tau, x_{i,j+3/2} \right) - F \left(U_{i,j+1}^{n+1/2}, U_{i,j}^{n+1/2}, (n+1)\tau, x_{i,j+1/2} \right) \right. \\ &\quad \left. + F \left(U_{i,j}^{n+1/2}, U_{i,j-1}^{n+1/2}, (n+1)\tau, x_{i,j+1/2} \right) - F \left(U_{i,j+1}^{n+1/2}, U_{i,j}^{n+1/2}, (n+1)\tau, x_{i,j+3/2} \right) \right], \end{aligned} \tag{2.3.9}$$

$$\begin{aligned} \mathcal{B}_{i,j}^n &= \lambda \left[F \left(U_{i,j+1}^{n+1/2}, U_{i,j}^{n+1/2}, (n+1)\tau, x_{i,j+3/2} \right) - F \left(U_{i,j+1}^{n+1/2}, U_{i,j}^{n+1/2}, (n+1)\tau, x_{i,j+1/2} \right) \right. \\ &\quad \left. + F \left(U_{i,j}^{n+1/2}, U_{i,j-1}^{n+1/2}, (n+1)\tau, x_{i,j-1/2} \right) - F \left(U_{i,j}^{n+1/2}, U_{i,j-1}^{n+1/2}, (n+1)\tau, x_{i,j+1/2} \right) \right]. \end{aligned} \tag{2.3.10}$$

From now on we omit the superscripts, $n+1/2$ or $n+1$, to enhance readability. Consider first the term $\mathcal{A}_{i,j}^n$ and use (2.2.4d) to obtain

$$\begin{aligned} \mathcal{A}_{i,j}^n &= U_{i,j+1} - U_{i,j} - \lambda \left[\frac{f(U_{i,j+2}) + f(U_{i,j+1}) - f(U_{i,j+1}) - f(U_{i,j})}{2} v_{i,j+3/2} \right. \\ &\quad \left. - \frac{f(U_{i,j+1}) + f(U_{i,j}) - f(U_{i,j}) - f(U_{i,j-1})}{2} v_{i,j+1/2} \right] \\ &\quad + \frac{1}{8} ((U_{i,j+2} - U_{i,j+1}) - 2(U_{i,j+1} - U_{i,j}) + (U_{i,j} - U_{i,j-1})) \\ &= \frac{3}{4} (U_{i,j+1} - U_{i,j}) + (U_{i,j+2} - U_{i,j+1}) \left(\frac{1}{8} - \frac{\lambda}{2} \frac{f(U_{i,j+2}) - f(U_{i,j+1})}{U_{i,j+2} - U_{i,j+1}} v_{i,j+3/2} \right) \\ &\quad + (U_{i,j} - U_{i,j-1}) \left(\frac{1}{8} + \frac{\lambda}{2} \frac{f(U_{i,j}) - f(U_{i,j-1})}{U_{i,j} - U_{i,j-1}} v_{i,j+1/2} \right) \end{aligned}$$

$$-\frac{\lambda}{2} [f(U_{i,j+1}) - f(U_{i,j})] \left(v_{i,j+3/2} - v_{i,j+1/2} \right).$$

Observe that both coefficients of $U_{i,j+2} - U_{i,j+1}$ and $U_{i,j} - U_{i,j-1}$ are positive. Then, summing the modulus of $\mathcal{A}_{i,j}^n$ over $i, j \in \mathbb{Z}$, using also (v), (2.3.1) and (2.3.3), yields:

$$\begin{aligned} & \sum_{i,j} h \left| \mathcal{A}_{i,j}^n \right| \\ & \leq \sum_{i,j} h \left| U_{i,j+1} - U_{i,j} \right| \left(1 + \frac{\lambda}{2} \frac{f(U_{i,j+1}) - f(U_{i,j})}{U_{i,j+1} - U_{i,j}} \left(v_{i,j+1/2} - v_{i,j+3/2} \right) \right) \\ & \quad + \sum_{i,j} h \frac{\lambda}{2} \left| U_{i,j+1} - U_{i,j} \right| \left| \frac{f(U_{i,j+1}) - f(U_{i,j})}{U_{i,j+1} - U_{i,j}} \right| \left| v_{i,j+3/2} - v_{i,j+1/2} \right| \\ & \leq \sum_{i,j} h \left| U_{i,j+1}^{n+1/2} - U_{i,j}^{n+1/2} \right| \left(1 + \tau \|\partial_u f\|_{\mathbf{L}^\infty} \left\| \partial_y v^{n+1} \right\|_{\mathbf{L}^\infty} \right) \\ & \leq \sum_{i,j} h \left| U_{i,j+1}^{n+1/2} - U_{i,j}^{n+1/2} \right| \left(1 + \tau K \|\partial_u f\|_{\mathbf{L}^\infty} \left\| w^{n+1} \right\|_{\mathbf{L}^\infty} \right). \end{aligned} \quad (2.3.11)$$

Pass now to $\mathcal{B}_{i,j}^n$. We continue omitting the superscripts.

$$\begin{aligned} \mathcal{B}_{i,j}^n &= \lambda \left[\frac{f(U_{i,j}) + f(U_{i,j+1})}{2} v_{i,j+3/2}^{n+1} - \frac{f(U_{i,j}) + f(U_{i,j+1})}{2} v_{i,j+1/2}^{n+1} \right. \\ & \quad \left. + \frac{f(U_{i,j-1}) + f(U_{i,j})}{2} v_{i,j-1/2}^{n+1} - \frac{f(U_{i,j-1}) + f(U_{i,j})}{2} v_{i,j+1/2}^{n+1} \right] \\ &= \frac{\lambda}{2} \left[f(U_{i,j}) \left(v_{i,j+3/2}^{n+1} - 2v_{i,j+1/2}^{n+1} + v_{i,j-1/2}^{n+1} \right) \right. \\ & \quad \left. + (f(U_{i,j+1}) - f(U_{i,j-1})) \left(v_{i,j+3/2}^{n+1} - v_{i,j+1/2}^{n+1} \right) + f(U_{i,j-1}) \left(v_{i,j+3/2}^{n+1} - 2v_{i,j+1/2}^{n+1} + v_{i,j-1/2}^{n+1} \right) \right]. \end{aligned}$$

Since $v = v_2$ is a smooth function and recalling that $v_{i,j+1/2}^{n+1} = \mathbf{v}((n+1)\tau, x_i, y_{j+1/2}) \cdot \mathbf{n}_{i,j+1/2}$ is the y -component of $\mathbf{v}$ at time $(n+1)\tau$ and point $x_{i,j+1/2}$, we obtain

$$v_{i,j+3/2}^{n+1} - 2v_{i,j+1/2}^{n+1} + v_{i,j-1/2}^{n+1} \leq h^2 \left\| \partial_{yy} v^{n+1} \right\|_{\mathbf{L}^\infty}.$$

Then, using also (v),

$$\frac{\lambda}{2} f \left(U_{i,j}^{n+1/2} \right) \left(v_{i,j+3/2}^{n+1} - 2v_{i,j+1/2}^{n+1} + v_{i,j-1/2}^{n+1} \right) \leq \frac{\lambda}{2} h^2 \|\partial_u f\|_{\mathbf{L}^\infty} \left| U_{i,j}^{n+1/2} \right| C \left(\left\| w^{n+1} \right\|_{\mathbf{L}^\infty} \right), \quad (2.3.12)$$

and similarly for the term with $f(U_{i,j-1})$, while the remaining term can be easily estimated as follows

$$\frac{\lambda}{2} \left(f(U_{i,j+1}^{n+1/2}) - f(U_{i,j-1}^{n+1/2}) \right) \left(v_{i,j+3/2}^{n+1} - v_{i,j+1/2}^{n+1} \right) \leq \frac{\lambda}{2} \|\partial_u f\|_{\mathbf{L}^\infty} \left| U_{i,j+1}^{n+1/2} - U_{i,j-1}^{n+1/2} \right| h \left\| \partial_y v^{n+1} \right\|_{\mathbf{L}^\infty}. \quad (2.3.13)$$

Hence, multiplying by h and summing over i and j , (2.3.12) and (2.3.13) yield

$$\sum_{i,j} h \left| \mathcal{B}_{i,j}^n \right| \leq \tau \|\partial_u f\|_{\mathbf{L}^\infty} \left(K \left\| w^{n+1} \right\|_{\mathbf{L}^\infty} \sum_{i,j} h \left| U_{i,j+1}^{n+1/2} - U_{i,j}^{n+1/2} \right| + C \left(\left\| w^{n+1} \right\|_{\mathbf{L}^\infty} \right) \left\| u^n \right\|_{\mathbf{L}^1} \right). \quad (2.3.14)$$

By (2.3.11) and (2.3.14) we have

$$\begin{aligned} \sum_{i,j} h \left| U_{i,j+1}^{n+1} - U_{i,j}^{n+1} \right| &\leq \sum_{i,j} h \left| U_{i,j+1}^{n+1/2} - U_{i,j}^{n+1/2} \right| \left(1 + 2 \tau K \|\partial_u f\|_{\mathbf{L}^\infty} \|w^{n+1}\|_{\mathbf{L}^\infty} \right) \\ &\quad + \tau \|\partial_u f\|_{\mathbf{L}^\infty} C \left(\|w^{n+1}\|_{\mathbf{L}^\infty} \right) \|u^n\|_{\mathbf{L}^1}. \end{aligned} \quad (2.3.15)$$

In a similar way we obtain

$$\begin{aligned} \sum_{i,j} h \left| U_{i+1,j}^{n+1} - U_{i,j}^{n+1} \right| &\leq \sum_{i,j} h \left| U_{i+1,j}^{n+1/2} - U_{i,j}^{n+1/2} \right| \left(1 + \tau K \|\partial_u f\|_{\mathbf{L}^\infty} \|w^{n+1}\|_{\mathbf{L}^\infty} \right) \\ &\quad + \tau K \|\partial_u f\|_{\mathbf{L}^\infty} \|w^{n+1}\|_{\mathbf{L}^\infty} \sum_{i,j} h \left| U_{i,j+1}^{n+1/2} - U_{i,j}^{n+1/2} \right| \\ &\quad + 2 \tau \|\partial_u f\|_{\mathbf{L}^\infty} C \left(\|w^{n+1}\|_{\mathbf{L}^\infty} \right) \|u^n\|_{\mathbf{L}^1}. \end{aligned} \quad (2.3.16)$$

By (2.3.15) and (2.3.16) we have therefore

$$\begin{aligned} \mathrm{TV}(U^{n+1}) &\leq \left(1 + 3 \tau K \|\partial_u f\|_{\mathbf{L}^\infty} \|w^{n+1}\|_{\mathbf{L}^\infty} \right) \mathrm{TV}(U^{n+1/2}) \\ &\quad + 3 \tau \|\partial_u f\|_{\mathbf{L}^\infty} C \left(\|w^{n+1}\|_{\mathbf{L}^\infty} \right) \|u^n\|_{\mathbf{L}^1}. \end{aligned} \quad (2.3.17)$$

Analogously to the estimate (2.3.17) for $\mathrm{TV}(U^{n+1})$, we obtain

$$\begin{aligned} \mathrm{TV}(U^{n+1/2}) &\leq \left(1 + 3 \tau K \|\partial_u f\|_{\mathbf{L}^\infty} \|w^{n+1}\|_{\mathbf{L}^\infty} \right) \mathrm{TV}(u^n) \\ &\quad + 3 \tau \|\partial_u f\|_{\mathbf{L}^\infty} C \left(\|w^{n+1}\|_{\mathbf{L}^\infty} \right) \|u^n\|_{\mathbf{L}^1}. \end{aligned} \quad (2.3.18)$$

Then,

$$\begin{aligned} \mathrm{TV}(u^{n+1}) &\leq e^{\tau \alpha} \|w^{n+1}\|_{\mathbf{L}^\infty} \left\{ \exp \left(6 \tau K \|\partial_u f\|_{\mathbf{L}^\infty} \|w^{n+1}\|_{\mathbf{L}^\infty} \right) \mathrm{TV}(u^n) \right. \\ &\quad + \tau \alpha e^{\tau \beta} \|U^{n+1}\|_{\mathbf{L}^\infty} \mathrm{TV}(w^{n+1}) \\ &\quad \left. + 3 \tau \|\partial_u f\|_{\mathbf{L}^\infty} C \left(\|w^{n+1}\|_{\mathbf{L}^\infty} \right) \|u^n\|_{\mathbf{L}^1} \left[1 + \exp \left(3 \tau K \|\partial_u f\|_{\mathbf{L}^\infty} \|w^{n+1}\|_{\mathbf{L}^\infty} \right) \right] \right\}. \end{aligned} \quad (2.3.19)$$

Collecting the estimates (2.3.19) and (2.3.7) of $\mathrm{TV}(u^{n+1})$ and $\mathrm{TV}(w^{n+1})$, we obtain now

$$\begin{aligned} &\mathrm{TV}(u^{n+1}) + \mathrm{TV}(w^{n+1}) \\ &\leq \left[e^{\tau \alpha} \|w^{n+1}\|_{\mathbf{L}^\infty} \exp \left(6 \tau K \|\partial_u f\|_{\mathbf{L}^\infty} \|w^{n+1}\|_{\mathbf{L}^\infty} \right) \right. \\ &\quad \left. + \left(1 + \tau \alpha e^{\tau \beta} e^{\tau \alpha} \|w^{n+1}\|_{\mathbf{L}^\infty} \|U^{n+1}\|_{\mathbf{L}^\infty} \right) \tau e^{\tau \gamma} e^{\tau \beta} \delta \|u^n\|_{\mathbf{L}^\infty} \delta \|w^n\|_{\mathbf{L}^\infty} \right] \mathrm{TV}(u^n) \end{aligned}$$

$$\begin{aligned}
& + e^{\tau \gamma} \left(1 + \tau \alpha e^{\tau \beta} e^{\tau \alpha} \|w^{n+1}\|_{\mathbf{L}^\infty} \|U^{n+1}\|_{\mathbf{L}^\infty} \right) \text{TV}(w^n) \\
& + 3 \tau e^{\tau \alpha} \|w^{n+1}\|_{\mathbf{L}^\infty} \|\partial_u f\|_{\mathbf{L}^\infty} C \left(\|w^{n+1}\|_{\mathbf{L}^\infty} \right) \|u^n\|_{\mathbf{L}^1} \left[1 + \exp \left(3 \tau K \|\partial_u f\|_{\mathbf{L}^\infty} \|w^{n+1}\|_{\mathbf{L}^\infty} \right) \right] \\
& \leq e^{\tau K_1} \text{TV}(u^n) + e^{\tau K_2} \text{TV}(w^n) + \tau K_3 e^{\tau K_4},
\end{aligned}$$

where K_l , $l = 1, \dots, 4$ are bounded functions depending on various norm of u^n , w^{n+1} and $\partial_u f$ as well as on all constants $\alpha, \beta, \gamma, \delta$ and K , defined in (v). Defining $K_3 = \max\{K_1, K_2\}$, $K_4 := K_3$ and $K_5 := K_4$ and using induction over n yields now

$$\text{TV}(u^n) + \text{TV}(w^n) \leq e^{n \tau K_3} \left[\text{TV}(u^o) + \text{TV}(w^o) + \frac{K_4}{K_3} e^{\tau K_5} \right].$$

This completes the proof. $\square$

2.3.4 Lipschitz Continuity in Time

Lemma 2.12. *Let assumptions (f), (v) and (0) hold. Then for all n the approximate solution (u^n, w^n) constructed by Algorithm 2.2 is such that, for any $n_1, n_2 \in \mathbb{N}$ with $n_1 \tau \leq T$ and $n_2 \tau \leq T$,*

$$\|u^{n_1} - u^{n_2}\|_{\mathbf{L}^1} \leq |n_1 - n_2| \tau K_6(T, \tau),$$

where the function $K_6(T, \tau)$ is uniformly bounded for all $n \leq \max\{n_1, n_2\}$ and depends on $\alpha, \beta, \gamma, \delta, K$, on various norms of $u, w, \partial_u f$, on the total variation of the initial datum and on the map C defined in (v).

Proof. Due to Lemma 2.9, u^n is uniformly bounded by some constant depending on T . Assumptions (f) and (v) guarantee therefore the Lipschitz continuity of the numerical flux function F defined in (2.2.4d). Using (2.2.4e), (2.2.4f) and (2.3.18), we can thus conclude

$$\begin{aligned}
& \|U^{n+1} - u^n\|_{\mathbf{L}^1} \\
& \leq \sum_{i,j} h^2 \left(\left| U_{i,j}^{n+1} - U_{i,j}^{n+1/2} \right| + \left| U_{i,j}^{n+1/2} - u_{i,j}^n \right| \right) \\
& \leq \tau \sum_{i,j} h \left[\left| F \left(U_{i,j+1}^{n+1/2}, U_{i,j}^{n+1/2}, (n+1) \tau, x_{i,j+1/2} \right) - F \left(U_{i,j}^{n+1/2}, U_{i,j-1}^{n+1/2}, (n+1) \tau, x_{i,j-1/2} \right) \right| \right. \\
& \quad \left. + \left| F \left(u_{i+1,j}^n, u_{i,j}^n, (n+1) \tau, x_{i+1/2,j} \right) - F \left(u_{i,j}^n, u_{i-1,j}^n, (n+1) \tau, x_{i-1/2,j} \right) \right| \right] \\
& \leq \tau \cdot 2L \sum_{i,j} h \left(\left| U_{i,j+1}^{n+1/2} - U_{i,j}^{n+1/2} \right| + \left| u_{i+1,j}^n - u_{i,j}^n \right| \right. \\
& \quad \left. + \left| v_{i+1/2,j}^{n+1} - v_{i-1/2,j}^{n+1} \right| + \left| v_{i,j+1/2}^{n+1} - v_{i,j-1/2}^{n+1} \right| \right) \\
& \leq \tau \cdot 2L \left(\left(2 + 3 \tau K \|\partial_u f\|_{\mathbf{L}^\infty} \|w^{n+1}\|_{\mathbf{L}^\infty} \right) \text{TV}(u^n) + \|\nabla v^{n+1}\|_{\mathbf{L}^1} \right)
\end{aligned}$$

$$+ 3 \tau \|\partial_u f\|_{\mathbf{L}^\infty} C \left(\|w^{n+1}\|_{\mathbf{L}^\infty} \right) \|u^n\|_{\mathbf{L}^1} \Big),$$

where L denotes the Lipschitz constant of F . Including the source term and defining T such that $\max\{n_1, n_2\} \tau \leq T < \infty$, we obtain by (2.2.4g)

$$\begin{aligned} \|u^{n+1} - u^n\|_{\mathbf{L}^1} &\leq \sum_{i,j} h^2 \left| U_{i,j}^{n+1} - u_{i,j}^n \right| + \tau \sum_{i,j} h^2 \left| U_{i,j}^{n+1} \right| \left| \left(\alpha w_{i,j}^{n+1} - \beta \right) \left(1 + \frac{\tau}{2} \left(\alpha w_{i,j}^{n+1} - \beta \right) \right) \right| \\ &\leq \|U^{n+1} - u^n\|_{\mathbf{L}^1} + \tau \alpha \|U^{n+1}\|_{\mathbf{L}^1} \|w^{n+1}\|_{\mathbf{L}^\infty} e^{\tau \alpha \|w^{n+1}\|_{\mathbf{L}^\infty}} \\ &\leq \tau \mathcal{K}_6(T, \tau), \end{aligned}$$

where $\mathcal{K}_6$ is uniformly bounded for all $n \leq \max\{n_1, n_2\}$ and all finite τ . $\square$

Remark 2.13. Using more refined estimates, the $\mathbf{L}^1$ bound for ∇v , necessary in the proof above, can be substituted by the $\mathbf{L}^\infty$ bound on ∇v widely used in the proofs of Lemmas 2.8–2.11. This allows to skip the assumption of ∇v being bounded in $\mathbf{L}^1$ in **(v)**.

2.4 Convergence

For each mesh width h , we define $N_\tau := \lfloor T/\tau \rfloor$ and

$$u_h = \sum_{n=0}^{N_\tau} \sum_{i,j} u_{i,j}^n \chi_{i,j}^n, \quad w_h = \sum_{n=0}^{N_\tau} \sum_{i,j} w_{i,j}^n \chi_{i,j}^n, \quad (2.4.1)$$

where $\chi_{i,j}^n$ is the characteristic function of $I_{i,j} \times [n\tau, (n+1)\tau]$, respectively $I_{i,j} \times [N_\tau\tau, T]$ for the last time step, with $I_{i,j}$ defined as in (2.2.3).

Theorem 2.14. Let assumptions **(f)**, **(v)** and **(0)** hold and fix $0 \leq T < \infty$. Let h_ℓ be a sequence of grid sizes such that $\lim_{\ell \rightarrow \infty} h_\ell = 0$ and fix $\lambda > 0$ such that the sequence $\tau_\ell := \lambda h_\ell$ fulfils (2.2.1) for all ℓ . Let u_{h_ℓ} and w_{h_ℓ} be given as in (2.4.1). Then the sequence (u_{h_ℓ}, w_{h_ℓ}) converges to the unique weak solution (u, w) of (2.1.1). More precisely, (u_{h_ℓ}) converges in $\mathbf{L}_{loc}^1$, while (w_{h_ℓ}) converges weakly* in $\mathbf{L}^\infty$.

Proof. Thanks to Lemma 2.9

$$\begin{aligned} \|w_h\|_{\mathbf{L}^\infty([0,T] \times \mathbb{R}^2; \mathbb{R})} &\leq e^{T\gamma} \|w^o\|_{\mathbf{L}^\infty(\mathbb{R}^2; \mathbb{R})} \\ \|u_h\|_{\mathbf{L}^\infty([0,T] \times \mathbb{R}^2; \mathbb{R})} &\leq \|u^o\|_{\mathbf{L}^\infty(\mathbb{R}^2; \mathbb{R})} \exp \left((2\mathcal{K}_1 + \mathcal{K}_2) \frac{1}{\gamma} e^{T\gamma} \right), \end{aligned}$$

so that the sequence (u_h, w_h) is bounded in $\mathbf{L}^\infty([0, T] \times \mathbb{R}^2; \mathbb{R}^2)$. This implies the existence of a subsequence (u_{h_k}, w_{h_k}) that converges weakly* in $\mathbf{L}^\infty([0, T] \times \mathbb{R}^2; \mathbb{R}^2)$ to (u, w) . Thanks to Lemma 2.10, (u_{h_k}) is also uniformly bounded in $\mathbf{L}^1([0, T] \times \mathbb{R}^2; \mathbb{R})$.

Furthermore, Lemma 2.11 and Lemma 2.12 yield a uniform bound for the space–time total variation of u_{h_k} , defined by

$$\mathrm{TV}_T(u_{h_k}) = \sum_{n=0}^{N_\tau} \left[\tau \, \mathrm{TV}(u_{h_k}^n) + \|u_{h_k}^{n+1} - u_{h_k}^n\|_{\mathbf{L}^1(\mathbb{R}^2; \mathbb{R})} \right]$$

We can thus apply [28, Theorem 1.7.3] and deduce the existence of $\bar{u} \in \mathbf{BV}_{loc}([0, T] \times \mathbb{R}^2; \mathbb{R})$ and a subsequence of (u_{h_k}) (still denoted by (u_{h_k})) such that

$$u_{h_k} \rightarrow \bar{u} \quad \text{in} \quad \mathbf{L}_{loc}^1([0, T] \times \mathbb{R}^2; \mathbb{R}), \quad (2.4.2)$$

$$u_{h_k}(t, x, y) \rightarrow \bar{u}(t, x, y) \quad \text{for a. e. } (t, x, y) \in [0, T] \times \mathbb{R}^2. \quad (2.4.3)$$

Due to the uniqueness of the limit $\bar{u}$ in $\mathbf{L}^1$, shown in Chapter 1 and [25] (see also Remark 2.1), we can conclude the convergence of the whole sequence (u_{h_k}) to $\bar{u}$.

From (2.4.3), it follows easily that u_{h_k} converges to $\bar{u}$ also in $\mathbf{L}^\infty([0, T] \times \mathbb{R}^2; \mathbb{R})$. Since strong convergence implies weak* convergence, we obtain that $u_{h_k} \xrightarrow{*} \bar{u}$ in $\mathbf{L}^\infty([0, T] \times \mathbb{R}^2; \mathbb{R})$. Due to the uniqueness of the weak* limit, we have that $u = \bar{u}$.

By (f), the continuity of the function f implies now that

$$f(u_{h_k}) \rightarrow f(u). \quad (2.4.4)$$

Note that Lemma 2.9 yields also

$$\|w_h(t, \cdot, \cdot)\|_{\mathbf{L}^\infty(\mathbb{R}^2; \mathbb{R})} \leq e^{T\gamma} \|w^0\|_{\mathbf{L}^\infty(\mathbb{R}^2; \mathbb{R})} \quad \text{for a. e. } t \in [0, T].$$

As above we can thus find a subsequence that converges weakly* in $\mathbf{L}^\infty(\mathbb{R}^2; \mathbb{R})$ for a. e. $t \in [0, T]$ and due to the uniqueness of the weak* limit, we have

$$w_{h_k}(t, \cdot, \cdot) \xrightarrow{*} w(t, \cdot, \cdot).$$

Recalling that $\eta \in \mathbf{L}^1(\mathbb{R}^2; \mathbb{R})$, it is now easy to prove that $(w_{h_k} * \eta)(t, \cdot, \cdot)$ converges (strongly) to $(w * \eta)(t, \cdot, \cdot)$ in $\mathbf{L}^1(\mathbb{R}^2; \mathbb{R})$ for a. e. $t \in [0, T]$. By (v), and in particular thanks to the fact that the Lipschitz constant of v is bounded, we obtain

$$v(w_{h_k} * \eta) \rightarrow v(w * \eta) \quad \text{in } \mathbf{L}^1(\mathbb{R}^2; \mathbb{R}) \quad \text{for a. e. } t \in [0, T]. \quad (2.4.5)$$

To prove that (u, w) are weak solutions of (2.1.1), we choose test functions $\psi \in \mathbf{C}_c^1([0, T], \mathbf{C}_c^2(\mathbb{R}^2; \mathbb{R}))$ and $\varphi \in \mathbf{C}_c^1([0, T] \times \mathbb{R}^2; \mathbb{R})$. Define now $\psi_{i,j}^{n,l} := \psi(t^{n,l}, x_{i,j})$, where $t^{n,l} = n\tau + l\tau_p$, and

$$\begin{aligned} \psi_h &= \sum_{n=0}^{N-1} \sum_{i,j} \sum_{l=0}^{m-1} \psi_{i,j}^{n,l} \chi_{i,j}^{n,l} \\ \delta_t \psi_h &= \sum_{n=0}^{N-1} \sum_{i,j} \sum_{l=0}^{m-1} \frac{\psi_{i,j}^{n,l} - \psi_{i,j}^{n,l-1}}{\tau_p} \chi_{i,j}^{n,l} \\ \Delta_h \psi_h &= \sum_{n=0}^{N-1} \sum_{i,j} \sum_{l=0}^{m-1} \frac{1}{h^2} \left(\psi_{i+1,j}^{n,l} + \psi_{i-1,j}^{n,l} + \psi_{i,j+1}^{n,l} + \psi_{i,j-1}^{n,l} - 4\psi_{i,j}^{n,l} \right) \chi_{i,j}^{n,l}. \end{aligned}$$

Here, $\chi_{i,j}^{n,l}$ is the characteristic function of $I_{i,j} \times [t^{n,l}, t^{n,l+1}]$, with $I_{i,j}$ defined as in (2.2.3). Note that $\delta_t \psi_h$ and $\Delta_h \psi_h$ are discrete versions of time derivative and Laplace operator. Due to the definition of ψ_h and its discrete derivatives, we have strong convergence in $\mathbf{L}^\infty([0, T] \times \mathbb{R}^2; \mathbb{R})$ for $\psi_h \rightarrow \psi$ as well as for the derivatives $\delta_t \psi_h \rightarrow \partial_t \psi$ and $\Delta_h \psi_h \rightarrow \Delta \psi$ as $h \rightarrow 0$.

Multiply (2.2.4b) by $h^2 \psi_{i,j}^{n,l}$ and sum over n, i, j and l to obtain

$$0 = \tau_p h^2 \sum_{n=0}^{N-1} \sum_{i,j} \sum_{l=0}^{m-1} W_{i,j}^{n,l} \left(\frac{\psi_{i,j}^{n,l} - \psi_{i,j}^{n,l-1}}{\tau_p} + \mu \frac{\psi_{i+1,j}^{n,l} + \psi_{i-1,j}^{n,l} + \psi_{i,j+1}^{n,l} + \psi_{i,j-1}^{n,l} - 4\psi_{i,j}^{n,l}}{h^2} \right)$$

$$+ \tau_p h^2 \sum_{n=0}^{N-1} \sum_{i,j} \sum_{l=0}^{m-1} \psi_{i,j}^{n,l} (\gamma - \delta u_{i,j}^n) \left[1 + \frac{\tau_p}{2} (\gamma - \delta u_{i,j}^n) \right] \frac{W_{i+1,j}^{n,l} + W_{i-1,j}^{n,l} + W_{i,j+1}^{n,l} + W_{i,j-1}^{n,l}}{4}.$$

Using the above convergence results, we can conclude

$$\int_0^T \int_{\mathbb{R}^2} w \partial_t \psi + \mu w \Delta \psi + w(\gamma - \delta u) \psi \, dx \, dy \, dt = 0.$$

Analogously as above we define

$$\begin{aligned} \varphi_h &= \sum_{n=0}^{N-1} \sum_{i,j} \varphi_{i,j}^n \chi_{i,j}^n & \delta_t \varphi_h &= \sum_{n=0}^{N-1} \sum_{i,j} \frac{\varphi_{i,j}^n - \varphi_{i,j}^{n-1}}{\tau} \chi_{i,j}^n \\ \delta_x^+ \varphi_h &= \sum_{n=0}^{N-1} \sum_{i,j} \frac{\varphi_{i+1,j}^n - \varphi_{i,j}^n}{h} \chi_{i,j}^n & \delta_x^- \varphi_h &= \sum_{n=0}^{N-1} \sum_{i,j} \frac{\varphi_{i,j}^n - \varphi_{i-1,j}^n}{h} \chi_{i,j}^n \\ \delta_y^+ \varphi_h &= \sum_{n=0}^{N-1} \sum_{i,j} \frac{\varphi_{i,j+1}^n - \varphi_{i,j}^n}{h} \chi_{i,j}^n & \delta_y^- \varphi_h &= \sum_{n=0}^{N-1} \sum_{i,j} \frac{\varphi_{i,j}^n - \varphi_{i,j-1}^n}{h} \chi_{i,j}^n \end{aligned}$$

and recall that we have $\varphi_h \rightarrow \varphi$ and $\delta_\ell^\pm \varphi_h \rightarrow \partial_\ell \varphi$ in $\mathbf{L}^\infty([0, T] \times \mathbb{R}^2; \mathbb{R})$ for $h \rightarrow 0$ and $\ell = t, x, y$. Multiplying (2.2.4e)–(2.2.4g) by $h^2 \varphi_{i,j}^n$ and summing over all n, i and j we obtain

$$\begin{aligned} 0 &= \tau h^2 \sum_{n=0}^{N-1} \sum_{i,j} u_{i,j}^n \frac{\varphi_{i,j}^n - \varphi_{i,j}^{n-1}}{\tau} + \tau h^2 \sum_{n=0}^{N-1} \sum_{i,j} (\alpha w_{i,j}^n - \beta) \varphi_{i,j}^n U_{i,j}^{n+1} \\ &\quad + \tau h^2 \sum_{n=0}^{N-1} \sum_{i,j} \frac{1}{2} f(u_{i,j}^n) \left(v_{i-1/2,j}^{n+1} \frac{\varphi_{i,j}^n - \varphi_{i-1,j}^n}{h} + v_{i+1/2,j}^{n+1} \frac{\varphi_{i+1,j}^n - \varphi_{i,j}^n}{h} \right) \\ &\quad + \tau h^2 \sum_{n=0}^{N-1} \sum_{i,j} \frac{1}{2} f(U_{i,j}^{n+1/2}) \left(v_{i,j-1/2}^{n+1} \frac{\varphi_{i,j}^n - \varphi_{i,j-1}^n}{h} + v_{i,j+1/2}^{n+1} \frac{\varphi_{i,j+1}^n - \varphi_{i,j}^n}{h} \right) \\ &\quad + h^2 \sum_{n=0}^{N-1} \sum_{i,j} \frac{h^2}{4} \left(u_{i,j}^n \frac{\varphi_{i-1,j}^n - 2\varphi_{i,j}^n + \varphi_{i+1,j}^n}{h^2} + U_{i,j}^{n+1/2} \frac{\varphi_{i,j-1}^n - 2\varphi_{i,j}^n + \varphi_{i,j+1}^n}{h^2} \right) \\ &\quad + \tau h^2 \sum_{n=0}^{N-1} \sum_{i,j} \frac{\tau}{2} (\alpha w_{i,j}^n - \beta)^2 \varphi_{i,j}^n U_{i,j}^{n+1}. \end{aligned}$$

Recall that w_h is uniformly bounded in $\mathbf{L}^\infty$ for all h . This directly implies that also w_h^2 is uniformly bounded in $\mathbf{L}^\infty$ and thus converges weakly* to some function $g \in \mathbf{L}^\infty([0, T] \times \mathbb{R}^2; \mathbb{R})$. Due to the smoothness of v and the convergence of u_h, w_h, w_h^2 and φ_h , we can thus deduce that the limit functions u, w and φ fulfil

$$\int_0^T \int_{\mathbb{R}^2} u \partial_t \varphi + f(u) v \cdot \nabla \varphi + (\alpha w - \beta) \varphi \, dx \, dy \, dt = 0.$$

We proved that (u, w) is a weak solution to (2.1.1). Since by Chapter 1, see also [25], and Remark 2.1 we know that the weak solution to (2.1.1) is unique, (u, w) is the unique weak solution. $\square$

2.5 Numerical Examples

To conclude the Chapter, we present some numerical examples that show on one hand the convergence of the scheme and on the other hand some qualitative properties of the system (2.1.1). In all examples, we make the following choice for the vector field v :

$$v(w) = \kappa \frac{\nabla(w * \eta)}{\sqrt{1 + \|\nabla(w * \eta)\|^2}}, \quad (2.5.1)$$

where the compactly supported kernel function η is chosen as follows

$$\eta(x, y) = \hat{\eta} \left(\ell^2 - \|(x, y)\|^2 \right)^3 \chi_{B(0, \ell)} \quad \text{with } \hat{\eta} \in \mathbb{R}^+ \text{ such that } \iint_{\mathbb{R}^2} \eta(x, y) \, dx \, dy = 1. \quad (2.5.2)$$

The positive parameter ℓ represents the maximal distance at which predators u feel the presence of prey w . It can be easily verified that (2.5.1) fulfils the assumption (v).

We compute the numerical solution on the domain

$$D = [0, 0.5] \times [0, 1]$$

and consider the following sizes of the space mesh:

$$h = 0.005, \quad h = 0.0025, \quad h = 0.00125.$$

Since no exact solutions are available, we use the numerical solution computed for $h = 0.000625$ as reference solution (u, w) . The corresponding time step sizes τ_p and τ are chosen according to the definition in Section 2.2, see especially (2.2.1) and the lines above.

Let (u_h, w_h) be the numerical solution associated to space mesh size h . The error is computed in the following way

$$\begin{aligned} \|u_h - u\|_{\mathbf{L}^1} &= \sup_{t \in [0, T]} \|u_h(t) - u(t)\|_{\mathbf{L}^1(D; \mathbb{R})}, \\ \|w_h - w\|_{\mathbf{L}^1} &= \sup_{t \in [0, T]} \|w_h(t) - w(t)\|_{\mathbf{L}^1(D; \mathbb{R})}. \end{aligned} \quad (2.5.3)$$

More precisely, we average the reference solution (u, w) on the coarse grid in order to compare it to the solution (u_h, w_h) .

We define EOC_u , respectively EOC_w , the experimental order of convergence for u , respectively for w , computed as follows:

$$\text{EOC}_u = \frac{\log \frac{\|u_1 - u\|_{\mathbf{L}^1}}{\|u_2 - u\|_{\mathbf{L}^1}}}{\log \frac{h_1}{h_2}}, \quad \text{EOC}_w = \frac{\log \frac{\|w_1 - w\|_{\mathbf{L}^1}}{\|w_2 - w\|_{\mathbf{L}^1}}}{\log \frac{h_1}{h_2}}, \quad (2.5.4)$$

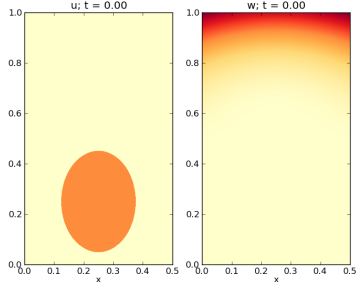
where $(u_1, w_1), (u_2, w_2)$ are solutions with grid size h_1 and h_2 respectively.

2.5.1 Example 1

In our first example, we consider the test case proposed in Paragraph 1.3.1, see also [25, Section 3.1], where the parameters are chosen as

$$\begin{aligned} \alpha &= 2 & \beta &= 1 & \kappa &= 1 \\ \gamma &= 1 & \delta &= 2 & \mu &= 0.5 & \ell &= 0.0375 \end{aligned} \quad (2.5.5)$$

with the following initial datum on D

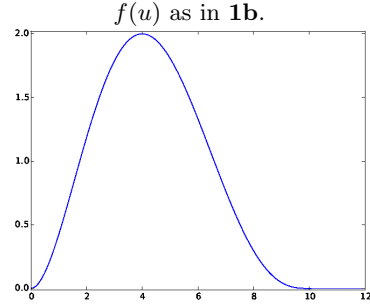


$$\begin{aligned}
 u_o(x, y) &= 4 \chi_A(x, y) \\
 w_o(x, y) &= 3(2y - 1) \max\{0, h(x, y)\} \chi_B(x, y) \\
 \text{where} & \\
 h(x, y) &= (4x - 1)^2 + (4y - 2)^2 - 0.25 \\
 A &= \{(x, y) \in \mathbb{R}^2 : (8x - 2)^2 + (1.25(4y - 1))^2 \leq 1\} \\
 B &= \{(x, y) \in \mathbb{R}^2 : y \geq 0.5\}.
 \end{aligned} \tag{2.5.6}$$

For this example we consider two hyperbolic flux functions

1a. $f(u) = u$, as in [25];

$$\mathbf{1b.} \quad f(u) = \begin{cases} \frac{u^2}{1728} (10 - u)^3 & \text{if } 0 \leq u \leq 10 \\ 0 & \text{elsewhere.} \end{cases}$$



It is easy to see that both functions fulfil assumption **(f)**. The constants in case **1b** are related to the initial datum. Indeed, this choice guarantees that $f(4) = 2$ is the maximal value of f . Note that $\|\partial_u f\|_{\mathbf{L}^\infty} = 5(3 + 8\sqrt{6})/144 \approx 0.78458 \leq 1$.

To evaluate the convolution without boundary effects, we compute the solution on a slightly bigger domain than D . More precisely, we enlarge the computational domain D by adding in all directions a constant quantity $\mathcal{C}(\eta) > 0$, related to the size of the support of the kernel function η . The enlarged domain is thus of the form

$$[-\mathcal{C}(\eta), 0.5 + \mathcal{C}(\eta)] \times [-\mathcal{C}(\eta), 1 + \mathcal{C}(\eta)] \supseteq D.$$

In particular, we choose $\mathcal{C}(\eta)$ large enough such that, when computing the convolution for a point on the boundary of D , the whole support of the kernel function η is inside the extended computational domain.

The boundary conditions for u and w are chosen to remain equal to the initial datum all along the boundary of this extended computational domain. For the balance law, this means to assume a constant value outside the computational domain and to compute the flux accordingly: whenever $\mathbf{v}(w)$ is pointing inward the domain, an inflow due to the boundary condition is considered. Concerning the parabolic equation, this choice of boundary conditions amounts to assume that the displayed solution is part of a solution defined on all $\mathbb{R}^2$ that gives constant inflow into the computational domain.

The solution is computed up to time $T_{\max} = 0.3$ in Example **1a** and up to time $T_{\max} = 0.5$ in Example **1b**.

In Table 2.1 and 2.2 we report the values of the $\mathbf{L}^1$ -error for the different mesh sizes and the corresponding experimental order of convergence for flux function **1a** and **1b** respectively. Figure 2.1 displays the error in logarithmic scale. The lines obtained connecting the values

for u , respectively w , can be easily compared with the line with slope 1, that represents the order of convergence we expect theoretically for smooth solutions.

h	$\ u_h - u\ _{L^1}$	EOC_u	$\ w_h - w\ _{L^1}$	EOC_w
0.005	0.56	—	0.303	—
0.0025	0.275	1.03	0.114	1.41
0.00125	0.106	1.38	0.034	1.74

Table 2.1: L^1 -error computed as in (2.5.3) and experimental order of convergence computed as in (2.5.4) for the solution to (2.1.1)–(2.5.1)–(2.5.5) with initial datum (2.5.6) and flux f as in **1a**.

h	$\ u_h - u\ _{L^1}$	EOC_u	$\ w_h - w\ _{L^1}$	EOC_w
0.005	0.579	—	0.43	—
0.0025	0.273	1.08	0.185	1.22
0.00125	0.101	1.43	0.064	1.54

Table 2.2: L^1 -error computed as in (2.5.3) and experimental order of convergence computed as in (2.5.4) for the solution to (2.1.1)–(2.5.1)–(2.5.5) with initial datum (2.5.6) and flux f as in **1b**.

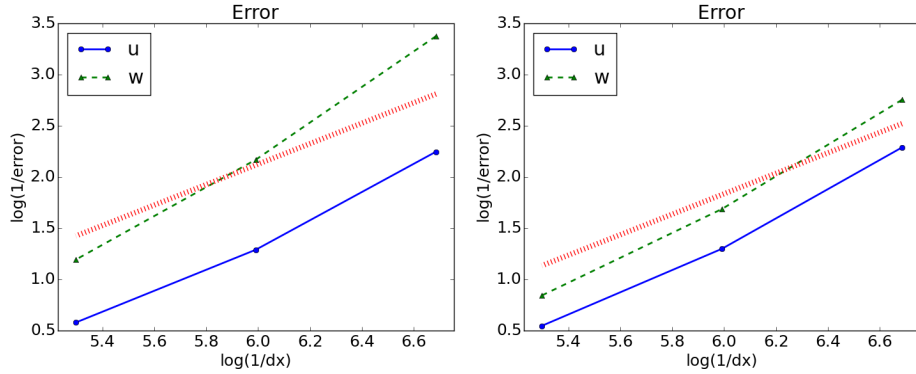


Figure 2.1: Plot of the L^1 -error the solution to (2.1.1), (2.5.1), (2.5.5) with initial datum (2.5.6): case **1a** on the left, case **1b** on the right. The dotted line has slope 1 and represents the order of convergence we expect theoretically.

To emphasise that the method is indeed positivity preserving as stated in the analytical part, we report the space–time minimum values of u_h and w_h in Table 2.3

h	$\min_{i,j,n} u_h$	$\min_{i,j,n} w_h$
0.005	0	0
0.0025	0	0
0.00125	0	0

h	$\min_{i,j,n} u_h$	$\min_{i,j,n} w_h$
0.005	0	0
0.0025	0	0
0.00125	0	0

Table 2.3: Space–time minimum values of u_h and w_h for the solution to (2.1.1)–(2.5.1)–(2.5.5) with initial datum (2.5.6) and flux function f as in **1a** (left) and **1b** (right).

Figures 2.2 and 2.3 show the results of the numerical integration for flux function f as in **1a** and **1b** respectively. The figures illustrate well also the spatial effect of the model:

thanks to the non locality of v , predators u move towards regions where the concentration of prey w is higher. Observe that boundary effects are negligible at least up to the maximal time of integration.

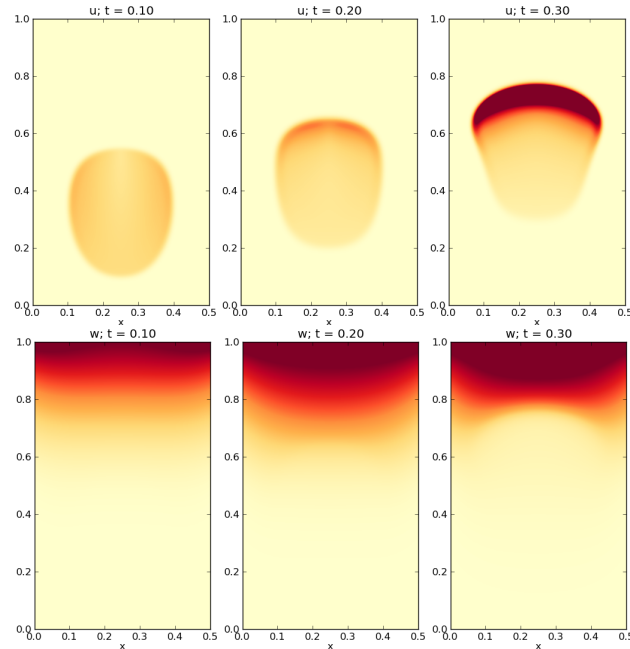


Figure 2.2: Numerical integration of (2.1.1), (2.5.1), (2.5.5) with initial datum (2.5.6) and f as in **1a** at time $t = 0.1, 0.2, 0.3$. This solution was obtained with $h = 0.000625$. Top row: u , bottom row: w . Darker colours indicate higher densities.

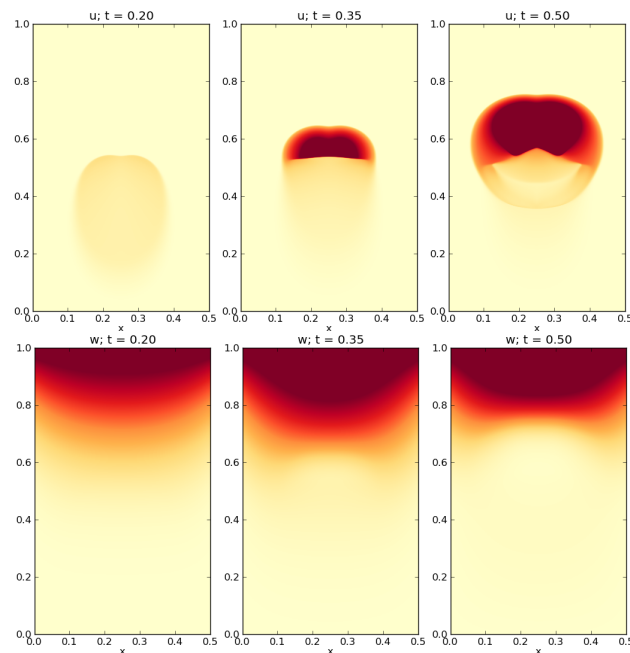


Figure 2.3: Numerical integration of (2.1.1), (2.5.1), (2.5.5) with initial datum (2.5.6) and f as in **1b** at time $t = 0.2, 0.35, 0.5$. This solution was obtained with $h = 0.000625$. Top row: u , bottom row: w . Darker colours indicate higher densities.

2.5.2 Example 2

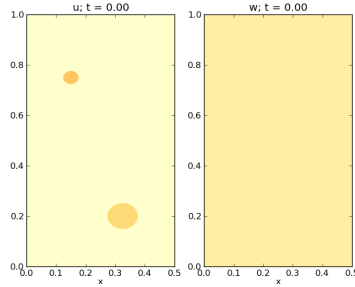
In this example, we modify the treatment of the boundary. In particular, we impose isolating boundary conditions for u and Neumann homogeneous boundary conditions for w . We consider

$$f(u) = u(1 - u), \quad (2.5.7)$$

and it is easy to see that it fulfils the assumption **(f)**. We set

$$\begin{aligned} \alpha &= 2 & \beta &= 0.8 & \kappa &= 1 \\ \gamma &= 0.8 & \delta &= 24 & \mu &= 0.1 & \ell &= 0.0625 \end{aligned} \quad (2.5.8)$$

with the following initial datum on D



$$\begin{aligned} u_o(x, y) &= 0.05 \left(5\chi_E(x, y) + 4\chi_F(x, y) \right) \\ w_o(x, y) &= 0.2 \end{aligned}$$

where

$$\begin{aligned} E &= \{(x, y) \in \mathbb{R}^2 : (4x - 0.6)^2 + (4y - 3)^2 \leq 0.01\} \\ F &= \{(x, y) \in \mathbb{R}^2 : (4x - 1.3)^2 + (4y - 0.8)^2 \leq 0.04\}. \end{aligned}$$

(2.5.9)

The solution is computed up to time $T_{\max} = 4$ on a mesh of width $h = 0.00125$. Figure 2.4 shows the evolution of the total mass (i.e. the $\mathbf{L}^1$ -norm in space) of predators u and preys w over time, while Figure 2.5 shows the spacial behaviour of the solution at different times.

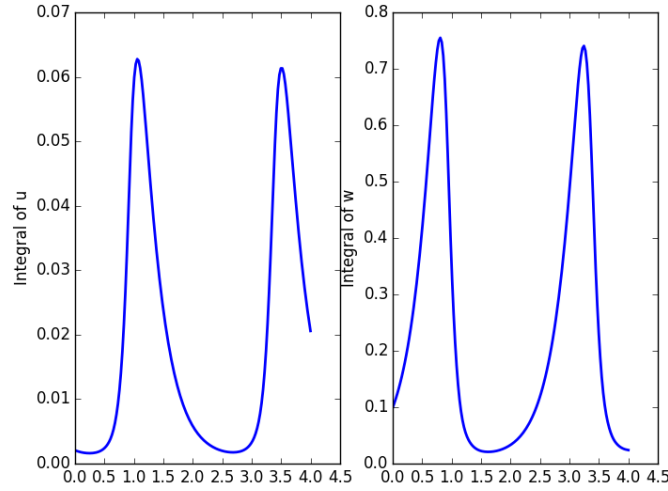


Figure 2.4: The typical Lotka–Volterra effect obtained for the solution to (2.1.1), (2.5.1), (2.5.7), (2.5.8) with initial datum (2.5.9) and a mesh of width $h = 0.00125$. The graphs display the integral of u (left), respectively w (right), representing the total mass of predators and prey.

In this example we can clearly see the typical Lotka–Volterra effect, see Figure 2.4, where the evolution of the total mass of predators and prey in time is shown. One population, in this case predators u , apparently almost disappear, then its mass rises again, due to feeding on prey and to newborns. At the same time the other population grows, until its mass reaches a sort of maximum point: from that instant on, predators eating prey produce a decrease in prey mass. However, when the total mass of prey is very low, predators have

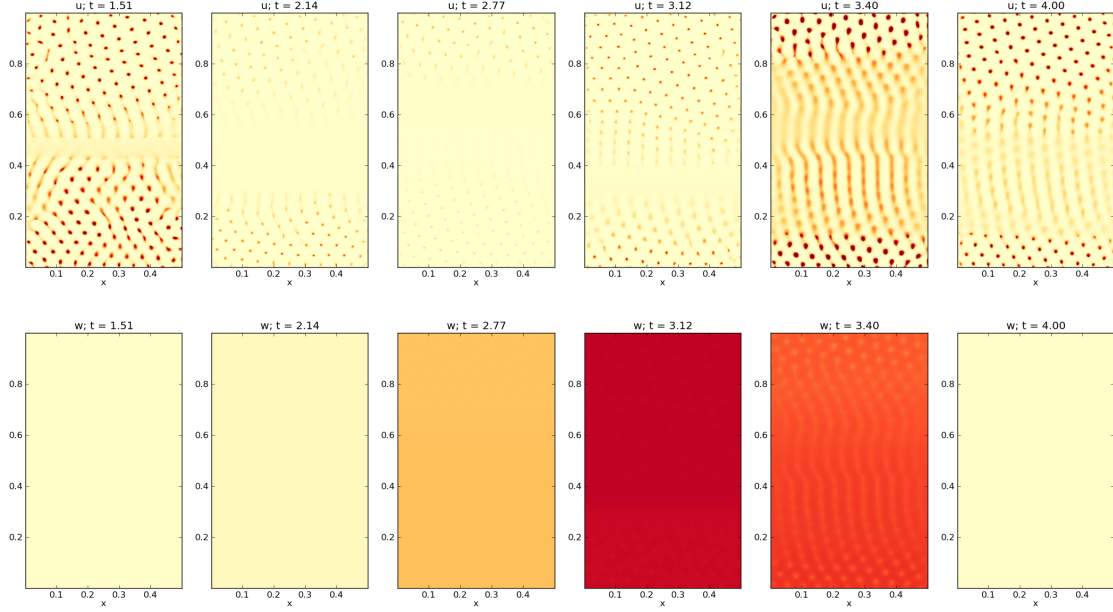


Figure 2.5: Numerical integration of (2.1.1), (2.5.1), (2.5.7), (2.5.8) with initial datum (2.5.9) and a mesh of width $h = 0.00125$. Top row: u , bottom row: w . Darker colours indicate higher densities. The first column shows the spacial distribution of u and w at a time where $\|w\|_{\mathbf{L}^1}$ attains its minimum, while $\|u\|_{\mathbf{L}^1}$ is already decreasing. The second column represents some intermediate time, where $\|u\|_{\mathbf{L}^1}$ is still decreasing while $\|w\|_{\mathbf{L}^1}$ has already started to increase. The third column shows a scenario where $\|u\|_{\mathbf{L}^1}$ is minimal, while the forth column shows u and w when $\|w\|_{\mathbf{L}^1}$ is maximal. In the fifth column, $\|u\|_{\mathbf{L}^1}$ is maximal, while $\|w\|_{\mathbf{L}^1}$ has started to decrease again. Finally, in the last column, $\|w\|_{\mathbf{L}^1}$ is again minimal, while $\|u\|_{\mathbf{L}^1}$ is still decreasing.

nothing left to eat, hence they decrease, while prey are free to increase, and the whole cycle begins again. This time-periodic behaviour can also be seen in the spacial behaviour, see Figure 2.5. There, almost the same spacial-patterns for u and w appear in every cycle.

As for the previous examples, we report the space-time minimum values of u_h and w_h in Table 2.4 to emphasise that the method is indeed positivity preserving.

h	$\min_{i,j,n} u_h$	$\min_{i,j,n} w_h$
0.00125	0	0.04

Table 2.4: Space-time minimum values of u_h and w_h for the solution to Example 2 using equations (2.1.1)–(2.5.1)–(2.5.7)–(2.5.8) and initial datum (2.5.9).

Chapter 3

Rigorous Estimates on Balance Laws in Bounded Domains

The content of this Chapter appeared on *Acta Mathematica Scientia* as [26].

3.1 Introduction

This Chapter is devoted to the well posedness of a general scalar balance law in an n -dimensional bounded domain, that is of

$$\begin{cases} \partial_t u + \nabla \cdot f(t, x, u) = F(t, x, u) & (t, x) \in I \times \Omega \\ u(0, x) = u_o(x) & x \in \Omega \\ u(t, \xi) = u_b(t, \xi) & (t, \xi) \in I \times \partial\Omega. \end{cases} \quad (3.1.1)$$

A key reference in this context is the classical paper by Bardos, Leroux and Nédélec [10]. There, the “correct” definition of solution to (3.1.1) is selected, in the spirit of the definition given by Kružkov in the case $\Omega = \mathbb{R}^n$, see [47, Definition 1]. A proof of the existence, uniqueness and continuous dependence of the solution on the initial data is described in [10] in the case $u_b = 0$.

For its relevance, since its publication, the well posedness of (3.1.1) proved in [10] was refined or explained in various text books, mostly in particular cases. For instance, the case $f = f(u)$, $F = 0$ and $u_b = 0$ is detailed in [28, Section 6.9], while non homogeneous boundary conditions are considered in [62, Section 15.1], always in the case $f = f(u)$, $F = 0$. A different type of boundary condition is considered, for instance, in [7].

Below, we aim for a presentation which covers the general case (3.1.1), which is self contained and with precise references to the elliptic or parabolic results required. Where possible, we also seek to underline which regularity is necessary at which step. As a result, we also obtain further estimates on the solution to (3.1.1).

As in [10] and [47], existence of solution is obtained through the vanishing viscosity technique. The usual term $\varepsilon \Delta u$ is added on the right hand side of the equation in (3.1.1), turning it into the parabolic problem

$$\begin{cases} \partial_t u_\varepsilon + \nabla \cdot f(t, x, u_\varepsilon) = F(t, x, u_\varepsilon) + \varepsilon \Delta u_\varepsilon & (t, x) \in I \times \Omega \\ u_\varepsilon(0, x) = u_o(x) & x \in \Omega \\ u_\varepsilon(t, \xi) = u_b(t, \xi) & (t, \xi) \in I \times \partial\Omega, \end{cases} \quad (3.1.2)$$

which is first considered under stricter conditions (regularity and compatibility of the data). Classical results from the parabolic literature [36, 48, 49] can then be applied to ensure the existence of a solution u_ε to (3.1.2) in the case $u_b = 0$. To pass to the limit $\varepsilon \rightarrow 0$, suitable bounds on u_ε are necessary. First, the $\mathbf{L}^\infty$ bound (3.3.3) is fundamental. In this connection, we note that the similar bound [10, Formula (9)] lacks a term that should be present also in the case $u_b = 0$ considered therein, refer to Section 3.3 for more details. Then, a tricky **BV** bound allows to prove that the family of solutions to (3.1.2) is relatively compact in $\mathbf{L}^1$, so that the limit of any convergent subsequence of the u_ε solves (3.1.1).

The next step is the problem with $u_b \neq 0$, a situation hardly considered in the literature. To rigorously extend the existence of solutions to the non homogeneous case, a time and space dependent translation in the u space of the solution to (3.1.1) is necessary. This leads on one side to the need of solving an elliptic problem and, on the other side, to prove that this translation does indeed give a solution to (3.1.1). An *ad hoc* adaptation of the doubling of variables technique from [47] makes this latter proof possible. However, to get the necessary estimates on the translated balance law (3.5.49), strict regularity requirements on the elliptic problem are necessary, see Lemma 3.15. All this leads to keep, in the present work, strict regularity assumptions on u_b and the condition that $u_b(0, \xi) = 0$ for all $\xi \in \partial\Omega$.

At this stage, the existence of solutions to (3.1.1) is proved, under rather strict conditions on initial and boundary data. A further use of the doubling of variables technique allows to prove the Lipschitz continuous dependence of the solution on the initial and boundary data. Remarkably, this technique allows to obtain a proof that essentially relies only on the definition of solution, in a generality wider than that available for the existence of solutions. Finally, we thus obtain at once also the uniqueness of solutions to (3.1.1) and to relax the necessary condition on the initial datum.

The bounds on the total variation of the solution have a key role throughout this work. First, they are obtained in the case $u_b = 0$, similarly to what is done in [10], see (3.4.1)–(3.4.2). This bound depends on the total variation of the initial datum and on various norms of the flow f and of the source F . The translation that allows to pass to the non homogeneous problem leads to consider a translated balance law, where the translated flow and source depend on an extension of the boundary data, see (3.5.49). Therefore, the bound on the total variation of the solution to the translated problem depends on high norms of the boundary datum, see (3.4.5), which in the end imposes to keep the condition $u_b(0, \xi) = 0$ for all $\xi \in \partial\Omega$.

The Chapter is organised as follows. The next section is devoted to the main result, which is obtained through estimates on the parabolic approximation (3.1.2) to (3.1.1), presented in Section 3.3. Then, Section 3.4 accounts for the hyperbolic results. All proofs are deferred to paragraphs 3.5.1, 3.5.2 and 3.5.3. The final paragraph gathers useful information on the trace operator.

3.2 Notations, Definitions and Main Result

Throughout, $\mathbb{R}^+ = [0, +\infty[$, $B(x, r)$ denotes the open ball centred at x with radius $r > 0$. The closed real interval $I = [0, T]$ is fixed, T being completely arbitrary. For the divergence of a vector field, possibly composed with another function, we use the notation

$$\nabla \cdot f(t, x, u(t, x)) = \operatorname{div} f(t, x, u(t, x)) + \partial_u f(t, x, u(t, x)) \cdot \nabla u(t, x).$$

The Lebesgue n dimensional measure of Ω is denoted $\mathcal{L}^n(\Omega)$, while the Hausdorff $n - 1$ dimensional measure of $\partial\Omega$ is $\mathcal{H}^{n-1}(\partial\Omega)$.

We use below the following standard assumptions, where $\ell \in \mathbb{N}$ and $\alpha \in [0, 1[$:

($\Omega_{\ell, \alpha}$) Ω is a bounded open subset of $\mathbb{R}^n$ with piecewise $\mathbf{C}^{\ell, \alpha}$ boundary $\partial\Omega$ and exterior unit normal vector ν .

(f) $f \in \mathbf{C}^2(\Sigma; \mathbb{R}^n)$, $\partial_u f \in \mathbf{L}^\infty(\Sigma; \mathbb{R}^n)$, $\partial_u \operatorname{div} f \in \mathbf{L}^\infty(\Sigma; \mathbb{R})$.

(F) $F \in \mathbf{C}^2(\Sigma; \mathbb{R})$, $\partial_u F \in \mathbf{L}^\infty(\Sigma; \mathbb{R})$.

(C) $u_o \in (\mathbf{L}^\infty \cap \mathbf{BV})(\Omega; \mathbb{R})$ and $u_b \in (\mathbf{L}^\infty \cap \mathbf{BV})(I \times \partial\Omega; \mathbb{R})$.

Above and in the sequel, we denote

$$\Sigma = I \times \bar{\Omega} \times \mathbb{R}.$$

Above, we followed the choice in [62, Chapter 10] of a boundary data with bounded total variation. Refer to [56, Section 2.6] and [58] for a generalisation to $\mathbf{L}^\infty$ boundary data. For a definition of functions of bounded variation on a manifold, refer for instance to [46, Definition 3.1].

Our starting point is the definition of solution, which originates in the work of Vol’pert [63], see also [13, 28, 47, 62]

Definition 3.1 ([10, p.1028]). *Let Ω satisfy $(\Omega_{2,0})$. Fix u_o and u_b satisfying **(C)**. A solution to (3.1.1) on I is a map $u \in (\mathbf{L}^\infty \cap \mathbf{BV})(I \times \Omega; \mathbb{R})$ such that for any test function $\varphi \in \mathbf{C}_c^2([-\infty, T[\times \mathbb{R}^n; \mathbb{R}^+)$ and for any $k \in \mathbb{R}$,*

$$\begin{aligned} & \int_I \int_\Omega \left\{ |u(t, x) - k| \partial_t \varphi(t, x) + \operatorname{sgn}(u(t, x) - k) (f(t, x, u) - f(t, x, k)) \cdot \nabla \varphi(t, x) \right. \\ & \quad \left. + \operatorname{sgn}(u(t, x) - k) (F(t, x, u) - \operatorname{div} f(t, x, k)) \varphi(t, x) \right\} dx dt \\ & \quad + \int_\Omega |u_o(x) - k| \varphi(0, x) dx \\ & - \int_I \int_{\partial\Omega} \operatorname{sgn}(u_b(t, \xi) - k) \left(f(t, \xi, (\operatorname{tr} u)(t, \xi)) - f(t, \xi, k) \right) \cdot \nu(\xi) \varphi(t, \xi) d\xi dt \geq 0. \end{aligned} \tag{3.2.1}$$

Above, $\operatorname{tr} u(t, \xi)$ denotes the trace of the map $x \rightarrow u(t, x)$ on $\partial\Omega$ evaluated at ξ . More information and references on the trace operator are collected in Paragraph 3.5.4.

Now, we recall consequences of Definition 3.1 specifying the sense in which the initial datum is attained.

Proposition 3.2. *Let $(\Omega_{2,0})$, **(f)**, **(F)** and **(C)** hold. Let $u \in (\mathbf{L}^\infty \cap \mathbf{BV})(I \times \Omega; \mathbb{R})$ be a solution to (3.1.1) in the sense of Definition 3.1. Then, there exists a set $\mathcal{E} \subset I$ of Lebesgue measure 0 such that*

$$\lim_{t \rightarrow 0+, t \in I \setminus \mathcal{E}} \int_\Omega |u(t, x) - u_o(x)| dx = 0.$$

The proof is deferred to Paragraph 3.5.3.

A further similar consequence of the above definition of solution and of the properties of the trace operator is the following Proposition. It gives information on the way in which the values of the boundary data are attained by the solution.

Proposition 3.3. *Let $(\Omega_{2,0})$, (f) , (F) and (C) hold. Let $u \in (\mathbf{L}^\infty \cap \mathbf{BV})(I \times \Omega; \mathbb{R})$ be a solution to (3.1.1) in the sense of Definition 3.1. Then, for all $k \in \mathbb{R}$ and for almost every $(t, \xi) \in \mathring{I} \times \partial\Omega$,*

$$\left[\operatorname{sgn}(\operatorname{tr} u(t, \xi) - k) - \operatorname{sgn}(u_b(t, \xi) - k) \right] \left[f(t, \xi, (\operatorname{tr} u)(t, \xi)) - f(t, \xi, k) \right] \cdot \nu(\xi) \geq 0. \quad (3.2.2)$$

Moreover, for almost every $(t, \xi) \in \mathring{I} \times \partial\Omega$

$$\min_{k \in \mathcal{I}(t, \xi)} \operatorname{sgn}(\operatorname{tr} u(t, \xi) - u_b(t, \xi)) \left[f(t, \xi, \operatorname{tr} u(t, \xi)) - f(t, \xi, k) \right] \cdot \nu(\xi) = 0, \quad (3.2.3)$$

where $\mathcal{I}(t, \xi) = \left\{ k \in \mathbb{R} : (u_b(t, \xi) - k)(k - \operatorname{tr} u(t, \xi)) \geq 0 \right\}$.

The proof is deferred to Paragraph 3.5.3. In other words, (3.2.3) states that $\operatorname{tr} u$ and u_b may differ whenever the jump between them gives rise to waves exiting Ω .

Recall now the classical concept of entropy – entropy flux pair, in the general case (3.1.1).

Definition 3.4. *An entropy – entropy flux pair for equation $\partial_t u + \nabla \cdot f(t, x, u) = F(t, x, u)$ is a pair of functions $(\mathcal{E}, \mathcal{F})$ such that:*

1. $\mathcal{E} \in \mathbf{C}^2(\mathbb{R}; \mathbb{R})$ and $\mathcal{F} \in \mathbf{C}^2(I \times \Omega \times \mathbb{R}; \mathbb{R}^n)$;
2. $\mathcal{E}$ is convex;
3. for all $(t, x, u) \in \Sigma$, $\mathcal{E}'(u) \partial_u f(t, x, u) = \partial_u \mathcal{F}(t, x, u)$.

In the case of the general balance law (3.1.1), the differential form of the entropy inequality is

$$\begin{aligned} \partial_t \mathcal{E}(u(t, x)) + \nabla \cdot \mathcal{F}(t, x, u(t, x)) &\leq \mathcal{E}'(u(t, x)) \left(F(t, x, u(t, x)) - \operatorname{div} f(t, x, u(t, x)) \right) \\ &\quad + \operatorname{div} \mathcal{F}(t, x, u(t, x)). \end{aligned}$$

Particular cases of this expression are considered, for instance, in [13, 14, 28, 38, 44, 62].

Definition 3.5. *Let Ω satisfy $(\Omega_{2,0})$. Fix u_o and u_b satisfying (C) . An entropy solution to (3.1.1) is a map $u \in (\mathbf{L}^\infty \cap \mathbf{BV})(I \times \Omega; \mathbb{R})$ such that for any entropy – entropy flux pair $(\mathcal{E}, \mathcal{F})$ and for any $\varphi \in \mathbf{C}_c^2([-\infty, T] \times \mathbb{R}^n; \mathbb{R}^+)$, the following inequality holds:*

$$\begin{aligned} &\int_I \int_\Omega \left\{ \mathcal{E}(u(t, x)) \partial_t \varphi(t, x) + \mathcal{F}(t, x, u(t, x)) \cdot \nabla \varphi(t, x) \right. \\ &\quad \left. + \left[\mathcal{E}'(u(t, x)) \left(F(t, x, u(t, x)) - \operatorname{div} f(t, x, u(t, x)) \right) + \operatorname{div} \mathcal{F}(t, x, u(t, x)) \right] \varphi(t, x) \right\} dx dt \\ &+ \int_\Omega \mathcal{E}(u_o(x)) \varphi(0, x) dx \\ &- \int_I \int_{\partial\Omega} \left[\mathcal{F}(t, \xi, u_b(t, \xi)) \right. \\ &\quad \left. - \mathcal{E}'(u_b(t, \xi)) \left(f(t, \xi, u_b(t, \xi)) - f(t, \xi, \operatorname{tr} u(t, \xi)) \right) \right] \cdot \nu(\xi) \varphi(t, \xi) d\xi dt \geq 0. \end{aligned} \quad (3.2.4)$$

Formally, Definition 3.1 is a “*particular*” case of Definition 3.5, obtained choosing as entropy – entropy flux pair the maps

$$\mathcal{E}(u) = |u - k| \quad \text{and} \quad \mathcal{F}(t, x, u) = \operatorname{sgn}(u - k) (f(t, x, u) - f(t, x, k)),$$

for $k \in \mathbb{R}$. However, the two definitions actually coincide.

Proposition 3.6. *Definitions 3.1 and 3.5 are equivalent for bounded solutions.*

This Proposition is well known and its proof is briefly sketched in Paragraph 3.5.3.

We are now ready to state the main result of this Chapter.

Theorem 3.7. *Let $T > 0$, $\alpha \in]0, 1[$, and assume that $(\Omega_{3,\alpha})$, $(\mathbf{f})$ and $(\mathbf{F})$ hold. In addition, suppose that $\partial_{uu}^2 f \in \mathbf{L}^\infty(I \times \Omega \times \mathbb{R}; \mathbb{R}^n)$. Fix an initial datum $u_o \in (\mathbf{L}^\infty \cap \mathbf{BV})(\Omega; \mathbb{R})$ and a boundary datum $u_b \in \mathbf{C}^{3,\alpha}(I \times \partial\Omega; \mathbb{R})$ with $u_b(0, \xi) = 0$ for all $\xi \in \partial\Omega$. Then, problem (3.1.1) admits a unique solution $u \in \mathbf{C}^{0,1}(I; \mathbf{L}^1(\Omega; \mathbb{R}))$. Moreover, the following estimates hold:*

$$\begin{aligned} \|u(t)\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} \leq & \left[\|u_o\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} + t \left(c_2 + c_1 \|u_b\|_{\mathbf{L}^\infty(I \times \partial\Omega; \mathbb{R})} \right. \right. \\ & \left. \left. + \|\partial_t u_b\|_{\mathbf{L}^\infty(I \times \partial\Omega; \mathbb{R})} + c_3 \|u_b\|_{\mathbf{C}^{2,\alpha}(I \times \partial\Omega; \mathbb{R})} \right) \right] \end{aligned} \quad (3.2.5)$$

$$\begin{aligned} \times \exp & \left(c_1 t + c_3 t \|u_b\|_{\mathbf{C}^{2,\alpha}(I \times \partial\Omega; \mathbb{R})} \right) + \|u_b\|_{\mathbf{L}^\infty([0,t] \times \partial\Omega; \mathbb{R})}, \\ \operatorname{TV}(u(t)) \leq & \mathcal{C}(\Omega, f, F, t) \left(1 + \|u_b\|_{\mathbf{C}^{3,\alpha}([0,t] \times \partial\Omega; \mathbb{R})} + \|u_b\|_{\mathbf{C}^{3,\alpha}([0,t] \times \partial\Omega; \mathbb{R})}^2 \right) \\ & \times \left(1 + t + \|u_o\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} + \operatorname{TV}(u_o) \right) \\ & \times \exp \left(\mathcal{C}(\Omega, f, F, t) (1 + \|u_b\|_{\mathbf{C}^{2,\alpha}([0,t] \times \partial\Omega; \mathbb{R})}) t \right) \end{aligned} \quad (3.2.6)$$

$$\|u(t) - u(s)\|_{\mathbf{L}^1(\Omega; \mathbb{R})} \leq \left(\sup_{\tau \in [s,t]} \operatorname{TV}(u(\tau)) \right) |t - s| \quad (3.2.7)$$

for $t, s \in I$, where c_1, c_2, c_3 and $\mathcal{C}(\Omega, f, F, t)$ are independent of the initial and boundary data, see (3.5.1), (3.5.74) and (3.5.75).

The proof consists of the lemmas and propositions in the sections below, together with the final bootstrap procedure presented in Paragraph 3.5.3. The Lipschitz continuous dependence of the solution on the initial and boundary data is stated and proved in Theorem 3.14.

Remark 3.8. The above estimate (3.2.5) shows that the solution u is in $\mathbf{L}^\infty(I \times \Omega; \mathbb{R})$. By (3.2.6), we also have $u(t) \in \mathbf{BV}(\Omega; \mathbb{R})$ for every $t \in I$. The Lipschitz continuity in time ensured by (3.2.7) then implies that $u \in (\mathbf{L}^\infty \cap \mathbf{BV})(I \times \Omega; \mathbb{R})$, as required in Definition 3.1. This can be proved using exactly the arguments in [13, Section 2.5, Proof of Theorem 2.6].

3.3 The Parabolic Problem (3.1.2)

All proofs of the statements in this Section are deferred to Paragraph 3.5.1. Note that the results in this section are obtained without requiring that $u_b = 0$.

The next Lemma provides the existence of classical solutions to the parabolic problem (3.1.2).

Lemma 3.9. Fix $\alpha \in]0, 1[$. Let conditions $(\Omega_{2,\alpha})$, (f) and (F) hold. Assume moreover that there exists a function $\bar{u} \in \mathbf{C}^{2,\delta}(I \times \bar{\Omega}; \mathbb{R})$, with $\delta \in]\alpha, 1[$, such that

$$\begin{aligned} \partial_t \bar{u}(0, \xi) + \nabla \cdot f(0, \xi, \bar{u}(0, \xi)) &= F(0, \xi, \bar{u}(0, \xi)) + \varepsilon \Delta \bar{u}(0, \xi) \\ u_o(\xi) &= \bar{u}(0, \xi) = u_b(0, \xi) \end{aligned} \quad \text{for all } \xi \in \partial\Omega. \quad (3.3.1)$$

Then, setting

$$u_o(x) = \bar{u}(0, x) \quad \text{for all } x \in \bar{\Omega} \quad \text{and} \quad u_b(t, \xi) = \bar{u}(t, \xi) \quad \text{for all } (t, \xi) \in I \times \partial\Omega, \quad (3.3.2)$$

there exists a unique solution u_ε to (3.1.2) of class $\mathbf{C}^{2,\gamma}(I \times \bar{\Omega}; \mathbb{R})$, for a suitable $\gamma \in]\delta, 1[$.

We now provide an $\mathbf{L}^\infty$ -estimate for the solution u_ε to (3.1.2). It is important to note that we obtain a bound that holds uniformly in ε , see (3.3.3).

Lemma 3.10. Fix $\alpha \in]0, 1[$. Let conditions $(\Omega_{2,\alpha})$, (f) and (F) hold. Assume moreover there exists a function $\bar{u} \in \mathbf{C}^{2,\delta}(I \times \bar{\Omega}; \mathbb{R})$, for $\delta \in]\alpha, 1[$, such that (3.3.1) holds. Let u_ε be a solution to (3.1.2) with u_o and u_b as in (3.3.2). Then, for all $t \in I$,

$$\|u_\varepsilon\|_{\mathbf{L}^\infty([0,t] \times \Omega; \mathbb{R})} \leq \left(\|u_o\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} + \|u_b\|_{\mathbf{L}^\infty([0,t] \times \partial\Omega; \mathbb{R})} + c_2 t \right) e^{c_1 t}, \quad (3.3.3)$$

where c_1, c_2 are constants depending on the $\mathbf{L}^\infty$ norms of $\operatorname{div} f$, $\partial_u \operatorname{div} f$, F and $\partial_u F$, as defined in (3.5.1).

We remark that, also in the case $u_b = 0$, due to the presence of the third addend in the right hand side, the above estimate (3.3.3) significantly differs from the $\mathbf{L}^\infty$ bound [10, Formula (9)], which can not be true. Indeed, the estimate [10, Formula (9)] implies that the solution to (3.1.2) with $u_o = 0$ and $u_b = 0$ is $u = 0$, which is false as, for instance, the case where $f(t, x, u) = -x$ and $F = 0$ clearly shows.

Consider now problem (3.1.2) with homogeneous boundary condition, i.e., $u_b(t, \xi) = 0$ for $(t, \xi) \in I \times \partial\Omega$. In the next Lemma we partly follow [10, Theorem 1], [28, Chapter 6, § 6.9] and [38, Chapter 4]. Introduce the notation

$$\begin{aligned} \mathcal{U}(t) &= [-\mathcal{M}(t), \mathcal{M}(t)] \quad \text{with} \\ \mathcal{M}(t) &= \left(\|u_o\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} + \|u_b\|_{\mathbf{L}^\infty([0,t] \times \partial\Omega; \mathbb{R})} + c_2 t \right) e^{c_1 t}, \end{aligned} \quad (3.3.4)$$

as in (3.3.3) and (3.5.1).

Lemma 3.11. Fix $\delta \in]0, 1[$. Let conditions $(\Omega_{2,\delta})$, (f) and (F) hold. Assume moreover that $u_o \in \mathbf{C}^{2,\delta}(\bar{\Omega}; \mathbb{R})$ is such that $u_o(\xi) = 0$ for all $\xi \in \partial\Omega$ and $u_b(t, \xi) = 0$ for $(t, \xi) \in I \times \Omega$. Let $u_\varepsilon \in \mathbf{C}^2(I \times \bar{\Omega}; \mathbb{R})$ be a solution to (3.1.2). Then,

$$\operatorname{TV}(u_\varepsilon(t)) \leq \mathcal{L}_\varepsilon(t) \quad (3.3.5)$$

$$\|u_\varepsilon(t) - u_\varepsilon(s)\|_{\mathbf{L}^1(\Omega; \mathbb{R})} \leq \mathcal{L}_\varepsilon(\max\{t, s\}) |t - s| \quad (3.3.6)$$

for $t, s \in I$, where

$$\mathcal{L}_\varepsilon(t) = \left(A_1 + A_2 t + A_3 \|\nabla u_o\|_{\mathbf{L}^1(\Omega; \mathbb{R}^n)} + \varepsilon \|\Delta u_o\|_{\mathbf{L}^1(\Omega; \mathbb{R})} \right) e^{A_4 t} \quad (3.3.7)$$

Above, A_1, A_2, A_3 and A_4 are constants depending on n , Ω and on norms of Df and F , see (3.5.30)–(3.5.31). In particular, they are independent of ε and of u_o .

3.4 The Hyperbolic Problem (3.1.1)

In the particular case of homogeneous boundary condition, we study the convergence of the sequence (u_ε) as ε tends to 0. We also prove that the limit function is a solution to problem (3.1.1), with homogeneous boundary condition.

Proposition 3.12. *Fix $\delta \in]0, 1[$. Let conditions $(\Omega_{2,\delta})$, $(\mathbf{f})$ and $(\mathbf{F})$ hold. Assume moreover that $u_o \in \mathbf{C}^{2,\delta}(\bar{\Omega}; \mathbb{R})$ is such that $u_o(\xi) = 0$ for $\xi \in \partial\Omega$, and $u_b(t, \xi) = 0$ for $(t, \xi) \in I \times \Omega$.*

Then, the family of solutions u_ε to (3.1.2) is relatively compact in $\mathbf{L}^1$. Any cluster point $u_\infty \in \mathbf{L}^1(I \times \Omega; \mathbb{R})$ of this family is a solution to (3.1.1), with $u_b = 0$, in the sense of Definition 3.1. Moreover, the following estimates hold:

$$\begin{aligned} \|u_\infty(t)\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} &\leq \left(\|u_o\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} + c_2 t \right) e^{c_1 t}, \\ \text{TV}(u_\infty(t)) &\leq \mathcal{L}(t) \\ \|u_\infty(t) - u_\infty(s)\|_{\mathbf{L}^1(\Omega; \mathbb{R})} &\leq \mathcal{L}(\max\{t, s\}) |t - s|, \end{aligned} \quad (3.4.1)$$

for $t, s \in I$, where

$$\mathcal{L}(t) = \left(A_1 + A_2 t + A_3 \|\nabla u_o\|_{\mathbf{L}^1(\Omega; \mathbb{R}^n)} \right) e^{A_4 t} \quad (3.4.2)$$

Above, $c_1, c_2, A_1, A_2, A_3, A_4$ are constants depending on n, Ω and on norms of Df and F , see (3.5.1) and (3.5.30)–(3.5.31), all independent of the initial datum.

Note that $u_\infty \in (\mathbf{L}^\infty \cap \mathbf{BV})(I \times \Omega; \mathbb{R})$, see Remark 3.8.

Theorem 3.13. *Fix $\alpha \in]0, 1[$. Let conditions $(\Omega_{3,\alpha})$, $(\mathbf{f})$ and $(\mathbf{F})$ hold. In addition, suppose that $\partial_{uu}^2 f \in \mathbf{L}^\infty(I \times \Omega \times \mathbb{R}; \mathbb{R}^n)$. Assume moreover that $u_b \in \mathbf{C}^{3,\alpha}(I \times \partial\Omega; \mathbb{R})$ and $u_o \in \mathbf{C}^{2,\delta}(\bar{\Omega}; \mathbb{R})$, with $\delta \in]\alpha, 1[$, are such that*

$$u_o(\xi) = 0 = u_b(0, \xi) \quad \text{for all } \xi \in \partial\Omega. \quad (3.4.3)$$

Then, there exists a solution $u \in \mathbf{C}^{0,1}(I; \mathbf{L}^1(\Omega; \mathbb{R}))$ to (3.1.1) in the sense of Definition 3.1. Moreover, the following bounds hold:

$$\begin{aligned} \|u(t)\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} &\leq \left[\|u_o\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} + t \left(c_2 + c_1 \|u_b\|_{\mathbf{L}^\infty(I \times \partial\Omega; \mathbb{R})} \right. \right. \\ &\quad \left. \left. + \|\partial_t u_b\|_{\mathbf{L}^\infty(I \times \partial\Omega; \mathbb{R})} + c_3 \|u_b\|_{\mathbf{C}^{2,\alpha}(I \times \partial\Omega; \mathbb{R})} \right) \right] \end{aligned} \quad (3.4.4)$$

$$\begin{aligned} &\times \exp \left(c_1 t + c_3 t \|u_b\|_{\mathbf{C}^{2,\alpha}(I \times \partial\Omega; \mathbb{R})} \right) + \|u_b\|_{\mathbf{L}^\infty([0,t] \times \partial\Omega; \mathbb{R})}, \\ \text{TV}(u(t)) &\leq \mathcal{C}(\Omega, f, F, t) \left(1 + \|u_b\|_{\mathbf{C}^{3,\alpha}([0,t] \times \partial\Omega; \mathbb{R})} + \|u_b\|_{\mathbf{C}^{3,\alpha}([0,t] \times \partial\Omega; \mathbb{R})}^2 \right) \\ &\times (1 + t + \text{TV}(u_o)) \end{aligned} \quad (3.4.5)$$

$$\|u(t) - u(s)\|_{\mathbf{L}^1(\Omega; \mathbb{R})} \leq \left(\sup_{\tau \in [s,t]} \text{TV}(u(\tau)) \right) |t - s| \quad (3.4.6)$$

for $t, s \in I$, where c_1, c_2, c_3 and $\mathcal{C}(\Omega, f, F, t)$ are independent of the initial and boundary data, see (3.5.1), (3.5.74) and (3.5.75).

Above, Remark 3.8 applies and guarantees that $u \in (\mathbf{L}^\infty \cap \mathbf{BV})(I \times \Omega; \mathbb{R})$.

Note that, up to now, we only have existence of a solution to (3.1.1), but not uniqueness. Following [10, Theorem 2], we extend [62, Theorem 15.1.5] to the case of balance laws with time and space dependent flow and source.

Theorem 3.14. *Let $(\Omega_{2,0})$, $(\mathbf{f})$ and $(\mathbf{F})$ hold. Set*

$$L_f = \|\partial_u f\|_{\mathbf{L}^\infty(\Sigma; \mathbb{R}^n)} \quad \text{and} \quad L_F = \|\partial_u F\|_{\mathbf{L}^\infty(\Sigma; \mathbb{R})}.$$

Assume that the initial data u_o, v_o and the boundary data u_b, v_b satisfy $(\mathbf{C})$. If u and v are the corresponding solutions to (3.1.1) in the sense of Definition 3.1, then, for all $t \in I$, the following estimate holds

$$\begin{aligned} \int_{\Omega} |u(t, x) - v(t, x)| \, dx &\leq e^{L_F t} \int_{\Omega} |u_o(x) - v_o(x)| \, dx \\ &\quad + L_f \int_0^t e^{L_F(t-\tau)} \int_{\partial\Omega} |u_b(\tau, \xi) - v_b(\tau, \xi)| \, d\xi \, d\tau. \end{aligned}$$

Theorem 3.14 guarantees then the uniqueness of the solution to (3.1.1).

3.5 Technical Details

3.5.1 Proofs Related to the Parabolic Problem

Proof of Lemma 3.9. To improve the readability, we write u instead of u_ε . We apply [36, Chapter 7, § 4, Theorem 9]. To this aim, in the notation of [36, § 4], we verify the required assumptions with reference to $Lu = \mathbf{f}(t, x, u, \nabla u)$, where

$$\begin{aligned} Lu &= \varepsilon \Delta u - \partial_t u \\ \mathbf{f}(t, x, u, w) &= \operatorname{div} f(t, x, u) + \partial_u f(t, x, u) \cdot w - F(t, x, u). \end{aligned}$$

The boundary and initial data ψ in [36] corresponds here to the function $\bar{u}$. The required $\mathbf{C}^{2,\alpha}$ regularity of $S = I \times \partial\Omega$ is ensured by the hypothesis. The parabolicity condition [36, Chapter 7, § 2, p.191, (A)] holds with $H_o = \varepsilon$. The condition [36, Chapter 7, § 4, p.204, (B')] on the coefficients of L is immediately satisfied: the only non-zero coefficient is the constant ε . By hypothesis, the function $\bar{u}$ is in $\mathbf{C}^{2,\delta}$, for $\alpha < \delta < 1$. The Hölder continuity of $\mathbf{f}$ follows from $(\mathbf{f})$ and $(\mathbf{F})$. Concerning [36, Chapter 7, § 2, p.203, Formula (4.10)], it reads:

$$\begin{aligned} u \mathbf{f}(t, x, u, 0) &= u (\operatorname{div} f(t, x, u) - F(t, x, u)) \\ &\leq |u| \left(\|\operatorname{div} f(\cdot, \cdot, 0)\|_{\mathbf{L}^\infty(I \times \bar{\Omega}; \mathbb{R})} + \|F(\cdot, \cdot, 0)\|_{\mathbf{L}^\infty(I \times \bar{\Omega}; \mathbb{R})} \right) \\ &\quad + u^2 \left(\|\partial_u \operatorname{div} f\|_{\mathbf{L}^\infty(I \times \bar{\Omega} \times \mathbb{R}; \mathbb{R})} + \|\partial_u F\|_{\mathbf{L}^\infty(I \times \bar{\Omega} \times \mathbb{R}; \mathbb{R})} \right) \\ &\leq A_1 u^2 + A_2 \end{aligned}$$

for suitable positive A_1, A_2 , by $(\mathbf{f})$ and $(\mathbf{F})$.

Passing to [36, Chapter 7, § 2, p.205, Formula (4.17)]

$$|\mathbf{f}(t, x, u, w)| \leq |\operatorname{div} f(t, x, u)| + \|\partial_u f(t, x, u)\| \|w\| + |F(t, x, u)|$$

$$\begin{aligned}
&\leq \left(\|\operatorname{div} f(\cdot, \cdot, 0)\|_{\mathbf{L}^\infty(I \times \bar{\Omega}; \mathbb{R})} + \|F(\cdot, \cdot, 0)\|_{\mathbf{L}^\infty(I \times \bar{\Omega}; \mathbb{R})} \right) \\
&\quad + \left(\|\partial_u \operatorname{div} f\|_{\mathbf{L}^\infty(I \times \bar{\Omega} \times \mathbb{R}; \mathbb{R})} + \|\partial_u F\|_{\mathbf{L}^\infty(I \times \bar{\Omega} \times \mathbb{R}; \mathbb{R})} \right) |u| \\
&\quad + \|\partial_u f\|_{\mathbf{L}^\infty(I \times \bar{\Omega} \times \mathbb{R}^n)} \|w\| \\
&\leq A(|u|) + \mu \|w\|
\end{aligned}$$

for a non decreasing function A and a positive scalar μ , by **(f)** and **(F)**. Lastly, the compatibility condition $L\bar{u}(0, x) = \mathbf{f}(0, x, \bar{u}, \nabla \bar{u})$ on $\partial\Omega$ holds by (3.3.1).

We can thus apply [36, Chapter 7, § 4, Theorem 9], obtaining the existence of a solution u_ε to (3.1.2) in the class $\mathbf{C}^{2,\gamma}(I \times \bar{\Omega}; \mathbb{R})$ for $0 < \gamma < 1$. Moreover, [36, Chapter 7, § 4, Theorem 6] ensures the uniqueness of the solution. The verification that the necessary assumptions are satisfied is here immediate. $\square$

Proof of Lemma 3.10. For the sake of readability, we write u instead of u_ε . We use [48, Chapter 1, § 2, Theorem 2.9], which refers to

$$\partial_t u - \sum_{i,j=1}^n a_{ij}(t, x, u, \nabla u) \partial_{ij}^2 u + a(t, x, u, \nabla u) = 0$$

where, in the present case,

$$\begin{aligned}
a_{ij}(t, x, u, p) &= \varepsilon \delta_{ij} \quad \text{for } i, j = 1, \dots, n, \\
a(t, x, u, p) &= \operatorname{div} f(t, x, u) + \partial_u f(t, x, u) \cdot p - F(t, x, u).
\end{aligned}$$

Condition $(\Omega_{2,\alpha})$ ensures the necessary regularity of the domain. By **(f)** and **(F)**, the regularity requirements on a_{ij} and a are met. Moreover,

$$\begin{aligned}
[48, \text{Formula (2.29)}]: \quad \sum_{i,j=1}^n a_{ij}(t, x, u, 0) \xi_i \xi_j &= \varepsilon \|\xi\|^2 \geq 0 \\
[48, \text{Formula (2.32)}]: \quad u a(t, x, u, 0) &= u \operatorname{div} f(t, x, u) - u F(t, x, u) \\
&\geq -\Phi(|u|) |u|
\end{aligned}$$

where $b_2 = 0$ in [48, Chapter 1, § 2, Formula (2.32)],

$$\begin{aligned}
\Phi(|u|) &= c_1 |u| + c_2 \quad \text{and} \quad \begin{aligned} c_1 &= 1 + \|\partial_u \operatorname{div} f\|_{\mathbf{L}^\infty(I \times \bar{\Omega} \times \mathbb{R}; \mathbb{R})} + \|\partial_u F\|_{\mathbf{L}^\infty(I \times \bar{\Omega} \times \mathbb{R}; \mathbb{R})}, \\ c_2 &= \|\operatorname{div} f(\cdot, \cdot, 0)\|_{\mathbf{L}^\infty(I \times \bar{\Omega}; \mathbb{R})} + \|F(\cdot, \cdot, 0)\|_{\mathbf{L}^\infty(I \times \bar{\Omega}; \mathbb{R})}. \end{aligned}
\end{aligned} \tag{3.5.1}$$

Note that Φ is nondecreasing, positive and condition [48, Chapter 1, § 2, Formula (2.33)] holds. Hence, [48, Chapter 1, § 2, Theorem 2.9] applies and the solution u to (3.1.2) satisfies [48, Formula (2.34)] with $\varphi(\xi) = (c_2/c_1)(\xi^{c_1} - 1)$, so that

$$\|u\|_{\mathbf{L}^\infty([0,t] \times \Omega; \mathbb{R})} \leq \left(\|u_o\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} + \|u_b\|_{\mathbf{L}^\infty([0,t] \times \partial\Omega; \mathbb{R})} \right) e^{c_1 t} + \frac{c_2}{c_1} (e^{c_1 t} - 1).$$

Observe that, for positive τ , $e^\tau - 1 \leq \tau e^\tau$. Hence, since $c_1 t \geq 0$, the inequality above becomes

$$\|u\|_{\mathbf{L}^\infty([0,t] \times \Omega; \mathbb{R})} \leq \left(\|u_o\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} + \|u_b\|_{\mathbf{L}^\infty([0,t] \times \partial\Omega; \mathbb{R})} \right) e^{c_1 t} + c_2 t e^{c_1 t},$$

completing the proof. $\square$

Proof of Lemma 3.11. First, define $w_\varepsilon \in \mathbf{C}^{2,\delta}(\bar{\Omega}; \mathbb{R})$ as solution to the elliptic problem

$$\begin{cases} \Delta w_\varepsilon &= -\Delta u_o + \frac{1}{\varepsilon} \operatorname{div} f(0, x, 0) + \frac{1}{\varepsilon} \partial_u f(0, x, 0) \cdot \nabla u_o(x) - \frac{1}{\varepsilon} F(0, x, 0) & x \in \Omega, \\ w_\varepsilon(\xi) &= 0 & \xi \in \partial\Omega. \end{cases}$$

The elliptic problem above admits a unique solution $w_\varepsilon \in \mathbf{C}^{2,\delta}(\bar{\Omega}; \mathbb{R})$ thanks to [50, Chapter 3, § 1, Theorem 1.3]. Indeed, with reference to the equation $Lw_\varepsilon(x) = \mathbf{f}(x)$ where

$$\begin{aligned} Lw_\varepsilon &= \sum_{i,j=1}^n a_{i,j}(x) \partial_{ij}^2 w_\varepsilon + \sum_{i=1}^n a_i(x) \partial_i w_\varepsilon + a(x) w_\varepsilon = \Delta w_\varepsilon \\ \mathbf{f}(x) &= -\Delta u_o + \frac{1}{\varepsilon} \operatorname{div} f(0, x, 0) + \frac{1}{\varepsilon} \partial_u f(0, x, 0) \cdot \nabla u_o(x) - \frac{1}{\varepsilon} F(0, x, 0), \end{aligned}$$

the hypotheses of [50, Chapter 3, § 1, Theorem 1.3] are all satisfied: the coefficients of L belong to $\mathbf{C}^\delta(\bar{\Omega}; \mathbb{R})$ and satisfy the ellipticity condition; we have $a(x) = 0$; the boundary $\partial\Omega$ is of class $\mathbf{C}^{2,\delta}$ by hypothesis; the function $\mathbf{f}$ is in $\mathbf{C}^\delta(\bar{\Omega}; \mathbb{R})$ thanks to the hypothesis on u_o , to $(\mathbf{f})$ and $(\mathbf{F})$; the homogeneous boundary condition implies that, in the notation of [50, Chapter 3, § 1], $\varphi = 0$, which is clearly in $\mathbf{C}^{2,\delta}(\partial\Omega; \mathbb{R})$.

Define now $\bar{u}_\varepsilon(t, x) = u_o(x) + w_\varepsilon(x)$ for every $(t, x) \in I \times \Omega$: this function $\bar{u}_\varepsilon$ belongs to $\mathbf{C}^{2,\delta}(I \times \bar{\Omega}; \mathbb{R})$ and it satisfies (3.3.1) and (3.3.2), with $u_b = 0$. Since $\partial\Omega$ is of class $\mathbf{C}^{2,\delta}$, it is also of class $\mathbf{C}^{2,\alpha}$ for any $\alpha \in]0, \delta[$. Hence, Lemma 3.9 yields that there exists a unique solution $u_\varepsilon \in \mathbf{C}^{2,\gamma}(I \times \bar{\Omega}; \mathbb{R})$, for a $\gamma \in]0, 1[$, to (3.1.2) with $u_b = 0$.

Following [10, 28, 38], for $\eta > 0$ introduce the functions

$$\sigma_\eta(z) = \begin{cases} z - \eta/2 & z > \eta \\ z^2/(2\eta) & z \in [-\eta, \eta] \\ -z + 3\eta/2 & z < -\eta, \end{cases} \quad \sigma'_\eta(z) = \begin{cases} 1 & z > \eta \\ z/\eta & z \in [-\eta, \eta] \\ -1 & z < -\eta. \end{cases} \quad (3.5.2)$$

Note that u_ε is of class $\mathbf{C}^2$, hence, by (3.1.2), Δu_ε is of class $\mathbf{C}^1$ and we can differentiate with respect to t the equation in (3.1.2):

$$\begin{aligned} &\partial_{tt}^2 u_\varepsilon(t, x) + \nabla \cdot \left(\partial_t f(t, x, u_\varepsilon(t, x)) + \partial_u f(t, x, u_\varepsilon(t, x)) \partial_t u_\varepsilon(t, x) \right) \\ &= \partial_t F(t, x, u_\varepsilon(t, x)) + \partial_u F(t, x, u_\varepsilon(t, x)) \partial_t u_\varepsilon(t, x) + \partial_t \Delta u_\varepsilon(t, x). \end{aligned} \quad (3.5.3)$$

Multiply by $\sigma'_\eta(\partial_t u_\varepsilon(t, x))$ and integrate over Ω each term above to obtain

$$\int_\Omega \partial_{tt}^2 u_\varepsilon(t, x) \sigma'_\eta(\partial_t u_\varepsilon(t, x)) \, dx = \frac{d}{dt} \int_\Omega \int_0^{\partial_t u_\varepsilon(t, x)} \sigma'_\eta(v) \, dv \, dx \stackrel{\eta \rightarrow 0}{=} \frac{d}{dt} \int_\Omega |\partial_t u_\varepsilon(t, x)| \, dx. \quad (3.5.4)$$

Concerning the second term on the first line of (3.5.3), we have

$$\begin{aligned} &\int_\Omega \nabla \cdot \partial_t f(t, x, u_\varepsilon(t, x)) \sigma'_\eta(\partial_t u_\varepsilon(t, x)) \, dx \\ &= \int_\Omega \left[\operatorname{div} \partial_t f(t, x, u_\varepsilon(t, x)) + \partial_u \partial_t f(t, x, u_\varepsilon(t, x)) \cdot \nabla u_\varepsilon(t, x) \right] \sigma'_\eta(\partial_t u_\varepsilon(t, x)) \, dx \\ &\geq -\mathcal{L}^n(\Omega) \|\operatorname{div} \partial_t f\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} - \|\nabla u_\varepsilon(t)\|_{\mathbf{L}^1(\Omega; \mathbb{R}^n)} \|\partial_u \partial_t f\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} \end{aligned} \quad (3.5.5)$$

and

$$\begin{aligned}
& \int_{\Omega} \nabla \cdot \left(\partial_u f(t, x, u_{\varepsilon}(t, x)) \partial_t u_{\varepsilon}(t, x) \right) \sigma'_{\eta}(\partial_t u_{\varepsilon}(t, x)) \, dx \\
&= \int_{\partial\Omega} \sigma'_{\eta}(\partial_t u_{\varepsilon}(t, \xi)) \partial_u f(t, \xi, u_{\varepsilon}(t, \xi)) \cdot \nu(\xi) \partial_t u_{\varepsilon}(t, \xi) \, d\xi \\
&\quad - \int_{\Omega} \partial_t u_{\varepsilon}(t, x) \partial_u f(t, x, u_{\varepsilon}(t, x)) \cdot \nabla \partial_t u_{\varepsilon}(t, x) \sigma''_{\eta}(\partial_t u_{\varepsilon}(t, x)) \, dx \\
&\geq - \|\partial_u f\|_{\mathbf{L}^{\infty}([0, t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} \int_{\partial\Omega} |\partial_t u_{\varepsilon}(t, \xi)| \, d\xi \\
&\quad - \|\partial_u f\|_{\mathbf{L}^{\infty}([0, t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} \int_{|\partial_t u_{\varepsilon}| \leq \eta} |\partial_t u_{\varepsilon}(t, x)| \nabla \left(\sigma'_{\eta}(\partial_t u_{\varepsilon}(t, x)) \right) \, dx \\
&= - \|\partial_u f\|_{\mathbf{L}^{\infty}([0, t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} \int_{\partial\Omega} |\partial_t u_b(t, \xi)| \, d\xi \\
&\quad - \|\partial_u f\|_{\mathbf{L}^{\infty}([0, t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} \int_{|\partial_t u_{\varepsilon}| \leq \eta} \nabla \partial_t u_{\varepsilon}(t, x) \, dx \\
&\stackrel{\eta \rightarrow 0}{=} - \|\partial_u f\|_{\mathbf{L}^{\infty}([0, t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} \int_{\partial\Omega} |\partial_t u_b(t, \xi)| \, d\xi \\
&\stackrel{u_b=0}{=} 0,
\end{aligned}$$

where, in the last limit, we used [10, Lemma 2].

To estimate the first two terms on the second line of (3.5.3), we compute:

$$\begin{aligned}
& \int_{\Omega} \left(\partial_t F(t, x, u_{\varepsilon}(t, x)) + \partial_u F(t, x, u_{\varepsilon}(t, x)) \partial_t u_{\varepsilon}(t, x) \right) \sigma'_{\eta}(\partial_t u_{\varepsilon}(t, x)) \, dx \\
&\leq \mathcal{L}^n(\Omega) \|\partial_t F\|_{\mathbf{L}^{\infty}([0, t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} + \|\partial_u F\|_{\mathbf{L}^{\infty}([0, t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} \|\partial_t u_{\varepsilon}(t)\|_{\mathbf{L}^1(\Omega; \mathbb{R}^n)}.
\end{aligned} \tag{3.5.6}$$

To bound the last term on the second line of (3.5.3), we proceed as follows:

$$\begin{aligned}
& \varepsilon \int_{\Omega} \partial_t \Delta u_{\varepsilon}(t, x) \sigma'_{\eta}(\partial_t u_{\varepsilon}(t, x)) \, dx \\
&= \varepsilon \int_{\partial\Omega} \partial_t \nabla u_{\varepsilon}(t, \xi) \cdot \nu(\xi) \sigma'_{\eta}(\partial_t u_{\varepsilon}(t, \xi)) \, d\xi - \int_{\Omega} \|\partial_t \nabla u_{\varepsilon}(t, x)\|^2 \sigma''_{\eta}(\partial_t u_{\varepsilon}(t, x)) \, dx \\
&\leq \varepsilon \int_{\partial\Omega} \partial_t \nabla u_{\varepsilon}(t, \xi) \cdot \nu(\xi) \sigma'_{\eta}(\partial_t u_{\varepsilon}(t, \xi)) \, d\xi \\
&= \varepsilon \int_{\partial\Omega} \nabla \sigma_{\eta}(\partial_t u_{\varepsilon}(t, \xi)) \cdot \nu(\xi) \, d\xi \\
&\stackrel{u_b=0}{=} 0.
\end{aligned}$$

Integrate (3.5.3) in time over $[0, t]$, using (3.5.4), (3.5.5) and (3.5.6) to obtain

$$\begin{aligned}
& \|\partial_t u_{\varepsilon}(t)\|_{\mathbf{L}^1(\Omega; \mathbb{R})} \\
&\leq \|\partial_t u_{\varepsilon}(0)\|_{\mathbf{L}^1(\Omega; \mathbb{R})} \\
&\quad + \mathcal{L}^n(\Omega) t \|\operatorname{div} \partial_t f\|_{\mathbf{L}^{\infty}([0, t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} \\
&\quad + \int_0^t \|\partial_u \partial_t f\|_{\mathbf{L}^{\infty}([0, \tau] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} \|\nabla u_{\varepsilon}(\tau)\|_{\mathbf{L}^1(\Omega; \mathbb{R}^n)} \, d\tau
\end{aligned}$$

$$\begin{aligned}
& + \mathcal{L}^n(\Omega) t \|\partial_t F\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} \\
& + \int_0^t \|\partial_u F\|_{\mathbf{L}^\infty([0,\tau] \times \Omega \times \mathcal{U}(\tau); \mathbb{R})} \|\partial_t u_\varepsilon(\tau)\|_{\mathbf{L}^1(\Omega; \mathbb{R})} d\tau \\
& \leq \|\partial_t u_\varepsilon(0)\|_{\mathbf{L}^1(\Omega; \mathbb{R})} + \mathcal{L}^n(\Omega) t \left(\|\operatorname{div} \partial_t f\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} + \|\partial_t F\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} \right) \\
& + \left(\|\partial_u \partial_t f\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} + \|\partial_u F\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} \right) \\
& \quad \times \int_0^t \left(\|\partial_t u_\varepsilon(\tau)\|_{\mathbf{L}^1(\Omega; \mathbb{R})} + \|\nabla u_\varepsilon(\tau)\|_{\mathbf{L}^1(\Omega; \mathbb{R}^n)} \right) d\tau. \tag{3.5.7}
\end{aligned}$$

Using the parabolic equation (3.1.2), we can estimate the first term in the right hand side above as follows:

$$\begin{aligned}
\|\partial_t u_\varepsilon(0)\|_{\mathbf{L}^1(\Omega; \mathbb{R})} & \leq \|\partial_u f\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} \|\nabla u_o\|_{\mathbf{L}^1(\Omega; \mathbb{R}^n)} \\
& + \mathcal{L}^n(\Omega) \left(\|\operatorname{div} f\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} + \|F\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} \right) \\
& + \varepsilon \|\Delta u_o\|_{\mathbf{L}^1(\Omega; \mathbb{R})}. \tag{3.5.8}
\end{aligned}$$

As noted above, Δu_ε is of class $\mathbf{C}^1$ and, for $j = 1, \dots, n$, we can differentiate the equation in (3.1.2) with respect to x_j to obtain

$$\partial_t \partial_j u_\varepsilon(t, x) + \nabla \cdot \frac{d}{dx_j} f(t, x, u_\varepsilon(t, x)) = \frac{d}{dx_j} F(t, x, u_\varepsilon(t, x)) + \varepsilon \Delta \partial_j u_\varepsilon(t, x). \tag{3.5.9}$$

Multiply by $\sigma'_\eta(\partial_j u_\varepsilon)$ and integrate each term in (3.5.9) over Ω :

$$\begin{aligned}
& \int_\Omega \partial_t \partial_j u_\varepsilon(t, x) \sigma'_\eta(\partial_j u_\varepsilon(t, x)) dx + \int_\Omega \nabla \cdot \frac{d}{dx_j} f(t, x, u_\varepsilon(t, x)) \sigma'_\eta(\partial_j u_\varepsilon(t, x)) dx \\
& = \int_\Omega \frac{d}{dx_j} F(t, x, u_\varepsilon(t, x)) \sigma'_\eta(\partial_j u_\varepsilon(t, x)) dx + \varepsilon \int_\Omega \Delta \partial_j u_\varepsilon(t, x) \sigma'_\eta(\partial_j u_\varepsilon(t, x)) dx. \tag{3.5.10}
\end{aligned}$$

Consider each term of (3.5.10) separately. The first term becomes

$$\int_\Omega \partial_t \partial_j u_\varepsilon(t, x) \sigma'_\eta(\partial_j u_\varepsilon(t, x)) dx = \int_\Omega \frac{d}{dt} \sigma_\eta(\partial_j u_\varepsilon(t, x)) dx = \frac{d}{dt} \int_\Omega \sigma_\eta(\partial_j u_\varepsilon(t, x)) dx. \tag{3.5.11}$$

To estimate the second term in the left hand side of (3.5.10), we follow [28, Chapter 6, Proof of Lemma 6.9.5], use the equality $\nabla \partial_j u_\varepsilon(t, x) \sigma'_\eta(\partial_j u_\varepsilon(t, x)) = \nabla \sigma'_\eta(\partial_j u_\varepsilon(t, x))$ and the Divergence Theorem:

$$\begin{aligned}
& \int_\Omega \nabla \cdot \frac{d}{dx_j} f(t, x, u_\varepsilon(t, x)) \sigma'_\eta(\partial_j u_\varepsilon(t, x)) dx \\
& = \int_\Omega \nabla \cdot \partial_j f(t, x, u_\varepsilon(t, x)) \sigma'_\eta(\partial_j u_\varepsilon(t, x)) dx \\
& \quad + \int_\Omega \nabla \cdot \left(\partial_u f(t, x, u_\varepsilon(t, x)) \partial_j u_\varepsilon(t, x) \right) \sigma'_\eta(\partial_j u_\varepsilon(t, x)) dx \\
& = \int_\Omega \operatorname{div} \partial_j f(t, x, u_\varepsilon(t, x)) \sigma'_\eta(\partial_j u_\varepsilon(t, x)) dx \\
& \quad + \int_\Omega \partial_u \partial_j f(t, x, u_\varepsilon(t, x)) \partial_j u_\varepsilon(t, x) \sigma'_\eta(\partial_j u_\varepsilon(t, x)) dx
\end{aligned}$$

$$\begin{aligned}
& + \int_{\Omega} \nabla \cdot \left(\partial_u f(t, x, u_{\varepsilon}(t, x)) \right) \partial_j u_{\varepsilon}(t, x) \sigma'_{\eta}(\partial_j u_{\varepsilon}(t, x)) \, dx \\
& + \int_{\Omega} \partial_u f(t, x, u_{\varepsilon}(t, x)) \cdot \nabla \sigma_{\eta}(\partial_j u_{\varepsilon}(t, x)) \, dx \\
& \geq -\mathcal{L}^n(\Omega) \|\nabla \operatorname{div} f\|_{\mathbf{L}^{\infty}([0, t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} \\
& \quad - \|\nabla \partial_u f\|_{\mathbf{L}^{\infty}([0, t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^{n \times n})} \int_{\Omega} \left| \partial_j u_{\varepsilon}(t, x) \sigma'_{\eta}(\partial_j u_{\varepsilon}(t, x)) \right| \, dx \\
& \quad + \int_{\Omega} \nabla \cdot \left(\partial_u f(t, x, u_{\varepsilon}(t, x)) \right) \partial_j u_{\varepsilon}(t, x) \sigma'_{\eta}(\partial_j u_{\varepsilon}(t, x)) \, dx \\
& \quad - \int_{\Omega} \nabla \cdot \left(\partial_u f(t, x, u_{\varepsilon}(t, x)) \right) \sigma_{\eta}(\partial_j u_{\varepsilon}(t, x)) \, dx \\
& \quad + \int_{\partial\Omega} \sigma_{\eta}(\partial_j u_{\varepsilon}(t, \xi)) \partial_u f(t, \xi, u_{\varepsilon}(t, \xi)) \cdot \nu(\xi) \, d\xi \\
& \geq -\mathcal{L}^n(\Omega) \|\nabla \operatorname{div} f\|_{\mathbf{L}^{\infty}([0, t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} \tag{3.5.12}
\end{aligned}$$

$$\begin{aligned}
& - \|\nabla \partial_u f\|_{\mathbf{L}^{\infty}([0, t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^{n \times n})} \int_{\Omega} |\partial_j u_{\varepsilon}(t, x)| \, dx \tag{3.5.13}
\end{aligned}$$

$$\begin{aligned}
& + \int_{\Omega} \nabla \cdot \left[\partial_u f(t, x, u_{\varepsilon}(t, x)) \right] \left[\partial_j u_{\varepsilon}(t, x) \sigma'_{\eta}(\partial_j u_{\varepsilon}(t, x)) - \sigma_{\eta}(\partial_j u_{\varepsilon}(t, x)) \right] \, dx \tag{3.5.14}
\end{aligned}$$

$$\begin{aligned}
& + \int_{\partial\Omega} \sigma_{\eta}(\partial_j u_{\varepsilon}(t, \xi)) \partial_u f(t, \xi, u_{\varepsilon}(t, \xi)) \cdot \nu(\xi) \, d\xi. \tag{3.5.15}
\end{aligned}$$

For later use, note that, for $\xi \in \partial\Omega$, (3.1.2) is the equality

$$\partial_u f(t, \xi, u_{\varepsilon}(t, \xi)) \cdot \nu(\xi) \partial_{\nu} u_{\varepsilon}(t, \xi) = \varepsilon \Delta u_{\varepsilon}(t, \xi) + F(t, \xi, u_{\varepsilon}(t, \xi)) - \operatorname{div} f(t, \xi, u_{\varepsilon}(t, \xi)).$$

Hence, thanks also to the fact that

$$\partial_{\nu} u_{\varepsilon} \nu_j = \partial_j u_{\varepsilon}, \tag{3.5.16}$$

we can now elaborate (3.5.15) as follows:

$$\begin{aligned}
& \int_{\partial\Omega} \sigma_{\eta}(\partial_j u_{\varepsilon}(t, \xi)) \partial_u f(t, \xi, u_{\varepsilon}(t, \xi)) \cdot \nu(\xi) \, d\xi \\
& = \int_{\partial\Omega} \frac{\sigma_{\eta}(\partial_j u_{\varepsilon}(t, \xi))}{\partial_j u_{\varepsilon}(t, \xi)} \left(\varepsilon \Delta u_{\varepsilon}(t, \xi) + F(t, \xi, u_{\varepsilon}(t, \xi)) - \operatorname{div} f(t, \xi, u_{\varepsilon}(t, \xi)) \right) \nu_j(\xi) \, d\xi. \tag{3.5.17}
\end{aligned}$$

Here we used the fact that $\sigma_{\eta}(z) = o(z)$ for $z \rightarrow 0$, so that the map $z \rightarrow \sigma_{\eta}(z)/z$ is well defined also at $z = 0$. Pass now to the first term in the right hand side of (3.5.10):

$$\begin{aligned}
& \int_{\Omega} \frac{d}{dx_j} F(t, x, u_{\varepsilon}(t, x)) \sigma'_{\eta}(\partial_j u_{\varepsilon}(t, x)) \, dx \\
& = \int_{\Omega} \left(\partial_j F(t, x, u_{\varepsilon}(t, x)) + \partial_u F(t, x, u_{\varepsilon}(t, x)) \partial_j u_{\varepsilon}(t, x) \right) \sigma'_{\eta}(\partial_j u_{\varepsilon}(t, x)) \, dx \\
& = \int_{\Omega} \partial_j F(t, x, u_{\varepsilon}(t, x)) \sigma'_{\eta}(\partial_j u_{\varepsilon}(t, x)) \, dx \\
& \quad + \int_{\Omega} \partial_u F(t, x, u_{\varepsilon}(t, x)) \partial_j u_{\varepsilon}(t, x) \sigma'_{\eta}(\partial_j u_{\varepsilon}(t, x)) \, dx
\end{aligned}$$

$$\leq \mathcal{L}^n(\Omega) \|\nabla F\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} + \|\partial_u F\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} \int_{\Omega} |\partial_j u_\varepsilon(t, x)| \, dx, \quad (3.5.18)$$

while the last term on the right hand side of (3.5.10) gives

$$\begin{aligned} & \varepsilon \int_{\Omega} \Delta \partial_j u_\varepsilon(t, x) \sigma'_\eta(\partial_j u_\varepsilon(t, x)) \, dx \\ &= \varepsilon \int_{\Omega} \operatorname{div} \nabla \partial_j u_\varepsilon(t, x) \sigma'_\eta(\partial_j u_\varepsilon(t, x)) \, dx \\ &= \varepsilon \int_{\partial\Omega} \sigma'_\eta(\partial_j u_\varepsilon(t, \xi)) \nabla \partial_j u_\varepsilon(t, \xi) \cdot \nu(\xi) \, d\xi - \varepsilon \int_{\Omega} \nabla \partial_j u_\varepsilon(t, x) \cdot \nabla \sigma'_\eta(\partial_j u_\varepsilon(t, x)) \, dx \\ &= \varepsilon \int_{\partial\Omega} \sigma'_\eta(\partial_j u_\varepsilon(t, \xi)) \nabla \partial_j u_\varepsilon(t, \xi) \cdot \nu(\xi) \, d\xi - \varepsilon \int_{\Omega} \sigma''_\eta(\partial_j u_\varepsilon(t, x)) \|\nabla \partial_j u_\varepsilon(t, x)\| \, dx \\ &\leq \varepsilon \int_{\partial\Omega} \sigma'_\eta(\partial_j u_\varepsilon(t, \xi)) \nabla \partial_j u_\varepsilon(t, \xi) \cdot \nu(\xi) \, d\xi \\ &\leq \varepsilon \int_{\partial\Omega} \sigma'_\eta(\partial_j u_\varepsilon(t, \xi)) \partial_\nu(\partial_j u_\varepsilon(t, \xi)) \, d\xi. \end{aligned} \quad (3.5.19)$$

Integrate (3.5.10) in time over $[0, t]$, and use (3.5.11), (3.5.12)–(3.5.14), (3.5.17), (3.5.18) and (3.5.19) to obtain

$$\begin{aligned} & \int_{\Omega} \sigma_\eta(\partial_j u_\varepsilon(t, x)) \, dx \quad (3.5.20) \\ &\leq \int_{\Omega} \sigma_\eta(\partial_j u_\varepsilon(0, x)) \, dx \\ &\quad + \mathcal{L}^n(\Omega) t \|\nabla \operatorname{div} f\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} \\ &\quad + \int_0^t \|\nabla \partial_u f\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^{n \times n})} \|\partial_j u_\varepsilon(\tau)\|_{\mathbf{L}^1(\Omega; \mathbb{R})} \, d\tau \\ &\quad - \int_0^t \int_{\Omega} \nabla \cdot [\partial_u f(\tau, x, u_\varepsilon(\tau, x))] [\partial_j u_\varepsilon(\tau, x) \sigma'_\eta(\partial_j u_\varepsilon(\tau, x)) - \sigma_\eta(\partial_j u_\varepsilon(\tau, x))] \, dx \, d\tau \\ &\quad - \int_0^t \int_{\partial\Omega} \frac{\sigma_\eta(\partial_j u_\varepsilon(\tau, \xi))}{\partial_j u_\varepsilon(\tau, \xi)} \left(\varepsilon \Delta u_\varepsilon(\tau, \xi) + F(\tau, \xi, u_\varepsilon(\tau, \xi)) - \operatorname{div} f(\tau, \xi, u_\varepsilon(\tau, \xi)) \right) \nu_j(\xi) \, d\xi \, d\tau \\ &\quad + \mathcal{L}^n(\Omega) t \|\nabla F\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} \\ &\quad + \int_0^t \|\partial_u F\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} \|\partial_j u_\varepsilon(\tau)\|_{\mathbf{L}^1(\Omega; \mathbb{R})} \, d\tau \\ &\quad + \int_0^t \int_{\partial\Omega} \varepsilon \sigma'_\eta(\partial_j u_\varepsilon(\tau, \xi)) \partial_\nu(\partial_j u_\varepsilon(\tau, \xi)) \, d\xi \, d\tau \\ &\leq \int_{\Omega} \sigma_\eta(\partial_j u_\varepsilon(0, x)) \, dx \\ &\quad + \mathcal{L}^n(\Omega) t \left(\|\nabla \operatorname{div} f\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} + \|\nabla F\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} \right) \\ &\quad + \left(\|\nabla \partial_u f\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^{n \times n})} + \|\partial_u F\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} \right) \int_0^t \|\partial_j u_\varepsilon(\tau)\|_{\mathbf{L}^1(\Omega; \mathbb{R})} \, d\tau \\ &\quad - \int_0^t \int_{\Omega} \nabla \cdot [\partial_u f(\tau, x, u_\varepsilon(\tau, x))] [\partial_j u_\varepsilon(\tau, x) \sigma'_\eta(\partial_j u_\varepsilon(\tau, x)) - \sigma_\eta(\partial_j u_\varepsilon(\tau, x))] \, dx \, d\tau \quad (3.5.21) \\ &\quad + \varepsilon \int_0^t \int_{\partial\Omega} \left(\sigma'_\eta(\partial_j u_\varepsilon(\tau, \xi)) \partial_\nu(\partial_j u_\varepsilon(\tau, \xi)) - \frac{\sigma_\eta(\partial_j u_\varepsilon(\tau, \xi))}{\partial_j u_\varepsilon(\tau, \xi)} \Delta u_\varepsilon(\tau, \xi) \nu_j(\xi) \right) \, d\xi \, d\tau \quad (3.5.22) \end{aligned}$$

$$- \int_0^t \int_{\partial\Omega} \frac{\sigma_\eta(\partial_j u_\varepsilon(\tau, \xi))}{\partial_j u_\varepsilon(\tau, \xi)} (F(\tau, \xi, 0) - \operatorname{div} f(\tau, \xi, 0)) \nu_j(\xi) \, d\xi \, d\tau. \quad (3.5.23)$$

To compute the limit $\eta \rightarrow 0$, consider first the latter three terms above separately:

$$\lim_{\eta \rightarrow 0} [(3.5.21)] = 0 \quad (3.5.24)$$

Concerning (3.5.22), following [28, Proof of Lemma 6.9.5], it is useful to recall the following relations, based on (3.5.16):

$$\partial_\nu(\partial_j u_\varepsilon) = \partial_{\nu\nu}^2 u_\varepsilon \nu_j + \mathcal{O}(1) \partial_\nu u_\varepsilon \quad \text{and} \quad \Delta u_\varepsilon = \partial_{\nu\nu}^2 u_\varepsilon + \mathcal{O}(1) \partial_\nu u_\varepsilon. \quad (3.5.25)$$

Here and in what follows, by $\mathcal{O}(1)$ we denote a constant dependent only on the geometry of Ω . In particular, $\mathcal{O}(1)$ is independent of the flow f , of the source F and of the initial datum u_o . Then, using (3.5.25) and the boundedness of $\sigma_\eta(z)/z$ and of σ'_η ,

$$\begin{aligned} & \sigma'_\eta(\partial_j u_\varepsilon(\tau, \xi)) \partial_\nu(\partial_j u_\varepsilon(t, \xi)) - \frac{\sigma_\eta(\partial_j u_\varepsilon(\tau, \xi))}{\partial_j u_\varepsilon(\tau, \xi)} \Delta u_\varepsilon(\tau, \xi) \nu_j(\xi) \\ &= \left(\sigma'_\eta(\partial_j u_\varepsilon(\tau, \xi)) - \frac{\sigma_\eta(\partial_j u_\varepsilon(\tau, \xi))}{\partial_j u_\varepsilon(\tau, \xi)} \right) \partial_{\nu\nu}^2 u_\varepsilon(\tau, \xi) \nu_j(\xi) + \mathcal{O}(1) \partial_\nu u_\varepsilon(\tau, \xi), \end{aligned}$$

whence, by [9, Lemma A.3], see also [38, Chapter 4],

$$\begin{aligned} \lim_{\eta \rightarrow 0} [(3.5.22)] &= \mathcal{O}(1) \varepsilon \int_0^t \int_{\partial\Omega} \partial_\nu u_\varepsilon(\tau, \xi) \, d\xi \, d\tau \\ &\leq \mathcal{O}(1) \varepsilon \int_0^t \int_{\partial\Omega} \|\nabla u_\varepsilon(\tau, \xi)\| \, d\xi \, d\tau \\ &\leq \mathcal{O}(1) \varepsilon \int_0^t \int_{\Omega} |\Delta u_\varepsilon(\tau, x)| \, dx \, d\tau \\ &\leq \mathcal{O}(1) \int_0^t \int_{\Omega} |\partial_t u_\varepsilon(\tau, x)| \, dx \, d\tau \\ &\quad + \mathcal{O}(1) \|\partial_u f\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} \int_0^t \int_{\Omega} \|\nabla u_\varepsilon(\tau, x)\| \, dx \, d\tau \\ &\quad + \mathcal{O}(1) \mathcal{L}^n(\Omega) t \left(\|\operatorname{div} f\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} + \|F\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} \right). \end{aligned} \quad (3.5.26)$$

Passing to (3.5.23), we have the following estimate that holds uniformly in η :

$$[(3.5.23)] \leq \mathcal{H}^{n-1}(\partial\Omega) t \left(\|\operatorname{div} f(\cdot, \cdot, 0)\|_{\mathbf{L}^\infty([0,t] \times \partial\Omega; \mathbb{R})} + \|F(\cdot, \cdot, 0)\|_{\mathbf{L}^\infty([0,t] \times \partial\Omega; \mathbb{R})} \right). \quad (3.5.27)$$

Insert now (3.5.24), (3.5.26) and (3.5.27) in (3.5.20)–(3.5.23) to obtain

$$\begin{aligned} & \int_{\Omega} |\partial_j u_\varepsilon(t, x)| \, dx \\ &\leq \int_{\Omega} |\partial_j u_\varepsilon(0, x)| \, dx \\ &\quad + \mathcal{L}^n(\Omega) t \left(\|\nabla \operatorname{div} f\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} + \|\nabla F\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} \right) \end{aligned}$$

$$\begin{aligned}
& + \mathcal{O}(1) \mathcal{L}^n(\Omega) t \left(\|\operatorname{div} f\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} + \|F\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} \right) \\
& + \mathcal{H}^{n-1}(\partial\Omega) t \left(\|\operatorname{div} f(\cdot, \cdot, 0)\|_{\mathbf{L}^\infty([0,t] \times \partial\Omega; \mathbb{R})} + \|F(\cdot, \cdot, 0)\|_{\mathbf{L}^\infty([0,t] \times \partial\Omega; \mathbb{R})} \right) \\
& + \left(\|\nabla \partial_u f\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^{n \times n})} + \|\partial_u F\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} \right) \int_0^t \|\partial_j u_\varepsilon(\tau)\|_{\mathbf{L}^1(\Omega; \mathbb{R})} d\tau \\
& + \mathcal{O}(1) \int_0^t \|\partial_t u_\varepsilon(\tau)\|_{\mathbf{L}^1(\Omega; \mathbb{R})} d\tau + \mathcal{O}(1) \|\partial_u f\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} \int_0^t \|\nabla u_\varepsilon(\tau)\|_{\mathbf{L}^1(\Omega; \mathbb{R}^n)} d\tau.
\end{aligned}$$

Summing over $j = 1, \dots, n$ and using the notation $\mathcal{O}(1)$, we get

$$\begin{aligned}
& \|\nabla u_\varepsilon(t)\|_{\mathbf{L}^1(\Omega; \mathbb{R}^n)} \\
& \leq \sqrt{n} \|\nabla u_\varepsilon(0)\|_{\mathbf{L}^1(\Omega; \mathbb{R}^n)}
\end{aligned} \tag{3.5.28}$$

$$\begin{aligned}
& + \mathcal{O}(1) t \left[\|\nabla \operatorname{div} f\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} + \|\nabla F\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} \right. \\
& \quad + \|\operatorname{div} f\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} + \|F\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} \\
& \quad \left. + \|\operatorname{div} f(\cdot, \cdot, 0)\|_{\mathbf{L}^\infty([0,t] \times \partial\Omega; \mathbb{R})} + \|F(\cdot, \cdot, 0)\|_{\mathbf{L}^\infty([0,t] \times \partial\Omega; \mathbb{R})} \right] \\
& + \mathcal{O}(1) \int_0^t \|\nabla u_\varepsilon(\tau)\|_{\mathbf{L}^1(\Omega; \mathbb{R}^n)} d\tau \\
& \times \left[\|\nabla \partial_u f\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^{n \times n})} + \|\partial_u F\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} + \|\partial_u f\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} \right] \\
& + \mathcal{O}(1) \int_0^t \|\partial_t u_\varepsilon(\tau)\|_{\mathbf{L}^1(\Omega; \mathbb{R})} d\tau.
\end{aligned} \tag{3.5.29}$$

Summing the inequalities (3.5.7), (3.5.8) and (3.5.28)–(3.5.29) we obtain the estimate

$$\begin{aligned}
& \|\partial_t u_\varepsilon(t)\|_{\mathbf{L}^1(\Omega; \mathbb{R})} + \|\nabla u_\varepsilon(t)\|_{\mathbf{L}^1(\Omega; \mathbb{R}^n)} \\
& \leq A_1 + A_2 t + A_3 \|\nabla u_o\|_{\mathbf{L}^1(\Omega; \mathbb{R}^n)} + \varepsilon \|\Delta u_o\|_{\mathbf{L}^1(\Omega; \mathbb{R})} \\
& \quad + A_4 \int_0^t \left[\|\partial_t u_\varepsilon(\tau)\|_{\mathbf{L}^1(\Omega; \mathbb{R})} + \|\nabla u_\varepsilon(\tau)\|_{\mathbf{L}^1(\Omega; \mathbb{R}^n)} \right] d\tau,
\end{aligned}$$

where

$$A_1 = \mathcal{O}(1) \left(\|\operatorname{div} f\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} + \|F\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} \right) \tag{3.5.30}$$

$$\begin{aligned}
A_2 = \mathcal{O}(1) \left[\|\nabla \operatorname{div} f\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} + \|\nabla F\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} \right. \\
+ \|\operatorname{div} f\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} + \|F\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} \\
+ \|\operatorname{div} \partial_t f\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} + \|\partial_t F\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} \\
\left. + \|\operatorname{div} f(\cdot, \cdot, 0)\|_{\mathbf{L}^\infty([0,t] \times \partial\Omega; \mathbb{R})} + \|F(\cdot, \cdot, 0)\|_{\mathbf{L}^\infty([0,t] \times \partial\Omega; \mathbb{R})} \right]
\end{aligned}$$

$$A_3 = \mathcal{O}(1) + \|\partial_u f\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)}$$

$$\begin{aligned}
A_4 = \mathcal{O}(1) \left[1 + \|\partial_t \partial_u f\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} + \|\partial_u F\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} \right. \\
\left. + \|\nabla \partial_u f\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^{n \times n})} + \|\partial_u f\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} \right].
\end{aligned} \tag{3.5.31}$$

Note that the A_i are increasing with t . Hence, an application of Gronwall Lemma yields

$$\begin{aligned} & \|\partial_t u_\varepsilon(t)\|_{\mathbf{L}^1(\Omega; \mathbb{R})} + \|\nabla u_\varepsilon(t)\|_{\mathbf{L}^1(\Omega; \mathbb{R}^n)} \\ & \leq \left(A_1 + A_2 t + A_3 \|\nabla u_o\|_{\mathbf{L}^1(\Omega; \mathbb{R}^n)} + \varepsilon \|\Delta u_o\|_{\mathbf{L}^1(\Omega; \mathbb{R})} \right) e^{A_4 t}. \end{aligned}$$

From the inequality above, (3.3.5) follows easily, introducing the notation (3.3.7). Noting that

$$\|u_\varepsilon(t) - u_\varepsilon(s)\|_{\mathbf{L}^1(\Omega; \mathbb{R})} \leq \int_s^t \|\partial_t u_\varepsilon(\tau)\|_{\mathbf{L}^1(\Omega; \mathbb{R})} d\tau,$$

we obtain (3.3.6), concluding the proof. $\square$

3.5.2 Proofs Related to the Hyperbolic Problem

Proof of Proposition 3.12. The family u_ε of solutions to (3.1.2) as constructed in Lemma 3.11 is uniformly bounded in $\mathbf{L}^1(I \times \Omega; \mathbb{R})$ by (3.3.3). It is also totally bounded in $\mathbf{L}^1(I \times \Omega; \mathbb{R})$ thanks to [41, Corollary 8], which can be applied by (3.3.5).

To prove that cluster point of the u_ε is a solution to (3.1.1) in the sense of Definition 3.1, we introduce $k \in \mathbb{R}$ and a test function $\varphi \in \mathbf{C}_c^2([-\infty, T] \times \mathbb{R}^n; \mathbb{R}^+)$. We multiply equation (3.1.2) by $\sigma'_\eta(u_\varepsilon(t, x) - k) \varphi(t, x)$, with $\eta > 0$ and σ'_η as in (3.5.2). Then, we integrate over $I \times \Omega$:

$$\begin{aligned} & \int_I \int_\Omega \left(\partial_t u_\varepsilon(t, x) + \nabla \cdot f(t, x, u_\varepsilon(t, x)) - F(t, x, u_\varepsilon(t, x)) \right) \sigma'_\eta(u_\varepsilon(t, x) - k) \varphi(t, x) dx dt \\ & = \int_I \int_\Omega \varepsilon \Delta u_\varepsilon(t, x) \sigma'_\eta(u_\varepsilon(t, x) - k) \varphi(t, x) dx dt. \end{aligned} \quad (3.5.32)$$

Consider each term in (3.5.32) separately. Integrate by part the first term:

$$\begin{aligned} & \int_I \int_\Omega \partial_t u_\varepsilon(t, x) \sigma'_\eta(u_\varepsilon(t, x) - k) \varphi(t, x) dx dt \\ & = \int_I \int_\Omega \frac{d}{dt} \sigma_\eta(u_\varepsilon(t, x) - k) \varphi(t, x) dx dt \\ & = - \int_\Omega \sigma_\eta(u_o(x) - k) \varphi(0, x) dx - \int_I \int_\Omega \sigma_\eta(u_\varepsilon(t, x) - k) \partial_t \varphi(t, x) dx dt. \end{aligned} \quad (3.5.33)$$

Concerning the second term in the left hand side of (3.5.32), first integrate by part, then add and subtract $\int_I \int_\Omega f(t, x, k) \cdot \nabla \left(\sigma'_\eta(u_\varepsilon(t, x) - k) \varphi(t, x) \right) dx dt$. After some rearrangements,

$$\begin{aligned} & \int_I \int_\Omega \nabla \cdot f(t, x, u_\varepsilon(t, x)) \sigma'_\eta(u_\varepsilon(t, x) - k) \varphi(t, x) dx dt \\ & = \int_I \int_{\partial\Omega} f(t, x, u_\varepsilon(t, \xi)) \sigma'_\eta(u_\varepsilon(t, \xi) - k) \varphi(t, \xi) \cdot \nu(\xi) d\xi dt \\ & \quad - \int_I \int_\Omega f(t, x, u_\varepsilon(t, x)) \cdot \nabla \left(\sigma'_\eta(u_\varepsilon(t, x) - k) \varphi(t, x) \right) dx dt \\ & = \int_I \int_{\partial\Omega} f(t, x, 0) \sigma'_\eta(-k) \varphi(t, \xi) \cdot \nu(\xi) d\xi dt \end{aligned}$$

$$\begin{aligned}
& - \int_I \int_{\Omega} \left(f(t, x, u_{\varepsilon}(t, x)) - f(t, x, k) \right) \cdot \nabla \left(\sigma'_{\eta}(u_{\varepsilon}(t, x) - k) \varphi(t, x) \right) dx dt \\
& - \int_I \int_{\Omega} f(t, x, k) \cdot \nabla \left(\sigma'_{\eta}(u_{\varepsilon}(t, x) - k) \varphi(t, x) \right) dx dt \\
& = \int_I \int_{\partial\Omega} f(t, x, 0) \sigma'_{\eta}(-k) \varphi(t, \xi) \cdot \nu(\xi) d\xi dt \\
& - \int_I \int_{\Omega} \left(f(t, x, u_{\varepsilon}(t, x)) - f(t, x, k) \right) \cdot \nabla \left(\sigma'_{\eta}(u_{\varepsilon}(t, x) - k) \varphi(t, x) \right) dx dt \\
& - \int_I \int_{\partial\Omega} f(t, \xi, k) \sigma'_{\eta}(-k) \varphi(t, \xi) \cdot \nu(\xi) d\xi dt \\
& + \int_I \int_{\Omega} \operatorname{div} f(t, x, k) \sigma'_{\eta}(u_{\varepsilon}(t, x) - k) \varphi(t, x) dx dt.
\end{aligned} \tag{3.5.34}$$

We do not modify the third term in the left hand side of (3.5.32). Passing to the right hand side of (3.5.32), we have:

$$\begin{aligned}
& \int_I \int_{\Omega} \varepsilon \Delta u_{\varepsilon}(t, x) \sigma'_{\eta}(u_{\varepsilon}(t, x) - k) \varphi(t, x) dx dt \\
& = \varepsilon \int_I \int_{\partial\Omega} \nabla u_{\varepsilon}(t, \xi) \sigma'_{\eta}(-k) \varphi(t, \xi) \cdot \nu(\xi) d\xi dt \\
& - \varepsilon \int_I \int_{\Omega} \nabla u_{\varepsilon}(t, x) \cdot \nabla \left(\sigma'_{\eta}(u_{\varepsilon}(t, x) - k) \varphi(t, x) \right) dx dt \\
& = \varepsilon \int_I \int_{\partial\Omega} \sigma'_{\eta}(-k) \varphi(t, \xi) \nabla u_{\varepsilon}(t, \xi) \cdot \nu(\xi) d\xi dt \\
& - \varepsilon \int_I \int_{\Omega} \sigma'_{\eta}(u_{\varepsilon}(t, x) - k) \nabla u_{\varepsilon}(t, x) \cdot \nabla \varphi(t, x) dx dt \\
& - \varepsilon \int_I \int_{\Omega} \|\nabla u_{\varepsilon}(t, x)\|^2 \sigma''_{\eta}(u_{\varepsilon}(t, x) - k) \varphi(t, x) dx dt.
\end{aligned} \tag{3.5.35}$$

Using (3.5.33), (3.5.34) and (3.5.35), equation (3.5.32) becomes

$$\begin{aligned}
& \int_I \int_{\Omega} \sigma_{\eta}(u_{\varepsilon}(t, x) - k) \partial_t \varphi(t, x) dx dt \\
& + \int_I \int_{\Omega} \sigma'_{\eta}(u_{\varepsilon}(t, x) - k) \left(f(t, x, u_{\varepsilon}(t, x)) - f(t, x, k) \right) \cdot \nabla \varphi(t, x) dx dt \\
& + \int_I \int_{\Omega} \sigma''_{\eta}(u_{\varepsilon}(t, x) - k) \left(f(t, x, u_{\varepsilon}(t, x)) - f(t, x, k) \right) \cdot \nabla u_{\varepsilon}(t, x) \varphi(t, x) dx dt \\
& + \int_I \int_{\Omega} \sigma'_{\eta}(u_{\varepsilon}(t, x) - k) \left(F(t, x, u_{\varepsilon}(t, x)) - \operatorname{div} f(t, x, k) \right) \varphi(t, x) dx dt \\
& + \int_{\Omega} \sigma_{\eta}(u_o(x) - k) \varphi(0, x) dx \\
& - \int_I \int_{\partial\Omega} \sigma'_{\eta}(-k) (f(t, \xi, 0) - f(t, \xi, k)) \varphi(t, \xi) \cdot \nu(\xi) d\xi dt \\
& = \varepsilon \int_I \int_{\Omega} \sigma'_{\eta}(u_{\varepsilon}(t, x) - k) \nabla u_{\varepsilon}(t, x) \cdot \nabla \varphi(t, x) dx dt \\
& + \varepsilon \int_I \int_{\Omega} \|\nabla u_{\varepsilon}(t, x)\|^2 \sigma''_{\eta}(u_{\varepsilon}(t, x) - k) \varphi(t, x) dx dt
\end{aligned} \tag{3.5.36}$$

$$\begin{aligned}
& - \int_I \int_{\Omega} F(t, x, u_{\varepsilon}(t, x)) \varphi(t, x) \Phi_h(x) \, dx \, dt \\
& + \varepsilon_m \int_I \int_{\Omega} \nabla u_{\varepsilon}(t, x) \cdot \nabla (\varphi(t, x) \Phi_h(x)) \, dx \, dt \\
& = \int_{\Omega} u_o(x) \varphi(0, x) \Phi_h(x) \, dx \\
& - \int_I \int_{\Omega} \left(u_{\varepsilon}(t, x) \partial_t \varphi(t, x) + f(t, x, u_{\varepsilon}(t, x)) \cdot \nabla \varphi(t, x) \right. \\
& \quad \left. + F(t, x, u_{\varepsilon}(t, x)) \varphi(t, x) - \varepsilon_m \nabla u_{\varepsilon}(t, x) \cdot \nabla \varphi(t, x) \right) \Phi_h(x) \, dx \, dt \quad (3.5.39) \\
& - \int_I \int_{\Omega} \left(f(t, x, u_{\varepsilon}(t, x)) - \varepsilon_m \nabla u_{\varepsilon}(t, x) \right) \cdot \nabla \Phi_h(x) \varphi(t, x) \, dx \, dt \\
& + \int_I \int_{\partial\Omega} f(t, \xi, 0) \varphi(t, \xi) \Phi_h(\xi) \cdot \nu(\xi) \, d\xi \, dt.
\end{aligned}$$

Let $m \rightarrow +\infty$:

$$\begin{aligned}
& \lim_{m \rightarrow +\infty} [(3.5.39)] \\
& = - \int_I \int_{\Omega} \left(u_{\infty}(t, x) \partial_t \varphi(t, x) + f(t, x, u_{\infty}(t, x)) \cdot \nabla \varphi(t, x) \right. \\
& \quad \left. + F(t, x, u_{\infty}(t, x)) \varphi(t, x) \right) \Phi_h(x) \, dx \, dt \quad (3.5.40) \\
& - \int_I \int_{\Omega} f(t, x, u_{\infty}(t, x)) \cdot \nabla \Phi_h(x) \varphi(t, x) \, dx \, dt \\
& + \int_{\Omega} u_o(x) \varphi(0, x) \Phi_h(x) \, dx + \int_I \int_{\partial\Omega} f(t, \xi, 0) \varphi(t, \xi) \cdot \nu(\xi) \, d\xi \, dt.
\end{aligned}$$

Now let $h \rightarrow 0$. Thanks to Lemma 3.22 and Lemma 3.20, we obtain

$$\lim_{h \rightarrow 0} [(3.5.40)] = \int_I \int_{\partial\Omega} \left(f(t, \xi, 0) - f(t, \xi, \text{tr } u_{\infty}(t, \xi)) \right) \varphi(t, \xi) \cdot \nu(\xi) \, d\xi \, dt.$$

Hence

$$\lim_{m \rightarrow +\infty} [(3.5.37)] = - \int_I \int_{\partial\Omega} \sigma'_{\eta}(-k) \left(f(t, \xi, 0) - f(t, \xi, \text{tr } u_{\infty}(t, \xi)) \right) \varphi(t, \xi) \cdot \nu(\xi) \, d\xi \, dt. \quad (3.5.41)$$

Therefore, in the limit $m \rightarrow +\infty$, we obtain that the equality (3.5.36)–(3.5.37) implies the inequality

$$\begin{aligned}
& \int_I \int_{\Omega} \sigma_{\eta}(u_{\infty}(t, x) - k) \partial_t \varphi(t, x) \, dx \, dt \\
& + \int_I \int_{\Omega} \sigma'_{\eta}(u_{\infty}(t, x) - k) \left(f(t, x, u_{\infty}(t, x)) - f(t, x, k) \right) \cdot \nabla \varphi(t, x) \, dx \, dt \\
& + \int_I \int_{\Omega} \sigma''_{\eta}(u_{\infty}(t, x) - k) \left(f(t, x, u_{\infty}(t, x)) - f(t, x, k) \right) \cdot \nabla u_{\infty}(t, x) \varphi(t, x) \, dx \, dt \\
& + \int_I \int_{\Omega} \sigma'_{\eta}(u_{\infty}(t, x) - k) \left(F(t, x, u_{\infty}(t, x)) - \text{div } f(t, x, k) \right) \varphi(t, x) \, dx \, dt \\
& + \int_{\Omega} \sigma_{\eta}(u_o(x) - k) \varphi(0, x) \, dx
\end{aligned}$$

$$\begin{aligned}
& - \int_I \int_{\partial\Omega} \sigma'_\eta(-k) (f(t, \xi, 0) - f(t, \xi, k)) \varphi(t, \xi) \cdot \nu(\xi) \, d\xi \, dt \\
& \geq \int_I \int_{\partial\Omega} \sigma'_\eta(-k) \left(f(t, \xi, 0) - f(t, \xi, \text{tr } u_\infty(t, \xi)) \right) \varphi(t, \xi) \cdot \nu(\xi) \, d\xi \, dt.
\end{aligned}$$

Let now $\eta \rightarrow 0$. Thanks to [10, Lemma 2], to the choice (3.5.2) of σ_η and of its derivative, we get

$$\begin{aligned}
& \int_I \int_{\Omega} |u_\infty(t, x) - k| \partial_t \varphi(t, x) \, dx \, dt \\
& + \int_I \int_{\Omega} \text{sgn}(u_\infty(t, x) - k) \left(f(t, x, u_\infty(t, x)) - f(t, x, k) \right) \cdot \nabla \varphi(t, x) \, dx \, dt \\
& + \int_I \int_{\Omega} \text{sgn}(u_\infty(t, x) - k) \left(F(t, x, u_\infty(t, x)) - \text{div } f(t, x, k) \right) \varphi(t, x) \, dx \, dt \\
& + \int_{\Omega} |u_o(x) - k| \varphi(0, x) \, dx \\
& - \int_I \int_{\partial\Omega} \text{sgn}(-k) \left(f(t, \xi, \text{tr } u_\infty(t, \xi)) - f(t, \xi, k) \right) \varphi(t, \xi) \cdot \nu(\xi) \, d\xi \, dt \\
& \geq 0,
\end{aligned}$$

that is (3.2.1) in the case $u_b = 0$. Hence u_∞ is a solution to (3.1.1) in the sense of Definition 3.1.

As a consequence of (3.3.3) in Lemma 3.10, u_∞ satisfies the $\mathbf{L}^\infty$ estimate

$$\|u_\infty\|_{\mathbf{L}^\infty([0, t] \times \Omega; \mathbb{R})} \leq \left(\|u_o\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} + c_2 t \right) e^{c_1 t},$$

where c_1, c_2 are defined in (3.5.1). Thanks to the lower semicontinuity in $\mathbf{L}^1$ of the total variation, see [5, Remark 3.5], the bound (3.3.5) in Lemma 3.11 gives

$$\text{TV}(u_\infty(t)) \leq \liminf_{\varepsilon \rightarrow 0} \text{TV}(u_\varepsilon(t)) \leq \liminf_{\varepsilon \rightarrow 0} \mathcal{L}_\varepsilon(t) = \mathcal{L}(t)$$

with $\mathcal{L}_\varepsilon(t)$ and $\mathcal{L}(t)$ as defined in (3.3.7) and (3.4.2).

From (3.3.6) in Lemma 3.11, we have for $t, s \in I$

$$\begin{aligned}
\|u_\infty(t) - u_\infty(s)\|_{\mathbf{L}^1(\Omega; \mathbb{R})} &= \lim_{\varepsilon \rightarrow 0} \|u_\varepsilon(t) - u_\varepsilon(s)\|_{\mathbf{L}^1(\Omega; \mathbb{R})} \\
&\leq \lim_{\varepsilon \rightarrow 0} \mathcal{L}_\varepsilon(\max\{t, s\}) |t - s| \\
&= \mathcal{L}(\max\{t, s\}) |t - s|,
\end{aligned}$$

concluding the proof. $\square$

The following Lemma will be of use in the proof of Theorem 3.13.

Lemma 3.15. *Let $k \in \mathbb{N}$ with $k \geq 2$, Ω satisfy $(\Omega_{k, \alpha})$ and fix $\psi \in \mathbf{C}^{k, \alpha}(I \times \partial\Omega; \mathbb{R})$. Then, the elliptic problem*

$$\begin{cases} \Delta z(t, x) = 0 & (t, x) \in I \times \Omega \\ z(t, \xi) = \psi(t, \xi) & (t, \xi) \in I \times \partial\Omega \end{cases} \quad (3.5.42)$$

admits a unique solution $z \in \mathbf{C}^{k, \alpha}(I \times \bar{\Omega}; \mathbb{R})$. Moreover,

$$\|z\|_{\mathbf{L}^\infty([0, t] \times \Omega; \mathbb{R})} \leq \|\psi\|_{\mathbf{L}^\infty([0, t] \times \partial\Omega; \mathbb{R})}, \quad (3.5.43)$$

$$\|\nabla z\|_{\mathbf{L}^\infty([0,t] \times \Omega; \mathbb{R}^n)} \leq \|\psi\|_{\mathbf{C}^{2,\alpha}([0,t] \times \partial\Omega; \mathbb{R})}, \quad (3.5.44)$$

$$(k \geq 3) \quad \|\partial_t z\|_{\mathbf{L}^\infty([0,t] \times \Omega; \mathbb{R})} \leq \|\partial_t \psi\|_{\mathbf{L}^\infty([0,t] \times \partial\Omega; \mathbb{R})}, \quad (3.5.45)$$

$$(k \geq 3) \quad \|Dz\|_{\mathbf{W}^{1,\infty}([0,t] \times \Omega; \mathbb{R})} \leq \mathcal{O}(1) \|\psi\|_{\mathbf{C}^{3,\alpha}(\partial\Omega; \mathbb{R})}. \quad (3.5.46)$$

Proof. We verify that the assumptions of [36, Chapter 3, § 8, Theorem 20], in the case $p = k + 2$, hold. With reference to the notation of [36, Chapter 3, § 8, Theorem 20], for any $t \in I$ and for $i = 0, \dots, k$, consider the problem

$$\begin{cases} Lz_i(t) = \mathbf{f} & \text{in } \Omega \\ z_i(t) = \partial_t^i \psi(t) & \text{in } \partial\Omega \end{cases} \quad \text{where} \quad \begin{aligned} &L = \Delta \text{ is an elliptic operator,} \\ &\mathbf{f} = 0 \text{ is of class } \mathbf{C}^{k-2,\alpha}, \\ &\partial\Omega \text{ is of class } \mathbf{C}^{k,\alpha}, \\ &\partial_t^i \psi(t) \text{ is of class } \mathbf{C}^{k,\alpha} \text{ in } x. \end{aligned} \quad (3.5.47)$$

Therefore, for any $t \in I$, (3.5.42) admits a solution $z_i(t) \in \mathbf{C}^{k,\alpha}(\bar{\Omega}; \mathbb{R})$.

Thanks to the form of L in (3.5.47) and to the continuity of $\partial_t^i \psi$, [36, Chapter 2, § 7, Theorem 20] can be applied, ensuring the uniqueness of the solution to (3.5.42).

Concerning the regularity in t , remark that $\partial_t^i \psi$ is of class $\mathbf{C}^{k-1,\alpha}$ in t . Hence, for $i = 0, \dots, k$, by the Maximum Principle [36, Chapter 2, § 7, Theorem 19] for any $x \in \Omega$, $t \in I$, h sufficiently small such that $t + h \in I$, considering separately the cases $i < k$ and $i = k$,

$$\begin{aligned} i < k : \quad & \left| \frac{z_i(t+h, x) - z_i(t, x)}{h} - z_{i+1}(t, x) \right| \leq \sup_{\xi \in \partial\Omega} \left| \frac{\partial_t^i \psi(t+h, \xi) - \partial_t^i \psi(t, \xi)}{h} - \partial_t^{i+1} \psi(t, h) \right| \\ & \leq \sup_{\xi \in \partial\Omega} \left| \partial_t^{i+1} \psi(t+h, \xi) - \partial_t^{i+1} \psi(t, \xi) \right| \\ & \leq \begin{cases} \sup_{\xi \in \partial\Omega} \left| \partial_t^{i+2} \psi \right| h & i+1 < k, \\ C h^\alpha & i+1 = k, \end{cases} \\ i = k : \quad & |z_k(t+h, x) - z_k(t, x)| \leq \sup_{\xi \in \partial\Omega} \left| \partial_t^k \psi(t+h, \xi) - \partial_t^k \psi(t, \xi) \right| \leq C h^\alpha. \end{aligned}$$

where C is the Hölder constant of $\partial_t^k \psi$. Hence, $z = z_0$ is of class $\mathbf{C}^{k,\alpha}$ in both t and x .

Concerning the bounds on z and on its derivatives, note that (3.5.43) immediately follow from the Maximum Principle [36, Chapter 2, Formula (7.5)]. The same result applies also to $\partial_t z = \Delta \partial_t z$, yielding (3.5.45), whenever $k \geq 3$. The Boundary Schauder Estimate [36, Chapter 3, p.86] provides the bound for ∇z , $\nabla^2 z$, $\nabla \partial_t z$ and $\partial_{tt}^2 z$, proving (3.5.44) and (3.5.46). $\square$

We recall the following result from [47], to be used in the proof below.

Lemma 3.16 ([47, Lemma 2]). *Fix positive r and choose $\rho \in [0, \min\{r, T\}]$. Let $w \in \mathbf{L}^\infty(I \times B(0, r); \mathbb{R})$. For $h \in]0, \rho[$, define*

$$A_h = \left\{ (t, X, s, Y) \in (I \times \mathbb{R}^N)^2 : \begin{array}{ll} |t - s| \leq h, & (t + s)/2 \in [\rho, T - \rho], \\ \|X - Y\| \leq h, & \|X + Y\|/2 \in [0, r - \rho] \end{array} \right\}.$$

Then, $\lim_{h \rightarrow 0+} \frac{1}{h^{1+N}} \int_{A_h} |w(t, X) - w(s, Y)| \, dt \, dX \, ds \, dY = 0$.

Proof of Theorem 3.13. Define z as the solution to (3.5.42) with $\psi(t, \xi) = u_b(t, \xi)$. Lemma 3.15 applies, ensuring the existence and uniqueness of a solution z of class $\mathbf{C}^{3,\alpha}$. Note that $z(0, x) = 0$ for all $x \in \bar{\Omega}$. For all $\check{k} \in \mathbb{R}$ and for all $\check{\varphi} \in \mathbf{C}_c^2([-\infty, T[\times \mathbb{R}^n; \mathbb{R}^+)$ the following equality holds

$$\begin{aligned} & \int_I \int_{\Omega} \left(\left| z(s, y) - \check{k} \right| \partial_s \check{\varphi}(s, y) + \operatorname{sgn} \left(z(s, y) - \check{k} \right) \partial_s z(s, y) \check{\varphi}(s, y) \right) dy ds \\ & + \int_{\Omega} \left| \check{k} \right| \check{\varphi}(0, y) dy = 0. \end{aligned} \quad (3.5.48)$$

We now apply Proposition 3.12 to the problem

$$\begin{cases} \partial_t v + \nabla \cdot g(t, x, v) = G(t, x, v) \\ v(0, x) = u_o(x) & x \in \Omega \\ v(t, \xi) = 0 & (t, \xi) \in I \times \partial\Omega \end{cases} \quad \text{where} \quad \begin{cases} g(t, x, v) = f(t, x, v + z(t, x)) , \\ G(t, x, v) = F(t, x, v + z(t, x)) - \partial_t z(t, x) . \end{cases} \quad (3.5.49)$$

To this aim, we verify the necessary assumptions. Clearly, $(\Omega_{2,\delta})$ holds. By assumption, $u_o \in \mathbf{C}^{2,\delta}(\bar{\Omega}; \mathbb{R})$ and $u_o(\xi) = 0$ for all $\xi \in \Omega$. By construction, the boundary data along $I \times \partial\Omega$ is zero. To verify that also **(f)** and **(F)** hold for g and G , simply use the assumptions on f , F and apply Lemma 3.15. Call v the solution to (3.5.49) as constructed in Proposition 3.12. By Definition 3.1, for all $\hat{k} \in \mathbb{R}$ and for all $\hat{\varphi} \in \mathbf{C}_c^2([-\infty, T[\times \mathbb{R}^n; \mathbb{R}^+)$,

$$\begin{aligned} 0 & \leq \int_I \int_{\Omega} \left[\left| v(t, x) - \hat{k} \right| \partial_t \hat{\varphi}(t, x) \right. \\ & \quad + \operatorname{sgn} \left[v(t, x) - \hat{k} \right] \left[g(t, x, v(t, x)) - g(t, x, \hat{k}) \right] \cdot \nabla \hat{\varphi}(t, x) \\ & \quad \left. + \operatorname{sgn} \left[v(t, x) - \hat{k} \right] \left[G(t, x, v(t, x)) - \operatorname{div} g(t, x, \hat{k}) \right] \hat{\varphi}(t, x) \right] dx dt \\ & \quad + \int_{\Omega} \left| u_o(x) - \hat{k} \right| \hat{\varphi}(0, x) dx \\ & \quad - \int_I \int_{\partial\Omega} \operatorname{sgn}(-\hat{k}) \left[g(t, \xi, \operatorname{tr} v(t, \xi)) - g(t, \xi, \hat{k}) \right] \cdot \nu(\xi) \hat{\varphi}(t, \xi) d\xi dt \\ & = \int_I \int_{\Omega} \left[\left| v(t, x) - \hat{k} \right| \partial_t \hat{\varphi}(t, x) \right. \end{aligned} \quad (3.5.50)$$

$$\begin{aligned} & \quad + \operatorname{sgn} \left[v(t, x) - \hat{k} \right] \left[f(t, x, v(t, x) + z(t, x)) - f(t, x, \hat{k} + z(t, x)) \right] \cdot \nabla \hat{\varphi}(t, x) \\ & \quad + \operatorname{sgn} \left[v(t, x) - \hat{k} \right] \left[F(t, x, v(t, x) + z(t, x)) - \partial_t z(t, x) - \operatorname{div} f(t, x, \hat{k} + z(t, x)) \right] \\ & \quad \quad \times \hat{\varphi}(t, x) \Big] dx dt \\ & \quad + \int_{\Omega} \left| u_o(x) - \hat{k} \right| \hat{\varphi}(0, x) dx \\ & \quad - \int_I \int_{\partial\Omega} \operatorname{sgn}(-\hat{k}) \left[f(t, \xi, \operatorname{tr} v(t, \xi) + z(t, \xi)) - f(t, \xi, \hat{k} + z(t, \xi)) \right] \cdot \nu(\xi) \hat{\varphi}(t, \xi) d\xi dt . \end{aligned} \quad (3.5.51)$$

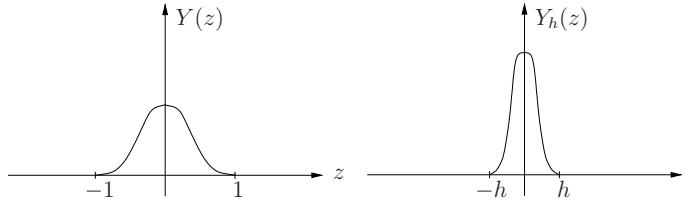
We now verify that the map

$$u(t, x) = v(t, x) + z(t, x)$$

is a solution to (3.1.1) in the sense of Definition 3.1. To this aim, we suitably modify the doubling of variables technique by Kruřkov, see [47]. Let $\tilde{k} = k - v(t, x)$ in (3.5.48) and $\hat{k} = k - z(s, y)$ in (3.5.50)–(3.5.51) for $k \in \mathbb{R}$. Integrate (3.5.48) with respect to t and x over $I \times \Omega$, integrate (3.5.50)–(3.5.51) in s and y over $I \times \Omega$. Add the resulting expressions, with as test function the map $\psi_h = \psi_h(t, x, s, y)$ defined by

$$\psi_h(t, x, s, y) = \varphi\left(\frac{t+s}{2}, x\right) Y_h(t-s) \prod_{i=1}^n Y_h(x_i - y_i), \quad (3.5.52)$$

with $\varphi \in \mathbf{C}_c^2([-\infty, T[\times \mathbb{R}^n; \mathbb{R}^+)$ and Y_h defined as follows. Let $Y \in \mathbf{C}_c^\infty(\mathbb{R}; \mathbb{R}^+)$ be such that

$$\begin{aligned} Y(-z) &= Y(z) \\ Y(z) &= 0 \text{ for } |z| \geq 1 \\ \int_{\mathbb{R}} Y(z) dz &= 1. \end{aligned} \quad (3.5.53)$$


and define $Y_h(z) = \frac{1}{h} Y\left(\frac{z}{h}\right)$. Obviously, $Y_h \in \mathbf{C}_c^\infty(\mathbb{R}; \mathbb{R}^+)$, $Y_h(-z) = Y_h(z)$, $Y_h(z) = 0$ for $|z| \geq h$, $\int_{\mathbb{R}} Y_h(z) dz = 1$ and $Y_h \rightarrow \delta_0$ as $h \rightarrow 0$, where δ_0 is the Dirac delta in 0.

We temporarily require also that

$$h \in]0, h_*[\quad \text{and} \quad \varphi(t, x) = 0 \quad \text{for all } x \quad \text{such that} \quad B(x, h_*) \cap (\mathbb{R}^n \setminus \Omega) \neq \emptyset \quad (3.5.54)$$

for a fixed positive h_* . We therefore obtain:

$$0 \leq \int_I \int_{\Omega} \int_I \int_{\Omega} \left[|v(t, x) + z(s, y) - k| (\partial_t \psi_h(t, x, s, y) + \partial_s \psi_h(t, x, s, y)) \right] \quad (3.5.55)$$

$$+ \operatorname{sgn} [v(t, x) + z(s, y) - k] \quad (3.5.56)$$

$$\times \left[f(t, x, u(t, x)) - f(t, x, z(t, x) - z(s, y) + k) \right] \cdot \nabla_x \psi_h(t, x, s, y) \quad (3.5.57)$$

$$+ \operatorname{sgn} [v(t, x) + z(s, y) - k] (\partial_s z(s, y) - \partial_t z(t, x)) \psi_h(t, x, s, y) \quad (3.5.58)$$

$$- \operatorname{sgn} [v(t, x) + z(s, y) - k] \operatorname{div} f(t, x, z(t, x) - z(s, y) + k) \psi_h(t, x, s, y) \quad (3.5.59)$$

$$+ \operatorname{sgn} [v(t, x) + z(s, y) - k] F(t, x, u(t, x)) \psi_h(t, x, s, y) \Big] dx dt dy ds \quad (3.5.60)$$

$$+ \int_I \int_{\Omega} \int_{\Omega} |u_o(x) + z(s, y) - k| \psi_h(0, x, s, y) dx dy ds \quad (3.5.61)$$

$$+ \int_I \int_{\Omega} \int_{\Omega} |v(t, x) - k| \psi_h(t, x, 0, y) dx dy dt. \quad (3.5.62)$$

To compute the limit as $h \rightarrow 0$, consider the terms above separately. First, proceeding as in [47, Formulæ (3.5)–(3.7)], thanks to (3.5.52), we have

$$\lim_{h \rightarrow 0} (3.5.55) = \int_I \int_{\Omega} |u(t, x) - k| \partial_t \varphi(t, x) dx dt. \quad (3.5.63)$$

To deal with (3.5.56)–(3.5.57) we simplify the notation by introducing the map

$$\Upsilon(t, x, s, y) = \operatorname{sgn} [v(t, x) + z(s, y) - k] \left[f(t, x, u(t, x)) - f(t, x, z(t, x) - z(s, y) + k) \right],$$

so that

$$\Upsilon(t, x, s, y) \cdot \nabla_x \psi_h(t, x, s, y) \quad (3.5.64)$$

$$= \Upsilon(t, x, t, x) \cdot \nabla \varphi(t, x) Y_h(t-s) \prod_{j=1}^n Y_h(x_j - y_j) \quad (3.5.65)$$

$$+ \Upsilon(t, x, t, x) \cdot \left(\nabla \varphi\left(\frac{t+s}{2}, x\right) - \nabla \varphi(t, x) \right) Y_h(t-s) \prod_{j=1}^n Y_h(x_j - y_j) \quad (3.5.66)$$

$$+ (\Upsilon(t, x, s, y) - \Upsilon(t, x, t, x)) \cdot \nabla \varphi\left(\frac{t+s}{2}, x\right) Y_h(t-s) \prod_{j=1}^n Y_h(x_j - y_j) \quad (3.5.67)$$

$$+ \sum_{i=1}^n \Upsilon_i(t, x, t, x) \varphi\left(\frac{t+s}{2}, x\right) Y_h(t-s) Y'_h(x_i - y_i) \prod_{j \neq i} Y_h(x_j - y_j) \quad (3.5.68)$$

$$+ \sum_{i=1}^n [\Upsilon_i(t, x, s, y) - \Upsilon_i(t, x, t, x)] \varphi\left(\frac{t+s}{2}, x\right) Y_h(t-s) Y'_h(x_i - y_i) \prod_{j \neq i} Y_h(x_j - y_j). \quad (3.5.69)$$

Then,

$$\int_I \int_{\Omega} \int_I \int_{\Omega} [(3.5.65)] \, dy \, ds \, dx \, dt = \int_I \int_{\Omega} \Upsilon(t, x, t, x) \cdot \nabla \varphi(t, x) \, dx \, dt.$$

To deal with (3.5.66), recall that $|Y_h| \leq (Y(0)/h) \chi_{[-h, h]}$ and apply Lemma 3.16 with

$$N = 2n + 1, \quad \begin{aligned} X &= (x, t, x), \\ Y &= (x, t, y), \end{aligned} \quad w(s, Y) = \frac{Y(0)^{n+1}}{h^{n+1}} \Upsilon(t, x, t, x) \cdot \nabla \varphi\left(\frac{t+s}{2}, x\right),$$

so that

$$\lim_{h \rightarrow 0} \int_I \int_{\Omega} \int_I \int_{\Omega} |[(3.5.66)]| \, dy \, ds \, dx \, dt = 0.$$

Similarly, to deal with (3.5.67), apply Lemma 3.16 with

$$N = 2n + 1, \quad \begin{aligned} X &= (x, t, x), \\ Y &= (x, t, y), \end{aligned} \quad w(s, Y) = \frac{Y(0)^{n+1} \|\nabla \varphi\|_{\mathbf{L}^\infty(I \times \mathbb{R}^n; \mathbb{R}^n)}}{h^{n+1}} \Upsilon(t, x, s, y),$$

so that

$$\lim_{h \rightarrow 0} \int_I \int_{\Omega} \int_I \int_{\Omega} |[(3.5.67)]| \, dy \, ds \, dx \, dt = 0.$$

The term (3.5.68) vanishes, since

$$\int_I \int_{\Omega} \int_I \int_{\Omega} [(3.5.68)] \, dy \, ds \, dx \, dt = \int \cdots \int_{x_i-h}^{x_i+h} Y'_h(x_i - y_i) \, dy_i \cdots \, dx \, dt = 0.$$

Finally, to estimate (3.5.69), recall that $|Y'_h| \leq (\|Y'\|_{\mathbf{L}^\infty(\mathbb{R}; \mathbb{R})}/h^2) \chi_{[-h, h]}$ and use Lemma 3.16 with

$$N = 2n + 1, \quad \begin{aligned} X &= (x, t, x), \\ Y &= (x, t, y), \end{aligned} \quad w(s, Y) = \frac{Y(0)^n \|Y'\|_{\mathbf{L}^\infty(\mathbb{R}; \mathbb{R})} \|\varphi\|_{\mathbf{L}^\infty(I \times \mathbb{R}^n; \mathbb{R})}}{h^{n+2}} \Upsilon(t, x, s, y),$$

so that, thanks to $2n + 1 \geq n + 2$,

$$\lim_{h \rightarrow 0} \int_I \int_\Omega \int_I \int_\Omega |[(3.5.69)]| \, dy \, ds \, dx \, dt = 0.$$

Hence

$$\begin{aligned} & \lim_{h \rightarrow 0} [(3.5.56) \times (3.5.57)] \\ &= \lim_{h \rightarrow 0} \int_I \int_\Omega \int_I \int_\Omega \Upsilon(t, x, s, y) \cdot \nabla_x \psi_h(t, x, s, y) \, dy \, ds \, dx \, dt \\ &= \int_I \int_\Omega \Upsilon(t, x, t, x) \cdot \nabla \varphi(t, x) \, dx \, dt \\ &= \int_I \int_\Omega \operatorname{sgn}(u(t, x) - k) \left(f(t, x, u(t, x)) - f(t, x, k) \right) \cdot \nabla \varphi(t, x) \, dx \, dt. \end{aligned} \quad (3.5.70)$$

Note that setting

$$N = n, \quad \begin{matrix} X=x \\ Y=y \end{matrix} \quad \text{and} \quad w(s, y) = \frac{Y(0)^{n+1}}{h^{n+1}} \|\varphi\|_{\mathbf{L}^\infty(I \times \mathbb{R}^n; \mathbb{R})} \partial_s z(s, y)$$

in Lemma 3.16, we obtain

$$\lim_{h \rightarrow 0} [(3.5.58)] = \lim_{h \rightarrow 0} \int_I \int_\Omega \int_I \int_\Omega |\partial_t z(t, x) - \partial_s z(s, y)| \psi_h(t, x, s, y) \, dy \, ds \, dx \, dt = 0.$$

Omitting now the integrals in (3.5.59), we have

$$\begin{aligned} [(3.5.59)] &= -\operatorname{sgn}[v(t, x) + z(s, y) - k] \operatorname{div} f(t, x, z(t, x) - z(s, y) + k) \psi_h(t, x, s, y) \\ &= -\operatorname{sgn}[u(t, x) - k] \operatorname{div} f(t, x, k) \varphi(t, x) Y_h(t - s) \prod Y_h(x_i - y_i) \\ &\quad - \operatorname{sgn}[u(t, x) - k] \operatorname{div} f(t, x, k) \left[\varphi\left(\frac{t+s}{2}, x\right) - \varphi(t, x) \right] Y_h(t - s) \prod Y_h(x_i - y_i) \\ &\quad - \operatorname{sgn}[u(t, x) - k] \left(\operatorname{div} f(t, x, z(t, x) - z(s, y) + k) - \operatorname{div} f(t, x, k) \right) \\ &\quad \times \varphi\left(\frac{t+s}{2}, x\right) Y_h(t - s) \prod Y_h(x_i - y_i) \\ &\quad - \left(\operatorname{sgn}[u(t, x) + z(s, y) - z(t, x) - k] - \operatorname{sgn}[u(t, x) - k] \right) \\ &\quad \times \operatorname{div} f(t, x, z(t, x) - z(s, y) + k) \varphi\left(\frac{t+s}{2}, x\right) Y_h(t - s) \prod Y_h(x_i - y_i). \end{aligned}$$

A repeated application of Lemma 3.16, together with standard estimates, yields

$$\lim_{h \rightarrow 0} [(3.5.59)] = - \int_I \int_\Omega \operatorname{sgn}[u(t, x) - k] \operatorname{div} f(t, x, k) \varphi(t, x) \, dx \, dt. \quad (3.5.71)$$

The term (3.5.60) is treated similarly, since

$$\begin{aligned} [(3.5.60)] &= \operatorname{sgn}[v(t, x) + z(s, y) - k] F(t, x, u(t, x)) \psi_h(t, x, s, y) \\ &= \operatorname{sgn}[u(t, x) - k] F(t, x, u(t, x)) \varphi(t, x) Y_h(t - s) \prod Y_h(x_i - y_i) \end{aligned}$$

$$\begin{aligned}
& + \operatorname{sgn} [u(t, x) - k] F(t, x, u(t, x)) \left[\varphi \left(\frac{t+s}{2}, x \right) - \varphi(t, x) \right] Y_h(t-s) \prod Y_h(x_i - y_i) \\
& + \left(\operatorname{sgn} [u(t, x) + z(s, y) - z(t, x) - k] - \operatorname{sgn} [u(t, x) - k] \right) F(t, x, u(t, x)) \\
& \quad \times \varphi \left(\frac{t+s}{2}, x \right) Y_h(t-s) \prod Y_h(x_i - y_i),
\end{aligned}$$

so that further applications of Lemma 3.16 lead to

$$\lim_{h \rightarrow 0} [(3.5.60)] = \int_I \int_{\Omega} \operatorname{sgn} [u(t, x) - k] F(t, x, u(t, x)) \varphi(t, x) dx dt. \quad (3.5.72)$$

To deal with (3.5.61) and (3.5.62), introduce the function

$$\Upsilon(x, s, y) = |u_o(x) + z(s, y) - k| + |v(s, x) - k|$$

and, exploiting the symmetry $Y(x) = Y(-x)$, we obtain

$$\begin{aligned}
& [(3.5.61) + (3.5.62)] \\
& = \int_I \int_{\Omega} \int_{\Omega} \Upsilon(x, s, y) \psi_h(0, x, s, y) dy dx ds \\
& = \int_I \int_{\Omega} \int_{\Omega} \Upsilon(x, 0, x) \varphi(0, x) Y_h(s) \prod Y_h(x_i - y_i) dy dx ds \\
& \quad + \int_I \int_{\Omega} \int_{\Omega} \Upsilon(x, 0, x) \left(\varphi \left(\frac{s}{2}, x \right) - \varphi(0, x) \right) Y_h(s) \prod Y_h(x_i - y_i) dy dx ds \\
& \quad + \int_I \int_{\Omega} \int_{\Omega} (\Upsilon(x, s, y) - \Upsilon(x, 0, x)) \varphi \left(\frac{s}{2}, x \right) Y_h(s) \prod Y_h(x_i - y_i) dy dx ds \\
& \leq \frac{1}{2} \int_{\Omega} \Upsilon(x, 0, x) \varphi(0, x) dx \\
& \quad + \frac{\|\partial_t \varphi\|_{\mathbf{L}^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{R})}}{2} \int_{\Omega} \Upsilon(x, 0, x) dx \int_0^h s Y_h(s) ds \\
& \quad + \|\varphi\|_{\mathbf{L}^\infty(\mathbb{R} \times \mathbb{R}^N; \mathbb{R})} \int_I \int_{\Omega} \int_{\Omega} |\Upsilon(x, s, y) - \Upsilon(x, 0, x)| Y_h(s) \prod Y_h(x_i - y_i) dy dx ds.
\end{aligned}$$

Both the two latter terms vanish in the limit $h \rightarrow 0$. Indeed, by Lemma 3.16, for a.e. $s \in [0, h]$, we have that $\int_{\Omega} \int_{\Omega} |\Upsilon(x, s, y) - \Upsilon(x, 0, x)| dy dx \rightarrow 0$. Hence,

$$\lim_{h \rightarrow 0} [(3.5.61) + (3.5.62)] = \int_{\Omega} |u_o(x) - k| \varphi(0, x) dx. \quad (3.5.73)$$

We can now summarise the computations: thanks to (3.5.63), (3.5.70), (3.5.71), (3.5.72) and (3.5.73), in the limit $h \rightarrow 0$ (3.5.55)–(3.5.62) becomes

$$\begin{aligned}
& \int_I \int_{\Omega} |u(t, x) - k| \partial_t \varphi(t, x) dx dt \\
& + \int_I \int_{\Omega} \operatorname{sgn} (u(t, x) - k) \left(f(t, x, u(t, x)) - f(t, x, k) \right) \cdot \nabla \varphi(t, x) dx dt \\
& + \int_I \int_{\Omega} \operatorname{sgn} (u(t, x) - k) \left(F(t, x, u(t, x)) - \operatorname{div} f(t, x, k) \right) \varphi(t, x) dx dt
\end{aligned}$$

$$+ \int_{\Omega} |u_o(x) - k| \varphi(0, x) \, dx \geq 0,$$

which holds under the choice (3.5.54) of φ . To pass to an arbitrary test function as in Definition 3.1, substitute $\varphi(t, x)$ with $(1 - \Phi_h(x)) \varphi(t, x)$, where $\varphi \in \mathbf{C}_c^2([-\infty, T] \times \mathbb{R}^n; \mathbb{R}^+)$ and Φ_h is as in (3.5.38):

$$\begin{aligned} & \int_I \int_{\Omega} |u(t, x) - k| \partial_t \varphi(t, x) (1 - \Phi_h(x)) \, dx \, dt \\ & + \int_I \int_{\Omega} \operatorname{sgn}(u(t, x) - k) \left(f(t, x, u(t, x)) - f(t, x, k) \right) \cdot \nabla \varphi(t, x) (1 - \Phi_h(x)) \, dx \, dt \\ & + \int_I \int_{\Omega} \operatorname{sgn}(u(t, x) - k) \left(F(t, x, u(t, x)) - \operatorname{div} f(t, x, k) \right) \varphi(t, x) (1 - \Phi_h(x)) \, dx \, dt \\ & + \int_{\Omega} |u_o(x) - k| \varphi(0, x) (1 - \Phi_h(x)) \, dx \\ & - \int_I \int_{\Omega} \operatorname{sgn}(u(t, x) - k) \left(f(t, x, u(t, x)) - f(t, x, k) \right) \cdot \nabla \Phi_h(x) \varphi(t, x) \, dx \, dt \geq 0. \end{aligned}$$

In the limit $h \rightarrow 0$, the first 4 lines above converge to the first 3 lines in the left hand side in (3.2.1) of Definition 3.1, by the Dominated Convergence Theorem. Concerning the latter term, use Lemma 3.22 and Lemma 3.20, which can be applied since the function $(w_1, w_2) \rightarrow \operatorname{sgn}(w_1 - w_2) (f(t, x, w_1) - f(t, x, w_2))$ is Lipschitz continuous, see [47, Lemma 3]. We therefore obtain that

$$\begin{aligned} & - \lim_{h \rightarrow 0} \int_I \int_{\Omega} \operatorname{sgn}(u(t, x) - k) \left(f(t, x, u(t, x)) - f(t, x, k) \right) \cdot \nabla \Phi_h(x) \varphi(t, x) \, dx \, dt \\ & = - \int_I \int_{\partial\Omega} \operatorname{sgn}(\operatorname{tr} u(t, \xi) - k) \left(f(t, \xi, \operatorname{tr} u(t, \xi)) - f(t, \xi, k) \right) \cdot \nu(\xi) \varphi(t, \xi) \, dx \, dt \\ & = - \int_I \int_{\partial\Omega} \operatorname{sgn}(u_b(t, \xi) - k) \left(f(t, \xi, \operatorname{tr} u(t, \xi)) - f(t, \xi, k) \right) \cdot \nu(\xi) \varphi(t, \xi) \, dx \, dt \\ & \quad - \int_I \int_{\partial\Omega} \left(\operatorname{sgn}(\operatorname{tr} u(t, \xi) - k) - \operatorname{sgn}(u_b(t, \xi) - k) \right) \\ & \quad \quad \times \left(f(t, \xi, \operatorname{tr} u(t, \xi)) - f(t, \xi, k) \right) \cdot \nu(\xi) \varphi(t, \xi) \, dx \, dt \\ & \leq - \int_I \int_{\partial\Omega} \left(\operatorname{sgn}(\operatorname{tr} u(t, \xi) - k) - \operatorname{sgn}(u_b(t, \xi) - k) \right), \end{aligned}$$

where to get to the last line, we used the following fact:

$$\begin{aligned} & - \int_I \int_{\partial\Omega} \left(\operatorname{sgn}(\operatorname{tr} u(t, \xi) - k) - \operatorname{sgn}(u_b(t, \xi) - k) \right) \\ & \quad \times \left(f(t, \xi, \operatorname{tr} u(t, \xi)) - f(t, \xi, k) \right) \cdot \nu(\xi) \varphi(t, \xi) \, dx \, dt \\ & = - \int_I \int_{\partial\Omega} \left(\operatorname{sgn}(\operatorname{tr} v(t, \xi) + z(t, \xi) - k) - \operatorname{sgn}(z(t, \xi) - k) \right) \\ & \quad \times \left(f(t, \xi, \operatorname{tr} v(t, \xi) + z(t, \xi)) - f(t, \xi, k) \right) \cdot \nu(\xi) \varphi(t, \xi) \, dx \, dt \\ & = - \int_I \int_{\partial\Omega} \left(\operatorname{sgn}(\operatorname{tr} v(t, \xi) - (k - z(t, \xi))) - \operatorname{sgn}(-(k - z(t, \xi))) \right) \\ & \quad \times \left(g(t, \xi, \operatorname{tr} v(t, \xi)) - g(t, \xi, k - z(t, \xi)) \right) \cdot \nu(\xi) \varphi(t, \xi) \, dx \, dt \end{aligned}$$

$$\leq 0,$$

since $\varphi \geq 0$ and by (3.2.2) in Proposition 3.3 applied to v as solution to (3.5.49)

$$\left(\operatorname{sgn} \left(\operatorname{tr} v(t, \xi) - \hat{k} \right) - \operatorname{sgn} \left(-\hat{k} \right) \right) \left(g(t, \xi, \operatorname{tr} v(t, \xi)) - g(t, \xi, \hat{k}) \right) \cdot \nu(\xi) \geq 0$$

for all $\hat{k} \in \mathbb{R}$ and for a.e. $(t, \xi) \in I \times \partial\Omega$. This completes the first part of the proof: the existence of a solution to (3.1.1) in the sense of Definition 3.1.

Consider now the $\mathbf{L}^\infty$ estimate. Recall (3.5.49), (3.5.1) and Proposition 3.12, so that

$$\begin{aligned} \|u(t)\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} &\leq \|v(t)\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} + \|z(t)\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} \\ &\leq \left[\|u_o\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} + t \left(c_2 + c_1 \|z\|_{\mathbf{L}^\infty(I \times \Omega; \mathbb{R})} + \|\partial_t z\|_{\mathbf{L}^\infty(I \times \Omega; \mathbb{R})} \right. \right. \\ &\quad \left. \left. + \left\| \partial_{uu}^2 f \right\|_{\mathbf{L}^\infty(I \times \Omega \times \mathbb{R}; \mathbb{R}^n)} \|\nabla z\|_{\mathbf{L}^\infty(I \times \Omega; \mathbb{R}^n)} \right) \right] \\ &\quad \times \exp \left(c_1 t + \left\| \partial_{uu}^2 f \right\|_{\mathbf{L}^\infty(I \times \Omega \times \mathbb{R}; \mathbb{R}^n)} \|\nabla z\|_{\mathbf{L}^\infty(I \times \Omega; \mathbb{R}^n)} t \right) \\ &\quad + \|z(t)\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} \\ &\leq M_u(t). \end{aligned}$$

Using also Lemma 3.15, we obtain

$$\begin{aligned} M_u(t) &= \left[\|u_o\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} + t \left(c_2 + c_1 \|u_b\|_{\mathbf{L}^\infty(I \times \partial\Omega; \mathbb{R})} + \|\partial_t u_b\|_{\mathbf{L}^\infty(I \times \partial\Omega; \mathbb{R})} + c_3 \|u_b\|_{\mathbf{C}^{2,\alpha}(I \times \partial\Omega; \mathbb{R})} \right) \right] \\ &\quad \times \exp \left(c_1 t + c_3 t \|u_b\|_{\mathbf{C}^{2,\alpha}(I \times \partial\Omega; \mathbb{R})} \right) + \|u_b\|_{\mathbf{L}^\infty([0,t] \times \partial\Omega; \mathbb{R})}, \end{aligned}$$

where

$$c_3 = \left\| \partial_{uu}^2 f \right\|_{\mathbf{L}^\infty(I \times \Omega \times \mathbb{R}; \mathbb{R}^n)}, \quad (3.5.74)$$

which proves the $\mathbf{L}^\infty$ estimate (3.4.4).

To obtain the TV bound, we use Proposition 3.12 to estimate $\operatorname{TV}(v)$ and standard estimates on elliptic problems to bound $\operatorname{TV}(z)$. To this aim, we call $A_i(g)$, for $i = 1, \dots, 4$, the quantities defined in (3.5.30)–(3.5.31), but with norms of g and G over $[0, t] \times \Omega \times \mathcal{V}(t)$, where $\mathcal{V}(t) = [-\mathcal{M}_v(t), \mathcal{M}_v(t)]$, with $\mathcal{M}_v(t)$ being an upper bound for $\|v\|_{\mathbf{L}^\infty([0,t] \times \Omega; \mathbb{R})}$ as in (3.3.4). Clearly, $\mathcal{V}(t) \subseteq \mathcal{U}(t) = [-M_u(t), M_u(t)]$. By (3.5.30)–(3.5.31), and Lemma 3.15, we have:

$$\begin{aligned} A_1(g) &\leq \mathcal{O}(1) \left[\|\operatorname{div} f\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} + \|F\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} \right. \\ &\quad \left. + \|\partial_u f\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} \|\nabla z\|_{\mathbf{L}^\infty([0,t] \times \Omega; \mathbb{R}^n)} + \|\partial_t z\|_{\mathbf{L}^\infty([0,t] \times \Omega; \mathbb{R})} \right] \\ &\leq \mathcal{O}(1) \left[\|Df\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^{n \times (2+n)})} + \|F\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} \right. \\ &\quad \left. + \left(1 + \|Df\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^{n \times (2+n)})} \right) \|u_b\|_{\mathbf{C}^{2,\alpha}([0,t] \times \partial\Omega; \mathbb{R})} \right] =: \mathcal{A}_1 \end{aligned}$$

$$\begin{aligned}
A_2(g) &\leq \mathcal{O}(1) \left[\|Df\|_{\mathbf{W}^{1,\infty}([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^{n \times (2+n)})} + \|F\|_{\mathbf{W}^{1,\infty}([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} \right. \\
&\quad + \left[1 + \|Df\|_{\mathbf{W}^{1,\infty}([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^{n \times (2+n)})} + \|\partial_u F\|_{\mathbf{W}^{1,\infty}([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} \right] \|Dz\|_{\mathbf{W}^{1,\infty}([0,t] \times \Omega; \mathbb{R})} \\
&\quad \left. + \left\| \partial_{uu}^2 f \right\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} \|Dz\|_{\mathbf{W}^{1,\infty}([0,t] \times \Omega; \mathbb{R})}^2 \right] \\
&\leq \mathcal{O}(1) \left[\|Df\|_{\mathbf{W}^{1,\infty}([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^{n \times (2+n)})} + \|F\|_{\mathbf{W}^{1,\infty}([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} \right. \\
&\quad + \left[1 + \|Df\|_{\mathbf{W}^{1,\infty}([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^{n \times (2+n)})} + \|\partial_u F\|_{\mathbf{W}^{1,\infty}([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} \right] \|u_b\|_{\mathbf{C}^{3,\alpha}([0,t] \times \partial\Omega; \mathbb{R})} \\
&\quad \left. + \left\| \partial_{uu}^2 f \right\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} \|u_b\|_{\mathbf{C}^{3,\alpha}([0,t] \times \partial\Omega; \mathbb{R})}^2 \right] =: \mathcal{A}_2 \\
A_3(g) &\leq \mathcal{O}(1) + \|\partial_u f\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} =: \mathcal{A}_3 \\
A_4(g) &\leq \mathcal{O}(1) \left[1 + \|Df\|_{\mathbf{W}^{1,\infty}([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^{n \times n})} + \|\partial_u F\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} \right. \\
&\quad \left. + \left\| \partial_{uu}^2 f \right\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} \left(\|\partial_t z\|_{\mathbf{L}^\infty([0,t] \times \Omega)} + \|\nabla z\|_{\mathbf{L}^\infty([0,t] \times \Omega)} \right) \right] \\
&\leq \mathcal{O}(1) \left[1 + \|Df\|_{\mathbf{W}^{1,\infty}([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^{n \times n})} + \|\partial_u F\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R})} \right. \\
&\quad \left. + \left\| \partial_{uu}^2 f \right\|_{\mathbf{L}^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} \|u_b\|_{\mathbf{C}^{2,\alpha}([0,t] \times \partial\Omega; \mathbb{R})} \right] =: \mathcal{A}_4
\end{aligned}$$

which proves the bound

$$\mathrm{TV}(v(t)) \leq (\mathcal{A}_1 + \mathcal{A}_2 t + \mathcal{A}_3 \mathrm{TV}(u_o)) e^{\mathcal{A}_4 t}. \quad (3.5.75)$$

Recall now that $\mathrm{TV}(u) \leq \mathrm{TV}(v) + \mathrm{TV}(z)$ and $\mathrm{TV}(z) \leq \mathcal{L}^n(\Omega) \|u_b\|_{\mathbf{C}^{2,\alpha}([0,t] \times \partial\Omega; \mathbb{R})}$ thanks to Lemma 3.15,.

The proof is completed. $\square$

Proof of Theorem 3.14. Assume preliminarily that $u_o \in \mathbf{C}^2(\bar{\Omega}; \mathbb{R})$ and $u_b \in \mathbf{C}^2(I \times \partial\Omega; \mathbb{R})$.

Let $\varphi \in \mathbf{C}_c^2([-\infty, T[\times \mathbb{R}^n; \mathbb{R}^+)$ be a test function as in Definition 3.1 with

$$\begin{aligned}
\varphi(0, x) &= 0 \quad \text{for all } x \in \mathbb{R}^n, \\
\varphi(t, x) &= 0 \quad \text{for all } t \in I \text{ and } x \text{ such that } B(x, h_*) \cap (\mathbb{R}^n \setminus \Omega) \neq \emptyset.
\end{aligned} \quad (3.5.76)$$

Define, for $h \in]0, h_*[$,

$$\psi_h(t, x, s, y) = \varphi\left(\frac{t+s}{2}, \frac{x+y}{2}\right) Y_h(t-s) \prod_{i=1}^n Y_h(x_i - y_i) \quad (3.5.77)$$

where Y_h is defined in (3.5.53). We now use the *doubling of variables* method, see [47]. In inequality (3.2.1), set $k = v(s, y)$ and use as test function the map $\psi_h = \psi_h(t, x, s, y)$ for a fixed point (s, y) and integrate over $I \times \Omega$ with respect to (s, y) :

$$\begin{aligned}
&\int_I \int_\Omega \int_I \int_\Omega \left\{ |u(t, x) - v(s, y)| \partial_t \psi_h(t, x, s, y) \right. \\
&\quad \left. + \operatorname{sgn}(u(t, x) - v(s, y)) \left[f(t, x, u(t, x)) - f(t, x, v(s, y)) \right] \cdot \nabla_x \psi_h(t, x, s, y) \right\}
\end{aligned}$$

$$\begin{aligned}
& + \operatorname{sgn} (u(t, x) - v(s, y)) \left[F(t, x, u(t, x)) - \operatorname{div} f(t, x, v(s, y)) \right] \psi_h(t, x, s, y) \Big\} dx dt dy ds \\
& + \int_I \int_{\Omega} \int_{\Omega} \psi_h(0, x, s, y) |u_o(x) - v(s, y)| dx dy ds \geq 0.
\end{aligned}$$

In the same way, starting from the inequality (3.2.1) for the function $v = v(s, y)$, set $k = u(t, x)$, consider the same test function $\psi_h = \psi_h(t, x, s, y)$ and integrate over $I \times \Omega$ with respect to the variables (t, x) :

$$\begin{aligned}
& \int_I \int_{\Omega} \int_I \int_{\Omega} \left\{ |v(s, y) - u(t, x)| \partial_s \psi_h(t, x, s, y) \right. \\
& + \operatorname{sgn} (v(s, y) - u(t, x)) \left[f(s, y, v(s, y)) - f(s, y, u(t, x)) \right] \cdot \nabla_y \psi_h(t, x, s, y) \\
& + \operatorname{sgn} (v(s, y) - u(t, x)) \left[F(s, y, v(s, y)) - \operatorname{div} f(s, y, u(t, x)) \right] \psi_h(t, x, s, y) \Big\} dy ds dx dt \\
& + \int_I \int_{\Omega} \int_{\Omega} \psi_h(t, x, 0, y) |v_o(y) - u(t, x)| dy dx dt \geq 0.
\end{aligned}$$

Summing the last two inequalities above, we obtain:

$$0 \leq \int_I \int_{\Omega} \int_I \int_{\Omega} \left\{ |u(t, x) - v(s, y)| (\partial_t \psi_h(t, x, s, y) + \partial_s \psi_h(t, x, s, y)) \right. \quad (3.5.78)$$

$$\begin{aligned}
& + \operatorname{sgn} (u(t, x) - v(s, y)) \left[f(t, x, u(t, x)) - f(t, x, v(s, y)) \right] \cdot \nabla_x \psi_h(t, x, s, y) \\
& + \operatorname{sgn} (v(s, y) - u(t, x)) \left[f(s, y, v(s, y)) - f(s, y, u(t, x)) \right] \cdot \nabla_y \psi_h(t, x, s, y) \\
& + \operatorname{sgn} (u(t, x) - v(s, y)) \left[F(t, x, u(t, x)) - F(s, y, v(s, y)) \right. \\
& \quad \left. + \operatorname{div} f(s, y, u(t, x)) - \operatorname{div} f(t, x, v(s, y)) \right] \psi_h(t, x, s, y) \Big\} dx dt dy ds \quad (3.5.79)
\end{aligned}$$

$$+ \int_I \int_{\Omega} \int_{\Omega} \psi_h(0, x, s, y) |u_o(x) - v(s, y)| dx dy ds \quad (3.5.80)$$

$$+ \int_I \int_{\Omega} \int_{\Omega} \psi_h(t, x, 0, y) |v_o(y) - u(t, x)| dy dx dt. \quad (3.5.81)$$

We follow the proof of [47, Theorem 1]. The first integral in the 5 lines [(3.5.78) $\cdots$ (3.5.79)], as h goes to 0, can be treated exactly as in [47], leading to the following analogue of [47, Formula (3.12)]:

$$\begin{aligned}
& \lim_{h \rightarrow 0+} [(3.5.78) \cdots (3.5.79)] \\
& = \int_I \int_{\Omega} \left\{ |u(t, x) - v(t, x)| \partial_t \varphi(t, x) \right. \\
& \quad + \operatorname{sgn} (u(t, x) - v(t, x)) \left[f(t, x, u(t, x)) - f(t, x, v(t, x)) \right] \cdot \nabla \varphi(t, x) \quad (3.5.82) \\
& \quad \left. + \operatorname{sgn} (u(t, x) - v(t, x)) \left[F(t, x, u(t, x)) - F(t, x, v(t, x)) \right] \varphi(t, x) \right\} dx dt.
\end{aligned}$$

To compute the second integral (3.5.80), observe preliminarily that

$$\begin{aligned}
|u_o(x) - v(s, y)| & \leq |u_o(x) - v(s, y)| - |u_o(x) - v(s, x)| \\
& \quad + |u_o(x) - v(s, x)| - |u_o(x) - v(0+, x)|
\end{aligned}$$

$$\begin{aligned}
& + |u_o(x) - v(0+, x)| \\
& \leq |v(s, y) - v(s, x)| + |v(s, x) - v(0+, x)| + |u_o(x) - v(0+, x)|.
\end{aligned}$$

Hence:

$$[(3.5.80)] \leq \int_I \int_{\Omega} \int_{\Omega} \psi_h(0, x, s, y) |v(s, y) - v(s, x)| \, dx \, dy \, ds \quad (3.5.83)$$

$$+ \int_I \int_{\Omega} \int_{\Omega} \psi_h(0, x, s, y) |v(s, x) - v(0+, x)| \, dx \, dy \, ds \quad (3.5.84)$$

$$+ \int_I \int_{\Omega} \int_{\Omega} \psi_h(0, x, s, y) |u_o(x) - v(0+, x)| \, dx \, dy \, ds. \quad (3.5.85)$$

Compute the limit as $h \rightarrow 0+$ of the three lines separately. First, apply Lemma 3.16 in the case of a function w depending only on the space variable to obtain

$$\lim_{h \rightarrow 0+} [(3.5.83)] = \lim_{h \rightarrow 0+} \int_I \int_{\Omega} \int_{\Omega} \psi_h(0, x, s, y) |v(s, y) - v(s, x)| \, dx \, dy \, ds = 0.$$

Second, by Lemma 3.18,

$$\lim_{h \rightarrow 0+} [(3.5.84)] = \int_{\Omega} \int_{\Omega} \left(\lim_{h \rightarrow 0+} \int_I \psi_h(0, x, s, y) |v(s, x) - v(0+, x)| \, ds \right) \, dx \, dy = 0.$$

Third, by the choice of the function Y_h and (3.5.76)

$$\lim_{h \rightarrow 0+} [(3.5.85)] = \int_{\Omega} \varphi(0, x) |u_o(x) - v(0+, x)| \, dx = 0,$$

proving that $\lim_{h \rightarrow 0+} (3.5.80) = 0$. The term (3.5.81) is treated exactly in the same way. Hence,

$$\lim_{h \rightarrow 0+} [(3.5.78) \cdots (3.5.81)] = [(3.5.82)],$$

so that

$$\begin{aligned}
0 \leq & \int_I \int_{\Omega} \left\{ |u(t, x) - v(t, x)| \, \partial_t \varphi(t, x) \right. \\
& + \operatorname{sgn}(u(t, x) - v(t, x)) \left[f(t, x, u(t, x)) - f(t, x, v(t, x)) \right] \cdot \nabla \varphi(t, x) \\
& \left. + \operatorname{sgn}(u(t, x) - v(t, x)) \left[F(t, x, u(t, x)) - F(t, x, v(t, x)) \right] \varphi(t, x) \right\} \, dx \, dt.
\end{aligned} \quad (3.5.86)$$

For $h > 0$, recall the function $\Phi_h \in \mathbf{C}_c^2(\mathbb{R}^n; [0, 1])$ defined in (3.5.38). Let $\Psi \in \mathbf{C}_c^2(]0, T[; \mathbb{R}^+)$ with $\Psi(0) = 0$. Note that for any $h > 0$ sufficiently small, the map

$$\varphi_h(t, x) = \Psi(t) (1 - \Phi_h(x)) \quad \text{for } (t, x) \in]-\infty, T[\times \mathbb{R}^n$$

satisfies (3.5.76). Introduce this test function φ_h in (3.5.86) and pass to the limit $h \rightarrow 0$ to obtain:

$$0 \leq \int_I \int_{\Omega} \left\{ |u(t, x) - v(t, x)| \, \Psi'(t) \right.$$

$$\begin{aligned}
& + \operatorname{sgn}(u(t, x) - v(t, x)) \left[F(t, x, u(t, x)) - F(t, x, v(t, x)) \right] \Psi(t) \Big\} dx dt \\
& - \int_I \int_{\partial\Omega} \operatorname{sgn}(\operatorname{tr} u(t, \xi) - \operatorname{tr} v(t, \xi)) \left[f(t, \xi, \operatorname{tr} u(t, \xi)) - f(t, \xi, \operatorname{tr} v(t, \xi)) \right] \cdot \nu(\xi) \Psi(t) d\xi dt,
\end{aligned} \tag{3.5.87}$$

where we used Lemma 3.22 and Lemma 3.20, which can be applied since the function $(u, v) \rightarrow \operatorname{sgn}(u - v) (f(t, x, u) - f(t, x, v))$ is Lipschitz continuous, see [47, Lemma 3].

To ease readability, we now omit the dependence on (t, ξ) of $f, \operatorname{tr} u, \operatorname{tr} v, u_b, v_b, \nu$. Apply (3.2.2) to u choosing $k = \operatorname{tr} v$ and to v choosing $k = \operatorname{tr} u$:

$$\begin{aligned}
& - \operatorname{sgn}(\operatorname{tr} u - \operatorname{tr} v) [f(\operatorname{tr} u) - f(\operatorname{tr} v)] \cdot \nu \leq - \operatorname{sgn}(u_b - \operatorname{tr} v) [f(\operatorname{tr} u) - f(\operatorname{tr} v)] \cdot \nu, \\
& - \operatorname{sgn}(\operatorname{tr} u - \operatorname{tr} v) [f(\operatorname{tr} u) - f(\operatorname{tr} v)] \cdot \nu \leq - \operatorname{sgn}(v_b - \operatorname{tr} u) [f(\operatorname{tr} v) - f(\operatorname{tr} u)] \cdot \nu.
\end{aligned}$$

Hence,

$$\begin{aligned}
& - \operatorname{sgn}(\operatorname{tr} u - \operatorname{tr} v) [f(\operatorname{tr} u) - f(\operatorname{tr} v)] \cdot \nu \\
& \leq \frac{1}{2} [\operatorname{sgn}(v_b - \operatorname{tr} u) - \operatorname{sgn}(u_b - \operatorname{tr} v)] [f(\operatorname{tr} u) - f(\operatorname{tr} v)] \cdot \nu.
\end{aligned} \tag{3.5.88}$$

The second line in (3.5.88) attains the following values:

	$u_b - \operatorname{tr} v > 0$	$u_b - \operatorname{tr} v = 0$	$u_b - \operatorname{tr} v < 0$
$v_b - \operatorname{tr} u > 0$	0	$\frac{1}{2} (f(\operatorname{tr} u) - f(\operatorname{tr} v)) \cdot \nu$	$(f(\operatorname{tr} u) - f(\operatorname{tr} v)) \cdot \nu$
$v_b - \operatorname{tr} u = 0$	$\frac{1}{2} (f(\operatorname{tr} v) - f(\operatorname{tr} u)) \cdot \nu$	0	$\frac{1}{2} (f(\operatorname{tr} u) - f(\operatorname{tr} v)) \cdot \nu$
$v_b - \operatorname{tr} u < 0$	$(f(\operatorname{tr} v) - f(\operatorname{tr} u)) \cdot \nu$	$\frac{1}{2} (f(\operatorname{tr} v) - f(\operatorname{tr} u)) \cdot \nu$	0

Clearly, we can reduce our study to two cases highlighted in the table above. Applying (3.2.3) to u with $k = u_b$ and to v with $k = v_b$ leads to

$$\operatorname{sgn}(\operatorname{tr} u - u_b) [f(\operatorname{tr} u) - f(u_b)] \cdot \nu \geq 0, \tag{3.5.89}$$

$$\operatorname{sgn}(\operatorname{tr} v - v_b) [f(\operatorname{tr} v) - f(v_b)] \cdot \nu \geq 0. \tag{3.5.90}$$

Let $J = J(t, \xi) = \left\{ k \in \mathbb{R} : (u_b(t, \xi) - k)(k - v_b(t, \xi)) \geq 0 \right\}$. Focus on each case separately.

Case I: $u_b - \operatorname{tr} v \leq 0$ and $v_b - \operatorname{tr} u \geq 0$.

1. If $u_b \leq \operatorname{tr} v \leq \operatorname{tr} u \leq v_b$ or $u_b \leq \operatorname{tr} u \leq \operatorname{tr} v \leq v_b$, then

$$(f(\operatorname{tr} u) - f(\operatorname{tr} v)) \cdot \nu \leq \sup_{s, r \in J} \|f(s) - f(r)\|,$$

where $\|\cdot\|$ represents the Euclidean norm in $\mathbb{R}^n$.

2. If $\operatorname{tr} u \leq v_b \leq u_b \leq \operatorname{tr} v$ or $\operatorname{tr} u \leq u_b \leq v_b \leq \operatorname{tr} v$, then

$$\text{by (3.5.89)} \Rightarrow f(\operatorname{tr} u) \cdot \nu \leq f(u_b) \cdot \nu, \quad \text{by (3.5.90)} \Rightarrow f(\operatorname{tr} v) \cdot \nu \geq f(v_b) \cdot \nu.$$

Hence we have

$$(f(\operatorname{tr} u) - f(\operatorname{tr} v)) \cdot \nu \leq (f(u_b) - f(v_b)) \cdot \nu \leq \sup_{s, r \in J} \|f(s) - f(r)\|.$$

3. If $u_b \leq \text{tr } u \leq v_b \leq \text{tr } v$, by (3.5.90) $f(\text{tr } v) \cdot \nu \geq f(v_b) \cdot \nu$, and using the fact that $\text{tr } u \in [u_b, v_b]$, we get

$$(f(\text{tr } u) - f(\text{tr } v)) \cdot \nu \leq (f(\text{tr } u) - f(v_b)) \cdot \nu \leq \sup_{s,r \in J} \|f(s) - f(r)\|.$$

4. If $\text{tr } u \leq u_b \leq \text{tr } v \leq v_b$, by (3.5.89) $f(\text{tr } u) \cdot \nu \leq f(u_b) \cdot \nu$, and using the fact that $\text{tr } v \in [u_b, v_b]$, we obtain

$$(f(\text{tr } u) - f(\text{tr } v)) \cdot \nu \leq (f(u_b) - f(\text{tr } v)) \cdot \nu \leq \sup_{s,r \in J} \|f(s) - f(r)\|.$$

Case II: $u_b - \text{tr } v \geq 0$ and $u_b - \text{tr } v \leq 0$.

1. If $v_b \leq \text{tr } v \leq \text{tr } u \leq u_b$ or $v_b \leq \text{tr } u \leq \text{tr } v \leq u_b$, then

$$(f(\text{tr } v) - f(\text{tr } u)) \cdot \nu \leq \sup_{s,r \in J} \|f(s) - f(r)\|.$$

2. If $\text{tr } v \leq v_b \leq u_b \leq \text{tr } u$ or $\text{tr } v \leq u_b \leq v_b \leq \text{tr } u$, then

$$\text{by (3.5.89)} \Rightarrow f(\text{tr } u) \cdot \nu \geq f(u_b) \cdot \nu, \quad \text{by (3.5.90)} \Rightarrow f(\text{tr } v) \cdot \nu \leq f(v_b) \cdot \nu.$$

Hence we have

$$(f(\text{tr } v) - f(\text{tr } u)) \cdot \nu \leq (f(v_b) - f(u_b)) \cdot \nu \leq \sup_{s,r \in J} \|f(s) - f(r)\|.$$

3. If $v_b \leq \text{tr } v \leq u_b \leq \text{tr } u$, by (3.5.89) $f(\text{tr } u) \cdot \nu \geq f(u_b) \cdot \nu$ and using the fact that $\text{tr } v \in [v_b, u_b]$, we get

$$(f(\text{tr } v) - f(\text{tr } u)) \cdot \nu \leq (f(\text{tr } v) - f(u_b)) \cdot \nu \leq \sup_{s,r \in J} \|f(s) - f(r)\|.$$

4. If $\text{tr } v \leq v_b \leq \text{tr } u \leq u_b$, by (3.5.90) $f(\text{tr } v) \cdot \nu \leq f(v_b) \cdot \nu$, and using the fact that $\text{tr } u \in [v_b, u_b]$, we obtain

$$(f(\text{tr } v) - f(\text{tr } u)) \cdot \nu \leq (f(v_b) - f(\text{tr } u)) \cdot \nu \leq \sup_{s,r \in J} \|f(s) - f(r)\|.$$

Hence, (3.5.88) can be estimated as follows:

$$\begin{aligned} & -\text{sgn}(\text{tr } u(t, \xi) - \text{tr } v(t, \xi)) \left[f(t, \xi, \text{tr } u(t, \xi)) - f(t, \xi, \text{tr } v(t, \xi)) \right] \cdot \nu(\xi) \\ & \leq \frac{1}{2} \left[\text{sgn}(v_b(t, \xi) - \text{tr } u(t, \xi)) - \text{sgn}(u_b(t, \xi) - \text{tr } v(t, \xi)) \right] \\ & \quad \times \left[f(t, \xi, \text{tr } u(t, \xi)) - f(t, \xi, \text{tr } v(t, \xi)) \right] \cdot \nu(\xi) \\ & \leq \sup_{s,r \in J(t,\xi)} \|f(t, \xi, s) - f(t, \xi, r)\| \\ & \leq \|\partial_u f\|_{\mathbf{L}^\infty(\Sigma; \mathbb{R}^n)} |u_b(t, \xi) - v_b(t, \xi)|. \end{aligned}$$

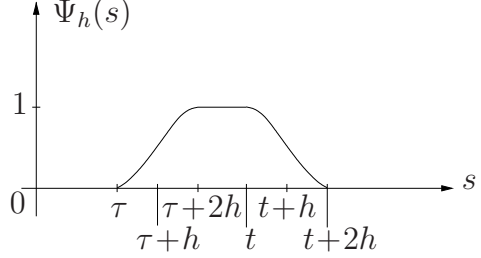
Since Ψ assumes only positive values, we can estimate (3.5.87) by

$$0 \leq [(3.5.87)] \leq \int_I \int_{\Omega} \left\{ |u(t, x) - v(t, x)| \Psi'(t) + \|\partial_u F\|_{\mathbf{L}^\infty(\Sigma; \mathbb{R})} |u(t, x) - v(t, x)| \Psi(t) \right\} dx dt \\ + \|\partial_u f\|_{\mathbf{L}^\infty(\Sigma; \mathbb{R}^n)} \int_I \int_{\partial\Omega} |u_b(t, \xi) - v_b(t, \xi)| \Psi(t) d\xi dt. \quad (3.5.91)$$

Introduce τ, t such that $0 < \tau < t < T$. Note that the map $s \rightarrow \Psi_h(s)$ defined by

$$\Psi_h(s) = \alpha_h(s - \tau - h) - \alpha_h(s - t - h),$$

where $\alpha_h(z) = \int_{-\infty}^z Y_h(\zeta) d\zeta$
and Y_h as in (3.5.53),



satisfies (3.5.76). Hence, we substitute Ψ_h for Ψ in (3.5.91). Observe that $\Psi_h \rightarrow \chi_{[\tau, t]}$ and $\Psi'_h \rightarrow \delta_\tau - \delta_t$ as h tends to 0. At the limit we obtain

$$0 \leq \int_{\Omega} |u(\tau, x) - v(\tau, x)| dx - \int_{\Omega} |u(t, x) - v(t, x)| dx \\ + \|\partial_u F\|_{\mathbf{L}^\infty(\Sigma; \mathbb{R})} \int_{\tau}^t \int_{\Omega} |u(s, x) - v(s, x)| dx ds \\ + \|\partial_u f\|_{\mathbf{L}^\infty(\Sigma; \mathbb{R}^n)} \int_{\tau}^t \int_{\partial\Omega} |u_b(s, \xi) - v_b(s, \xi)| d\xi ds.$$

A Gronwall type argument yields

$$\int_{\Omega} |u(t, x) - v(t, x)| dx \\ \leq e^{\|\partial_u F\|_{\mathbf{L}^\infty(\Sigma; \mathbb{R})}(t-\tau)} \int_{\Omega} |u(\tau, x) - v(\tau, x)| dx \\ + \|\partial_u f\|_{\mathbf{L}^\infty(\Sigma; \mathbb{R}^n)} \int_{\tau}^t e^{(t-\tau-s)\|\partial_u F\|_{\mathbf{L}^\infty(\Sigma; \mathbb{R})}} \int_{\partial\Omega} |u_b(s, \xi) - v_b(s, \xi)| d\xi ds. \quad (3.5.92)$$

In the limit $\tau \rightarrow 0$ for a.e. τ , an application of Proposition 3.2 completes the proof when $u_o \in \mathbf{C}^2(\bar{\Omega}; \mathbb{R})$ and $u_b \in \mathbf{C}^2(I \times \partial\Omega; \mathbb{R})$. The general case now follows by a straightforward regularisation argument. $\square$

3.5.3 Proofs Related to Section 3.2

Proof of Proposition 3.2. Let $M = \max\{\|u\|_{\mathbf{L}^\infty(I \times \Omega; \mathbb{R})}, \|u_o\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})}, \|u_b\|_{\mathbf{L}^\infty(I \times \partial\Omega; \mathbb{R})}\}$. We first prove that choosing $k \in]-\infty, -M[\cap]M, +\infty[$, the terms containing k in the left hand side in (3.2.1) vanish. Indeed, assuming $k < -M$, observe that

$$\begin{aligned} |u(t, x) - k| &= u(t, x) - k & \text{sgn}(u(t, x) - k) &= 1 \\ |u_o(x) - k| &= u_o(x) - k & \text{sgn}(u_b(t, \xi) - k) &= 1. \end{aligned} \quad (3.5.93)$$

Therefore, the terms containing k in the left hand side in (3.2.1) are:

$$\begin{aligned}
& \int_I \int_{\Omega} \left(-k \partial_t \varphi(t, x) - f(t, x, k) \cdot \nabla \varphi(t, x) - \operatorname{div} f(t, x, k) \varphi(t, x) \right) dx dt \\
& \quad - \int_{\Omega} k \varphi(0, x) dx + \int_I \int_{\partial\Omega} f(t, \xi, k) \cdot \nu(\xi) \varphi(t, \xi) d\xi dt \\
& = \int_I \int_{\Omega} -\operatorname{div} (f(t, x, k) \varphi(t, x)) dx dt + \int_I \int_{\partial\Omega} f(t, \xi, k) \cdot \nu(\xi) \varphi(t, \xi) d\xi dt \\
& \quad = 0.
\end{aligned}$$

The inequality (3.2.1) now reads

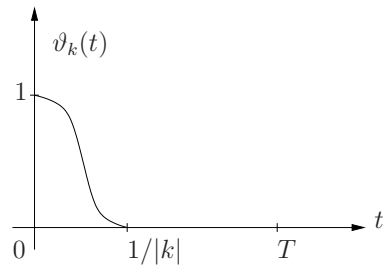
$$\begin{aligned}
0 & \leq \int_I \int_{\Omega} \left\{ u(t, x) \partial_t \varphi(t, x) + f(t, x, u) \cdot \nabla \varphi(t, x) + F(t, x, u) \varphi(t, x) \right\} dx dt \\
& \quad + \int_{\Omega} \varphi(0, x) u_o(x) dx - \int_I \int_{\partial\Omega} f(t, \xi, (\operatorname{tr} u)(t, \xi)) \cdot \nu(\xi) \varphi(t, \xi) d\xi dt \\
& = \int_I \int_{\Omega} \left\{ u(t, x) \partial_t \varphi(t, x) + f(t, x, u) \cdot \nabla \varphi(t, x) + F(t, x, u) \varphi(t, x) \right\} dx dt \\
& \quad + \int_{\Omega} \varphi(0, x) u_o(x) dx - \int_I \int_{\partial\Omega} \operatorname{tr} f(t, \xi, u(t, \xi)) \cdot \nu(\xi) \varphi(t, \xi) d\xi dt \\
& = \int_I \int_{\Omega} \left\{ u(t, x) \partial_t \varphi(t, x) - \varphi(t, x) \operatorname{div} f(t, x, u) + F(t, x, u) \varphi(t, x) \right\} dx dt \\
& \quad + \int_{\Omega} \varphi(0, x) u_o(x) dx,
\end{aligned}$$

by Lemma 3.20 and the Divergence Theorem. Choose $\varphi_k \in \mathbf{C}_c^2[-\infty, T[\times \mathbb{R}^n; \mathbb{R}^+)$ as

$$\varphi_k(t, x) = \vartheta_k(t) \psi(x),$$

where $\vartheta_k \in \mathbf{C}_c^2([0, T[; [0, 1])$ is such that

$$\begin{aligned}
& \vartheta_k(0) = 1, \\
& \vartheta_k(t) = 0 \text{ for all } t \geq 1/|k|, \\
& \sup_k \frac{1}{|k|} \|\vartheta'_k\|_{\mathbf{C}^0} < +\infty,
\end{aligned}$$



while $\psi \in \mathbf{C}_c^2(\mathbb{R}^n; \mathbb{R}^+)$. Hence,

$$\begin{aligned}
0 & \leq \int_I \int_{\Omega} \left\{ u(t, x) \vartheta'_k(t) \psi(x) + (F(t, x, u) - \operatorname{div} f(t, x, u)) \vartheta_k(t) \psi(x) \right\} dx dt \\
& \quad + \int_{\Omega} \psi(x) u_o(x) dx.
\end{aligned}$$

Pass now to the limit for $k \rightarrow -\infty$. Observe that, by the Dominated Convergence Theorem,

$$\lim_{k \rightarrow +\infty} \int_I \int_{\Omega} (F(t, x, u) - \operatorname{div} f(t, x, u)) \vartheta_k(t) \psi(x) dx dt = 0.$$

Thanks to Lemma 3.21 and to the Dominated Convergence Theorem we also have

$$\lim_{k \rightarrow +\infty} \int_I \int_{\Omega} u(t, x) \vartheta'_k(t) \psi(x) \, dx \, dt = - \int_{\Omega} u(0+, x) \psi(x) \, dx.$$

Then, in the case $k < -M$, (3.2.1) reduces to

$$0 \leq \int_{\Omega} \psi(x) u_o(x) \, dx - \int_{\Omega} u(0+, x) \psi(x) \, dx.$$

If $k > M$ the signs in (3.5.93) are opposite and analogous computations show that (3.2.1) reduces to

$$0 \geq \int_{\Omega} \psi(x) u_o(x) \, dx - \int_{\Omega} u(0+, x) \psi(x) \, dx.$$

Hence,

$$\int_{\Omega} \psi(x) (u_o(x) - u(0+, x)) \, dx = 0 \quad \text{for all } \psi \in \mathbf{C}_c^2(\mathbb{R}^n; \mathbb{R}^+).$$

We then obtain that

$$\begin{aligned} 0 &= \int_{\Omega} \psi(x) (u_o(x) - u(0+, x)) \, dx \\ &= \int_{\Omega} \psi(x) \left(u_o(x) - \lim_{t \rightarrow 0+} \frac{1}{t} \int_0^t u(\tau, x) \, d\tau \right) \, dx \\ &= \lim_{t \rightarrow 0+} \frac{1}{t} \int_{\Omega} \int_0^t \psi(x) (u_o(x) - u(\tau, x)) \, d\tau \, dx \\ &= \lim_{t \rightarrow 0+} \frac{1}{t} \int_0^t \int_{\Omega} \psi(x) (u_o(x) - u(\tau, x)) \, dx \, d\tau. \end{aligned}$$

Therefore, there exists a set $\mathcal{E} \subset I$ with measure 0 such that

$$\lim_{t \rightarrow 0+, t \in I \setminus \mathcal{E}} \int_{\Omega} \psi(x) (u_o(x) - u(t, x)) \, dx = 0$$

and, by the arbitrariness of ψ , the proof is completed. $\square$

Proof of Proposition 3.3. Let $\Psi \in \mathbf{C}_c^2([0, T] \times \mathbb{R}^n; \mathbb{R}^+)$ and Φ_h as in (3.5.38). Write (3.2.1) with $\varphi(t, x) = \Psi(t, x) \Phi_h(x)$ and take the limit as $h \rightarrow 0$. For all $k \in \mathbb{R}$:

$$\begin{aligned} \lim_{h \rightarrow 0} \int_I \int_{\Omega} |u(t, x) - k| \partial_t \Psi(t, x) \Phi_h(x) \, dx \, dt &= 0; \\ \lim_{h \rightarrow 0} \int_I \int_{\Omega} \operatorname{sgn}(u(t, x) - k) (f(t, x, u) - f(t, x, k)) \cdot \nabla \Psi(t, x) \Phi_h(x) \, dx \, dt &= 0; \\ \lim_{h \rightarrow 0} \int_I \int_{\Omega} \operatorname{sgn}(u(t, x) - k) (f(t, x, u) - f(t, x, k)) \cdot \nabla \Phi_h(x) \Psi(t, x) \, dx \, dt &= \\ \int_I \int_{\partial \Omega} \operatorname{sgn}(\operatorname{tr} u(t, \xi) - k) \left(f(t, \xi, \operatorname{tr} u(t, \xi)) - f(t, \xi, k) \right) \cdot \nu(\xi) \Psi(t, \xi) \, d\xi \, dt; \\ \lim_{h \rightarrow 0} \int_I \int_{\Omega} \operatorname{sgn}(u(t, x) - k) (F(t, x, u) - \operatorname{div} f(t, x, k)) \Psi(t, x) \Phi_h(x) \, dx \, dt &= 0; \\ \lim_{h \rightarrow 0} \int_{\Omega} \Psi(0, x) \Phi_h(x) |u_o(x) - k| \, dx &= 0; \end{aligned}$$

$$\begin{aligned} \lim_{h \rightarrow 0} \iint_{I \times \partial\Omega} \operatorname{sgn}(u_b(t, \xi) - k) \left[f(t, \xi, \operatorname{tr} u(t, \xi)) - f(t, \xi, k) \right] \cdot \nu(\xi) \Psi(t, \xi) \Phi_h(\xi) \, d\xi \, dt = \\ \iint_{I \times \partial\Omega} \operatorname{sgn}(u_b(t, \xi) - k) \left(f(t, \xi, \operatorname{tr} u(t, \xi)) - f(t, \xi, k) \right) \cdot \nu(\xi) \Psi(t, \xi) \, d\xi \, dt, \end{aligned}$$

where we used the Dominated Convergence Theorem, Lemma 3.22 and Lemma 3.20. The latter Lemma can be used since the function $(u, k) \rightarrow \operatorname{sgn}(u - k) (f(t, x, u) - f(t, x, k))$ is Lipschitz continuous by [47, Lemma 3]. Therefore,

$$\begin{aligned} \iint_{I \times \partial\Omega} \left[\operatorname{sgn}(\operatorname{tr} u(t, \xi) - k) - \operatorname{sgn}(u_b(t, \xi) - k) \right] \left[f(t, \xi, \operatorname{tr} u(t, \xi)) - f(t, \xi, k) \right] \\ \cdot \nu(\xi) \Psi(t, x) \, d\xi \, dt \geq 0. \end{aligned}$$

Hence,

$$\left[\operatorname{sgn}(\operatorname{tr} u(t, \xi) - k) - \operatorname{sgn}(u_b(t, \xi) - k) \right] \left[f(t, \xi, \operatorname{tr} u(t, \xi)) - f(t, \xi, k) \right] \cdot \nu(\xi) \geq 0 \quad (3.5.94)$$

almost everywhere on $]0, T[\times \partial\Omega$ for all $k \in \mathbb{R}$. Inequality (3.5.94) is reduced to (3.2.3) by taking k in the interval $\mathcal{I}(t, \xi)$. $\square$

Proof of Proposition 3.6. Let u satisfy Definition 3.5. Then, choose for instance $\mathcal{E}_m(u) = \sqrt{\frac{1}{m} + (u - k)^2}$ for $k \in \mathbb{R}$ and $m \in \mathbb{N}$. The entropy flux is then defined by 3. in Definition 3.4. A standard limiting procedure allows to obtain (3.2.1) in the limit $m \rightarrow +\infty$.

Conversely, let u solve problem (3.1.1) in the sense of Definition 3.1 and assume that $\|u\|_{\mathbf{L}^\infty(I \times \Omega; \mathbb{R})} \leq M$. Then, clearly, u satisfies (3.2.4) with $\mathcal{E}(u) = \alpha|u - k| + \beta$, for any $\alpha > 0$ and $k, \beta \in \mathbb{R}$. Further, note that if u satisfies (3.2.4) with two pairs $(\mathcal{E}_1, \mathcal{F}_1)$ and $(\mathcal{E}_2, \mathcal{F}_2)$ (for continuous maps $\mathcal{E}_1, \mathcal{E}_2, \mathcal{F}_1, \mathcal{F}_2$), then it satisfies the same inequality also with respect to $(\mathcal{E}_1 + \mathcal{E}_2, \mathcal{F}_1 + \mathcal{F}_2)$. Inductively, u satisfies (3.2.4) for any pair $(\mathcal{E}, \mathcal{F})$ with $\mathcal{E}$ piecewise linear and continuous on $[-M, M]$. Remark also that if u satisfies (3.2.4) with respect to the continuous pairs $(\mathcal{E}_n, \mathcal{F}_n)$ and the $\mathcal{E}_n$ are uniformly convergent to $\mathcal{E}$ on $[-M, M]$, then u satisfies (3.2.4) also with respect to the pair $(\mathcal{E}, \mathcal{F})$, where $\mathcal{F}$ is given by 3. in Definition 3.5. Finally, since any convex entropy $\mathcal{E}$ is the uniform limit on $[-M, M]$ of piecewise linear and continuous functions, we obtain the proof. $\square$

Proof of Theorem 3.7. The proof consists in regularising the initial datum through a sequence u_o^m . Applying Theorem 3.13, we have a sequence of solutions u_m . Theorem 3.14 allows to prove that u_m satisfies the Cauchy condition, hence converges to a map u , which is proved to solve (3.1.1).

To approximate the initial datum, using [39, Formula (1.8) and Proposition 1.15] introduce a sequence $\tilde{u}_m \in \mathbf{C}^\infty(\Omega; \mathbb{R})$ such that

$$\lim_{m \rightarrow +\infty} \|u_o - \tilde{u}_m\|_{\mathbf{L}^1(\Omega; \mathbb{R})} = 0, \quad \|\tilde{u}_m\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} \leq \|u_o\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} \quad \text{and} \quad \lim_{m \rightarrow +\infty} \operatorname{TV}(\tilde{u}_m) = \operatorname{TV}(u_o).$$

Define now $\Psi_m = 1 - \Phi_{1/m}$, with $\Phi_{1/m}$ as in (3.5.38). Let

$$u_o^m(x) = \Psi_m(x) \tilde{u}_m(x) \quad \text{for all } x \in \Omega. \quad (3.5.95)$$

By construction, $\lim_{m \rightarrow +\infty} \|u_o^m - u_o\|_{\mathbf{L}^1(\Omega; \mathbb{R})} = 0$, so that u_o^m is a Cauchy sequence in $\mathbf{L}^1(\Omega; \mathbb{R})$. We have also the uniform bounds

$$\|u_o^m\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} \leq \|u_o\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})}, \quad (3.5.96)$$

$$\begin{aligned} \text{TV}(u_o^m) &\leq \|\nabla \Psi_m\|_{\mathbf{L}^1(\Omega; \mathbb{R}^n)} \|\tilde{u}_m\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} + \|\Psi_m\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} \|\nabla \tilde{u}_m\|_{\mathbf{L}^1(\Omega; \mathbb{R}^n)} \\ &\leq \mathcal{O}(1) \|u_o\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} + \text{TV}(u_o). \end{aligned} \quad (3.5.97)$$

Since for any $m \in \mathbb{N} \setminus \{0\}$ we have that $u_o^m(\xi) = u_b(0, \xi) = 0$, Theorem 3.13 applies to (3.1.1) with initial datum u_o^m and boundary datum u_b , yielding the existence of a solution u_m to (3.1.1) in the sense of Definition 3.1, which satisfies the estimates (3.4.4), (3.4.5) and (3.4.6). Theorem 3.14 then implies that

$$\int_{\Omega} |u_{m'}(t, x) - u_{m''}(t, x)| \, dx \leq e^{L_F t} \int_{\Omega} |u_o^{m'}(x) - u_o^{m''}(x)| \, dx,$$

proving that the sequence $u_m(t)$ satisfies the Cauchy condition in $\mathbf{L}^1(\Omega; \mathbb{R})$ uniformly in $t \in I$.

Call $u = \lim_{m \rightarrow \infty} u_m$. We now verify that u solves (3.1.1). By Proposition 3.6, each u_m satisfies (3.2.4) for any $\mathbf{C}^2$ entropy $\mathcal{E}_l(u) = \sqrt{\frac{1}{l} + (u - k)^2}$ and for any $\varphi \in \mathbf{C}_c^2([-\infty, T] \times \mathbb{R}^n; \mathbb{R}^+)$:

$$\begin{aligned} &\int_I \int_{\Omega} \left\{ \mathcal{E}_l(u_m(t, x)) \partial_t \varphi(t, x) + \mathcal{F}_l(t, x, u_m(t, x)) \cdot \nabla \varphi(t, x) \right. \\ &\quad \left. + \left[\mathcal{E}_l'(u_m(t, x)) \left(F(t, x, u_m(t, x)) - \text{div } f(t, x, u_m(t, x)) \right) \right. \right. \\ &\quad \left. \left. + \text{div } \mathcal{F}_l(t, x, u_m(t, x)) \right] \varphi(t, x) \right\} \, dx \, dt \\ &\quad + \int_{\Omega} \mathcal{E}_l(u_o^m(x)) \varphi(0, x) \, dx \\ &- \int_I \int_{\partial\Omega} \left[\mathcal{F}_l(t, \xi, u_b(t, \xi)) - \mathcal{E}_l'(u_b(t, \xi)) \left(f(t, \xi, u_b(t, \xi)) - f(t, \xi, \text{tr } u_m(t, \xi)) \right) \right] \cdot \nu(\xi) \\ &\quad \times \varphi(t, \xi) \, d\xi \, dt \geq 0. \end{aligned}$$

In the limit $m \rightarrow +\infty$, since u_m converges in $\mathbf{L}^1$ to u , $\text{tr } u_m$ converges to $\text{tr } u$ by Lemma 3.18, which can be applied thanks to the estimate (3.4.5). Hence, we have

$$\begin{aligned} &\int_I \int_{\Omega} \left\{ \mathcal{E}_l(u(t, x)) \partial_t \varphi(t, x) + \mathcal{F}_l(t, x, u(t, x)) \cdot \nabla \varphi(t, x) \right. \\ &\quad \left. + \left[\mathcal{E}_l'(u(t, x)) \left(F(t, x, u(t, x)) - \text{div } f(t, x, u(t, x)) \right) + \text{div } \mathcal{F}_l(t, x, u(t, x)) \right] \varphi(t, x) \right\} \, dx \, dt \\ &\quad - \int_I \int_{\partial\Omega} \left[\mathcal{F}_l(t, \xi, u_b(t, \xi)) - \mathcal{E}_l'(u_b(t, \xi)) \left(f(t, \xi, u_b(t, \xi)) - f(t, \xi, \text{tr } u(t, \xi)) \right) \right] \cdot \nu(\xi) \\ &\quad \times \varphi(t, \xi) \, d\xi \, dt \geq 0. \end{aligned}$$

In the limit $l \rightarrow +\infty$, we have the convergences $\mathcal{E}_l \rightarrow \mathcal{E}$ and $\mathcal{F}_l \rightarrow \mathcal{F}$, with $\mathcal{E}(u) = |u - k|$ and $\mathcal{F}(t, x, u) = \text{sgn}(u - k) (f(t, x, u) - f(t, x, k))$, so that

$$\begin{aligned} &\int_I \int_{\Omega} \left\{ |u(t, x) - k| \partial_t \varphi(t, x) + \text{sgn}(u(t, x) - k) (f(t, x, u) - f(t, x, k)) \cdot \nabla \varphi(t, x) \right. \\ &\quad \left. + \text{sgn}(u(t, x) - k) (F(t, x, u) - \text{div } f(t, x, k)) \varphi(t, x) \right\} \, dx \, dt \end{aligned}$$

$$\begin{aligned}
& + \int_{\Omega} |u(0, x) - k| \varphi(0, x) \, dx \\
& - \int_I \int_{\partial\Omega} \operatorname{sgn}(u_b(t, \xi) - k) \left(f(t, \xi, (\operatorname{tr} u)(t, \xi)) - f(t, \xi, k) \right) \cdot \nu(\xi) \varphi(t, \xi) \, d\xi \, dt \geq 0.
\end{aligned}$$

Finally, observe that

$$\begin{aligned}
\int_{\Omega} |u(0, x) - k| \varphi(0, x) \, dx & \leq \int_{\Omega} |u_o(x) - k| \varphi(0, x) \, dx \\
& + \int_{\Omega} |u_o(x) - u_o^m(x)| \varphi(0, x) \, dx \xrightarrow{m \rightarrow +\infty} 0 \text{ by (3.5.95)} \\
& + \int_{\Omega} |u_o^m(x) - u_m(0, x)| \varphi(0, x) \, dx \xrightarrow{m \rightarrow +\infty} 0 \text{ by Proposition 3.2} \\
& + \int_{\Omega} |u_m(0, x) - u(0, x)| \varphi(0, x) \, dx \xrightarrow{m \rightarrow +\infty} 0 \text{ since } u_m \rightarrow u \text{ in } \mathbf{L}^1,
\end{aligned}$$

concluding the existence proof.

The $\mathbf{L}^\infty$ and TV bounds and the Lipschitz continuity in time directly follow from (3.4.4), (3.4.5) and (3.4.6), thanks to the properties (3.5.96) and (3.5.97) of the sequence u_o^m . $\square$

3.5.4 The Trace Operator

A relevant role is played by the trace operator which we recall here from [31, Paragraph 5.3].

Definition 3.17. *Let $A \subset \mathbb{R}^n$ be bounded with Lipschitz boundary. The trace operator is the map $\operatorname{tr}_A: \mathbf{BV}(A; \mathbb{R}) \rightarrow \mathbf{L}^1(\partial A; \mathbb{R})$ such that for all $\varphi \in \mathbf{C}^1(\mathbb{R}^n; \mathbb{R}^n)$ and for all $w \in \mathbf{BV}(A; \mathbb{R})$,*

$$\int_{\partial A} ((\operatorname{tr}_A w)(\xi)) \varphi(\xi) \cdot \nu(\xi) \, d\xi = \int_A w(x) \operatorname{div} \varphi(x) \, dx + \int_A \varphi(x) \, d(\nabla w(x)).$$

Below, when no misunderstanding arises, we omit the dependence of the trace operator on the set. First of all, we recall without proof the following two lemmas.

Lemma 3.18 ([31, Paragraph 5.3, Theorem 1 and Theorem 2]). *Let $A \subset \mathbb{R}^n$ be bounded with Lipschitz boundary. Fix $w \in \mathbf{BV}(A; \mathbb{R})$. Then, the trace operator is a bounded linear operator and for $\mathcal{H}^{n-1}$ -a.e. $\xi \in \partial A$,*

$$\lim_{r \rightarrow 0+} \frac{1}{\mathcal{L}^n(B(\xi, r) \cap A)} \int_{B(\xi, r) \cap A} |w(x) - (\operatorname{tr} w)(\xi)| \, dx = 0.$$

Lemma 3.19 ([31, Paragraph 5.3, Remark to Theorem 2]). *Let $A \subset \mathbb{R}^n$ be bounded with Lipschitz boundary. Fix $w \in \mathbf{BV}(A; \mathbb{R}) \cap \mathbf{C}^0(\bar{A}; \mathbb{R})$. Then, $(\operatorname{tr} w)(\xi) = w(\xi)$ for $\mathcal{H}^{n-1}$ -a.e. $\xi \in \partial A$.*

Recall also the following property.

Lemma 3.20. *Let $A \subset \mathbb{R}^n$ be bounded with Lipschitz boundary. Fix $w \in \mathbf{BV}(A; \mathbb{R})$ and $h \in \mathbf{C}^{0,1}(\mathbb{R}; \mathbb{R})$. Then, $\operatorname{tr}(h \circ w) = h \circ (\operatorname{tr} w)$ for $\mathcal{H}^{n-1}$ -a.e. $\xi \in \partial A$.*

Proof. For any $\xi \in \partial A$ and for $r > 0$ sufficiently small, compute:

$$\begin{aligned}
& \left| (\operatorname{tr}(h \circ w))(\xi) - (h \circ (\operatorname{tr} w))(\xi) \right| \\
& \leq \frac{1}{\mathcal{L}^n(B(\xi, r) \cap A)} \int_{B(\xi, r) \cap A} \left| (h \circ w)(x) - (\operatorname{tr}(h \circ w))(\xi) \right| dx \\
& \quad + \frac{1}{\mathcal{L}^n(B(\xi, r) \cap A)} \int_{B(\xi, r) \cap A} \left| (h \circ w)(x) - h((\operatorname{tr} w)(\xi)) \right| dx \\
& \leq \frac{1}{\mathcal{L}^n(B(\xi, r) \cap A)} \int_{B(\xi, r) \cap A} \left| (h \circ w)(x) - (\operatorname{tr}(h \circ w))(\xi) \right| dx \\
& \quad + \frac{\mathbf{Lip}(h)}{\mathcal{L}^n(B(\xi, r) \cap A)} \int_{B(\xi, r) \cap A} |w(x) - (\operatorname{tr} w)(\xi)| dx
\end{aligned}$$

and both addends in the right hand side above vanish as $r \rightarrow 0+$ by Lemma 3.18. $\square$

The next two Lemmas relate the values attained by the trace of u with limits at the boundary of integrals of u .

Lemma 3.21. *Let $T > 0$ and $u \in \mathbf{BV}([0, T]; \mathbb{R})$. Choose a sequence $\varphi_k \in \mathbf{C}_c^1([0, T]; [0, 1])$ such that $\varphi_k(0) = 1$, $\varphi_k(t) = 0$ for all $t \geq 1/k$ and $\sup_k \frac{1}{k} \|\varphi'_k\|_{\mathbf{L}^\infty([0, T]; \mathbb{R})} < +\infty$. Then,*

$$\lim_{k \rightarrow +\infty} \int_I u(t) \varphi'_k(t) dt = -u(0+).$$

Above, we used the standard notation $u(0+) = \operatorname{tr}_{[0, T]} u(0)$.

Proof. Denote $c = \sup_k \frac{1}{k} \|\varphi'_k\|_{\mathbf{L}^\infty([0, T]; \mathbb{R}^n)}$. Compute:

$$\begin{aligned}
\left| \int_I u(t) \varphi'_k(t) dt + u(0+) \right| &= \left| \int_0^{1/k} u(t) \varphi'_k(t) dt + u(0+) \right| \\
&\leq \int_0^{1/k} |u(t) - u(0+)| |\varphi'_k(t)| dt + \left| u(0+) \int_0^{1/k} \varphi'_k(t) dt + u(0+) \right| \\
&\leq \frac{c}{1/k} \int_0^{1/k} |u(t) - u(0+)| dt
\end{aligned}$$

which vanishes as $k \rightarrow +\infty$ by Lemma 3.18. $\square$

Lemma 3.22. *Let Ω satisfy $(\Omega_{2,0})$ and $u \in \mathbf{BV}(\Omega; \mathbb{R})$. Choose a sequence $\chi_k \in \mathbf{C}_c^1(\mathbb{R}^n; [0, 1])$ such that $\chi_k(\xi) = 1$ for all $\xi \in \partial\Omega$, $\chi_k(x) = 0$ for all $x \in \Omega$ with $B(x, 1/k) \subseteq \Omega$ and moreover $\sup_k \frac{1}{k} \|\nabla \chi_k\|_{\mathbf{L}^\infty(\Omega; \mathbb{R}^n)} < +\infty$. Then,*

$$\lim_{k \rightarrow +\infty} \int_\Omega u(x) \nabla \chi_k(x) dx = \int_{\partial\Omega} \operatorname{tr}_\Omega u(\xi) \nu(\xi) d\xi.$$

Proof. Let e_i be the i -th vector of the standard basis in $\mathbb{R}^n$. Set

$$\Omega_k = \{x \in \Omega: d(x, \partial\Omega) \leq 1/k\}.$$

Then:

$$\left(\int_\Omega u(x) \nabla \chi_k(x) dx - \int_{\partial\Omega} \operatorname{tr}_\Omega u(\xi) \nu(\xi) d\xi \right) \cdot e_i$$

$$\begin{aligned} &= \int_{\Omega} u(x) \partial_i \chi_k(x) \, dx - \int_{\partial\Omega} \operatorname{tr} u(\xi) \chi_k(\xi) e_i \cdot \nu(\xi) \, d\xi \\ &= \int_{\Omega_k} u(x) \partial_i \chi_k(x) \, dx - \int_{\partial\Omega_k} \operatorname{tr} u(\xi) \chi_k(\xi) e_i \cdot \nu(\xi) \, d\xi \\ &= - \int_{\Omega_k} \chi_k(x) \, d(\nabla u)_i(x), \end{aligned}$$

where Definition 3.17 was used to obtain the last expression. By the Dominated Convergence Theorem, $\lim_{k \rightarrow +\infty} \int_{\Omega_k} \chi_k(x) \, d(\nabla u)_i(x) = 0$, completing the proof. $\square$

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