# Concentrated Patterns in Biological Systems

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# Contents

CHAPTER 1

# Introduction

This thesis is concerned with pattern formation in biological systems, in particular those having the property that they concentrate at a boundary point in a bounded smooth domain.

We study this problem within the framework of deterministic reaction-diffusion systems. As a prototype system, which on the one hand captures the essential biological behavior and on the other hand is not too complex for a rigorous and explicit mathematical analysis, we consider the Gierer-Meinhardt system (see [**29**]). After a suitable rescaling, this system can be written as follows:

$$(GM) \quad \begin{cases} A_t = D_a \Delta A - A + \frac{A^2}{H}, \quad A > 0 \quad \text{in } \Omega, \\ \tau H_t = D_h \Delta H - H + A^2, \quad H > 0 \quad \text{in } \Omega, \\ \frac{\partial A}{\partial \nu} = \frac{\partial H}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

The unknowns $A = A(x,t)$ and $H = H(x,t)$ represent the concentrations of the biochemicals called activator and inhibitor, respectively, at a point $x \in \Omega \subset \mathbb{R}^2$ and at a time $t > 0$, respectively; the diffusion coefficients $D_a$, $D_h$ and the time relaxation constant $\tau$ are positive constants (independent of $x \in \Omega$ and $t > 0$) with $D_a << D_h$ and $\tau$ independent of $D_a$, $D_h$; we also use the notation $D_a = \epsilon^2$ and $D_h = D$; $\Delta := \sum_{j=1}^2 \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator in $\mathbb{R}^2$; $\Omega$ is a smooth bounded domain in $\mathbb{R}^2$; $\nu(x)$ is the outward unit normal vector at $x \in \partial\Omega$; $\frac{\partial}{\partial \nu_x}$ denotes the normal derivative at $x \in \partial\Omega$ (for simplicity we will mostly drop the index $x$).

To understand how (1.1) arises as a model of a biochemical reaction in a living organism note that there is an autocatalytic production of the activator $A$ via the $\frac{A^2}{H}$ term. The inhibitor $H$ is activated by $A$ due to the term $A^2$ in the second equation, but $H$ inhibits the production of the activator $A$ as $\frac{A^2}{H}$

decreases for increasing $H$. This summarizes the reaction part of the system. Adding the physical phenomenon of diffusion to the system we arrive at (1.1).

The classical Turing type linearized analysis shows that the homogeneous steady states are unstable. In the present study we perform an analysis in the neighborhood of certain inhomogeneous steady states and prove their existence and (linearized) stability rigorously.

The system (1.1) was first introduced by A. Gierer and H. Meinhardt, scientists at the Max-Planck Institute for Developmental Biology in Tübingen, to study the problem of formation of new heads for hydra, the orientation of the head and the arms/legs in embryotic growth, and lately the beautiful patterns on sea shells [**49**], [**50**]. It has been successfully used to predict patterns and understand the mechanism of their formation. Typically, in these examples the problem is posed in a two-dimensional domain (except in the study of sea shells which is considered as a system on a one-dimensional interval with the time axis corresponding to the direction of shell growth in time). Actually, they study the more general system

$$(GGM) \qquad \begin{cases} \frac{\partial A}{\partial t} = D_a \Delta A - A + \frac{A^p}{H^q}, \\ \tau \frac{\partial H}{\partial t} = D_H \Delta H - H + \frac{A^r}{H^s}, \end{cases}$$

$$p > 1, \quad q > 0, \quad r > 0, \quad s \geq 0,$$

$$0 < \frac{p-1}{q} < \frac{r}{s+1}.$$

Using the same mathematical methods but putting more effort into the notation one could generalize many of our results to this more general system. For example, the existence result, Theorem 1.1, can be extended to the generalized system without any technical difficulty. For the stability result, Theorem 1.2, there should be some restrictions on $(p, q, r, s)$. See [**14**], [**57**], [**58**], [**75**], and [**87**] for related studies on nonlocal eigenvalue problems (NLEPs).

For simplicity and readability in this thesis we restrict our attention to the simpler system (1.1).

To start the discussion of the mathematical behavior of (1.1), we first recall Turing's idea of a *diffusion-driven instability* [**65**]. Therefore, we first drop the diffusion terms and consider the kinetic system instead:

$$\begin{cases} A_t = -A + \frac{A^2}{H}, \\ \tau H_t = -H + A^2. \end{cases} \tag{1.2}$$

This system has a unique constant steady state $A \equiv 1, H \equiv 1$. For $0 < \tau < 1$ it is easy to see that the constant solution $A \equiv 1, H \equiv 1$ is (linearly) stable as a steady state of (1.2). However, if in (1.1) $\frac{D_a}{D_h}$ is small, then the constant steady state $A \equiv 1, H \equiv 1$ becomes unstable. This example shows that in the case of a system of partial differential equations diffusion may lead to instability of homogeneous states contrary to the common knowledge that diffusion is a smoothing and trivializing process. This intuition fails since we deal with a system of partial differential equations rather than a single equation. Therefore the maximum principle or energy methods are not available for the analysis. This example also shows that the size of the diffusion coefficients $D_a$ and $D_h$ is essential for the behavior of (1.1). Throughout this thesis we will assume that $D_a = \epsilon^2$ is small. Furthermore, we will assume that $D_h \sim \epsilon^{-1}$, where the notation $A \sim B$ means that $\lim \frac{A}{B} = C > 0$.

In this thesis, we will show that there is a critical growth rate for the inhibitor diffusivity $D_h$ given by

$$\lim_{\epsilon \to 0} D_h \epsilon = c_0^{-1}, \quad c_0 > 0,$$

such that there exists a steady state for (1.1) whose shape is given by a boundary spike the location of which is determined by the interaction of boundary curvature and Green's function effects.

Before stating the result in full detail, we first introduce some notation for $P \in \partial\Omega$,

$$\nabla_{\tau(P)} := \frac{\partial}{\partial \tau(P)}$$

with $\frac{\partial}{\partial \tau(P)}$ denoting the tangential derivative with respect to $P$ at $P \in \partial\Omega$. We will sometimes drop the argument $P$ if this can be done without causing confusion. Let $G_0(x, \xi)$ be the Green function which satisfies the following nonlocal linear boundary value problem for $\xi \in \Omega$:

$$\begin{cases} \Delta G_0(x,\xi) - \frac{1}{|\Omega|} + \delta_\xi(x) = 0 & \text{for } x \text{ in } \Omega, \\ \int_\Omega G_0(x,\xi)\,dx = 0, & \\ \frac{\partial G_0(x,\xi)}{\partial \nu_x} = 0 & \text{for } x \text{ on } \partial\Omega, \end{cases} \tag{1.3}$$

Then

$$H_0(Q,P) = \frac{1}{2\pi} \log \frac{1}{|Q-P|} - G_0(Q,P), \quad P, Q \in \Omega$$

is the regular part of the Green function for which the limit $H_0(P,P) = \lim_{Q\to P} H_0(Q,P)$ exists.

For $P \in \partial\Omega$ the behavior of $G_0(Q,P)$ is different. We define

$$H_0(Q,P) = \frac{1}{\pi} \log \frac{1}{|Q-P|} - G_0(Q,P), \quad Q \in \Omega, P \in \partial\Omega,$$

and then the limit $H_0(P,P) = \lim_{Q\to P, Q\in\Omega} H_0(Q,P)$ exists.

Let $w$ be the unique solution in $H^1(\mathbb{R}^2)$ of the problem

$$\begin{cases} \Delta w - w + w^2 = 0, \quad w > 0 \quad \text{in } \mathbb{R}^2, \\ w(0) = \max_{y\in\mathbb{R}^2} w(y), \quad w(y) \to 0 \text{ as } |y| \to \infty. \end{cases} \tag{1.4}$$

For existence and uniqueness of the solution of (1.4) we refer to [**28**] and [**43**]. We also recall that $w$ is radially symmetric and

$$w(y) \sim |y|^{-1/2} e^{-|y|} \quad \text{as } |y| \to \infty. \tag{1.5}$$

Finally, we introduce two negative constants $\nu_1$ and $\nu_2$ which are given by

$$\nu_1 = -\frac{1}{3} \int_{\mathbb{R}} \left( \frac{\partial w(y_1,0)}{\partial y_1} \right)^2 y_1^2\,dy_1 < 0 \tag{1.6}$$

and

$$\nu_2 = -\frac{1}{6} |\Omega| \int_{\mathbb{R}^2_+} w^3(y)\,dy, \tag{1.7}$$

respectively.

Now our fist main result, which is on existence, may be stated as follows:

THEOREM 1.1. *Let*

$$\lim_{\epsilon\to 0} D_h \epsilon = c_0^{-1}, \quad c_0 > 0.$$

*For $P \in \partial\Omega$, we define*

$$F(P) = \nu_1 h(P) + c_0 \nu_2 H_0(P,P), \tag{1.8}$$

*where $h$ denotes the curvature of $\partial\Omega$ at $P$, $H_0(P,P)$ denotes the regular part of the Green function, and $\nu_1$ and $\nu_2$ were defined in (1.6) and (1.7), respectively.*

*Suppose that $P_0$ is a nondegenerate critical point of $F(P)$ along the boundary, i.e. for $P = P_0$,*

$$\nabla_{\tau(P_0)} F(P_0) = 0,$$

$$(\nabla_{\tau(P_0)})^2 F(P_0)) \neq 0.$$

*Then, for $\epsilon$ small enough, problem (GM) has a steady state $(A_\epsilon, H_\epsilon)$ with the following properties:*

*(1) $A_\epsilon(x) = \xi_\epsilon(w(\frac{x-P^\epsilon}{\epsilon}) + O(\epsilon))$ uniformly for $x \in \bar{\Omega}$; $P^\epsilon \to P_0$ as $\epsilon \to 0$,*

*(2) $H_\epsilon(x) = \xi_\epsilon(1 + O(\epsilon))$ uniformly for $x \in \bar{\Omega}$,*

*where $w$ was defined in (1.4) and*

$$\xi_\epsilon = \frac{|\Omega|}{\epsilon^2 \int_{\mathbb{R}^2_+} w^2(y)\,dy} + O\left(\epsilon^{-1}\log\frac{1}{\epsilon}\right). \tag{1.9}$$

Next, we study the stability or instability of the boundary peak solutions constructed in Theorem 1.1. To this end, we need to study the following eigenvalue problem

$$\mathcal{L}_\epsilon \begin{pmatrix} \phi_\epsilon \\ \\ \psi_\epsilon \end{pmatrix} = \begin{pmatrix} \epsilon^2 \Delta\phi_\epsilon - \phi_\epsilon + 2\frac{A_\epsilon}{H_\epsilon}\phi_\epsilon - \frac{A_\epsilon^2}{H_\epsilon^2}\psi_\epsilon \\ \frac{1}{\tau}\left(\frac{1}{\beta^2}\Delta\psi_\epsilon - \psi_\epsilon + 2A_\epsilon\phi_\epsilon\right) \end{pmatrix} = \lambda_\epsilon \begin{pmatrix} \phi_\epsilon \\ \psi_\epsilon \end{pmatrix}, \tag{1.10}$$

where $(A_\epsilon, H_\epsilon)$ is the solution constructed Theorem 1.1 and $\lambda_\epsilon \in \mathbb{C}$, the set of complex numbers.

We say that the steady state $(A_\epsilon, H_\epsilon)$ is **linearly stable** if the spectrum $\sigma(\mathcal{L}_\epsilon)$ of $\mathcal{L}_\epsilon$ lies in the left half plane $\{\lambda \in \mathbb{C} : \ \mathrm{Re}(\lambda) < 0\}$. The steady state $(A_\epsilon, H_\epsilon)$ is called **linearly unstable** if there exists an eigenvalue $\lambda_\epsilon$ of $\mathcal{L}_\epsilon$ with $\mathrm{Re}(\lambda_\epsilon) > 0$. (From now on, we use the notations linearly stable and linearly unstable as defined above.)

Our second main result, which is on stability, is stated as follows.

THEOREM 1.2. *Let $P_0$ be a nondegenerate critical point of $F$ along the boundary, i.e., $\nabla_{\tau(P_0)}F(P_0) = 0$. Assume further that*

$$(*) \qquad P_0 \text{ is a nondegenerate local maximum point of } F(P),$$

*i.e., $(\nabla_{\tau(P_0)})^2 F(P_0) < 0$. Then there exists a unique $\tau_1 > 0$ such that for $\tau < \tau_1$ the steady state $(A_\epsilon, H_\epsilon)$ introduced in Theorem 1.1 is linearly stable, while for $\tau > \tau_1$ it is linearly unstable.*

*Moreover, if $\lambda_\epsilon \to 0$, then we have the following asymptotic behavior of $\lambda_\epsilon$:*

$$\frac{\lambda_\epsilon}{\epsilon^2} \to \eta_0 (\nabla_{\tau(P_0)})^2 F(P_0), \tag{1.11}$$

*where*

$$\eta_0 = \frac{1}{\int_{\mathbb{R}^2_+} \left(\frac{\partial w}{\partial y_1}\right)^2 dy} > 0, \tag{1.12}$$

*the eigenfunction corresponding to $\lambda_\epsilon$ satisfies*

$$\phi^\epsilon = \epsilon \nabla_{\tau(P^\epsilon)} w \left(\frac{x - P^\epsilon}{\epsilon}\right) + o(1), \tag{1.13}$$

*and $P^\epsilon \to P_0$ as $\epsilon \to 0$.*

**Remark.** The condition $(*)$ on the location $P_0$ arises in the study of small ($o(1)$) eigenvalues. The continuous function $F(P)$ obviously attains its global maximum at $P \in \partial\Omega$ for any smooth bounded domain $\Omega \subset \mathbb{R}^2$. If this global maximum point is also a nondegenerate critical point, then condition $(*)$ is satisfied for this global maximum point. We believe that for **generic** domains this global maximum point $P_0$ is nondegenerate.

The first term in $F(P)$ is related to the local geometry of $\partial\Omega$ near $P$ and enters into the analysis through the first equation of (GM), whereas the second term is related to the global geometry of $\Omega$ and enters into the analysis through the second equation. Let us mention that to our knowledge this result is the first of its kind for reaction-diffusion systems, where the boundary curvature and the Green function interact in such an additive way.

We are not aware of results of such a coupling phenomenon in the setting of a single second-order elliptic partial differential equation. Thus it appears that this coupling behavior is typical for systems but does not arise for a single equation.

This two-dimensional result is potentially important for applications in mathematical biology, for which the Gierer-Meinhardt has been designed.

The key references for this thesis are [**77**], where to the best of our knowledge Liapunov-Schmidt reduction was used for the first time to construct boundary spikes for a nonlinear elliptic partial differential equation in higher dimensions, and [**82**], where the existence and stability of (interior) multiple spike solutions for the Gierer-Meinhardt system were established. We combine those two approaches to construct steady states with a single boundary spike and prove their (in)stability.

Now we comment on some related work.

Numerical studies by Meinhardt [**49**] and more recently by Holloway [**36**] and Maini and McInerney [**48**] have revealed that when $\epsilon$ is small and $D$ is finite, (GM) seems to have stable stationary solutions with the property that the activator is localized around a finite number of points in $\overline{\Omega}$. Moreover, as $\epsilon \to 0$, the pattern exhibits a *"point condensation phenomenon"*. By this we mean that the activator is localized in narrower and narrower regions of size $O(\epsilon)$ around these points and eventually shrinks to the set of points itself as $\epsilon \to 0$. Furthermore, the maximum value of the activator and the inhibitor, respectively, diverges to $+\infty$.

Although it has been observed numerically that these patterns are stable, until recently it has been an open problem to give a rigorous proof of these facts. Namely, how can one rigorously construct these solutions? Where are the spikes located? Are these solutions stable?

Recall that the stationary system for (GM) is the following system of elliptic equations:

$$\begin{cases} \epsilon^2 \Delta A - A + \frac{A^2}{H} = 0, \quad A > 0 \quad \text{in } \Omega, \\ D_h \Delta H - H + A^2 = 0, \quad H > 0 \quad \text{in } \Omega, \\ \frac{\partial A}{\partial \nu} = \frac{\partial H}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.14}$$

There are a number of recent results for interior spikes solutions for this system in the case $\Omega \subset \mathbb{R}^2$. In [**80**], for the strong-coupling case, i.e., $D_h \sim$

1, Wei and the author constructed single interior spike solutions to (1.14) (without loss of generality, we assumed that $D_h = 1$). Then in [**83**] we continued that study: After constructing interior $K$-spike solutions (for any integer $K \geq 1$), we also proved that they are (linearly) stable for $\tau = 0$ provided that the limiting spike locations $\mathbf{P}^0 = (P_1^0, ..., P_K^0)$ constitute a nondegenerate local maximum point of the following functional

$$F_1(\mathbf{P}) = \sum_{k=1}^{K} H_1(P_k, P_k) - \sum_{i,j=1,\ldots,K, i\neq j} G_1(P_i, P_j), \tag{1.15}$$

where $G_1(P, x)$ is the Green function of $-\Delta+1$ under the Neumann boundary condition, i.e., $G_1$ satisfies

$$\begin{cases} -\Delta G_1 + G_1 = \delta_P & \text{in } \Omega, \\ \frac{\partial G_1}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Here $\delta_P$ is the Dirac delta distribution at a point $P \in \Omega$ and

$$H_1(x, P) = K_1(|x - P|) - G_1(x, P),$$

where $K_1$ is the fundamental solution of $-\Delta + 1$ in $\mathbb{R}^2$ with singularity at 0. Therefore for any finite $D_h \sim 1$, the stability of interior $K-$spike solutions does not depend on $D_h$ but on the spike locations only.

For the weak-coupling case, i.e., $D_h \to \infty$, Wei and the author proved in [**82**] that, for $\epsilon << 1$, if a condition similar to $(*)$ holds and if $\tau$ is large or $K > 1$, then there are stability thresholds

$$D_1(\epsilon) > D_2(\epsilon) > D_3(\epsilon) > \ldots > D_K(\epsilon) > \ldots$$

such that

$$\text{if } \lim_{\epsilon\to 0} \tfrac{D_K(\epsilon)}{D_h} > 1, \text{ the } K\text{-peaked solution is stable}$$

$$\text{and if } \lim_{\epsilon\to 0} \tfrac{D_K(\epsilon)}{D_h} < 1, \text{ the } K\text{-peaked solution is unstable.}$$

Furthermore, we proved that

$$\lim_{\epsilon\to 0} \frac{|\Omega|}{2\pi D_K(\epsilon)} \log \frac{\sqrt{|\Omega|}}{\epsilon} = K.$$

Note that $D_K(\epsilon)$ is invariant under scaling of the domain $\Omega$.

For this case stability depends on the growth rate of the inhibitor diffusivity and on the spike locations (which is related to the condition similar to $(*)$).

Very recently, Wei and the author [**85**] gave a bifurcation analysis, which proves for the weak-coupling case that there are transitions from solutions with $K$ interior spikes of equal heights to solutions with $K$ interior spikes, where $k_1$ of them are low and $k_2$ are high (with $k_1 + k_2 = K$). We also show that at most two different heights are possible. Concerning stability, we prove that these solutions with two different heights can be stable within a narrow range of parameters if a condition like $(*)$ holds. To be more explicit, existence is possible if $\epsilon << 1$, the limit $\lim_{\epsilon\to 0} \frac{|\Omega|}{2\pi D_h} \log \frac{\sqrt{|\Omega|}}{\epsilon}$ exists, and

$$2\sqrt{k_1 k_2} < \lim_{\epsilon\to 0} \frac{|\Omega|}{2\pi D_h} \log \frac{\sqrt{|\Omega|}}{\epsilon} \neq K. \tag{1.16}$$

Stability is possible if $\epsilon << 1$, $k_1 > k_2$, and

$$2\sqrt{k_1 k_2} < \lim_{\epsilon\to 0} \frac{|\Omega|}{2\pi D_h} \log \frac{\sqrt{|\Omega|}}{\epsilon} < K. \tag{1.17}$$

For all these results on (1.14) we assume that $\Omega \subset \mathbb{R}^2$. For $\Omega \subset \mathbb{R}$, similar results have been proved by Iron, Ward, and Wei [**38**], [**67**] (formal asymptotics), and by Wei and the author [**86**] (rigorous proofs). It is desirable to understand how functionals like $F_1(\mathbf{P})$ are related to the geometry of $\Omega$. For the case of one interior spike in two dimensions there are very recent results by Kolokolnikov and Ward [**41**].

Let me also mention the papers [**18**], [**19**], [**20**], [**21**], where existence and stability results for steady states the system on the real line line have been given by using a dynamical systems approach.

For the dynamics of an interior spike, I refer to [**11**].

Another issue of recent investigation of Wei and the author concerns the existence and stability of cluster solutions for large reaction-diffusion systems [**81**], [**84**]. A typical example is the hypercyclical reaction-diffusion system which arises as a spatial model concerning the origin of life similar to the one introduced by Eigen and Schuster [**22**] – [**24**], [**25**]. It arises in the modeling of catalytic networks in the case that a number of RNA-like polymers ("components") catalyse the replication of each other in a cyclic way. Examples in nature include the Krebs cycle for biosynthesis in the living cell and the Bethe-Weizsäcker cycle for high-rate energy production in massive stars. Eigen and Schuster argue that the hypercycle satisfies important criteria of natural selection: 1. Selective stability of each component due to favorable competition with error copies, 2. Cooperative behavior of the components integrated into the hypercycle, and 3. Favorable competition of the hypercycle unit with other less efficient systems.

We show rigorously that this may lead to compartmentation (i.e., the build-up of spatially small and essentially closed subsystems) due to spontaneous formation of clusters (or "spikes").

Generally speaking, system (1.14) is quite difficult to study since it has neither a variational structure nor a priori estimates. A way to get a hold on (1.14), which is been very successful, is by examining its so-called *shadow system*. Namely, we let $D_h \to +\infty$ first and assume that $\Delta H$ is bounded. Then the second equation in (1.14) implies (at least formally) that $H =$ const. in $\Omega$. Then the second equation results in the integral relation

$$H|\Omega| = \int_\Omega A^2\,dx. \tag{1.18}$$

Substituting (1.18) into the first equation gives

$$\epsilon^2 \Delta A - A + \frac{A^2}{\int_\Omega A^2\,dx}|\Omega| = 0,\ A > 0 \quad \text{in } \Omega.$$

It is well-known (see [**39**], [**56**], [**60**], [**70**]) that the study of the shadow system amounts to the study of the following single equation for $p = 2$:

$$\begin{cases} \epsilon^2 \Delta u - u + u^p = 0, \quad u > 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.19}$$

Equation (1.19), which has a variational structure, has been studied by numerous authors. It is known that equation (1.19) has both boundary spike solutions and interior spike solutions. For boundary spike solutions, see [**6**], [**16**], [**31**], [**34**], [**44**], [**54**], [**55**], [**56**], [**70**], [**77**], [**79**], and the references therein. (When $p = \frac{N+2}{N-2}, N \geq 3$, boundary spike solutions of (1.19) have been studied in [**2**], [**3**], [**4**], [**33**], [**32**], [**52**], etc.) For interior spike solutions, please see [**5**], [**9**], [**35**], [**42**], [**71**], [**72**], [**78**]. For the stability of spike solutions, please see [**37**], [**76**] and [**73**]. For dynamics, I refer to [**10**].

The structure of this thesis is as follows. In Chapter 2, we provide some preliminaries, namely some elementary results for two linear operators in half space and the calculation of the height of the spike.

In Chapter 3, we rigorously prove the existence of boundary spike solutions to (GM) for the critical growth rate. We proceed as follows: In Section 1, we construct a suitable approximation to the solutions and prove some asymptotic expansions with rigorous error estimates for this expansion. In Section 2, we use the Liapunov-Schmidt method to reduce the problem to a finite-dimensional problem, where the variable is a point

$$P \in \overline{\Lambda},$$

with

$$\Lambda = \{P \in \partial\Omega \,:\, |P - P_0| < r\}$$

for some small number $r > 0$. (Recall that $P_0$ is a nondegenerate critical point along the boundary of the function $F(P)$ which was defined in (1.8).) In Section 3, we show that this finite-dimensional reduced problem actually has a (unique) solution and finish the proof of Theorem 1.1.

In Chapter 4, we rigorously study the stability of solutions with one boundary spike to (GM) for the critical growth rate, $D_h \sim \epsilon^{-1}$, under the assumption $(*)$ of Theorem 1.2. In Section 1, we study the possible limits (or accumulation points) of the large eigenvalues $\lambda_\epsilon$ with $\lambda_\epsilon \to \lambda_0 \neq 0$ as $\epsilon \to 0$. We will show that $\lambda_0$ solves a nonlocal eigenvalue problem. In Section 2, we study the small eigenvalues $\lambda_\epsilon$ with $\lambda_\epsilon \to 0$ as $\epsilon \to 0$.

In Chapter 5, in five appendices, we provide some technial results. They are separated from the main body of the text to improve readability. In Appendix A, we give rigorous proofs for the two important asymptotic expansions, (3.13) and (3.17), which are related to the local geometry of $\partial\Omega$ near a given point $P \in \partial\Omega$. In Appendix B, we prove a trace inequality and an elliptic regularity estimate, which have been used in Appendix A. In Appendix C, we prove Proposition 3.3 and Proposition 3.4, which are the key results for the Liapunov-Schmidt reduction method. In Appendix D, we prove Proposition 3.9, which gives an important error estimate for the Liapunov-Schmidt reduction method. In Appendix E, we estimate a quantity $J$, which is needed in (4.48) of Chapter 4, Section 2 for the computation of the small eigenvalues.

In Chapter 6, we conclude this thesis with a brief discussion of the results which have been obtained in this thesis and we mention some open problems.

Throughout this thesis $C$ will denote a generic constant which may change from line to line.

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CHAPTER 2

# Preliminaries

In this chapter we provide some preliminary results which will be important for the rest of this thesis. We begin by considering two linear operators in half space. Then we calculate the heights of the spikes.

## 1. Two linear operators in half space

In this section we introduce and study two linear operators in half space which will be important for the further analysis.

Let

$$L_0\phi = \Delta\phi - \phi + 2w\phi, \quad \phi \in H^2_N(\mathbb{R}^2_+), \tag{2.1}$$

where

$$\mathbb{R}^2_+ = \{(y_1, y_2) \in \mathbb{R}^2 \, : \, y_1 > 0\}$$

and

$$H^2_N(\mathbb{R}^2_+) = \left\{ u \in H^2(\mathbb{R}^2_+) \, : \, \frac{\partial u}{\partial y_2} = 0 \text{ on } \partial\mathbb{R}^2_+ \right\}.$$

As the operator $L_0$ is self-adjoint, its kernel and co-kernel are given by

$$C_0 = K_0 = \operatorname{span}\left\{ \frac{\partial w}{\partial y_1} \right\}.$$

Then we define the orthogonal complements of kernel and co-kernel, respectively:

$$C_0^\perp = \text{orthogonal complement of } C_0 \text{ in } L^2(\mathbb{R}^2_+)$$

and

$$K_0^\perp = C_0^\perp \cap H^2_N(\mathbb{R}^2_+).$$

By standard Fredholm theory the self-adjoint operator

$$L_0 : K_0^\perp \to C_0^\perp$$

is a one-to-one map.

Furthermore, let $\pi_0$ be the following $L^2$-projection: $\pi_0 : L^2(\mathbb{R}^2_+) \to C_0^\perp$.

Now we discuss a nonlocal, not self-adjoint operator which is related to $L_0$. Let

$$L_1\phi := \Delta\phi - \phi + 2w\phi - 2\frac{\int_{\mathbb{R}^2_+} w\phi\, dy}{\int_{\mathbb{R}^2_+} w^2\, dy} w^2 : H^2_N(\mathbb{R}^2_+) \to L^2(\mathbb{R}^2_+).$$

Then we have

Lemma 2.1.

$$Ker(L_1) = Ker(L_0).$$

**Proof.** Since $w$ is an even function in $y_1$ and $\frac{\partial w}{\partial y_1}$ is an odd function in $y_1$ we get

$$\int_{\mathbb{R}^2_+} w \frac{\partial w}{\partial y_1}\, dy = 0. \tag{2.2}$$

This implies that $\mathrm{Ker}(L_0) \subseteq \mathrm{Ker}(L_1)$. For $\phi \in H_N(\mathbb{R}^2_+)$

$$L_1\phi = (\Delta - 1 + 2w)\left(\phi - 2\frac{\int_{\mathbb{R}^2_+} w\phi\, dy}{\int_{\mathbb{R}^2_+} w^2\, dy} w\right) = 0$$

is equivalent to

$$\phi - 2\frac{\int_{\mathbb{R}^2_+} w\phi\, dy}{\int_{\mathbb{R}^2_+} w^2\, dy} w \in \mathrm{Ker}(L_0).$$

Written explicitly, we get

$$\phi - 2\frac{\int_{\mathbb{R}^2_+} w\phi\, dy}{\int_{\mathbb{R}^2_+} w^2\, dy} w = c\frac{\partial w}{\partial y_1}, \quad c \in \mathbb{R}. \tag{2.3}$$

Multiplying (2.3) by $w$, integrating over $\mathbb{R}^2_+$, and using (2.2), we get

$$(1-2)\int_{\mathbb{R}^2_+} w\phi\, dy = 0.$$

Therefore $\int_{\mathbb{R}^2_+} w\phi\, dy = 0$ and (2.3) implies that $\phi \in \mathrm{Ker}(L_0)$. Thus $\mathrm{Ker}(L_1) \subseteq \mathrm{Ker}(L_0)$. Lemma 2.1 is proved.

□

## 2. Calculating the heights of the spikes

The calculations in this section will guide us in the choice of suitable approximate solutions. Let

$$\begin{cases} A_\epsilon = \xi_\epsilon(w(\frac{x-P}{\epsilon}) + O(\epsilon)) \quad \text{in } H^2(\Omega_{\epsilon,P}), \\ \qquad H_\epsilon = \xi_\epsilon(1 + O(\epsilon)) \quad \text{in } H^2(\Omega), \end{cases} \tag{2.4}$$

where $w$ is the unique solution in $H^1(\mathbb{R}^2)$ of problem (1.4). We will determine $\xi_\epsilon$ in leading order. The precise choice of $\xi_\epsilon$ will be given in Section 3.1.

Throughout this paper, we always assume that $P \in \bar{\Lambda}$ where

$$\Lambda = \{P \in \partial\Omega \,:\, |P - P_0| < r\}$$

with some small number $r > 0$ and that $P_0$ is a critical point of the function $F(P)$ along the boundary.

We shall frequently use the following technical lemma.

Lemma 2.2. *Let $u$ be a solution of*

$$\Delta u - u + f = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

*Suppose*

$$|f(x)| \leq \eta e^{-\frac{\alpha|x-P|}{\epsilon}}$$

*for some $\alpha > 0$. Then we have*

$$|u(P)| \leq C_1 \eta \epsilon^2 \log \frac{1}{\epsilon}, \tag{2.5}$$

*and*

$$|u(P) - u(x)| \leq C_2 \eta \epsilon^2 \log\left(\frac{|x-P|}{\epsilon} + 1\right), \tag{2.6}$$

*where $C_1 > 0, C_2 > 0$ are generic constants (independent of $\epsilon > 0$ and $\eta > 0$).*

**Proof.** By the representation formula, we calculate

$$u(x) = \int_\Omega G(x,z) f(z) dz$$

and

$$u(P) = \int_\Omega G(P,z) f(z) dz = \epsilon^2 \int_{\Omega_{\epsilon,P}} G(P, P + \epsilon y) \eta \mathrm{e}^{-\alpha|y|} dy$$

$$\leq C_1\eta\epsilon^2\log\frac{1}{\epsilon}.$$

Similarly, we can obtain (2.6).

□

Let $\beta>0$. For $x\in\Omega$ and $\xi\in\Omega$ let $G_\beta(x,\xi)$ be the Green function

$$\begin{cases}\Delta G_\beta-\beta^2G_\beta+\delta_\xi=0 & \text{in }\Omega,\\ \frac{\partial G_\beta}{\partial\nu}=0 & \text{on }\partial\Omega.\end{cases}\tag{2.7}$$

Recall that we have defined $G_0(x,\xi)$ in (1.3) as the unique solution in $H^1(\Omega)$ of the problem

$$\begin{cases}\Delta G_0-\frac{1}{|\Omega|}+\delta_\xi=0 & \text{in }\Omega,\\ \int_\Omega G_0(x,\xi)\,d\xi=0, & \\ \frac{\partial G_\beta}{\partial\nu}=0 & \text{on }\partial\Omega.\end{cases}$$

Then we can derive a relation between $G_\beta$ and $G_0$ as follows: From (2.7) we get

$$\int_\Omega G_\beta(x,\xi)\,dx=\beta^{-2}.$$

Set

$$G_\beta(x,\xi)=\frac{\beta^{-2}}{|\Omega|}+\overline{G}_\beta(x,\xi).$$

Then

$$\begin{cases}\Delta\bar{G}_\beta-\beta^2\bar{G}_\beta-\frac{1}{|\Omega|}+\delta_\xi=0 & \text{in }\Omega,\\ \int_\Omega\bar{G}_\beta(x,\xi)\,dx=0, & \\ \frac{\partial\bar{G}_\beta}{\partial\nu}=0 & \text{on }\partial\Omega.\end{cases}\tag{2.8}$$

Equations (1.3) and (2.8) imply that

$$\bar{G}_\beta(x,\xi)=G_0(x,\xi)+O(\beta^2)\qquad\text{in the operator norm }L^2(\Omega)\to H^2(\Omega).$$

Hence

$$G_\beta(x,\xi)=\frac{\beta^{-2}}{|\Omega|}+G_0(x,\xi)+O(\beta^2)\qquad\text{in the operator norm }L^2(\Omega)\to H^2(\Omega).\tag{2.9}$$

Then, from the equation for $H$,

$$\Delta H-\beta^2H+\beta^2A^2=0,$$

we get for $P\in\partial\Omega$:

$$H_\epsilon(P)=\int_\Omega G_\beta(P,\xi)\beta^2A^2(\xi)\,d\xi$$

$$= \int_\Omega \left( \frac{\beta^{-2}}{|\Omega|} + G_0(P,\xi) + O(\beta^2) \right) \beta^2 \xi_\epsilon^2 \left( w\left(\frac{\xi - P}{\epsilon}\right) + O(\epsilon) \right)^2 \, d\xi$$
$$= \int_\Omega \left( \frac{1}{|\Omega|} + \beta^2 G_0(P,\xi) + O(\beta^4) \right) \xi_\epsilon^2 \left( w\left(\frac{\xi - P}{\epsilon}\right) + O(\epsilon) \right)^2 \, d\xi.$$

Thus, by choosing

$$\xi_\epsilon = H_\epsilon(P),$$

we get

$$\xi_\epsilon = \xi_\epsilon^2 \left( \frac{\epsilon^2}{|\Omega|} \int_{\mathbb{R}^2_+} w^2(y)\, dy + \beta^2 \int_\Omega G_0(P,\xi) w^2 \left(\frac{\xi - P}{\epsilon}\right) d\xi + O(\epsilon^3) \right). \tag{2.10}$$

Inserting the expansion for $G_0$ and the relation $\beta^2 = c_0\epsilon$ into (2.10) gives

$$\xi_\epsilon^{-1} = \frac{\epsilon^2}{|\Omega|} \int_{\mathbb{R}^2_+} w^2(y)\, dy$$
$$+ c_0\epsilon \int_\Omega \left( \frac{1}{\pi} \log \frac{1}{|P - \xi|} - H_0(P,\xi) \right) w^2 \left(\frac{\xi - P}{\epsilon}\right) d\xi + O(\epsilon^3)$$
$$= \frac{\epsilon^2}{|\Omega|} \int_{\mathbb{R}^2_+} w^2(y)\, dy + \frac{c_0}{\pi} \epsilon^3 \log \frac{1}{\epsilon} \int_{\mathbb{R}^2_+} w^2(y)\, dy + O(\epsilon^3). \tag{2.11}$$

Note that $H_0 \in C^2(\overline{\Omega} \times \partial\Omega)$.

The leading terms in (2.10) give

$$\xi_\epsilon^{-1} = \frac{\epsilon^2}{|\Omega|} \int_{\mathbb{R}^2_+} w^2(y)\, dy + O\left(\epsilon^3 \log \frac{1}{\epsilon}\right). \tag{2.12}$$

This implies

$$\xi_\epsilon = \frac{|\Omega|}{\epsilon^2 \int_{\mathbb{R}^2_+} w^2(y)\, dy} + O\left(\epsilon^{-1} \log \frac{1}{\epsilon}\right). \tag{2.13}$$

In this section, we have calculated the height of the spike of the steady state in leading order under the assumption that its shape is given a priori by a boundary spike. In the next chapter, we will prove rigorously that these one-peak steady states indeed do exist.

CHAPTER 3

# Existence

In this chapter we prove the existence in the critical case

$$\frac{1}{\beta^2} = D_h = \frac{1}{c_0\epsilon} > 0,$$

which is the threshold between the weak coupling case (see [**82**]) and the shadow system case, where the system can be reduced to a single equation (see [**42**], [**73**]).

## 1. Construction of the Approximate Solutions

In this section, we introduce functions which are good approximations to boundary one-spike steady states.

Motivated by the results in Section 1, we rescale

$$\hat{A}(y) = \frac{1}{\xi_\epsilon} A(\epsilon y + P), \quad y \in \Omega_{\epsilon,P},$$

$$\hat{H}(x) = \frac{1}{\xi_\epsilon} H(x), \quad x \in \Omega,$$

where $\xi_\epsilon$ in leading order is given by (1.9) and will be given exactly in (3.10) below. (Recall that

$$\xi_\epsilon = \frac{|\Omega|}{\epsilon^2 \int_{\mathbb{R}^2_+} w^2(y)\, dy} + O\left(\epsilon^{-1} \log \frac{1}{\epsilon}\right).)$$

Then $\hat{A}$, $\hat{H}$ are solutions to (1.14) if and only if they solve the following rescaled form of the stationary Gierer-Meinhardt system

$$\Delta_y \hat{A} - \hat{A} + \frac{\hat{A}^2}{\hat{H}} = 0, \quad y \in \Omega_{\epsilon,P},$$

$$\Delta_x \hat{H} - \beta^2 \hat{H} + \beta^2 \xi_\epsilon \hat{A}^2 = 0, \quad x \in \Omega. \tag{3.1}$$

For a function $\hat{A} \in H^1(\Omega_{\epsilon,P})$, let $T[\hat{A}^2]$ be the unique solution in $H^1(\Omega)$ of the problem

$$\Delta T[\hat{A}^2] - \beta^2 T[\hat{A}^2] + \beta^2(\xi_\epsilon \hat{A}^2) = 0 \text{ in } \Omega, \ \frac{\partial T[\hat{A}^2]}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{3.2}$$

Then $T$ is a linear integral operator which can be represented as follows:

$$T[\hat{A}^2](x) = \beta^2 \int_\Omega G_\beta(x,\xi)\xi_\epsilon \hat{A}^2\left(\frac{\xi - P}{\epsilon}\right) d\xi. \tag{3.3}$$

The system (3.1) is equivalent to the following equation in operator form:

$$S_\epsilon(\hat{A},\hat{H}) = \begin{pmatrix} S_1(\hat{A},\hat{H}) \\ S_2(\hat{A},\hat{H}) \end{pmatrix} = 0, \quad H_N^2(\Omega_{\epsilon,P}) \times H_N^2(\Omega) \to L^2(\Omega_{\epsilon,P}) \times L^2(\Omega), \tag{3.4}$$

where

$$S_1(\hat{A},\hat{H}) = \Delta_y \hat{A} - \hat{A} + \frac{\hat{A}^2}{\hat{H}} : \quad H_N^2(\Omega_{\epsilon,P}) \times H_N^2(\Omega) \to L^2(\Omega_{\epsilon,P}),$$

$$S_2(\hat{A},\hat{H}) = \Delta_x \hat{H} - \beta^2 \hat{H} + \beta^2 \xi_\epsilon \hat{A}^2 : \quad H_N^2(\Omega_{\epsilon,P}) \times H_N^2(\Omega) \to L^2(\Omega).$$

Note that (3.4) is also equivalent to the single equation

$$S_1(\hat{A}, T[\hat{A}^2]) = 0, \quad \hat{A} \in H_N^2(\Omega_{\epsilon,P}). \tag{3.5}$$

Throughout this thesis we will use these equivalent formatulions whichever is most convenient.

For an open and smooth subset $U$ of $\mathbb{R}^2$, we define the projection of a function $W \in H^2(U)$ onto

$$\mathcal{P}_U W \in H_N^2(U) := \left\{ u \in H^2(U) : \frac{\partial u}{\partial \nu_x} = 0 \quad \text{for all } x \in \partial\Omega \right\}$$

by $\mathcal{P}_U W = W - \mathcal{Q}_U W$, where $\mathcal{Q}_U W$ satisfies

$$\begin{cases} \Delta \mathcal{Q}_U W - \mathcal{Q}_U W = 0 & \text{in } U, \\ \frac{\partial \mathcal{Q}_U W}{\partial \nu} = \frac{\partial W}{\partial \nu} & \text{on } \partial U. \end{cases} \tag{3.6}$$

We further introduce the notations

$$w_{\epsilon,P}(P + \epsilon z) = P_{\Omega_{\epsilon,P}} w(z), \qquad z \in \Omega_{\epsilon,P}, \tag{3.7}$$

and

$$w_\epsilon(x) = w\left(\frac{x-P}{\epsilon}\right), \qquad x \in \mathbb{R}^2, \tag{3.8}$$

where $w$ was defined in (1.4). By the maximum principle, $w_{\epsilon,P}(z) > 0$ for $z \in \Omega_{\epsilon,P}$.

Our ansatz for $(\hat{A}, \hat{H})$ is

$$\begin{cases} \hat{A}(z) = A_{\epsilon,P}(z) = w_{\epsilon,P}(z), \quad z \in \Omega_{\epsilon,P} \\ \hat{H}(x) = H_{\epsilon,P}(x) = T[w_{\epsilon,P}^2](x), \quad x \in \Omega \end{cases} \tag{3.9}$$

and we choose $\xi_\epsilon$ such that

$$H_{\epsilon,P}(P) = T[w_{\epsilon,P}^2](P) = 1. \tag{3.10}$$

(Recall that by definition $T[w_{\epsilon,P}^2](P) = \beta^2 \xi_\epsilon \int_\Omega G_\beta(P,\xi) w_{\epsilon,P}^2\left(\frac{\xi - P}{\epsilon}\right) d\xi$ is linear in $\xi_\epsilon$.)

We will use the standard inner products

$$< u, v >= \int_\Omega uv\,dx,$$

$$\|u\|_0^2 =< u, u >,$$

$$\|u\|_1^2 = \|u\|_0^2 + \sum_{i=1}^{2} \|D_i u\|_0^2,$$

$$\text{and} \qquad \|u\|_2^2 = \|u\|_0^2 + \sum_{i=1}^{2} \|D_i u\|_0^2 + \sum_{i,j=1}^{2} \|D_{ij} u\|_0^2.$$

In this thesis, we will frequently rescale the domain, and the norm will also have rescaled forms:

$$< u, v >_{0,\epsilon}= \epsilon^{-2} \int_\Omega uv\,dx,$$

$$\|u\|_{0,\epsilon}^2 = \epsilon^{-2} < u, u >,$$

$$\|u\|_{1,\epsilon}^2 = \epsilon^{-2}\left(\|u\|_0^2 + \epsilon^2 \sum_{i=1}^{2} \|D_i u\|_0^2\right),$$

$$\text{and} \qquad \|u\|_{2,\epsilon}^2 = \epsilon^{-2}\left(\|u\|_0^2 + \epsilon^2 \sum_{i=1}^{2} \|D_i u\|_0^2 + \epsilon^4 \sum_{i,j=1}^{2} \|D_{ij} u\|_0^2\right).$$

We note that (3.6) implies that $\varphi_\epsilon(z) := w_\epsilon(z) - w_{\epsilon,P}(z)$ satisfies

$$\Delta_z \varphi_\epsilon - \varphi_\epsilon = 0 \quad \text{in } \Omega_{\epsilon,P},$$

$$\frac{\partial \varphi_\epsilon}{\partial \nu} = \frac{\partial w_\epsilon}{\partial \nu} \quad \text{on } \partial\Omega_{\epsilon,P}. \tag{3.11}$$

We introduce a diffeomorphism which straightens the boundary in a neighborhood of $P \in \partial\Omega$. By a rotation of the coordinate system, we may assume that the normal to $\partial\Omega$ at $P$ is $(0,1)$. Let $B'(\delta) = (-\delta, \delta)$. Fix $P = (P_1, P_2) \in \partial\Omega$. Then we can find $\delta_1 > 0$ and a smooth function $\rho : B'(\delta_1) \to R$ such that for some neighborhood $N$ of $P$

(1) $\rho(0) = 0$, $\rho'(0) = 0$,
(2) $\Omega \cap N = \{(x_1, x_2) \in N : x_2 - P_2 > \rho(x_1 - P_1)\}$
and $\partial\Omega \cap N = \{(x_1, x_2) \in N : x_2 - P_2 = \rho(x_1 - P_1)\}$.

Now we define a mapping $X = T(x) = (T_1(x), T_2(x))$ for $x \in B(P, \delta_1)$ by

$$T_1(x) = x_1 - P_1, \quad T_2(x) = x_2 - P_2 - \rho(x_1 - P_1).$$

We also define a mapping $T_\epsilon$ which is a rescaling of $T$:

$$T_\epsilon(x) = (T_{\epsilon,1}(x), T_{\epsilon,2}(x)) = \frac{1}{\epsilon} T(x)$$

and we denote

$$y = T_\epsilon(x).$$

We can calculate that $DT(P) = I$, the identity mapping. Thus $T$ has an inverse mapping $x = T^{-1}(X)$ for $X \in B(\delta_2) \subset T(B(P, \delta_1))$, where $\delta_2$ is some positive constant. Let $T^{-1}(B(\delta_2)) \cap \Omega = \Omega_P$. Then we have defined a local diffeomorphism $T : \Omega_P \to B(\delta_2)$ such that

(1) $T(0) = 0$, $DT(0) = I$,
(2) $T(\Omega_P) = \mathbb{R}^2_+ \cap B(\delta_2)$ and $T(\partial\Omega \cap \overline{\Omega_P}) = \partial\mathbb{R}^2_+ \cap \overline{B(\delta_2)}$.

Later in the paper, if we want to specify the location of the diffeomorphism, we will sometimes use the notations $\rho^P$, $T_P$ or $T_{\epsilon,P}$. And we will drop the letter $P$ if this can be done without causing confusion.

Note that the curvature $h(P)$ of $\partial\Omega$ at $P$ satisfies $h(P) = (\rho^P)''(0)$. Since we assume that $\partial\Omega$ is smooth, Taylor expansion gives us

$$\rho^P(a) = \frac{1}{2}(\rho^P)''(0)a^2 + \frac{1}{6}(\rho^P)'''(0)a^3 + O(|a|^4)$$

for $a \in \mathbb{R}$ small.

Let $\Omega_P$ be the neighborhood of $P$ introduced above, and let $M$ and $N$ also be neighborhoods of $P$ satisfying $M \subset\subset N \subset \Omega_P$. Then we define a cut-off function $\chi(x) : \mathbb{R}^2 \to [0,1]$ such that

$$\begin{aligned} &(1) \quad \chi \in C_0^\infty(\Omega), \\ &(2) \quad \chi(x) = 1 \text{ for } x \in M, \\ &(2) \quad \chi(x) = 0 \text{ for } x \in \overline{\Omega} \setminus N. \end{aligned} \tag{3.12}$$

Recalling the notation $y = T_\epsilon(x)$ from above, we have proved in [**77**] that

$$\varphi_\epsilon(x) = \Big(\epsilon v_1(y) + \epsilon^2 v_2(y) + \epsilon^2 v_3(y)\Big)\chi(x) + \epsilon^3 \varphi^0_{\epsilon,P}(x), \tag{3.13}$$

where $v_1$ is the unique solution in $H^1(\mathbb{R}^2_+)$ of

$$\begin{cases} \Delta v - v = 0 & \text{in } \mathbb{R}^2_+, \\ \frac{\partial v}{\partial y_2} = -\frac{1}{2}\frac{\partial w}{\partial y_1}\rho''(0)y_1 & \text{on } \partial\mathbb{R}^2_+, \end{cases} \tag{3.14}$$

$v_2$ is the unique solution in $H^1(\mathbb{R}^2_+)$ of

$$\begin{cases} \Delta v - v - 2\rho''(0)y_1\frac{\partial^2 v_1}{\partial y_1 \partial y_2} - \rho''(0)\frac{\partial v}{\partial y_2} = 0 & \text{in } \mathbb{R}^2_+, \\ \frac{\partial v}{\partial y_2} = \rho''(0)y_1\frac{\partial v_1}{\partial y_1} & \text{on } \partial\mathbb{R}^2_+, \end{cases} \tag{3.15}$$

$v_3$ is the unique solution in $H^1(\mathbb{R}^2_+)$ of

$$\begin{cases} \Delta v - v = 0 & \text{in } \mathbb{R}^2_+, \\ \frac{\partial v}{\partial y_2} = -\frac{1}{3}\frac{\partial w}{\partial y_1}\rho'''(0)y_1^2 & \text{on } \partial\mathbb{R}^2_+, \end{cases} \tag{3.16}$$

and

$$\|\varphi^0_{\epsilon,P}\|_{2,\epsilon} \le C.$$

Note that $v_1$, $v_2$ are even functions in $y_1$ and $v_3$ is an odd function in $y_1$, i.e. $v_1(y_1,y_2) = v_1(-y_1,y_2)$, $v_2(y_1,y_2) = v_2(-y_1,y_2)$, $v_3(y_1,y_2) = -v_3(-y_1,y_2)$.

Moreover, it follows from the maximum principle that

$$|v_1(y)|, |v_2(y)|, |v_3(y)| \le Ce^{-\mu|y|} \quad \text{for some} \quad 0 < \mu < 1.$$

These symmetry properties will be essential for the analysis of the Liapunov-Schmidt reduction in Section 3 and the reduced problem in Section 4, respectively.

We next analyze $\frac{\partial w_{\epsilon,P}(x)}{\partial \tau(P)}$. By our choice of the coordinate system, we may assume that

$$\frac{\partial}{\partial \tau(P)} = \frac{\partial}{\partial P_1} = -\frac{\partial}{\partial x_1}.$$

Then $\frac{\partial}{\partial P_1} h_{\epsilon,P}(x)$ satisfies

$$\begin{cases} \epsilon^2 \Delta v - v = 0 & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = \frac{\partial}{\partial \nu}\frac{\partial}{\partial P_1} w\left(\frac{x-P}{\epsilon}\right) & \text{on } \partial\Omega, \end{cases}$$

Furthermore, [**77**] tells us that

$$\left[\frac{\partial w_\epsilon}{\partial \tau(P)} - \frac{\partial w_{\epsilon,P}}{\partial \tau(P)}\right]\left(\frac{x-P}{\epsilon}\right) = w_1(y)\chi(x) + \epsilon w^\epsilon_{2,P}(x), \tag{3.17}$$

where $w_1$ is the unique solution in $H^1(\mathbb{R}^2_+)$ of

$$\begin{cases} \Delta v - v = 0 & \text{in } \mathbb{R}^2_+, \\ \frac{\partial v}{\partial y_2} = \frac{1}{2}\left(\frac{\partial^2 w}{\partial y_1^2} y_1 + \frac{\partial w}{\partial y_1}\right)\rho''(0) & \text{on } \partial\mathbb{R}^2_+ \end{cases} \tag{3.18}$$

and

$$\|w^\epsilon_{2,P}\|_{2,\epsilon} \le C.$$

It follows from the maximum principle that $|w_1(y)| \le C\exp(-\mu|y|)$ for some $0 < \mu < 1$ and $w_1$ is an odd function in $y_1$, i.e., $w_1(y_1, y_2) = -w_1(-y_1, y_2)$.

We include the proofs of (3.13) and (3.17) in Appendix A.

We conclude this section with some computations and estimates for $S_1(A_{\epsilon,P}, H_{\epsilon,P})$, which will be essential for the rest of this thesis.

We begin with a technical lemma:

Lemma 3.1. *For $x \in \Omega_P$, we have*

$$\frac{\partial w_{\epsilon,P}}{\partial \tau(P)}(x) = -\frac{1}{\epsilon}\frac{\partial w(y)}{\partial y_1} + \chi(x)\Bigg[\frac{|y|w''(|y|) - w'(|y|)}{2|y|^3}\rho''(0)y_1^3 y_2 + \rho''(0)y_1\frac{\partial w(y)}{\partial y_2} - w_1(y)\Bigg] + \epsilon r_\epsilon(x), \tag{3.19}$$

*where $\|r_\epsilon\|_{2,\epsilon} \le C$. Furthermore, for $x \in \Omega_P$,*

$$w_{\epsilon,P}(x) = w(y) - \epsilon\chi(x)\frac{w'(|y|)}{|y|}\rho''(0)y_1^2 y_2 + O(\epsilon^2 e^{-C|y|}). \tag{3.20}$$

**Remark.** Lemma 3.1 immediately implies

$$(w_{\epsilon,P}(z))^2 = w^2(y) - 2\epsilon\chi(x)w(y)\frac{w'(|y|)}{|y|}\rho''(0)y_1^2 y_2 + O(\epsilon^2 e^{-C|y|}). \tag{3.21}$$

**Proof of Lemma 3.1.** First, we compute

$$|z| - |y| = \frac{|x-P|^2 - |T(x)|^2}{\epsilon(|x-P| + |T(x)|)} = \frac{\epsilon^2 y_2^2 - (\epsilon y_2 + \rho(x_1 - P_1))^2}{\epsilon(|x-P| + |T(x)|)}$$

$$= -\frac{2\epsilon y_2 \rho(x_1 - P_1) - \rho^2(x_1 - P_1)}{\epsilon^2(|y| + |z|)}$$

$$= -\frac{\epsilon}{2|y|}\rho''(0) y_1^2 y_2 + O(\epsilon^2 |y|^2). \tag{3.22}$$

By (3.17), we have

$$\frac{\partial w_{\epsilon,P}}{\partial \tau(P)}(x) = \frac{\partial w_\epsilon}{\partial \tau(P)}(x) - \chi(x) w_1(y) - \epsilon w_{2,P}^\epsilon(x).$$

By (3.22),

$$\frac{\partial w_\epsilon}{\partial \tau(P)}(x) = -\frac{1}{\epsilon}\left(\frac{\partial w(y)}{\partial y_1} - \rho'(x_1 - P_1)\frac{\partial w(y)}{\partial y_2}\right)$$

$$= -\frac{1}{\epsilon}\frac{\partial w(y)}{\partial y_1} - \frac{1}{\epsilon}\frac{|y|w''(|y|) - w'(|y|)}{|y|^2} y_1(|z| - |y|) + \rho''(0) y_1 \frac{\partial w(y)}{\partial y_2} + O(\epsilon e^{-k|y|})$$

$$= -\frac{1}{\epsilon}\frac{\partial w(y)}{\partial y_1} + \frac{|y|w''(|y|) - w'(|y|)}{2|y|^3}\rho''(0) y_1^3 y_2$$

$$+\rho''(0) y_1 \frac{\partial w(y)}{\partial y_2} + O(\epsilon e^{-k|y|}).$$

This implies

$$\frac{\partial w_{\epsilon,P}}{\partial \tau(P)}(x) = -\frac{1}{\epsilon}\frac{\partial w(y)}{\partial y_1} + \chi(x)\Bigg[\frac{|y|w''(|y|) - w'(|y|)}{2|y|^3}\rho''(0) y_1^3 y_2$$

$$+\rho''(0) y_1 \frac{\partial w(y)}{\partial y_2} - w_1(y)\Bigg] + \epsilon r_\epsilon(x)$$

and (3.19) is proved. Similarly, using (3.13), we derive

$$(w_{\epsilon,P}(x))^2 = w^2(y) - 2\epsilon\chi(x) w(y)\frac{w'(|y|)}{|y|}\rho''(0) y_1^2 y_2$$

$$+O(\epsilon^2 e^{-C|y|})$$

and (3.20) is proved.

This concludes the proof of Lemma 3.1.

□

We substitute our ansatz into the rescaled Gierer-Meinhardt system and calculate by (3.13)

$$
\begin{aligned}
S_1(A_{\epsilon,P}, H_{\epsilon,P}) &= \Delta_z A_{\epsilon,P} - A_{\epsilon,P} + \frac{A^2_{\epsilon,P}}{H_{\epsilon,P}} \\
&= -w_\epsilon^2 + \frac{w^2_{\epsilon,P}}{H_{\epsilon,P}(P)} - \frac{w^2_{\epsilon,P}}{(H_{\epsilon,P}(P))^2}(H_{\epsilon,P}(x) - H_{\epsilon,P}(P)) + O(\epsilon^2) \\
&= -2w_\epsilon\varphi_\epsilon + w^2_{\epsilon,P} - w^2_{\epsilon,P}(H_{\epsilon,P}(x) - H_{\epsilon,P}(P)) + O(\epsilon^2)
\end{aligned} \tag{3.23}
$$

in $L^2(\Omega_{\epsilon,P})$.

Using Lemma 3.1 and equation (3.22), we calculate for $x = T_\epsilon^{-1}(y) = P + \epsilon z$, $|\epsilon y| < \delta_2$:

$$
\begin{aligned}
H_{\epsilon,P}(P+\epsilon z) - H_{\epsilon,P}(P) &= c_0\epsilon \int_\Omega [G_\beta(P+\epsilon z,\xi) - G_\beta(P,\xi)]\xi_\epsilon A^2_{\epsilon,P} d\xi \\
&= c_0\epsilon\xi_\epsilon \int_\Omega [G_\beta(P+\epsilon z,\xi) - G_\beta(P,\xi)] w^2_{\epsilon,P} d\xi \\
&= c_0\epsilon\xi_\epsilon \int_\Omega [G_0(P+\epsilon z,\xi) - G_0(P,\xi)] w^2_\epsilon d\xi + O(\epsilon^3|y|) \\
&= c_0\epsilon^3\xi_\epsilon \int_{\mathbb{R}^2_+} \frac{1}{\pi}\log\frac{|y|}{|z-y|} w^2(y) dy \\
&\quad -c_0\epsilon^4\xi_\epsilon \frac{1}{2}\frac{\partial H_0(P,P)}{\partial\tau(P)} z_1 \int_{\mathbb{R}^2_+} w^2(y)\,dy + O(\epsilon^5\xi_\epsilon|z|).
\end{aligned} \tag{3.24}
$$

**Remark.** Note that, by the symmetry of $H(P,Q)$,

$$
\left.\frac{\partial H_0(P,Q)}{\partial\tau(P)}\right|_{P=Q} = \frac{1}{2}\frac{\partial H_0(P,P)}{\partial\tau(P)}.
$$

Substituting (3.24) and (3.13) into (3.23), and keeping in mind that the terms $v_1$ and $v_2$ in (3.13) are radially symmetric in $y$, whereas $v_3$ is odd in $y_1$, we have the following key estimate

LEMMA 3.2. *For $x = T_\epsilon^{-1}(y) = P + \epsilon z$, $|\epsilon y| < \delta_2$ we have*

$$
S_1(A_{\epsilon,P}, H_{\epsilon,P}) = S_{1,1} + S_{1,2}, \tag{3.25}
$$

*where*

$$
\begin{aligned}
S_{1,1}(y) = &-c_0\epsilon^4\xi_\epsilon \frac{1}{2}\frac{\partial H_0(P,P)}{\partial\tau(P)} w^2(y) y_1 \int_{\mathbb{R}^2_+} w^2(y)\,dy \\
&- 2\epsilon^2 w(y) v_3(y) + O\left(\epsilon^3 e^{-C|y|}\right),
\end{aligned} \tag{3.26}
$$

$$S_{1,2}(y) = \epsilon w^2(y) R(|y|) - 2\epsilon w(y) v_1(y) + O(\epsilon^2 e^{-C|y|}) \tag{3.27}$$

*For $S_{1,2}(y)$, the term $R(|y|)$ is radially symmetric in $y$ with $R(|y|) = O(\log(1+|y|))$ and the error term is also radially symmetric in $y$.*

*Furthermore, $\|S_1(A_{\epsilon,P}, H_{\epsilon,P})\|_{0,\epsilon} = O(\epsilon)$.*

The above estimates will be important in the following sections, where (3.4) is solved exactly.

## 2. The Liapunov-Schmidt Reduction Method

In this section we study the linearized operator defined by

$$\tilde{L}_{\epsilon,P} := DS_\epsilon \begin{pmatrix} A_{\epsilon,P} \\ H_{\epsilon,P} \end{pmatrix},$$

$$\tilde{L}_{\epsilon,P} : H_N^2(\Omega_{\epsilon,P}) \times H_N^2(\Omega) \to L^2(\Omega_{\epsilon,P}) \times L^2(\Omega).$$

We will introduce the suitable functional-analytical framework, which is provided by the Liapunov-Schmidt reduction method.

First, we introduce the non-selfadjoint and non-local linear operator

$$L^\epsilon \phi := \Delta\phi - \phi + 2A_{\epsilon,P} H_{\epsilon,P}^{-1} \phi - 2\frac{\int_{\Omega_{\epsilon,P}} A_{\epsilon,P}\phi\, dz}{\int_{\Omega_{\epsilon,P}} A_{\epsilon,P}^2\, dz} A_{\epsilon,P}^2 \,:\, H_N^2(\Omega_{\epsilon,P}) \to L^2(\Omega_{\epsilon,P}),$$

which will turn out to be the limit of (the first line of) the operators $\tilde{L}_{\epsilon,P}$ as $\epsilon \to 0$.

We introduce an approximation to the kernel and the co-kernel of $\tilde{L}_{\epsilon,P}$, respectively. Their most important properties will be given in the Propositions 3.6 and 3.7. They are defined as follows:

$$K_{\epsilon,P} = C_{\epsilon,P} := \text{span}\ \left\{ \epsilon \frac{\partial w_{\epsilon,P}}{\partial \tau(P)} \right\},$$

where $\frac{\partial}{\partial \tau(P)}$ is the tangential derivative at $P$. Their orthogonal complements are defined as follows:

$$C_{\epsilon,P}^\perp = L^2\text{-orthogonal complement of } C_{\epsilon,P},$$

$$K_{\epsilon,P}^\perp = C_{\epsilon,P}^\perp \cap H_N^2(\Omega_{\epsilon,P}).$$

Finally, we define the following projection operator:

$$\hat{\pi}_{\epsilon,P} \,:\, L^2(\Omega_{\epsilon,P}) \to C_{\epsilon,P}^\perp.$$

If we restrict the domain of $L^\epsilon$ to $K_{\epsilon,P}^\perp$ and project the range into $C_{\epsilon,P}^\perp$ we get the operator

$$L_{\epsilon,P} := \hat{\pi}_{\epsilon,P} \circ L^\epsilon : K_{\epsilon,P}^\perp \to C_{\epsilon,P}^\perp.$$

The most important property of $L_{\epsilon,P}$ will be that it is a well-posed operator in the sense given in Proposition 3.5. We remark that since $w_{\epsilon,P}(y) = w(y)(1+O(\epsilon))$ in $H^2(\Omega_{\epsilon,P})$, for small $\epsilon$ the self-adjoint linear operator

$$l_{\epsilon,P} := \hat{\pi}_{\epsilon,P} \circ (\Delta - 1 + 2w_{\epsilon,P}) : K_{\epsilon,P}^\perp \to C_{\epsilon,P}^\perp$$

is a one-to-one map with uniformly bounded inverse. This result follows from the following two lemmas, which are Propositions 3.1 and 3.2 of [**77**]. We include the proofs of these two propositions in Appendix C.

These are the key results for applying Liapunov-Schmidt reduction. This method has been taylored to construct concentrated solutions for of the Schrödinger equation in $\mathbb{R}^N$ for $N \geq 1$ [**27**], [**62**], [**63**]. To our knowledge it has first been used in [**77**] for boundary spikes in semilinear partial differential equations in bounded multi-dimensional domains. Further results can be found for example in [**78**] and [**34**].

Proposition 3.3. *There exist positive constants $\overline{\epsilon}, \lambda$ such that for all $\epsilon \in (0, \overline{\epsilon})$*

$$\|l_{\epsilon,P}\Phi\|_{L^2(\Omega_{\epsilon,P})} \geq \lambda \|\Phi\|_{H^2(\Omega_{\epsilon,P})} \tag{3.28}$$

*for all $\Phi \in K_{\epsilon,P}^\perp$.*

Proposition 3.4. *There exists a positive constant $\overline{\overline{\epsilon}}$ such that for all $\epsilon \in (0, \overline{\overline{\epsilon}})$ and $P \in \overline{\Lambda}$ the map*

$$l_{\epsilon,P} \,:\, K_{\epsilon,P}^\perp \to C_{\epsilon,P}^\perp$$

*is surjective.*

The strategy of proof in this section is to generalize that result first to the projection $L_{\epsilon,P} = \hat{\pi}_{\epsilon,P} \circ L^\epsilon$ of the non-local operator $L^\epsilon$ (see Proposition 3.5) and then to the projection of the linearized operator of the system $\tilde{L}_{\epsilon,P}$ (see

Proposition 3.6). We remark that this is the first result where the Liapunov-Schmidt reduction method is applied to construct boundary spike solutions for systems of partial differential equations.

The following proposition is the key estimate in applying the Liapunov-Schmidt reduction method.

PROPOSITION 3.5. *For $\epsilon$ sufficiently small, the map $L_{\epsilon,P}$ one-to-one. Moreover, for $\epsilon$ sufficiently small, the inverse of $L_{\epsilon,P}$ exists and its norm is bounded uniformly with respect to $\epsilon$.*

**Proof.** We first show that there exist constants $C > 0, \bar{\epsilon} > 0$ such that

$$\|L_{\epsilon,P}\phi\|_{0,\epsilon} \geq C\|\phi\|_{2,\epsilon} \tag{3.29}$$

for all $\epsilon \in (0,\bar{\epsilon})$, $P \in \overline{\Lambda}$, $\phi \in K^{\perp}_{\epsilon,P}$.

Suppose that (3.29) is false. Then there exist sequences $\{\epsilon_k\}$, $\{P^k\}$, and $\{\phi_k\}$ with $\epsilon_k \to 0$, $P^k \in \overline{\Lambda}$, $\phi_k \in K^{\perp}_{\epsilon_k,P^k}$ such that

$$\|L_{\epsilon_k,P^k}\phi_k\|_{0,\epsilon_k} \to 0, \tag{3.30}$$

$$\|\phi_k\|_{2,\epsilon_k} = 1, \qquad k = 1,\, 2,\, \dots . \tag{3.31}$$

Written more explicitly, we have the following situation

$$\Delta_y\phi_k - \phi_k + 2\frac{A_{\epsilon_k,P^k}}{H_{\epsilon_k,P^k}}\phi_k - 2\frac{\int_{\Omega_{\epsilon_k}} A_{\epsilon_k,P^k}\phi_k\,dz}{\int_{\Omega_{\epsilon_k,P^k}} A^2_{\epsilon_k,P^k}\,dz}A^2_{\epsilon_k,P^k} = f_k, \tag{3.32}$$

where

$$\|\hat{\pi}_{\epsilon,P}(f_k)\|_{0,\epsilon_k} \to 0$$

$$\phi_k \in K^{\perp}_{\epsilon_k,P^k}, \quad \|\phi_k\|_{2,\epsilon_k} = 1. \tag{3.33}$$

We now show that this is impossible. Set $A_k = A_{\epsilon_k,P^k}, \Omega_k = \Omega_{\epsilon_k,P^k}$.

Note that

$$H_{\epsilon_k,P^k} = 1 + o(1) \ \text{ in } L^\infty(\Omega),$$

$$(\Delta_y - 1 + 2A_k)A_k = A_k^2 + o(1) \text{ in } L^2(\Omega_k).$$

Thus we have from (3.32)

$$(\Delta_y - 1 + 2A_k)\left(\phi_k - 2\frac{\int_{\Omega_k} A_k\phi_k\,dz}{\int_{\Omega_k} A_k^2\,dz}A_k\right) = f_k + o(1) \text{ in } L^2(\Omega_k).$$

Since

$$\int_{\Omega_k} A_k \epsilon_k \frac{\partial A_k}{\partial \tau(P^k)}\, dz = o(1)$$

in $H^2(\Omega_k)$ and since the operator $l_{\epsilon_k,P^k}$ is a one-to-one map with uniformly bounded inverse with respect to $\epsilon_k$ from $K^\perp_{\epsilon_k,P^k}$ to $C^\perp_{\epsilon_k,P^k}$ (see the Propositions 3.3 and 3.4) we have

$$\phi_k - 2\frac{\int_{\Omega_k} A_k \phi_k\, dz}{\int_{\Omega_k} A_k^2\, dz} A_k = o(1) \text{ in } H^2(\Omega_k). \tag{3.34}$$

Multiplying (3.34) by $A_k$ and integrating over $\Omega_k$ implies that

$$\int_{\Omega_k} A_k \phi_k\, dz = o(1)$$

and thus since the second term on the l.h.s. (3.34) is $o(1)$ in $H^2(\Omega_k)$ we conclude

$$\|\phi_k\|_{2,\epsilon_k} = o(1).$$

This is a contradiction.

Therefore (3.29) holds and $L_{\epsilon,P}$ is an injective map. Next we show that $L_{\epsilon,P}$ is also surjective. To this end, we just need to show that the conjugate of $L_{\epsilon,P}$ (denoted by $L^*_{\epsilon,P}$) is an injective operator from $K^\perp_{\epsilon,P}$ to $C^\perp_{\epsilon,P}$.

Suppose not. Then there exist sequences $\{\epsilon_k\}$, $\{P^k\}$, and $\{\phi_k\}$ with $\epsilon_k \to 0$, $P^k \in \overline{\Lambda}$, $\phi_k \in K^\perp_{\epsilon_k,P^k}$ such that

$$\|L^*_{\epsilon_k,P^k}\phi_k\|_{0,\epsilon_k} \in C_{\epsilon_k,P^k}, \tag{3.35}$$

$$\|\phi_k\|_{2,\epsilon_k} = 1, \qquad k = 1, 2, \ldots . \tag{3.36}$$

Explicitly, we have

$$\Delta_y \phi_k - \phi_k + 2A_k H^{-1}_{\epsilon_k,P^k}\phi_k - 2\frac{\int_{\Omega_{\epsilon_k,P^k}} A_k^2 \phi_k\, dz}{\int_{\Omega_{\epsilon_k,P^k}} A_k^2\, dz} A_k \in C_{\epsilon_k,P^k}. \tag{3.37}$$

Multiplying (3.37) by $A_k$ and integrating over $\Omega_{\epsilon_k,P^k}$, we obtain arguing in the same way as before that

$$\int_{\Omega_{\epsilon_k,P^k}} A_k^2 \phi_k\, dz = o(1).$$

Hence the non-local term in (3.37) vanishes and $\phi_k$ satisfies

$$\Delta_y \phi_k - \phi_k + 2A_k H^{-1}_{\epsilon_k,P^k}\phi_k + o(1) \in C_{\epsilon,P}, \quad \phi \in K^\perp_{\epsilon,P}$$

which implies that $\|\phi_k\|_{H^2(\Omega_{\epsilon_k,P^k})} = o(1)$ as the operator $l_{\epsilon,P}$ is injective. This is a contradiction !

Therefore $L^*_{\epsilon,P}$ is injective and $L_{\epsilon,P}$ is surjective.

□

We now deal with the linearized operator of system (3.1), namely the operator $\tilde{L}_{\epsilon,P}$ utilizing the results on $L_{\epsilon,P}$.

Again, $\tilde{L}_{\epsilon,P}$ is not uniformly invertible in $\epsilon$, which is now due to the approximate kernel

$$\mathcal{K}_{\epsilon,P} := K_{\epsilon,P} \oplus \{0\} \subset H^2_N(\Omega_{\epsilon,P}) \times H^2_N(\Omega).$$

and the approximate cokernel

$$\mathcal{C}_{\epsilon,P} := C_{\epsilon,P} \oplus \{0\} \subset L^2(\Omega_{\epsilon,P}) \times L^2(\Omega).$$

We also define

$$\mathcal{K}^\perp_{\epsilon,P} := K^\perp_{\epsilon,P} \oplus H^2_N(\Omega) \subset H^2_N(\Omega_{\epsilon,P}) \times H^2_N(\Omega),$$

$$\mathcal{C}^\perp_{\epsilon,P} := C^\perp_{\epsilon,P} \oplus L^2(\Omega) \subset L^2(\Omega_{\epsilon,P}) \times L^2(\Omega).$$

Let $\pi_{\epsilon,P} : L^2(\Omega_{\epsilon,P}) \times L^2(\Omega) \to \mathcal{C}^\perp_{\epsilon,P}$ denote the $L^2$-orthogonal projection. (Here the second component of the projection is the identity map.) Our goal in this section is to show that for $\epsilon$ small enough the equation

$$\pi_{\epsilon,P} \circ S_\epsilon \begin{pmatrix} A_{\epsilon,P} + \Phi_{\epsilon,P} \\ H_{\epsilon,P} + \Psi_{\epsilon,P} \end{pmatrix} = 0$$

has the unique solution $\Sigma_{\epsilon,P} = \begin{pmatrix} \Phi_{\epsilon,P}(y) \\ \Psi_{\epsilon,P}(x) \end{pmatrix}$ in $\mathcal{K}^\perp_{\epsilon,P}$.

As a preparation in the following two propositions we show the invertibility of the corresponding linearized operator

$$\mathcal{L}_{\epsilon,P} = \pi_{\epsilon,P} \circ \tilde{L}_{\epsilon,P}, \quad \mathcal{K}^\perp_{\epsilon,P} \to \mathcal{C}^\perp_{\epsilon,P}.$$

Proposition 3.6. *There exist positive constants $\bar{\epsilon}, \lambda$ such that for all $\epsilon \in (0, \bar{\epsilon})$ and for all $\Sigma \in \mathcal{K}^\perp_{\epsilon,P}$*

$$\|\mathcal{L}_{\epsilon,P}\Sigma\|_{L^2(\Omega_{\epsilon,P})\times L^2(\Omega)} \geq \lambda \|\Sigma\|_{H^2(\Omega_{\epsilon,P})\times H^2(\Omega)}. \tag{3.38}$$

PROPOSITION 3.7. *There exists a positive constant* $\overline{\overline{\epsilon}}$ *such that for all* $\epsilon \in (0, \overline{\overline{\epsilon}})$ *the map* $\mathcal{L}_{\epsilon,P}$ *is surjective.*

Note that Proposition 3.6 implies that $\mathcal{L}_{\epsilon,P}$ is injective.

**Proof of Proposition 3.6.** This proposition follows from Proposition 3.5. In fact, suppose that (3.38) is false. Then there exist sequences $\{\epsilon_k\}$, $\{P^k\}$, and $\{\Sigma_k\}$ with $\epsilon_k \to 0$, $P^k \in \overline{\Lambda}$, $\Sigma_k = \begin{pmatrix} \phi_k(y) \\ \psi_k(x) \end{pmatrix} \in \mathcal{K}^{\perp}_{\epsilon_k,P^k}$ such that

$$\|\mathcal{L}_{\epsilon_k,P^k}\Sigma_k\|_{L^2(\Omega_{\epsilon_k,P^k})\times L^2(\Omega)} \to 0, \tag{3.39}$$

$$\|\Sigma_k\|_{H^2(\Omega_{\epsilon_k,P^k})\times H^2(\Omega)} = 1, \qquad k = 1,\, 2,\, \dots\,. \tag{3.40}$$

Namely, we have the following situation

$$\Delta_y\phi_k - \phi_k + 2A_{\epsilon_k,P^k}H^{-1}_{\epsilon_k,P^k}\phi_k - A^2_{\epsilon_k,P^k}H^{-2}_{\epsilon_k,P^k}\psi_k = f_k, \tag{3.41}$$

$$\|\hat{\pi}_{\epsilon_k,P^k}(f_k)\|_{L^2(\Omega_{\epsilon_k,P^k})} \to 0, \tag{3.42}$$

$$\Delta_x\psi_k - \beta_k^2\psi_k + 2\beta_k^2\xi_{\epsilon_k}A_{\epsilon_k,P^k}\phi_k = \beta_k^2 g_k, \tag{3.43}$$

$$\|g_k\|_{L^2(\Omega)} \to 0,$$

$$\phi_k \in K^{\perp}_{\epsilon_k,P^k}, \tag{3.44}$$

$$\|\phi_k\|^2_{H^2(\Omega_{\epsilon_k,P^k})} + \|\psi_k\|^2_{H^2(\Omega)} = 1. \tag{3.45}$$

(Note that $\beta_k^2 = c_0\epsilon_k$.)

We now show that this will give a contradiction. Set $A_k = A_{\epsilon_k,P^k}$, $\Omega_k = \Omega_{\epsilon_k,P^k}$, $\xi_k = \xi_{\epsilon_k}$.

We first note that by (3.43) we have by a straightforward application of the Green function as in Section 2.2 that

$$\overline{\psi_k} := \frac{1}{|\Omega|}\int_\Omega \psi_k\,dx = 2\xi_k\int_\Omega A_k\phi_k\,dx \tag{3.46}$$

and

$$\|\psi_k - \overline{\psi_k}\|_{L^\infty(\Omega)} \le \|\psi_k - \overline{\psi_k}\|_{H^2(\Omega)} \le C\beta_k^2. \tag{3.47}$$

Thus

$$\|A_k^2(\psi_k - \overline{\psi_k}))\|_{L^2(\Omega_k)} \to 0 \tag{3.48}$$

as $k \to \infty$. Recalling the definition of $\xi_k$ in (1.9), (3.46) gives

$$\overline{\psi_k} = 2\frac{\int_\Omega A_k\phi_k\,dx}{\int_\Omega A_k^2\,dx} + o(1).$$

By (3.47) we also get

$$\left\|\psi_k - 2\frac{\int_\Omega A_k\phi_k\,dx}{\int_\Omega A_k^2\,dx}\right\|_{L^\infty(\Omega)} \to 0.$$

Thus by (3.41) we have

$$L_{\epsilon_k,P^k}\phi_k = o(1) \quad \text{in} \quad L^2(\Omega_k), \quad \phi_k \in K^\perp_{\epsilon_k,P^k}. \tag{3.49}$$

By Proposition 3.6, we conclude $\|\phi_k\|_{H^2(\Omega_k)} = o(1)$. Hence by (3.46) and (3.47) we have $\|\psi_k\|_{H^2}(\Omega) = o(1)$.

This contradicts the assumption (3.45) and the proof of Proposition 3.6 is completed.

□

**Proof of Proposition 3.7.** We need to show that the conjugate operator of $\mathcal{L}_{\epsilon,P}$ (denoted by $\mathcal{L}^*_{\epsilon,P}$) is injective from $\mathcal{K}^\perp_{\epsilon,P}$ to $\mathcal{C}^\perp_{\epsilon,P}$ if $\epsilon$ is small enough. Suppose not. Then there exist sequences $\{\epsilon_k\}$, $\{P^k\}$, $\{\phi_k\}$, and $\{\psi_k\}$ with $\epsilon_k \to 0$, $P^k \in \overline{\Lambda}$ such that $\phi_k \in K^\perp_{\epsilon_k,P^k}$, $\psi \in H^2_N(\Omega)$ and (using the same simplified notation as in the proof of Proposition 3.6)

$$\Delta_y\phi_k - \phi_k + 2A_k\phi_k + 2\xi_k\beta_k^2 A_k\psi_k \in C^\perp_{\epsilon,P}, \quad y \in \Omega_k, \tag{3.50}$$

$$\Delta_x\psi_k - \beta_k^2\psi_k - A_k^2\phi_k = 0, \tag{3.51}$$

$$\|\phi_k\|_{H^2(\Omega_k^2)} + \xi_k\beta_k^2\|\psi_k\|^2_{H^2(\Omega)} = 1. \tag{3.52}$$

Similar as in the proof of Proposition 3.6, (3.51) implies that

$$\xi_k\beta_k^2\overline{\psi_k} = -(1+o(1))\frac{\int_{\Omega_k} A_k^2\phi_k\,dz}{\int_{\Omega_k} A_k^2\,dz}.$$

Substituting this result into (3.50) we obtain

$$L_{\epsilon_k,P^k}\phi_k + o(1) \in C_{\epsilon,P}, \quad \phi_k \in K^\perp_{\epsilon_k,P^k}.$$

By Proposition 3.5, $\|\phi_k\|_{H^2(\Omega_k)} = o(1)$ and by (3.51) also $\|\psi_k\|_{H^2(\Omega)} = o(1)$. This is a contradiction with (3.52).

□

Now we are in a position to solve the equation

$$\pi_{\epsilon,P} \circ S_\epsilon \begin{pmatrix} A_{\epsilon,P} + \phi \\ H_{\epsilon,P} + \psi \end{pmatrix} = 0. \tag{3.53}$$

Since $\mathcal{L}_{\epsilon,P}$ is invertible (call the inverse $\mathcal{L}_{\epsilon,P}^{-1}$) we can rewrite (1.14) as follows:

$$\Sigma = -(\mathcal{L}_{\epsilon,P}^{-1} \circ \pi_{\epsilon,P})(S_\epsilon \begin{pmatrix} A_{\epsilon,P} \\ H_{\epsilon,P} \end{pmatrix}) - (\mathcal{L}_{\epsilon,P}^{-1} \circ \pi_{\epsilon,P}) N_{\epsilon,P}(\Sigma) \equiv M_{\epsilon,P}(\Sigma), \tag{3.54}$$

where

$$N_{\epsilon,P}(\Sigma) = S_\epsilon \begin{pmatrix} A_{\epsilon,P} + \phi \\ H_{\epsilon,P} + \psi \end{pmatrix} - S_\epsilon \begin{pmatrix} A_{\epsilon,P} \\ H_{\epsilon,P} \end{pmatrix} - S'_\epsilon \begin{pmatrix} A_{\epsilon,P} \\ H_{\epsilon,P} \end{pmatrix} \begin{bmatrix} \phi \\ \psi \end{bmatrix}$$

and the operator $M_{\epsilon,P}$ is defined by (3.54) for $\Sigma = (\phi, \psi) \in H^2_N(\Omega_{\epsilon,P}) \times H^2_N(\Omega)$. We are going to show that the operator $M_{\epsilon,P}$ is a contraction on

$$B_{\epsilon,\delta} \equiv \{\Sigma \in H^2_N(\Omega_{\epsilon,P}) \times H^2_N(\Omega) \, : \, \|\Sigma\|_{H^2(\Omega_{\epsilon,P}) \times H^2(\Omega)} < \delta\}$$

if $\delta$ is small enough. We have by the estimate (3.23) and Propositions 3.6 and 3.7

$$\begin{aligned} \|M_{\epsilon,P}(\Sigma)\|_{H^2(\Omega_{\epsilon,P}) \times H^2(\Omega)} \leq \lambda^{-1} \Bigg( & \|\pi_{\epsilon,P} \circ N_{\epsilon,P}(\Sigma)\|_{L^2(\Omega_{\epsilon,P}) \times L^2(\Omega)} \\ & + \left\| \pi_{\epsilon,P} \circ S_\epsilon \begin{pmatrix} A_{\epsilon,P} \\ H_{\epsilon,P} \end{pmatrix} \right\|_{L^2(\Omega_{\epsilon,P}) \times L^2(\Omega)} \Bigg) \\ \leq \lambda^{-1} C (c(\delta)\delta + \epsilon), & \end{aligned}$$

where $\lambda > 0$ is independent of $\epsilon > 0$ and $\delta > 0$. Furthermore, it holds that $c(\delta) \to 0$ as $\delta \to 0$. Similarly we show

$$\|M_{\epsilon,P}(\Sigma) - M_{\epsilon,P}(\Sigma')\|_{H^2(\Omega_{\epsilon,P}) \times H^2(\Omega)} \leq \lambda^{-1} c(\delta) \|\Sigma - \Sigma'\|_{H^2(\Omega_{\epsilon,P}) \times H^2(\Omega)}$$

where $c(\delta) \to 0$ as $\delta \to 0$ uniformly in $\epsilon$ if $\epsilon$ is small enough . If we choose $\delta$ and $\epsilon$ small enough, then $M_{\epsilon,P}$ is a contraction on $B_{\epsilon,\delta}$. The existence of a fixed point $\Sigma_{\epsilon,P}$ now follows from the Contraction Mapping Principle and $\Sigma_{\epsilon,P}$ is a solution of (3.54).

We have thus proved

PROPOSITION 3.8. *There exist* $\overline{\epsilon} > 0$ *such that for every pair* $\epsilon$, $P$ *with* $0 < \epsilon < \overline{\epsilon}$ *and* $P \in \overline{\Lambda}$ *there exists a unique* $(\Phi_{\epsilon,P}, \Psi_{\epsilon,P}) \in \mathcal{K}^{\perp}_{\epsilon,P}$ *satisfying* $S_\epsilon\left(\left(\begin{array}{c} A_{\epsilon,P} + \Phi_{\epsilon,\mathbf{P}} \\ H_{\epsilon,P} + \Psi_{\epsilon,P} \end{array}\right)\right) \in \mathcal{C}_{\epsilon,P}$ *and*

$$\|(\Phi_{\epsilon,P}, \Psi_{\epsilon,P})\|_{H^2(\Omega_{\epsilon,P})\times H^2(\Omega)} \leq C\epsilon. \tag{3.55}$$

We need another statement about the asymptotic behavior of the function $\Phi_{\epsilon,P}$ as $\epsilon \to 0$, which gives an asymptotic expansion in $\epsilon$ with a rigorous error estimate and is stated as follows.

PROPOSITION 3.9. *We have*

$$\Phi_{\epsilon,P}(x) = \epsilon(\Phi_0(y)\chi(x)) + \epsilon^2\varphi^1_{\epsilon,P}(x), \tag{3.56}$$

*where*

$$\|\varphi^1_{\epsilon,P}\|_{2,\epsilon} \leq C$$

*and* $\Phi_0$ *is the unique solution in* $K_0^{\perp}$ *of*

$$\begin{aligned} \Delta\Phi_0 - \Phi_0 + 2w\Phi_0 - 2\frac{\int_{\mathbb{R}^2_+} w\Phi_0\,dy}{\int_{\mathbb{R}^2_+} w^2\,dy}w^2 - 2wv_1 \\ +R(|y|)w^2 = 0 \qquad &\text{in } \mathbb{R}^2_+, \\ \frac{\partial\Phi_0}{\partial y_2} = 0 \qquad &\text{on } \partial\mathbb{R}^2_+, \end{aligned}$$

*where* $R(|y|)$ *was introduced in Lemma 3.2.*

Note that $\Phi_0$ is a radially symmetric function, i.e., $\Phi_0(y) = \Phi_0(|y|)$ for $y \in B(\delta_2)$. Recall that the uniqueness of $\Phi_0$ follows from Lemma 2.1. A proof of Proposition 3.9 will be given in Appendix D.

## 3. The Reduced Problem

In this section we solve the reduced problem and prove our main existence theorem.

By Proposition 3.8 there exists a unique solution $(\Phi_{\epsilon,P}, \psi_{\epsilon,P}) \in \mathcal{K}^{\perp}_{\epsilon,P}$ such that

$$S_\epsilon\left(\begin{array}{c} A_{\epsilon,P} + \Phi_{\epsilon,P} \\ H_{\epsilon,P} + \Psi_{\epsilon,P} \end{array}\right) = \left(\begin{array}{c} v_{\epsilon,P} \\ 0 \end{array}\right) \in \mathcal{C}_{\epsilon,P}.$$

Our goal in this section is to find $P \in \overline{\Lambda}$ such that

$$S_\epsilon \begin{pmatrix} A_{\epsilon,P} + \Phi_{\epsilon,P} \\ H_{\epsilon,P} + \Psi_{\epsilon,P} \end{pmatrix} \perp \mathcal{C}_{\epsilon,P}.$$

This implies that

$$S_\epsilon \begin{pmatrix} A_{\epsilon,P} + \Phi_{\epsilon,P} \\ H_{\epsilon,P} + \Psi_{\epsilon,P} \end{pmatrix} = 0.$$

For $P \in \overline{\Lambda}$, let

$$W_\epsilon(P) := \frac{1}{\epsilon} \int_{\Omega_{\epsilon,P}} S_1(A_{\epsilon,P} + \Phi_{\epsilon,P}, H_{\epsilon,P} + \Psi_{\epsilon,P}) \frac{\partial w_{\epsilon,P}}{\partial \tau(P)}\, dz.$$

Then $W_\epsilon(P)$ is a map which is continuous in $P$ and our problem is reduced to finding a zero of $W_\epsilon(P)$.

We calculate

$$\begin{aligned}
&\int_{\Omega_{\epsilon,P}} S_1(A_{\epsilon,P} + \Phi_{\epsilon,P}, H_{\epsilon,P} + \Psi_{\epsilon,P}) \frac{\partial w_{\epsilon,P}}{\partial \tau(P)}\, dz \\
&= \int_{\Omega_{\epsilon,P}} \left[ \Delta(A_{\epsilon,P} + \Phi_{\epsilon,P}) - (A_{\epsilon,P} + \Phi_{\epsilon,P}) + \frac{(A_{\epsilon,P} + \Phi_{\epsilon,P})^2}{H_{\epsilon,P} + \Psi_{\epsilon,P}} \right] \frac{\partial w_{\epsilon,P}}{\partial \tau(P)}\, dz \\
&= \int_{\Omega_{\epsilon,P}} \left[ \Delta(A_{\epsilon,P} + \Phi_{\epsilon,P}) - (A_{\epsilon,P} + \Phi_{\epsilon,P}) + \frac{(A_{\epsilon,P} + \Phi_{\epsilon,P})^2}{H_{\epsilon,P}(P) + \Psi_{\epsilon,P}(P)} \right] \frac{\partial w_{\epsilon,P}}{\partial \tau(P)}\, dz \\
&\quad + \int_{\Omega_{\epsilon,P}} \left[ \frac{(A_{\epsilon,P} + \Phi_{\epsilon,P})^2}{H_{\epsilon,P} + \Psi_{\epsilon,P}} - \frac{(A_{\epsilon,P} + \Phi_{\epsilon,P})^2}{H_{\epsilon,P}(P) + \Psi_{\epsilon,P}(P)} \right] \frac{\partial w_{\epsilon,P}}{\partial \tau(P)}\, dz \\
&=: I_1 + I_2,
\end{aligned}$$

where $I_1$ and $I_2$ are defined by the last equation. Furthermore, we calculate

$$\begin{aligned}
I_1 &= \int_{\Omega_{\epsilon,P}} \left[ \Delta(A_{\epsilon,P} + \Phi_{\epsilon,P}) - (A_{\epsilon,P} + \Phi_{\epsilon,P}) + \frac{(A_{\epsilon,P} + \Phi_{\epsilon,P})^2}{1 + \Psi_{\epsilon,P}(P)} \right] \frac{\partial w_{\epsilon,P}}{\partial \tau(P)}\, dz \\
&= \int_{\Omega_{\epsilon,P}} [\Delta(w_{\epsilon,P} + \Phi_{\epsilon,P}) - (w_{\epsilon,P} + \Phi_{\epsilon,P}) + (w_{\epsilon,P} + \Phi_{\epsilon,P})^2] \frac{\partial w_{\epsilon,P}}{\partial \tau(P)}\, dz \\
&\quad + \left( \frac{1}{1 + \Psi_{\epsilon,P}(P)} - 1 \right) \int_{\Omega_{\epsilon,P}} (w_{\epsilon,P} + \Phi_{\epsilon,P})^2 \frac{\partial w_{\epsilon,P}}{\partial \tau(P)}\, dz \\
&= \int_{\Omega_{\epsilon,P}} (w_{\epsilon,P} + \Phi_{\epsilon,P})^2 \frac{\partial w_{\epsilon,P}}{\partial \tau(P)}\, dz + \int_{\Omega_{\epsilon,P}} [\Delta \frac{\partial w_{\epsilon,P}}{\partial \tau(P)} - \frac{\partial w_{\epsilon,P}}{\partial \tau(P)}](w_{\epsilon,P} + \Phi_{\epsilon,P})\, dz \\
&\quad + \left( \frac{1}{1 + \Psi_{\epsilon,P}(P)} - 1 \right) \int_{\Omega_{\epsilon,P}} (w_{\epsilon,P} + \Phi_{\epsilon,P})^2 \frac{\partial w_{\epsilon,P}}{\partial \tau(P)}\, dz \\
&= \int_{\Omega_{\epsilon,P}} (w_{\epsilon,P} + \Phi_{\epsilon,P})^2 \frac{\partial w_{\epsilon,P}}{\partial \tau(P)} + \int_{\Omega_{\epsilon,P}} [-\frac{\partial (w_\epsilon)^2}{\partial \tau(P)}](w_{\epsilon,P} + \Phi_{\epsilon,P})\, dz
\end{aligned}$$

$$+\left(\frac{1}{1+\Psi_{\epsilon,P}(P)}-1\right)\int_{\Omega_{\epsilon,P}}(w_{\epsilon,P}+\Phi_{\epsilon,P})^2\,\frac{\partial w_{\epsilon,P}}{\partial\tau(P)}\,dz$$

by (3.7). Using the identity

$$\int_{\Omega_{\epsilon,P}}\frac{\partial(w_\epsilon)^2}{\partial\tau(P)}w_{\epsilon,P}\,dz$$
$$=\int_{\Omega_{\epsilon,P}}[\frac{\partial}{\partial\tau(P)}(-\Delta w_{\epsilon,P}+w_{\epsilon,P})]w_{\epsilon,P}\,dz$$
$$=\int_{\Omega_{\epsilon,P}}[(-\Delta\frac{\partial w_{\epsilon,P}}{\partial\tau(P)})w_{\epsilon,P}+\frac{\partial w_{\epsilon,P}}{\partial\tau(P)}w_{\epsilon,P}]\,dz$$
$$=\int_{\Omega_{\epsilon,P}}[-\frac{\partial w_{\epsilon,P}}{\partial\tau(P)}\Delta w_{\epsilon,P}+\frac{\partial w_{\epsilon,P}}{\partial\tau(P)}w_{\epsilon,P}]\,dz$$
$$=\int_{\Omega_{\epsilon,P}}(-\Delta w_{\epsilon,P}+w_{\epsilon,P})\frac{\partial w_{\epsilon,P}}{\partial\tau(P)}\,dz$$
$$=\int_{\Omega_{\epsilon,P}}w_\epsilon^2\frac{\partial w_{\epsilon,P}}{\partial\tau(P)}\,dz,$$

adding and subtracting further terms, and after some re-ordering, we get

$$I_1=\int_{\Omega_{\epsilon,P}}[(w_{\epsilon,P}+\Phi_{\epsilon,P})^2-(w_{\epsilon,P})^2-2w_{\epsilon,P}\Phi_{\epsilon,P})]\frac{\partial w_{\epsilon,P}}{\partial\tau(P)}\,dz$$
$$+\int_{\Omega_{\epsilon,P}}2[w_{\epsilon,P}\frac{\partial w_{\epsilon,P}}{\partial\tau(P)}-w_\epsilon\frac{\partial w_\epsilon}{\partial\tau(P)}]\Phi_{\epsilon,P}\,dz$$
$$+\left(\frac{1}{1+\Psi_{\epsilon,P}(P)}-1\right)\int_{\Omega_{\epsilon,P}}(w_{\epsilon,P}+\Phi_{\epsilon,P})^2\,\frac{\partial w_{\epsilon,P}}{\partial\tau(P)}\,dz$$
$$+\int_{\Omega_{\epsilon,P}}[(w_{\epsilon,P})^2-(w_\epsilon)^2]\frac{\partial w_{\epsilon,P}}{\partial\tau(P)}\,dz$$
$$=:I_{1,a}+I_{1,b}+I_{1,c}+J^\epsilon.$$

We now calculate these three terms separately. We will see that $J^\epsilon$ is the leading term and the other two terms are of higher order. It will be essential to use the expansions given in (3.13), (3.17), (3.56) and the symmetries associated with these.

We first calculate $I_{1,b}$ by using (3.56):

$$I_{1,b}=\int_{\Omega_{\epsilon,P}}2\left[w_{\epsilon,P}\frac{\partial w_{\epsilon,P}}{\partial\tau(P)}-w_\epsilon\frac{\partial w_\epsilon}{\partial\tau(P)}\right]\Phi_{\epsilon,P}\,dz$$
$$=\epsilon\int_{\Omega_{\epsilon,P}}\left[2w_{\epsilon,P}\frac{\partial w_{\epsilon,P}}{\partial\tau(P)}-2w_\epsilon\frac{\partial w_\epsilon}{\partial\tau(P)}\right]\Phi_0\chi\,dz$$

$$+\epsilon^2 \int_{\Omega_{\epsilon,P}} \left[2w_{\epsilon,P}\frac{\partial w_{\epsilon,P}}{\partial\tau(P)} - 2w_\epsilon \frac{\partial w_\epsilon}{\partial\tau(P)}\right] \varphi^1_{\epsilon,P}\, dz$$
$$= \epsilon I_{1,b,a} + \epsilon^2 I_{1,b,b}.$$

Note that

$$\begin{aligned} &2w_{\epsilon,P}\frac{\partial w_{\epsilon,P}}{\partial\tau(P)} - 2w_\epsilon \frac{\partial w_\epsilon}{\partial\tau(P)} \\ &= [2w_{\epsilon,P} - 2w_\epsilon]\,\frac{\partial w_{\epsilon,P}}{\partial\tau(P)} \\ &+ 2w_\epsilon \left[\frac{\partial w_{\epsilon,P}}{\partial\tau(P)} - \frac{\partial w_\epsilon}{\partial\tau(P)}\right] \end{aligned} \tag{3.57}$$

and therefore by (3.13) and (3.17)

$$I_{1,b,a} = \epsilon^2 \int_{\Omega_{\epsilon,P}} \left[2v_3 \frac{\partial w_\epsilon}{\partial\tau(P)} + 2w_\epsilon w^\epsilon_{2,P}\right] \Phi_0\chi\, dz + O(\epsilon^2)$$
$$= O(\epsilon)$$

using symmetry. By Proposition 3.9 we have

$$\int_{\Omega_{\epsilon,P}} |\varphi^1_{\epsilon,P}|^2\, dz \le C.$$

Hence, we conclude that

$$|I_{1,b,b}| = O(1)$$

and

$$|I_{1,b}| = O(\epsilon^2).$$

Next we estimate $I_{1,a}$:

$$|I_{1,a}| \le C \int_{\Omega_{\epsilon,P}} 2\Phi^2_{\epsilon,P}\frac{\partial w_{\epsilon,P}}{\partial\tau(P)}\, dz$$
$$= 2\int_{\Omega_{\epsilon,P}} \epsilon^2[\Phi_0^2\chi^2 + 2\epsilon\Phi_0\chi\varphi^1_{\epsilon,P}]\frac{\partial w_\epsilon}{\partial\tau(P)}\, dz + O(\epsilon^3)$$
$$= O(\epsilon^2)$$

since $\Phi_0$ is even in $y_1$. Since $\Psi_{\epsilon,P}(P) = O(\epsilon)$, in the same way it follows that $I_{1,c} = O(\epsilon^2)$.

Finally, we compute the main term $J^\epsilon$.

$$J^\epsilon = \int_{\Omega_{\epsilon,P}} [(w_{\epsilon,P})^2 - (w_\epsilon)^2]\frac{\partial w_{\epsilon,P}}{\partial\tau(P)}\, dz$$
$$= \int_{\Omega_{\epsilon,P}} 2w_\epsilon(w_{\epsilon,P} - w_\epsilon)\frac{\partial w_{\epsilon,P}}{\partial\tau(P)}\, dz + \int_{\Omega_{\epsilon,P}} (w_{\epsilon,P} - w_\epsilon)^2\frac{\partial w_{\epsilon,P}}{\partial\tau(P)}\, dz$$

$$= -\epsilon \int_{\mathbb{R}^2_+} \left(2w(|y|) - 2\epsilon\chi \frac{w'(|y|)}{|y|}\rho''(0)y_1^2 y_2\right) (v_1\chi + \epsilon(v_2\chi + v_3\chi) + \epsilon^2 \varphi^0_{\epsilon,P})$$

$$\times \Bigg( -\frac{1}{\epsilon}\frac{\partial w}{\partial y_1} + \chi(x) \left[ \frac{|y|w''(|y|) - w'(|y|)}{2|y|^3} \rho''(0) y_1^3 y_2 + \rho''(0) y_1 \frac{\partial w(y)}{\partial y_2} + w_1(y) \right]$$

$$+\epsilon w^\epsilon_{2,P}(x)\Bigg)\, dy - \epsilon \int_{\mathbb{R}^2_+} (v_1^2\chi^2) \left(\frac{\partial w}{\partial y_1}\right) dy + O(\epsilon^2)$$

$$= -\epsilon \int_{\mathbb{R}^2_+} 2w_\epsilon v_3 \chi \frac{\partial w_\epsilon}{\partial y_1}\, dy + O(\epsilon^2)$$

$$= -\epsilon \int_{\mathbb{R}^2_+} 2wv_3 \frac{\partial w}{\partial y_1}\, dy + O(\epsilon^2).$$

We now use (compare (3.16))

$$\begin{cases} \Delta v_3 - v_3 = 0 & \text{in } \mathbb{R}^2_+, \\ \frac{\partial v_3}{\partial y_2} = -\frac{\partial w}{\partial y_1}\frac{1}{3}\rho'''(0)y_1^2 & \text{on } \partial\mathbb{R}^2_+ \end{cases}$$

and have

$$\int_{\mathbb{R}^2_+} 2wv_3 \frac{\partial w}{\partial y_1}\, dy$$

$$= \int_{\mathbb{R}^2_+} \frac{\partial (w^2)}{\partial y_1} v_3\, dy$$

$$= \int_{\mathbb{R}^2_+} \left(\Delta \frac{\partial w}{\partial y_1} - \frac{\partial w}{\partial y_1}\right) v_3\, dy$$

by (1.4). We further calculate

$$\int_{\mathbb{R}^2_+} 2wv_3 \frac{\partial w}{\partial y_1}\, dy$$

$$= \int_{\mathbb{R}^2_+} \frac{\partial w}{\partial y_1} (\Delta v_3 - v_3)\, dy + \int_{\partial\mathbb{R}^2_+} \left( \frac{\partial v_3}{\partial y_2}\frac{\partial w}{\partial y_1} - v_3 \frac{\partial^2 w}{\partial y_1 \partial y_2}\right) dy_1$$

$$= -\frac{1}{3}\int_{\mathbb{R}} \left(\frac{1}{y_1}\frac{\partial w(y_1,0)}{\partial y_1}\right)^2 \rho'''(0) y_1^4 dy_1$$

(by (3.16) and since $\frac{\partial^2 w}{\partial y_2 \partial y_1} = 0$ for $y_2 = 0$)

$$= \nu_1 \rho'''(0)$$

$$= \nu_1 \frac{\partial h(P)}{\partial \tau(P)},$$

where $\nu_1$ is given by (1.6). (Recall that

$$\nu_1 = -\frac{1}{3}\int_{\mathbb{R}} \left(\frac{\partial w(y_1,0)}{\partial y_1}\right)^2 y_1^2\, dy_1 < 0$$

and $h(P)$ is the curvature of $\partial\Omega$ at $P$.) In summary, we get

$$I_1 = \epsilon\nu_1 \frac{\partial h(P)}{\partial \tau(P)} + o(\epsilon), \tag{3.58}$$

where $\nu_1$ was defined in (1.6).

For $I_2$, we have

$$\begin{aligned} I_2 = &-\int_{\Omega_{\epsilon,P}} \frac{(A_{\epsilon,P}+\Phi_{\epsilon,P})^2}{H^2_{\epsilon,P}(P)}(H_{\epsilon,P}-H_{\epsilon,P}(P))\frac{\partial w_{\epsilon,P}}{\partial\tau(P)}\, dz \\ &-\int_{\Omega_{\epsilon,P}} \frac{(A_{\epsilon,P}+\Phi_{\epsilon,P})^2}{H^2_{\epsilon,P}(P)}(\Psi_{\epsilon,P}-\Psi_{\epsilon,P}(P))\frac{\partial w_{\epsilon,P}}{\partial\tau(P)}\, dz + O(\epsilon^2) \\ &=: I_{2,1} + I_{2,2}. \end{aligned}$$

Then

$$I_{2,1} = \int_{\Omega_{\epsilon,P}} (w_{\epsilon,P}+\Phi_{\epsilon,P})^2(z)(H_{\epsilon,P}(P+\epsilon z)-H_{\epsilon,P}(P))\frac{\partial w_{\epsilon,P}(z)}{\partial\tau(P)}\, dz + O(\epsilon^2).$$

Now, by (3.24),

$$\begin{aligned} I_{2,1} &= \int_{\Omega_{\epsilon,P}} w^2_{\epsilon,P}(z)(H_{\epsilon,P}(P+\epsilon z)-H_{\epsilon,P}(P))\frac{\partial w_{\epsilon,P}(z)}{\partial\tau(P)}\, dz + O(\epsilon^2) \\ &= c_0\epsilon^2\xi_\epsilon \frac{1}{2}\frac{\partial H_0(P,P)}{\partial\tau(P)}\int_{\mathbb{R}^2_+} w^2(y)y_1\frac{\partial w}{\partial y_1}\, dy \int_{\mathbb{R}^2_+} w^2(y)\, dy + O(\epsilon^2) \\ &= -\frac{c_0}{6}\epsilon^3\xi_\epsilon \frac{\partial H_0(P,P)}{\partial\tau(P)}\int_{\mathbb{R}^2_+} w^3(y)\, dy \int_{\mathbb{R}^2_+} w^2(y)\, dy + O(\epsilon^2). \end{aligned} \tag{3.59}$$

Similar to the calculations for $I_{2,1}$, we can estimate $I_{2,2}$:

$$\begin{aligned} I_{2,2} &= \int_{\Omega_{\epsilon,P}} w^2_{\epsilon,P}(z)(\Psi_{\epsilon,P}(P+\epsilon z)-\Psi_{\epsilon,P}(P))\frac{\partial w_{\epsilon,P}(z)}{\partial\tau(P)}\, dz + o(\epsilon) \\ &= -\int_{\Omega_{\epsilon,P}} \frac{1}{3}\frac{\partial (w^3_{\epsilon,P}(z))}{\partial\tau(P)}(\Psi_{\epsilon,P}(P+\epsilon z)-\Psi_{\epsilon,P}(P))\, dz + o(\epsilon). \end{aligned} \tag{3.60}$$

Now we recall that, for $|\epsilon y| < \delta_2$, $\Psi_{\epsilon,P}$ satisfies

$$\Delta\Psi_{\epsilon,P} - \beta^2\Psi_{\epsilon,P} + 2\beta^2\xi_\epsilon A_{\epsilon,P}\Phi_{\epsilon,P} + \beta^2\xi_\epsilon\Phi^2_{\epsilon,P} = 0.$$

Similar computations as those leading to (3.24) show that

$$
\begin{aligned}
&\Psi_{\epsilon,P}(P+\epsilon z) - \Psi_{\epsilon,P}(P) \\
&= \int_\Omega (G_\beta(P+\epsilon z,\xi) - G_\beta(P,\xi))\beta^2 \xi_\epsilon \\
&\times \left(2A_{\epsilon,P}\left(\frac{\xi-P}{\epsilon}\right)\Phi_{\epsilon,P}\left(\frac{\xi-P}{\epsilon}\right) + \Phi_{\epsilon,P}^2\left(\frac{\xi-P}{\epsilon}\right)\right) d\xi \\
&= o(\epsilon|\nabla_{\tau(P)}H(P,P)|\,|y|) + \epsilon R_1(|y|), \qquad (3.61)
\end{aligned}
$$

where $R_1(|y|)$ is a radially symmetric function.

Substituting (3.61) into (3.60), we obtain that

$$
\begin{aligned}
I_{2,2} &= o(\epsilon) - o(\epsilon|\nabla_{\tau(P)}H(P,P)|)\frac{1}{3}\int_{\mathbb{R}^2_+} w^3(y)\,dy \int_{\mathbb{R}^2_+} w^2(y)\,dy \\
&= o(\epsilon).
\end{aligned}
$$

In summary, we get

$$
I_2 = c_0 \epsilon \nu_2 \frac{\partial H(P,P)}{\partial \tau(P)} + o(\epsilon), \qquad (3.62)
$$

where $\nu_2$ is given by (1.7). We now combine $I_1$ and $I_2$. This gives

$$
W_\epsilon(P) = \nabla_{\tau(P)}(\nu_1 h(P) + c_0\nu_2 H_0(P,P)) + o(1) = \nabla_{\tau(P)}F(P) + o(1). \qquad (3.63)
$$

Suppose that, at $P = P_0$, we have $\nabla_{\tau(P)}W_\epsilon(P_0) = 0, \det((\nabla_{\tau(P)})^2 W_\epsilon(P_0) \neq 0$. Then, since $W_\epsilon$ is continuous and, for $\epsilon$ small enough, maps balls $B(P_0,\delta)$ into balls of the same kind with possibly a larger $\delta$, the standard Brouwer's fixed point theorem shows that for $\epsilon << 1$ there exists a $P^\epsilon \in \overline{\Lambda}$ such that $W_\epsilon(P^\epsilon) = 0$ and $P^\epsilon \to P_0$.

Thus we have proved the following proposition.

PROPOSITION 3.10. *For $\epsilon$ sufficiently small there exist points $P^\epsilon \in \overline{\Lambda}$ such that $W_\epsilon(P^\epsilon) = 0$ and $P_\epsilon \to P_0$.*

Finally, we prove Theorem 1.1.

**Proof of Theorem 1.1.** By Proposition 3.10, there exist $P^\epsilon \in \overline{\Lambda}$ such that $P^\epsilon \to P_0$ and $W_\epsilon(P^\epsilon) = 0$. In other words, we have $S_1(A_{\epsilon,P^\epsilon} + \Phi_{\epsilon,P^\epsilon}, H_{\epsilon,P^\epsilon} + \Psi_{\epsilon,P^\epsilon}) = 0$ and therefore also $S_\epsilon(A_{\epsilon,P^\epsilon}, H_{\epsilon,P^\epsilon}) = 0$. Let $A_\epsilon = \xi_\epsilon(A_{\epsilon,P^\epsilon} + \Phi_{\epsilon,P^\epsilon}), H_\epsilon = \xi_\epsilon(H_{\epsilon,P^\epsilon} + \Psi_{\epsilon,P^\epsilon})$. Since $A_{\epsilon,P^\epsilon}(y) \sim w(y)$ for $y \in \Omega_{\epsilon,P}$ it follows by solving the second equation of (1.14) that $H_\epsilon = 1 + O(\epsilon) > 0$. By the maximum principle it follows that $A_\epsilon > 0$: First, there has to be $A_\epsilon \geq 0$ since otherwise there has to be a local minimum point $y_{min} \in \Omega$ such that $A_\epsilon(y_{min}) < 0$ which gives a contradiction by considering the first equation of (1.14) at $y_{min}$. The more difficult case of a minimum point $y_{min} \in \Omega$ such that $A_\epsilon(y_{min}) = 0$ also leads to a contradiction since this can only be the case if $A_\epsilon(y) = 0$ for all $y$ in a neighborhood of $y_{min}$.

Similar as in the proof of Theorem 1.2 of [**54**], we conclude $P^\epsilon \in \overline{\Lambda}$, and there is only one such $P^\epsilon$.

Therefore $(A_\epsilon, H_\epsilon)$ satisfies all the claims, and the proof of Theorem 1.1 is completed.

□

CHAPTER 4

# Stability

We consider the stability of $(A_\epsilon, H_\epsilon)$ constructed in Theorem 1.1.

Linearizing the system (GM) around the equilibrium states $(A_\epsilon, H_\epsilon)$ we obtain the following eigenvalue problem

$$\begin{cases} \Delta_y \phi_\epsilon - \phi_\epsilon + 2\frac{A_\epsilon}{H_\epsilon}\phi_\epsilon - \frac{A_\epsilon^2}{H_\epsilon^2}\psi_\epsilon = \lambda_\epsilon \phi_\epsilon, \\ \frac{1}{\beta^2}\Delta\psi_\epsilon - \psi_\epsilon + 2A_\epsilon\phi_\epsilon = \tau\lambda_\epsilon\psi_\epsilon. \end{cases} \tag{4.1}$$

Here $D = \frac{1}{\beta^2}$, $\lambda_\epsilon$ is some complex number, and

$$\phi_\epsilon \in H_N^2(\Omega_\epsilon, P^\epsilon),\ \psi_\epsilon \in H_N^2(\Omega). \tag{4.2}$$

Let

$$\hat{A}_\epsilon = \xi_\epsilon^{-1} A_\epsilon = A_{\epsilon,P^\epsilon} + \Phi_{\epsilon,P^\epsilon}, \hat{H}_\epsilon = \xi_\epsilon^{-1} H_\epsilon = H_{\epsilon,P^\epsilon} + \Psi_{\epsilon,P^\epsilon}. \tag{4.3}$$

Then (4.1) becomes

$$\begin{cases} \Delta_y \phi_\epsilon - \phi_\epsilon + 2\frac{\hat{A}_\epsilon}{\hat{H}_\epsilon}\phi_\epsilon - \frac{\hat{A}_\epsilon^2}{\hat{H}_\epsilon^2}\psi_\epsilon = \lambda_\epsilon \phi_\epsilon, \\ \frac{1}{\beta^2}\Delta\psi_\epsilon - \psi_\epsilon + 2\xi_\epsilon\hat{A}_\epsilon\phi_\epsilon = \tau\lambda_\epsilon\psi_\epsilon. \end{cases} \tag{4.4}$$

## 1. The Large Eigenvalues

In this section, we study the large eigenvalues, i.e., we assume that $|\lambda_\epsilon| \geq c > 0$ for $\epsilon$ small. Furthermore, without loss of generality, we may assume that

$$(1+\tau)c < \frac{1}{2}. \tag{4.5}$$

If $\mathrm{Re}(\lambda_\epsilon) \leq -c$, we are done. (Then $\lambda_\epsilon$ is a stable large eigenvalue.)

We first prove that there exists $C > 0$ such that

$$|\lambda_\epsilon| \leq C. \tag{4.6}$$

We multiply the first equation in (4.4) by $\overline{\phi_\epsilon}$ and integrate. Without loss of generality, let us assume that

$$\|\phi_\epsilon\|^2_{H^2(\Omega_{\epsilon,P^\epsilon})} + \|\psi_\epsilon\|^2_{H^2(\Omega)} = 1.$$

Then we have

$$\int_{\Omega_{\epsilon,P^\epsilon}} \left[ \frac{1}{2}|\nabla\phi_\epsilon|^2 - \frac{1}{2}|\phi_\epsilon|^2 + 2\frac{\hat{A}_\epsilon}{\hat{H}_\epsilon}|\phi_\epsilon|^2 \right.$$

$$\left. -\lambda_\epsilon|\phi_\epsilon|^2 - \frac{\hat{A}^2_\epsilon}{\hat{H}^2_\epsilon}\overline{\phi_\epsilon}\psi_\epsilon \right] dz = 0. \tag{4.7}$$

By the second equation in (4.4), we have

$$\|\psi_\epsilon\|_{L^\infty(\Omega)} \le C\|\phi\|_{L^2(\Omega_{\epsilon,P^\epsilon})}.$$

Substituting this into (4.7) gives

$$|\lambda_\epsilon|O(1) + O(1) = 0.$$

Therefore (4.6) is proved.

Therefore, we may assume that $\mathrm{Re}(\lambda_\epsilon) \ge -c$ and, because of (4.6), there is a subsequence $\epsilon \to 0$ such that $\lambda_\epsilon \to \lambda_0 \ne 0$. We shall derive the limiting eigenvalue problem which is a NLEP.

We introduce the following notation:

$$\beta_{\lambda_\epsilon} = \beta\sqrt{1+\tau\lambda_\epsilon}, \tag{4.8}$$

where in $\sqrt{1+\tau\lambda_\epsilon}$ we take the principal part of the square root. (This means that the real part of $\sqrt{1+\tau\lambda_\epsilon}$ is positive, which is possible since by (4.5) we have $\tau c < \frac{1}{2}$ and therefore Re $(1+\tau\lambda_\epsilon) \ge \frac{1}{2}$.)

We cut off $\phi_\epsilon$ as follows: Introduce

$$\phi_{\epsilon,1}(x) = \phi_\epsilon(x)\chi(x),$$

where $\chi(x)$ was introduced in (3.12).

From (4.4), $\mathrm{Re}(\lambda_\epsilon) \ge -c$, the exponential decay of $w$ (see (1.5)), and by (3.13), (3.17), it follows that

$$\hat{A}_\epsilon\phi_\epsilon = \phi_{\epsilon,1}w_\epsilon + o(1) \quad \text{in } H^2(\Omega_{\epsilon,P^\epsilon}). \tag{4.9}$$

Then, by a standard procedure (see [**30**], Section 7.12), we extend $\phi_{\epsilon,1}$ to a function defined on $\mathbb{R}^2$ such that

$$\|\phi_{\epsilon,1}\|_{H^2(\mathbb{R}^2)} \le C\|\phi_{\epsilon,1}\|_{H^2(\Omega_{\epsilon,P^\epsilon})}.$$

Since $\|\phi_\epsilon\|_{H^2(\Omega_{\epsilon,P})} = 1$, $\|\phi_{\epsilon,1}\|_{H^2(\Omega_{\epsilon,P})} \le C$. By taking a subsequence of $\epsilon$, we may also assume that $\phi_{\epsilon,1} \to \phi_1$ as $\epsilon \to 0$ in $H^1(\mathbb{R}^2)$ for some $\phi_1 \in H^1(\mathbb{R}^2)$.

We have by (4.4)

$$\psi_\epsilon(x) = \int_\Omega 2\beta^2 \xi_\epsilon G_{\beta_{\lambda_\epsilon}}(x,\xi)\hat{A}_\epsilon\left(\frac{\xi - P^\epsilon}{\epsilon}\right)\phi_\epsilon(\xi)\,d\xi. \tag{4.10}$$

At $x = P^\epsilon$, we calculate

$$\psi_\epsilon(P^\epsilon) = 2\beta^2\int_\Omega G_{\beta_{\lambda_\epsilon}}(P^\epsilon,\xi)\xi_\epsilon w\left(\frac{\xi - P^\epsilon}{\epsilon}\right)\phi_\epsilon(\xi)\,d\xi\,(1+o(1))$$

$$= 2\beta^2\int_\Omega \left(\frac{(\beta_{\lambda_\epsilon})^{-2}}{|\Omega|} + G_0(P^\epsilon,\xi) + O(\epsilon)\right)\xi_\epsilon w\left(\frac{\xi - P^\epsilon}{\epsilon}\right)\phi_{\epsilon,1}(\xi)\,d\xi\,(1+o(1))$$

$$= 2\int_\Omega \left(\frac{1}{|\Omega|(1+\tau\lambda_\epsilon)} + \beta^2 G_0(P^\epsilon,\xi) + O(\beta^4)\right)\xi_\epsilon w\left(\frac{x - P^\epsilon}{\epsilon}\right)\phi_{\epsilon,1}(\xi)\,d\xi\,(1+o(1))$$

$$= \Bigg(2\frac{1}{|\Omega|(1+\tau\lambda_\epsilon)}\xi_\epsilon\int_\Omega w\left(\frac{\xi - P^\epsilon}{\epsilon}\right)\phi_{\epsilon,1}(\xi)\,d\xi$$

$$+ 2\xi_\epsilon\frac{\beta^2}{\pi}\log\frac{1}{\epsilon}\int_\Omega w\left(\frac{\xi - P^\epsilon}{\epsilon}\right)\phi_{\epsilon,1}(\xi)\,d\xi\Bigg)(1+o(1)). \tag{4.11}$$

Considering the leading order term in (4.11), we get

$$\psi_\epsilon(P^\epsilon) = 2\frac{1}{|\Omega|\,(1+\tau\lambda_\epsilon)}\xi_\epsilon\int_\Omega w\phi_{\epsilon,1}\,(1+o\,(1))\;dx. \tag{4.12}$$

We substitute (4.12) into (4.4) and get

$$\Delta\phi_{\epsilon,1} - \phi_{\epsilon,1} + 2w\phi_{\epsilon,1} - w^2\cdot 2\frac{1}{|\Omega|(1+\tau\lambda_\epsilon)}\xi_\epsilon\int_\Omega w\phi_{\epsilon,1}\,dx = \lambda_\epsilon\phi_{\epsilon,1}\,(1+o(1)).$$

After letting $\epsilon \to 0$ and using (1.9) this gives the nonlocal eigenvalue problem (NLEP)

$$\Delta\phi_1 - \phi_1 + 2w\phi_1 - \frac{2}{(1+\tau\lambda_0)}\frac{\int_{\mathbb{R}^2_+} w\phi_1\,dy}{\int_{\mathbb{R}^2_+} w^2\,dy}w^2 = \lambda_0\phi_1. \tag{4.13}$$

Now we recall the following well-known result:

LEMMA 4.1. *The eigenvalue problem*

$$L_0\phi = \mu\phi, \quad \phi \in H^2_N(\mathbb{R}^2_+), \tag{4.14}$$

*admits the following set of eigenvalues*

$$\mu_1 > 0, \quad \mu_2 = 0, \quad \mu_3 < 0, \ldots . \tag{4.15}$$

*Without loss of generality, the eigenfunction $\Phi_0$ corresponding to $\mu_1$ is positive and radially symmetric.*

**Proof.** This lemma follows from Theorem 2.1 of [**45**] and Lemma C of [**55**]. □

Next, we consider the following eigenvalue problem:

$$L\phi := \Delta\phi - \phi + 2w\phi - \gamma\frac{\int_{\mathbb{R}^2_+} w\phi\,dy}{\int_{\mathbb{R}^2_+} w^2\,dy}w^2 = \lambda_0\phi,\ \phi \in H^2_N(\mathbb{R}^2_+), \tag{4.16}$$

where

$$\gamma = \frac{\mu}{1+\tau\lambda_0}, \quad \mu > 0,\, \tau \geq 0.$$

This will be the key to studying the large eigenvalues of the Gierer-Meinhardt system.

The result is as follows:

THEOREM 4.2. *Let $\gamma = \frac{\mu}{1+\tau\lambda_0}$ where $\mu > 0, \tau \geq 0$ and $L$ be defined in (4.16).*

*(1) Suppose that $\mu > 1$. Then there exists a unique $\tau = \tau_1 > 0$ such that for $\tau < \tau_1$, (4.16) admits a positive eigenvalue, and for $\tau > \tau_1$, all eigenvalues of problem (4.16) satisfy $Re(\lambda) < 0$. At $\tau = \tau_1$, $L$ has a Hopf bifurcation.*

*(2) Suppose that $\mu < 1$. Then $L$ admits a positive eigenvalue.*

**Proof of Theorem 4.2.**

Theorem 4.2 will be proved by the following two lemmas.

LEMMA 4.3. *If $\mu < 1$, then $L$ has a positive eigenvalue $\lambda_0 > 0$.*

**Proof.** First, we may assume that $\phi$ is a radially symmetric function, namely, $\phi \in H^2_r(\mathbb{R}^2_+) = \{u \in H^2(\mathbb{R}^2_+) : u = u(|y|)\}$. Let $L_0$ be given by (2.1). Then, by Lemma 2.1, $L_0$ is invertible in $H_r^2(\mathbb{R}_+^2)$. Let us denote its inverse by $L_0^{-1}$. On the other hand, by Lemma 4.1, $L_0$ has a unique positive eigenvalue. We denote this eigenvalue by $\mu_1$. Let us assume that $\lambda_0 \neq \mu_1$.

Then $\lambda_0$ is an eigenvalue of (4.16) if and only if it satisfies the following algebraic equation:

$$\int_{\mathbb{R}_+^2} w^2\,dy = \frac{\mu}{1+\tau\lambda_0}\int_{\mathbb{R}_+^2}[((L_0-\lambda_0)^{-1}w^2)w]\,dy. \tag{4.17}$$

Equation (4.17) can be simplified to

$$\rho(\lambda_0) := ((\mu-1)-\tau\lambda_0)\int_{\mathbb{R}_+^2} w^2\,dy + \mu\lambda_0\int_{\mathbb{R}_+^2}[((L_0-\lambda_0)^{-1}w)w]\,dy = 0. \tag{4.18}$$

Note that $\rho(0) = (\mu-1)\int_{\mathbb{R}_+^2} w^2\,dy < 0$. On the other hand, since $\lambda_0 \to \mu_1$ and $\lambda_0 < \mu_1$, we have $\int_{\mathbb{R}_+^2}((L_0-\lambda_0)^{-1}w)w\,dy \to +\infty$ and hence $\rho_0(\lambda_0) \to +\infty$. By continuity, there exists a $\lambda_0 \in (0,\mu_1)$ such that $\rho(\lambda_0) = 0$. Such a positive $\lambda_0$ is an eigenvalue of $L$.

□

Next we consider the case $\mu > 1$. By Theorem 1.4 of [**73**], for $\tau = 0$ (and by continuity, for $\tau$ small), all eigenvalues lie on the left half plane. By [**14**], for $\tau$ large, there exist unstable eigenvalues.

Note that the eigenvalues will not cross through zero: in fact, if $\lambda_0 = 0$, then we have

$$L_0\phi - \mu\frac{\int_{\mathbb{R}_+^2} w\phi\,dy}{\int_{\mathbb{R}_+^2} w^2\,dy}w^2 = 0$$

which implies that

$$L_0\left(\phi - \mu\frac{\int_{\mathbb{R}_+^2} w\phi\,dy}{\int_{\mathbb{R}_+^2} w^2\,dy}w\right) = 0$$

and hence by Lemma 2.1

$$\phi - \mu\frac{\int_{\mathbb{R}_+^2} w\phi\,dy}{\int_{\mathbb{R}_+^2} w^2\,dy}w \in K_0.$$

This is impossible since $\phi$ and $w$ are radially symmetric and $\phi \neq cw$ for all $c \in \mathbb{R}$.

Thus there must be a point $\tau_1$ at which $L$ has a Hopf bifurcation, i.e., $L$ has a purely imaginary eigenvalue $\alpha = \sqrt{-1}\alpha_I$. To prove Theorem 4.2 (1), all we need to show is that $\tau_1$ is unique. That is

Lemma 4.4. *Let $\mu > 1$. Then there exists a unique $\tau_1 > 0$ such that $L$ has a Hopf bifurcation.*

**Proof.** Let $\lambda_0 = \sqrt{-1}\alpha_I$ be an eigenvalue of $L$. Without loss of generality, we may assume that $\alpha_I > 0$. (Note that $-\sqrt{-1}\alpha_I$ is also an eigenvalue of $L$.) Let $\phi_0 = (L_0 - \sqrt{-1}\alpha_I)^{-1}w^2$. Then (4.16) becomes

$$\frac{\int_{\mathbb{R}^2_+} w\phi_0\,dy}{\int_{\mathbb{R}^2_+} w^2\,dy} = \frac{1+\tau\sqrt{-1}\alpha_I}{\mu}. \tag{4.19}$$

Let $\phi_0 = \phi_0^R + \sqrt{-1}\phi_0^I$. Then from (4.19), we obtain the two equations

$$\frac{\int_{\mathbb{R}^2_+} w\phi_0^R\,dy}{\int_{\mathbb{R}^2_+} w^2\,dy} = \frac{1}{\mu}, \tag{4.20}$$

$$\frac{\int_{\mathbb{R}^2_+} w\phi_0^I\,dy}{\int_{\mathbb{R}^2_+} w^2\,dy} = \frac{\tau\alpha_I}{\mu}. \tag{4.21}$$

Note that (4.20) is independent of $\tau$.

Let us now compute $\int_{\mathbb{R}^2_+} w\phi_0^R\,dy$. Observe that $(\phi_0^R, \phi_0^I)$ satisfies

$$L_0\phi_0^R = w^2 - \alpha_I\phi_0^I, \quad L_0\phi_0^I = \alpha_I\phi_0^R.$$

So $\phi_0^R = \alpha_I^{-1}L_0\phi_0^I$ and

$$\phi_0^I = \alpha_I(L_0^2+\alpha_I^2)^{-1}w^2, \quad \phi_0^R = L_0(L_0^2+\alpha_I^2)^{-1}w^2. \tag{4.22}$$

Substituting (4.22) into (4.20) and (4.21), we obtain

$$\frac{\int_{\mathbb{R}^2_+} [wL_0(L_0^2+\alpha_I^2)^{-1}w^2]\,dy}{\int_{\mathbb{R}^2_+} w^2\,dy} = \frac{1}{\mu}, \tag{4.23}$$

$$\frac{\int_{\mathbb{R}^2_+} [w(L_0^2+\alpha_I^2)^{-1}w^2]\,dy}{\int_{\mathbb{R}^2_+} w^2\,dy} = \frac{\tau}{\mu}. \tag{4.24}$$

Let $h(\alpha_I) = \frac{\int_{\mathbb{R}^2_+} wL_0(L_0^2+\alpha_I^2)^{-1}w^2\,dy}{\int_{\mathbb{R}^2_+} w^2\,dy}$. Then integration by parts gives $h(\alpha_I) = \frac{\int_{\mathbb{R}^2_+} w^2(L_0^2+\alpha_I^2)^{-1}w^2\,dy}{\int_{\mathbb{R}^2_+} w^2\,dy}$. Note that $h'(\alpha_I) = -2\alpha_I\frac{\int_{\mathbb{R}^2_+} w^2(L_0^2+\alpha_I^2)^{-2}w^2\,dy}{\int_{\mathbb{R}^2_+} w^2\,dy} < 0$. So since

$$h(0) = \frac{\int_{\mathbb{R}^2_+} w(L_0^{-1}w^2)\,dy}{\int_{\mathbb{R}^2_+} w^2\,dy} = 1,$$

$h(\alpha_I) \to 0$ as $\alpha_I \to \infty$, and since $\mu > 1$, there exists a unique $\alpha_I > 0$ such that (4.23) holds. Substituting this unique $\alpha_I$ into (4.24), we obtain a unique $\tau = \tau_1 > 0$.

Lemma 4.4 is thus proved.

□

Theorem 4.2 now follows from Lemma 4.3 and Lemma 4.4.

□

By Theorem 4.2, problem (4.13) is stable if $\tau < \tau_1$, which implies that the large eigenvalues of (4.4) are stable.

If $\tau > \tau_1$, by Theorem 4.2, problem (4.13) has an eigenvalue $\lambda_0$ such that Re $(\lambda_0) \geq a_0 > 0$ for some $a_0$. We now claim that problem (4.4) also admits an eigenvalue $\lambda_\epsilon$ with $\lambda_\epsilon = \lambda_0 + o(1)$, which implies that problem (4.4) is unstable. To this end, we follow the argument given in Section 2 of [**14**], where the following eigenvalue problem was studied:

$$\begin{cases} \epsilon^2 \Delta h - h + p u_\epsilon^{p-1} h - \frac{qr}{s+1+\tau\lambda_\epsilon} \frac{\int_\Omega u_\epsilon^{r-1} h\, dx}{\int_\Omega u_\epsilon^r\, dx} u_\epsilon^p = \lambda_\epsilon h \text{ in } \Omega, \\ h = 0 \text{ on } \partial\Omega, \end{cases} \tag{4.25}$$

where $u_\epsilon$ is a solution of the single equation

$$\begin{cases} \epsilon^2 \Delta u_\epsilon - u_\epsilon + u_\epsilon^p = 0 \text{ in } \Omega, \\ u_\epsilon > 0 \text{ in } \Omega,\ u_\epsilon = 0 \text{ on } \partial\Omega. \end{cases}$$

Here $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$ and $1 < p < +\infty$ if $N = 1, 2$, $\frac{qr}{(s+1)(p-1)} > 1$ and $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain.

If $u_\epsilon$ is a single interior peak solution, then it can be shown ([**73**]) that the limiting eigenvalue problem is a NLEP

$$\Delta\phi - \phi + p w^{p-1}\phi - \frac{qr}{s+1+\tau\lambda_0} \frac{\int_{\mathbb{R}^N} w^{r-1}\phi\, dy}{\int_{\mathbb{R}^N} w^r\, dy} w^p = \lambda_0 \phi \tag{4.26}$$

where $w$ is the corresponding ground state solution in $\mathbb{R}^N$:

$$\Delta w - w + w^p = 0, \quad w > 0 \text{ in } \mathbb{R}^N, \quad w = w(|y|) \in H^1(\mathbb{R}^N).$$

Dancer in [**14**] showed that if $\lambda_0 \neq 0$, $\mathrm{Re}(\lambda_0) > 0$ is an unstable eigenvalue of (4.26), then there exists an eigenvalue $\lambda_\epsilon$ of (4.25) such that $\lambda_\epsilon \to \lambda_0$.

We now follow his idea. Let $\lambda_0 \neq 0$ be an eigenvalue of problem (4.13) with $\mathrm{Re}(\lambda_0) > 0$. We first note that from the equation for $\psi_\epsilon$ we can express

$\psi_\epsilon$ in terms of $\phi_\epsilon$. Now we write the first equation for $\phi_\epsilon$ as follows:

$$\phi_\epsilon = \mathcal{R}_\epsilon(\lambda_\epsilon)\left[2\frac{\hat{A}_\epsilon}{\hat{H}_\epsilon}\phi_\epsilon - \frac{\hat{A}_\epsilon^2}{\hat{H}_\epsilon^2}\psi_\epsilon\right], \tag{4.27}$$

where $\mathcal{R}_\epsilon(\lambda_\epsilon)$ is the inverse of $-\Delta + (1+\lambda_\epsilon)$ in $H_N^2(\Omega_\epsilon)$ (which exists if $\mathrm{Re}(\lambda_\epsilon) > -1$ or $\mathrm{Im}(\lambda_\epsilon) \neq 0$) and $\psi_\epsilon = \mathcal{F}[\phi_\epsilon]$ is given by (4.10), where $\mathcal{F}$ is a compact operator of $\phi_\epsilon$. The important thing is that $\mathcal{R}_\epsilon(\lambda_\epsilon)$ is a compact operator if $\epsilon$ is sufficiently small. The rest of the argument follows exactly that in [**14**]. We omit the details here.

This finishes the proof of Theorem 1.2 in the large eigenvalue case.

## 2. The Small Eigenvalues

We now study (4.4) for small eigenvalues. Namely, we assume that $\lambda_\epsilon \to 0$ as $\epsilon \to 0$. We will show that the small eigenvalues are related to $(\nabla_{\tau(P_0)})^2 F(P_0)$.

Let us assume that condition $(*)$ holds true. That is, $(\nabla_{\tau(P_0)})^2 F(P_0) < 0$. Our main result in this section says that if $\lambda_\epsilon \to 0$, then

$$\lambda_\epsilon \sim \epsilon^2 (\nabla_{\tau(P_0)})^2 F(P_0). \tag{4.28}$$

From (4.28), we see that all small eigenvalues of the operator $\mathcal{L}_\epsilon$ defined in (1.10) are stable, provided that condition $(*)$ holds.

Again, let $(A_\epsilon, H_\epsilon)$ be the equilibrium state of (1.14) and let $(\hat{A}_\epsilon, \hat{H}_\epsilon)$ be the rescaled solution given by (4.3).

We first recall the system (4.4):

$$\begin{cases} \epsilon^2\Delta\phi_\epsilon - \phi_\epsilon + 2\frac{\hat{A}_\epsilon}{\hat{H}_\epsilon}\phi_\epsilon - \frac{(\hat{A}_\epsilon)^2}{(\hat{H}_\epsilon)^2}\psi_\epsilon = \lambda_\epsilon\phi_\epsilon, \\ \frac{1}{\beta^2}\Delta\psi_\epsilon - \psi_\epsilon + 2\xi_\epsilon\hat{A}_\epsilon\phi_\epsilon = \tau\lambda_\epsilon\psi_\epsilon, \end{cases}$$

where

$$\phi_\epsilon \in H_N^2(\Omega_\epsilon),\ \psi_\epsilon \in H_N^2(\Omega).$$

Suppose that $\lambda_\epsilon \to 0$ and $\|\phi_\epsilon\|_\epsilon = 1$. We now look for the leading term in the expansion of $\phi_\epsilon$.

After rescaling and taking a subsequence, we have that

$$\tilde{\phi}_\epsilon(z) = \phi_\epsilon(P^\epsilon + \epsilon z) \to \phi_0,$$

where $\phi_0$ is a solution of

$$\begin{cases} \Delta\phi_0 - \phi_0 + 2w\phi_0 - \dfrac{\int_{\mathbb{R}^2_+} w(y)\phi_0(y)\,dy}{\int_{\mathbb{R}^2_+} w^2(y)\,dy} w^2 = 0 \ \text{ in } \mathbb{R}^2_+, \\ \phi_0 \in H^1(\mathbb{R}^2_+), \frac{\partial \phi_0}{\partial y_2} = 0 \text{ on } \partial\mathbb{R}^2_+. \end{cases} \tag{4.29}$$

By Lemma 2.5, there exists $\alpha_0 > 0$ such that $\phi_0 = \alpha_0 \frac{\partial w}{\partial y_1}$.

Since we expect that the eigenfunctions with small ($o(1)$) eigenvalues are related to the translation mode, we make the ansatz

$$\phi_\epsilon = \alpha_\epsilon \epsilon \frac{\partial w_{\epsilon,P^\epsilon}}{\partial \tau(P^\epsilon)} + \bar{\phi}_\epsilon, \tag{4.30}$$

where $\alpha_\epsilon \in \mathbb{R}, \phi_\epsilon^\perp \in K_{\epsilon,P}^\perp$. Suppose that $\|\phi_\epsilon\|_{2,\epsilon} = 1$. Then $|\alpha_\epsilon| \le 1$, since

$$\alpha_\epsilon = \frac{\int_\Omega \phi_\epsilon \epsilon \frac{\partial w_{\epsilon,P^\epsilon}}{\partial \tau(P^\epsilon)}\,dx}{\int_\Omega \left(\epsilon \frac{\partial w_{\epsilon,P^\epsilon}}{\partial \tau(P^\epsilon)}\right)^2 dx}.$$

We first need the following expansion:

LEMMA 4.5. *Let $\Phi_{\epsilon,P}$ be defined by Lemma 3.8. Then for $\epsilon$ sufficiently small and $P \in \overline{\Lambda}$, we have*

$$\frac{\partial}{\partial \tau(P)} \Phi_{\epsilon,P} = -\epsilon \frac{\partial \Phi_0((x-P)/\epsilon)}{\partial x_1} + \frac{\frac{\partial}{\partial \tau(P)}\xi_\epsilon}{\xi_\epsilon} w^2 + O(\epsilon) \qquad \text{in } H^2(\Omega_{\epsilon,P^\epsilon}). \tag{4.31}$$

*Proof.* Recall that $\Phi_{\epsilon,P}$ is a solution of the equation

$$\hat{\pi}_{\epsilon,P} \circ S_1(w_{\epsilon,P} + \Phi_{\epsilon,P}, T[(w_{\epsilon,P} + \Phi_{\epsilon,P})^2]) = 0 \tag{4.32}$$

such that

$$\Phi_{\epsilon,P} \in K_{\epsilon,P}^\perp. \tag{4.33}$$

We rewrite (4.32) in a shorter form as

$$\hat{\pi}_{\epsilon,P} \circ S^\epsilon(w_{\epsilon,P} + \Phi_{\epsilon,P}) = 0, \tag{4.34}$$

where $S^\epsilon(A) := S_1(A, T[A^2])$. By differentiating equation (3.6) for $W = w$ twice with respect to $\frac{\partial}{\partial \tau(P)}$, we conclude that the functions $w_{\epsilon,P}$ and $\frac{\partial w_{\epsilon,P}}{\partial \tau(P)}$ are $C^1$ in $P$. This implies that the projection $\hat{\pi}_{\epsilon,P}$ is $C^1$ in $P$. Applying $\frac{\partial}{\partial \tau(P)}$ to (4.34) gives

$$\hat{\pi}_{\epsilon,P} \circ DS^\epsilon(w_{\epsilon,P} + \Phi_{\epsilon,P}) \left( \frac{\partial w_{\epsilon,P}}{\partial \tau(P)} + \frac{\partial \Phi_{\epsilon,P}}{\partial \tau(P)} \right)$$

$$-\hat{\pi}_{\epsilon,P} \circ \left[ \frac{(w_{\epsilon,P} + \Phi_{\epsilon,P})^2}{T[(w_{\epsilon,P} + \Phi_{\epsilon,P})^2]} \frac{\frac{\partial \xi_\epsilon}{\partial \tau(P)}}{\xi_\epsilon} \right]$$

$$+\frac{\partial \hat{\pi}_{\epsilon,P}}{\partial \tau(P)} \circ S^\epsilon(w_{\epsilon,P} + \Phi_{\epsilon,P}) = 0, \tag{4.35}$$

where

$$DS^\epsilon(w_{\epsilon,P} + \Phi_{\epsilon,P})\phi = \Delta\phi - \phi + 2\frac{(w_{\epsilon,P} + \Phi_{\epsilon,P})\phi}{T[(w_{\epsilon,P} + \Phi_{\epsilon,P})^2]}$$

$$-\frac{(w_{\epsilon,P} + \Phi_{\epsilon,P})^2}{(T[(w_{\epsilon,P} + \Phi_{\epsilon,P})^2])^2} T[2(w_{\epsilon,P} + \Phi_{\epsilon,P})\phi].$$

Note that

$$\frac{(w_{\epsilon,P} + \Phi_{\epsilon,P})^2}{T[(w_{\epsilon,P} + \Phi_{\epsilon,P})^2]} \frac{\frac{\partial \xi_\epsilon}{\partial \tau(P)}}{\xi_\epsilon} \sim w^2 \frac{\frac{\partial \xi_\epsilon}{\partial \tau(P)}}{\xi_\epsilon}.$$

Now we conclude the proof in the same way as in Lemma 4.2 of [**76**].

□

Now we consider the function

$$\phi_{\epsilon,1}(y) = \frac{\partial w_{\epsilon,P^\epsilon}}{\partial \tau(P^\epsilon)}. \tag{4.36}$$

We decompose $\phi_\epsilon$ as follows:

$$\phi_\epsilon = \alpha_\epsilon \phi_{\epsilon,1} + \bar{\phi}_\epsilon. \tag{4.37}$$

The proof is divided into two steps. In Step 1 we will give an expansion for $\bar{\phi}_\epsilon$ with a rigorous and explicit error estimate. In Step 2 we will give a rigorous derivation of the asymptotic behavior of the small eigenvalues.

**Step 1.** Expansion and estimate for $\bar{\phi}_\epsilon$.

Observe that $\bar{\phi}_\epsilon \in \mathcal{K}^\perp_{\epsilon,P^\epsilon}$ satisfies

$$\Delta\bar{\phi}_\epsilon - \bar{\phi}_\epsilon + 2\frac{\hat{A}_\epsilon \bar{\phi}_\epsilon}{\hat{H}_\epsilon} - \frac{\hat{A}_\epsilon^2}{\hat{H}_\epsilon^2} T_\tau[2\hat{A}_\epsilon \bar{\phi}_\epsilon]$$

$$+\alpha_\epsilon \left( 2w_\epsilon \frac{\partial w_\epsilon}{\partial y_1} + 2\hat{A}_\epsilon \epsilon \frac{\partial w_{\epsilon,P^\epsilon}}{\partial \tau(P^\epsilon)} \right.$$

$$+2\hat{A}_\epsilon \epsilon \frac{\partial w_{\epsilon,P^\epsilon}}{\partial \tau(P^\epsilon)} \Big( (\hat{H}_\epsilon - H_{\epsilon,P^\epsilon}) + (H_{\epsilon,P^\epsilon} - H_{\epsilon,P^\epsilon}(P^\epsilon)) + O(\epsilon^2) \Big)$$

$$\left. -\frac{\hat{A}_\epsilon^2}{\hat{H}_\epsilon^2} T_\tau[2\hat{A}_\epsilon \epsilon \frac{\partial w_{\epsilon,P^\epsilon}}{\partial \tau(P^\epsilon)}] \right)$$

$$= \lambda_\epsilon \alpha_\epsilon \epsilon \frac{\partial}{\partial \tau(P^\epsilon)} w_{\epsilon,P^\epsilon} + \lambda_\epsilon \bar{\phi}_\epsilon, \quad \text{in } H^2(\Omega_{\epsilon,P^\epsilon}), \tag{4.38}$$

where

$$T_\tau[2\hat{A}_\epsilon \phi_\epsilon] = 2\xi_\epsilon \beta^2 \int_\Omega G_{\sqrt{1+\tau\lambda_\epsilon}\beta}(P^\epsilon, \xi) \hat{A}_\epsilon \left( \frac{\xi - P^\epsilon}{\epsilon} \right) \phi_\epsilon \, d\xi.$$

This implies

$$\begin{aligned} & T_\tau[2\xi_\epsilon \hat{A}_\epsilon \bar{\phi}_\epsilon] - T[2\xi_\epsilon \hat{A}_\epsilon \bar{\phi}_\epsilon] \\ = {} & 2\xi_\epsilon \beta^2 \int_\Omega \left( G_{\sqrt{1+\tau\lambda_\epsilon}\beta}(P^\epsilon, \xi) - G_\beta(P^\epsilon, \xi) \right) \hat{A}_\epsilon \left( \frac{\xi - P^\epsilon}{\epsilon} \right) \bar{\phi}_\epsilon \, d\xi \\ = {} & 2\xi_\epsilon \frac{1}{|\Omega|} \frac{-\tau\lambda_\epsilon}{1+\tau\lambda_\epsilon} \int_\Omega \hat{A}_\epsilon \bar{\phi}_\epsilon \, d\xi + O\left( \epsilon^2 \xi_\epsilon \beta^4 |1 - \sqrt{1+\tau\lambda_\epsilon}| \right) \\ = {} & O((\|\bar{\phi}_\epsilon\|_{2,\epsilon} + \epsilon)\tau|\lambda_\epsilon|). \end{aligned}$$

(Note that $\xi_\epsilon \int_\Omega \hat{A}_\epsilon \bar{\phi}_\epsilon \, d\xi = O(\|\bar{\phi}_\epsilon|_{2,\epsilon} + \epsilon)$ .) In the same way, we can prove

$$\begin{aligned} & T_\tau[2\xi_\epsilon \hat{A}_\epsilon \epsilon \frac{\partial w_{\epsilon,P^\epsilon}}{\partial \tau(P^\epsilon)}] - T[2\xi_\epsilon \hat{A}_\epsilon \epsilon \frac{\partial w_{\epsilon,P^\epsilon}}{\partial \tau(P^\epsilon)}] \\ & = O(\epsilon^2 \tau |\lambda_\epsilon|). \end{aligned}$$

By Propositions 3.6 and 3.7, we have

$$\|\bar{\phi}_\epsilon\|_{H^2(\Omega_\epsilon, P^\epsilon)} \le C\epsilon|\alpha_\epsilon|. \tag{4.39}$$

On the other hand, by Proposition 3.8, for $P \in \overline{\Lambda}$, we have

$$S^\epsilon(w_{\epsilon,P} + \Phi_{\epsilon,P}) = \gamma(P) \frac{\partial}{\partial \tau(P)} w_{\epsilon,P}.$$

Applying $\frac{\partial}{\partial \tau(P)}$ to the above equation and setting $P = P^\epsilon$, we have

$$DS^\epsilon(w_{\epsilon,P^\epsilon} + \Phi_{\epsilon,P^\epsilon}) \frac{\partial}{\partial \tau(P^\epsilon)} (w_{\epsilon,P^\epsilon} + \Phi_{\epsilon,P^\epsilon}) - \frac{(w_{\epsilon,P^\epsilon} + \Phi_{\epsilon,P^\epsilon})^2}{\hat{H}_\epsilon^2} \frac{\frac{\partial}{\partial \tau(P^\epsilon)} \xi_\epsilon}{\xi_\epsilon}$$

$$-\gamma(P^\epsilon) \frac{\partial^2}{\partial \tau(P^\epsilon)^2} w_{\epsilon,P^\epsilon} \in C_{\epsilon,P^\epsilon}.$$

Since $\hat{A}_\epsilon = w_{\epsilon,P^\epsilon} + \Phi_{\epsilon,P^\epsilon}$ is a solution, $\gamma(P^\epsilon) = 0$. Thus we have

$$DS^\epsilon(w_{\epsilon,P^\epsilon} + \Phi_{\epsilon,P^\epsilon}) \frac{\partial}{\partial \tau(P^\epsilon)} (w_{\epsilon,P^\epsilon} + \Phi_{\epsilon,P^\epsilon}) - \frac{(w_{\epsilon,P^\epsilon} + \Phi_{\epsilon,P^\epsilon})^2}{\hat{H}_\epsilon^2} \frac{\frac{\partial}{\partial \tau(P^\epsilon)} \xi_\epsilon}{\xi_\epsilon} \in C_{\epsilon,P^\epsilon},$$

which implies that

$$\Delta \frac{\partial \Phi_{\epsilon,P^\epsilon}}{\partial \tau(P^\epsilon)} - \frac{\partial \Phi_{\epsilon,P^\epsilon}}{\partial \tau(P^\epsilon)} + 2 \frac{\hat{A}_\epsilon}{\hat{H}_\epsilon} \frac{\partial \Phi_{\epsilon,P^\epsilon}}{\partial \tau(P^\epsilon)} - \frac{\hat{A}_\epsilon^2}{\hat{H}_\epsilon^2} T[2\hat{A}_\epsilon \frac{\partial \Phi_{\epsilon,P^\epsilon}}{\partial \tau(P^\epsilon)}]$$

$$+2w_\epsilon \frac{\partial w_\epsilon}{\partial y_1} + 2\hat{A}_\epsilon \epsilon \frac{\partial w_{\epsilon,P^\epsilon}}{\partial \tau(P^\epsilon)}$$

$$+2\hat{A}_\epsilon \epsilon \frac{\partial w_{\epsilon,P^\epsilon}}{\partial \tau(P^\epsilon)} \left( (\hat{H}_\epsilon - H_{\epsilon,P^\epsilon}) + (H_{\epsilon,P^\epsilon} - H_{\epsilon,P^\epsilon}(P^\epsilon)) + O(\epsilon^2) \right)$$

$$-\frac{\hat{A}_\epsilon^2}{\hat{H}_\epsilon^2} T[2\hat{A}_\epsilon \frac{\partial w_{\epsilon,P^\epsilon}}{\partial \tau(P^\epsilon)}] - \frac{(w_{\epsilon,P^\epsilon} + \Phi_{\epsilon,P^\epsilon})^2}{\hat{H}_\epsilon^2} \frac{\frac{\partial}{\partial \tau(P^\epsilon)} \xi_\epsilon}{\xi_\epsilon} \in C_{\epsilon,P^\epsilon}. \tag{4.40}$$

Comparing (4.40) and (4.38), Propositions 3.6 and 3.7 imply that

$$\bar{\phi}_\epsilon = \alpha_\epsilon \epsilon \frac{\partial \Phi_{\epsilon,P^\epsilon}}{\partial \tau(P^\epsilon)} + \frac{\frac{\partial}{\partial \tau(P^\epsilon)} \xi_\epsilon}{\xi_\epsilon} w_{\epsilon,P^\epsilon} + O(\epsilon |\lambda_\epsilon \alpha_\epsilon|). \tag{4.41}$$

Hence

$$\phi_\epsilon = \alpha_\epsilon \left( \frac{\partial}{\partial \tau(P^\epsilon)} \epsilon (w_{\epsilon,P^\epsilon} + \Phi_{\epsilon,P^\epsilon}) - \frac{\frac{\partial}{\partial \tau(P^\epsilon)} \xi_\epsilon}{\xi_\epsilon} w_{\epsilon,P^\epsilon} \right) + O(\epsilon |\lambda_\epsilon \alpha_\epsilon|). \tag{4.42}$$

**Step 2.** An algebraic equation for $\alpha_\epsilon$.

Multiplying both sides of (4.38) by $\frac{\partial w_{\epsilon,P^\epsilon}}{\partial \tau(P^\epsilon)}$ and integrating over $\Omega_{\epsilon,P^\epsilon}$, we obtain

$$r.h.s. = \lambda_\epsilon \alpha_\epsilon \int_{\Omega_{\epsilon,P^\epsilon}} \epsilon^2 \left( \frac{\partial w_{\epsilon,P^\epsilon}}{\partial \tau(P^\epsilon)} \right)^2 dz + \lambda_\epsilon \int_{\Omega_{\epsilon,P^\epsilon}} \epsilon \left( \bar{\phi}_\epsilon \frac{\partial w_{\epsilon,P^\epsilon}}{\partial \tau(P^\epsilon)} \right) dz$$

$$= \lambda_\epsilon \alpha_\epsilon \int_{\mathbb{R}^2_+} \left( \frac{\partial w}{\partial y_1} \right)^2 dy \, (1 + o(1)) \tag{4.43}$$

by (4.41) and Lemma 4.5.

On the other hand,

$$\begin{aligned} l.h.s. = {} & \alpha_\epsilon \int_{\Omega_{\epsilon,P^\epsilon}} \left( 2w_\epsilon \frac{\partial w_\epsilon}{\partial y_1} + 2\hat{A}_\epsilon \epsilon \frac{\partial w_{\epsilon,P^\epsilon}}{\partial \tau(P^\epsilon)} \right) \frac{\partial w_{\epsilon,P^\epsilon}}{\partial \tau(P^\epsilon)} \, dz \\ & + \alpha_\epsilon \int_{\Omega_{\epsilon,P^\epsilon}} 2\hat{A}_\epsilon \epsilon \frac{\partial w_{\epsilon,P^\epsilon}}{\partial \tau(P^\epsilon)} (\hat{H}_\epsilon - H_{\epsilon,P^\epsilon}) \frac{\partial w_{\epsilon,P^\epsilon}}{\partial \tau(P^\epsilon)} \, dz \\ & - \alpha_\epsilon \int_{\Omega_{\epsilon,P^\epsilon}} \frac{\hat{A}_\epsilon^2}{\hat{H}_\epsilon^2} T_\tau [2\hat{A}_\epsilon \epsilon \frac{\partial w_{\epsilon,P^\epsilon}}{\partial \tau(P^\epsilon)}] \frac{\partial w_{\epsilon,P^\epsilon}}{\partial \tau(P^\epsilon)} \, dz \\ & + \int_{\Omega_{\epsilon,P^\epsilon}} \left( \Delta \bar{\phi}_\epsilon - \bar{\phi}_\epsilon + 2 \frac{\hat{A} \bar{\phi}_\epsilon}{\hat{H}_\epsilon} - \frac{\hat{A}_\epsilon^2}{\hat{H}_\epsilon^2} T_\tau [2\hat{A}_\epsilon \bar{\phi}_\epsilon] \right) \frac{\partial w_{\epsilon,P^\epsilon}}{\partial \tau(P^\epsilon)} \, dz \\ & + \alpha_\epsilon \int_{\Omega_{\epsilon,P^\epsilon}} 2\hat{A}_\epsilon \epsilon \frac{\partial w_{\epsilon,P^\epsilon}}{\partial \tau(P^\epsilon)} \left( (H_{\epsilon,P^\epsilon} - 1) + O(\epsilon^2) \right) \frac{\partial w_{\epsilon,P^\epsilon}}{\partial \tau(P^\epsilon)} \, dz \\ = {} & I_3 + I_4 + O(\epsilon |\lambda_\epsilon \alpha_\epsilon| + \epsilon^3) \end{aligned} \tag{4.44}$$

by (4.41) as in Section 3.3.

Now, to compute $I_3$, we the results of [**76**] about the spectrum of a boundary spike for a single equation. For the convenience of the reader, we provide a sketch of the proof. To compute $I_4$, we will proceed similar to [**82**]. For $\phi \in H^2_N(\Omega)$, let $u_\epsilon = w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon}$, where

$$\hat{S}_\epsilon(u_\epsilon) := \epsilon^2 \Delta u_\epsilon - u_\epsilon + u_\epsilon^2 = C\epsilon^3 \rho'''(0) \nabla_{\tau(P^\epsilon)} w_{\epsilon,P^\epsilon}$$

and

$$L_\epsilon(u_\epsilon)\phi := \epsilon^2 \Delta \phi - \phi + 2u_\epsilon \phi.$$

Then we calculate using integration by parts in the same way as in Section 39 that

$$\begin{aligned}
\epsilon I_3 = \int_\Omega \left(\epsilon^2 \Delta \nabla_{\tau(P^\epsilon)} w_{\epsilon,P^\epsilon} - \nabla_{\tau(P^\epsilon)} w_{\epsilon,P^\epsilon} + \nabla_{\tau(P^\epsilon)} w^2_{\epsilon,P^\epsilon}\right) \frac{\partial w_{\epsilon,P^\epsilon}}{\partial \tau(P^\epsilon)}\, dx + O(\epsilon^4) \\
= \int_\Omega [L_\epsilon(\nabla_{\tau(P^\epsilon)}(w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon}))\nabla_{\tau(P^\epsilon)}(w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon})]\, dx + O(\epsilon^4) \\
= \int_\Omega [L_\epsilon(\nabla_{\tau(P^\epsilon)}(w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon}))(w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon})]\, dx + O(\epsilon^4) \\
- \int_\Omega \hat{S}_\epsilon(w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon})\nabla_{\tau(P^\epsilon)}(w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon})\nabla_{\tau(P^\epsilon)}(w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon})\, dx + O(\epsilon^4) \\
= -\nabla^2_{\tau(P^\epsilon)} K_\epsilon(P^\epsilon) + O(\epsilon^4),
\end{aligned}$$

(since

$$\begin{aligned}
\int_\Omega \hat{S}_\epsilon(w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon})\nabla^2_{\tau(P^\epsilon)}(w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon})\, dx \\
= C\epsilon^3 \rho'''(0) \int_\Omega \nabla_{\tau(P^\epsilon)}(w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon})\nabla^2_{\tau(P^\epsilon)}(w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon})\, dx \\
= \frac{1}{2} C\epsilon^3 \rho'''(0) \int_\Omega \nabla_{\tau(P^\epsilon)}[(\nabla_{\tau(P^\epsilon)}(w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon})]^2\, dx \\
= O(\epsilon^4))
\end{aligned}$$

where

$$K_\epsilon(P) := J_\epsilon(w_{\epsilon,P} + \phi_{\epsilon,P}) : \overline{\Lambda} \to R. \tag{4.45}$$

Let

$$J_\epsilon(u) = \frac{\epsilon^2}{2} \int_\Omega |\nabla u|^2\, dx + \frac{1}{2} \int_\Omega u^2\, dx - \frac{1}{3} \int_\Omega u^3\, dx \tag{4.46}$$

be the energy of $u \in H^1(\Omega)$ and

$$J(w) = \frac{1}{2} \int_{R^N} (|\nabla w|^2 + w^2)\, dy - \frac{1}{3} \int_{R^N} w^3\, dy \tag{4.47}$$

be the energy of $w$.

We now estimate$\nabla^2_{\tau(P^\epsilon)} K_\epsilon$.

Let

$$D_1 = \frac{\partial}{\partial \tau(P^\epsilon)} + \frac{\partial}{\partial x_1}.$$

Since $D_1 g_1(w(\frac{x-P^\epsilon}{\epsilon})) = 0$ and $D_1 g_2(w'(\frac{x-P^\epsilon}{\epsilon})) = 0$, where $g_1$ and $g_2$ are any $C^1$ functions, we can prove that

$$\begin{aligned} J := \int_\Omega D_1 [\frac{\partial}{\partial x_1} u_\epsilon \cdot \frac{\partial}{\partial x_1} \nabla_{\tau(P^\epsilon)} (w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon}) + u_\epsilon \nabla_{\tau(P^\epsilon)} (w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon}) \\ -(w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon})^2 (\nabla_{\tau(P^\epsilon)} (w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon}))]\, dx \\ = o(\epsilon^3). \end{aligned} \tag{4.48}$$

The proof of (4.48) requires some work. This is done in Appendix E.

We can now estimate $\nabla^2_{\tau(P^\epsilon)} K_\epsilon(P^\epsilon)$. The main idea is to use $D_1$ to reduce integrating in $\Omega$ to integrating on $\partial\Omega$.

By definition, we have

$$\begin{aligned} \nabla^2_{\tau(P^\epsilon)} K_\epsilon(P^\epsilon) &= \epsilon^2 < \nabla_{\tau(P^\epsilon)} (w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon}), \nabla_{\tau(P^\epsilon)} (w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon}) >_{1,\epsilon} \\ &- \int_\Omega 2(w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon}) \nabla_{\tau(P^\epsilon)} (w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon}) \nabla_{\tau(P^\epsilon)} (w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon})\, dx \\ &+\epsilon^2 < w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon}, \nabla^2_{\tau(P^\epsilon)} (w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon}) >_{1,\epsilon} \\ &- \int_\Omega (w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon})^2 \nabla^2_{\tau(P^\epsilon)} (w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon})\, dx \\ &= \int_\Omega \nabla_{\tau(P^\epsilon)} [\frac{\partial}{\partial x_1} u_\epsilon \frac{\partial}{\partial x_1} \nabla_{\tau(P^\epsilon)} (w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon}) + u_\epsilon \nabla_{\tau(P^\epsilon)} (w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon}) \\ &\quad -(w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon})^2 (\nabla_{\tau(P^\epsilon)} (w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon}))]\, dx \\ &= -\int_\Omega \frac{\partial}{\partial x_1} [\frac{\partial}{\partial x_1} u_\epsilon \frac{\partial}{\partial x_1} \nabla_{\tau(P^\epsilon)} (w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon}) + u_\epsilon \nabla_{\tau(P^\epsilon)} (w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon}) \\ &\quad -(w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon})^2 (\nabla_{\tau(P^\epsilon)} (w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon}))]\, dx \\ &+ \int_\Omega D_1 [\frac{\partial}{\partial x_1} u_\epsilon \frac{\partial}{\partial x_1} \nabla_{\tau(P^\epsilon)} (w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon}) + u_\epsilon \nabla_{\tau(P^\epsilon)} (w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon}) \\ &\quad -(w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon})^2 (\nabla_{\tau(P^\epsilon)} (w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon}))]\, dx \\ &= -\int_{\partial\Omega} \nabla_{\tau(P^\epsilon)} [\frac{1}{2}(\epsilon^2 |\nabla u_\epsilon|^2 + u_\epsilon^2) - \frac{1}{3}(w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon})^3] \nu_1(x) dx + o(\epsilon^3) \end{aligned}$$

(integrating by parts and using (4.48))

$$= \int_{\partial\Omega} \frac{\partial}{\partial x_1} [\frac{1}{2}(\epsilon^2 |\nabla u_\epsilon|^2 + u_\epsilon^2) - \frac{1}{3}(w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon})^3] \nu_1(x) dx + o(\epsilon^3)$$

(using $D_1$)

$$= \int_{\partial\Omega} \frac{\partial}{\partial x_1}[\frac{1}{2}(\epsilon^2|\nabla u_\epsilon|^2 + u_\epsilon^2) - \frac{1}{3}(w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon})^3]\nu_1(x)dx + o(\epsilon^3)$$

$$- \int_{\partial\Omega}[\frac{1}{2}(\epsilon^2|\nabla u_\epsilon|^2 + u_\epsilon^2) - \frac{1}{3}(w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon})^3]\nu_{1,1}(x)dx + o(\epsilon^3)$$

$$= I_{3,1} + I_{3,2}$$

where $I_{3,1}$ and $I_{3,2}$ are defined by the last equality.

We first compute $I_{3,1}$.

$$I_{3,1} = \epsilon \int_{\mathbb{R}} \frac{\partial}{\epsilon \partial y_1}[(\frac{1}{2}[|\nabla w|^2 + w^2] - \frac{1}{3}w^3)\nu_1(\epsilon y)]\sqrt{1+(\rho')^2}dy + o(\epsilon^3)$$

$$= -\epsilon \int_{\mathbb{R}}[(\frac{1}{2}[|\nabla w|^2 + w^2] - \frac{1}{3}w^3)\nu_1(\epsilon y)]\frac{\partial}{\partial x_1}\sqrt{1+(\rho')^2}dy + o(\epsilon^3)$$

$$= -\epsilon^3 \int_{\mathbb{R}}[(\frac{1}{2}[|\nabla w|^2 + w^2] - \frac{1}{3}w^3)\rho'' y_1](\rho'')^2 y_1 dy + o(\epsilon^3)$$

$$= -\epsilon^3(\rho'')^3 \int_{\mathbb{R}}[(\frac{1}{2}[|\nabla w|^2 + w^2] - \frac{1}{3}w^3)]y_1^2 dy + o(\epsilon^3).$$

(Here the function $\rho$ and its derivatives are computed at $P^\epsilon$.)

To estimate $I_{3,2}$, we note that

$$\nu_{1,1}(x) = \frac{1}{\sqrt{1+(\rho')^2}}\left(\rho'' - \frac{(\rho')^2\rho''}{1+(\rho')^2}\right)$$

Since $\nu_2 = -1/\sqrt{1+(\rho')^2}$, we have

$$\nu_{i,j}(x) + \rho''(0)\nu_2 \tag{4.49}$$

$$= \rho''' x_1 + \frac{1}{2}\rho'''' x_1^2 - (\rho'')^3 x_1^2 + O(|x|^3).$$

Applying (2.14) of Lemma 2.3 of [**71**], we have

$$\int_{\partial\Omega}[\frac{1}{2}\epsilon^2|\nabla u_\epsilon|^2 + \frac{1}{2}u_\epsilon^2 - \frac{1}{3}u_\epsilon^3]\,\nu_2\,dx = o(\epsilon^3). \tag{4.50}$$

By (4.50), we have

$$I_{3,2} = -\int_{\partial\Omega}[\frac{1}{2}\epsilon^2|\nabla u_\epsilon|^2 + \frac{1}{2}u_\epsilon^2 - \frac{1}{3}u_\epsilon^3]\nu_{1,1}(x)dx + O(e^{-\delta/\epsilon})$$

$$= -\int_{\partial\Omega}[\frac{1}{2}\epsilon^2|\nabla u_\epsilon|^2 + \frac{1}{2}u_\epsilon^2 - \frac{1}{3}u_\epsilon^3][\nu_{1,1}(x) + \rho''(0)\nu_2]\,dx + o(\epsilon^3)$$

$$= -\int_{\partial\Omega}[\frac{1}{2}\epsilon^2|\nabla u_\epsilon|^2 + \frac{1}{2}u_\epsilon^2 - \frac{1}{3}u_\epsilon^3]\rho''' x_1\,dx$$

$$-\frac{1}{2}\int_{\partial\Omega}[\frac{1}{2}\epsilon^2|\nabla u_\epsilon|^2 + \frac{1}{2}u_\epsilon^2 - \frac{1}{3}u_\epsilon^3]\rho'''' x_1^2\,dx$$

$$+\int_{\partial\Omega}[\frac{1}{2}\epsilon^2|\nabla u_\epsilon|^2+\frac{1}{2}u_\epsilon^2-\frac{1}{3}u_\epsilon^3](\rho'')^3x_1^2\,dx+o(\epsilon^3)\ (\text{by }(4.49))$$
$$=-\epsilon^3\frac{1}{2}\rho''''\int_{\mathbb{R}}(\frac{1}{2}|\nabla w|^2+\frac{1}{2}w^2-\frac{1}{3}w^3)y_1^2\,dy$$
$$+\epsilon^3(\rho'')^3\int_{\mathbb{R}}(\frac{1}{2}|\nabla w|^2+\frac{1}{2}w^2-\frac{1}{3}w^3)y_1^2\,dy+o(\epsilon^3).$$

Combining the estimates for $I_{3,1}$ and $I_{3,2}$, we obtain

$$I_{3,1}+I_{3,2}=-\frac{1}{2}\rho''''\epsilon^3\int_{\mathbb{R}}(\frac{1}{2}|\nabla w|^2+\frac{1}{2}w^2-\frac{1}{3}w^3)y_1^2\,dy+o(\epsilon^3)$$
$$=-\frac{1}{2}\nabla^2_{\tau(P^\epsilon)}H(P^\epsilon)\epsilon^3\int_{\mathbb{R}}(\frac{1}{2}|\nabla w|^2+\frac{1}{2}w^2-\frac{1}{3}w^3)y_1^2dy+o(\epsilon^3)$$
$$=-\frac{1}{2}\nabla^2_{\tau(P^\epsilon)}H(P^\epsilon)\epsilon^3\int_{\mathbb{R}}(\frac{1}{2}|\nabla w|^2+\frac{1}{2}w^2-\frac{1}{3}w^3)|y|^2dy+o(\epsilon^3).$$

By Lemma 3.3 of [**54**],

$$B=\frac{1}{3}\int_{\mathbb{R}^2_+}(w'(|z|))^2z_2dz$$
$$=\frac{1}{2}\int_{\mathbb{R}^2_+}(\frac{1}{2}(|\nabla w|^2+w^2)-\frac{1}{3}w^3)\,z_2\,dz.$$

Since $w$ is radially symmetric, it is easy to see that

$$\int_{\mathbb{R}^2_+}(\frac{1}{2}(|\nabla w|^2+w^2)-\frac{1}{3}w^3)z_2dz=\int_{\mathbb{R}}(\frac{1}{2}(|\nabla w|^2+w^2)-\frac{1}{3}w^3)|z|^2dz.$$

Thus

$$B=\frac{1}{2}\int_{\mathbb{R}}(\frac{1}{2}|\nabla w|^2+\frac{1}{2}w^2-\frac{1}{3}w^3)|y|^2dy$$

and

$$I_3=-\epsilon^2B\nabla^2_{\tau(P^\epsilon)}H(P^\epsilon)+o(\epsilon^3).$$

We now calculate $I_4$.

$$I_4=\alpha_\epsilon\int_{\Omega_{\epsilon,P^\epsilon}}2\hat{A}_\epsilon\frac{\partial w_{\epsilon,P^\epsilon}}{\partial\tau(P^\epsilon)}(\hat{H}-H_{\epsilon,P^\epsilon})\,dz$$
$$=\epsilon\alpha_\epsilon\int_{\Omega_{\epsilon,P^\epsilon}}\frac{(\hat{A}_\epsilon)^2}{(\hat{H}_\epsilon)^2}\left[-\frac{1}{\epsilon}\psi_\epsilon+\frac{\partial\hat{H}_\epsilon}{\partial x_1}\right]\frac{\partial\hat{A}_\epsilon}{\partial y_1}\,dz+o(\epsilon^2|\alpha_\epsilon|),$$

where

$$\psi_\epsilon(x)=-2\epsilon c_0\epsilon^2\xi_\epsilon\int_\Omega H_0(x,\xi)\hat{A}_\epsilon\left(\frac{\xi-P^\epsilon}{\epsilon}\right)\frac{\partial\hat{A}_\epsilon}{\partial x_1}\left(\frac{\xi-P^\epsilon}{\epsilon}\right)\,d\xi$$

and

$$\frac{\partial\hat{H}_\epsilon}{\partial x_1}(x)=-\xi_\epsilon c_0\epsilon^2\int_\Omega\frac{\partial}{\partial x_1}H_0(x,\xi)\hat{A}_\epsilon^2\left(\frac{\xi-P^\epsilon}{\epsilon}\right)\,d\xi.$$

Therefore, we obtain

$$I_4 = \epsilon^2 \int_{\mathbb{R}^2_+} w^2 \frac{\partial w}{\partial y_1} y_1 \alpha_\epsilon \left( -\frac{\partial^2}{\partial \tau (P^\epsilon)^2} H_0(P^\epsilon, P^\epsilon)(P^\epsilon) \right) dy$$
$$+ o(\epsilon^2 |\alpha_\epsilon|). \tag{4.51}$$

Note that

$$\int_{\mathbb{R}^2_+} w^2(y) \frac{\partial w(y)}{\partial y_1} y_1 \, dy = -\frac{1}{3} \int_{\mathbb{R}^2} w^3(y) \, dy.$$

Thus we have

$$I_4 = \frac{\epsilon^2}{3} \int_{\mathbb{R}^2_+} w^3(y) \, dy \alpha_\epsilon \left( \frac{\partial^2}{\partial \tau (P^\epsilon)^2} H_0(P^\epsilon, P^\epsilon) \right) \tag{4.52}$$
$$+ o(\epsilon^2 |\alpha_\epsilon|).$$

Combining the *l.h.s.* with the *r.h.s.*, we have

$$I_3 + I_4 + o(\epsilon^2 |\alpha_\epsilon|)$$
$$= \lambda_\epsilon \alpha_\epsilon \int_{\mathbb{R}^2_+} \left( \frac{\partial w}{\partial y_1} \right)^2 dy \, (1 + o(1)). \tag{4.53}$$

This concludes Step 2.

From (4.53), we see that the small eigenvalues with $\lambda_\epsilon \to 0$ satisfy $|\lambda_\epsilon| \sim \epsilon^2 \nabla^2_{\tau(P^\epsilon)} F(P^\epsilon)$.

By the condition $(*)$, $(\nabla_{\tau(P_0)})^2 F(P_0) < 0$, it follows that $\mathrm{Re}(\lambda_\epsilon) < 0$. Therefore the small eigenvalues $\lambda_\epsilon$ are stable for (4.4) if $\epsilon$ is small enough.

**Completion of the proof of Theorem 1.2:**

Theorem 1.2 now follows from the results of Section 1 and Section 2 in this Chapter.

$\square$

CHAPTER 5

# Appendices

## 1. Appendix A

In this appendix we give rigorous proofs for the expansions (3.13) and (3.17).

Recalling from Section 3.1 the following notation: For $x \in \Omega_P$ we have introduced the mapping $T(x) = (T_1(x), T_2(x))$ with

$$\begin{cases} T_1(x_1) = x_1 - P_1, \\ T_2(x_1) = x_2 - P_2 - \rho(x_1) \end{cases}$$

and the rescaled mapping

$$y = \frac{1}{\epsilon} T(x).$$

The Laplace operator and the boundary derivative operator become

$$\epsilon^2 \Delta_x = \Delta_y + (\rho')^2 \frac{\partial^2}{\partial y_2^2} - 2\rho' \frac{\partial^2}{\partial y_1 \partial y_2} - \epsilon \rho'' \frac{\partial}{\partial y_2} \quad \text{for } x \in \Omega_P,$$

$$\sqrt{1 + (\rho')^2} \frac{\partial}{\partial \nu_x} = \frac{1}{\epsilon} \left\{ \rho_1 \frac{\partial}{\partial y_1} - (1 + (\rho')^2) \frac{\partial}{\partial y_2} \right\} \quad \text{for } x \in \overline{\Omega_P} \cap \partial\Omega.$$

As a preparation for the proof of (3.13), we begin with

LEMMA 5.1. *Let $u$ be a solution of*

$$\begin{cases} \epsilon^2 \Delta u - u + f = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

*Assume that $\int_\Omega |f|^2 \, dx \le C\epsilon^2$, $\int_{\partial\Omega} |g|^2 \, dx \le C\epsilon$. Then*

$$\|u\|_{1,\epsilon} \le C.$$

**Proof.** Multiplying (5.1) by $u$ and integrating over $\Omega$, we have after integration by parts

$$\epsilon^2 \int_\Omega |\nabla u|^2 \, dx + \int_\Omega u^2 \, dx = \int_\Omega f u \, dx + \epsilon^2 \int_{\partial\Omega} g u \, dx.$$

Lemma 2.2 follows by the following trace inequality which will be proved in Lemma 5.3

$$\|u\|_{L^2(\partial\Omega_{\epsilon,P})} \leq C\|u\|_{H^1(\Omega_{\epsilon,P})},$$

where $\Omega_{\epsilon,P} = \{z|\ x = P + \epsilon z \in \Omega\}$ for a fixed $P \in \partial\Omega$ and by the Cauchy-Schwarz inequality.

□

**Proof of (3.13).** We first compute the equation for $\varphi^0_{\epsilon,P}(x)$:

$$\begin{aligned}
&-\epsilon^2\Delta_x\varphi^0_{\epsilon,P}(x) + \varphi^0_{\epsilon,P}(x) \\
&= \frac{1}{\epsilon^3}\left[\epsilon^2\left\{\Delta_x(\epsilon v_1\chi + \epsilon^2(v_2\chi + v_3\chi))\right\} - \epsilon v_1\chi - \epsilon^2 v_2\chi - \epsilon^2 v_3\chi\right] \\
&= \frac{1}{\epsilon^2}\Bigg[\left\{\Delta_y v_1 + (\rho')^2\frac{\partial^2 v_1}{\partial y_2^2} - 2\rho'\frac{\partial^2 v_1}{\partial y_1\partial y_2} - \epsilon\rho''\frac{\partial v_1}{\partial y_2} - v_1\right\} \\
&\quad +\epsilon\left\{\Delta_y v_2 + (\rho')^2\frac{\partial^2 v_2}{\partial y_2^2} - 2\rho'\frac{\partial^2 v_2}{\partial y_1\partial y_2} - \epsilon\rho''\frac{\partial v_2}{\partial y_2} - v_2\right\} \\
&\quad +\epsilon\left\{\Delta_y v_3 + (\rho')^2\frac{\partial^2 v_3}{\partial y_2^2} - 2\rho'\frac{\partial^2 v_3}{\partial y_1\partial y_2} - \epsilon\rho''\frac{\partial v_3}{\partial y_2} - v_3\right\}\Bigg]\chi + E_\epsilon \\
&= \frac{1}{\epsilon^2}\Bigg[\left\{(\rho')^2\frac{\partial^2 v_1}{\partial y_2^2} - 2(\rho'(0) - \epsilon\rho''(0)y_1)\frac{\partial^2 v_1}{\partial y_1\partial y_2}\right\} \\
&\quad +\epsilon\left\{(\rho')^2\frac{\partial^2 v_2}{\partial y_2^2} - 2\rho'\frac{\partial^2 v_2}{\partial y_1\partial y_2} - \epsilon\rho''\frac{\partial v_2}{\partial y_2} - (\rho'' - \rho''(0))\frac{\partial v_1}{\partial y_2}\right\} \\
&\quad +\epsilon\left\{(\rho')^2\frac{\partial^2 v_3}{\partial y_2^2} - 2\rho'\frac{\partial^2 v_3}{\partial y_1\partial y_2} - \epsilon\rho''\frac{\partial v_3}{\partial y_2}\right\}\Bigg]\chi + \frac{1}{\epsilon^2}E_\epsilon \\
&=: f_\epsilon
\end{aligned}$$

where $E_\epsilon$ denotes all the terms involving derivatives of $\chi$. Here we have used (3.14), (3.15), and (3.16).

Now recall that $|v_1(y)|, |v_2(y)|, |v_3(y)| \leq \exp(-\mu|y|)$ for some $\mu < 1$. Therefore

$$\|E_\epsilon\|_{L^\infty(\Omega)} = O(e^{-C/\epsilon}).$$

Using Taylor expansion for $\rho$ there exists some $k > 0$ such that

$$\|f_\epsilon\|_0^2 \leq C\int_{\text{supp }\chi\cap\Omega}\exp\left(\frac{-2k|x-P|}{\epsilon}\right) \leq C\epsilon^2.$$

Next we estimate $\frac{\partial\varphi^0_{\epsilon,P}}{\partial\nu_x}$. For $x\in\partial\Omega$, we have

$$\frac{\partial\varphi^0_{\epsilon,P}}{\partial\nu_x}=\frac{1}{\epsilon^3}\left\{\frac{\partial w\left(\left|\frac{x-P}{\epsilon}\right|\right)}{\partial\nu_x}-\epsilon\frac{\partial(v_1\chi)}{\partial\nu_x}-\epsilon^2\left(\frac{\partial(v_2\chi)}{\partial\nu_x}+\frac{\partial(v_3\chi)}{\partial\nu_x}\right)\right\}.$$

Note that

$$\begin{aligned}\chi\frac{\partial w\left(\left|\frac{x-P}{\epsilon}\right|\right)}{\partial\nu_x}\sqrt{1+(\rho')^2}&=\chi w'\left(\left|\frac{x-P}{\epsilon}\right|\right)\frac{<x-P,\nu>}{\epsilon|x-P|}\sqrt{1+(\rho')^2}\\&=\chi w'\left(\frac{x-P}{\epsilon}\right)\frac{1}{\epsilon|x-P|}\left\{\frac{1}{2}\rho''(0)(x_1-P_1)^2\right.\\&\quad\left.+\frac{1}{3}\rho'''(0)(x_1-P_1)^3+O(|x'-P'|^4)\right\}\\&=\chi\frac{w'(|z|)}{|z|}\left\{\frac{1}{2}\rho''(0)y_1^2+\frac{\epsilon}{3}\rho'''(0)y_1^3\right\}+O(\epsilon^2\exp(-\mu|z|)).\end{aligned}$$

We now need the following lemma, which is Lemma A.2 of [**6**]:

Lemma 5.2. *Let* $x\in\overline{\Omega_P}\cap\partial\Omega$, $y=\frac{1}{\epsilon}T(x)$, $z=\frac{x-P}{\epsilon}$. *Then*

$$\frac{w'(|z|)}{|z|}=\frac{w'(|y|)}{|y|}+\epsilon^2\frac{|y|w''(|y|)-w'(|y|)}{8|y|^3}\left(\rho''(0)y_1^2\right)^2+O(\epsilon^3e^{-k|y|})\tag{5.2}$$

*for some* $k>0$.

**Proof.** By Taylor expansion, we have

$$\frac{w'(|z|)}{|z|}=\frac{w'(|y|)}{|y|}+\frac{|y|w''(|y|)-w'(|y|)}{|y|^2}(|z|-|y|)+O\left((|z|-|y|)^2e^{-k|y|}\right).\tag{5.3}$$

Since $x\in\overline{\Omega_P}\cap\partial\Omega$, $y_2=\epsilon^{-1}(x_2-P_2-\rho(x_1-P_1))$, we calculate

$$\begin{aligned}|z|-|y|&=\frac{1}{\epsilon}(|x-P|-|T(x)|)=\frac{\rho(x_1-P_1)^2}{\epsilon(|x-P|+|T(x)|)}\\&=\frac{\epsilon^2}{8|y|}\left(\rho''(0)y_1^2\right)^2+O(\epsilon^3|y|^4).\end{aligned}\tag{5.4}$$

Therefore (5.2) can be obtained by combining (5.3) and (5.4).

□

By Lemma 5.2 we conclude

$$\chi\frac{\partial w\left(\left|\frac{x-P}{\epsilon}\right|\right)}{\partial\nu_x}\sqrt{1+(\rho')^2}$$

$$= \chi\left(\frac{w'(|z|)}{|z|}\left\{\frac{1}{2}\rho''(0)y_1^2 + \frac{\epsilon}{3}\rho'''(0)y_1^3\right\} + O(\epsilon^2\exp(-k|y|))\right).$$

(Without loss of generality we may assume that $k < \mu$.)

Furthermore,

$$\chi\sqrt{1+(\rho')^2}\frac{\partial v_1}{\partial \nu_x} = \chi\frac{1}{\epsilon}\left\{\rho'\frac{\partial v_1}{\partial y_1} - (1+(\rho')^2)\frac{\partial v_1}{\partial y_2}\right\},$$

$$\begin{aligned}\epsilon\frac{\partial\varphi^0_{\epsilon,P}}{\partial\nu_x}(x) = \chi\frac{1}{\sqrt{1+(\rho')^2\epsilon^2}}\Bigg[&\frac{w'(|y|)}{|y|}\left\{\frac{1}{2}\rho''(0)y_1^2 + \frac{\epsilon}{3}\rho'''(0)y_1^3\right\}\\ &+\Bigg\{-\rho'\frac{\partial v_1}{\partial y_1} + \frac{\partial v_1}{\partial y_2} + (\rho')^2\frac{\partial v_1}{\partial y_2}\\ &-\epsilon\rho'\frac{\partial v_2}{\partial y_1} + \epsilon\frac{\partial v_2}{\partial y_2} + \epsilon(\rho')^2\frac{\partial v_2}{\partial y_2}\\ &-\epsilon\rho'\frac{\partial v_3}{\partial y_1} + \epsilon\frac{\partial v_3}{\partial y_2} + (\rho')^2\frac{\partial v_3}{\partial y_2}\Bigg\}\Bigg]\\ &+E_\epsilon + O(\epsilon^2(\exp(-k|y|)))\\ &=: g_\epsilon,\end{aligned}$$

where again $E_\epsilon$ denotes all the terms involving derivatives of $\chi$. Here we have used (3.14), (3.15), and (3.16).

Now recall that $|v_1(y)|, |v_2(y)|, |v_3(y)| \le \exp(-\mu|y|)$ for some $\mu < 1$. Therefore

$$\|E_\epsilon\|_{L^\infty(\partial\Omega)} = O(e^{-C/\epsilon}).$$

Using Taylor expansion for $\rho$ we have

$$|g_\epsilon| \le \frac{\chi}{\epsilon^2\sqrt{1+(\rho')^2}}O(\epsilon^2\exp(-k|y|).$$

This implies

$$\|g_\epsilon\|^2_{L^2(\partial\Omega)} \le C\int_{\text{supp }\chi\cap\partial\Omega}\exp\left(\frac{-2k|x-P|}{\epsilon}\right) \le C\epsilon.$$

In summary: The error function $\varphi^0_{\epsilon,P}$ satisfies satisfies

$$\epsilon^2\Delta_x\varphi^0_{\epsilon,P} - \varphi^0_{\epsilon,P} + f_\epsilon = 0 \qquad \text{in } \Omega,$$

$$\frac{\partial\varphi^0_{\epsilon,P}}{\partial\nu_x} = g_\epsilon \qquad \text{on } \partial\Omega,$$

where $f_\epsilon \in L^2(\Omega)$, $g_\epsilon \in L^2(\partial\Omega)$ and $\|f_\epsilon\|^2_{L^2(\Omega} \le C\epsilon^2$, $\|g_\epsilon\|_{L^2(\partial\Omega)} \le C\epsilon$. Therefore by Lemma 5.1

$$\|\varphi^0_{\epsilon,P}\|_{1,\epsilon} \le C.$$

By using the elliptic regularity estimate given in Lemma 5.4 the proof of (3.13) is completed.

□

**Proof of (3.17).** First,

$$\begin{aligned}
&-\epsilon^2\Delta_x w^\epsilon_{2,P} + w^\epsilon_{2,P} \\
&= \frac{1}{\epsilon}\left[\epsilon^2\Delta_x \frac{\partial}{\partial\tau(P)}\varphi_\epsilon - \frac{\partial}{\partial\tau(P)}\varphi_\epsilon + \epsilon^2\Delta_x(\chi w_1) - \chi w_1\right] \\
&= \frac{1}{\epsilon}\chi\left[\epsilon^2\Delta_x w_1 - w_1\right] + E_\epsilon \\
&= \frac{1}{\epsilon}\chi\left[\Delta_y w_1 + (\rho')^2\frac{\partial^2 w_1}{\partial y_2^2} - 2\rho'\frac{\partial^2 w_1}{\partial y_1\partial y_2} - \epsilon\rho''\frac{\partial w_1}{\partial y_2} - w_1\right] + E_\epsilon, \\
&= \frac{1}{\epsilon}\chi\left[(\rho')^2\frac{\partial^2 w_1}{\partial y_2^2} - 2\rho'\frac{\partial^2 w_1}{\partial y_1\partial y_2} - \epsilon\rho''\frac{\partial w_1}{\partial y_2}\right] + E_\epsilon \\
&:= h_\epsilon,
\end{aligned}$$

where again $E_\epsilon$ denotes all the terms involving derivatives of $\chi$. Here we have used (3.18).

Again, since $|w_1(y)| \le \exp(-\mu|y|)$ for some $\mu < 1$. we have

$$\|E_\epsilon\|_{L^\infty(\Omega)} = O(e^{-C/\epsilon}).$$

Using Taylor expansion for $\rho$ we have

$$|h_\epsilon| \le C\chi\exp\left(-k\frac{|x-P|}{\epsilon}\right).$$

This implies

$$\|h_\epsilon\|^2_0 \le C\int_{\text{supp }\chi\cap\partial\Omega}\exp\left(\frac{-2k|x-P|}{\epsilon}\right) \le C\epsilon^2.$$

On the other hand, by (3.13), for $x \in \partial\Omega$,

$$\frac{\partial w^\epsilon_{2,P}}{\partial\nu_x} = \frac{1}{\epsilon}\left[\frac{\partial}{\partial\nu_x}\frac{\partial\varphi_\epsilon}{\partial\tau(P)} - \frac{\partial(\chi w_1)}{\partial\nu_x}\right]$$

$$= \frac{1}{\epsilon}\left[\epsilon\frac{\partial}{\partial\nu_x}\frac{\partial(\chi v_1)}{\partial\tau(P)} - \frac{\partial(\chi w_1)}{\partial\nu_x}\right] + \left[\epsilon\frac{\partial}{\partial\nu_x}\frac{\partial(\chi(v_2+v_3))}{\partial\tau(P)} + \epsilon^2\frac{\partial}{\partial\nu_x}\frac{\partial\phi^0_{\epsilon,P}}{\partial\tau(P)}\right].$$

By (3.14) and (3.18) we have

$$\frac{\partial v_1(y)}{\partial y_1} = -w_1(y).$$

Hence

$$\frac{1}{\epsilon}\left[\epsilon\frac{\partial}{\partial\nu_x}\frac{\partial(\chi v_1)}{\partial\tau(P)} - \frac{\partial(\chi w_1)}{\partial\nu_x}\right]$$

$$= -\frac{1}{\epsilon}\chi\frac{\partial}{\partial\nu_x}\left[\frac{\partial v_1(y)}{\partial y_1} - w_1(y)\right] - \frac{1}{\epsilon}\frac{\partial\chi}{\partial\nu_x}\left[\frac{\partial v_1(y)}{\partial y_1} - w_1(y)\right] = 0$$

and

$$\left\|\epsilon\frac{\partial w^2_{\epsilon,P}}{\partial\nu_x}\right\|_{L^2(\partial\Omega)} = \left\|\left[\epsilon^2\frac{\partial}{\partial\tau(P)}\frac{\partial(\chi(v_2+v_3))}{\partial\nu_x} + \epsilon^3\frac{\partial}{\partial\tau(P)}\frac{\partial\varphi^0_{\epsilon,P}}{\partial\nu_x}\right]\right\|_{L^2(\partial\Omega)} \leq C\epsilon.$$

Then by Lemma 5.1 and the elliptic regularity estimate given in Lemma 5.4, we have $\|w^2_{\epsilon,P}\|_{2,\epsilon} \leq C$. The proof of (3.17) is completed.

□

## 2. Appendix B

In this section we prove a trace inequality and an elliptic regularity estimate.

We begin with the following trace inequality:

Lemma 5.3. *Let $0 < \epsilon \leq 1$. Then*

$$\|\Phi\|_{L^2(\partial\Omega_{\epsilon,P})} \leq C\|\Phi\|_{H^1(\Omega_{\epsilon,P})} \tag{5.5}$$

*for all $\Phi \in H^1(\Omega_{\epsilon,P})$, where the constant $C$ is independent of $\epsilon$.*

Note that the constant $C$ in (5.5) is required to be independent of $\epsilon$. Therefore Lemma 5.3 is special although trace inequalities are quite standard.

**Proof of Lemma 5.3.** For $\Phi \in H^1(\Omega_{\epsilon,P})$, we define $\Psi \in H^1(\Omega)$ by a linear transformation:

$$\Psi(x) = \Phi(z), \qquad \text{where } z = \frac{x-P}{\epsilon}.$$

Observe that $\|\Phi\|^2_{L^2(\partial\Omega_{\epsilon,P})} = \epsilon^{-1}\|\Psi\|^2_{L^2(\partial\Omega)}$, $\|\Phi\|^2_{L^2(\Omega_{\epsilon,P})} = \epsilon^{-2}\|\Psi\|^2_{L^2(\Omega)}$, and $\|\nabla\Phi\|^2_{L^2(\Omega_{\epsilon,P})} = \|\nabla\Psi\|^2_{L^2(\Omega)}$. Therefore (5.5) is equivalent to

$$\|\Psi\|^2_{L^2(\partial\Omega)} \leq C\left(\epsilon\|\nabla\Psi\|^2_{L^2(\Omega)} + \frac{1}{\epsilon}\|\Psi\|^2_{L^2(\Omega)}\right) \tag{5.6}$$

for all $\Psi \in H^1(\Omega)$ and $0 < \epsilon \leq 1$, where $C$ is independent of $\epsilon$. The proof of (5.6) is standard and is omitted here (see for example the proof of Theorem 3.1 in [**1**]).

□

We now prove the following elliptic regularity estimate:

LEMMA 5.4. *Let $0 < \epsilon \leq \epsilon_0$ for $\epsilon$ small enough. Then*

$$\|\Phi\|_{H^2(\Omega_{\epsilon,P})} \leq C(\|\Delta\Phi\|_{L^2(\Omega_{\epsilon,P})} + \|\Phi\|_{H^1(\Omega_{\epsilon,P})}) \tag{5.7}$$

*for all $\Phi \in H^2_N(\Omega_{\epsilon,P})$, where $\Omega_{\epsilon,P}$ is as defined in Section 2 and $C$ is a constant independent of $\epsilon$.*

**Proof of Lemma 5.4.** Observe that for given $\delta > 0$ we can find $R_1 > 0$ and $\epsilon_0$ such that for $0 < \epsilon \leq \epsilon_0$

$$\|(\rho')^2\|_{L^\infty(B(R_0/\epsilon))} \leq \delta,\ \|\rho\|_{L^\infty(B(R_0/\epsilon))} \leq \delta,$$
$$\|\epsilon\rho''\|_{L^\infty(B(R_0/\epsilon))} \leq \delta. \tag{5.8}$$

Recall that the normal derivative operator is transformed as follows:

$$\begin{aligned}\epsilon\frac{\partial}{\partial\nu_x} &= \{1+(\rho')^2\}^{-1/2}\left\{\rho'\frac{\partial}{\partial y_1} - (1+(\rho')^2)\frac{\partial}{\partial y_2}\right\}\\ &= -\frac{\partial}{\partial y_2} + B^\epsilon,\end{aligned}$$

where $B^\epsilon$ is a differential operator on $B(R_1/\epsilon) \cup \{y_2 = 0\}$ with coefficients which are uniformly bounded in $L^\infty$ for $0 < \epsilon \leq \epsilon_0$ (compare section 2). From $\{\Omega_P : P \in \partial\Omega\}$ we select a finite subcovering of $\partial\Omega$ and denote it by $\{U_1, \dots, U_n\}$. Choosing $U_0 = \Omega$ the set $\{U_0, \dots, U_n\}$ is a finite covering of $\overline{\Omega}$ consisting of open sets. From now on we keep this covering fixed. Let $\{\theta_0, \dots, \theta_n\}$ be a partition of unity subordinate to this open covering. Denote $\theta_i^\epsilon(y) = \theta_i \circ T^{-1}(\epsilon y)$. Since

$$u = \sum_{i=0}^n \theta_i^\epsilon u$$

we have

$$\|u\|^2_{H^2(\Omega_{\epsilon,P})} \leq \|\theta_0^\epsilon u\|^2_{H^2(\Omega_{\epsilon,P})} + \sum_{i=1}^n \|\theta_i^\epsilon u\|^2_{H^2(\Omega_{\epsilon,P})}. \tag{5.9}$$

Since $\theta_0^\epsilon$ has compact support in $\mathbb{R}^2$, we have

$$\|\theta_0^\epsilon u\|^2_{H^2(\mathbb{R}^2)} = \|\Delta(\theta_0^\epsilon u)\|^2_{L^2(\mathbb{R}^2)} + \|\theta_0^\epsilon u\|^2_{H^1(\mathbb{R}^2)}$$

(see for example [**30**], Corollary 9.10). Because of

$$\Delta(\theta_0^\epsilon) = \theta_0^\epsilon \Delta u + 2\nabla u \cdot \theta_0^\epsilon + u\Delta\theta_0^\epsilon$$

and

$$\|\nabla\theta_0^\epsilon\|_{L^\infty(\mathbb{R}^2)} \leq C\epsilon, \quad \|\Delta\theta_0^\epsilon\|_{L^\infty(\mathbb{R}^2)} \leq C\epsilon^2,$$

we obtain

$$\|\theta_0^\epsilon u\|^2_{H^2(\Omega_{\epsilon,P})} \leq C(\|\theta_0^\epsilon \Delta u\|^2_{L^2(\Omega_{\epsilon,P})} + \|u\|^2_{H^1(\Omega_{\epsilon,P})}). \tag{5.10}$$

We are now going to estimate $\theta_i^\epsilon u$, $i = 1, \dots, n$. Note that

$$\frac{1}{C}\|(\theta_i^\epsilon u)^*\|_{H^k(\mathbb{R}^2_+)} \leq \|\theta_i^\epsilon u\|_{H^k(\Omega_{\epsilon,P})} \leq C\|(\theta_i^\epsilon u)^*\|_{H^k(\mathbb{R}^2_+)} \tag{5.11}$$

where $k = 0, 1$, or $2$ and

$$v^*(y) \equiv v(\frac{1}{\epsilon}T^{-1}(\epsilon y))$$

for $v \in H^2(U_i^\epsilon)$. Then

$$\|(\theta_i^\epsilon u)^*\|^2_{H^2(\mathbb{R}^2_+)} \leq \overline{C}\Bigg(\|\Delta(\theta_i^\epsilon u)^*\|^2_{L^2(\mathbb{R}^2_+)} + \left\|\frac{\partial}{\partial y_2}(\theta_i^\epsilon u)^*\right\|^2_{H^{1/2}(\mathbb{R}\times\{0\})} + \|(\theta_i^\epsilon u)^*\|^2_{H^1(\mathbb{R}^2_+)}\Bigg) \tag{5.12}$$

(see for example [**46**], Theorem 4.1). Now (5.8) implies that

$$\|A^\epsilon(\theta_i^\epsilon u)^*\|^2_{L^2(\mathbb{R}^2_+)} \leq \delta^2\|(\theta_i^\epsilon u)^*\|^2_{H^1(\mathbb{R}^2_+)}.$$

Therefore from (5.12) we get

$$(1-\overline{C}\delta^2)\|(\theta_i^\epsilon u)^*\|^2_{H^2(\mathbb{R}^2_+)}$$
$$\leq \overline{C}\Bigg(\|(\Delta + A^\epsilon)(\theta_i^\epsilon u)^*\|^2_{L^2(\mathbb{R}^2_+)} + \left\|\frac{\partial}{\partial y_2}(\theta_i^\epsilon u)^*\right\|^2_{H^{1/2}(\mathbb{R}\times\{0\})}$$

$$+\|(\theta_i^\epsilon u)^*\|^2_{H^1(\mathbb{R}^2_+)}\Bigg).$$

For the operator $B^\epsilon$ we can calculate in an analogous way. The trace theorem implies

$$(1-\tilde{C}\delta^2)\|(\theta_i^\epsilon u)^*\|^2_{H^2(\mathbb{R}^2_+)}$$
$$\leq C\Bigg(\|(\Delta+A^\epsilon)(\theta_i^\epsilon u)^*\|^2_{L^2(\mathbb{R}^2_+)}+\left\|\left(\frac{\partial}{\partial y_2}+B^\epsilon\right)(\theta_i^\epsilon u)^*\right\|^2_{H^{1/2}(\mathbb{R}\times\{0\})}$$
$$+\|(\theta_i^\epsilon u)^*\|^2_{H^1(\mathbb{R}^2_+)}\Bigg).$$

Since $\tilde{C}$ is by construction independent of $\epsilon$ we can choose $\delta$ so small that $1-\tilde{C}\delta^2\geq 1/2$. This implies

$$\|\theta_i^\epsilon u\|^2_{H^2(\Omega_{\epsilon,P})}\leq C\Bigg(\|\Delta(\theta_i^\epsilon u)\|^2_{L^2(\Omega_{\epsilon,P})}+\left\|\frac{\partial}{\partial\nu_\epsilon}(\theta_i^\epsilon u)\right\|^2_{H^{1/2}(\partial\Omega_{\epsilon,P})}$$
$$\|\theta_i^\epsilon u\|^2_{H^1(\Omega_{\epsilon,P})}\Bigg). \tag{5.13}$$

Similarly as before

$$\|\Delta(\theta_i^\epsilon u)\|^2_{L^2(\Omega_{\epsilon,P})}\leq C(\|\theta_i^\epsilon\Delta u\|^2_{L^2(\Omega_{\epsilon,P})}+\|u\|^2_{H^1(\Omega_{\epsilon,P})}) \tag{5.14}$$

and

$$\left\|\frac{\partial}{\partial\nu_\epsilon}(\theta_i^\epsilon u)\right\|^2_{H^{1/2}(\partial\Omega_{\epsilon,P})}\leq C\|u\|^2_{H^1(\Omega_{\epsilon,P})} \tag{5.15}$$

because of $\partial u/\partial\nu_\epsilon=0$. Combining (5.13) – (5.15), we get

$$\|\theta_i^\epsilon u\|^2_{H^2(\Omega_{\epsilon,P})}\leq C(\|\theta_i^\epsilon\Delta u\|^2_{L^2(\Omega_{\epsilon,P})}+\|u\|^2_{H^1(\Omega_{\epsilon,P})}). \tag{5.16}$$

We conclude, using (5.9), (5.10), and (5.16), that

$$\|u\|^2_{H^2(\Omega_{\epsilon,P})}\leq C\left(\sum_{i=0}^n\|\theta_\epsilon^i\Delta u\|^2_{L^2(\Omega_{\epsilon,P})}+(n+1)\|u\|^2_{H^1(\Omega_{\epsilon,P})}\right)$$
$$\leq C_n(\|\Delta u\|^2_{L^2(\Omega_{\epsilon,P})}+\|u\|^2_{H^1(\Omega_{\epsilon,P})})$$

where $C_n$ depends on $n$. Since $n$ is independent of $\epsilon$ the proof of Lemma 5.4 is completed.

□

## 3. Appendix C

**Proof of Proposition 3.3.** Suppose that (3.28) is false. Then there exist sequences $\{\epsilon_k\}$, $\{P^k\}$, and $\{\Phi_k\}$ with $\epsilon_k \to 0$, $P^k \in \overline{\Lambda}$, $\Phi_k \in K^{\perp}_{\epsilon_k,P^k}$ such that

$$\|l_{\epsilon_k,P^k}\Phi_k\|_{0,\epsilon} \to 0, \tag{5.17}$$

$$\|\Phi_k\|_{2,\epsilon} = 1, \qquad k = 1,\, 2,\, \ldots \tag{5.18}$$

We omit the argument $\Omega_{\epsilon_k,P^k}$ where this can be done without confusion. Denote

$$e_k = \left(\left\|\frac{\partial}{\tau(P^k)}P_{\Omega_{\epsilon_k,P^k}}w\right\|_{2,\epsilon}\right)^{-1}\frac{\partial}{\tau(P^k)}P_{\Omega_{\epsilon_k,P^k}}w.$$

We define $\Phi_{0k}(x) = (1-\chi(x-P))\Phi_k(x)$ and $\Phi_{1k}(x) = \chi(x-P)\Phi_k(x)$. We also define $\varphi_k(z) = \Phi_k(\epsilon_k z)$ and $\varphi_{ik}(z) = \Phi_{ik}(\epsilon_k z)$ for $i = 1, 2$ and $z = \frac{x-P^k}{\epsilon_k} \in \Omega_{\epsilon_k,P^k}$. By assumption, we have

$$\|\varphi_k\|_{H^2(\Omega_{\epsilon_k,P^k})} = 1, \quad \|\varphi\|_{H^2(\Omega_{\epsilon_k,P^k})} \le C.$$

We can extend $\varphi_k(z)$ and $\varphi_{ik}(z)$ from $\Omega_{\epsilon_k,P^k}$ to $\mathbb{R}^2$ such that

$$1 \le \|\varphi_k\|_{H^2(\mathbb{R}^2)} \le C, \quad \|\varphi_{ik}\|_{H^2(\mathbb{R}^2)} \le C$$

for some constant $C$ independent of $\epsilon_k$ (see [**30**], Lemma 6.37 and Theorem 7.25). Therefore there exists a subsequence of $\varphi_k$ (still denoted by $\varphi_k$) which converge to a limit $\varphi_\infty$ weakly in $H^2(\mathbb{R}^2)$. There also exist subsequences to $\varphi_{ik}$ which converge to a limit $\varphi_{i\infty}$ weakly in $H^2(\mathbb{R}^2)$. Obviously $\varphi_{1\infty}+\varphi_{2\infty} = \varphi_\infty$.

The plan for the rest of the proof is as follows: We first show that $\varphi_\infty = 0$. Then we prove that $\|\Phi_k\|_{2,\epsilon_k} \to 0$, which is a contradiction to our assumptions and will complete the proof.

To this end, we begin by showing that

$$\|l_{\epsilon_k,P^k}\Phi_{ik}\|_{0,\epsilon_k} \to 0 \quad \text{ as } k \to \infty \tag{5.19}$$

for $i = 0, 1$. We first prove (5.19) for $i = 0$. We calculate

$$\begin{aligned}\|l_{\epsilon_k,P^k}\Phi_{0k}\|_{0,\epsilon_k} &= \|\hat{\pi}_{\epsilon_k,P^k}\hat{l}_{\epsilon_k,P^k}((1-\chi)\Phi_k)\|_{0,\epsilon_k}\\ &= \|\hat{\pi}_{\epsilon_k,P^k}(1-\chi)\hat{l}_{\epsilon_k,P^k}\Phi_k\|_{0,\epsilon_k} + O(\epsilon_k\|\Phi_k\|_{2,\epsilon_k}),\end{aligned}$$

since the other terms involve gradients in $\chi$. Now observe that

$$\|\hat{l}_{\epsilon_k,P^k}\Phi_k\| \leq C\|\Phi_k\|_{2,\epsilon_k} \leq C. \tag{5.20}$$

Furthermore,

$$\|\hat{l}_{\epsilon_k,P^k}\Phi_k\|^2_{0,\epsilon_k} \leq C\left(\|l_{\epsilon_k,P^k}\Phi_k\|^2_{0,\epsilon_k} + \|(1-\hat{\pi}_{\epsilon_k,P^k})\hat{l}_{\epsilon_k,P^k}\Phi_k\|^2_{0,\epsilon_k}\right)$$
$$=: J_1 + J_2.$$

Now $J_1 \leq \|l_{\epsilon_k,P^k}\Phi_k\|^2_{0,\epsilon_k} \to 0$. To estimate $J_2$, we calculate

$$(1-\hat{\pi}_{\epsilon_k,P^k})\hat{l}_{\epsilon_k,P^k}\Phi_k = \alpha_k < \hat{l}_{\epsilon_k,P^k}\Phi_k, e_k >_{0,\epsilon_k} \;\; e_k.$$

Since $\|e_k\|_{L^\infty(supp(1-\chi))} = O(e^{-C/\epsilon_k})$, $|\alpha_k| \leq C$ and because of (5.20), we calculate $I_2 = O(e^{-C/\epsilon_k}) \to 0$. This implies (5.19) for $i = 0$.

We note that

$$\|l_{\epsilon_k,P^k}\Phi_{1k}\|^2_{0,\epsilon_k} = \|l_{\epsilon_k,P^k}\Phi_k\|^2_{0,\epsilon_k} - \|l_{\epsilon_k,P^k}\Phi_{0k}\|^2_{0,\epsilon_k} + O(e^{-C/\epsilon_k})$$
$$= o(1)$$

by our assumption and by (5.19) for $i = 0$. This proves (5.19) for $i = 1$.

We now prove $\varphi_\infty = 0$. We define $\psi_{ik} : \mathbb{R}^2_+ \to R$ by

$$\psi_{ik}(y) = \varphi_{ik}(\epsilon^{-1}T^{P^k}(\epsilon y)) \quad \text{ for } |\epsilon y| \leq \delta_2$$

and $\psi_{ik}(y) = 0$ for all other $y \in \mathbb{R}^2_+$. Since $T$ and $T^{-1}$ have bounded derivatives,

$$\|\psi_{ik}\|_{H^2(\mathbb{R}^2_+)} \leq C, \quad k = 1, 2, \dots .$$

Therefore, there exists a subsequence which converges to a limit $\psi_{i\infty}$ weakly in $H^2(\mathbb{R}^2_+)$ as $k \to \infty$. To prove that $\varphi_{i\infty} = 0$, it suffices to prove $\psi_{i\infty} = 0$.

For $i = 1$, we calculate

$$\frac{1}{\epsilon_k}\int_\Omega \Phi_{1,k}(x)e_k(x)\,dx = \frac{1}{\epsilon_k}\int_\Omega \Phi_k(x)e_k(x)\,dx - \int_\Omega (1-\chi)\Phi_k(x)e_k(x)\,dx$$
$$= 0 - O(e^{-C/\epsilon_k})$$

and

$$\frac{1}{\epsilon_k}\int_\Omega \Phi_{1,k}(x)e_k(x)\,dx$$
$$= \int_{\mathbb{R}^2_+} \psi_{1,k}(y)e_k(y)\,dy$$

$$\to \left( \left\| \frac{\partial w}{\partial y_1} \right\|_{0,\epsilon}^2 \right)^{-1} \int_{\mathbb{R}^2_+} \psi_{1,\infty}(y) \frac{\partial w}{\partial y_1}(y)\, dy \quad \text{as } k \to \infty$$

since in the last integrand $\psi_{1,\infty}$ converges weakly in $L^2(\mathbb{R}^2_+)$ and $\frac{\partial w}{\partial y_1}$ converges strongly in $L^2(\mathbb{R}^2_+)$. Therefore $\psi_{1\infty} \in K^{\perp}_{\epsilon,P}$.

The operators $\hat{l}_{\epsilon,P}$, $\hat{\pi}_{\epsilon,P}$, and $l_{\epsilon,P}$ can be transformed to $y$-coordinates by the diffeomorphism $T_\epsilon$. The transformed operators are denoted by $\tilde{\hat{l}}_{\epsilon,P}$, $\tilde{\hat{\pi}}_{\epsilon,P}$, and $\tilde{l}_{\epsilon,P}$, respectively. To this end the kernel is spanned by the transformed functions which are cut off outside the range of $T_\epsilon$. Since $T$ and $T^{-1}$ have bounded derivatives, we have

$$c\|l_{\epsilon_k,P^k}\Phi_{1k}\|_{0,\epsilon_k} \le \|\tilde{l}_{\epsilon_k,P^k}\psi_{1k}\|_{L^2(\mathbb{R}^2_+)} \le C\|l_{\epsilon_k,P^k}\Phi_{1k}\|_{0,\epsilon_k}$$

for some constants $0 < c < C$. In particular, by (5.19),

$$\|\tilde{l}_{\epsilon_k,P^k}\psi_{1k}\|_{L^2(\mathbb{R}^2_+)} \to 0 \quad \text{as } k \to \infty. \tag{5.21}$$

Since

$$e_k \to \left( \left\| \frac{\partial w}{\partial y_1} \right\|_{0,\epsilon}^2 \right)^{-1} \frac{\partial w}{\partial y_1}$$

we have for the projection in $y$-coordinates

$$\tilde{\hat{\pi}}_{\epsilon_k,P^k} \to \pi_0$$

and for the linear operator in $y$-coordinates

$$\tilde{\hat{l}}_{\epsilon_k,P^k} \to \Delta - 1 + w^2 = L_0.$$

(For details of this convergence, please see [**54**]). By (5.21), $\pi_0 L_0(\psi_{1\infty}) = 0$. Therefore, $L_0(\psi_{1\infty}) \in K_0$. On the other hand, for any $\varphi \in K_0$, we have

$$\int_{\mathbb{R}^2_+} L_0(\psi_{1\infty})\varphi\, dy = \int_{\mathbb{R}^2_+} L_0(\varphi)\psi_{1\infty}\, dy = 0.$$

This implies $L_0(\psi_{1\infty}) = 0$ and thus $\psi_{1\infty} \in K_0$ by Lemma 4.1.

Since $\psi_{1\infty} \in K_0$ and $\psi_{1\infty} \in K_0^{\perp}$, we know that ğ $\psi_{1\infty} = 0$ and therefore $\varphi_{1\infty} = 0$.

To prove that $\varphi_{0\infty} = 0$, note that by (5.19) and since

$$\|\hat{\pi}_{\epsilon_k,P^k}\Phi_{0k}\|_{0,\epsilon_k} = O(e^{-C/\epsilon_k})$$

we get by the same reasoning as for $\varphi_{1\infty}$ that

$$\begin{cases} \Delta\varphi_{0\infty} - \varphi_{0\infty} = 0 & \text{in } \mathbb{R}^2_+, \\ \frac{\partial \varphi_{0\infty}}{\partial \nu} = 0 & \text{on } \partial\mathbb{R}^2_+, \\ \varphi_{0\infty} \in H^2(\mathbb{R}^2_+). \end{cases}$$

Therefore $\varphi_{0\infty} = 0$ and we have proved that

$$\varphi_\infty = \varphi_{0\infty} + \varphi_{1\infty} = 0.$$

We finally prove that $\|\Phi_k\|_{2,\epsilon_k} \to 0$ as $k \to \infty$. First, note that

$$< \tilde{l}_{\epsilon_k, P^k} \Phi_k, e_k >_{0,\epsilon_k} \to 0$$

as $\varphi_k$ converges to 0 weakly in $H^2(\mathbb{R}^2_+)$ and $e_k$ converges strongly in $H^2(\mathbb{R}^2_+)$. Therefore also $\|(1 - \hat{\pi}_{\epsilon,P})\hat{l}_{\epsilon,P}\Phi_k\|_{0,\epsilon_k} \to 0$ as $k \to \infty$. By the assumption we have $\|\hat{l}_{\epsilon,P}\Phi_k\|_{0,\epsilon_k} \to 0$ as $k \to \infty$. By a simple cut-off argument and since

$$\int_{B(R)} w^2_{\epsilon_k,P^k}(z)\varphi_k^2(z)\,dz$$

$$\leq C \int_{B(R)} \varphi_k^2(z)\,dz \to 0$$

as weak convergence in $H^2(\mathbb{R}^2_+)$ implies strong convergence in $L^2(B(R))$, we have

$$\|(\epsilon^2\Delta - 1)\Phi_k\|_{0,\epsilon_k} \to 0,$$

and

$$c\|\Phi_k\|^2_{1,\epsilon_k} \leq \epsilon_k^{-2}\left[\epsilon_k^2 \int_\Omega |\nabla\Phi_k|^2\,dx + \int_{\Omega\Phi_k^2} dx\right]$$

$$-\epsilon_k^{-2}\int_\Omega((\epsilon_k^2\Delta - 1)\Phi_k)\Phi_k\,dx$$

$$\leq C\|(\epsilon_k^2\Delta - 1)\Phi_k\|_{0,\epsilon_k}\|\Phi_k\|_{0,\epsilon_k} \to 0.$$

In summary, we have $\|\epsilon_k^2\Delta\Phi_k\|_{0,\epsilon_k} \to 0$ and $\|\Phi_k\|_{1,\epsilon_k} \to 0$. By the elliptic regularity estimate given in Lemma 5.4, we have

$$\|\Phi_k\|_{2,\epsilon_k} \leq C(\|\epsilon_k^2\Delta\Phi_k\|_{0,\epsilon_k} + \|\Phi_k\|_{1,k}),$$

where $C > 0$ is independent of $\epsilon > 0$. Therefore, finally, $\|\Phi_k\|_{2,\epsilon_k} \to 0$ as $k \to \infty$, which contradicts to the assumption that $\|\Phi_k\|_{2,\epsilon_k} = 1$ for $k = 1, 2, \ldots$. This completes the proof of Proposition 3.3.

□

**Proof of Proposition 3.4.** By Lemma 3.38 we know that $\mathrm{Ker}(l_{\epsilon,P}) = \{0\}$. Therefore it suffices to show that $l_{\epsilon,P}$ is a Fredholm operator with index 0.

Now $l_{\epsilon,P} : K^{\perp}_{\epsilon,P} \to C^{\perp}_{\epsilon,P}$ is a closed operator, whose domain $K^{\perp}_{\epsilon,P} \subset C^{\perp}_{\epsilon,P}$ is dense in $C^{\perp}_{\epsilon,P}$. Then $\mathrm{Range}(l_{\epsilon,P})$ is closed. Moreover,

$$l_{\epsilon,P} = \hat{l}_{\epsilon,P} - (1 - \hat{\pi}_{\epsilon,P})\hat{l}_{\epsilon,P},$$

It is standard to show that $\hat{l}_{\epsilon,P}$ is a Fredholm operator with index 0, and $(1-\hat{\pi}_{\epsilon,P})\hat{l}_{\epsilon,P}$ is relatively compact with respect to $\hat{l}_{\epsilon,P}$ in the sense of Kato (see [**42**], p. 194), since $K_{\epsilon,P}$ has finite dimension. Therefore, by [**42**], Theorem 5.26, $l_{\epsilon,P}$ is also a Fredholm operator with the same range as $\hat{l}_{\epsilon,P}$. This implies that codim $\mathrm{Range}(l_{\epsilon,P}) = \dim\ \mathrm{Ker}(l_{\epsilon,P}) = 0$ and $\mathrm{Range}(l_{\epsilon,P}) = C^{\perp}_{\epsilon,P}$.

□

## 4. Appendix D

In this section we prove Proposition 3.9.

**Proof of Proposition 3.9.** To begin with, Proposition 3.8 says that

$$\hat{\pi}_{\epsilon,P}(S_1(A_{\epsilon,P} + \Phi_{\epsilon,P}), T[(A_{\epsilon,P} + \Phi_{\epsilon,P})^2]) = 0.$$

Written in more explicit terms, this means that

$$\begin{aligned}
0 &= \hat{\pi}_{\epsilon,P}\left[\epsilon^2\Delta\Phi_{\epsilon,P} - \Phi_{\epsilon,P} + \epsilon^2\Delta A_{\epsilon,P} - A_{\epsilon,P} + \frac{(A_{\epsilon,P} + \Phi_{\epsilon,P})^2}{T[(A_{\epsilon,P} + \Phi_{\epsilon,P})^2]}\right] \\
&= \hat{\pi}_{\epsilon,P}\left[\epsilon^2\Delta\Phi_{\epsilon,P} - \Phi_{\epsilon,P} + 2\frac{A_{\epsilon,P}\Phi_{\epsilon,P}}{T[A^2_{\epsilon,P}]} - 2\frac{A^2_{\epsilon,P}}{(T[A_{\epsilon,P}])^2}T[A_{\epsilon,P}\Phi_{\epsilon,P}]\right. \\
&\qquad \left. + \frac{A^2_{\epsilon,P}}{T[A^2_{\epsilon,P}]} - w^2\right] + O(\epsilon^2) \\
&= \hat{\pi}_{\epsilon,P}\left[L_\epsilon\Phi_{\epsilon,P} + \frac{A^2_{\epsilon,P}}{T[A^2_{\epsilon,P}]} - w^2\right] + O(\epsilon^2) \\
&= \hat{\pi}_{\epsilon,P}\left[L_\epsilon\Phi_{\epsilon,P} + S_{1,2}\right] + O(\epsilon^2)
\end{aligned} \tag{5.22}$$

in $L^2(\Omega_{\epsilon,P})$ by Lemma 3.2.

The notations $\Omega_P\, \chi, \rho$, and $T$ have the same meanings as in Section 3. Our strategy is to decompose $\varphi^1_{\epsilon,P}$ into three parts and show that each of them is bounded in $\|\cdot\|_{H^2(\Omega_{\epsilon,P})}$ as $\epsilon \to 0$. That means we make the ansatz

$$\varphi^1_{\epsilon,P}(x) = \varphi^{1,1}_{\epsilon,P}(x) + \varphi^{1,2}_{\epsilon,P}(x) + \varphi^{1,3}_{\epsilon,P}(x), \tag{5.23}$$

where the functions $\varphi^{1,1}_{\epsilon,P}$, $\varphi^{1,2}_{\epsilon,P}$, $\varphi^{1,3}_{\epsilon,P}$ will be defined as follows. Let $\varphi^{1,1}_{\epsilon,P}$ be the unique solution in $H^1(\Omega)$ of

$$\epsilon^2 \Delta \varphi^{1,1}_{\epsilon,P} - \varphi^{1,1}_{\epsilon,P} = 0 \qquad \text{in } \Omega,$$

$$\frac{\partial \varphi^{1,1}_{\epsilon,P}}{\partial \nu} = g_\epsilon \qquad \text{on } \partial\Omega, \tag{5.24}$$

where

$$g_\epsilon(x) = -\frac{\partial}{\partial \nu_x}[\Phi_0(y)\chi(x)].$$

Define $\varphi^{1,2}_{\epsilon,P}$ to be the unique solution in $H^1(\Omega)$ of

$$\varphi^{1,2}_{\epsilon,P} = -\frac{1}{\epsilon}(1-\hat{\pi}_{\epsilon,P})\Phi_0(y)\chi(x) - (1-\hat{\pi}_{\epsilon,P})\varphi^{1,1}_{\epsilon,P}. \tag{5.25}$$

Finally, define $\varphi^{1,3}_{\epsilon,P}(x)$ as follows:

$$\varphi^{1,3}_{\epsilon,P}(x) = \varphi^1_{\epsilon,P}(x) - \varphi^{1,1}_{\epsilon,P}(x) - \varphi^{1,2}_{\epsilon,P}(x). \tag{5.26}$$

By definition, $\varphi^{1,3}_{\epsilon,P} \in K^\perp_{\epsilon,P}$.

Since

$$|\Phi_0| \le C \exp(-\mu|y|) \qquad \text{for } 0 < \mu < 1$$

we have

$$\left\| \epsilon \frac{\partial \varphi^{1,1}_{\epsilon,P}}{\partial \nu} \right\|^2_{L^2(\partial\Omega)} \le C\epsilon.$$

Then by Lemma 5.1,

$$\|\varphi^{1,1}_{\epsilon,P}\|_{1,\epsilon} \le C.$$

Furthermore, by (5.24),

$$\|\epsilon^2 \Delta \varphi^{1,1}_{\epsilon,P}\|_{0,\epsilon} = \|\varphi^{1,1}_{\epsilon,P}\|_{0,\epsilon}.$$

Now, by Lemma 5.4,

$$\|\varphi^{1,1}_{\epsilon,P}\|_{0,\epsilon}\|_{2,\epsilon} \le C(\epsilon^2 \|\Delta \varphi^{1,1}_{\epsilon,P}\|_{0,\epsilon} + \|\varphi^{1,1}_{\epsilon,P}\|_{1,\epsilon}) \le C.$$

For $\varphi^{1,2}_{\epsilon,P}$ we calculate

$$\|\varphi^{1,2}_{\epsilon,P}\|_{0,\epsilon} = -\alpha \left\langle \frac{1}{\epsilon}\Phi_0(y)\chi(x) + \varphi^{1,1}_{\epsilon,P}, e_1 \right\rangle_{0,\epsilon} e_1.$$

As $\|e_1\|_{2,\epsilon} \le C$, it remains to be shown that

$$\left\langle \frac{1}{\epsilon}\Phi_0(y)\chi(x) + \varphi^{1,1}_{\epsilon,P}, e_1 \right\rangle_{0,\epsilon} \le C.$$

We have

$$\begin{aligned}
&\left\langle \frac{1}{\epsilon}\Phi_0(y)\chi(x) + \varphi^{1,1}_{\epsilon,P}, e_1 \right\rangle_{0,\epsilon} \\
&\le \frac{1}{\epsilon}\|\Phi_0(y)\chi(x)\|_0\|e_1\|_0 + \|\varphi^{1,1}_{\epsilon,P}\|_0\|e_1\|_0 \\
&= \frac{1}{\epsilon}O(\epsilon)O(1) + O(\epsilon)O(1) = O(1).
\end{aligned}$$

Finally, we prove that $\|\varphi^{1,3}_{\epsilon,P}\|_{0,\epsilon} \le C$. Note that, since $\varphi^{1,3}_{\epsilon,P} \in K^{\perp}_{\epsilon,P}$, it satisfies

$$L_{\epsilon,P}\varphi^{1,3}_{\epsilon,P} = -\frac{1}{\epsilon^2} f_\epsilon,$$

where

$$f_\epsilon(x) = L_{\epsilon,P}(\varphi^{1}_{\epsilon,P}(x) - \varphi^{1,1}_{\epsilon,P}(x) - \varphi^{1,2}_{\epsilon,P}(x)).$$

By Lemma 3.5 we know that

$$\|L_{\epsilon,P}\varphi^{1,3}_{\epsilon,P}\|_{0,\epsilon} \ge \lambda\|\varphi^{1,3}_{\epsilon,P}\|_{2,\epsilon}.$$

Set

$$\hat{f}_\epsilon = L_\epsilon(\Phi_{\epsilon,P} - \epsilon\Phi_0\chi - \epsilon^2\varphi^{1,1}_{\epsilon,P} - \epsilon^2\varphi^{1,2}_{\epsilon,P}) =: \hat{f}_{\epsilon,1} - \hat{f}_{\epsilon,2} - \hat{f}_{\epsilon,3} - \hat{f}_{\epsilon,4}.$$

It remains to be shown that $\|f_\epsilon\|_{0,\epsilon} = \|\hat{\pi}_{\epsilon,P}\hat{f}_\epsilon\|_{0,\epsilon} \le C\epsilon^2$. Now

$$\begin{aligned}
\hat{f}_{\epsilon,3} = \epsilon^2 L_\epsilon\varphi^{1,1}_{\epsilon,P} &= \epsilon^2\left[\epsilon^2\Delta\varphi^{1,1}_{\epsilon,P} - \varphi^{1,1}_{\epsilon,P} + 2A_{\epsilon,P}H^{-1}_{\epsilon,P}\varphi^{1,1}_{\epsilon,P} - 2\frac{\int_\Omega A_{\epsilon,P}\varphi^{1,1}_{\epsilon,P}\,dx}{\int_\Omega A^2_{\epsilon,P}\,dx}A^2_{\epsilon,P}\right] \\
&= \epsilon^2\left[2A_{\epsilon,P}H^{-1}_{\epsilon,P}\varphi^{1,1}_{\epsilon,P} - 2\frac{\int_\Omega A_{\epsilon,P}\varphi^{1,1}_{\epsilon,P}\,dx}{\int_\Omega A^2_{\epsilon,P}\,dx}A^2_{\epsilon,P}\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|\hat{\pi}_{\epsilon,P}\hat{f}_{\epsilon,3}\|_{0,\epsilon} \le \|\hat{f}_{\epsilon,3}\|_{0,\epsilon} \\
&\le C\epsilon^2(\|\varphi^{1,1}_{\epsilon,P}\|_{0,\epsilon})^{1/2}(\|A_{\epsilon,P}\|_{0,\epsilon})^{1/2} \\
&\le C\epsilon^2.
\end{aligned}$$

By definition $\varphi^{1,2}_{\epsilon,P} \in K_{\epsilon,P}$. Therefore, it can be represented as follows:

$$\varphi^{1,2}_{\epsilon,P}(x) = \alpha_\epsilon \frac{\partial w_{\epsilon,P}}{\partial \tau(P)}, \quad \alpha \in R.$$

Note that by (5.25), $|\alpha_\epsilon| \le C$. This implies

$$\hat{f}_{\epsilon,3} = \epsilon^2 L_\epsilon \varphi^{1,2}_{\epsilon,P} = \epsilon^2 \left[ \epsilon^2 \Delta \varphi^{1,2}_{\epsilon,P} - \varphi^{1,2}_{\epsilon,P} + 2A_{\epsilon,P} H^{-1}_{\epsilon,P} \varphi^{1,2}_{\epsilon,P} - 2\frac{\int_\Omega A_{\epsilon,P}\varphi^{1,2}_{\epsilon,P}\,dx}{\int_\Omega A^2_{\epsilon,P}\,dx} A^2_{\epsilon,P} \right]$$

$$= \epsilon^2 \alpha_\epsilon \left[ \epsilon^2 \Delta \frac{w_{\epsilon,P}}{\partial \tau(P)} - \frac{\partial w_{\epsilon,P}}{\partial \tau(P)} + 2A_{\epsilon,P} H^{-1}_{\epsilon,P} \frac{w_{\epsilon,P}}{\partial \tau(P)} - 2\frac{\int_\Omega A_{\epsilon,P}\frac{\partial w_{\epsilon,P}}{\partial \tau(P)}\,dx}{\int_\Omega A^2_{\epsilon,P}\,dx} A^2_{\epsilon,P} \right]$$

$$= \epsilon^2 \alpha_\epsilon \left[ -2w_\epsilon \frac{\partial w_\epsilon}{\partial \tau(P)} + 2A_{\epsilon,P} H^{-1}_{\epsilon,P} \frac{w_{\epsilon,P}}{\partial \tau(P)} - 2\frac{\int_\Omega A_{\epsilon,P}\frac{\partial w_{\epsilon,P}}{\partial \tau(P)}\,dx}{\int_\Omega A^2_{\epsilon,P}\,dx} A^2_{\epsilon,P} \right].$$

Therefore, by (3.13) and (3.17),

$$\|\hat{\pi}_{\epsilon,P} \hat{f}_{\epsilon,3}\|_{0,\epsilon} \le \|\hat{f}_{\epsilon,3}\|_{0,\epsilon} \le C\epsilon^2.$$

Finally, we estimate $\hat{\pi}_{\epsilon,P}(\hat{f}_{\epsilon,1} - \hat{f}_{\epsilon,2})$. Note that, by (3.13) and (5.22),

$$\begin{aligned}
\hat{\pi}_{\epsilon,P} \hat{f}_{\epsilon,1} &= \hat{\pi}_{\epsilon,P} L_\epsilon \Phi_{\epsilon,P} \\
&= -\hat{\pi}_{\epsilon,P}\left[S_{1,2}\right] + O(\epsilon^2) \\
&= \epsilon \hat{\pi}_{\epsilon,P}\left(\chi\left[2wv_1 - w^2 R(|y|)\right]\right) + O(\epsilon^2)
\end{aligned}$$

in $L^2(\Omega_{\epsilon,P})$, where $R(|y|)$ was defined in (3.27).

On the other hand,

$$\begin{aligned}
\hat{f}_{\epsilon,2} &= \epsilon L_\epsilon \Phi_0 \chi \\
&= \epsilon \left[ \epsilon^2 \Delta(\Phi_0\chi) - \Phi_0\chi + 2A_{\epsilon,P} H^{-1}_{\epsilon,P} \Phi_0\chi - 2\frac{\int_\Omega A_{\epsilon,P}\Phi_0\chi\,dx}{\int_\Omega A^2_{\epsilon,P}\,dx} A^2_{\epsilon,P} \right] \\
&= \epsilon\chi \left[ \epsilon^2 \Delta\Phi_0 - \Phi_0 + 2A_{\epsilon,P} H^{-1}_{\epsilon,P} \Phi_0 - 2\frac{\int_\Omega A_{\epsilon,P}\Phi_0\chi\,dx}{\int_\Omega A^2_{\epsilon,P}\,dx} A^2_{\epsilon,P} + E_\epsilon \right] \\
&= \epsilon\chi \Bigg[ \epsilon^2 \Delta_y \Phi_0 + (\rho')^2 \frac{\partial^2 \Phi_0}{\partial y_2^2} - 2\rho' \frac{\partial^2 \Phi_0}{\partial y_1 \partial y_2} \\
&\quad -\epsilon\rho'' \frac{\partial \Phi_0}{\partial y_2} - \Phi_0 + 2A_{\epsilon,P} H^{-1}_{\epsilon,P} \Phi_0 - 2\frac{\int_\Omega A_{\epsilon,P}\Phi_0\,dx}{\int_\Omega A^2_{\epsilon,P}\,dx} A^2_{\epsilon,P} \Bigg] + O(\epsilon^2) \\
&= \epsilon\chi \Bigg[ \Delta_y \Phi_0 - \Phi_0 + (\rho')^2 \frac{\partial^2 \Phi_0}{\partial y_2^2} - 2\rho' \frac{\partial^2 \Phi_0}{\partial y_1 \partial y_2}
\end{aligned}$$

$$-\epsilon\rho''\frac{\partial\Phi_0}{\partial y_2}+2A_{\epsilon,P}H^{-1}_{\epsilon,P}\Phi_0-2\frac{\int_\Omega A_{\epsilon,P}\Phi_0\,dx}{\int_\Omega A^2_{\epsilon,P}\,dx}A^2_{\epsilon,P}\Bigg]+O(\epsilon^2)$$

$$=\epsilon\chi\left[\Delta_y\Phi_0-\Phi_0+2w\Phi_0-2\frac{\int_{\mathbb{R}^2_+}w\Phi_0\,dy}{\int_{\mathbb{R}^2_+}w^2(y)\,dy}w^2\right]+O(\epsilon^2)$$

$$=\chi S_{1,2}$$

$$=\epsilon\chi[-2wv_1+w^2R(|y|)]+O(\epsilon^2)$$

in $L^2(\mathbb{R}^2_+)$ by the definition of $\Phi_0$ in Proposition 3.9 and (3.13), where $E_\epsilon$ denotes all terms with derivatives of $\chi$. (Note that $\|E_\epsilon\|_{0,\epsilon}\le C\epsilon$.)

Therefore, we have

$$\|\hat{\pi}_{\epsilon,P}(f_{\epsilon,1}-f_{\epsilon,2})\|_{0,\epsilon}\le C\epsilon^2.$$

This concludes the proof of Proposition 3.9.

□

## 5. Appendix E

In this appendix, we give the proof of the estimate for $J$ which is needed in (4.48). Some of the calculations are long, straightforward and similar to those in Appendix A. We shall therefore omit most of the details.

**Estimate A:**

$$D_1w_{\epsilon,P^\epsilon}=-\epsilon\tilde{v}_1\Big(\frac{x-P^\epsilon}{\epsilon}\Big)+O(\epsilon^2)$$

where $\tilde{v}_1$ is the unique solution in $H^1(\mathbb{R}^2_+)$ of

$$\begin{cases}\Delta v-v=0 & \text{in } \mathbb{R}^2_+,\\ \frac{\partial v}{\partial y_2}=-\frac{1}{2}\frac{\partial w}{\partial y_1}\rho'''(0)y_1 & \text{on } \partial\mathbb{R}^2_+.\end{cases}\tag{5.27}$$

**Proof.** By direct computations similar to the proof of (3.14). □

**Estimate B:**

$$\epsilon^2<D_1(w_{\epsilon,P^\epsilon}+\phi_{\epsilon,P^\epsilon}),\partial_1(w_{\epsilon,P^\epsilon}+\phi_{\epsilon,P^\epsilon})>_{1,\epsilon}$$
$$-\int_\Omega D_1(w_{\epsilon,P^\epsilon}+\phi_{\epsilon,P^\epsilon})^2\nabla_{\tau(P^\epsilon)}(w_{\epsilon,P^\epsilon}+\phi_{\epsilon,P^\epsilon})\,dx=o(\epsilon^3).$$

**Proof.**

$$l.h.s.=\int_\Omega[2w\nabla_{\tau(P^\epsilon)}wD_1(w_{\epsilon,P^\epsilon}+\phi_{\epsilon,P^\epsilon})$$
$$-2(w_{\epsilon,P^\epsilon}+\phi_{\epsilon,P^\epsilon})D_1(w_{\epsilon,P^\epsilon}+\phi_{\epsilon,P^\epsilon})\nabla_{\tau(P^\epsilon)}(w_{\epsilon,P^\epsilon}+\phi_{\epsilon,P^\epsilon})]\,dx$$

$$= o(\epsilon^3)$$

since the first term in $D_1(w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon})$ is an even function.

□

**Estimate C:**

$$D_1 \frac{\partial}{\partial \tau(P^\epsilon)} w_{\epsilon,P^\epsilon} = \tilde{w}_1 + O(\epsilon),$$

where $\tilde{w}_1$ is the unique solution in $H^1(\mathbb{R}^2_+)$ of

$$\begin{cases} \Delta v - v = 0 & \text{in } \mathbb{R}^2_+, \\ \frac{\partial v}{\partial y_2} = \frac{1}{2}\left(\frac{\partial^2 w}{\partial y_1^2} y_1 + \frac{\partial w}{\partial y_1}\right)\rho'''(0) & \text{on } \partial\mathbb{R}^2_+. \end{cases} \tag{5.28}$$

**Proof.** By direct computations similar to the proof of (3.18). □

**Estimate D:**

$$\epsilon^2 < (w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon}), D_1 \nabla_{\tau(P^\epsilon)}(w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon}) >_{1,\epsilon}$$
$$- \int_\Omega (w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon})^2 D_1 \nabla_{\tau(P^\epsilon)}(w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon})\, dx = o(\epsilon^3).$$

**Proof.**

$$l.h.s. = \int_\Omega [w^2 - (w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon})^2] D_1 \nabla_{\tau(P^\epsilon)}(w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon})\, dx.$$

Note that the first term in the expansion of $w^2 - (w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon})^2$ is an even function and by Estimate C the first term in the expansion of $D_1 \nabla_{\tau(P^\epsilon)}(w_{\epsilon,P^\epsilon} + \phi_{\epsilon,P^\epsilon})$ is an odd function. Therefore, the whole expression is of the order $o(\epsilon^3)$.

□

Combining Estimates B and D, we obtain the estimate for $J$.

CHAPTER 6

# Discussion

## 1. Discussion

Let us discuss what has been achieved in this paper and which important questions are still left open. We have investigated the Gierer-Meinhardt system which is a well-known reaction-diffusion system within the class of Turing systems. We consider the case of a particular growth rate of the inhibitor diffusivity namely, $D_h \sim \epsilon^{-1}$ for small activator diffusivity $\epsilon^2$. In a bounded domain we rigorously prove the existence of a solution with a single spike at the boundary and are able to locate the spike in terms of the tangential derivatives of the curvature of the boundary and the Green function $G_0$. Furthermore, we derive rigorous results on linear stability. We have $o(1)$ eigenvalues which are given to leading order in terms of the second tangential derivatives of the curvature of the boundary plus the Green function $G_0$. We also have $O(1)$ eigenvalues which are given as eigenvalues of related nonlocal eigenvalue problems in $\mathbb{R}^2_+$. We show that there exists a $\tau_1 > 0$ such that for $0 \leq \tau < \tau_1$, these $O(1)$ eigenvalues lie on the left half of the complex plane, while for $\tau > \tau_1$ at least on of them lies on the right half plane.

It would be desirable to find conditions on the $o(1)$ eigenvalues which are not given in terms of the Green function and its tangential derivatives but rather in terms of the domain $\Omega$. For recent progress in this direction, we refer to [**41**].

There are almost no analytical results in either the weak or the strong coupling case on the dynamics of the full Gierer-Meinhardt system in a two-dimensional domain including the critical growth rate studied in this thesis except for [**11**] where the dynamics of an interior spike in the weak coupling case is studied.

Furthermore, there are no analytical results at all about the existence or stability of $K$-peaked solutions in a three- or higher-dimensional domain.

All these important questions are still open and deserve further attention in the future.

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