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SURVEY ARTICLE

Mathematical Challenges for the Theory of Hyperbolic Balance Laws in Fluid Mechanics: Complexity, Scales, Randomness

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Abstract
Understanding the dynamics of hyperbolic balance laws is of paramount interest in the realm of fluid mechanics. Nevertheless, fundamental questions on the analysis and the numerics for distinctive hyperbolic features related to turbulent flow motion remain vastly open. Recent progress on the mathematical side reveals novel routes to face these concerns. This includes findings about the failure of the entropy principle to ensure uniqueness, the use of structure-preserving concepts in high-order numerical methods, and the advent of tailored probabilistic approaches. Whereas each of these three directions on hyperbolic modelling are of completely different origin they are all linked to small- or subscale features in the solutions which are either enhanced or depleted by the hyperbolic nonlinearity. Thus, any progress in the field might contribute to a deeper understanding of turbulent flow motion on the basis of the continuum-scale mathematical models. We present an overview on the mathematical state-of-the-art in the field and relate it to the scientific work in the DFG Priority Research Programme 2410. As such, the survey is not necessarily targeting at readers with comprehensive knowledge on hyperbolic balance laws but instead aims at a general audience of reseachers which are interested to gain an overview on the field and associated challenges in fluid mechanics.

Keywords Hyperbolic balance laws · Shock wave theory · Structure- and asymptotic-preserving numerics · Stochastic modelling and uncertainty quantification

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1 Introduction

Since its inception in the mid-twentieth century, the mathematical theory of hyperbolic balance laws is evolving as a major subfield of nonlinear partial differential equations (PDEs). It has its origin in the study of the Euler equations which govern the flow of an ideal fluid, i.e., when higher-order effects like viscosity or heat conduction are neglected. In fact, hyperbolic structures became irreplaceable for describing time-dependent transport in all classical fields of physics and turned out to be even indispensable for quantitatively correct predictions about the spatial dynamics in biological and social sciences.

Whereas the models for each single application might appear diverse, they can be expressed as first-order PDE systems in divergence form and subsumed into a general form. For $d \in \mathbb{N}$ spatial dimensions, let fluxes $\mathbf{f}_i = \mathbf{f}_i(\mathbf{u})$, $i = 1, \dots, d$, and a source $\mathbf{b} = \mathbf{b}(\mathbf{u})$ be given. We then search in a time-space domain $(0, T) \times D$ for some vector-valued unknown $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$ that maps to the state space $\mathcal{U}$ and satisfies the first-order system

$$\mathbf{u}_t + \mathbf{f}_1(\mathbf{u})_{x_1} + \cdots + \mathbf{f}_d(\mathbf{u})_{x_d} = \mathbf{b}(\mathbf{u}). \tag{1.1}$$

This system is called a *hyperbolic system of balance laws in* $\mathcal{U}$ if the flux Jacobian, that is the matrix $\sum_{i=1,\dots,d} n_i D\mathbf{f}_i(\mathbf{u})$, is diagonalizable over the set of real numbers for all $\mathbf{n} = (n_1, \dots, n_d)^T \in \mathcal{S}^{d-1}$ and all $\mathbf{u} \in \mathcal{U}$. Hyperbolicity is the necessary solvability condition for initial value problems for (1.1). For $\mathbf{b} \equiv \mathbf{0}$ we obtain a system of *hyperbolic conservation laws*. In Sect. 2 we will see that for fluidmechanical systems the eigenvalues typically depend on the local fluid velocity and the speed of sound as a property of the fluid under consideration.

The most intriguing property which demarcates hyperbolic models from other types of evolutionary PDEs is the possible loss of regularity of solutions in finite time via the on-set of shock waves: namely, when the fields in the balance laws are nonlinear functions of the state variables, smooth wave profiles may get compressed and eventually break, giving rise to discontinuities that propagate and might develop instabilities via a cascade of largely non-smooth flow fields. As a consequence, the mathematical theory has to deal with weak solutions. Actually the set of weak solutions is too big to ensure well-posedness of initial-value problems for (1.1). Therefore, and motivated by the rôle of the second law of thermodynamics in fluid mechanical problems, weak solutions are constrained by an additional entropy condition. For a convex function $\eta = \eta(\mathbf{u})$, the *(mathematical) entropy*, and the *entropy flux* $\mathbf{q} = (q_1(\mathbf{u}), \dots, q_d(\mathbf{u}))^T$, that is supposed to satisfy the compatibility condition

$$(\nabla_{\mathbf{u}} \eta(\mathbf{u}))^T D f_i(\mathbf{u}) = (\nabla_{\mathbf{u}} q_i(\mathbf{u}))^T \quad (i = 1, \dots, d,\ \mathbf{u} \in \mathcal{U}), \tag{1.2}$$

the entropy condition writes as

$$\eta(\mathbf{u})_t + q_1(\mathbf{u})_{x_1} + \cdots + q_d(\mathbf{u})_{x_d} \leq (\nabla_{\mathbf{u}} \eta(\mathbf{u}))^T \mathbf{b}(\mathbf{u}) \text{ in } \mathcal{D}'((0, T) \times D). \tag{1.3}$$

For most hyperbolic systems in the context of fluid mechanics a mathematical entropy is given by the negative of the physical entropy such that (1.3) is just a version of the

second law of thermodynamics. In this paper some caution has to be taken on the use of the expressions energy and entropy. For our guiding example of the barotropic Euler equations (see Sect. 2) the total energy is actually a mathematical entropy such that (1.3) plays at the same time the rôle of an energy estimate.

The entropy approach has been partially successful having led to the weak-strong uniqueness principle of Dafermos & DiPerna [17]. For low-regularity cases, well-posedness is only ensured for scalar equations in multiple space dimensions by the seminal work of Kružkov [47] and by the contributions of Bressan and co-workers for one-dimensional systems [9]. However, for multi-dimensional systems, the global well-posedness analysis is vastly incomplete up to now, cf. [7, 9, 17, 41] for comprehensive accounts of the field.

Currently, the field of hyperbolic balance laws is subject to a massive transformation that started from new findings in mathematical fluid mechanics, in particular for the Euler equations which can be cast for the compressible case in the form (1.1) of a hyperbolic conservation law. Let us outline the three most important changes in the field focussing on the Euler equations. The spectacular findings about the failure of well-posedness of entropy solutions for the Euler system in multiple space dimensions bear completely new ensemble-averaged solution concepts. The age-old problems about the deficiencies of the Euler system for intermediate-scale and asymptotic flow regimes get new drive by unifying multilevel modelling concepts and the associated design of structure-preserving numerics. Deterministic balance laws get replaced by inherently stochastic models that can be accessed thanks to the rapidly evolving progress in stochastic analysis.

These new pathways for hyperbolic modelling look multifaceted and disconnected, but they are linked by strong methodological issues. They are of fundamental relevance to relate statistical theories for small-scale compressible turbulence to the Euler equations (and of course the Navier-Stokes equations in the high-Reynolds number regime).

In 2023 the German Research Foundation (DFG) established the Priority Programme SPP-2410 *Hyperbolic Balance Laws in Fluid Mechanics: Complexity, Scales, Randomness (CoScaRa)*. In more than twenty sub-projects CoScaRa researchers from mathematics, fluid mechanics and theoretical physics in Germany and Switzerland work on research questions related to new trends in hyperbolic modelling for the Euler equations. In this note we aim to give an overview on some of these questions focusing mainly on the analytical and numerical aspects. The highly interesting embedding into the theory of turbulent flows will be discussed in a forthcoming outlet. The same applies for the wide field of stochastic balance laws. We restrict ourselves mostly to the barotropic Euler equations, see (2.3) below. This is a severe simplification from the fluidmechanical point of view but makes the presentation more easy accessible for non-specialists.

In Sect. 2 we set the stage and introduce the Euler equations as an instance of a hyperbolic balance law. The remainder of the paper is then grouped in Sect. 3 on analytical topics and in Sect. 4 on numerical topics. In each of these sections we address several, partly only loosely connected topics of interest. However we want to emphasize the tight connection between the analysis and the numerics in the field of hyperbolic balance laws: each of the topics discussed in the analytical Sect. 3 is mirrored in the numerical Sect. 4.

With our survey paper we aim at an audience which is interested to get a first impression on the state-of-the-art in the field of hyperbolic balance laws in fluid mechanics. In what follows we mostly do not require any expert knowledge in neither hyperbolic balance laws nor mathematical fluid mechanics. However, some general background in the analysis of partial differential equations and the associated numerics is of avail. Moreover, some details might be understandable on first sight only for specialists but the authors hope that numerous remarks help to make the content accessible for a broader communitiy. Whenever we refer to results or methods that cannot be explained due to the limited space we provide precice links to the pertinent literature.

2 Hyperbolic Balance Laws in Fluid Mechanics

Let us consider the motion of a compressible ideal fluid on a macroscopic continuum-mechanical scale. For time $t \in (0, T)$, $T > 0$, and space variable $\mathbf{x} \in D \subseteq \mathbb{R}^d$ the motion is governed by the system of Euler equations, which –for unknowns density $\rho = \rho(t, \mathbf{x})$, velocity $\mathbf{v} = \mathbf{v}(t, \mathbf{x}) = (v_1(t, \mathbf{x}), \dots, v_d(t, \mathbf{x}))^T$ and total energy $E = E(t, \mathbf{x})$– writes as (see e.g. [16])

$$\begin{aligned} \rho_t + \operatorname{div}(\rho \mathbf{v}) &= 0, \\ (\rho \mathbf{v})_t + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p &= \mathbf{0}, \qquad \text{in } (0, \infty) \times D. \\ E_t + \operatorname{div}((E + p)\mathbf{v}) &= 0 \end{aligned} \tag{2.1}$$

Here p denotes the pressure function, which depends as given constitutive relation on the thermodynamical unknowns. Initial and boundary conditions depending on the choice of the domain D will be specified later. For an appropriate thermodynamical setting the Euler equations (2.1) are a system of hyperbolic balance law of type (1.1). If one accounts for the viscous parts of the stress tensor and Fourier's law of heat conduction the first-order system (2.1) turns into the hyperbolic-parabolic Navier-Stokes-Fourier system.

We consider in this paper the barotropic Euler system with $p = p(\rho)$, i.e., the pressure is a function of density alone. In fact, we restrict ourselves to the explicit γ-law pressure function

$$p(\rho) = a\rho^\gamma \tag{2.2}$$

with $\gamma \in (1, 2)$ and $a > 0$. All results in this survey can also be formulated for general monotone-increasing pressure functions p with $p \in C^1([0, \infty)) \cap C^\infty((0, \infty))$, $p(\rho) > 0$ for $\rho > 0$ and $p(0) = 0$. We stick to the γ-law to avoid technicalities, e.g. in connection with the vacuum case $\rho = 0$.

Now, in the barotropic case the equation for the total energy decouples and we are left with the $(d + 1)$-dimensional first-order system

$$\begin{aligned} \rho_t + \operatorname{div}(\rho \mathbf{v}) &= 0, \\ (\rho \mathbf{v})_t + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p(\rho) &= \mathbf{0} \end{aligned} \qquad \text{in } (0, T) \times D. \tag{2.3}$$

For momentum $\mathbf{m} = (m_1, \dots, m_d)^T = (\rho v_1, \dots, \rho v_d)^T$, we define the state space

$$\mathcal{U} := \left\{ \mathbf{u} = (\rho, \mathbf{m}^T = (m_1, \dots, m_d))^T \,\middle|\, \rho > 0,\ \mathbf{m} \in \mathbb{R}^d \right\}.$$

Note that the vacuum state is excluded. We can then recast (2.3) in the form of the conservation law (1.1) with unknown $\mathbf{u} \in \mathcal{U}$ and flux functions given by

$$\mathbf{f}_i(\mathbf{u}) = \begin{pmatrix} m_i \\ \frac{1}{\rho} m_1 m_i + p(\rho)\delta_{i1} \\ \vdots \\ \frac{1}{\rho} m_d m_i + p(\rho)\delta_{id} \end{pmatrix} \qquad (i \in \{1, \dots, d\}). \tag{2.4}$$

In (2.4), we used the Kronecker symbol δ_{ij} with $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$.

For $\mathbf{n} = (n_1, \dots, n_d)^T \in \mathcal{S}^{d-1}$, the eigenvalues of the flux Jacobian $\sum_{i=1,\dots,d} n_i D\mathbf{f}_i(\mathbf{u}) \in \mathbb{R}^{(d+1)\times(d+1)}$ are then readily computed for $\mathbf{u} \in \mathcal{U}$ as

$$\begin{aligned} \lambda_1(\mathbf{u}; \mathbf{n}) = \mathbf{n} \cdot \mathbf{v} - \sqrt{p'(\rho)} < \lambda_2(\mathbf{u}; \mathbf{n}) = \cdots \\ = \lambda_d(\mathbf{u}; \mathbf{n}) = \mathbf{n} \cdot \mathbf{v} < \lambda_{d+1}(\mathbf{u}; \mathbf{n}) = \mathbf{n} \cdot \mathbf{n} + \sqrt{p'(\rho)}. \end{aligned} \tag{2.5}$$

Due to $p'(\rho) > 0$ for $\rho > 0$, we observe that $\lambda_1(\mathbf{u}; \mathbf{n}), \dots, \lambda_{d+1}(\mathbf{u}; \mathbf{n})$ are real numbers and come with a complete set of eigenvectors. As explained in Sect. 1 we observe that the barotropic Euler system (2.3) is hyperbolic in the state space $\mathcal{U}$.

The system (2.3) is equipped with an entropy η and a compatible entropy flux $\mathbf{q}$ (see (1.2)). The entropy is given by the total energy function

$$\eta(\mathbf{u}) = \frac{1}{2\rho} |\mathbf{m}|^2 + W(\rho), \tag{2.6}$$

and the entropy flux is defined as

$$\mathbf{q}(\mathbf{u}) = \frac{\mathbf{m}}{\rho} \big(\eta(\mathbf{u}) + p(\rho) \big). \tag{2.7}$$

The function W in (2.6) is the Helmholtz free energy determined from $\rho W''(\rho) = p'(\rho)$. Since p is monotone increasing we readily see that η is strictly convex in $\mathcal{U}$. In passing we note that the strict convexity already implies the hyperbolicity of (2.3), see e.g. [34, 40]. In the sequel we use the term energy only to indicate statements on estimates for the kinetic and/or the free-energy part of η but not in the context of entropy constraints like (1.3).

The drafted entropy setting provides the grounds to establish the local well-posedness for classical solutions of initial-value problems for (2.3) for sufficiently small end time (and sufficiently restricted initial datum), see [44, 51]. By a classical solution we mean a function $\mathbf{u}$ that is continuous in the closed set $[0, T] \times \bar{D}$ and such that all derivatives in (2.3) exist as continuous functions.

In general, we cannot expect to have classical solutions due to the occurrence of singularities like shock or contact waves. As sketched in Sect. 1 we require a weak solution concept that allows to integrate the entropy constraint (1.3). In the following

we only consider the cases $D = \mathbb{R}^d$ or $D = \mathbb{T}^d$. For the barotropic Euler equations (2.3) with a γ-law pressure this leads to

Definition 2.1 (Weak and entropy solutions) Let the initial function $\mathbf{u}_0 = (\rho_0, \mathbf{m}_0^T)^T : D \to \mathcal{U}$ with $\rho_0 \geq 0$ a.a. in Ω and $\eta(\mathbf{u}_0) \in L^1(D)$ be given. A function $\mathbf{u} = (\rho, \mathbf{m}^T)^T : [0,T] \times D \to \mathcal{U}$ with $\rho \in C_{\text{weak}}(0,T; L^\gamma(D))$, $\rho \geq 0$ a.e. in Ω, $\mathbf{m} \in (C_{\text{weak}}(0,T; L^{\frac{\gamma}{2\gamma+1}}(D)))^d$ and $\eta(\mathbf{u}) \in L^\infty(0,T; L^1(D))$ is called a *weak solution of the initial-value problem for* (2.3) if

$$\begin{aligned} \int_0^T \int_D \rho\varphi_t + \mathbf{m} \cdot \nabla\varphi \, d\mathbf{x}dt &= -\int_D \rho_0\varphi(0,\mathbf{x}) \, d\mathbf{x}, \\ \int_0^T \int_D \mathbf{m} \cdot \boldsymbol{\varphi}_t + \frac{1}{\rho}\mathbf{m} \otimes \mathbf{m} : \nabla\boldsymbol{\varphi} + p(\rho)\nabla \cdot \boldsymbol{\varphi} \, d\mathbf{x}dt &= -\int_D \mathbf{m}_0 \cdot \boldsymbol{\varphi}(0,\mathbf{x}) \, d\mathbf{x} \end{aligned} \tag{2.8}$$

is satisfied for all test functions $\varphi \in C_0^1([0,T) \times D)$, and $\boldsymbol{\varphi} \in (C_0^1([0,T) \times D))^d$.

A weak solution is called *entropy solution* if in addition

$$\int_0^T \int_D \eta(\mathbf{u})\psi_t + \mathbf{q}(\mathbf{u}) \cdot \nabla\psi \, d\mathbf{x}dt \geq -\int_D \eta(\mathbf{u}_0)\psi(0,\mathbf{x}) \, d\mathbf{x} \tag{2.9}$$

holds for all test functions $\psi \in C_0^1([0,T) \times D)$ with $\psi \geq 0$.

Remark 2.2 (i) In Definition 2.1 we denote for $p \in [1,\infty) \cup \infty$, $m \in \mathbb{N}$, and some set $E \subseteq \mathbb{R}^m$ by $L^p(E)$ the usual Lebesque spaces. The space $C_0^1(E)$ denotes the space of differentiable functions which have compact support in E. The space $C_{\text{weak}}([0,T); L^p(D))$ is defined by

$$C_{\text{weak}}([0,T); L^p(D)) = \left\{ f \in C([s,t]; L^p(D)) \,\middle|\, 0 \leq s \leq t < T, \int_D fg \, \mathrm{d}\mathbf{x} \in C([0,T)) \,\forall\, g \in L^{p'}(D) \right\}.$$

Here p' is conjugate to p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$.

(ii) The required integral bounds on the initial datum and the weak solutions in Definition 2.1 are sufficient to ensure the existence of all terms in (2.8). In fact, also the weaker conditions $\rho \in L^\infty(0,T; L^\gamma(D))$ and $\mathbf{m} \in L^\infty(0,T; L^{\frac{2\gamma}{\gamma+1}}(D)))^d$ would be enough but the usage of spaces with some continuity in time provides the right setting for (the proof of) Theorem 3.10 and other related results.

(iii) The condition (2.9) corresponds to the general condition (1.3). For (2.9) we have tacitly assumed the existence of the integrals. Note however that the listed assumptions do not guarantee the existence of the integral involving the entropy flux $\mathbf{q}$ which contains a cubic $|\mathbf{m}|^3$-term. Imposing weak time-continuity properties of $\mathbf{u}$ one can deduce for $0 \leq s \leq t < T$ the global energy estimate

$$\int_D \eta(\mathbf{u}(t,\mathbf{x})) \, d\mathbf{x} \leq \int_D \eta(\mathbf{u}(s,\mathbf{x})) \, d\mathbf{x} \leq \int_D \eta(\mathbf{u}_0(\mathbf{x})) \, d\mathbf{x} < \infty \tag{2.10}$$

for all $t \in [0, T)$. Such global conditions can substitute (2.9) to achieve certain analytical results on weak solutions, see e.g. [27].

Moreover, the global energy estimate induces, by the convexity of η, that the density is nonnegative (but not positive!) and that momentum vanishes for vacuum states. Thus, in general we have $\rho \geq 0$ a.e. in Ω, which implies that if $\rho = 0$ then $\mathbf{m} = 0$.

Except for the case $d = 1$ almost nothing is known for the existence of weak or entropy solutions in the sense of Definition 2.1. The recent falsification of the well-posedness hypothesis of entropy solutions led to various new notions of solutions which are spotlighted below.

3 New Directions in the Analysis for Hyperbolic Balance Laws

In the sequel we discuss a number of current trends in the analysis of hyperbolic balance laws that are in one or the other way related to the overall goal of a well-posedness theory for (2.3) in multiple space dimensions. With the choice of topics we try to cover most of the analysis research directions in CoScaRa. The subsequent Sect. 4 is devoted to corresponding trends in the numerics. If not explicitly stated otherwise we will always assume that $d > 1$ holds.

3.1 The Weak-Strong Uniqueness Principle and Relative Entropies

To introduce the seminal weak-strong uniqueness result in Theorem 3.1 below we require the notion of a relative entropy for the barotropic Euler system (2.3). Following the formulation for general conservation laws from [17] it is given for two functions $\mathbf{u} = (\rho, \mathbf{m}^T)^T, \bar{\mathbf{u}} = (\bar{\rho}, \bar{\mathbf{m}}^T)^T : [0, T] \times D \to \mathcal{U}$ by

$$E[\mathbf{u}|\bar{\mathbf{u}}] := \eta(\mathbf{u}) - \eta(\bar{\mathbf{u}}) - \nabla_{\mathbf{u}}\eta(\bar{\mathbf{u}}) \cdot (\mathbf{u} - \bar{\mathbf{u}}). \tag{3.1}$$

If $\mathbf{u}, \bar{\mathbf{u}}$ take values in a compact and convex subset of $\mathcal{U}$ it is straightforward to exploit the convexity of η to show that the relative entropy $E[\mathbf{u}|\bar{\mathbf{u}}]$ is bounded from below and above by multiples of the term $|\mathbf{u} - \bar{\mathbf{u}}|^2$. Thus, the relative entropy $E[\mathbf{u}|\bar{\mathbf{u}}]$ vanishes if and only if $\mathbf{u} = \bar{\mathbf{u}}$ holds. For the barotropic Euler system (2.3) we deduce by (2.6) after a short calculation the formula

$$E[\mathbf{u}|\bar{\mathbf{u}}] = \frac{1}{2}\Big(\frac{|\mathbf{m}|^2}{\rho} - 2\mathbf{m}\frac{\bar{\mathbf{m}}}{\bar{\rho}} + \rho\frac{|\bar{\mathbf{m}}|^2}{\bar{\rho}^2}\Big) + W(\rho) - W(\bar{\rho}) - W'(\bar{\rho})(\rho - \bar{\rho}).$$

Now, the idea of the weak-strong uniqueness principle is to choose an entropy solution $\mathbf{u}$ and a classical solution $\bar{\mathbf{u}}$ on the minimum of the respective life spans. We state a typical comparison result as it has been pioneered by Dafermos & DiPerna, see [17].

Theorem 3.1 *Let $\mathcal{K} \subset \mathcal{U}$ be a compact and convex set. Consider for $D = \mathbb{T}^d$ initial functions $\bar{\mathbf{u}}_0 \in (C^1(D))^{d+1}$ and $\mathbf{u}_0 \in (L^\infty(D))^{d+1}$ with $\bar{\mathbf{u}}_0(\mathbf{x})$, $\mathbf{u}_0(\mathbf{x}) \in \mathcal{K}$ a.e. in D.*

Furthermore assume that a classical solution $\bar{\mathbf{u}} \in (C^1([0,T] \times D))^{d+1}$ *of the initial-value problem for* (2.3) *exists and satisfies* $\bar{\mathbf{u}}(t,\mathbf{x}) \in \mathcal{K}$ *for* $(t,\mathbf{x}) \in [0,T] \times D$.

Then, we have for an entropy solution $\mathbf{u}$ *of* (2.3), *which satisfies* $\mathbf{u}(t,\mathbf{x}) \in \mathcal{K}$ *a.e. in* $(0,T) \times D$, *the following statements.*

(i) *There is a constant* $C = C\big(T, \|\bar{\mathbf{u}}\|_{W^{1,\infty}((0,T)\times D)}\big) \geq 0$ *such that*

$$\int_D E[\mathbf{u}|\bar{\mathbf{u}}](t,\mathbf{x})\,d\mathbf{x} - \int_D E[\mathbf{u}_0|\bar{\mathbf{u}}_0](\mathbf{x})\,d\mathbf{x} \leq C \int_0^t \int_D E[\mathbf{u}|\bar{\mathbf{u}}](s,\mathbf{x})\,d\mathbf{x}ds \quad (3.2)$$

holds for almost all $t \in (0,T)$.

(ii) *For* $\mathbf{u}_0 = \bar{\mathbf{u}}_0$ *in* D *it holds*

$$\mathbf{u} = \bar{\mathbf{u}} \text{ a.e. in } (0,T) \times D. \quad (3.3)$$

Remark 3.2 The property (3.3) is referred to as the *weak-strong uniqueness principle*. Our formulation in Theorem 3.1 can be generalized to other spatial domains and considerably sharpened for the fluidmechanical equations. For more details we refer to [27, 63]. The classical and the entropy solution in Theorem 3.1 take values in the compact set $\mathcal{K} \subset \mathcal{U}$. This implies in particular that vacuum states with $\rho \equiv 0$ are excluded.

The result (3.2) and the definition of relative entropies indicate that the L^2-difference of two solutions to (2.3) or more general hyperbolic balance laws like relaxation systems is dissipated as time tends to infinity. More precise quantitative statements on asymptotic stability can be obtained combining the relative entropy analysis and structural properties of the flux Jacobians like the Shizuta-Kawashima condition, see e.g. [45]. In the framework of CoScaRa we refer to the recent works [32] for relativistic flows.

Even beyond purely analytical investigations relative entropies turned out to be useful for applications in the realm of numerics, see Sect. 4.1. The stability property (3.2) is also the starting point for various further investigations, in particular to analyze singular limits.

3.2 Asymptotic Regimes and Singular Limits

A particular challenge for the mathematical analysis is the investigation of asymptotic flow regimes. Examples include the passage from the compressible regime to the incompressible flow regime if the Mach number tends to zero, the high-Reynolds number limit when viscous effects diminish and the Navier-Stokes equations pass over to the Euler equations, or the high-Knudsen number regime when the validity of the continuum assumption breaks down. Mathematically speaking all these limit passages come with a change of type of the underlying PDE models which hinders to apply well-established tools that typically work only within one class of evolution equations. In turn, the relative entropy approach turns out be a very flexible to deal with different types of fluidmechanical systems if they come with a proper entropy structure.

We exemplify the usage of the relative entropy approach on the high-friction limit for a (barotropic) Euler-Darcy system that leads to a parabolic limit model. The numerical counterpart is presented in Sect. 4.2. For a small parameter $\varepsilon > 0$ our approximate Euler-Darcy system is given by

$$\begin{aligned} \rho_t^\varepsilon + \operatorname{div}(\mathbf{m}^\varepsilon) &= 0, \\ \mathbf{m}_t^\varepsilon + \operatorname{div}\left(\frac{1}{\rho^\varepsilon}\mathbf{m}^\varepsilon \otimes \mathbf{m}^\varepsilon\right) + \frac{1}{\varepsilon^2}\nabla p(\rho^\varepsilon) &= -\frac{1}{\varepsilon^2} M \mathbf{m}^\varepsilon \end{aligned} \qquad \text{in } (0,T)\times D. \tag{3.4}$$

Here $M > 0$ is the so-called mobility and the system (3.4) can be used to describe the compressible motion of a gas in a homogeneous porous medium with low permeability. Obvious scaling arguments suggest that a limit density $\bar{\rho}$ satisfies the parabolic porous-medium equation

$$\bar{\rho}_t - \frac{1}{M}\operatorname{div}\left(p'(\bar{\rho})\nabla\bar{\rho}\right) = 0 \qquad \text{in } (0,T)\times D. \tag{3.5}$$

Note that $p' > 0$ holds on the state space. Let us formally rewrite (3.5) into the equivalent pseudo first-order model

$$\bar{\rho}_t + \operatorname{div}(\bar{\mathbf{m}}) = 0, \quad \nabla p(\bar{\rho}) = -\bar{\mathbf{m}} \text{ in } (0,T)\times D. \tag{3.6}$$

In the sequel we consider the initial-value problems for (3.4) and (3.6) for prepared smooth initial data $\mathbf{u}_0 = \bar{\mathbf{u}}_0 = \left(\rho_0, -M^{-1}\nabla p(\rho_0)^T\right)^T$. In view of (2.6) and (2.10) it is then straightforward to verify for $D = \mathbb{T}^d$, that an entropy solution $\mathbf{u}^\varepsilon$ of the initial-value problem for (3.4) and a classical solution $\bar{\mathbf{u}} = \left(\bar{\rho}, \bar{\mathbf{m}}^T := -M^{-1}\nabla p(\bar{\rho})^T\right)^T$ of the initial-value problem for (3.6) obey for almost all $t \in (0,T)$ the global energy estimates

$$\begin{aligned} \int_D \eta(\rho^\varepsilon, \varepsilon^2\mathbf{m}^\varepsilon)\,d\mathbf{x} + M\int_0^t\int_D \frac{|\mathbf{m}^\varepsilon|^2}{\rho}\,d\mathbf{x}ds &\le \int_D \eta(\rho_0, \varepsilon^2\mathbf{m}^\varepsilon)\,d\mathbf{x}, \\ \int_D \eta(\bar{\rho}, \mathbf{0})\,d\mathbf{x} + \frac{1}{M}\int_0^t\int_D \bar{\rho}|\nabla W'(\bar{\rho})|^2\,d\mathbf{x}ds &\le \int_D \eta(\rho_0, \mathbf{0})\,d\mathbf{x}. \end{aligned} \tag{3.7}$$

This structure motivates to define for (3.4) and (3.6) the relative entropy $E[\mathbf{u}^\varepsilon|\bar{\mathbf{u}}]$, exactly as in (3.1). In [49, 57] the following result has been shown.

Theorem 3.3 *Let $\mathcal{K} \subset \mathcal{U}$ be a compact and convex set. Consider prepared initial data $\mathbf{u}_0 \in (C^3(D))^{d+1}$ with $\mathbf{u}_0(\mathbf{x}) \in \mathcal{K}$ for $\mathbf{x} \in D$.*

Furthermore assume that an entropy solution $\mathbf{u}^\varepsilon$ of the initial-value problem for (3.4) and a classical solution $\bar{\mathbf{u}}$ of (3.6) with $\bar{\rho} \in C^3([0,T]\times D)$ exist. They are supposed to satisfy $\mathbf{u}^\varepsilon(t,\mathbf{x}), \bar{\mathbf{u}}(t,\mathbf{x}) \in \mathcal{K}$ for almost all $(t,\mathbf{x}) \in (0,T)\times D$.

Then, there is a constant $C = C\left(T, \mathcal{K}, \bar{\rho}\right) \ge 0$ such that

$$\int_D E[\mathbf{u}^\varepsilon|\bar{\mathbf{u}}](t,\mathbf{x})\,d\mathbf{x} \le C\varepsilon^4 \tag{3.8}$$

holds for almost all $t \in (0,T)$.

Remark 3.4 (i) By the equivalence of the relative entropy with the quadratic difference between $\mathbf{u}^\varepsilon$ and $\bar{\mathbf{u}}$ we obtain the L^2-convergence of $\mathbf{u}^\varepsilon$ to $\bar{\mathbf{u}}$ in the case of well-prepared data. Note that the assumption of well-prepared data can be relaxed but simplifies the formulation of (3.8).

(ii) In Theorem 3.3, the solution $\bar{\mathbf{u}}$ of the limit problem takes the rôle of the smooth solution in the weak-strong uniqueness principle in Theorem (3.3). This is natural since we can expect solutions of the parabolic equation (3.5) to be more regular. Note that the first inequality in (3.7) provides an ε-independent a-priori estimate on the kinetic energy which stems from the friction term whereas the kinetic energy contribution in η depends on ε. The uniform estimate is crucial to derive the convergence statement.

The application of relative entropies to handle singular limits is a very active field. CoScaRa scientists apply the technique for singular limits related to the Navier-Stokes-Korteweg system, artificial-compressibility approximations and relaxation systems. Let us cite [61] among many others.

3.3 Entropy Dissipation and the Onsager Conjecture

In Sects. 3.1, 3.2 we used entropy dissipation properties of entropy solutions mostly in a qualitative way, e.g. to obtain a-priori estimates. Determining or even controlling the exact entropy dissipation rate for hyperbolic conservation laws is a much more intricate issue that is of outstanding importance in fluid mechanics. We highlight its relevance in the context of the famous Onsager relation. The original version of Onsager's conjecture refers to the incompressible Euler/Navier-Stokes system and is related to the dissipative anomaly for turbulent flows [55]. In the modern mathematical language the first part of the conjecture states that a weak solution of the Euler system will conserve the kinetic energy as long as the weak solution is regular enough, precisely as long as it is Hölder-continuous in space with Hölder exponent $1/3$. Solutions which are less regular than those from this critical function space will then actually dissipate kinetic energy. In the incompressible realm the complete Onsager conjecture has been rigorously confirmed, see e.g. [10, 15].

Among others, the fact that the entropy η from (2.6) allows the exchange of kinetic and free energy fractions, the verification of Onsager-like statements for compressible flow is considerably more difficult. For the barotropic equations (2.3) we present a result on entropy conservation and associated regularity properties from [26]. We choose $D = \mathbb{T}^d$. The rôle of the critical function space is now taken for some $\alpha \geq 0$ and $p \in [1, \infty) \cup \{\infty\}$ by the Besov space $B_p^{\alpha,\infty}((0,T) \times D)$, i.e., those functions f on the time-space domain $(0,T) \times D$ such that

$$\begin{aligned} &\|f\|_{B_3^{\alpha,\infty}((0,T)\times D)} \\ &\quad := \|f\|_{L^p((0,T)\times D)} \\ &\qquad + \sup_{\xi := (t,\mathbf{x}) \in (0,T)\times D} \frac{\|f(\cdot+\xi) - f\|_{L^p((0,T)\times D)\setminus\{\tilde{\xi}-\xi \,|\, \tilde{\xi}\in(0,T)\times D\}}}{|\xi|^\alpha} < \infty \end{aligned}$$

holds.

Theorem 3.5 *For some compact set $\mathcal{K} \subset \mathcal{U}$ let $\mathbf{u} : [0, T] \times D \to \mathcal{U}$ with $\mathbf{u}(t, \mathbf{x}) \in \mathcal{K}$ for almost all $(t, \mathbf{x}) \in D$ be a weak solution of the initial-value problem for (2.3).*

If $\mathbf{u}$ satisfies for $\alpha, \beta \in [0, 1]$ with

$$\beta > \max\left\{1 - 2\alpha, (1 - \alpha)/2\right\} \tag{3.9}$$

the regularity property

$$\begin{aligned} \mathbf{v} \in \left(B_3^{\alpha,\infty}((0, T) \times D)\right)^d,\ \rho \in B_3^{\beta,\infty}((0, T) \times D), \\ \mathbf{m} \in \left(B_3^{\beta,\infty}((0, T) \times D)\right)^d, \end{aligned} \tag{3.10}$$

the condition (2.10) holds with equality.

Remark 3.6 (i) The property $\mathbf{u}(t, \mathbf{x}) \in \mathcal{K}$ ensures that the density component of the weak solutions is bounded away from zero, and thus $\mathbf{v} = \frac{\mathbf{m}}{\rho}$ is well-defined. It is possible to prove Theorem 3.5 even without excluding vacuum states [5].

(ii) Let us consider an entropy solution of (2.3) that is of bounded total variation and includes shock waves. Then the entropy constraint (2.10) holds with strict sign. Notably the space of bounded functions that have bounded total variation embeds into the Besov space $B_3^{\alpha,\infty}((0, T) \times D)$ with $\alpha = 1/3$. We recover condition (3.9) with equality. In this critical case the entropy conservation statement of Theorem 3.5 fails.

Current work in CoScaRa is devoted to a complete proof (or falsification) of Onsager's conjecture even for the full Euler system (2.1) or closely related hyperbolic transport systems, e.g. [1]. This contributes to an upcoming regularity theory for inviscid flow. Particular challenges are the characterization of wild solutions (cf. Sect. 3.4) with respect to the entropy dissipation rate, and the design of numerical schemes that obey the dissipation rates on the discrete level (cf. Sect. 4.3). Such questions rise again for the probabilistic solution concepts in Sect. 3.5.

3.4 Non-uniqueness of Entropy Solutions and Convex Integration

For a long time entropy-based solution concepts like in Definition 2.1 have been taken for granted to ensure the global well-posedness for hyperbolic equations. In fact, this belief turned out to be true for one space dimension [9]. Direct numerical simulations have led to heavy doubts about a sole focus to entropy constraints. The results for several high-resolution schemes indicate non-convergence in the presence of shear flow. Mesh refinement bears the development of ever finer structures contradicting strong convergence to a unique limit (see [39]). In Fig. 1 we display the density field for a forward-facing step geometry which displays the development of such vortex structures. All these numerical observations have been confirmed by the groundbreaking discovery of intriguingly wild non-uniqueness cases within the class of entropy solutions rendering this corner stone of the hyperbolic theory to be insufficient for multiple space dimensions, see e.g. [13, 18].

To illustrate the striking failure of the entropy criterium we present the following result from [38] for the periodic case $D = \mathbb{T}^3$.

Fig. 1 Kelvin-Helmholtz instability in numerical simulation of a forward-facing step problem [31]

Theorem 3.7 *For any* $\delta \in [0, \frac{1}{7})$ *and any initial density* $\rho_0 \in C^\infty(D)$ *mapping to some compact interval in* $(0, \infty)$, *there exists an initial momentum* $\mathbf{m}_0 \in (C^\delta(D))^3$ *such that there are infinitely many distinct entropy solutions* $\mathbf{u} = (\rho, \mathbf{m}^T)^T \in C^\infty([0, T] \times D) \times \left(C^\delta([0, T] \times D)\right)^3$ *of the barotropic Euler equations* (2.3) *that satisfy the entropy inequality* (2.6) *with a strict sign.*

We note that the striking Theorem 3.7 does not invoke discontinuous shock wave solutions but detects a non-uniqueness within (time-space) Hölder continuous entropy solutions. With other words, Theorem 3.7 shows that strict entropy decay for compressible flow is not necessarily connected to the occurence of shock solutions. It is then tempting to connect such wild solutions with the onset of turbulent structures complementing the dissipative anomaly as formulated with Onsager's conjecture.

The proof of Theorem 3.7 relies on Gromov's theory of convex integration. Starting with the pioneering works of De Lellis & Székelyhidi [18, 19], this technique has been widely used to construct wild entropy solutions. For instance, within CoScaRa a focus is on the construction of multiple shock wave solutions starting from Riemann initial data, see [52].

3.5 Probabilistic Solution Concepts

The Onsager conjecture aims to describe turbulent flow features by the investigation of the dynamics of a reduced quantity of interest: the scalar entropy η. The Euler equations display vast ill-posedness in the framework of weak entropy solutions. The set of entropy solutions can contain wildly oscillating functions creating finer and finer temporal and spatial scales. Both of these observations, motivate to advance purely deterministic solution concepts for single solution trajectories to probabilistic notions and associated ensemble-averaged quantities or momenta of probability distributions.

For a probabilistic framework, two directions are emerging for compressible flow. Following the survey [36], there are on the one hand measure-valued solutions that describe the solution behaviour of (2.3) by a space-time probability measure that can be interpreted as a one-point statistics, see e.g. [27]. Then, there are statistical solutions that contain spatial or temporal correlation information and thus represent a much stronger notion of solution than the measure-valued concept, see e.g. [30]. Reduced descriptions that contain multi-point statistics can be described by kinetic hierarchies for probability density functions that refer to selected quantities of interest, see e.g. [58].

To spotlight measure-valued solutions we need the notion of a Young measure. A Young measure $\nu^{\mathbf{u}}$ is a weak-$*$ measurable time-space parameterized family of

probability measures $\nu^{\mathbf{u}} = \{\nu^{\mathbf{u}}_{(t,\mathbf{x})}\}_{(t,\mathbf{x})\in(0,T)\times D}$, i.e., the map

$$(t,\mathbf{x}) \mapsto \left\langle \nu^{\mathbf{u}}_{(t,\mathbf{x})}, f(\mathbf{u})\right\rangle := \int_{\mathcal{U}} f(\mathbf{u})\, d\nu^{\mathbf{u}}_{(t,\mathbf{x})}$$

is measurable for all $f \in C^0(\mathcal{U})$. The theorem of Banach-Alaoglu ensures that each subsequence of a uniformly L^p-bounded, $1 < p \le \infty$, and thus weak-$\star$ convergent sequence of functions $\{\mathbf{u}_n\}_{n\in\mathbb{N}}$ on $(0,T)\times D$ generates a Young measure $\nu^{\mathbf{u}}$. It is then fundamental for our application to the nonlinear system (2.3) to note that the generated Young measure is compatible with weak convergence under nonlinear functions: For any $f \in C^0(\mathcal{U})$ we have

$$\begin{aligned}&\lim_{n\to\infty}\int_0^T\int_D f(\mathbf{u}_n)\varphi(t,\mathbf{x})\,d\mathbf{x}dt\\ &\quad=\int_0^T\int_D \left\langle \nu^{\mathbf{u}}_{(t,\mathbf{x})}, f(\mathbf{u})\right\rangle\varphi(t,\mathbf{x})\,d\mathbf{x}dt\ \forall\,\varphi\in L^\infty((0,T)\times D).\end{aligned}$$

We can now present the definition of a special type of measure-valued solutions with either $D=\mathbb{R}^d$ or $D=\mathbb{T}^d$.

Definition 3.8 (Dissipative measure-valued solution) Let the initial function $\mathbf{u}_0 : D \to \mathcal{U}$ with $\mathbf{u}_0 \in (L^1(D))^{d+1}$, $\rho_0 \ge 0$ a.e. in Ω and $\eta(\mathbf{u}_0) \in L^1(D)$ be given.

A Young measure $\nu^{\mathbf{u}}_{(t,\mathbf{x})}$ with $\langle \nu^{\mathbf{u}}_{(t,\mathbf{x})}, \rho\rangle \in C_{\text{weak}}\big(0,T;L^\gamma(D)\big)$, $\langle \nu^{\mathbf{u}}_{(t,\mathbf{x})}, \mathbf{m}\rangle \in \big(C_{\text{weak}}(0,T;(L^{\frac{2\gamma}{\gamma+1}}(D))\big)^d$ and $\langle \nu^{\mathbf{u}}_{(t,\mathbf{x})}, \eta(\mathbf{u})\rangle \in L^\infty\big(0,T;L^1(D)\big)$ is called a *dissipative measure-valued solution of the initial value problem for* (2.3) if we have

$$\int_0^T\int_D \left\langle \nu^{\mathbf{u}}_{(t,\mathbf{x})}, \rho\right\rangle\varphi_t + \left\langle \nu^{\mathbf{u}}_{(t,\mathbf{x})}, \mathbf{m}\right\rangle\cdot\nabla\varphi\,d\mathbf{x}dt = -\int_D \rho_0\varphi(0,\mathbf{x})\,d\mathbf{x}$$

for all test functions $\varphi \in C_0^1([0,T)\times D)$,

and if there is a measure (called Reynolds defect) $\mathcal{R} \in L^\infty(0,T;\mathcal{M}^+(D,\mathbb{R}^{d\times d}))$ such that

$$\begin{aligned}&\int_0^T\int_D \left\langle \nu^{\mathbf{u}}_{(t,\mathbf{x})}, \mathbf{m}\right\rangle\cdot\boldsymbol{\varphi}_t + \left\langle \nu^{\mathbf{u}}_{(t,\mathbf{x})}, \frac{1}{\rho}\mathbf{m}\otimes\mathbf{m}\right\rangle : \nabla\boldsymbol{\varphi} + \left\langle \nu^{\mathbf{u}}_{(t,\mathbf{x})}, p(\rho)\right\rangle\nabla\cdot\boldsymbol{\varphi}\,d\mathbf{x}dt\\ &\quad=-\int_0^T\int_D \nabla\boldsymbol{\varphi} : d\mathcal{R}(t)dt - \int_D \mathbf{m}_0\cdot\boldsymbol{\varphi}(0,\mathbf{x})\,d\mathbf{x}\end{aligned}$$

holds for all $\boldsymbol{\varphi} \in (C_0^1([0,T)\times D))^d$,

and if there is a nonnegative dissipation function $\mathcal{D} \in L^\infty(0,T)$ with $|\mathrm{Tr}(\mathcal{R}(t))(D)| \le c\mathcal{D}(t)$ for some $c \ge 0$ and

$$\int_D \left\langle \nu^{\mathbf{u}}_{(t,\mathbf{x})}, \eta(\mathbf{u})\right\rangle d\mathbf{x} \le -\mathcal{D}(t) + \int_D \eta(\mathbf{u}_0)\psi(0,\mathbf{x})\,d\mathbf{x} \tag{3.11}$$

for almost all $t \in (0,T)$. Here Tr denotes the trace operation for a matrix.

Remark 3.9 (i) If the measure $\mathcal{R}$ in Definition 3.8 vanishes and moreover the Young measure $\nu^{\mathbf{u}}_{(t,\mathbf{x})}$ turns out to be a Dirac measure we recover the notion of a weak solution from Definition 2.1.

(ii) As compared to the definition of weak solutions the Reynolds defect measure $\mathcal{R}$ looks strange on first glance. Dissipative measure-valued solutions can be typically obtained by analyzing only weak convergence of approximate solutions (e.g. obtained from the vanishing viscosity limit in the Navier-Stokes system or numerical methods that obey a discrete version of the entropy condition). In the limit process one exploits the noted compatibility of weak-$\star$ convergence with the flux nonlinearities. The Reynolds defect can then be separated from the limit expression as a tensor-valued measure resulting from possible oscillations or concentrations in the approximate sequence. Accordingly, the Reynolds stress vanishes as soon as the approximations converge strongly. It is an interesting interpretation that the Reynolds defect captures the effect of turbulent structures, i.e., wild oscillations, in the solutions of the approximate Navier-Stokes system.

(iii) Note that the entropy condition in Definition 3.8 is only formulated in a global way without specifying local flux balances. In any case it ensures that it is almost sure that either the density is positive or otherwise the momentum vanishes simultaneously.

The most clear-cut advantage of the notion of dissipative measure-valued solutions is that for every initial data a dissipative measure-valued solution is known to exist globally. Recall that this remains a completely open question for weak solutions.

Theorem 3.10 *Let the initial function* $\mathbf{u}_0 : D \to \mathcal{U}$ *with* $\rho_0 \geq 0$ *a.e. in* Ω *and* $\eta(\mathbf{u}_0) \in L^1(D)$ *be given.*

Then there exists a dissipative measure-valued solution for the initial-value problem for the barotropic Euler equations (2.3).

Proofs can be found in [27, 36]. As mentioned in Remark 3.9 the result is proven by analyzing the vanishing-viscosity limit for the Navier-Stokes equations. In this case precise characterizations of the Reynolds defect and the dissipation function can be given.

For dissipative measure-valued solutions one can also verify a weak-strong uniqueness result. Of course one cannot expect any more uniqueness results by enlarging the space of solutions as we have done with the notion of dissipative measure-valued solutions. Moreover, a practical computation of Young measures is not very feasible. In e.g. [27] a concept of dissipative solutions is introduced which leads to solution functions which can be interpreted as expected value of the underlying Young measure and makes no reference to a specific approximating sequence. We will discuss these concepts in the context of numerical methods in Sect. 4.4.

The analysis and numerics for probabilistic concepts for the Euler equations are an important part of the scientific work in CoScaRa. The projects strive to extend the concepts to other compressible flow descriptions and to exploit possible relations between all the up-coming formulations. This includes also approaches to track single solution trajectories by time-accurate fine control of the Reynolds defect and the entropy dissipation via variational techniques, see [12, 24].

3.6 Random Perturbations of Hyperbolic Balance Laws

The study of the influence of parametric noise on the solutions of evolutionary PDEs has become a discipline its own in the last decade, in particular what concerns the numerical aspects of uncertainty quantification. The uncertainty quantification for hyperbolic balance laws with parametric uncertainties faces distinctive difficulties which root already in the analysis of the random systems: propagating shock waves induce low-regularity solutions in physical and stochastic space. We outline the formulation of random Euler equations and present the seminal stochastic-Galerkin method in the sequel.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where Ω is the set of all elementary events $\omega \in \Omega$, $\mathcal{F}$ is a σ-algebra on Ω and $\mathbb{P}$ is a probability measure. We restrict ourselves to scalar absolutely-continuous random variables $\xi : \Omega \to \mathbb{R}$ with density function p_ξ and choose $D = \mathbb{T}^d$.

Then, we focus on the initial-value problem for (2.3) with random initial data, i.e.,

$$\mathbf{u}(0, \mathbf{x}, \xi(\omega)) = \mathbf{u}_0(\mathbf{x}, \xi(\omega)) \quad ((\mathbf{x}, \omega) \in D \times \Omega). \tag{3.12}$$

We can easily adapt the definitions for classical solutions and weak solutions to the random case. For instance, a *random weak solution of* (2.3), (3.12) is a random field $\mathbf{u} \in L^2\big(\Omega; (L^\infty(0, T; L^1(D))^{d+1}\big)$ such that (2.8) holds $\mathbb{P}$-a.s. $\omega \in \Omega$ for all test functions $\varphi \in C_0^\infty([0, T) \times D)$ and $\boldsymbol{\varphi} \in (C_0^\infty([0, T) \times D))^d$. The random entropy solution is defined in an analogous way. The existence of random entropy solutions is discussed in [3] for hyperbolic conservation laws but like in the deterministic case almost nothing is known for (2.3), (3.12).

The polynomial chaos expansion is a generic way to derive deterministic models for the stochastic modes of random fields. We showcase the stochastic-Galerkin method as one instance for such fully intrusive approaches. We expand $\mathbf{u}$ into a generalized Fourier series using an orthonormal polynomial basis. Let $\{\Psi_i\}_{i \in \mathbb{N}_0} : \Omega \to \mathbb{R}$ be a $L^2(\Omega)$-orthonormal basis with respect to the density function p_ξ, i.e., for $i, j \in \mathbb{N}_0$ we have

$$\langle \Psi_i, \Psi_j \rangle := \int_{\mathbb{R}} \Psi_i(z) \Psi_j(z) p_\xi(z)\, dz = \delta_{ij}. \tag{3.13}$$

The random entropy solution $\mathbf{u}$ of (2.3), (3.12) can be expressed as the Fourier-like series

$$\mathbf{u}(t, \mathbf{x}, \xi(\omega)) = \sum_{n=0}^{\infty} \mathbf{u}_n(t, \mathbf{x}) \Psi_n(\xi(\omega)). \tag{3.14}$$

From the deterministic Fourier modes we can immediately extract the expectation and variance of components of $\mathbf{u}$. We truncate (3.14) and define for $N \in \mathbb{N}$

$$\mathbf{u}^N(t, \mathbf{x}, \xi(\omega)) = \sum_{n=0}^{N} \mathbf{u}_n^N(t, \mathbf{x}) \Psi_n(\xi(\omega)). \tag{3.15}$$

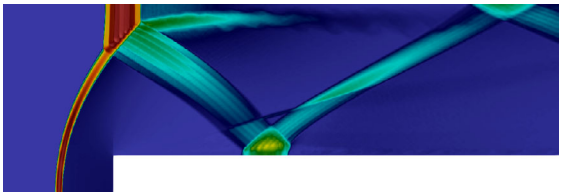

Fig. 2 Variance for forward-facing step with random inflow [31]

Starting from the classical form (2.3) with fluxes from (2.4) and using (3.13) we arrive at the purely deterministic *stochastic-Galerkin system* for the Fourier modes given by

$$(\mathbf{u}_n^N)_t + \left\langle \mathbf{f}_1(\mathbf{u}^N(\cdot,\xi(\omega)),\Psi_n(\xi(\omega))\right\rangle_{x_1} + \cdots + \left\langle \mathbf{f}_d(\mathbf{u}^N(\cdot,\xi(\omega)),\Psi_n(\xi(\omega))\right\rangle_{x_d} = \mathbf{0} \tag{3.16}$$

in $(0,T)\times D$, $n=1\ldots,N$. The corresponding initial condition is $\mathbf{u}_n^N(0,\cdot) = \langle \mathbf{u}_0,\Psi_n\rangle$.

The strongly-coupled system (3.16) is a first-order system of the large (!) dimension $N(d+1)$. Its structure depends on the chosen polynomial basis. As for many other related approaches the major question is the hyperbolicity of (3.16) which can fail for the barotropic Euler equations. For recent contributions from CoScaRa-scientists in this direction we refer to e.g. [37] or [53]. Most of the works tackle the numerical aspects for solving random hyperbolic balance laws by intrusive or non-intrusive polynomial-chaos approaches or statistical Monte-Carlo methods. A specific example will be discussed in Sect. 4.5. Here we conclude with a numerical result in Fig. 2. The forward-facing-step configuration is the same as in Fig. 1 but with random inflow data. Figure 2 displays the approximate variance of density which can be directly computed from the truncated Fourier modes.

4 Directions in Numerical Methods for Hyperbolic Balance Laws

The new trends in analysis outlined in Sect. 3 are often triggered by simulation results and vice versa provide the grounds not only to analyze numerical discretization methods with regard to convergence but also to trigger completely new approaches on the discrete level. In this section we will emphasize this point of view and mirror the sub-sections in Sect. 3. By now there is a vast world of numerical methods for hyperbolic balance laws. The development of upwinding techniques using exact Riemann solvers or wave tracking methods enables the robust computation of discontinuous weak solutions. Using finite-volume or Discontinuous-Galerkin discretisations in space, even high-order entropy-stable methods have been developed. Most notably, these tools moved mainstream as the basis of commerical software and (open-source) implementations in academia. To focus more on the underlying ideas we restrict ourselves in Sect. 4 to simple finite-volume approximations for the barotropic Euler equations (2.3). If not stated otherwise we will always consider the multi-dimensional case $d>1$.

4.1 Relative Entropies and Numerical Methods

We start with a short description of the finite-volume method as the most classical numerical method for hyperbolic conservation laws and in particular the system (2.3).

To avoid issues about boundary conditions we consider $D = \mathbb{R}^d$ or $D = \mathbb{T}^d$. Let a conforming triangulation $\mathcal{T}_h$ of D be given that consists of a set of simplices, such that $\operatorname{diam}(K) \le h$, $|K| \ge ch^d$ holds for all cells $K \in \mathcal{T}_h$ and some $c > 0$. By $\mathcal{E}(K)$ we denote the set of facets of $K \in \mathcal{T}_h$. We assume that for each $K \in \mathcal{T}_h$ and each facet e of K, there exists exactly one neighboring cell $K_e \in \mathcal{T}_h$. For the sake of simplicity we consider the semi-discrete case.

Definition 4.1 (Finite-volume method) For $t \in [0, T]$, let the family $\{\mathbf{u}_K = \mathbf{u}_K(t)\}_{K \in \mathcal{T}_h} \subset \mathcal{U}$ be given by the *finite-volume method*

$$\begin{aligned} \mathbf{u}_K'(t) &= -\frac{1}{|K|} \sum_{e \in \mathcal{E}(K)} |e| \mathbf{g}_{K,e}\big(\mathbf{u}_K(t), \mathbf{u}_{K_e}(t)\big), \\ \mathbf{u}_K(0) &= \frac{1}{|K|} \int_K \mathbf{u}_0(\mathbf{x})\, d\mathbf{x}. \end{aligned} \tag{4.1}$$

From the iteratives $\mathbf{u}_K$ we define the piecewise constant approximation $\mathbf{u}_h : [0, T] \times D \to \mathcal{U}$ by

$$\mathbf{u}_h(t, \mathbf{x}) = \mathbf{u}_K(t) \text{ for } \mathbf{x} \in K,\ K \in \mathcal{T}_h.$$

In (4.1), the function $\mathbf{g}_{K,e} : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^{d+1}$ is a consistent numerical flux function to the flux in (2.3), that is

$$\mathbf{g}_{K,e}(\mathbf{w}, \mathbf{w}) = \sum_{i=1}^{d} n_i^{K,e} \mathbf{f}_i(\mathbf{w}) \quad (\mathbf{w} \in \mathcal{U}),$$

where $\mathbf{n}^{K,e} = (n_1^{K,e}, \dots, n_d^{K,e})^T$ is the unit outward normal to $e \in \mathcal{E}(K)$. Furthermore we impose the conservation property $\mathbf{g}_{K,e}(\mathbf{w}, \tilde{\mathbf{w}}) = -\mathbf{g}_{K_e,e}(\tilde{\mathbf{w}}, \mathbf{w})$ for $\mathbf{w}, \tilde{\mathbf{w}} \in \mathcal{U}$.

For the Euler equations there exists a large variety of numerical flux functions including the Lax-Friedrichs and Riemann-solver based fluxes like the Godunov flux, see e.g. [46]. We assume tacitly that the iteratives $\mathbf{u}_K$ exist and take values in the state space $\mathcal{U}$. But note that the latter invariance property can very rarely been verified and is usually connected to a discrete version of the entropy inequality.

The foremost goal of numerical analysis is to verify the convergence of the finite-volume approximation $\mathbf{u}_h$ if the mesh parameter h tends to 0. This requires the initial-value problem for (2.3) to be well-posed which is not the case for entropy solutions as we have seen in Sect. 3. Therefore we assume that the initial-value problem for (2.3) has a classical solution. Recalling the weak-strong uniqueness principle as in Theorem 3.1 one can exploit the stability statement (3.2) to obtain for $D = \mathbb{T}^d$ a convergence result [43]. f

Theorem 4.2 *Let $\mathcal{K} \subset \mathcal{U}$ be a compact and convex set. Consider an initial function $\mathbf{u}_0 \in (C^1(D))^{d+1}$ with $\mathbf{u}(x) \in \mathcal{K}$ for $\mathbf{x} \in D$. Furthermore assume that a classical solution $\bar{\mathbf{u}} \in (C^1([0,T] \times D))^{d+1}$ of the initial-value problem for (2.3) is given and satisfies $\bar{\mathbf{u}}(t,\mathbf{x}) \in \mathcal{K}$ for $(t,\mathbf{x}) \in [0,T] \times D$.*

Then, for a finite-volume approximation $\mathbf{u}_h$, that obeys the cell entropy inequality

$$\begin{aligned} &\eta(\mathbf{u}_K(t))' + \frac{1}{|K|} \sum_{e \in \mathcal{E}(K)} |e| \mathbf{q}_{K,e}\big(\mathbf{u}_K(t), \mathbf{u}_{K_e}(t)\big) + \frac{1}{|K|} \mathcal{D}_h(t) \leq 0, \\ &\mathcal{D}_h(t) := \frac{1}{|K|} \sum_{e \in \mathcal{E}(K)} |e| \Big| \mathbf{g}_{K,e}\big(\mathbf{u}_K(t), \mathbf{u}_{K_e}(t)\big) - \mathbf{g}_{K,e}\big(\mathbf{u}_K(t), \mathbf{u}_K(t)\big) \Big|^2, \end{aligned} \tag{4.2}$$

and $\mathbf{u}_h(t,\mathbf{x}) \in \mathcal{K}$ for $(t,\mathbf{x}) \in [0,T] \times D$ for all $h > 0$, the a-priori error estimate

$$\int_0^T \int_D e^{-\alpha t} |\bar{\mathbf{u}} - \mathbf{u}_h|^2 \, d\mathbf{x} dt + \sum_{K \in \mathcal{T}_h} \mathcal{D}_h(t) \leq Ch \tag{4.3}$$

holds for constants $C, \alpha \geq 0$ which do not depend on the mesh.

In the cell entropy inequality (4.2) the function $\mathbf{q}_{K,e} : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ is a consistent numerical entropy flux function to the entropy flux $\mathbf{q}$ in (2.9).

Remark 4.3 (i) Theorem 4.2 provides an L^2-error estimate $\|\bar{\mathbf{u}} - \mathbf{u}_h\|_{L^2((0,T)\times D)} = \mathcal{O}(h^{\frac{1}{2}})$. This is not the expected optimal linear convergence rate for smooth solutions, the proof of which is open in general.

(ii) The proof of Theorem 4.2 relies on the comparison estimate (3.2) for the relative entropies E in Theorem (3.3) and the equivalence of the relative energy and the square difference. The rôle of the entropy solution is taken by the finite-volume approximation $\mathbf{u}_h$ that satisfies the cell entropy inequality (4.2) but of course not the exact entropy inequality (2.9). The residual term gives rise to the energy dissipation term $\mathcal{D}_h(t)$ which is the discrete analogue to $\mathcal{D}(t)$ in (2.9) and in fact ensures a weak BV-like estimate for $\mathbf{u}_h$.

(iii) The condition $\mathbf{u}_h(t,\mathbf{x}) \in \mathcal{K}$ hides a uniform L^∞-bound for the finite-volume approximation.

(iv) A cell entropy inequality (4.2) can be realized for some choices of numerical fluxes. We refer to Theorem 4.5 below and outline the Lax-Friedrichs/Rusanov flux given by

$$\mathbf{g}^{cLxF}_{K,e}(\mathbf{u}_1, \mathbf{u}_2) = \sum_{i=1}^d n_i^{K,e} \frac{\mathbf{f}_i(\mathbf{u}_1) + \mathbf{f}_i(\mathbf{u}_2)}{2} + \frac{1}{\lambda_{K,e}} (\mathbf{u}_1 - \mathbf{u}_2) \qquad (\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{U}). \tag{4.4}$$

The numbers $\lambda_{K,e}$ are chosen to be the inverse of the maximum of the absolute values of the eigenvalues of the $\mathbf{n}^{K,e}$-rotated flux Jacobian in $\mathbf{u}_K$ (Lax-Friedrichs) or as the maximum taken over absolute values of the eigenvalues of the $\mathbf{n}^{K,e}$-rotated flux Jacobian in $\mathbf{u}_K$ and $\mathbf{u}_{K_e}$ (Rusanov). In both cases the entropy inequality holds, see [62].

Relative entropies are also used to provide a-posteriori error estimates as the basis for adaptive algorithms, see [20]. Current research within CoScaRa aims at the numerical analysis for higher-order discretizations (see below) and for the more complex initial/boundary-value problems [25], which might lead to very bad time-step conditions if different-size cells are used.

4.2 Structure-Preserving Methods for Extreme Flow Regimes

The numerical simulation of compressible flow becomes intricate in extreme regimes due to largely differing transport speeds like for the low-Mach number limit or due to very stiff balance terms as for the Euler-Darcy model (3.4). Indeed, it is on the numerical side where the main progress is evolving right now. To get over these barriers new structure-preserving numerical approaches are emerging which guarantee properties of the underlying PDE model on the discrete level. This approach includes the asymptotic-preserving schemes (cf. [42, 59] for overviews), which we will explain using the friction limit for the Euler-Darcy equations (3.4), see Sect. 3.2. To avoid a complicated nomenclature we choose here $D = \mathbb{R}$. Let the mesh be given by ordered equal-length cells $K_i = (x_{i-1/2}, x_{i+1/2})$, $x_i = ih$ for $i \in \mathbb{Z}$ and mesh parameter $h = |K_i| > 0$. By adapting standard notations for the one-dimensional case we introduce the following two-step finite-volume iteration after an obvious operator splitting for the system (3.4).

$$\begin{aligned} \mathbf{u}_i^{\varepsilon\prime}(t) &= -\frac{1}{|K_i|}\Big(\mathbf{g}^{\mathrm{LxF}}(\mathbf{u}_i^{\varepsilon}(t), \mathbf{u}_{i+1}^{\varepsilon}(t)) - \mathbf{g}^{\mathrm{LxF}}(\mathbf{u}_{i-1}^{\varepsilon}(t), \mathbf{u}_i^{\varepsilon}(t))\Big), \\ \mathbf{u}_i^{\varepsilon\prime}(t) &= \Big(0, -\frac{1}{\varepsilon^2} M m_i^{\varepsilon}(t)\Big)^T. \end{aligned} \tag{4.5}$$

We use the Lax-Friedrichs flux from (4.4) with $\lambda_{K,e} = \lambda$ for the flux discretization in $(4.5)_1$. In the fully-discrete version the stiff ordinary differential equation $(4.5)_2$ is solved by some implicit scheme which does not introduce limitations on the time step. The time discretization for the finite-volume discretization $(4.5)_1$ will be chosen in an explicit manner. Since the eigenvalues of the flux Jacobian depend critically on ε this may lead to severe time step restrictions. In fact, this can be avoided by a further splitting of the pressure term between $(4.5)_1$ and $(4.5)_2$ which we omit here for the sake of simplicity.

Now, we know from Theorem 3.3 that solutions of the approximate model (3.4) approach the parabolic limit model (3.5) if ε tends to zero. The numerical method for the approximate model, here (4.5), is called *asymptotic-preserving* if the limits $h \to 0$ and $\varepsilon \to 0$ commute in a certain sense: in the limit $\varepsilon \to 0$ in the numerical discretization for the approximate model one recovers a consistent discretization of the limit problem. For the Euler-Darcy equations we get

Theorem 4.4 *The semi-discrete numerical method* (4.5) *for the initial-value problem for the Euler-Darcy equations* (3.4) *is asymptotic-preserving, i.e., for* $\varepsilon \to 0$ *one obtains a consistent discretization of the limit model* (3.5) (*see also* (3.6)), *that is given*

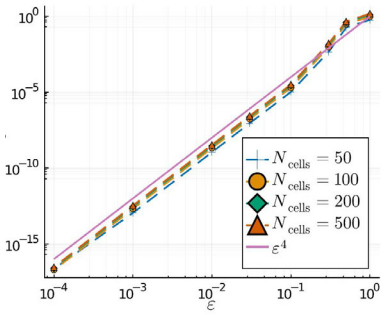

Fig. 3 Experimental ε-convergence of the relative entropy for the asymptotic-preserving scheme (4.5)

by

$$\begin{aligned}\bar{\rho}_i'(t) &= -\frac{1}{2|K_i|}\big(\overline{m}_{i+1}(t) - \overline{m}_{i-1}(t)\big) + \frac{1}{\lambda|K|}\big(\bar{\rho}_{i+1}(t) - 2\bar{\rho}_i(t) + \bar{\rho}_{i-1}(t)\big),\\ \overline{m}_i(t) &= -\frac{1}{2|K_i|M}\big(p(\bar{\rho}_{i+1}(t)) - p(\bar{\rho}_{i-1}(t))\big).\end{aligned} \tag{4.6}$$

The limit-model discretization in (4.6) comes with the prize that the stencil for the discretization of the nonlinear diffusion term in (3.5) includes not only the next-neighbour cells. This complicates the generalization to multiple space dimensions, at least on unstructured meshes.

In Fig. 3 we display the quantitiy $\int_D E[\mathbf{u}_h^\varepsilon|\bar{\mathbf{u}}]\,d\mathbf{x}$ at some fixed time for various meshes (indicated by the used numbers of cells N_{cells}) and values for the small parameter ε. The function $\mathbf{u}_h^\varepsilon$ is computed with a fully-discrete version of (4.5) for prepared initial data, and $\bar{\mathbf{u}}$ is a (manufactured) smooth solution of the limit model (3.5). For $\varepsilon \to 0$, we recover fourth-order behaviour confirming the analytical result of Theorem 3.3.

Main research directions for the further development of asymptotic-preserving schemes are the efficient extensions to systems in multiple space dimensions and the design of robust, possibly higher-order time discretizations. These are typically mixed implicit-explicit approaches. Concerning the design of new time-discretization methods within CoScaRa let us refer to e.g. [21, 56, 60]. In fact asymptotic-preserving methods can be seen as a subclass of the much wider family of *structure-preserving* methods. This includes also well-balancing techniques to preserve equilibrium solutions on the discrete level, see e.g. [8]. Yet another group are constraint-preserving methods that maintain algebraic or differential side conditions, steer entropy, free energy, positivity, or symmetry constraints while preserving high-order accuracy. This brings us directly to the next topic.

4.3 Discrete Control of the Entropy Dissipation

In Sect. 3.3 we observed in the framework of the discussion of the Onsager hypothesis that an entropy solution $\mathbf{u}$ of the initial-value problem for (2.3) conserves the entropy η exactly as long as $\mathbf{u}$ is regular enough, see Theorem 3.5. Otherwise the entropy might be dissipated. On the discrete level it is essential to mimic the exact dissipation

rate as exactly as possible. Moreover, the cell entropy inequality with a positive term $\mathcal{D}_h$ is essential for the construction of a robust and convergent numerical scheme (see Theorem 4.7 below). How to realize a fine tuning of the entropy dissipation rate on the discrete level? Depending on the used discretization method several ideas for entropy dissipation control including exact entropy conservation have been developped.

We sketch here the approach of Tadmor [62] applied to the Euler equations (2.3). The starting point is the observation that the convexity of η induces a change of variables in the convex state space $\mathcal{U}$ via the mapping

$$\mathbf{u} \mapsto \tilde{\mathbf{u}} := \nabla_{\mathbf{u}} \eta(\mathbf{u}) \qquad (\mathbf{u} \in \mathcal{U}).$$

With the transformed flux $\tilde{\mathbf{f}}_i(\tilde{\mathbf{u}}) := \mathbf{f}_i(\mathbf{u})$, $i = 1, \dots, d$, (see e.g. [54] for the precise form of $\tilde{\mathbf{f}}_i$ for the barotropic Euler system (2.3)) we equivalently rewrite (2.3) in the form

$$\mathbf{u}_t + \tilde{\mathbf{f}}(\tilde{\mathbf{u}})_{x_1} + \cdots + \tilde{\mathbf{f}}(\tilde{\mathbf{u}})_{x_d} = 0 \text{ in } (0, T) \times D,$$

which is the basis for our finite-volume scheme. We consider a special consistent numerical flux function $\tilde{\mathbf{g}}_{K,e} : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^{d+1}$: for $\tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2 \in \tilde{\mathcal{U}}$, $\alpha \in \{0, 1\}$, and some $\lambda_{K,e} > 0$ with $\lambda_{K,e} = \lambda_{K_e,e}$ we define

$$\begin{aligned} \tilde{\mathbf{g}}_{K,e}(\tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2) &= \sum_{i=1}^{d} n_i^{K,e} \tilde{\mathbf{g}}_i^*(\tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2) + \frac{\alpha}{\lambda_{K,e}} (\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2), \\ \tilde{\mathbf{g}}_i^*(\tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2) &= \int_0^1 \tilde{\mathbf{f}}_i(\tilde{\mathbf{u}}_1 - s(\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2)\, ds. \end{aligned} \tag{4.7}$$

The one-dimensional central numerical flux $\tilde{\mathbf{g}}_i^*$ is called Tadmor flux. The last term in $(4.7)_1$ stands for artifical dissipation as we can observe from the following theorem (cf. [43]).

Theorem 4.5 *We consider the finite-volume method* (4.1) *with the numerical flux function* $\tilde{\mathbf{g}}_{K,e} : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^{d+1}$ *from* (4.7).

It satisfies the cell entropy inequality (4.2) *for some consistent numerical entropy flux to the entropy flux* $\mathbf{q}$ *with*

$$\mathcal{D}_h(t) = \frac{1}{|K|} \sum_{e \in \mathcal{E}(K)} |e| \frac{\alpha}{2\lambda_{K,e}} \left| \tilde{\mathbf{u}}_{K,e}(t) - \tilde{\mathbf{u}}_K(t) \right|^2.$$

Notably, for $\alpha = 0$, *we obtain a scheme that exactly conserves the entropy.*

Remark 4.6 (i) The result of Theorem 4.5 has been generalized in many directions. In particular, it has been used to construct all kinds of entropy-dissipative Discontinuous-Galerkin schemes, see e.g. [33].

(ii) The choice of the parameter $\lambda_{K,e}$ might not only depend on space but also on the finite-volume discretization itself. In the constant case the scheme resembles the design of artificial viscosity in the Lax-Friedrichs flux. Like in the Rusanov flux typically a dependence on the local eigenvalues is introduced.

The finite-volume approximation $\mathbf{u}_h$ obtained by using the numerical flux from (4.7) is of first order if $\lambda_{K,e}$ is chosen to be a constant. However, the pure Tadmor flux, which is obtained by setting $\alpha = 0$, leads to a second order method in space. A major trend in the numerics for hyperbolic balance laws is to design high-order (in space and time) schemes that satisfy a discrete cell entropy inequality like (4.2) (and obey other structure-preserving properties like being asymptotic-preserving or well-balanced!). Let us mention [48] for the investigation of algebraic entropy corrections within finite-element discretizations. A completely other idea is pursued by sub-cell resolution approaches which mix methods of different order to control artificial viscosity and thus the entropy dissipation rate [35]. Very promising are the latest developments on the third-order active-flux method ()see the survey paper [14]). For the active flux method, one propagates not only the cell average but also point values on the boundaries. It has been further generalized to include adaptive mesh refinement [11] or to enable arbitrary order of consistency [2].

4.4 Numerical Methods for Probabilistic Solution Concepts

Beyond the convergence of finite-volume approximations in the case of classical well-posedness of the initial-value problem for (2.3) one can also verify quite general convergence statements for the probabilistic solution concepts from Sect. 3.5. We showcase a recent result for a sequence of numerical solutions generating a Young measure that represents a dissipative measure-valued solution.

Let us consider again the finite-volume method from Definition 4.1 using $D = \mathbb{T}^d$. We choose the Lax-Friedrichs numerical flux function from (4.4) in Remark 4.3(iv). In what follows it is assumed that the initial data and the finite-volume approximation $\mathbf{u}_h = (\rho_h, \mathbf{m}_h^T)^T$ is contained in a compact subset of $\mathcal{U}$ that is independent of the mesh parameter h. This implies in particular by the mass conservation property of a finite-volume scheme that the discrete density ρ_h satisfies the uniform a-priori bound

$$\rho_h \in L^\infty(0, T; L^1(D)). \tag{4.8}$$

The discrete energy bound (4.2) ensures then $\eta(\mathbf{u}_h) \in L^\infty(0, T; L^1(D))$ uniformly in h. Then, the definition of η from (2.6) and the choice of the γ-law pressure induce the uniform bounds

$$\begin{gathered} \rho_h \in L^\infty(0, T; L^\gamma(D)), \ \frac{\mathbf{m}_h}{\sqrt{\rho_h}} \in (L^\infty(0, T; L^1(D)))^d, \\ \mathbf{m}_h \in (L^\infty(0, T; L^{\frac{2\gamma}{\gamma+1}}(D)))^d. \end{gathered} \tag{4.9}$$

The bounds (4.8), (4.9) might be the optimal uniform a-priori bounds that can be achieved for entropy-stable finite-volume approximations as long as one does not exploit the entropy dissipation rate. They do not suffice to ensure convergence in any strong topology but they allow to generate a Young measure that represents a dissipative measure-valued solution.

Theorem 4.7 *Let $\mathcal{K} \subset \mathcal{U}$ be a compact and convex set. Consider an initial function $\mathbf{u}_0 \in (C^1(D))^{d+1}$ with $\mathbf{u}_0(\mathbf{x}) \in \mathcal{K}$ for $\mathbf{x} \in D$.*

Let $\mathbf{u}_h$ be a finite-volume approximation (4.1) using the numerical flux $\mathbf{g}_{K,e}$ from (4.4) and assume that $\mathbf{u}_h(t,\mathbf{x}) \in \mathcal{K}$ holds for $(t,\mathbf{x}) \in [0,T] \times D$ and $h > 0$.

Then, there is a subsequence of $\{\mathbf{u}_h\}_{h>0}$ (still denoted by $\{\mathbf{u}_h\}_{h>0}$), which generates a Young measure $\nu^{\mathbf{u}}$ such that

$$\lim_{h\to 0} \int_0^T \int_D \mathbf{u}_h \varphi(t,\mathbf{x})\, d\mathbf{x}dt = \int_0^T \int_D \left\langle \nu^{\mathbf{u}}_{(t,\mathbf{x})}, \mathbf{u} \right\rangle \varphi(t,\mathbf{x})\, d\mathbf{x}dt \quad \forall\, \varphi \in L^1((0,T) \times D).$$

The Young measure $\nu^{\mathbf{u}}$ is a dissipative measure-valued solution of the initial-value problem for (2.3), see Definition 3.8.

Remark 4.8 (i) For the initial conditions and the finite-volume approximation a uniform L^∞-bound is required as in Theorem 4.2 for classical solutions. The verification of this bound is a missing link in the entire convergence theory.

(i) Assumptions on the initial data and numerical solutions in Theorem 4.7 can be relaxed in the following way. The initial data satisfy $\rho_0 > 0$ and $\eta(\mathbf{u}_0) \in L^1(\Omega)$. As shown in, e.g., [27, Sect. 12] a finite-volume method with a numerical flux based on viscous upwinding converges weakly-$\star$ to a dissipative measure-valued solution. Note that no additional assumption on this finite-volume solution is required.

(ii) Theorem 4.7 provides a convergence statement but does not include a convergence rate. It is unlikely to obtain a convergence rate in view of the fact that dissipative measure-valued solutions cannot expected to be unique. However, if a classical solution $\mathbf{u}$ exists the weak-strong uniqueness principle for dissipative measure-valued solutions can be used to show that the finite-volume approximation $\mathbf{u}_h$ converges strongly to $\mathbf{u}$.

(iii) In [28], weak-$\star$ convergence of finite-volume approximations $\{\mathbf{u}_h\}_{h>0}$ has been strengthened to the strong convergence applying an ensemble-averaged approach, the so-called $\mathcal{K}$-convergecnce. More precisely, the Césaro averages of $\{\mathbf{u}_h\}_{h>0}$ converge strongly in $L^q(0,T; L^\gamma(\Omega) \times L^{\frac{2\gamma}{\gamma+1}}(\Omega))$, $1 \le q < \infty$, to a dissipative measure-valued solution $\langle \nu^{\mathbf{u}}_{(t,\mathbf{x})}, \mathbf{u} \rangle$.

The result of Theorem 4.7 can also be obtained for the full Euler system (2.1), see, e.g., [27, Sect. 10] and [4, 50]. Currently, in CoScaRa the convergence results are extended to higher-order schemes and other types of schemes that obey a cell entropy inequality. Completely in parallel there are initiatives to analyze the convergence of numerical methods towards statistical solutions, see [30]. In view of the missing well-posedness for entropy solutions it is also important to develop schemes that provide pathwise solutions for e.g. random initial data, see [6, 29] and the following text.

4.5 Uncertainty Quantification for Hyperbolic Balance Laws

The quantification of uncertainties in solutions of random PDEs is a most rapidly expanding research field at the interface between numerics and computational statis-

tics. Although uncertainty quantification for hyperbolic balance laws is less developed than for e.g. elliptic PDEs, basic approaches have found their way into the field, see e.g. [3]. With regard to CoScaRa, preparatory work concerning the design of stochastic-Galerkin methods has been sketched already in Sect. 3.6. A particular challenge for the quantification of uncertainties in hyperbolic transport are possibly oscillating approximations in the spatial-stochastic domain due to loss of regularity. The transfer of the well-known limiting and filtering methodologies for purely spatial irregularities is just starting to become a trend in the field.

In this concluding sub-section we will not focus on the development of methods and their mathematical analysis. Rather, following [23] we showcase what can be done with a state-of-the-art method for the quantification of uncertainties. The SG-approach in [23] builds upon the stochastic-Galerkin system from (3.16) but utilizes a multi-element idea. To address the low-regularity issues the stochastic space Ω is devided into disjoint random elements and a local approximation on each random element is used. The discretization of the resulting deterministic system is then realized within the open-source software framework FLEXI [31]. The FLEXI code relies on a high-order Discontinuous-Galerkin method in space and an explicit Runge-Kutta time-stepping scheme. Thanks to its compact stencil and the explicit time-stepping scheme the method is well-suited for applications on high-performance computing systems. To avoid problems associated with the failure of hyperbolicity of (3.16) a hyperbolicity-preserving limiter from [22] is employed.

In a first numerical example we consider the flow over the NACA 0012 airfoil, which is governed by the compressible Navier-Stokes equations. We have two sources of normally distributed randomness, namely the dynamic viscosity of the fluid and the angle of attack of the airfoil. For the specific set-up we refer the interested reader to Sect. 5.2 of [23]. Concerning the numerical parameters we use 10 cells in the stochastic space and choose polynomial degree 3 for the Discontinuous-Galerkin space discretization using the numerical flux (4.4). The spatial computational mesh consists of 652 quadrilateral cells. Simulation results for this demanding setting are shown in Fig. 4. To provide some validation the outcome of a computation for the same setting using an alternative non-intrusive method is also displayed. One observes acceptable coincidence for the expectation whereas there are visible differences for the standard deviation plot.

Finally we report on simulation results for a three-dimensional problem, namely the ideal flow around a space capsule with an uncertain Mach number at the inflow boundary using the Euler equations (2.1). The results are again taken from [23], see Sect. 5.4. For the spatial mesh $71,819$ cells are used. To capture strong shock waves in a robust way the polynomial degree is reduced from 3 to one (finite-volume method) when a singularity is detected. The simulation run has been done on 4800 cores at the Cray XC40 system Hazel Hen at the High Performance Computing Center Stuttgart. In Fig. 5 we show the approximate expectation and standard deviation of the Mach number (plotted on the spacecraft's surface) and the pressure in a slice through the flow field. In front of the capsule a bow shock forms that triggers a complex wave pattern which is displayed on the capsule's surface. Yet another shock structure develops at the thruster's outlets and emerges to the far field. As expected

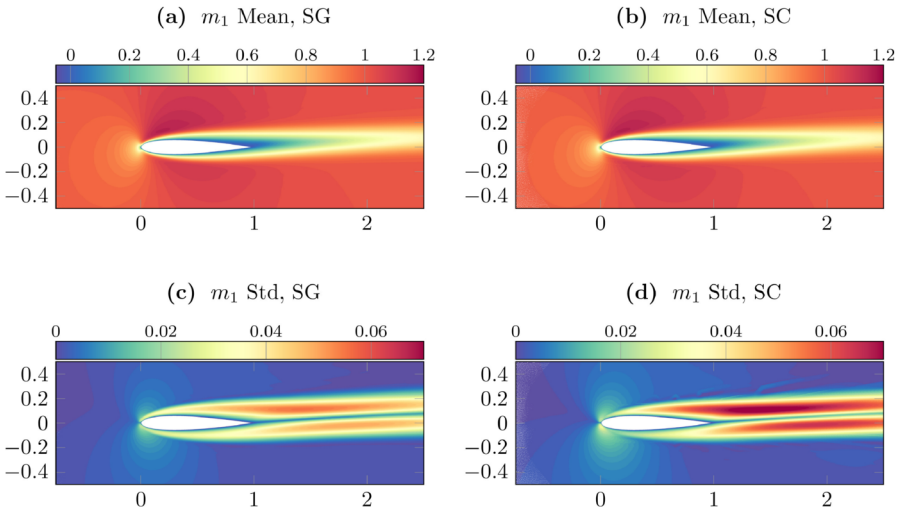

Fig. 4 Comparison of (approximate) expectation and standard deviation for the first momentum component between the *SG*-approach (left column) and a non-intrusive collocation method as benchmark

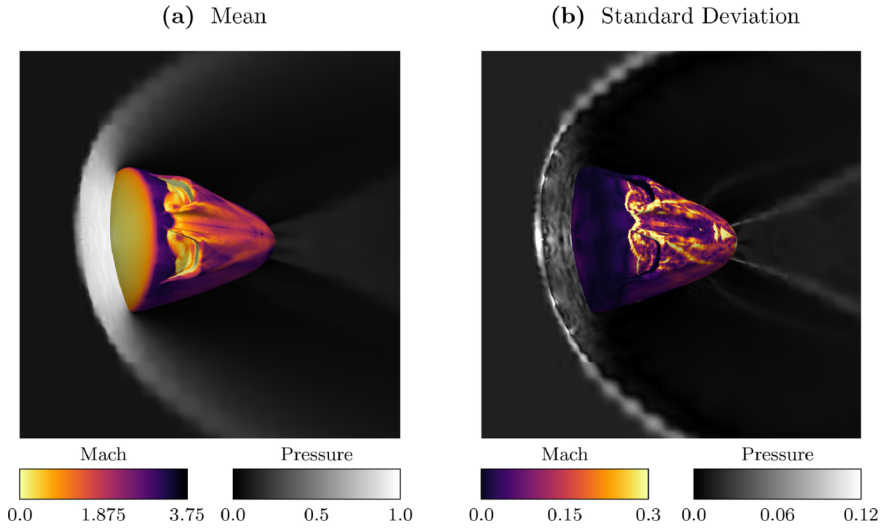

Fig. 5 Approximate expectation and standard deviation for the Mach number (on the capsule's surface) and the pressure field. The coincidence of shock patterns and maxima of the standard deviations are obvious

the approximated variance is highest where shock positions vary with changing inflow Mach number.

The results underline the potential of current methods of uncertainty quantification for real-world applications. Nevertheless there are many open challenges including

the treatment of high-dimensional noise, the handling of rough random perturbations, or noise in the flux operators.

5 Conclusions

At the example of the barotropic Euler equations, we highlighted recent developments and challenges for the analysis and the numerics for hyperbolic balance laws. These trends are triggered by the renewed interest in the importance of hyperbolic features for turbulent fluid motion. Most prominently, the discovery that the entropy concept does not suffice to ensure the well-posedness of the basic equations of fluid mechanics has opened new avenues not only for the analysis but also for our approach to the numerics for hyperbolic balance laws. This fundamental finding is related to the completely unsatisfactory understanding how to ground turbulent flow patterns using the equations of fluid mechanics. It is the vision of the Priority Programme SPP-2410 *Hyperbolic Balance Laws in Fluid Mechanics: Complexity, Scales, Randomness* to gain more insight into this challenge by providing new tailored mathematical tools. It targets at novel ensemble-averaged solution concepts, which can be accessed by latest discretization methods, at accuracy-controlled multiscale model selection along with structure-preserving numerics and at a stochastic model environment for precision-controlled uncertainty quantification.

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