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Subordinated Gaussian random fields in elliptic partial differential equations

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Abstract
To model subsurface flow in uncertain heterogeneous or fractured media an elliptic equation with a discontinuous stochastic diffusion coefficient—also called random field—may be used. In case of a one-dimensional parameter space, Lévy processes allow for jumps and display great flexibility in the distributions used. However, in various situations (e.g. microstructure modeling), a one-dimensional parameter space is not sufficient. Classical extensions of Lévy processes on two parameter dimensions suffer from the fact that they do not allow for spatial discontinuities [see for example Barth and Stein (Stoch Part Differ Equ Anal Comput 6(2):286–334, 2018)]. In this paper a new subordination approach is employed [see also Barth and Merkle (Subordinated gaussian random fields. ArXiv e-prints, arXiv:2012.06353 [math.PR], 2020)] to generate Lévy-type discontinuous random fields on a two-dimensional spatial parameter domain. Existence and uniqueness of a (pathwise) solution to a general elliptic partial differential equation is proved and an approximation theory for the diffusion coefficient and the corresponding solution provided. Further, numerical examples using a Monte Carlo approach on a Finite Element discretization validate our theoretical results.

Keywords Stochastic partial differential equations · Lévy fields · Finite element methods · Circulant embedding · Subordination · Discontinuous random fields

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1 Introduction

Over the last decade partial differential equations with stochastic operators\ data\ domain became a widely studied object. This branch of research is oftentimes called uncertainty quantification. Especially for problems where data is sparse or measurement errors are unavoidable, like subsurface flow problems, the theory provides an approach to quantify this uncertainty. There are two main approaches to discretize the uncertain problem: intrusive and non-intrusive methods. The former require the solution of a high dimensional partial differential equation, where the dimensionality depends on the smoothness of the random field or process (see for example [7, 22, 30] and the references therein). The latter consist of (essentially) sampling methods and require multiple solutions of a low dimensional problem (see, among others [1, 9, 11, 12, 29, 34]). Up to date mainly Gaussian random fields were used to model the diffusivity in an elliptic equation (as a model for a subsurface flow problem). Gaussian random fields have the advantage that they may be used in both approaches and that they are stochastically very well understood objects. A great disadvantage is however, that the distributions underlying the field are Gaussian and therefore the model lacks flexibility, in the sense that the field cannot have pointwise marginal distributions with heavy-tails. Further, Gaussian random fields that are commonly used in applications, like Matérn fields, have $\mathbb{P}$-almost surely spatial continuous paths. In view of these limitations, some extensions of Gaussian models for the diffusivity in elliptic PDEs have been explored in the literature: The paper [29] presents a multilevel Monte Carlo method for elliptic equations with jump-diffusion coefficients for two-phase random media, where the diffusion coefficient assumes two different values on random domains. Discontinuous models for the diffusion coefficient in elliptic PDEs have also been used in the context of Bayesian inversion, where geometric priors or level-set priors are used as a model for the diffusion, see e.g. [27, 28]. In the recent article [20], the diffusion coefficient of an elliptic PDE is modeled as a transformation of smoothed Lévy noise fields. The authors investigate the existence, integrability and approximation of the corresponding stochastic PDE solution. The presented model yields a high distributional flexibility of the diffusion coefficient, however, the considered fields are continuous in space.

In this paper we propose a two-dimensional subordinated Gaussian random field as stochastic diffusion coefficient in an elliptic equation. The subordinated Gaussian random field is a type of a (discontinuous) Lévy field. Different subordinators display unique patterns in the discontinuities and have varied marginal distributions (see [8]). Naturally the spatial regularity of a subordinated Gaussian random field depends on the subordinator. We prove existence and uniqueness of a solution to the elliptic equation in a pathwise sense and provide different discretization schemes.

We structured the rest of the paper as follows: in Sect. 2 we introduce a general pathwise existence and uniqueness result for a stochastic elliptic equation under mild assumptions on the coefficient. These assumptions accommodate the subordinated Gaussian random fields we introduce in Sect. 3. In Sect. 4 we approximate the specific diffusion coefficient which is used in this paper and show convergence of the elliptic equation with the approximated coefficient to the unapproximated solution in Sect. 5. Section 6 provides spatial approximation methods and in Sect. 7 numerical examples are presented.

2 The stochastic elliptic problem

In this section we introduce the framework of the general stochastic elliptic boundary value problem which allows for discontinuous diffusion coefficients. For the general setting and pathwise existence theory we follow Barth and Stein [11]. In the following, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Let $(B, \|\cdot\|_B)$ be a Banach space and Z be a B valued random variable, i.e. a measurable function $Z : \Omega \to B$. The space $L^p(\Omega; B)$ contains all strongly measurable B-valued random variables with $\|Z\|_{L^p(\Omega;B)} < +\infty$, for $p \in [1, +\infty]$, where the norm is defined by

$$\|Z\|_{L^p(\Omega;B)} = \begin{cases} \mathbb{E}(\|Z\|_B^p)^{\frac{1}{p}} & , \text{ if } 1 \le p < +\infty, \\ \underset{\omega\in\Omega}{\text{ess sup}} \|Z\|_B & , \text{ if } p = +\infty. \end{cases}$$

2.1 Problem formulation

Let $\mathcal{D} \subset \mathbb{R}^d$ for $d \in \mathbb{N}$ be a bounded, connected Lipschitz domain. We consider the equation

$$-\nabla \cdot (a(\omega, \underline{x})\nabla u(\omega, \underline{x})) = f(\omega, \underline{x}) \text{ in } \Omega \times \mathcal{D}, \tag{2.1}$$

where $a : \Omega \times \mathcal{D} \to \mathbb{R}_+$ is a stochastic (jump-diffusion) coefficient and $f : \Omega \times \mathcal{D} \to \mathbb{R}$ is a (measurable) random source function. Further, we impose the following boundary conditions

$$u(\omega, \underline{x}) = 0 \text{ on } \Omega \times \Gamma_1, \tag{2.2}$$

$$a(\omega, \underline{x})\overrightarrow{n} \cdot \nabla u(\omega, \underline{x}) = g(\omega, \underline{x}) \text{ on } \Omega \times \Gamma_2, \tag{2.3}$$

where we assume to have a decomposition $\partial\mathcal{D} = \Gamma_1 \dot{\cup} \Gamma_2$ with two $(d-1)$-dimensional manifolds Γ_1, Γ_2 such that the exterior normal derivative $\overrightarrow{n} \cdot \nabla u$ on Γ_2 is well-defined for every $u \in C^1(\overline{\mathcal{D}})$. Here, $\overrightarrow{n}$ is the outward unit normal vector to Γ_2 and $g : \Omega \times \Gamma_2 \to \mathbb{R}$ a measurable function. Note that we just reduce the theoretical analysis to the case of homogeneous Dirichlet boundary conditions to simplify notation. The extension to the inhomogeneous case is straightforward.

We now state assumptions under which the elliptic boundary value problem has a unique solution.

Assumption 2.1 Let $H := L^2(\mathcal{D})$. We assume that

i. for any fixed $\underline{x} \in \mathcal{D}$ the mapping $\omega \mapsto a(\omega, \underline{x})$ is measurable, i.e. $a(\cdot, \underline{x})$ is a (real-valued) random variable,
ii. for any fixed $\omega \in \Omega$ the mapping $a(\omega, \cdot)$ is $\mathcal{B}(\mathcal{D}) - \mathcal{B}(\mathbb{R}_+)$-measurable and it holds $a_-(\omega) := \underset{\underline{x}\in\mathcal{D}}{\text{ess inf}}\, a(\omega, \underline{x}) > 0$ and $a_+(\omega) := \underset{\underline{x}\in\mathcal{D}}{\text{ess sup}}\, a(\omega, \underline{x}) < +\infty$,
iii. $\frac{1}{a_-} \in L^p(\Omega; \mathbb{R})$, $f \in L^q(\Omega; H)$ and $g \in L^q(\Omega; L^2(\Gamma_2))$ for some $p, q \in [1, +\infty]$ such that $r := (\frac{1}{p} + \frac{1}{q})^{-1} \ge 1$.

Remark 2.2 Note that Assumption 2.1 implies that the real-valued mappings $a_-, a_+ : \Omega \to \mathbb{R}$ are measurable. This can be seen as follows: For fixed $p \geq 1$ consider the mapping

$$I_p : \Omega \to \mathbb{R},\ \omega \mapsto \|a(\omega, \cdot)\|_{L^p(\mathcal{D})} = (\int_{\mathcal{D}} a(\omega, \underline{x})^p d\underline{x}))^{1/p},$$

which is well-defined by Assumption 2.1. It follows from the definition of the Lebesgue integral and Assumption 2.1(i) that the mapping $\omega \mapsto I_p(\omega)$ is $\mathcal{F}-\mathcal{B}(\mathbb{R})$ measurable. For a fixed $\omega \in \Omega$, by the embedding theorem for L^p spaces (see [2, Theorem 2.14]), we get

$$a_+(\omega) = \lim_{m\to\infty} \|a(\omega, \cdot)\|_{L^m(\mathcal{D})}.$$

Since this holds for all $\omega \in \Omega$ we obtain by Aliprantis and Border [4, Lemma 4.29] that the mapping

$$\omega \mapsto a_+(\omega),$$

is $\mathcal{F} - \mathcal{B}(\mathbb{R})$-measurable. The measurability of $\omega \mapsto a_-(\omega)$ follows analogously. Note that we do not treat the random coefficient $a : \Omega \times \mathcal{D} \to \mathbb{R}$ as a $L^\infty(\mathcal{D})$-valued random variable, since $L^\infty(\mathcal{D})$ is not separable and therefore the strong measurability of the mapping $a : \Omega \to L^\infty(\mathcal{D})$ is only guaranteed in a very restrictive setting. Nevertheless, the measurability of the functions a_+ and a_- allows taking expectations of these real-valued random variables. In order to avoid confusion about that, we use the notation $\mathbb{E}(\text{ess sup}_{\underline{x}\in\mathcal{D}} |\cdot|^s)^{1/s}$.

2.2 Weak solution

We denote by $H^1(\mathcal{D})$ the Sobolev space on $\mathcal{D}$ with the norm $\|v\|_{H^1(\mathcal{D})} := (\|v\|^2_{L^2(\mathcal{D})} + \|\nabla v\|^2_{L^2(\mathcal{D})})^{1/2}$, for $v \in H^1(\mathcal{D})$ (see for example [21, Section 5.2]). Further, we denote by T the trace operator $T : H^1(\mathcal{D}) \to H^{\frac{1}{2}}(\partial\mathcal{D})$ where $Tv = v|_{\partial\mathcal{D}}$ for $v \in C^\infty(\overline{\mathcal{D}})$ (see [18]). We define the subspace $V \subset H^1(\mathcal{D})$ as

$$V := \{v \in H^1(\mathcal{D}) \mid Tv|_{\Gamma_1} = 0\},$$

with the standard Sobolev norm, i.e. $\|\cdot\|_V := \|\cdot\|_{H^1(\mathcal{D})}$. We identify $H = L^2(\mathcal{D})$ with its dual space H' and work on the Gelfand triplet $V \subset H \simeq H' \subset V'$. Hence, Assumption 2.1 guarantees that $f(\omega, \cdot) \in V'$ and $g(\omega, \cdot) \in H^{-\frac{1}{2}}(\Gamma_2)$ for $\mathbb{P}$-almost every $\omega \in \Omega$. We multiply Eq. (2.1) by a test function $v \in V$, integrate by parts and use the boundary conditions (2.2) and (2.3) to obtain

$$\int_{\mathcal{D}} -\nabla \cdot (a(\omega, \underline{x}) \nabla u(\omega, \underline{x})) v(\underline{x}) d\underline{x} = \int_{\mathcal{D}} a(\omega, \underline{x}) \nabla u(\omega, \underline{x}) \cdot \nabla v(\underline{x}) d\underline{x} - \int_{\Gamma_2} g(\omega, \underline{x}) [T v](\underline{x}) d\underline{x}.$$

This leads to the following pathwise weak formulation of the problem: For any $\omega \in \Omega$, given $f(\omega, \cdot) \in V'$ and $g(\omega, \cdot) \in H^{-\frac{1}{2}}(\Gamma_2)$, find $u(\omega, \cdot) \in V$ such that

$$B_{a(\omega)}(u(\omega, \cdot), v) = F_\omega(v) \tag{2.4}$$

for all $v \in V$. The function $u(\omega, \cdot)$ is then called *pathwise weak solution* to problem (2.1)–(2.3). Here, the bilinear form $B_{a(\omega)}$ and the operator F_ω are given by

$$B_{a(\omega)} : V \times V \to \mathbb{R},\ (u, v) \mapsto \int_{\mathcal{D}} a(\omega, \underline{x}) \nabla u(\underline{x}) \cdot \nabla v(\underline{x}) d\underline{x},$$

and

$$F_\omega : V \to \mathbb{R},\ v \mapsto \int_{\mathcal{D}} f(\omega, x) v(\underline{x}) d\underline{x} + \int_{\Gamma_2} g(\omega, \underline{x}) [T v](\underline{x}) d\underline{x},$$

for fixed $\omega \in \Omega$, where the integrals in F_ω are understood as the duality pairings:

$$\int_{\mathcal{D}} f(\omega, \underline{x}) v(\underline{x}) d\underline{x} = {}_{V'}\langle f(\omega, \cdot), v \rangle_V \text{ and}$$
$$\int_{\Gamma_2} g(\omega, \underline{x}) [T v](\underline{x}) d\underline{x} = {}_{H^{-\frac{1}{2}}(\Gamma_2)}\langle g(\omega, \cdot), T v \rangle_{H^{\frac{1}{2}}(\Gamma_2)},$$

for $v \in V$.

Theorem 2.3 (See [11, Theorem 2.5]) *Under Assumption 2.1, there exists a unique pathwise weak solution $u(\omega, \cdot) \in V$ to problem (2.4) for very $\omega \in \Omega$. Furthermore, $u \in L^r(\Omega; V)$ and*

$$\|u\|_{L^r(\Omega;V)} \le C(a_-, \mathcal{D}, p)(\|f\|_{L^q(\Omega;H)} + \|g\|_{L^q(\Omega;L^2(\Gamma_2))}),$$

where $C(a_-, \mathcal{D}, p) > 0$ is a constant depending on a_-, p and the volume of $\mathcal{D}$.

In addition to the existence of the solution, the following remark gives a rigorous justification for the measurability of the solution mapping

$$u : \Omega \to V,\ \omega \mapsto u(\omega, \cdot),$$

which maps any $\omega \in \Omega$ on the corresponding pathwise weak PDE solution.

Remark 2.4 Let $(v_n,\ n \in \mathbb{N}) \subset V$ be an orthonormal basis of the separable Hilbert space V. For every $n \in \mathbb{N}$ we define the mapping

$$\begin{aligned} J_n : \Omega \times V &\to \mathbb{R} \\ (\omega, v) &\mapsto \int_{\mathcal{D}} a(\omega, \underline{x}) \nabla v(\underline{x}) \cdot \nabla v_n(\underline{x}) d\underline{x} - \int_{\mathcal{D}} f(\omega, \underline{x}) v_n(\underline{x}) d\underline{x} \\ &\quad - \int_{\Gamma_2} g(\omega, \underline{x}) [T v_n](\underline{x}) d\underline{x}. \end{aligned}$$

It is easy to see that this mapping is Carathéodory for any $n \in \mathbb{N}$, i.e. $J_n(\cdot, v)$ is $\mathcal{F}$-$\mathcal{B}(\mathbb{R})$-measurable for any fixed $v \in V$ and $J_n(\omega, \cdot)$ is continuous on V for any fixed $\omega \in \Omega$. We define the correspondences

$$\phi_n(\omega) := \{v \in V \mid J_n(\omega, v) = 0\},$$

for every $n \in \mathbb{N}$. It follows from Aliprantis and Border [4, Corollary 18.8] that this correspondence has a measurable graph, i.e.

$$\{(\omega, v) \in \Omega \times V \mid v \in \phi_n(\omega)\} \in \mathcal{F} \otimes \mathcal{B}(V).$$

Further, by Assumption 2.1 and the Lax–Milgram lemma (see for example [26, Lemma 6.97] and [11, Theorem 2.5]) we know that for every fixed $\omega \in \Omega$, there exists a unique solution $u(\omega, \cdot) \in V$ satisfying (2.4) for every $v \in V$. Therefore, we obtain for the graph of the solution mapping:

$$\begin{aligned} \{(\omega, u(\omega, \cdot)) \mid \omega \in \Omega\} &= \{(\omega, v) \in \Omega \times V \mid J_n(\omega, v) = 0,\ \text{for all } n \in \mathbb{N}\} \\ &= \bigcap_{n \in \mathbb{N}} \{(\omega, v) \in \Omega \times V \mid v \in \phi_n(\omega)\} \in \mathcal{F} \otimes \mathcal{B}(V). \end{aligned}$$

This implies for an arbitrary measurable set $\tilde{V} \in \mathcal{B}(V)$

$$\{(\omega, u(\omega, \cdot)) \mid \omega \in \Omega\} \cap \Omega \times \tilde{V} \in \mathcal{F} \otimes \mathcal{B}(V)$$

and therefore

$$\{\omega \in \Omega \mid u(\omega, \cdot) \in \tilde{V}\} \in \mathcal{F},$$

by the projection theorem (see [4, Theorem 18.25]), which gives the measurability of the solution mapping.

3 Subordinated Gaussian random fields

A random field $W : \Omega \times \mathcal{D} \to \mathbb{R}$ is called an $\mathbb{R} -$ *valued Gaussian random field (GRF)* if for any tuple $(\underline{x}_1, \ldots, \underline{x}_n) \subset \mathcal{D}$ and any number $n \in \mathbb{N}$ the $\mathbb{R}^n$-valued random variable

$$[W(\underline{x}_1), \dots, W(\underline{x}_n)]^T : \Omega \to \mathbb{R}^n$$

is multivariate normally distributed (see [3, Section 1.2]). Here $\underline{x}^T$ denotes the transpose of the vector $\underline{x}$. We denote by

$$m(\underline{x}) := \mathbb{E}(W(\underline{x})), \quad \underline{x} \in \mathcal{D}, \quad \text{and}$$
$$q(\underline{x}, \underline{y}) := Cov(W(\underline{x}), W(\underline{y})), \quad \underline{x}, \underline{y} \in \mathcal{D},$$

the associated mean and covariance function. The *covariance operator* $Q : L^2(\mathcal{D}) \to L^2(\mathcal{D})$ of W is defined by

$$Q(\psi)(\underline{x}) = \int_{\mathcal{D}} q(\underline{x}, \underline{y})\psi(\underline{y})d\underline{y} \text{ for } \underline{x} \in \mathcal{D}.$$

Further, if $\mathcal{D} \subset \mathbb{R}^d$ is compact and W is centered, i.e. $m \equiv 0$, there exists a decreasing sequence $(\lambda_i,\ i \in \mathbb{N})$ of real eigenvalues of Q with corresponding eigenfunctions $(e_i,\ i \in \mathbb{N}) \subset L^2(\mathcal{D})$ which form an orthonormal basis of $L^2(\mathcal{D})$ (see [3, Section 3.2] and [35, Theorem VI.3.2 and Chapter II.3]).

Example 3.1 One important class of continuous GRFs is given by the Matérn family: for a given smoothness parameter $\nu > 1/2$, correlation parameter $r > 0$ and variance $\sigma^2 > 0$, the Matérn-ν covariance function is given by $q_M(\underline{x}, \underline{y}) = \rho_M(\|\underline{x} - \underline{y}\|_2)$, for $(\underline{x}, \underline{y}) \in \mathbb{R}_+^d \times \mathbb{R}_+^d$, with

$$\rho_M(s) = \sigma^2 \frac{2^{1-\nu}}{\Gamma(\nu)} \Big(\frac{2s\sqrt{\nu}}{r}\Big)^{\nu} K_\nu\Big(\frac{2s\sqrt{\nu}}{r}\Big), \text{ for } s \geq 0,$$

where $\Gamma(\cdot)$ is the Gamma function and $K_\nu(\cdot)$ is the modified Bessel function of the second kind (see [23, Section 2.2 and Proposition 1]). Here, $\|\underline{x}\|_2 := (\sum_{i=1}^d \underline{x}_i^2)^{\frac{1}{2}}$ denotes the Euclidean norm of the vector $\underline{x} \in \mathbb{R}^d$. A Matérn-$\nu$ GRF is a centered GRF with covariance function q_M.

3.1 Construction of subordinated GRFs

Let $D > 0$ be fixed and $\mathcal{I} = [0, D]$ or $\mathcal{I} = \mathbb{R}_+$ with $\mathbb{R}_+ := [0, +\infty)$. A real-valued stochastic process $l = (l(t),\ t \in \mathcal{I})$ is said to be a *Lévy process* on $\mathcal{I}$ if $l(0) = 0$ $\mathbb{P}$-a.s., l has independent and stationary increments and l is stochastically continuous (see [5, Section 1.3]). One of the most important properties of Lévy processes is the so called *Lévy–Khinchin formula*.

Theorem 3.2 (Lévy–Khinchin formula, see [5, Th. 1.3.3]) *Let l be a real-valued Lévy process on the interval $\mathcal{I} \subseteq \mathbb{R}_+$. There exist constants, $\gamma_l \in \mathbb{R}$, $\sigma_l^2 \in \mathbb{R}_+$ and a*

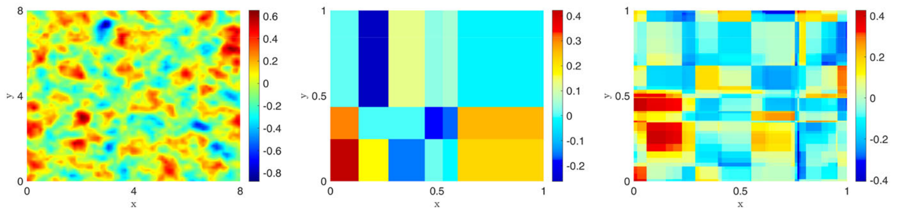

Fig. 1 Sample of Matérn-1.5-GRF (left), Poisson-subordinated GRF (middle) and Gamma-subordinated GRF (right)

measure ν on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that the characteristic function $\phi_{l(t)}$, for $t \in \mathcal{I}$, admits the representation

$$\phi_{l(t)}(u) := \mathbb{E}(\exp(iul(t))) = \exp\left(t\left(i\gamma_l u - \frac{\sigma_l^2}{2}u^2 + \int_{\mathbb{R}\setminus\{0\}} e^{iuy} - 1 - iuy\mathbb{1}_{\{|y|\leq 1\}}\nu(dy)\right)\right).$$

Motivated by Theorem 3.2 we denote by $(\gamma_l, \sigma_l^2, \nu)$ the characteristic triplet of the Lévy process l. A (Lévy-)subordinator is a Lévy process which is *non-decreasing* $\mathbb{P}$-a.s. By [5, Theorem 1.3.15] it follows that the Lévy triplet of a Lévy-subordinator always admits the form $(\gamma_l, 0, \nu)$ with a measure ν on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ satisfying

$$\nu(-\infty, 0) = 0 \text{ and } \int_0^\infty \min(y, 1)\, \nu(dy) < \infty.$$

Remark 3.3 Let $B = (B(t),\ t \geq 0)$ be a standard Brownian motion and $l = (l(t),\ t \geq 0)$ be a Lévy subordinator. The stochastic process defined by

$$L(t) := B(l(t)),\ t \geq 0,$$

is called *subordinated Brownian motion* and is again a Lévy process (see [5, Theorem 1.3.25]).

In [8] the authors propose a new approach to extend standard subordinated Lévy processes on a higher dimensional parameter space. Motivated by the rich class of subordinated Brownian motions, the authors construct discontinuous random fields by subordinating a GRF on a d-dimensional parameter domain by d one-dimensional Lévy subordinators. In case of a two-dimensional parameter space the construction is as follows: For a GRF $W : \Omega \times \mathbb{R}_+^2 \to \mathbb{R}$ and two (Lévy-)subordinators $l_1,\ l_2$ on $[0, D]$, with a finite $D > 0$, we define the real-valued random field

$$L(x, y) := W(l_1(x), l_2(y)),\ \text{for } x, y \in [0, D].$$

Figure 1 shows samples of a GRF with Martérn-1.5 covariance function and the corresponding subordinated field where we used Poisson and Gamma processes as subordinators.

This construction yields a rich class of discontinuous random fields which also admit a Lévy–Khinchin-type formula. Further, the newly constructed random fields are also interesting for practical reasons, since formulas for the covariance functions can be derived which is very useful for applications, e.g. in statistical fitting. For a theoretical investigation of the constructed random fields we refer to Barth and Merkle [8].

3.2 Subordinated GRFs as diffusion coefficients in elliptic problems

In the following, we define the specific diffusion coefficient that we consider in problem (2.1)–(2.3). In order to allow discontinuities, we incorporate a subordinated GRF in the coefficient additionally to a Gaussian component. The construction of the coefficient is done so that Theorem 2.3 is applicable and, at the same time, the coefficient is as versatile as possible.

Definition 3.4 We consider the domain $\mathcal{D} = (0, D)^2$ with $D < +\infty$.[1] We define the jump-diffusion coefficient a in problem (2.1)–(2.3) with $d = 2$ as

$$a : \Omega \times \mathcal{D} \to (0, +\infty),\ (\omega, x, y) \mapsto \overline{a}(x, y) + \Phi_1(W_1(x, y)) + \Phi_2(W_2(l_1(x), l_2(y))), \tag{3.1}$$

where

- $\overline{a} : \mathcal{D} \to (0, +\infty)$ is deterministic, continuous and there exist constants $\overline{a}_+, \overline{a}_- > 0$ with $\overline{a}_- \le \overline{a}(x, y) \le \overline{a}_+$ for $(x, y) \in \mathcal{D}$.
- $\Phi_1,\ \Phi_2 : \mathbb{R} \to [0, +\infty)$ are continuous.
- W_1 and W_2 are zero-mean GRFs on $\mathcal{D}$ respectively on $[0, +\infty)^2$ with $\mathbb{P} - a.s.$ continuous paths.
- l_1 and l_2 are Lévy subordinators on $[0, D]$ with Lévy triplets $(\gamma_1, 0, \nu_1)$ and $(\gamma_2, 0, \nu_2)$ which are independent of the GRFs W_1 and W_2.

Remark 3.5 The first two assumptions ensure that the diffusion coefficient a is positive over the domain $\mathcal{D}$. To show the convergence of the approximated diffusion coefficient in Sect. 5.1 we have to impose independence of the GRFs W_1 and W_2 (see Assumption 5.2 and the proof of Theorem 5.5). This assumption is in the sense natural as also one-dimensional Lévy processes admit an additive decomposition into a continuous part and a pure-jump part which are stochastically independent (Lévy–Itô decomposition, see e.g. [5, Theorem 2.4.11]). For the same reason the assumption that the Lévy subordinators are independent of the GRFs is also natural (see for example [5, Section 1.3.2]).

In order to verify Assumption 2.1(i) and (ii) we need the following Lemma. For a proof we refer to Barth and Merkle [8, Lemma 3.2 and Lemma 3.3]).

Lemma 3.6 *For fixed $(x, y) \in \mathcal{D}$ the mapping $\omega \mapsto a(\omega, x, y)$ is $\mathcal{F} - \mathcal{B}(\mathbb{R}_+)$-measurable. Further, for fixed $\omega \in \Omega$, the mapping $(x, y) \mapsto a(\omega, x, y)$ is $\mathcal{B}(\mathcal{D}) - \mathcal{B}(\mathbb{R}_+)$-measurable.*

[1] For simplicity we chose a square domain, recangular ones may be considered in the same way

Definition 3.4 guarantees the existence of a pathwise weak solution to problem (2.1), as we prove in the following theorem.

Theorem 3.7 *Let a be as in Definition 3.4 and let $f \in L^q(\Omega; H)$, $g \in L^q(\Omega; L^2(\Gamma_2))$ for some $q \in [1, +\infty)$. Then there exists a unique pathwise weak solution $u(\omega, \cdot) \in V$ to problem (2.1) for $\mathbb{P}$-almost every $\omega \in \Omega$. Furthermore, $u \in L^r(\Omega; V)$ for all $r \in [1, q)$ and*

$$\|u\|_{L^r(\Omega;V)} \leq C(\overline{a}_-, \mathcal{D})(\|f\|_{L^q(\Omega;H)} + \|g\|_{L^q(\Omega;L^2(\Gamma_2))}),$$

where $C(\overline{a}_-, \mathcal{D}) > 0$ is a constant depending only on the indicated parameter and the volume of $\mathcal{D}$.

Proof In order to apply Theorem 2.3 we have to verify Assumption 2.1. We have $a_-(\omega) = \inf_{x\in\mathcal{D}} a(\omega, x) \geq \overline{a}_-$ for every fixed $\omega \in \Omega$ by Definition 3.4. Further, $W_2(\omega)$ is continuous on $K(\omega) := [0, l_1(\omega, D)] \times [0, l_2(\omega, D)]$ and therefore

$$\begin{aligned} a_+(\omega) = \sup_{(x,y)\in\mathcal{D}} a(\omega, x, y) \leq \\ \overline{a}_+ + \sup_{(x,y)\in\mathcal{D}} \Phi_1(W_1(\omega, x, y)) + \sup_{(x,y)\in K(\omega)} \Phi_2(W_2(\omega, x, y)) < +\infty. \end{aligned}$$

For $1 \leq r < q$ define $p := (\frac{1}{r} - \frac{1}{q})^{-1} > 0$. We observe that

$$0 \leq \frac{1}{a_-(\omega)} \leq \frac{1}{\overline{a}_-} < \infty,$$

$\mathbb{P}$-a.s. and hence $1/a_- \in L^p(\Omega; \mathbb{R})$. Therefore, Assumption 2.1 holds with $r = (1/p + 1/q)^{-1}$ and the assertion follows by Theorem 2.3. □

4 Approximation of the diffusion coefficient

To simulate the solution to the elliptic equation we need to define a tractable approximation of the diffusion coefficient. Here, we face a new challenge regarding the GRF W_2 which is subordinated by the Lévy processes l_1 and l_2: due to the fact that the Lévy subordinators in general can attain any value in $[0, +\infty)$ we have to consider (and approximate) the GRF W_2 on the *unbounded* domain $[0, +\infty)$. In most cases where elliptic PDEs of the form (2.1) have been considered with a random coefficient, the problem is stated on a bounded domain, see e. g. [11, 13, 14, 23, 25]. Many regularity results for GRFs formulated for a bounded parameter space cannot easily be transferred to an unbounded parameter space (see also [3, Chapter 1], especially the discussion on p. 13). Even the Karhunen–Loève expansion of a GRF requires compactness of the domain (see e.g. [3, Section 3.2]). We avoid an unbounded parameter domain by bounding the subordinators from above in Sect. 4.1. Furthermore, to show convergence of the solution in Sect. 5 we need to bound the diffusion coefficient from above by a deterministic upper bound A (see Theorem 5.11 and Remark 5.9). Note that the weaker assumption of L^p integrability of the pathwise upper bound a_+ instead of a deterministic upper bound A of the diffusion coefficient is not enough in this setting. The reason being that the absence of a deterministic upper bound A would lead to an additional ω-dependence in the regularity constants in Assumption 5.7 which would be difficult to handle in the convergence theorem in Sect. 5.2 (see also Remark 5.9). Therefore, we cut off the diffusion coefficient at a deterministic level A (see Subsection 4.2). Subsequently we show that this induces an error in the solution approximation which can be controlled and which vanishes for growing A (see Sect. 5.1 and Theorem 5.1). Summarized, we derive an approximation in three steps: First, we bound the subordinators, and, second, we cut off the diffusion coefficient. Finally, we consider approximations of the GRFs and the subordinators and prove the convergence of this approximation of the diffusion coefficient under suitable assumptions.

4.1 First approximation: bounding the Lévy subordinators

For a fixed $K \in (0, +\infty)$, we define the cut-function $\chi_K : [0, +\infty) \to [0, K]$ as $\chi_K(z) := \min(z, K)$ for $z \in [0, +\infty)$. Instead of problem (2.1) we consider the following modified problem

$$-\nabla \cdot (a_K(\omega, \underline{x})\nabla u_K(\omega, \underline{x})) = f(\omega, \underline{x}) \text{ in } \Omega \times \mathcal{D}, \tag{4.1}$$

and impose the boundary conditions

$$u_K(\omega, \underline{x}) = 0 \text{ on } \Omega \times \Gamma_1, \tag{4.2}$$

$$a_K(\omega, \underline{x})\overrightarrow{n} \cdot \nabla u_K(\omega, \underline{x}) = g(\omega, \underline{x}) \text{ on } \Omega \times \Gamma_2. \tag{4.3}$$

Here, the diffusion coefficient is defined by

$$\begin{aligned} a_K : \Omega \times \mathcal{D} \to (0, +\infty),\ (\omega, x, y) \mapsto \overline{a}(x, y) &+ \Phi_1(W_1(x, y)) \\ &+ \Phi_2(W_2(\chi_K(l_1(x)), \chi_K(l_2(y)))). \end{aligned} \tag{4.4}$$

For functions $f \in L^q(\Omega; H)$ and $g \in L^q(\Omega; L^2(\Gamma_2))$ with $q \in [1, +\infty)$, there exists a weak solution $u_K \in L^r(\Omega; V)$ to problem (4.1)–(4.3) for $r \in [1, q)$ (see Theorem 3.7).[2]

Remark 4.1 We note that the influence of this problem modification can be controlled: one may choose $K > 0$ such that

$$\mathbb{P}\left(\max(\sup_{x\in[0,D]} l_1(x), \sup_{y\in[0,D]} l_2(y)) \geq K\right) = \mathbb{P}(\max(l_1(D), l_2(D)) \geq K) < \varepsilon$$

[2] For simplicity we assume one fixed K for all spatial dimensions. The results in the subsequent sections hold for individual independent values in each spatial dimension as well.

for any $\varepsilon > 0$. In other words, pathwise the modified problem coincides with the original one up to a set of samples, whose probability can be made arbitrarily small.

4.2 Second modification: diffusion cut-off

We consider again the cut function $\chi_A(z) := \min(z, A)$ for $z \in [0, +\infty)$ with a fixed positive number $A > 0$ and consider the following problem

$$-\nabla \cdot (a_{K,A}(\omega, \underline{x})\nabla u_{K,A}(\omega, \underline{x})) = f(\omega, \underline{x}) \text{ in } \Omega \times \mathcal{D}, \tag{4.5}$$

where we impose the boundary conditions

$$u_{K,A}(\omega, \underline{x}) = 0 \text{ on } \Omega \times \Gamma_1, \tag{4.6}$$

$$a_{K,A}(\omega, \underline{x})\overrightarrow{n} \cdot \nabla u_{K,A}(\omega, \underline{x}) = g(\omega, \underline{x}) \text{ on } \Omega \times \Gamma. \tag{4.7}$$

The diffusion coefficient $a_{K,A}$ is defined by

$$\begin{aligned} a_{K,A} &: \Omega \times \mathcal{D} \to (0, +\infty), \\ &(\omega, x, y) \mapsto \chi_A\Big(\overline{a}(x, y) + \Phi_1(W_1(x, y)) + \Phi_2(W_2(\chi_K(l_1(x)), \chi_K(l_2(y))))\Big). \end{aligned} \tag{4.8}$$

Again, Theorem 3.7 applies in this case and yields the existence of a pathwise weak solution $u_{K,A} \in L^r(\Omega; V)$ for $r \in [1, q)$ if $f \in L^q(\Omega; H)$ and $g \in L^q(\Omega; L^2(\Gamma_2))$. The error of the modification vanishes for growing A as shown in the end of Sect. 5.1.

4.3 Approximation of GRF and subordinators

In the following, we show how to approximate the modified diffusion coefficient $a_{K,A}$ using approximations $W_1^{\varepsilon_W} \approx W_1$, $W_2^{\varepsilon_W} \approx W_2$ of the GRFs and $l_1^{\varepsilon_l} \approx l_1$, $l_2^{\varepsilon_l} \approx l_2$ of the Lévy subordinators. We aim to approximate the GRFs W_1 and W_2 in the diffusion coefficient by sampling on a discrete grid. These approximations may be extended to the whole domain using linear interpolation (see [24, 25]). Certain regularity assumptions on the GRFs then allow for a quantification of the corresponding approximation error (see Lemma 4.4). In [10], the authors considered approximations of general Lévy processes and quantified the approximation error in an $L^s(\Omega)$-sense uniformly in $x \in [0, D]$. Such an approximation may be obtained by piecewise constant approximations constructed from samples of the process $(l_j(x_i),\ i = 1, \dots, N_l)$ on a discrete grid $(x_i,\ i = 1, \dots, N_l) \subset [0, D]$ with $N_l \in \mathbb{N}$ grid points, for $j = 1, 2$ (see Sect. 7, [10, 33]). Motivated by this, we formulate the following working assumptions on the covariance operators of the Gaussian fields, the subordinators and the data of the elliptic problem.

Assumption 4.2 Let W_1 be a zero-mean GRF on $[0, D]^2$ and W_2 be a zero-mean GRF on $[0, K]^2$. We denote by $q_1 : [0, D]^2 \times [0, D]^2 \to \mathbb{R}$ and $q_2 : [0, K]^2 \times$

$[0, K]^2 \to \mathbb{R}$ the covariance functions of these random fields and by Q_1, Q_2 the associated covariance operators defined by

$$Q_j\phi = \int_{[0,z_j]^2} q_j((x, y), (x', y'))\phi(x', y')d(x', y'),$$

for $\phi \in L^2([0, z_j]^2)$ with $z = (D, K)$ and $j = 1, 2$. We denote by $(\lambda_i^{(1)}, e_i^{(1)},\ i \in \mathbb{N})$ resp. $(\lambda_i^{(2)}, e_i^{(2)},\ i \in \mathbb{N})$ the eigenpairs associated to the covariance operators Q_1 and Q_2. In particular, $(e_i^{(1)},\ i \in \mathbb{N})$ resp. $(e_i^{(2)},\ i \in \mathbb{N})$ are ONBs of $L^2([0, D]^2)$ resp. $L^2([0, K]^2)$.

i. We assume that the eigenfunctions are continuously differentiable and there exist positive constants $\alpha,\ \beta,\ C_e,\ C_\lambda > 0$ such that for any $i \in \mathbb{N}$ it holds

$$\|e_i^{(1)}\|_{L^\infty([0,D]^2)},\ \|e_i^{(2)}\|_{L^\infty([0,K]^2)} \le C_e,$$
$$\|\nabla e_i^{(1)}\|_{L^\infty([0,D]^2)},\ \|\nabla e_i^{(2)}\|_{L^\infty([0,K]^2)} \le C_e i^\alpha,$$
$$\sum_{i=1}^{\infty}(\lambda_i^{(1)} + \lambda_i^{(2)})i^\beta \le C_\lambda < +\infty.$$

ii. There exist constants $\phi,\ \psi, C_{lip} > 0$ such that the continuous functions $\Phi_1,\ \Phi_2 : \mathbb{R} \to [0, +\infty)$ from Definition 3.4 satisfy

$$|\Phi_1'(x)| \le \phi \exp(\psi|x|),\ |\Phi_2(x) - \Phi_2(y)| \le C_{lip}\,|x - y| \text{ for } x, y \in \mathbb{R}.$$

In particular, $\Phi_1 \in C^1(\mathbb{R})$.

iii. $f \in L^q(\Omega; H)$ and $g \in L^q(\Omega; L^2(\Gamma_2))$ for some $q \in (1, +\infty)$.

iv. $\overline{a} : \mathcal{D} \to (0, +\infty)$ is deterministic, continuous and there exist constants $\overline{a}_+, \overline{a}_- > 0$ with $\overline{a}_- \le \overline{a}(x, y) \le \overline{a}_+$ for $(x, y) \in \mathcal{D}$.

v. l_1 and l_2 are Lévy subordinators on $[0, D]$ with Lévy triplets $(\gamma_1, 0, \nu_1)$ and $(\gamma_2, 0, \nu_2)$ which are independent of the GRFs W_1 and W_2. Further, we assume that we have approximations $l_1^{(\varepsilon_l)},\ l_2^{(\varepsilon_l)}$ of these processes and there exist constants $C_l > 0$ and $\eta > 1$ such that for every $s \in [1, \eta)$ it holds

$$\mathbb{E}(|l_j(x) - l_j^{(\varepsilon_l)}(x)|^s) \le C_l\varepsilon_l,$$

for $\varepsilon_l > 0$, $x \in [0, D]$ and $j = 1, 2$.

Remark 4.3 Note that the first assumption on the eigenpairs of the GRFs is natural (see [11, 23]). For example, the case that $Q_1,\ Q_2$ are Matérn covariance operators are included. Assumption 4.2(ii) is necessary to be able to quantify the error of the approximation of the diffusion coefficient. Assumption 4.2(iii) is necessary to ensure the existence of a solution and has already been formulated in Assumption 2.1. The last assumption ensures that we can approximate the Lévy subordinators in an L^s-sense. This can always be achieved under appropriate assumptions on the tails of the distribution of the subordinators, see [10, Assumption 3.6, Assumption 3.7 and Theorem 3.21].

For any numerical simulation we have to approximate the GRF as well as the subordinating Lévy processes, which results in an additional approximation of the coefficient $a_{K,A}$ given in Eq. (4.8). In the following we want to quantify the error induced by this approximation.

It follows by an application of the Kolmogorov–Chentsov theorem ([16, Theorem 3.5]) that W_1 and W_2 can be assumed to have Hölder-continuous paths with Hölder exponent $b \in (0, (2\gamma k - 2)/(2k))$ for $0 < \gamma \leq \min(1, \beta/(2\alpha))$ and every $k \in \mathbb{N}$ (see the proof of [11, Lemma 3.5]). Further, it follows by an application of the Sobolev embedding theorem that $W_1 \in L^n(\Omega; C^{0,\gamma}([0, D]^2))$ and $W_2 \in L^n(\Omega; C^{0,\gamma}([0, K]^2))$, i.e.

$$\mathbb{E}\left(\left(\sup_{z\neq z'\in[0,D]^2} \frac{|W_1(z) - W_1(z')|}{|z - z'|_2^{\gamma}}\right)^n\right), \mathbb{E}\left(\left(\sup_{z\neq z'\in[0,K]^2} \frac{|W_2(z) - W_2(z')|}{|z - z'|_2^{\gamma}}\right)^n\right) < +\infty, \tag{4.9}$$

for every $n \in [1, +\infty)$ and $\gamma < \min(1, \beta/(2\,\alpha))$ (see [13, Proposition 3.1]).

Next, we prove a bound on the error of the approximated diffusion coefficient, where the GRFs are approximated by a discrete evaluation and (bi-)linear interpolation between these points (see [24, 25]).

Lemma 4.4 *We consider the discrete grids $G_1^{(\varepsilon_W)} = \{(x_i, x_j)|\ i, j = 0, \dots, M_{\varepsilon_W}^{(1)}\}$ on $[0, D]^2$ and $G_2^{(\varepsilon_W)} = \{(y_i, y_j)|\ i, j = 0, \dots, M_{\varepsilon_W}^{(2)}\}$ on $[0, K]^2$ where $(x_i,\ i = 0, \dots, M_{\varepsilon_W}^{(1)})$ is an equidistant grid on $[0, D]$ with maximum step size ε_W and $(y_i,\ i = 0, \dots, M_{\varepsilon_W}^{(2)})$ is an equidistant grid on $[0, K]$ with maximum step size ε_W. Further, let $W_1^{(\varepsilon_W)}$ and $W_2^{(\varepsilon_W)}$ be approximations of the GRFs $W_1,\ W_2$ on the discrete grids $G_1^{(\varepsilon_W)}$ resp. $G_2^{(\varepsilon_W)}$ which are constructed by point evaluation of the random fields W_1 and W_2 on the grids and linear interpolation between the grid points. Under Assumption 4.2(i) it holds for $n \in [1, +\infty)$:*

$$\|W_1 - W_1^{(\varepsilon_W)}\|_{L^n(\Omega;L^\infty([0,D]^2))} \leq C(D, n)\varepsilon_W^{\gamma}$$
$$\|W_2 - W_2^{(\varepsilon_W)}\|_{L^n(\Omega;L^\infty([0,K]^2))} \leq C(K, n)\varepsilon_W^{\gamma}$$

for $\gamma < \min(1, \beta/(2\alpha))$ where β and α are the parameters from Assumption 4.2.

Proof Note that for any fixed $\omega \in \Omega$ and $\mathcal{D}_{ij} := [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ with $i, j \in \{1, \dots, M_{\varepsilon_W}^{(1)}\}$, it holds

$$\begin{aligned}&\max_{(x,y)\in\mathcal{D}_{ij}} W_1^{(\varepsilon_W)}(x, y),\ \min_{(x,y)\in\mathcal{D}_{ij}} W_1^{(\varepsilon_W)}(x, y)\\ &\quad \in \{W_1(x_i, y_j), W_1(x_{i+1}, y_j), W_1(x_i, y_{j+1}), W_1(x_{i+1}, y_{j+1})\}.\end{aligned}$$

This holds since $W_1^{(\varepsilon_W)}$ is constructed by (bi-)linear interpolation of the GRF W_1 and the piecewise linear interpolants attain their maximum and minimum at the corners (the Hessian evaluated at the (unique) stationary point of the bilinear basis functions is always indefinite). Therefore, for a fixed $(x, y) \in [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ it follows from the intermediate value theorem that $W_1^{(\varepsilon_W)}(x, y) = W_1(x', y')$ for appropriate $(x', y') \in [x_i, x_{i+1}] \times [y_j, y_{j+1}]$. Using this observation we estimate

$$
\begin{aligned}
&\|W_1 - W_1^{(\varepsilon_W)}\|^n_{L^n(\Omega;L^\infty([0,D]^2))}\\
&= \mathbb{E}\left(\sup_{(x,y)\in[0,D]^2} |W_1(x, y) - W_1^{(\varepsilon_W)}(x, y)|^n\right)\\
&\le \mathbb{E}\left(\sup_{\substack{(x,y),(x',y')\in[0,D]^2,\\ |(x,y)^T-(x',y')^T|_2\le\sqrt{2}\varepsilon_W}} |W_1(x, y) - W_1(x', y')|^n\right)\\
&= \varepsilon_W^{n\gamma}\, \mathbb{E}\left(\sup_{\substack{(x,y),(x',y')\in[0,D]^2,\\ |(x,y)^T-(x',y')^T|_2\le\sqrt{2}\varepsilon_W}} \frac{|W_1(x, y) - W_1(x', y')|^n}{\varepsilon_W^{n\gamma}}\right)\\
&\le 2^{\frac{n\gamma}{2}} \varepsilon_W^{n\gamma}\, \mathbb{E}\left(\left(\sup_{\substack{(x,y)\neq(x',y')\in[0,D]^2,\\ |(x,y)^T-(x',y')^T|_2\le\sqrt{2}\varepsilon_W}} \frac{|W_1(x, y) - W_1(x', y')|}{|(x, y)^T - (x', y')^T|_2^\gamma}\right)^n\right)\\
&\le 2^{\frac{n\gamma}{2}} \varepsilon_W^{n\gamma}\, \mathbb{E}\left(\left(\sup_{(x,y)\neq(x',y')\in[0,D]^2} \frac{|W_1(x, y) - W_1(x', y')|}{|(x, y)^T - (x', y')^T|_2^\gamma}\right)^n\right)\\
&\le C(D)\varepsilon_W^{n\gamma},
\end{aligned}
$$

where we used Eq. (4.9) in the last step. Equivalently the error bound for W_2 follows. □

Remark 4.5 Note that Lemma 4.4 immediately implies for $m \in [1, +\infty)$

$$
\begin{aligned}
\|W_1 - W_1^{(\varepsilon_W)}\|_{L^n(\Omega;L^m([0,D]^2))} &= \mathbb{E}\Big(\Big(\int_{[0,D]^2} |W_1(x, y) - W_1^{(\varepsilon_W)}(x, y)|^m d(x, y)\Big)^{\frac{n}{m}}\Big)^{\frac{1}{n}}\\
&\le D^{\frac{2}{m}}\mathbb{E}\left(\sup_{(x,y)\in[0,D]^2} |W_1(x, y) - W_1^{(\varepsilon_W)}(x, y)|^n\right)^{\frac{1}{n}}\\
&= D^{\frac{2}{m}} \|W_1 - W_1^{(\varepsilon_W)}\|_{L^n(\Omega;L^\infty([0,D]^2))} \le C(D, m, n)\varepsilon_W^\gamma.
\end{aligned}
$$

4.4 Convergence to the modified diffusion coefficient

Given some approximations $W_1^{(\varepsilon_W)} \approx W_1$, $W_2^{(\varepsilon_W)} \approx W_2$ as in Lemma 4.4 and approximations $l_1^{(\varepsilon_l)} \approx l_1$, $l_2^{(\varepsilon_l)} \approx l_2$ as in Assumption 4.2 v as well as some fixed constants $K, A > 0$, we approximate the diffusion coefficient $a_{K,A}$ in (4.8) by $a_{K,A}^{(\varepsilon_W,\varepsilon_l)} : \Omega \times \mathcal{D} \to (0, +\infty)$ with

$$a_{K,A}^{(\varepsilon_W,\varepsilon_l)}(x, y) = \chi_A\Big(\overline{a}(x, y) + \Phi_1(W_1^{(\varepsilon_W)}(x, y)) + \Phi_2(W_2^{(\varepsilon_W)}(\chi_K(l_1^{(\varepsilon_l)}(x)), \chi_K(l_2^{(\varepsilon_l)}(y))))\Big) \tag{4.10}$$

for $(x, y) \in \mathcal{D}$. To prove a convergence result for this approximated coefficient (Theorem 4.8) we need the following two technical lemmas. The second can be proved by the use of Sato [32, Proposition 1.16]. For a detailed proof we refer to Barth and Merkle [8].

Lemma 4.6 *For $n, m \in [1, +\infty)$ with $n \geq m$ and $\varphi \in L^n([0, D]^2 \times \Omega; \mathbb{R}, \lambda \otimes \mathbb{P})$), where λ denotes the Lebesgue measure on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$, it holds*

$$\|\varphi\|_{L^n(\Omega;L^m([0,D]^2))} \leq C(D, n, m)\|\varphi\|_{L^n([0,D]^2\times\Omega;\mathbb{R}))}.$$

Proof The case $n = m$ is trivial. For $n > m$ we use Hölder's inequality and obtain

$$\begin{aligned}
\|\varphi\|_{L^n(\Omega;L^m([0,D]^2))} &= \left(\int_\Omega \left(\int_{[0,D]^2} |\varphi(x, y)|^m d(x, y)\right)^{\frac{n}{m}} d\mathbb{P}\right)^{\frac{1}{n}} \\
&\leq \left(\int_\Omega \left(\int_{[0,D]^2} |\varphi(x)|^n d(x, y)\right) D^{\frac{2(n-m)}{m}} d\mathbb{P}\right)^{\frac{1}{n}} \\
&= D^{\frac{2}{m}-\frac{2}{n}} \|\varphi\|_{L^n([0,D]^2\times\Omega;\mathbb{R}))}.
\end{aligned}$$

□

Lemma 4.7 *Let $W : \Omega \times \mathbb{R}_+^d \to \mathbb{R}$ be a $\mathbb{P}$-a.s. continuous random field and let $Z : \Omega \to \mathbb{R}_+^d$ be a $\mathbb{R}_+^d$-valued random variable which is independent of the random field W. Further, let $\varphi : \mathbb{R} \to \mathbb{R}$ be a deterministic, continuous function. It holds*

$$\mathbb{E}(\varphi(W(Z)) = \mathbb{E}(\zeta(Z)),$$

where $\zeta(z) := \mathbb{E}(\varphi(W(z))$ for deterministic $z \in \mathbb{R}_+^d$.

Theorem 4.8 *Let $W_1^{(\varepsilon_W)} \approx W_1$, $W_2^{(\varepsilon_W)} \approx W_2$ be approximations of the GRFs on discrete grids as in Lemma 4.4. Further, let $1 \leq t \leq s < \eta$ and $0 < \gamma < min(1, \beta/(2\alpha))$ such that $s\gamma \geq 2$. Under Assumption 4.2 we get the following error bound for the approximation of the diffusion coefficient:*

$$\|a_{K,A} - a_{K,A}^{(\varepsilon_W,\ \varepsilon_l)}\|_{L^s(\Omega;L^t([0,D]^2))} \leq C(\varepsilon_W^\gamma + \varepsilon_l^{\frac{1}{s}}),$$

with a constant C which does not depend on the discretization parameters ε_W and ε_l.

Proof Since the cut function χ_A is Lipschitz continuous with Lipschitz constant 1 we calculate

$$\begin{aligned}
&\|a_{K,A} - a_{K,A}^{(\varepsilon_W,\ \varepsilon_l)}\|_{L^s(\Omega;L^t([0,D]^2))} \\
&\quad \leq \|\Phi_1(W_1) - \Phi_1(W_1^{(\varepsilon_W)})\|_{L^s(\Omega;L^t([0,D]^2))} \\
&\qquad + \|\Phi_2(W_2(\chi_K(l_1), \chi_K(l_2))) - \Phi_2(W_2^{(\varepsilon_W)}(\chi_K(l_1^{(\varepsilon_l)}), \chi_K(l_2^{(\varepsilon_l)})))\|_{L^s(\Omega;L^t([0,D]^2))} \\
&\quad =: I_1 + I_2
\end{aligned}$$

First, we consider I_1 and use Assumption 4.2(ii) and the same calculation as in Remark 4.5 to get

$$I_1 \leq D^{\frac{2}{t}} \|\Phi_1(W_1) - \Phi_1(W_1^{(\varepsilon_W)})\|_{L^s(\Omega;L^\infty([0,D]^2))}.$$

The mean value theorem yields, for fixed $(x, y) \in [0, D]^2$ and an appropriately chosen value $\xi \in (\min(W_1(x, y), W_1^{(\varepsilon_W)}(x, y)), \max(W_1(x, y), W_1^{(\varepsilon_W)}(x, y)))$,

$$\begin{aligned}
&|\Phi_1(W_1(x, y)) - \Phi_1(W_1^{(\varepsilon_W)}(x, y))| \\
&\quad = |\Phi_1'(\xi)|\,|W_1(x, y) - W_1^{(\varepsilon_W)}(x, y)| \\
&\quad \leq \phi\, \exp(\psi|\xi|)|W_1(x, y) - W_1^{(\varepsilon_W)}(x, y)| \\
&\quad \leq \phi\, \max\{\exp(\psi|W_1(x, y)|), \exp(\psi|W_1^{(\varepsilon_W)}(x, y)|)\}\,|W_1(x, y) - W_1^{(\varepsilon_W)}(x, y)|,
\end{aligned}$$

for $\mathbb{P}$-almost every $\omega \in \Omega$. As already mentioned in the proof of Lemma 4.4, for any $\omega \in \Omega$ and fixed $\mathcal{D}_{ij} := [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ with $i, j \in \{1, \ldots, M_{\varepsilon_W}^{(1)}\}$, it holds

$$\begin{aligned}
&\max_{(x,y)\in\mathcal{D}_{ij}} W_1^{(\varepsilon_W)}(x, y), \quad \min_{(x,y)\in\mathcal{D}_{ij}} W_1^{(\varepsilon_W)}(x, y) \\
&\quad \in \{W_1(x_i, y_j), W_1(x_{i+1}, y_j), W_1(x_i, y_{j+1}), W_1(x_{i+1}, y_{j+1})\}.
\end{aligned}$$

Therefore, we obtain the pathwise estimate

$$\begin{aligned}
&\|\Phi_1(W_1) - \Phi_1(W_1^{(\varepsilon_W)})\|_{L^\infty([0,D]^2)} \\
&\quad \leq \max_{(x,y)\in[0,D]^2} \phi\, \max\{\exp(\psi|W_1(x, y)|), \\
&\qquad \exp(\psi|W_1^{(\varepsilon_W)}(x, y)|)\} \max_{(x,y)\in[0,D]^2} |W_1(x, y) - W_1^{(\varepsilon_W)}(x, y)| \\
&\quad \leq \phi \exp\Big(\psi\, \max\{\max_{(x,y)\in[0,D]^2} |W_1(x, y)|, \max_{(x,y)\in[0,D]^2} |W_1^{(\varepsilon_W)}(x, y)|\}\Big) \\
&\qquad \times \max_{(x,y)\in[0,D]^2} |W_1(x, y) - W_1^{(\varepsilon_W)}(x, y)| \\
&\quad = \phi\, \exp\Big(\psi \max_{(x,y)\in[0,D]^2} |W_1(x, y)|\Big) \max_{(x,y)\in[0,D]^2} |W_1(x, y) - W_1^{(\varepsilon_W)}(x, y)|.
\end{aligned}$$

Finally, we obtain for any $n_1, n_2 \in [1, +\infty)$ with $1/n_1 + 1/n_2 = 1$ by Hölder's inequality

$$\begin{aligned} I_1 &\leq D^{\frac{2}{t}} \| \Phi_1(W_1) - \Phi_1(W_1^{(\varepsilon_W)}) \|_{L^s(\Omega; L^\infty([0,D]^2))} \\ &\leq D^{\frac{2}{t}} \phi \, \| \exp(\psi |W_1|) \|_{L^{sn_1}(\Omega; L^\infty([0,D]^2))} \, \| W_1 - W_1^{(\varepsilon_W)} \|_{L^{sn_2}(\Omega; L^\infty([0,D]^2))} \\ &\leq C(D) \varepsilon_W^\gamma, \end{aligned}$$

where we used Lemma 4.4 and the fact that $\| \exp(\psi |W_1|) \|_{L^{sn_1}(\Omega; L^\infty([0,D]^2))} < \infty$ (see [3, Theorem 2.1.1] and the proof of Theorem 5.5 for more details).

For the second summand we calculate:

$$\begin{aligned} I_2 &\leq \| \Phi_2(W_2(\chi_K(l_1), \chi_K(l_2))) - \Phi_2(W_2(\chi_K(l_1^{(\varepsilon_l)}), \chi_K(l_2^{(\varepsilon_l)}))) \|_{L^s(\Omega; L^t([0,D]^2))} \\ &\quad + \| \Phi_2(W_2(\chi_K(l_1^{(\varepsilon_l)}), \chi_K(l_2^{(\varepsilon_l)}))) - \Phi_2(W_2^{(\varepsilon_W)}(\chi_K(l_1^{(\varepsilon_l)}), \chi_K(l_2^{(\varepsilon_l)}))) \|_{L^s(\Omega; L^t([0,D]^2))} \\ &= I_3 + I_4. \end{aligned}$$

We use the same calculation as in Remark 4.5 and the Lipschitz continuity of Φ_2 to calculate for the summand I_4

$$\begin{aligned} I_4 &\leq D^{\frac{2}{t}} \| \Phi_2(W_2(\chi_K(l_1^{(\varepsilon_l)}), \chi_K(l_2^{(\varepsilon_l)}))) - \Phi_2(W_2^{(\varepsilon_W)}(\chi_K(l_1^{(\varepsilon_l)}), \chi_K(l_2^{(\varepsilon_l)}))) \|_{L^s(\Omega; L^\infty([0,D]^2))} \\ &\leq C_{lip} D^{\frac{2}{t}} \| W_2(\chi_K(l_1^{(\varepsilon_l)}), \chi_K(l_2^{(\varepsilon_l)})) - W_2^{(\varepsilon_W)}(\chi_K(l_1^{(\varepsilon_l)}), \chi_K(l_2^{(\varepsilon_l)})) \|_{L^s(\Omega; L^\infty([0,D]^2))} \\ &\leq C_{lip} D^{\frac{2}{t}} \| W_2 - W_2^{(\varepsilon_W)} \|_{L^s(\Omega; L^\infty([0,K]^2))} \\ &\leq C_{lip} D^{\frac{2}{t}} C(K, s) \varepsilon_W^\gamma, \end{aligned}$$

where we used Lemma 4.4. It remains to bound the summand I_3. We estimate using Lemma 4.6

$$\begin{aligned} I_3 &= \| \Phi_2(W_2(\chi_K(l_1), \chi_K(l_2))) - \Phi_2(W_2(\chi_K(l_1^{(\varepsilon_l)}), \chi_K(l_2^{(\varepsilon_l)}))) \|_{L^s(\Omega; L^t([0,D]^2))} \\ &\leq D^{\frac{2}{t} - \frac{2}{s}} \| \Phi_2(W_2(\chi_K(l_1), \chi_K(l_2))) - \Phi_2(W_2(\chi_K(l_1^{(\varepsilon_l)}), \chi_K(l_2^{(\varepsilon_l)}))) \|_{L^s([0,D]^2 \times \Omega)} \\ &= D^{\frac{2}{t} - \frac{2}{s}} \Big(\int_{[0,D]^2} \mathbb{E}\Big(|\Phi_2\big(W_2(\chi_K(l_1(x)), \chi_K(l_2(y)))\big) \\ &\quad - \Phi_2(W_2(\chi_K(l_1^{(\varepsilon_l)}(x)), \chi_K(l_2^{(\varepsilon_l)}(y))))|^s \Big) d(x,y) \Big)^{\frac{1}{s}} \\ &\leq C_{lip} D^{\frac{2}{t} - \frac{2}{s}} \Big(\int_{[0,D]^2} \mathbb{E}(|W_2(\chi_K(l_1(x)), \chi_K(l_2(y))) \\ &\quad - W_2(\chi_K(l_1^{(\varepsilon_l)}(x)), \chi_K(l_2^{(\varepsilon_l)}(y)))|^s) d(x,y) \Big)^{\frac{1}{s}}. \end{aligned}$$

We know by Lemma 4.7 that it holds for $(x, y) \in [0, D]^2$

$$\begin{aligned} &\mathbb{E}(|W_2(\chi_K(l_1(x)), \chi_K(l_2(y))) - W_2(\chi_K(l_1^{(\varepsilon_l)}(x)), \chi_K(l_2^{(\varepsilon_l)}(y)))|^s) \\ &\quad = \mathbb{E}(\varphi(\chi_K(l_1(x)), \chi_K(l_2(y)), \chi_K(l_1^{(\varepsilon_l)}(x)), \chi_K(l_2^{(\varepsilon_l)}(y)))) \end{aligned}$$

where

$$\varphi(x, y, v, w) := \mathbb{E}(|W_2(x, y) - W_2(v, w)|^s).$$

For $(x, y) = (v, w)$ it holds $\varphi(x, y, v, w) = 0$ and for $(x, y) \neq (v, w) \in [0, K]^2$ we get

$$\begin{aligned}
\varphi(x, y, v, w) &= |(x, y)^T - (v, w)^T|_2^{\gamma s}\, \mathbb{E}\Big(\frac{|W_2(x, y) - W_2(v, w)|^s}{|(x, y)^T - (v, w)^T|_2^{\gamma s}}\Big) \\
&\leq |(x, y)^T - (v, w)^T|_2^{\gamma s} \mathbb{E}\left(\left(\sup_{z \neq z' \in [0,K]^2} \frac{|W_2(z) - W_2(z')|}{|z - z'|_2^{\gamma}}\right)^s\right) \\
&\leq C(K)|(x, y)^T - (v, w)^T|_2^{\gamma s}
\end{aligned}$$

by Eq. (4.9). Further, we know from Hölder's inequality for $\gamma s \geq 2$ that it holds

$$\begin{aligned}
|(x, y)^T - (v, w)^T|_2^{\gamma s} &= ((x - v)^2 + (y - w)^2)^{\frac{\gamma s}{2}} \\
&\leq 2^{\frac{\gamma s}{2} - 1}(|x - v|^{\gamma s} + |y - w|^{\gamma s})
\end{aligned}$$

and therefore we calculate

$$\begin{aligned}
&\mathbb{E}(|W_2(\chi_K(l_1(x)), \chi_K(l_2(y))) - W_2(\chi_K(l_1^{(\varepsilon_l)}(x)), \chi_K(l_2^{(\varepsilon_l)}(y)))|^s) \\
&\quad \leq 2^{\frac{\gamma s}{2} - 1} C(K)\mathbb{E}(|\chi_K(l_1(x)) - \chi_K(l_1^{(\varepsilon_l)}(x))|^{\gamma s} \\
&\qquad + |\chi_K(l_2(y)) - \chi_K(l_2^{(\varepsilon_l)}(y))|^{\gamma s}) \\
&\quad \leq 2^{\frac{\gamma s}{2} - 1} C(K)\mathbb{E}(|l_1(x) - l_1^{(\varepsilon_l)}(x)|^{\gamma s} + |l_2(y) - l_2^{(\varepsilon_l)}(y)|^{\gamma s}) \\
&\quad \leq 2^{\frac{\gamma s}{2}} C(K) C_l \varepsilon_l
\end{aligned}$$

where we used the Lipschitz continuity of χ_K and Assumption 4.2(v) in the last step. Therefore, we finally obtain

$$I_3 \leq C_{lip} D^{\frac{2}{t}} 2^{\frac{\gamma}{2}} C(K)^{\frac{1}{s}} C_l^{\frac{1}{s}} \varepsilon_l^{\frac{1}{s}} =: C(C_{lip}, D, t, \gamma, s, K, C_l)\varepsilon_l^{\frac{1}{s}}$$

which proves that

$$\|a_{K,A} - a_{K,A}^{(\varepsilon_W,\, \varepsilon_l)}\|_{L^s(\Omega; L^t([0,D]^2))} \leq C(D, K, C_{lip}, t, \gamma, s, C_l)(\varepsilon_W^{\gamma} + \varepsilon_l^{\frac{1}{s}}).$$

□

5 Convergence analysis

In this section we derive an error bound for the approximation of the solution. We split the error in two components: the first component is associated with the cut-off of the diffusion coefficient we described in Sect. 4.2. The second error contributor corresponds to the approximation of the GRFs and the Lévy subordinators we considered in Sect. 4.4.

Let $r \in [1, q)$ with q as in Assumption 4.2(iii) and denote by $u_K \in L^r(\Omega; V)$ the weak solution to problem (4.1)–(4.4). Further, let $u_{K,A}^{(\varepsilon_W,\varepsilon_l)} \in L^r(\Omega; V)$ be the weak solution to the problem

$$-\nabla \cdot (a_{K,A}^{(\varepsilon_W,\varepsilon_l)}(\omega, \underline{x})\nabla u_{K,A}^{(\varepsilon_W,\varepsilon_l)}(\omega, \underline{x})) = f(\omega, \underline{x}) \text{ in } \Omega \times \mathcal{D}, \tag{5.1}$$

with boundary conditions

$$u_{K,A}^{(\varepsilon_W,\varepsilon_l)}(\omega, \underline{x}) = 0 \text{ on } \Omega \times \Gamma_1, \tag{5.2}$$

$$a_{K,A}^{(\varepsilon_W,\varepsilon_l)}(\omega, \underline{x})\overrightarrow{n} \cdot \nabla u_{K,A}^{(\varepsilon_W,\varepsilon_l)}(\omega, x) = g(\omega, x) \text{ on } \Omega \times \Gamma_2. \tag{5.3}$$

Note that Theorem 3.7 also applies to the elliptic problem with coefficient $a_{K,A}^{(\varepsilon_W,\varepsilon_l)}$. The aim of this section is to quantify the error of the approximation $u_{K,A}^{(\varepsilon_W,\varepsilon_l)} \approx u_K$.[3] By the triangle inequality we obtain

$$\|u_K - u_{K,A}^{(\varepsilon_W,\varepsilon_l)}\| \leq \|u_K - u_{K,A}\| + \|u_{K,A} - u_{K,A}^{(\varepsilon_W,\varepsilon_l)}\| =: E_1 + E_2 \tag{5.4}$$

for any suitable norm $\|\cdot\|$. Here, $u_{K,A}$ is the solution to the truncated problem (4.5)–(4.8). We consider the two error contributions E_1 and E_2 separately in this section. The results presented in Sects. 5.1 and 5.2 prove the following theorem.

Theorem 5.1 *Under Assumptions 5.2 and 5.7 it holds*

$$\|u_K - u_{K,A}^{(\varepsilon_W,\varepsilon_l)}\|_{L^r(\Omega;V)} \to 0 \textit{ for } A \to +\infty,\ \varepsilon_W, \varepsilon_l \to 0,$$

for admissible values of $r \geq 2$ (see Remark 5.6 and Theorem 5.11). As a consequence, we obtain

$$\mathbb{P}(\|u_K - u_{K,A}^{(\varepsilon_W,\varepsilon_l)}\|_V > \delta) \to 0 \textit{ for } A \to +\infty,\ \varepsilon_W, \varepsilon_l \to 0,$$

for any $\delta > 0$. Furthermore, it holds

$$\begin{aligned}&\mathbb{P}(\{\omega \in \Omega \mid u_K(\omega) \neq u_{K,A} \textit{ in } V\} \\ &= \mathbb{P}(\{\omega \in \Omega \mid \operatorname*{ess\,sup}_{(x,y)\in\mathcal{D}} a_K(\omega, x, y)\} > A\}) \to 0, \textit{ for } A \to +\infty.\end{aligned}$$

Therefore, the solution $u_{K,A}$ to the modified problem (4.5)–(4.7) coincides with the solution u_K to problem (4.1)–(4.3) up to a set of samples, whose probability can be made arbitrarily small for growing A. With Remark 4.1 we conclude that the solution

[3] The error of the approximation $u_K \approx u$ may be controlled as in Remark 4.1 but cannot be quantified for the solution.

$u_{K,A}$ coincides with the solution u to problem (2.1)–(2.3) with diffusion coefficient a defined in Eq. (3.1) up to a set of samples, whose probability can be made arbitratily small for growing thresholds K and A.

5.1 Bound on E_1

Assumption 5.2 We assume that the GRFs W_1 and W_2 occurring in the diffusion coefficient (3.1) are stochastically independent.

The aim of this subsection is to show that the first error contributor E_1 in Eq. (5.4) vanishes for increasing cut-off threshold A. The strategy consists of two separated steps: in the first step we show the stability of the solution, which means that the value E_1 can be controlled by the quality of the approximation of the diffusion coefficient $a_{K,A} \approx a_K$. In the second step, we show that the quality of the approximation of the diffusion coefficient can be controlled by the cut-off threshold A. The first step is given by Theorem 5.4. In order to prove it we need the following lemma.

Lemma 5.3 *For fixed cut-off levels A and K we consider the solution $u_K \in L^r(\Omega; V)$ and its approximation $u_{K,A} \in L^r(\Omega; V)$ for $r \in [1, q)$. It holds the pathwise estimate*

$$\|u_K - u_{K,A}\|_V \leq a_{K,+}C(\overline{a}_-, \mathcal{D})\|\nabla u_K - \nabla u_{K,A}\|_{L^2(\mathcal{D})},$$

for $\mathbb{P}$-almost every $\omega \in \Omega$. Here, the constant $C(\overline{a}_-, \mathcal{D})$ only depends on the indicated parameters and we define $a_{K,+}(\omega) := \max\{1, \operatorname{ess\,sup}_{(x,y)\in\mathcal{D}} a_K(\omega, x, y)\} < \infty$ for $\omega \in \Omega$.

Proof For a fixed $\omega \in \Omega$ we consider the variational problem: find a unique $\hat{w} \in V$ such that

$$B_{a_K}(\hat{w}, v) = \langle u_K - u_{K,A}, v\rangle_{L^2(\mathcal{D})},$$

for all $v \in V$. By the Lax–Milgram theorem there exists a unique solution $\hat{w} \in V$ with

$$\|\hat{w}\|_V \leq C'(\overline{a}_-, \mathcal{D})\|u_K - u_{K,A}\|_{L^2(\mathcal{D})},$$

(see Theorem 3.7 and [11, Theorem 2.5]). Therefore, we obtain by Hölder's inequality

$$\begin{aligned}
\|u_K - u_{K,A}\|^2_{L^2(\mathcal{D})} &= B_{a_K}(\hat{w}, u_K - u_{K,A}) \\
&= \langle a_K \nabla\hat{w}, \nabla u_K - \nabla u_{K,A}\rangle_{L^2(\mathcal{D})} \\
&\leq a_{K,+}\|\nabla\hat{w}\|_{L^2(\mathcal{D})}\|\nabla u_K - \nabla u_{K,A}\|_{L^2(D)} \\
&\leq \|u_K - u_{K,A}\|_{L^2(\mathcal{D})} a_{K,+}C'(\overline{a}_-, \mathcal{D})\|\nabla u_K - \nabla u_{K,A}\|_{L^2(\mathcal{D})} \\
&\leq \frac{1}{2}\|u_K - u_{K,A}\|^2_{L^2(\mathcal{D})} + a^2_{K,+}C'(\overline{a}_-, \mathcal{D})^2/2 \\
&\qquad \|\nabla u_K - \nabla u_{K,A}\|^2_{L^2(\mathcal{D})},
\end{aligned}$$

where we used Young's inequality in the last step. Finally, we obtain

$$\begin{aligned}\|u_K - u_{K,A}\|_V^2 &= \|u_K - u_{K,A}\|_{L^2(\mathcal{D})}^2 + \|\nabla u_K - \nabla u_{K,A}\|_{L^2(\mathcal{D})}^2 \\ &\leq (1 + a_{K,+}^2 C'(\overline{a}_-, \mathcal{D})^2)\|\nabla u_K - \nabla u_{K,A}\|_{L^2(\mathcal{D})}^2.\end{aligned}$$

□

Theorem 5.4 *Let $f \in L^q(\Omega; H)$ and $g \in L^q(\Omega; L^2(\Gamma_2))$ for some $q \in [1, +\infty)$. Further, for a given number $t \in (1, +\infty)$ we define the dual number $n := \frac{t}{t-1}$. Then, for any for $r \in [1, q/n)$, holds*

$$\begin{aligned}\|u_K - u_{K,A}\|_{L^r(\Omega;V)} &\leq C(\mathcal{D}, \overline{a}_-, r)\,(\|f\|_{L^q(\Omega;H)} \\ &\quad + \|g\|_{L^q(\Omega;\Gamma_2))})\,\mathbb{E}\left(\operatorname*{ess\,sup}_{\underline{x}\in\mathcal{D}} |a_K(\underline{x}) - a_{K,A}(\underline{x})|^{rt}\right)^{\frac{1}{rt}}\end{aligned}$$

Proof By a direct calculation we obtain

$$\|\nabla u_K - \nabla u_{K,A}\|_{L^2(\mathcal{D})}^2 \leq \frac{1}{\overline{a}_-}\int_{\mathcal{D}} a_{K,A}|\nabla u_K - \nabla u_{K,A}|_2^2 d(x, y).$$

Since u_K and $u_{K,A}$ are weak solutions of problem (4.1)–(4.4) resp. (4.5)–(4.8) it holds

$$\begin{aligned}\int_{\mathcal{D}} a_{K,A}\nabla u_{K,A}\cdot\nabla u_K d(x, y) &= \int_{\mathcal{D}} a_K|\nabla u_K|_2^2 d(x, y),\\ \int_{\mathcal{D}} a_{K,A}|\nabla u_{K,A}|_2^2 d(x, y) &= \int_{\mathcal{D}} a_K\nabla u_K\cdot\nabla u_{K,A} d(x, y),\end{aligned}$$

$\mathbb{P}$-almost surely and therefore

$$\int_{\mathcal{D}} a_{K,A}|\nabla u_K - \nabla u_{K,A}|_2^2 d(x, y) = \int_{\mathcal{D}} (a_{K,A} - a_K)\nabla u_K(\nabla u_K - \nabla u_{K,A}) d(x, y).$$

We estimate using Hölder's inequality

$$\begin{aligned}\|\nabla u_K - \nabla u_{K,A}\|_{L^2(\mathcal{D})}^2 &\leq \frac{1}{\overline{a}_-}\|a_K - a_{K,A}\|_{L^\infty(\mathcal{D})}\|\nabla u_K\|_{L^2(\mathcal{D})}\|\nabla u_K - \nabla u_{K,A}\|_{L^2(\mathcal{D})}\\ &\leq \frac{1}{\overline{a}_-}\|a_K - a_{K,A}\|_{L^\infty(\mathcal{D})}\|u_K\|_V\|\nabla u_K - \nabla u_{K,A}\|_{L^2(\mathcal{D})}\end{aligned}$$

and therefore we obtain

$$\|\nabla u_K - \nabla u_{K,A}\|_{L^2(\mathcal{D})} \leq \frac{1}{\overline{a}_-}\|a_K - a_{K,A}\|_{L^\infty(\mathcal{D})}\|u_K\|_V.$$

Using Lemma 5.3 we obtain the pathwise estimate

$$\|u_K - u_{K,A}\|_V \leq C(\overline{a}_-, \mathcal{D}) a_{K,+} \|a_K - a_{K,A}\|_{L^\infty(\mathcal{D})} \|u_K\|_V.$$

Using again Hölder's inequality we have

$$\begin{aligned}
&\|u_K - u_{K,A}\|_{L^r(\Omega;V)} \\
&\quad \leq C(\overline{a}_-, \mathcal{D}) \mathbb{E}\left(\operatorname*{ess\,sup}_{\underline{x}\in\mathcal{D}} |a_K(\underline{x}) - a_{K,A}(\underline{x})|^{rt}\right)^{\frac{1}{rt}} \mathbb{E}(a_{K,+}^{nr} \|u_K\|_V^{nr})^{\frac{1}{nr}}.
\end{aligned}$$

By assumption it holds $nr < q$. Therefore, we can choose a real number $\rho > 1$ such that $nr\rho < q$. We define the dual number $\rho' := \frac{\rho}{\rho - 1} \in (1, +\infty)$ and use Hölder's inequality to calculate

$$\begin{aligned}
&\|u_K - u_{K,A}\|_{L^r(\Omega;V)} \\
&\quad \leq C(\overline{a}_-, \mathcal{D}) \mathbb{E}\left(\operatorname*{ess\,sup}_{\underline{x}\in\mathcal{D}} |a_K(\underline{x}) - a_{K,A}(\underline{x})|^{rt}\right)^{\frac{1}{rt}} \mathbb{E}(a_{K,+}^{nr\rho'})^{\frac{1}{nr\rho'}} \|u_K\|_{L^{nr\rho}(\Omega;V)}.
\end{aligned}$$

Obviously Theorem 3.7 applies also to problem (4.1)–(4.4). Therefore, since $nr\rho < q$ by assumption we conclude

$$\begin{aligned}
&\|u_K - u_{K,A}\|_{L^r(\Omega;V)} \\
&\quad \leq C(\mathcal{D}, \overline{a}_-, r) \mathbb{E}\left(\operatorname*{ess\,sup}_{\underline{x}\in\mathcal{D}} |a_K(\underline{x}) - a_{K,A}(\underline{x})|^{rt}\right)^{\frac{1}{rt}} (\|f\|_{L^q(\Omega;H)} + \|g\|_{L^q(\Omega;\Gamma_2)}),
\end{aligned}$$

where we additionally used the fact that $\mathbb{E}(a_{K,+}^{nr\rho'})^{\frac{1}{nr\rho'}} < +\infty$ (see Theorem 5.5). □

In other words, finding a bound for the error contribution E_1 in Eq. (5.4) reduces to quantifying the quality of the approximation of the diffusion coefficient $a_{K,A} \approx a_K$. For readability the proof of the following theorem can be found in Appendix A.

Theorem 5.5 *For any $n \in (1, +\infty)$ it holds*

$$\mathbb{E}\left(\operatorname*{ess\,sup}_{\underline{x}\in\mathcal{D}} a_K(\underline{x})^n\right)^{\frac{1}{n}} < +\infty.$$

Further, for any $\delta > 0$ there exists a constant $A = A(\delta, n) > 0$ such that

$$\mathbb{E}\left(\operatorname*{ess\,sup}_{\underline{x}\in\mathcal{D}} |a_K(\underline{x}) - a_{K,A}(\underline{x})|^n\right)^{1/n} < \delta.$$

We close this subsection with the following remark on the convergence of the error contributor E_1 in Eq. (5.4).

Remark 5.6 From Theorem 5.4 together with Theorem 5.5 we obtain

$$\|u_K - u_{K,A}\|_{L^r(\Omega;V)} \to 0, \text{ for } A \to \infty,$$

for every $r < q$.

5.2 Bound on E_2

The aim is to bound the second term of Eq. (5.4) given by

$$E_2 = \|u_{K,A} - u_{K,A}^{(\varepsilon_W,\varepsilon_l)}\|$$

in an appropriate norm. For technical reasons we have to impose an additional assumption on the solution of the truncated problem. The subsequent remarks discuss situations under which this assumption is fulfilled.

Assumption 5.7 We assume that there exist constants $j_{reg} > 0$ and $k_{reg} \geq 2$ such that

$$C_{reg} := \mathbb{E}(\|\nabla u_{K,A}\|_{L^{2+j_{reg}}(\mathcal{D})}^{k_{reg}}) < +\infty. \tag{5.5}$$

Since $u_{K,A} \in H^1(\mathcal{D})$ we already know that $\nabla u_{K,A} \in L^2(\mathcal{D})$. Assumption 5.7 requires a slightly higher integrability over the spatial domain. Since this is an assumption on the regularity of the solution $u_{K,A}$ we denote the above constant by C_{reg}.

Remark 5.8 Note that Assumption 5.7 is fulfilled if there exists $\theta \in (0, 1)$ such that

$$\|u_{K,A}\|_{H^{1+\theta}(\mathcal{D})} \leq C(f, a) \tag{5.6}$$

with some constant $C(f, a)$ with $\mathbb{E}(C(f, a)^{k_{reg}}) < \infty$. This is true since for any $\rho \geq 2$ and an arbitrary function $\varphi \in H^{1+\theta}(\mathcal{D})$ the inequality

$$\|\nabla\varphi\|_{L^\rho(\mathcal{D})} \leq C\|\nabla\varphi\|_{H^{1-\frac{2}{\rho}(\mathcal{D})}(\mathcal{D})} \leq C\|\varphi\|_{H^{2-\frac{2}{\rho}}(\mathcal{D})}$$

holds for $\theta := 1-\frac{2}{\rho}$ (see [17, Theorem 6.7]). Here, the constant $C = C(\mathcal{D}, \theta)$ depends only on the indicated parameters. Hence, the condition (5.6) implies Eq. (5.5) with $j_{reg} = \frac{2\theta}{1-\theta}$.

Remark 5.9 Note that there are several results about higher integrability of the gradient of the solution to an elliptic PDE of the form (4.5)–(4.8). For instance [31] yields that the solution $u_{K,A}$ has $H^{1+\delta/(2\pi)}$ regularity with $\delta = min(1, \overline{a}_-, A^{-1})$ under mixed boundary conditions and under the assumption that $a_{K,A}$ is piecewise constant (see [31, Theorem 7.3]). This corresponds to the case where no Gaussian noise is considered (i.e. $\Phi_1 \equiv 0$) and $\overline{a}$ is constant. Another important result is given in [19]. It follows by Dreyfuss [19, Theorem 1] that under the assumption that there exists $q > 2$ with $f \in L^q(\mathcal{D})$ $\mathbb{P} - a.s.$ there exists a constant $C = C(\mathcal{D}, \|f\|_{L^q(\mathcal{D})}, \overline{a}_-, A)$ and a positive number $\vartheta = \vartheta(\mathcal{D}, \|f\|_{L^q(\mathcal{D})}, \overline{a}_-, A) > 0$ only depending on the indicated parameters, such that:

$$\|\nabla u_{K,A}\|_{L^{2+\vartheta}(\mathcal{D})} \leq C. \tag{5.7}$$

In particular, if the right hand side f of the problem is deterministic, then ϑ and the constant C in (5.7) are deterministic and one immediately obtains

$$\mathbb{E}(\|\nabla u_{K,A}\|_{L^{2+\vartheta}(\mathcal{D})}^{k_{reg}}) < +\infty,$$

for any $k_{reg} \geq 1$ and a deterministic, positive constant $\vartheta > 0$.

Next, we show that for a given approximation of the diffusion coefficient, the resulting error contributor E_2 is bounded by the approximation error of the diffusion coefficient. Similar to the corresponding assertion we gave in Sect. 5.1 we need the following Lemma for the proof of this error bound. For a proof we refer to Lemma 5.3.

Lemma 5.10 *For fixed cut-off levels A, K and fixed approximation parameters ε_W, ε_l we consider the PDE solutions $u_{A,K} \in L^r(\Omega; V)$ and $u_{K,A}^{(\varepsilon_W,\varepsilon_l)} \in L^r(\Omega; V)$ for $r \in [1, q)$. It holds the pathwise estimate*

$$\|u_{K,A} - u_{K,A}^{(\varepsilon_W,\varepsilon_l)}\|_V \leq a_{K,+} C(\overline{a}_-, \mathcal{D}) \|\nabla u_{K,A} - \nabla u_{K,A}^{(\varepsilon_W,\varepsilon_l)}\|_{L^2(\mathcal{D})},$$

for $\mathbb{P}$-almost every $\omega \in \Omega$. Here, constant $C(\overline{a}_-, \mathcal{D})$ depends only on the indicated parameters and $a_{K,+}(\omega) := \max\{1, \operatorname{ess\,sup}_{(x,y)\in\mathcal{D}} a_K(\omega, x, y)\} < \infty$ for $\omega \in \Omega$.

Theorem 5.11 *Let $r \geq 2$ and $b, c \in [1, +\infty]$ be given such that it holds*

$$rc\gamma \geq 2 \text{ and } 2b \leq rc < \eta$$

with a fixed real number $\gamma \in (0, min(1, \beta/(2\alpha))$. Here, the parameters η, α and β are determined by the GRFs W_1, W_2 and the Lévy subordinators l_1, l_2 (see Assumption 4.2). Let $m, n \in [1, +\infty]$ be real numbers such that

$$\frac{1}{m} + \frac{1}{c} = \frac{1}{n} + \frac{1}{b} = 1,$$

and let $k_{reg} \geq 2$ and $j_{reg} > 0$ be the regularity specifiers given by Assumption 5.7. If it holds that

$$n < 1 + \frac{j_{reg}}{2} \text{ and } rm < k_{reg},$$

then the approximated solution $u_{K,A}^{(\varepsilon_W,\varepsilon_l)}$ *converges to the solution* $u_{K,A}$ *of the truncated problem for* $\varepsilon_W, \varepsilon_l \to 0$ *and it holds*

$$\begin{aligned}\|u_{K,A} - u_{K,A}^{(\varepsilon_W,\varepsilon_l)}\|_{L^r(\Omega;V)} &\le C(\overline{a}_-, \mathcal{D}, r) C_{reg} \|a_{K,A}^{(\varepsilon_W,\varepsilon_l)} - a_{K,A}\|_{L^{rc}(\Omega;L^{2b}(\mathcal{D}))} \\ &\le C_{reg} C(\overline{a}_-, \mathcal{D}, r)(\varepsilon_W^{\gamma} + \varepsilon_l^{\frac{1}{rc}}).\end{aligned}$$

Proof By a direct calculation we obtain the pathwise estimate

$$\|\nabla u_{K,A} - \nabla u_{K,A}^{(\varepsilon_W,\varepsilon_l)}\|_{L^2(\mathcal{D})}^2 \le \frac{1}{\overline{a}_-} \int_{\mathcal{D}} a_{K,A}^{(\varepsilon_W,\varepsilon_l)} (|\nabla u_{K,A} - \nabla u_{K,A}^{(\varepsilon_W,\varepsilon_l)}|_2^2) d\underline{x}.$$

Since $u_{K,A}$ (resp. $u_{K,A}^{(\varepsilon_W,\varepsilon_l)}$) is the weak solution to problem (4.5)–(4.8) [resp. (5.1)–(5.3)] we have

$$\int_D a_{K,A}^{(\varepsilon_W,\varepsilon_l)} \nabla u_{K,A}^{(\varepsilon_W,\varepsilon_l)} \cdot \nabla u_{K,A} d\underline{x} = \int_{\mathcal{D}} a_{K,A} |\nabla u_{K,A}|_2^2 d\underline{x},$$
$$\int_{\mathcal{D}} a_{K,A}^{(\varepsilon_W,\varepsilon_l)} |\nabla u_{K,A}^{(\varepsilon_W,\varepsilon_l)}|_2^2 d\underline{x} = \int_{\mathcal{D}} a_{K,A} \nabla u_{K,A} \cdot \nabla u_{K,A}^{(\varepsilon_W,\varepsilon_l)} d\underline{x}$$

$\mathbb{P}$-a.s. and therefore

$$\begin{aligned}&\int_D a_{K,A}^{(\varepsilon_W,\varepsilon_l)} |\nabla u_{K,A} - \nabla u_{K,A}^{(\varepsilon_W,\varepsilon_l)}|_2^2 d\underline{x} \\ &\quad = \int_{\mathcal{D}} (a_{K,A}^{(\varepsilon_W,\varepsilon_l)} - a_{K,A}) \nabla u_{K,A} \cdot (\nabla u_{K,A} - \nabla u_{K,A}^{(\varepsilon_W,\varepsilon_l)}) d\underline{x}.\end{aligned}$$

Using Hölder's inequality we calculate

$$\begin{aligned}&\|\nabla u_{K,A} - \nabla u_{K,A}^{(\varepsilon_W,\varepsilon_l)}\|_{L^2(\mathcal{D})}^2 \\ &\quad \le \frac{1}{\overline{a}_-} \|(a_{K,A}^{(\varepsilon_W,\varepsilon_l)} - a_{K,A}) \nabla u_{K,A}\|_{L^2(\mathcal{D})} \|\nabla u_{K,A} - \nabla u_{K,A}^{(\varepsilon_W,\varepsilon_l)}\|_{L^2(\mathcal{D})}\end{aligned}$$

and therefore

$$\|\nabla u_{K,A} - \nabla u_{K,A}^{(\varepsilon_W,\varepsilon_l)}\|_{L^2(\mathcal{D})} \le \frac{1}{\overline{a}_-} \|(a_{K,A}^{(\varepsilon_W,\varepsilon_l)} - a_{K,A}) \nabla u_{K,A}\|_{L^2(\mathcal{D})},$$

Next, we apply Lemma 5.10 to obtain the following estimate.

$$\|u_{K,A} - u_{K,A}^{(\varepsilon_W,\varepsilon_l)}\|_V \le C(\overline{a}_-, \mathcal{D}) a_{K,+} \|(a_{K,A}^{(\varepsilon_W,\varepsilon_l)} - a_{K,A}) \nabla u_{K,A}\|_{L^2(\mathcal{D})}$$

Hence, it remains to bound the norm $\|(a_{K,A}^{(\varepsilon_W,\varepsilon_l)} - a_{K,A}) \nabla u_{K,A}\|_{L^2(\mathcal{D})}$. By Hölder's inequality we obtain

$$\|(a_{K,A}^{(\varepsilon_W,\varepsilon_l)} - a_{K,A}) \nabla u_{K,A}\|_{L^2(\mathcal{D})}$$

$$\leq \|a_{K,A}^{(\varepsilon_W,\varepsilon_l)} - a_{K,A}\|_{L^{2b}(\mathcal{D})}\|\nabla u_{K,A}\|_{L^{2n}(\mathcal{D})}.$$

Applying Hölder's inequality once more we estimate

$$\begin{aligned}
&\|u_{K,A} - u_{K,A}^{(\varepsilon_W,\varepsilon_l)}\|_{L^r(\Omega;V)} \\
&\quad\leq C(\overline{a}_-,\mathcal{D})\|a_{K,A}^{(\varepsilon_W,\varepsilon_l)} - a_{K,A}\|_{L^{rc}(\Omega;L^{2b}(\mathcal{D}))}\mathbb{E}(a_{K,+}^{rm}\|\nabla u_{K,A}\|_{L^{2n}(\mathcal{D})}^{rm})^{\frac{1}{rm}}
\end{aligned}$$

By assumption it holds $rm < k_{reg}$. Hence, we can choose a real number $\rho > 1$ such that $rm\rho < k_{reg}$. We define the dual number $\rho' := \frac{\rho}{1-\rho} \in (1,+\infty)$ and use Hölder's inequality to obtain

$$\begin{aligned}
&\mathbb{E}(a_{K,+}^{rm}\|\nabla u_{K,A}\|_{L^{2n}(\mathcal{D})}^{rm})^{\frac{1}{rm}} \\
&\quad\leq \mathbb{E}(a_{K,+}^{rm\rho'})^{\frac{1}{rm\rho'}}\|\nabla u_{K,A}\|_{L^{rm\rho}(\Omega;L^{2n}(\mathcal{D}))} \leq C(\mathcal{D},r)C_{reg},
\end{aligned}$$

where we again used the fact that $\mathbb{E}(a_{K,+}^{nr\rho'})^{\frac{1}{nr\rho'}} < +\infty$ (see Theorem 5.5) together with Assumption 5.7. Finally we obtain the estimate

$$\begin{aligned}
\|u_{K,A} - u_{K,A}^{(\varepsilon_W,\varepsilon_l)}\|_{L^r(\Omega;V)} &\leq C(\overline{a}_-,\mathcal{D},r)C_{reg}\|a_{K,A}^{(\varepsilon_W,\varepsilon_l)} - a_{K,A}\|_{L^{rc}(\Omega;L^{2b}(\mathcal{D}))} \\
&\leq C_{reg}C(\overline{a}_-,\mathcal{D},r)(\varepsilon_W^{\gamma} + \varepsilon_l^{\frac{1}{rc}}),
\end{aligned}$$

where we applied Theorem 4.8 in the last estimate. □

We close this section with a remark on how to choose the parameters A, ε_W and ε_l to obtain an approximation error smaller than any given threshold $\delta > 0$.

Remark 5.12 For any given parameter K large enough (see Remark 4.1), we choose a positive number $A > 0$ such that the first error contributor satisfies $E_1 = \|u_K - u_{K,A}\|_{L^r(\Omega;V)} < \delta/2$ (see Theorems 5.4 and 5.5). Afterwards, under the assumptions of Theorem 5.11, we may choose the approximation parameters ε_W and ε_l small enough, such that the second error contributor satisfies $E_2 = \|u_{K,A} - u_{K,A}^{(\varepsilon_W,\varepsilon_l)}\|_{L^r(\Omega;V)} < \delta/2$. Hence, we get an overall error smaller than δ [see Eq. (5.4)].

6 Pathwise sample-adapted finite element approximation

We want to approximate the solution u to the problem (2.1)–(2.3) with diffusion coefficient a given by Eq. (3.1) using a pathwise Finite Element (FE) approximation of the solution $u_{K,A}^{(\varepsilon_W,\varepsilon_l)}$ of problem (5.1)–(5.3) where the approximated diffusion coefficient $a_{K,A}^{(\varepsilon_W,\varepsilon_l)}$ is given by (4.10). Therefore, for almost all $\omega \in \Omega$, we have to find a function $u_{K,A}^{(\varepsilon_W,\varepsilon_l)}(\omega,\cdot) \in V$ such that it holds

$$B_{a_{K,A}^{(\varepsilon_W,\varepsilon_l)}(\omega)}(u_{K,A}^{(\varepsilon_W,\varepsilon_l)}(\omega,\cdot), v)$$

$$
\begin{aligned}
&:= \int_{\mathcal{D}} a_{K,A}^{(\varepsilon_W,\varepsilon_l)}(\omega,\underline{x})\nabla u_{K,A}^{(\varepsilon_W,\varepsilon_l)}(\omega,\underline{x})\cdot\nabla v(\underline{x})d\underline{x}\\
&= \int_{\mathcal{D}} f(\omega,\underline{x})v(\underline{x})d\underline{x} + \int_{\Gamma_2} g(\omega,\underline{x})[Tv](\underline{x})d\underline{x} =: F_\omega(v), \qquad (6.1)
\end{aligned}
$$

for every $v \in V$. Here, $K, A, \varepsilon_W, \varepsilon_l$ are fixed approximation parameters. In order to solve this variational problem numerically we consider a standard Galerkin approach with linear elements and assume $\mathcal{V} = (V_\ell,\ \ell \in \mathbb{N}_0)$ to be a sequence of finite-dimensional subspaces $V_\ell \subset V$ with $\dim(V_\ell) = d_\ell$ for $\ell \in \mathbb{N}_0$. We denote by $(h_\ell,\ \ell \in \mathbb{N}_0)$ the corresponding sequence of refinement sizes which is assumed to decrease monotonically to zero for $\ell \to \infty$. Let $\ell \in \mathbb{N}_0$ be fixed and denote by $\{v_1^{(\ell)},\dots,v_{d_\ell}^{(\ell)}\}$ a basis of V_ℓ. The (pathwise) discrete version of (6.1) reads:

Find $u_{K,A,\ell}^{(\varepsilon_W,\varepsilon_l)}(\omega,\cdot) \in V_\ell$ such that

$$
B_{a_{K,A}^{(\varepsilon_W,\varepsilon_l)}(\omega)}(u_{K,A,\ell}^{(\varepsilon_W,\varepsilon_l)}(\omega,\cdot), v_\ell^{(i)}) = F_\omega(v_\ell^{(i)}) \text{ for all } i = 1,\dots,d_\ell.
$$

We expand the function $u_{K,A,\ell}^{(\varepsilon_W,\varepsilon_l)}(\omega,\cdot)$ with respect to the basis $\{v_1^{(\ell)},\dots,v_{d_\ell}^{(\ell)}\}$:

$$
u_{K,A,\ell}^{(\varepsilon_W,\varepsilon_l)}(\omega,\cdot) = \sum_{i=1}^{d_\ell} c_i v_i^{(\ell)},
$$

where the coefficient vector $\mathbf{c} = (c_1,\dots,c_{d_\ell})^T \in \mathbb{R}^{d_\ell}$ is determined by the linear equation system

$$
\mathbf{B}(\omega)\mathbf{c} = \mathbf{F}(\omega),
$$

with a stochastic stiffness matrix $\mathbf{B}(\omega)_{i,j} = B_{a_{K,A}^{(\varepsilon_W,\varepsilon_l)}(\omega)}(v_i^{(\ell)}, v_j^{(\ell)})$ and load vector $\mathbf{F}(\omega)_i = F_\omega(v_i^{(\ell)})$ for $i, j = 1,\dots,d_\ell$.

Remark 6.1 Let $(\mathcal{K}_\ell,\ \ell \in \mathbb{N}_0)$ be a sequence of triangulations on $\mathcal{D}$ and denote by $\theta_\ell > 0$ the minimum interior angle of all triangles in $\mathcal{K}_\ell$. We assume $\theta_\ell \geq \theta > 0$ for a positive constant θ and define the maximum diameter of the triangulation $\mathcal{K}_\ell$ by $h_\ell := \max_{K\in\mathcal{K}_\ell} \operatorname{diam}(K)$, for $\ell \in \mathbb{N}_0$. Further, we define the finite dimensional subspaces by $V_\ell := \{v \in V \mid v|_K \in \mathcal{P}_1,\ K \in \mathcal{K}_\ell\}$, where $\mathcal{P}_1$ denotes the space of all polynomials up to degree one. The convergence of the FE method is determined by the regularity of the solution: If we assume that for $\mathbb{P}-$almost all $\omega \in \Omega$ it holds $u_{K,A}^{(\varepsilon_W,\varepsilon_l)}(\omega,\cdot) \in H^{1+\kappa_a}(\mathcal{D})$ for some positive number $\kappa_a > 0$, the pathwise discretization error is bounded by Céa's lemma $\mathbb{P}$-a.s. by

$$
\begin{aligned}
&\|u_{K,A}^{(\varepsilon_W,\varepsilon_l)}(\omega,\cdot) - u_{K,A,\ell}^{(\varepsilon_W,\varepsilon_l)}(\omega,\cdot)\|_V\\
&\quad\leq C_{\theta,\mathcal{D}}\frac{A}{\overline{a}_-}\|u_{K,A}^{(\varepsilon_W,\varepsilon_l)}(\omega,\cdot)\|_{H^{1+\kappa_a}(\mathcal{D})} h_\ell^{\min(\kappa_a,1)},
\end{aligned}
$$

(see [11, Section 4] and [26, Chapter 8]). If the bound $\|u_{K,A}^{(\varepsilon_W,\varepsilon_l)}\|_{L^2(\Omega;H^{1+\kappa_a}(\mathcal{D}))} \leq C_u = C_u(K, A)$ is finite for the fixed approximation parameters K, A, we immediately obtain

$$\begin{aligned}\|u_{K,A}^{(\varepsilon_W,\varepsilon_l)} - u_{K,A,\ell}^{(\varepsilon_W,\varepsilon_l)}\|_{L^2(\Omega;V)} \\ \leq C_{\theta,\mathcal{D}}\frac{A}{\underline{a}_-}C_u h_\ell^{\min(\kappa_a,1)}.\end{aligned}$$

We note that, by construction of our random field, we always obtain an interface geometry with fixed angles and bounded jump height, which have great influence on the solution regularity, see e.g. [31].

For general elliptic jump-diffusion problems, one obtains a discretization error of order $\kappa_a \in (1/2, 1)$. In general, we cannot expect the full order of convergence $\kappa_a = 1$ for discontinuous diffusion coefficients. Without special treatment of the interfaces with respect to the triangulation, one cannot expect a convergence rate which is higher than $\kappa_a = 1/2$ for the deterministic interface problem (see [6, 11]). The convergence of the FE method may be improved by the use of triangulations which are adapted to the discontinuities. This is explained in more detail in the following subsection.

6.1 Sample-adapted triangulations

In [11], the authors suggest sample-adapted triangulations to improve the convergence rate of the FE approximation: Consider a fixed $\omega \in \Omega$ and assume that the discontinuities of the diffusion coefficient are described by the partition $\mathcal{T}(\omega) = (\mathcal{T}_i,\ i = 1, \dots, \tau(\omega))$ of the domain $\mathcal{D}$ where $\tau(\omega)$ describes the number of elements in the partition. We consider finite-dimensional subspaces $\hat{V}_\ell(\omega) \subset V$ with (stochastic) dimension $\hat{d}_\ell(\omega) \in \mathbb{N}$. We denote by $\hat{\theta}_\ell(\omega)$ the minimal interior angle within $\mathcal{K}_\ell(\omega)$ and assume the existence of a positive number $\theta > 0$ such that $\inf\{\hat{\theta}_\ell(\omega) \mid \ell \in \mathbb{N}_0\} \geq \theta$ for $\mathbb{P}$-almost all $\omega \in \Omega$. Assume that $\mathcal{K}_\ell(\omega)$ is a triangulation of $\mathcal{D}$ which is adjusted to the partition $\mathcal{T}(\omega)$ in the sense that for every $i = 1, \dots, \tau(\omega)$ it holds

$$\mathcal{T}_i \subset \bigcup_{\kappa\in\mathcal{K}_\ell(\omega)} \kappa \text{ and } \hat{h}_\ell(\omega) := \max_{K\in\mathcal{K}_\ell(\omega)} diam(K) \leq \overline{h}_\ell,$$

for all $\ell \in \mathbb{N}_0$, where $(\overline{h}_\ell,\ \ell \in \mathbb{N}_0)$ is a deterministic, decreasing sequence of refinement thresholds which converges to zero (see Fig. 2).

Sample-adapted triangulations lead to an improved convergence rate for elliptic problems with discontinuous coefficients (see [11, Section 4.1] and Sect. 7). This is especially true for jump-diffusion coefficients with polygonal jump geometry, which is the case for the subordinated GRF considered in this paper (see Fig. 2, [11, 15]). The question arises whether mean squared convergence rates can be derived analytically (cf. Assumption 7.1). While it is not possible to prove convergence rates for the mean squared error for our diffusion coefficient in general due to the regularity of the stochastic solution, in practice one can (at least) recover the convergence rates of the

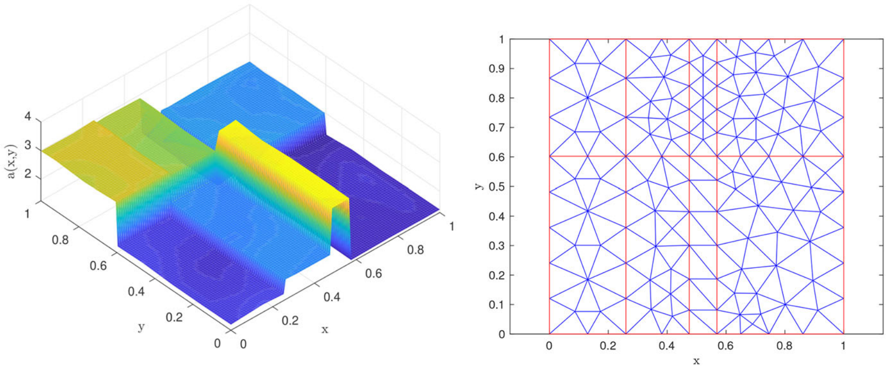

Fig. 2 A sample of a Poisson-subordinated Matérn-1.5-GRF (left) with corresponding sample-adapted triangulations (right)

deterministic jump-diffusion problem in the strong error (see Sect. 7, [11, Section 4.1] and [15, Section 2]). In fact, in the non-adapted case it is possible to get even better convergence rates than expected for some examples.

7 Numerical examples

In the following experiments we work on the domain $\mathcal{D} = (0, 1)^2$ and use a FE method with hat-function basis as described in Sect. 6. Here, we distinguish between the *standard* FEM approach with standard triangulations and the *sample-adapted* FEM approach. The aim of our numerical experiments is to compare the sample-adapted with the non-adapted approach and investigate how different Lévy subordinators and GRFs influence the strong convergence rate. In our first example we use Poisson processes with low intensity to investigate the superiority of the presented sample-adapted triangulation. In the second example we use Poisson subordinators with a significantly higher intensity. Besides Poisson subordinators we also use Gamma processes which have infinite activity.

7.1 Strong error approximation

Remark 6.1 and Sect. 6.1 motivate the following assumption on the approximation error of the FE method for non-adapted and adapted triangulations for the rest of this paper (see [11, Assumption 4.4]).

Assumption 7.1 There exist deterministic constants $\hat{C}_u, C_u, \hat{\kappa}_a, \kappa_a > 0$ such that for any $\varepsilon_W, \varepsilon_l > 0$ and any $\ell \in \mathbb{N}_0$, the Finite Element approximation errors of $\hat{u}_{K,A,\ell}^{(\varepsilon_W,\varepsilon_l)} \approx u_{K,A}^{(\varepsilon_W,\varepsilon_l)}$ in the (sample-adapted) subspaces $\hat{V}_\ell$, respectively $u_{K,A,\ell}^{(\varepsilon_W,\varepsilon_l)} \approx u_{K,A}^{(\varepsilon_W,\varepsilon_l)}$ in V_ℓ, are bounded by

$$\|u_{K,A}^{(\varepsilon_W,\varepsilon_l)} - \hat{u}_{K,A,\ell}^{(\varepsilon_W,\varepsilon_l)}\|_{L^2(\Omega;V)} \le \hat{C}_u \mathbb{E}(\hat{h}_\ell^{2\hat{\kappa}_a})^{1/2}, \text{ respectively,}$$

$$\|u_{K,A}^{(\varepsilon_W,\varepsilon_l)} - u_{K,A,\ell}^{(\varepsilon_W,\varepsilon_l)}\|_{L^2(\Omega;V)} \le C_u h_\ell^{\kappa_a},$$

where the constants $\hat{C}_u, C_u$ may depend on a, f, g, K, A but are independent of $\hat{h}_\ell, h_\ell, \hat{\kappa}_a$ and κ_a.

In each numerical experiment we choose a problem dependent cut-off level K for the subordinators in (4.4) large enough so that its influence is negligibly (see Remark 4.1). Further, we choose the cut-off level for the diffusion coefficient A in (4.8) large enough such that it has no influence in numerical experiments, $e.g.$ $A = 50$ and therefore the error induced by the error contributor E_1 in (5.4) can be neglected in our experiments. We estimate the strong error using a standard Monte Carlo estimator. Assume that a sequence of (sample-adapted) finite-dimensional subspaces $(\hat{V}_\ell,\ \ell \in \mathbb{N}_0) \subset V$ is given where we use the notation of Sect. 6. For readability we only treat the case of pathwise sample-adapted Finite Element approximations in the rest of the theoretical consideration in this subsection. We would like to point out, however, that similar arguments lead to the corresponding results for standard FE approximations.

Under the assumptions of Theorem 5.11 and Assumption 7.1 we obtain

$$\begin{aligned}\|u_{K,A} - \hat{u}_{K,A,\ell}^{(\varepsilon_W,\varepsilon_l)}\|_{L^2(\Omega;V)} &\le \|u_{K,A} - u_{K,A}^{(\varepsilon_W,\varepsilon_l)}\|_{L^2(\Omega;V)} \\ &\quad + \|u_{K,A}^{(\varepsilon_W,\varepsilon_l)} - \hat{u}_{K,A,\ell}^{(\varepsilon_W,\varepsilon_l)}\|_{L^2(\Omega;V)} \\ &\le C(\varepsilon_W^\gamma + \varepsilon_l^{\frac{1}{2c}} + \mathbb{E}(\hat{h}_\ell^{2\hat{\kappa}_a})^{1/2}),\end{aligned} \tag{7.1}$$

with a constant $C = C(C_{reg}, D, \overline{a}_-, \hat{C}_u)$. Therefore, in order to equilibrate all error contributions, we choose the approximation parameters ε_W and ε_l in the following way:

$$\varepsilon_W \simeq \mathbb{E}(\hat{h}_\ell^{2\hat{\kappa}_a})^{1/(2\gamma)} \quad \text{and} \quad \varepsilon_l \simeq \mathbb{E}(\hat{h}_\ell^{2\hat{\kappa}_a})^c. \tag{7.2}$$

For readability, we omit the cut-off parameters K and A in the following and use the notation $\hat{u}_{\ell,\varepsilon_W,\varepsilon_l} = \hat{u}_{K,A,\ell}^{(\varepsilon_W,\varepsilon_l)}$. Choosing the approximation parameters $\varepsilon_W, \varepsilon_l$ according to (7.2), we can investigate the strong error convergence rate by a Monte Carlo estimation of the left hand side of (7.1): for a fixed natural number $M \in \mathbb{N}$ we approximate

$$\begin{aligned}\|u_{K,A} - \hat{u}_{K,A,\ell}^{(\varepsilon_W,\varepsilon_l)}\|_{L^2(\Omega;V)}^2 &= \|u_{K,A} - \hat{u}_{\ell,\varepsilon_W,\varepsilon_l}\|_{L^2(\Omega;V)}^2 \\ &\approx \frac{1}{M}\sum_{i=1}^M \|u_{ref}^{(i)} - \hat{u}_{\ell,\varepsilon_W,\varepsilon_l}^{(i)}\|_V^2,\end{aligned} \tag{7.3}$$

where $(u_{ref}^{(i)},\ i = 1,\dots,M)$ are i.i.d. realizations of the stochastic reference solution $u_{ref} \approx u_{K,A}$ and $(\hat{u}_{\ell,\varepsilon_W,\varepsilon_l}^{(i)},\ i = 1\dots,M)$ are i.i.d. realizations of the FE approximation $\hat{u}_{\ell,\varepsilon_W,\varepsilon_l}$ of the PDE solution on the FE subspace $\hat{V}_\ell$. In all examples we choose the sample number M so that the standard deviation of the MC samples is smaller than 10% of the MC estimator itself.

7.2 PDE parameters

In all of our numerical examples we choose $\overline{a} \equiv 1/10$, $f \equiv 10$, $\Phi_1 = 1/100 \exp(\cdot)$ and $\Phi_2 = 5\,|\cdot|$. Further, we impose mixed Dirichlet–Neumann boundary conditions if nothing else is explicitly mentioned. To be precise, we split the domain boundary $\partial\mathcal{D}$ by $\Gamma_1 = \{0,1\} \times [0,1]$ and $\Gamma_2 = (0,1) \times \{0,1\}$ and impose the pathwise mixed Dirichlet–Neumann boundary conditions

$$u_{K,A}(\omega,\cdot) = \begin{cases} 0.1 & on\ \{0\} \times [0,1] \\ 0.3 & on\ \{1\} \times [0,1] \end{cases} \quad \text{and} \quad a_{K,A}\overrightarrow{n} \cdot \nabla u_{K,A} = 0 \text{ on } \Gamma_2,$$

for $\omega \in \Omega$.

We choose W_1 to be a Matérn-1.5-GRF on $\mathcal{D}$ with correlation length $r_1 = 0.5$ and different variance parameters σ_1^2. Further, we set W_2 to be a Matérn-1.5-GRF on $[0,K]^2$ which is independent of W_1 with different variances σ_2^2 and correlation lengths r_2. We use a reference grid with 800×800 equally spaced points on the domain $\mathcal{D}$ for interpolation and prolongation.

7.3 Poisson subordinators

In this section we use Poisson processes to subordinate the GRF W_2 in the diffusion coefficient in (3.1). We consider both, high and low intensity Poisson processes and vary the boundary conditions. Further, using Poisson subordinators allows for a detailed investigation of the approximation error caused by approximating the Lévy subordinators l_1 and l_2 according to Assumption 4.2(v).

7.3.1 The two approximation methods

Using Poisson processes as subordinators allows for two different simulation approaches in the numerical examples: the first approach is an exact and grid-independent simulation of a Poisson process using for example the *Method of Exponential Spacings* or the *Uniform Method* (see [33, Section 8.1.2]). On the other hand, one may also work with approximations of the Poisson processes satisfying Assumption 4.2(v).

We recall that a Poisson(λ)-process l is a Lévy process with $l(t) \sim Poiss(t\lambda)$ for $t > 0$, i.e. it admits the discrete density

$$\mathbb{P}(l(t) = k) = e^{-t\lambda}\frac{(t\lambda)^k}{k!}, \ \text{for } k \in \mathbb{N}_0.$$

We sample the values of the Poisson(λ)-processes l_1 and l_2 on an equidistant grid $\{x_i,\ i = 0,\dots,N_l\}$ with $x_0 = 0$ and $x_{N_l} = 1$ and step size $|x_{i+1} - x_i| \le \varepsilon_l \le 1$ for all $i = 0,\dots,N_l - 1$. Further, we approximate the stochastic processes by a piecewise constant extension $l_j^{(\varepsilon_l)} \approx l_j$ of the values on the grid:

$$l_j^{(\varepsilon_l)}(x) = \begin{cases} l_j(x_i) & x \in [x_i, x_{i+1}) \text{ for } i = 0, \dots, N_l - 1, \\ l_j(x_{N_l-1}) & x = 1. \end{cases}$$

for $j = 1, 2$. Since the Poisson process has independent, Poisson distributed increments, values of the Poisson process at the discrete points $\{x_i,\ i = 0, \dots, N_l\}$ can be generated by adding independent Poisson distributed random variables. In the following we refer to this approach as the *approximation approach* to simulate a Poisson process. Note that in this case Assumption 4.2(v) holds with $\eta = +\infty$. In fact, for any $s \in [1, +\infty)$ we obtain for $j = 1, 2$ and an arbitrary $x \in [0, 1)$ with $x \in [x_i, x_{i+1})$:

$$\mathbb{E}(|l_j(x) - l_j^{(\varepsilon_l)}(x)|^s) = \mathbb{E}(|l_j(x) - l_j(x_i)|^s) \le \mathbb{E}(|l_j(x_{i+1} - x_i)|^s) \le \mathbb{E}(|l_j(\varepsilon_l)|^s),$$

which is independent of the specific $x \in [0, 1)$. Note that this also holds for $x = D = 1$ and therefore

$$\sup_{x \in [0,1]} \mathbb{E}(|l_j(x) - l_j^{(\varepsilon_l)}(x)|^s) \le \mathbb{E}(|l_j(\varepsilon_l)|^s).$$

For a Poisson process with parameter λ we obtain

$$\mathbb{E}(|l_j(\varepsilon_l)|^s) = e^{-\lambda\varepsilon_l} \sum_{k=0}^{\infty} k^s \frac{(\lambda\varepsilon_l)^k}{k!} \le \varepsilon_l \sum_{k=1}^{\infty} k^s \frac{\lambda^k}{k!} \le C_l \varepsilon_l,$$

where the series converges by the ratio test.

Since the Poisson process allows for both approaches - approximation and exact simulation of the process - the use of these processes are suitable to investigate the additional error in the approximation of the PDE solution resulting from an approximation of the subordinators. Note that the main difference of the proposed approaches is that the approximation approach is grid dependent and the exact simulation of the Poisson process is not. The differences in the computational costs for the two approaches are insignificant in our numerical examples.

7.3.2 Poisson subordinators: low intensity and mixed boundary conditions

In this example we choose l_1 and l_2 to be Poisson(1)-subordinators. Further, the variance parameter of the GRF W_1 is set to be $\sigma_1^2 = 1.5^2$ and the variance and correlation parameters of the GRF W_2 are given by $\sigma_2^2 = 0.3^2$ and $r_2 = 1$.

For independent Poisson(1)-subordinators l_1 and l_2 we choose $K = 8$ as the cut-off parameter [see (4.4)]. With this choice we obtain

$$\mathbb{P}\left(\sup_{t \in [0,1]} l_j(t) > K\right) = \mathbb{P}(l_j(1) > K) \approx 1.1252e{-}06,$$

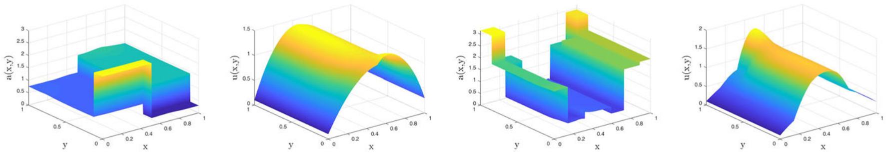

Fig. 3 Different samples of the diffusion coefficient with Poisson(1)-subordinators and the corresponding PDE solutions with mixed Dirichlet–Neumann boundary conditions

for $j = 1, 2$, such that this cut-off has no influence in the numerical example. Note that for Matérn-1.5-GRFs we can expect $\gamma = 1$ in Eq. (4.9) (see e.g. [12, Chapter 5], [23, Proposition 9] and [14, Section 2.3]).

We approximate the GRFs W_1 and W_2 by the circulant embedding method (see [24, 25]) to obtain approximations $W_1^{(\varepsilon_W)} \approx W_1$ and $W_2^{(\varepsilon_W)} \approx W_2$ as in Lemma 4.4. Since $\eta = +\infty$ and $f \in L^q(\Omega; H)$ for every $q \geq 1$ we choose for any positive $\delta > 0$

$$r = 2,\ c = b = 1 + \delta$$

to obtain from Theorem 5.11

$$\|u_{K,A} - u_{K,A}^{(\varepsilon_W,\varepsilon_l)}\|_{L^2(\Omega;V)} \leq C_{reg}\frac{C(D)}{\overline{a}_-}(\varepsilon_W + \varepsilon_l^{\frac{1}{2c}}),$$

where we have to assume that $j_{reg} \geq 2((1+\delta)/\delta - 1)$ and $k_{reg} \geq 2(1+\delta)/\delta$ for the regularity constants j_{reg}, k_{reg} given in Assumption 5.7. For $\delta = 0.05$ we obtain

$$\|u_{K,A} - u_{K,A}^{(\varepsilon_W,\varepsilon_l)}\|_{L^2(\Omega;V)} \leq C_{reg}\frac{C(D)}{\overline{a}_-}(\varepsilon_W + \varepsilon_l^{\frac{1}{2.01}}). \tag{7.4}$$

Therefore, we get $\gamma = 1$ and $c = 2.01/2$ in the equilibration formula (7.2).

Figure 3 shows two different samples of the diffusion coefficient and the corresponding FE approximations of the PDE solution.

The FE discretization parameters are given by $h_\ell = 0.4 \cdot 2^{-(\ell-1)}$ for $l = 1, \ldots, 7$. We set $u_{ref} = \hat{u}_{7,\varepsilon_W,\varepsilon_l}$, where the approximation parameters ε_W and ε_l are choosen according to (7.2) and we compute $M = 100$ samples to estimate the strong error by the Monte Carlo estimator [see (7.3)]. In this experiment we investigate the strong error convergence rate for the sample-adapted FE approach as well as convergence rate for the non-adapted FE approach (see Sect. 6). In Sect. 7.3.1 we described two approaches to simulate Poisson subordinators. We run this experiment with both approaches: first, we approximate the Poisson process via sampling on an equidistant level-dependent grid and, in a second run of the experiment, we simulate the Poisson subordinators exactly using the Uniform Method described in [33, Section 8.1.2]. The convergence results for the both approaches for this experiment are given in the Fig. 4.

We see a convergence rate of approximately 0.7 for the standard FEM discretization and full order convergence ($\hat{\kappa}_a \approx 1$) for the sample-adapted approach. On the right hand side of Fig. 4 one sees that the sample-adapted approach is more efficient in terms of computational effort if we consider the error-to-(averaged)DOF-plot. Only

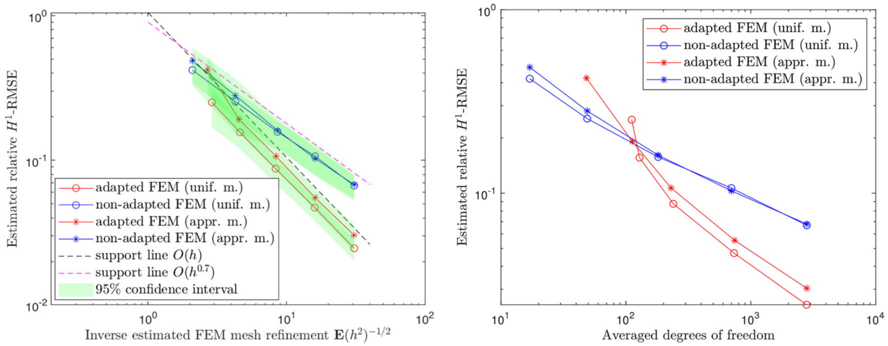

Fig. 4 Convergence results for Poisson(1)-subordinators using the approximation approach and the Uniform Method with mixed Dirichlet–Neumann boundary conditions

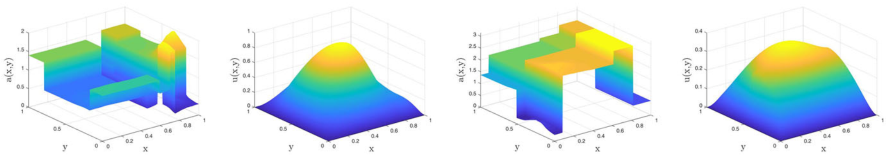

Fig. 5 Different samples of the diffusion coefficient with Poisson(1)-subordinators and the corresponding PDE solutions with homogeneous Dirichlet boundary conditions

on the first level the standard FEM approach seems to be more efficient (pre-asymptotic behaviour). If we compare the results for the approximation method with the Uniform Method (see Sect. 7.3.1), we find that, while the convergence rates are the same, the constant of the error in the sample-adapted approach is slightly smaller for the Uniform Method. This shift is exactly the additional error resulting from an approximation of the subordinators in the approximation approach. We also see that, compared to the approximation approach, on the lower levels the averaged degrees of freedom in the sample-adapted FEM approach is slightly higher if we simulate the Poisson subordinators exactly. This is caused by the fact that in this case we do not approximate the discontinuities of the field which are generated by the Poisson processes. This results in a higher average number of degrees of freedom on the lower levels because discontinuities are more likely close to each other.

7.3.3 Poisson subordinators: low intensity and homogeneous Dirichlet boundary conditions

Next, we consider the elliptic PDE under homogeneous Dirichlet boundary conditions. All other parameters remain as in Sect. 7.3.2. Figure 5 shows samples of the diffusion coefficient and the corresponding FE approximation of the PDE solution.

We estimate the strong error convergence rate for this problem in the same way as in the previous example using $M = 250$ samples and we use the approximation approach to simulate the Poisson subordinators (see Sect. 7.3.1). Convergence results are given in Fig. 6.

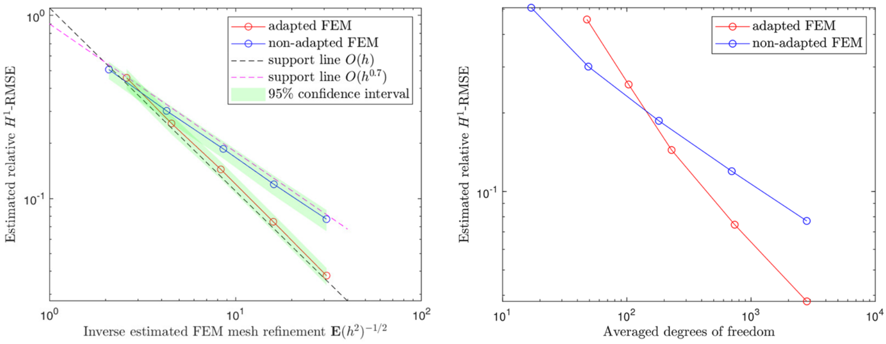

Fig. 6 Convergence results for Poisson(1)-subordinators using the approximation approach and homogeneous Dirichlet boundary conditions

As in the experiment with mixed Dirichlet-Neumann boundary conditions we obtain convergence order of $\kappa_a \approx 0.7$ for the standard FEM approach and full order convergence for the sample-adapted approach. Also in case of homogeneous Dirichlet boundary conditions the sample-adapted FEM is more efficient in terms of the averaged number of degrees of freedom.

7.3.4 Poisson subordinators: high intensity and mixed boundary conditions

In this section we want to consider subordinators with higher intensity, resulting in a higher number of discontinuities in the diffusion coefficient. Therefore, we consider l_1 and l_2 to be Poisson(5) processes. We set $K = 1$ in the diffusion coefficient (4.4) and consider the downscaled process

$$\tilde{l}_j(t) = \frac{1}{15} l_j(t),$$

for $t \in [0, 1]$ and $j = 1, 2$. With this choice it is reasonable to expect that this cut-off has no numerical influence since

$$\mathbb{P}\left(\sup_{t\in[0,1]} \frac{l_j(t)}{15} > 1\right) = \mathbb{P}(l_j(1) > 15) \approx 6.9008e{-}05,$$

for $j = 1, 2$. The reason we consider the downscaled process is that otherwise we would have to simulate the GRF W_2 on the domain $[0, 15]^2$ which is very time consuming. Note that the downscaling of the subordinators has no effect on the jump activity. The variance parameter of the field W_1 is chosen to be $\sigma_1^2 = 1$ and the parameters of the GRF W_2 are set to be $\sigma_2^2 = 0.3^2$ and $r_2 = 0.5$. Figure 7 shows samples of the coefficient and the corresponding pathwise FEM solution.

As in the first experiment, we again run this experiment using both methods described in Sect. 7.3.1: the approximation approach using Poisson-distributed increments and the Uniform Method. We use the discretization steps $h_\ell = 0.1 \cdot 1.7^{-(\ell-1)}$ for $\ell = 1, \dots, 7$ and $M = 150$ samples.

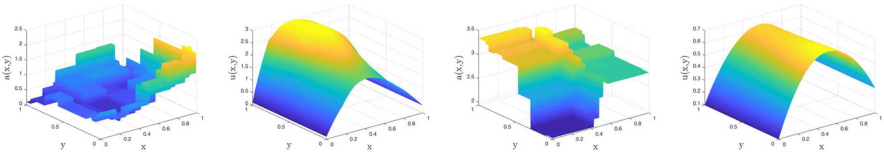

Fig. 7 Different samples of the diffusion coefficient with Poisson(5)-subordinators (top) and the corresponding PDE solutions with mixed Dirichlet–Neumann boundary conditions (bottom)

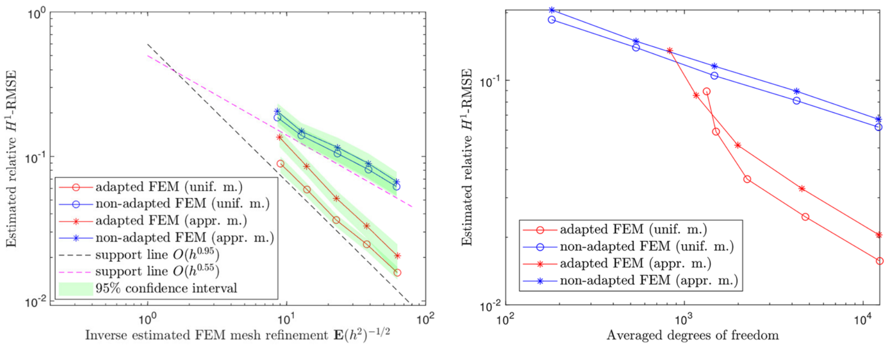

Fig. 8 Convergence results for Poisson(5)-subordinators using the approximation approach and the Uniform Method with mixed Dirichlet–Neumann boundary conditions

In Fig. 8 we see that we get almost full order convergence for the sample-adapted FE method for both approximation approaches of the Poisson processes. Compared to the low-intensity examples with Poisson(1)-subordinators given in Sects. 7.3.2 and 7.3.3, we get a slightly lower convergence rate of approximately 0.55 for the standard FEM approach. This holds for both approximation methods of the Poisson subordinators. Hence, we see that the way how the Poisson-subordinators are simulated seems to have no effect on the convergence rate.

7.3.5 Poisson subordination of a GRF with short correlation length: high intensity and mixed boundary conditions

In our construction of the jump-diffusion coefficient, the jumps are generated by the subordinated GRF. To be precise, the number of spatial jumps is determined by the subordinators and the jump intensities (in terms of the differences in height between the jumps) are essentially determined by the GRF W_2. This fact allows to control the jump intensities of the diffusion coefficient by the correlation parameter of the underlying GRF W_2. In the following experiment we want to investigate the influence of the jump intensities of the diffusion coefficient on the convergence rates.

In Sect. 7.3.4 we subordinated a Matérn-1.5-GRF with pointwise standard deviation $\sigma_2^2 = 0.3^2$ and a correlation length of $r_2 = 0.5$. In the following experiment we set the standard deviation of the GRF W_2 to $\sigma_2^2 = 0.5^2$ and the correlation length to $r_2 = 0.1$ and leave all the other parameters unchanged. Figure 9 compares the resulting GRF with the field W_2 with parameters $\sigma_2^2 = 0.3^2$ and $r_2 = 0.5$ which we used in Sect. 7.3.4.

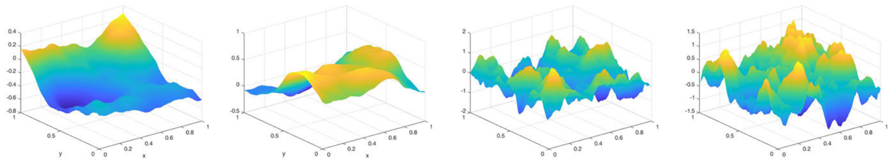

Fig. 9 Samples of a Matérn-1.5-GRF with $\sigma_2^2 = 0.3^2$, $r_2 = 0.5$ (left) and with parameters $\sigma_2^2 = 0.5^2$, $r_2 = 0.1$ (right)

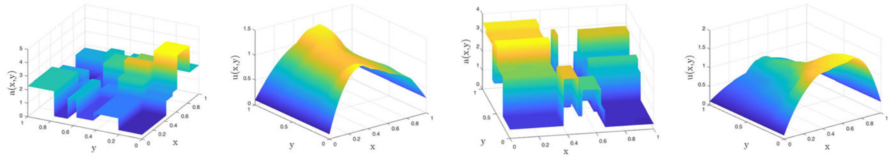

Fig. 10 Different samples of the diffusion coefficient with Poisson(5)-subordinators and the corresponding PDE solutions with mixed Dirichlet-Neumann boundary conditions and small correlation length $r_2 = 0.1$ of the GRF W_2

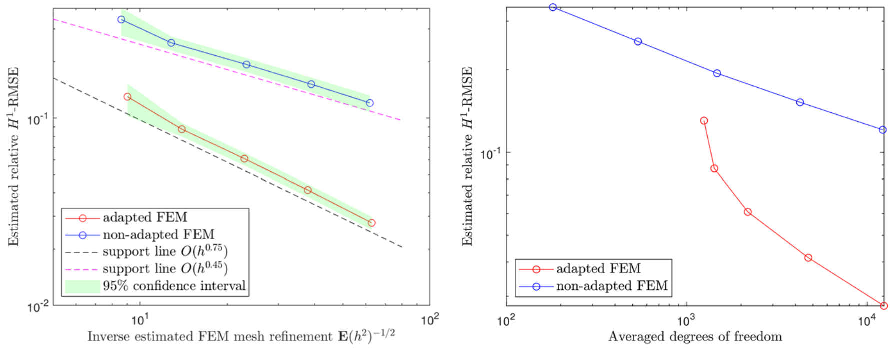

Fig. 11 Convergence results for Poisson(5)-subordinators using the Uniform Method with mixed Dirichlet–Neumann boundary conditions and GRF parameters $\sigma_2^2 = 0.5^2$ and $r_2 = 0.1$

Subordinating the GRF with small correlation length (right plots in Fig. 9) result in higher jump intensities in the diffusion coefficient as the subordination of the GRF with higher correlation length (left plots in Fig. 9). Figure 10 shows samples of the diffusion coefficient and the corresponding PDE solutions where the parameters of W_2 are $\sigma_2^2 = 0.5^2$ and $r_2 = 0.1$.

As expected, the resulting jump coefficient shows jumps with a higher intensity compared to the jump coefficient in the previous experiment where we used the GRF W_2 with parameters $\sigma_2^2 = 0.3^2$ and $r_2 = 0.5$ (see Fig. 7).

We estimate the strong error convergence rate using this high-intensity jump coefficient using $M = 200$ samples and approximate the Poisson subordinators by the Uniform Method.

Figure 11 shows that for the GRF W_2 with small correlation length the convergence rates are reduced for both approaches: the standard FEM approach and the sample-adapted version. We cannot preserve full order convergence in the sample-adapted FEM but observe a convergence rate of approximately 0.75. In the non-adapted approach we obtain a convergence rate of approximately 0.45. Looking at the error-to-(averaged)DOF-plot on the right hand side of Fig. 11 we see that still the sample-adapted approach is by a large margin more efficient in terms of computational effort. This experiment confirms our expectations since the FEM convergence rate has been shown to be strongly influenced by the regularity of the jump-diffusion coefficient (see e.g. [11, 31]).

7.4 Gamma subordinators

In order to also consider Lévy subordinators with infinite activity we take Gamma processes to subordinate the GRF in the remaining numerical examples. We set the standard deviation of the GRF W_1 to be $\sigma_1^2 = 1.5^2$ and we choose $\sigma_2^2 = 0.3^2$ and $r_2 = 1$ for the Matérn-1.5-GRF W_2 and leave the other parameters unchanged. For $a_G, b_G > 0$, a $Gamma(a_G, b_G)$-distributed random variable admits the density function

$$x \mapsto \frac{b_G^{a_G}}{\Gamma(a_G)} x^{a_G-1} \exp(-x b_G), \quad \text{for } x > 0,$$

where $\Gamma(\cdot)$ denotes the Gamma function. A Gamma process $(l(t), t \geq 0)$ has independent Gamma distributed increments. Being precise, $l(t) - l(s) \sim Gamma(a_G \cdot (t-s), b_G)$ for $0 < s < t$ (see [33, Chapter 8]). The following lemma is essential to approximate the Gamma processes.

Lemma 7.2 *Let Z be a $Gamma(a_G, b_G)$ distributed random variable for positive parameters $a_G, b_G > 0$. It holds*

$$\mathbb{E}(Z^n) = b_G^{-n} \frac{\Gamma(a_G + n)}{\Gamma(a_G)},$$

for all $n \in \mathbb{N}_0$.

Proof We calculate

$$\begin{aligned}
\mathbb{E}(Z^n) &= \frac{b_G^{a_G}}{\Gamma(a_G)} \int_0^\infty x^{n+a_G-1} exp(-x b_G) dx \\
&= \frac{b_G^{a_G}}{\Gamma(a_G)} \frac{\Gamma(a_G+n)}{b_G^{a_G+n}} \int_0^\infty \frac{b_G^{a_G+n}}{\Gamma(a_G+n)} x^{n+a_G-1} exp(-x b_G) dx \\
&= b_G^{-n} \frac{\Gamma(a_G+n)}{\Gamma(a_G)},
\end{aligned}$$

where we used that the integral over the Gamma density equals one in the last step. □

In our numerical experiments we choose l_j to be a $Gamma(4, 10)$-process for $j = 1, 2$. Since increments of a Gamma process are Gamma-distributed random variables it is straightforward to generate values of a Gamma process on grid points

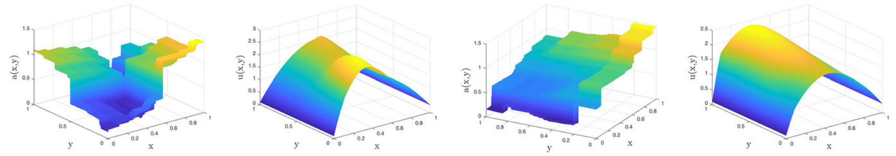

Fig. 12 Different samples of the diffusion coefficient with Gamma(4, 10)-subordinators and the corresponding PDE solutions with mixed Dirichlet–Neumann boundary conditions

$(x_i)_{i=0}^{N_l} \subset [0, 1]$ with $|x_{i+1} - x_i| \leq \varepsilon_l$ for $i = 0, \ldots, N_l - 1$. We then use the piecewise constant extension of the simulated values $\{l_j(x_i),\ i = 0, \ldots, N_l - 1,\ j = 1, 2\}$ to approximate the Lévy subordinators:

$$l_j^{(\varepsilon_l)}(x) = \begin{cases} l_j(x_i) & x \in [x_i, x_{i+1}) \text{ for } i = 0, \ldots, N_l - 1, \\ l_j(x_{N_l - 1}) & x = 1. \end{cases}$$

for $j = 1, 2$. Note that in this case Assumption 4.2(v) is fulfilled for any fixed $\eta < +\infty$. To see that we consider a fixed $s \in \mathbb{N}$ with $s \leq \eta$ and calculate for an arbitrary $x \in [0, 1)$ with $x \in [x_i, x_{i+1})$:

$$\begin{aligned} \mathbb{E}(|l_j(x) - l_j^{(\varepsilon_l)}(x)|^s) &\leq \mathbb{E}(|l_j(x_{i+1}) - l_j(x_i)|^s) \\ &\leq \mathbb{E}(|l_j(\varepsilon_l)|^s) \\ &= b_G^{-s} \frac{\Gamma(a_G \varepsilon_l + s)}{\Gamma(a_G \varepsilon_l)} \\ &= b_G^{-s} \prod_{i=1}^{s-1} (a_G \varepsilon_l + i) a_G \varepsilon_l \\ &\leq C_l \varepsilon_l. \end{aligned}$$

Figure 12 shows samples of the jump-diffusion coefficient with Gamma(4, 10)-subordinator and corresponding FE solution where we used mixed Dirichlet-Neumann boundary conditions.

We set the diffusion cut-off to $K = 2$ since in this case we obtain

$$\mathbb{P}\left(\sup_{t \in [0,1]} l_j(t) \geq 2\right) = \mathbb{P}(l_j(1) \geq 2) \approx 3.2042e{-}06,$$

for $j = 1, 2$. The use of infinite-activity Gamma subordinators in the diffusion coefficient does not allow anymore for a sample-adapted approach to solve the PDE problem. Hence, we only use the standard FEM approach to solve the PDE samplewise and estimate the strong error convergence. We use $M = 200$ samples to estimate the strong error on the levels $\ell = 1, \ldots, 5$ where we set the non-adaptive FEM solution $u_{7,\varepsilon_W,\varepsilon_l}$ on level $L = 7$ to be the reference solution. We choose the FEM discretization steps to be $h_\ell = 0.4 \cdot 2^{-(\ell-1)}$ for $\ell = 1, \ldots, 7$.

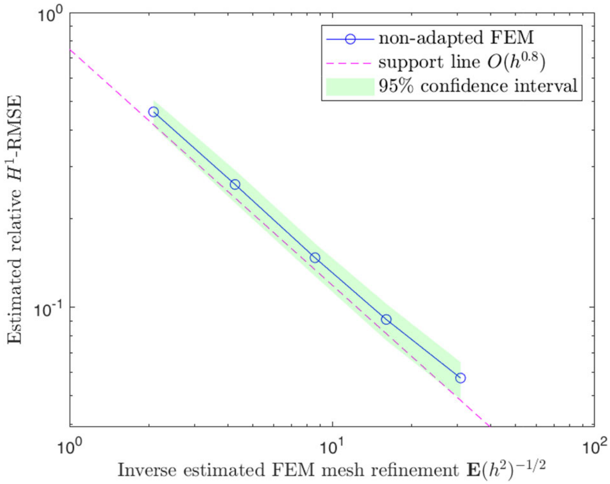

Fig. 13 Convergence results for Gamma(4, 10)-subordinators with mixed Dirichlet–Neumann boundary conditions

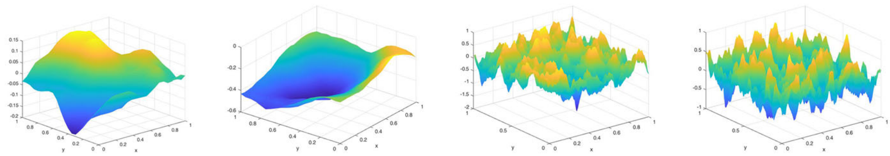

Fig. 14 Samples of a Matérn-1.5-GRF with correlation length $r_2 = 1$ (left) and with correlation length $r_2 = 0.05$ (right)

Figure 13 shows a convergence rate of approximately 0.8 for the standard-FEM approach. Since we do not treat the discontinuities in a special way we cannot expect full order convergence. In fact, the given convergence is comparably good since in general we cannot prove a higher convergence order than 0.5 for the standard deterministic FEM approach without special treatment of the discontinuities (see [6, 11]). The convergence rate of approximately 0.8 in this example is based on the comparatively large correlation length of the underlying GRF W_2 (see Sect. 7.3.5).

In Sect. 7.3.5 we investigated the effect of a rougher diffusion coefficient on the convergence rate for Poisson(5)-subordinators. In the following experiment we follow a similar strategy and use a shorter correlation length in the GRF W_2 which is subordinated by Gamma processes. Therefore, we choose the parameters of the Matérn-1.5-GRF W_2 to be $\sigma_2^2 = 0.3^2$ and $r_2 = 0.05$. Figure 14 shows a comparison of the resulting GRFs W_2 with the different correlation lengths.

In Fig. 15, the GRF with small correlation length results in higher jumps of the diffusion coefficient and stronger deformations of the corresponding PDE solution compared to the previous example (see Fig. 12).

We estimate the strong error taking $M = 200$ samples where we use the non-adapted FEM solution $u_{9,\varepsilon_W,\varepsilon_l}$ on level $L = 9$ as reference solution and choose the

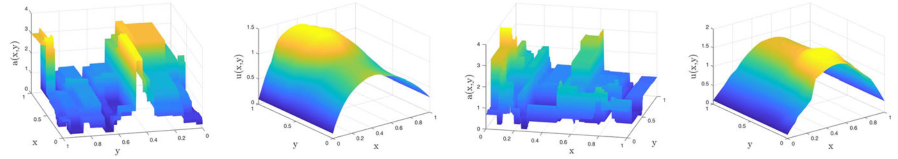

Fig. 15 Different samples of the diffusion coefficient with Gamma(4, 10)-subordinators and the corresponding PDE solutions with mixed Dirichlet–Neumann boundary conditions where the correlation length of W_2 is $r_2 = 0.05$

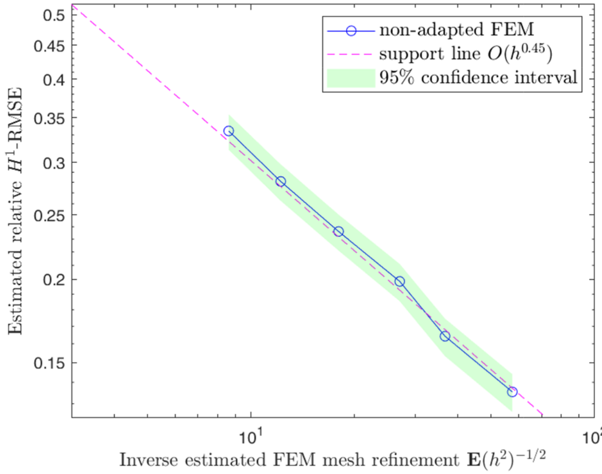

Fig. 16 Convergence results for Gamma(4, 10)-subordinators with mixed Dirichlet–Neumann boundary conditions where the correlation length of W_2 is $r_2 = 0.05$

FEM discretization steps to be $h_\ell = 0.1 \cdot 1.5^{-(\ell-1)}$ for $\ell = 1, \ldots, 9$. Figure 16 shows the convergence on the levels $\ell = 1, \ldots, 6$.

We observe a convergence rate of approximately 0.45 which is significantly smaller than the rate of approximately 0.8 we obtained in the example where we used a GRF W_2 with correlation length $r_2 = 1$ (see Fig. 13). This again confirms that, for subordinated GRFs, the convergence rate of the FE method is highly dependent on the correlation length of the underlying GRF W_2 and the resulting jump-intensity of the diffusion coefficient.

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Availability of data and materials All data used in our experiments have been produced with MATLAB random number generators and no external datasets have been used. The datasets generated and analysed during the current study are available from the corresponding author on reasonable request.

Appendix A Proof of Theorem 5.5

Theorem A.1 (Theorem 5.5) *For any* $\delta > 0$ *and any* $n \in (1, +\infty)$ *there exists a constant* $A = A(\delta, n) > 0$ *such that*

$$\mathbb{E}\left(\operatorname*{ess\,sup}_{\underline{x}\in\mathcal{D}} |a_K(\underline{x}) - a_{K,A}(\underline{x})|^n\right)^{1/n} < \delta.$$

Proof *Step 1*: Tail estimation for the coefficient

By Assumption 4.2(ii), the functions Φ_1, Φ_2 from Definition 3.4 fulfill

$$|\Phi_1'(x)| \leq \phi \exp(\psi|x|),\ |\Phi_2(x) - \Phi_2(y)| \leq C_{lip}|x - y|,$$

for $x, y \in \mathbb{R}$. By the mean value theorem, for any $x \in \mathbb{R}$ there exists a real number $\xi \in \mathbb{R}$ with $|\xi| \leq |x|$ such that it holds

$$\begin{aligned} |\Phi_1(x)| &\leq C(1 + |x|\Phi_1'(\xi)) \leq C(1 + |x|\phi \exp(\psi|\xi|)) \leq C(1 + |x|\phi \exp(\psi|x|)) \\ &\leq \tilde{\phi} \exp(\tilde{\psi}|x|), \end{aligned} \tag{A.1}$$

for positive constants $\tilde{\phi}$ and $\tilde{\psi}$ which are independent of $x \in \mathbb{R}$.

Since the GRFs W_1 and W_2 are $\mathbb{P} - a.s.$ bounded on $\mathcal{D}$ resp. on $[0, K]^2$ it follows from Adler and Taylor [3, Theorem 2.1.1] that $\mu_1 := \mathbb{E}\left(\sup_{(x,y)\in\mathcal{D}} W_1(x, y)\right) < +\infty$ and

$$\mathbb{P}(\|W_1\|_{L^\infty(\mathcal{D})} > m) \leq 2\,\mathbb{P}\left(\sup_{(x,y)\in\mathcal{D}} W_1(x, y) > m\right) \leq 2\exp\Big(-\frac{(m - \mu_1)^2}{2\sigma_{\mathcal{D}}^2}\Big), \tag{A.2}$$

for $m > \mu_1$ with a finite constant $\sigma_{\mathcal{D}}^2$ defined by

$$\sigma_{\mathcal{D}}^2 := \sup_{(x,y)\in\mathcal{D}} \mathbb{E}(W_1(x, y)^2) = \sum_{i=1}^\infty \lambda_i^{(1)} e_i^{(1)}(x, y)^2 \leq C_e^2 \sum_{i=1}^\infty \lambda_i^{(1)} < +\infty,$$

by Assumption 4.2 *i*. For a given $\varepsilon \in (0,1)$ we choose the real number A such that it holds

$$A > 3\,\tilde{\phi}\,\exp\Big(\tilde{\psi}\big(\sqrt{2\sigma_{\mathcal{D}}^2|\ln(\varepsilon/2)|}+\mu_1\big)\Big). \tag{A.3}$$

With this choice we obtain the bound

$$\mathbb{P}\left(\sup_{(x,y)\in\mathcal{D}}\Phi_1(W_1(x,y)) > A/3\right) \le \varepsilon. \tag{A.4}$$

This can be seen by the following calculation

$$\begin{aligned}
&\mathbb{P}\left(\sup_{(x,y)\in\mathcal{D}}\Phi_1(W_1(x,y)) > A/3\right)\\
&\le \mathbb{P}\left(\sup_{(x,y)\in\mathcal{D}}\tilde{\phi}\,\exp(\tilde{\psi}|W_1(x,y)|) > A/3\right)\\
&= \mathbb{P}(\|W_1\|_{L^\infty(\mathcal{D})} > 1/\tilde{\psi}\,\ln(A/(3\tilde{\phi})))\\
&\le 2\,\exp\Big(-\frac{(1/\tilde{\psi}\,\ln(A/(3\tilde{\phi}))-\mu_1)^2}{2\sigma_{\mathcal{D}}^2}\Big)\\
&\le \varepsilon,
\end{aligned}$$

where we used (A.1) in the first step, the estimate (A.2) in the third step and condition (A.3) in the last step.

Obviously, an estimation as in Eq. (A.2) holds for the GRF W_2:

$$\mathbb{P}(\|W_2\|_{L^\infty([0,K]^2)} > m) \le 2\,\mathbb{P}\left(\sup_{(x,y)\in[0,K]^2} W_2(x,y) > m\right) \le 2\exp\Big(-\frac{(m-\mu_2)^2}{2\sigma_K^2}\Big) \tag{A.5}$$

for $m > \mu_2$ with $\mu_2 := \mathbb{E}\left(\sup_{(x,y)\in[0,K]^2} W_2(x,y)\right) < +\infty$ and

$$\sigma_K^2 := \sup_{(x,y)\in[0,K]^2}\mathbb{E}(W_2(x,y)^2) = \sum_{i=1}^{\infty}\lambda_i^{(2)}e_i^{(2)}(x,y)^2 \le C_e^2\sum_{i=1}^{\infty}\lambda_i^{(2)} < +\infty.$$

By the Lipschitz continuity of Φ_2 we conclude the existence of a constant $\phi_2 > 0$ such that

$$|\Phi_2(x)| \le \phi_2(1+|x|), \tag{A.6}$$

for $x \in \mathbb{R}$. If we again fix a positive $\varepsilon \in (0, 1)$ and choose the real number A such that

$$A > 3\phi_2\Big(\sqrt{2\sigma_K^2|\ln(\varepsilon/2)|} + \mu_2 + 1\Big),$$

we obtain the following bound

$$\mathbb{P}\left(\sup_{(x,y)\in[0,K]^2} \Phi_2(W_2(x,y)) > A/3\right) \leq \varepsilon. \tag{A.7}$$

This can be seen by the following calculation:

$$\begin{aligned}
&\mathbb{P}\left(\sup_{(x,y)\in[0,K]^2} \Phi_2(W_2(x,y)) > A/3\right)\\
&\leq \mathbb{P}(\phi_2(1 + \|W_2\|_{L^\infty([0,K]^2)}) > A/3)\\
&\leq \mathbb{P}(\|W_2\|_{L^\infty([0,K]^2)} > A/(3\phi_2) - 1)\\
&\leq 2\exp\Big(-\frac{(A/(3\phi_2) - 1 - \mu_2)^2}{2\sigma_K^2}\Big)\\
&\leq \varepsilon.
\end{aligned}$$

Step 2: Finite moments of the coefficient
In this step we want to show that for any $n \in [1, +\infty)$ it holds

$$\mathbb{E}\left(\operatorname*{ess\,sup}_{\underline{x}\in\mathcal{D}} |a_K(\underline{x})|^n\right) =: C_{a_K}(n, K, D) < +\infty. \tag{A.8}$$

We use the definition of the coefficient a_K in (4.4) and Hölder's inequality to calculate

$$\begin{aligned}
&\mathbb{E}\left(\operatorname*{ess\,sup}_{\underline{x}\in\mathcal{D}} |a_K(\underline{x})|^n\right)\\
&\leq \mathbb{E}(|\overline{a}_+ + \sup_{(x,y)\in\mathcal{D}} \Phi_1(W_1(x,y)) + \sup_{(x,y)\in[0,K]^2} \Phi_2(W_2(x,y))|^n)\\
&\leq 3^{\frac{n-1}{n}}\Big(\overline{a}_+^n + \mathbb{E}\left(\sup_{(x,y)\in\mathcal{D}} \Phi_1(W_1(x,y))^n\right) + \mathbb{E}\left(\sup_{(x,y)\in[0,K]^2} \Phi_2(W_2(x,y))^n\right)\Big)\\
&= 3^{\frac{n-1}{n}}(\overline{a}_+^n + I_1 + I_2).
\end{aligned}$$

Therefore, it remains to show that it holds $I_1, I_2 < +\infty$.

By Fubini's theorem, for every nonnegative random variable X it holds

$$\mathbb{E}(X) = \int_\Omega X\,d\mathbb{P} = \int_\Omega \int_0^\infty \mathbb{1}_{\{X\geq c\}} dc\, d\mathbb{P} = \int_0^\infty \mathbb{P}(X \geq c) dc$$

if the right hand side exists. We use this fact and Eq. (A.1) to estimate for I_1:

$$\begin{aligned}
I_1 &\le \tilde{\phi}^n \mathbb{E}(\exp(\tilde{\psi} n \|W_1\|_{L^\infty(\mathcal{D})}) \\
&= \tilde{\phi}^n \int_0^\infty \mathbb{P}(\exp(\tilde{\psi} n \|W_1\|_{L^\infty(\mathcal{D})}) > c) dc \\
&= \tilde{\phi}^n \int_0^\infty \mathbb{P}(\|W_1\|_{L^\infty(\mathcal{D})} > \ln(c)/(\tilde{\psi} n)) dc \\
&= \tilde{\phi}^n \tilde{\psi} n \int_{-\infty}^{+\infty} \exp(\tilde{\psi} n c) \mathbb{P}(\|W_1\|_{L^\infty(\mathcal{D})} > c) dc \\
&\le \tilde{\phi}^n \tilde{\psi} n \left(\frac{1}{\tilde{\psi} n} + \mu_1 \exp(\tilde{\psi} n \mu_1) + \int_{\mu_1}^\infty 2 \exp\left(\tilde{\psi} n c - \frac{(c-\mu_1)^2}{2\sigma_{\mathcal{D}}^2} \right) dc \right) < +\infty,
\end{aligned}$$

where we split the integral and used Eq. (A.2) in the last step. In a similar way, we use Eq. (A.6) to calculate for the second summand I_2:

$$\begin{aligned}
I_2 &\le \phi_2^n \mathbb{E}((1 + \|W_2\|_{L^\infty([0,K]^2)})^n) \\
&= \phi_2^n \int_0^\infty \mathbb{P}((1 + \|W_2\|_{L^\infty([0,K]^2)})^n > c) dc \\
&= \phi_2^n \int_0^\infty \mathbb{P}(\|W_2\|_{L^\infty([0,K]^2)} > c^{\frac{1}{n}} - 1) dc \\
&= \phi_2^n n \int_{-1}^\infty (c+1)^{n-1} \mathbb{P}(\|W_2\|_{L^\infty([0,K]^2)} > c) dc \\
&\le \phi_2^n n \left((\mu_2 + 1)^n + 2 \int_{\mu_2}^\infty (c+1)^{n-1} \exp\left(-\frac{(c-\mu_2)^2}{2\sigma_K^2} \right) dc \right) < +\infty,
\end{aligned}$$

where we used Eq. (A.5) in the last step. This proves Eq. (A.8).

Step 3: Estimate for the approximation of the diffusion coefficient.

Now, let $\delta \in (0, 1)$ be arbitrary. Choose $A = A(\delta) > 0$ such that

$$A > \max\left\{ 3\overline{a}_+, 3\,\tilde{\phi}\, \exp\left(\tilde{\psi}(\sqrt{2\sigma_{\mathcal{D}}^2 |\ln(\varepsilon/2)|} + \mu_1) \right), 3\phi_2\left(\sqrt{2\sigma_K^2 |\ln(\varepsilon/2)|} + \mu_2 + 1 \right) \right\}$$

for $\varepsilon := 1 - \sqrt{1 - (\delta^{2s}/C_{a_K}(2s, K, D))}$.

We estimate using Hölder's inequality:

$$\begin{aligned}
&\mathbb{E}\left(\operatorname*{ess\,sup}_{\underline{x} \in \mathcal{D}} |a_K(\underline{x}) - a_{K,A}(\underline{x})|^s \right) \\
&\quad \le \mathbb{E}(\|a_K\|_{L^\infty(\mathcal{D})}^s \mathbb{1}_{\{\|a_K\|_{L^\infty(\mathcal{D})}) \ge A\}}) \\
&\quad \le \mathbb{E}(\|a_K\|_{L^\infty(\mathcal{D})}^{2s})^{\frac{1}{2}} \mathbb{P}\left(\operatorname*{ess\,sup}_{\underline{x} \in \mathcal{D}} |a_K(\underline{x})| \ge A \right)^{\frac{1}{2}}
\end{aligned}$$

$$= C_{a_K}(2s, K, D)^{\frac{1}{2}} \mathbb{P}\left(\operatorname*{ess\,sup}_{\underline{x}\in\mathcal{D}} |a_K(\underline{x})| \geq A\right)^{\frac{1}{2}}$$

For the second factor we estimate using the independence of W_1 and W_2

$$\begin{aligned}
&\mathbb{P}\left(\operatorname*{ess\,sup}_{\underline{x}\in\mathcal{D}} |a_K(\underline{x})| \geq A\right) \\
&\quad = 1 - \mathbb{P}\left(\operatorname*{ess\,sup}_{\underline{x}\in\mathcal{D}} |a_K(\underline{x})| \leq A\right) \\
&\quad \leq 1 - \mathbb{P}(\|\Phi_1(W_1)\|_{L^\infty(\mathcal{D})} \leq A/3) \cdot \mathbb{P}(\|\Phi_2(W_2)\|_{L^\infty([0,K]^2)} \leq A/3) \\
&\quad \leq 1 - (1 - \mathbb{P}(\|\Phi_1(W_1)\|_{L^\infty(\mathcal{D})} \geq A/3)) \cdot (1 - \mathbb{P}(\|\Phi_2(W_2)\|_{L^\infty([0,K]^2)} \geq A/3)) \\
&\quad \leq 1 - (1-\varepsilon)^2 \\
&\quad \leq \delta^{2s}/C_{a_K}(2s, K, D),
\end{aligned}$$

where we used Eqs. (A.4) and (A.7) in the fourth step and therefore we obtain

$$\mathbb{E}\left(\operatorname*{ess\,sup}_{\underline{x}\in\mathcal{D}} |a_K(\underline{x}) - a_{K,A}(\underline{x})|^s\right)^{1/s} \leq \delta.$$

□

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