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Muskat–Leverett Two-Phase Flow in Thin Cylindric Porous Media: Asymptotic Approach

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ABSTRACT

A reduced-dimensional asymptotic modeling approach is presented for the analysis of two-phase flow in a thin cylinder with an aperture of order $\mathcal{O}(\varepsilon)$, where ε is a small positive parameter. We consider a nonlinear Muskat–Leverett two-phase flow model expressed in terms of a fractional flow formulation and Darcy's law, with saturation and reduced pressure as unknown. The given flow seeps through the lateral surface of the cylinder. This exchange process leads to a nonhomogeneous Neumann boundary condition with an intensity factor ε^α ($\alpha \geq 1$) that controls mass transport. Furthermore, the absolute permeability tensor comprises the intensity coefficient ε^β , $\beta \in \mathbb{R}$, in the transverse direction. The asymptotic behavior of the solution is studied as $\varepsilon \rightarrow 0$, that is, when the thin cylinder shrinks into an interval. Two qualitatively distinct cases were discovered in the asymptotic behavior of the solution: $\alpha = 1$ and $\beta < 2$, and $\alpha > \beta - 1$ and $\alpha > 1$. In each of these cases, two-term asymptotic approximations are constructed for both reduced pressure and saturation, accompanied by rigorous asymptotic estimates. These approximations were then used to derive approximations for the pressure and flow velocity of each phase. Two one-dimensional models corresponding to the two-phase Muskat–Leverett flow are derived, depending on the values of parameters α and β (each model is a nonlinear elliptic–parabolic system of two differential equations).

1 | Introduction

Understanding the effective dynamics of a multiphase fluid in a porous medium, which may consist of, for example, gas, various liquids, and sometimes solids, is of primary importance in a number of fields, including the oil and gas industry, geothermal energy, environmental engineering, soil science, and chemical engineering (see, e.g., [1, 2]). Therefore, it is essential to identify and comprehend the principal parameters and their impact on flow dynamics in porous media, as this has important implications for the optimization of engineering processes.

It is often the case that porous media comprise a multitude of heterogeneous fractures, which in turn exhibit further characteristics that render them porous media [3]. Compared to their length, the width is small, spanning a range from micrometers to centimeters. In contrast, the length of a fracture can extend to tens of meters.

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This study examines a nonlinear Muskat–Leverett two-phase flow model that describes the flow of two immiscible and incompressible fluids at the Darcy scale, taking into account the effect of capillary pressure at the interfaces between them. We consider this model in a thin cylinder, designated as Ω_ε , as an example of a simple fracture. Here, the small parameter ε characterizes the dimensionless cross-sectional diameter of the cylinder $\Omega_\varepsilon \subset \mathbb{R}^3$, with the longitudinal variable $x_1 \in (0, \ell)$ and transversal variables $\bar{x}_1 = (x_2, x_3)$.

One of the most significant attributes of flow dynamics in porous media is the absolute permeability, which quantifies the intrinsic capacity of a rock's pore network (via its effective cross-sectional area) to transmit fluid. Because pore geometry often varies with direction (anisotropy), the absolute permeability is represented by a symmetric, second-order tensor $\mathbb{K}$. In our model,

$$\mathbb{K} = \begin{pmatrix} k_1(x_1) & 0 & 0 \\ 0 & \varepsilon^\beta k_{22}\left(x_1, \frac{\bar{x}_1}{\varepsilon}\right) & \varepsilon^\beta k_{23}\left(x_1, \frac{\bar{x}_1}{\varepsilon}\right) \\ 0 & \varepsilon^\beta k_{32}\left(x_1, \frac{\bar{x}_1}{\varepsilon}\right) & \varepsilon^\beta k_{33}\left(x_1, \frac{\bar{x}_1}{\varepsilon}\right) \end{pmatrix}, \quad (1)$$

where $\beta \in \mathbb{R}$ and k_{ij} are the permeability components in the i – j direction. The block structure of the permeability tensor occurs in many physical scenarios (see, e.g., [4, 5]); layered sedimentary rocks and shale formations also exhibit anisotropic permeability due to their stratified structures.

After non-dimensionalization, the governing equations exhibit a range of coefficients, both relatively small and large. By fixing one parameter and comparing it with other coefficients, it is possible to identify intensity multipliers near certain coefficients. In our problem, one such intensity factor is the term ε^β in the absolute permeability tensor (1).

Including anisotropic permeability tensors is crucial for accurate macroscopic flow modeling. Several recent studies have numerically examined their effect on fluid flow; see, for example, [4–6] and the references therein. In this regard, we would like to mention the papers [7, 8] (see also references there) that have examined the asymptotic behavior of elliptic boundary-value problems in composites with highly anisotropic fibers exhibiting high conductivity in the longitudinal direction and low conductivity in the transverse directions. These works demonstrate that such anisotropy can significantly affect the limit behavior.

The other intensity factor, ε^α , appears in the boundary conditions

$$\vec{V}_w \cdot \nu_\varepsilon = \varepsilon^\alpha b Q\left(x_1, \frac{\bar{x}_1}{\varepsilon}\right) \quad \text{and} \quad \vec{V}_o \cdot \nu_\varepsilon = \varepsilon^\alpha (1 - b) Q\left(x_1, \frac{\bar{x}_1}{\varepsilon}\right) \quad (2)$$

on the lateral surface of the cylinder Ω_ε , which connect the flows in the cylinder with those in the surrounding environment, where $\vec{V}_w$ and $\vec{V}_o$ are the unknown velocities of the two components of the flow, ν_ε is the outward unit normal to the lateral surface, b is a fractional flow function of the w -phase, and Q is the specified flow rate (injection/extraction) of the mixture. A rigorous statement of the problem, written in a fractional flow formulation, that is, in terms of saturation and reduced (global) pressure, as well as the main assumptions, are given in Section 2, see in particular (18)–(24).

The aim of this work is to study the asymptotic behavior of the solution as the small parameter ε approaches zero, that is, as the thin cylinder Ω_ε shrinks into an interval, and to identify the influence of α and β on the flow. This analysis supports the development of appropriate one-dimensional models for different parameter regimes.

To achieve this goal, we use asymptotic multiscale approximation methodology, a powerful tool for perturbed problems, which allows us to capture the influence of additional model parameters by incorporating higher order terms. The correct choice of asymptotic scaling is crucial and depends on the problem parameters.

In our problem, the exponent parameter β quantifies the relative strength of transversal transport versus longitudinal flow. In thin fracture-like geometries, the physically relevant range is $\beta \geq 0$: $\beta > 0$ corresponds to reduced transversal permeability; $\beta = 0$ to comparable magnitude. Values $\beta < 0$ imply asymptotically large transversal permeability (strong transversal flow) and are not physically relevant for our interpretation of Ω_ε as a simple porous fracture, since the fracture orientation aligns with the dominant direction of the pore structures (in our case, the longitudinal direction). Nevertheless, owing to the block structure of $\mathbb{K}$ and the regularity properties of the problem (see Section 2), our asymptotic constructions and error estimates remain valid for $\beta < 0$ within the regimes studied. We therefore keep $\beta < 0$ in the mathematical formulation as an admissible extension. The parameter α in (2) controls the intensity of mass exchange between the fracture and the surrounding medium.

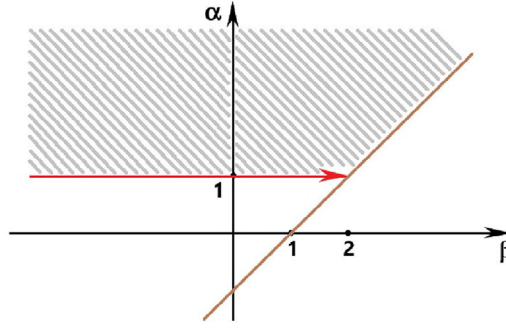


FIGURE 1 | The first case corresponds to the red ray, and the second case to the gray shaded area.

The asymptotic behavior depends on the balance between ε^α (lateral exchange) and ε^β (transversal transport). We focus on two qualitatively distinct regimes (see Figure 1):

1. $\alpha = 1, \beta < 2$: The lateral exchange influences the leading-order limit and β appears in the first correction;
2. $\alpha > 1, \beta < \alpha + 1$: Both α and β affect only higher order terms.

In these parameter regimes, we construct two-term asymptotic approximations for reduced pressure and saturation, accompanied by rigorous estimates in various norms, including the energy and uniform pointwise norms. The first case is treated in Sections 3 and 4, the second in Section 5. The results obtained in these sections are then used to construct asymptotic approximations for the corresponding pressures and velocities $\vec{V}_w$ and $\vec{V}_o$ in Section 6.

It is crucial to highlight the significance of error estimates and convergence rates in substantiating the adequacy of one-dimensional models designed to describe real processes in three-dimensional thin bodies in applied problems. In our context, these estimates provide compelling evidence of the influence of the parameters α and β : In the first case, α already affects the leading term of the asymptotics, while β enters in the next term; in the second case, both parameters influence only higher order terms. Convergence alone can be insufficient, as the limit may lose important information from the original problem; such features are recovered only by constructing further terms in the asymptotics.

In both regimes, our analysis shows that the two-phase flow in the thin cylinder Ω_ε is predominantly longitudinal. This follows from the asymptotic balance: Transversal velocity components are of higher order in ε compared to the longitudinal component, so the leading term of the asymptotics of the total flow velocity $\vec{V} = \vec{V}_o + \vec{V}_w$ has the form $\left(\lambda(s_0(x_1, t)) k_1(x_1) \partial_{x_1} p_0(x_1, t), 0, 0 \right)^\top$, where $\top$ denotes transposition. Physically, this reflects the fact that in slender fracture-like domains with the chosen scalings and block-structure permeability, longitudinal transport dominates, while transversal effects enter only through higher order corrections. In the first regime, the functions p_0 and s_0 satisfy the one-dimensional two-phase Muskat–Leverett problem

$$\begin{cases} \partial_{x_1} \left(\lambda(s_0) k_1(x_1) \partial_{x_1} p_0 \right) = \widehat{Q}(x_1, t) & \text{in } (0, \ell) \times (0, T), \\ \phi \partial_t s_0 = \partial_{x_1} \left(\Lambda(s_0) k_1(x_1) \partial_{x_1} s_0 + \lambda_w(s_0) k_1(x_1) \partial_{x_1} p_0 \right) - b(s_0) \widehat{Q} & \text{in } (0, \ell) \times (0, T), \\ p_0(0, t) = q_0(t) \quad \text{and} \quad p_0(\ell, t) = q_\ell(t), & t \in [0, T], \\ s_0(0, t) = S^0(0, t) \quad \text{and} \quad s_0(\ell, t) = S^0(\ell, t), & t \in [0, T], \\ s_0(x_1, 0) = S^0(x_1, 0), & x_1 \in [0, \ell], \end{cases}$$

where the function $\widehat{Q}$ is the limit transformation of the function Q from the boundary conditions (2); for details see Sections 2 and 3. In the second regime, the function $\widehat{Q}$ is absent in this problem (see (107)).

In the borderline case $\alpha \geq 1, \beta = \alpha + 1$, the lateral exchange and transverse transport both influence the leading term at the same order, which requires a different asymptotic approach. Other parameter values lead to unbounded terms in our asymptotic ansatzes as $\varepsilon \rightarrow 0$, indicating instability and the need for alternative modeling strategies. These cases are not treated here; the roles of α and β are further analyzed in Section 7.

As this is a pioneering work to study the influence of such diverse parameters on two-phase flow dynamics via an asymptotic approach, we have simplified the problem formulation (omitted the acceleration of gravity in Darcy's law, considered the problem within a rectilinear cylinder, and made additional assumptions, particularly regarding the non-singularity of the capillary pressure; see Section 2). For comparison, the Buckley–Leverett model, widely studied in the

literature, neglects both capillary pressure and gravity. Section 7 discusses these constraints and outlines directions for future research.

Literature Review

Mathematical modeling of the flow of two incompressible and immiscible fluids in porous media has been the subject of intense study over the past several decades. Of particular interest are the existence and uniqueness of solutions of such models owing to permeability degeneracy (see Section 2 for further details).

Several models for the interaction of single-phase flows in fractured porous media and dimensionally reduced models are described in [9]. Here, the authors proposed a new dimensionally reduced single-phase flow model in fractured porous media, which introduced the simulation of co-dimensional fracture crossings.

The modeling and numerical simulation of two-phase Darcy flow in fractured porous media have advanced along three main paths: hybrid-dimensional and discrete-fracture frameworks that couple lower dimensional fractures with the surrounding matrix [10–14]; adaptive mesh strategies, such as the finite-volume moving-mesh method for capturing real-time fracture nucleation and propagation [15]; and reduced-order/upscaling techniques that simplify complex fracture networks [16, 17]. These targeted innovations have improved the accuracy and efficiency of the models, but the rich legacy of foundational theories and methods in this field continues to influence modern research. A detailed overview of these developments is provided in the Introduction of [15].

In addition, we cite several publications that employed diverse asymptotic techniques. In [18], convergence results ($\varepsilon \rightarrow 0$) were obtained for a single-phase Darcy flow in a porous medium in a region containing a narrow channel with a width $\mathcal{O}(\varepsilon)$ and high permeability $\mathcal{O}(\varepsilon^{-1})$. In [19], five different reduced models of the Richards equation in fractured media were derived, depending on the ratio of fracture-to-matrix porosity and fracture-to-matrix absolute hydraulic conductivity, which are scalars; the results apply only to nondegenerate cases, where the medium is strictly unsaturated. A steady-state single-phase Darcy flow problem governed by elliptic differential equations in a domain with fractures of aperture ε was analyzed in [20], namely, by scaling the fracture conductivity as ε^α and letting $\varepsilon \rightarrow 0$, the authors prove convergence to distinct discrete-fracture limit models. A formal asymptotic approach applied to two-phase flow in a medium containing a single fracture was presented in [21], and a similar formal asymptotic technique was employed for steady-state single-phase Darcy flow in [22], with numerical examples in both papers validating several of the resulting reduced models.

Homogenization of two-phase flows in periodic porous media in different settings was obtained in [23–28] (see also some relevant links therein). Namely, [23–26] derive rigorous Darcy-scale homogenized systems using two-scale convergence techniques, with [26] specialized to double-porosity/fractured media; a multiparameter expansion for a steady nonlinear two-phase equation with high-contrast permeabilities was developed in [28]; and paper [27] presents a formal phase-field homogenization that links pore-scale Navier–Stokes–Cahn–Hilliard dynamics to Darcy-scale laws.

Only a few papers have conducted asymptotic studies of fluid flows in thin fractured-porous media. Using the method of asymptotic expansions with respect to the thickness of the thin circular cylinder, and assuming that the fluids are strictly separated at $t = 0$ and that only one phase is in contact with the rigid lateral wall, one-dimensional equations were formally identified in [29] from the classical Navier–Stokes system for two incompressible immiscible flows separated by a free boundary with small surface tension effects. A similar model in a two-dimensional thin strip of varying width was considered in [30], but the free interface separating the two fluids is traversal to the flow direction, which depends on the velocities of the fluids and the surface tension between the fluids. A single phase flow in a thin periodic fractured-porous layer with high contrast permeability (the so-called “dual porosity” regime) was considered in [31]. The authors modeled this process using a linear parabolic equation with a rapidly oscillating high-contrast periodic coefficient in an elliptic operator and considered this equation in a thin rectangle (or thin rectangular parallelepiped) with a homogeneous Neumann boundary condition. The corresponding homogenized problems were derived, and the convergence results were proved when the thin rectangle vanished in the limit, employing the two-scale convergence method. To the best of our knowledge, no asymptotic studies have been conducted to investigate two-phase flow in thin-fractured porous media.

2 | Problem Statement

Let $\varpi \subset \mathbb{R}^2$ be a bounded, simply connected domain with the smooth boundary, such that the origin lies in ϖ . The thin cylinder Ω_ε is defined as follows:

$$\Omega_\varepsilon = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < x_1 < \ell, \quad \varepsilon^{-1}\bar{x}_1 \in \varpi\},$$

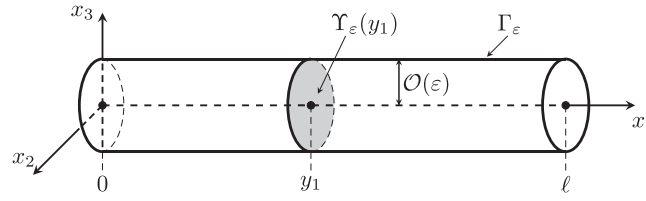


FIGURE 2 | Schematic illustration of a thin cylindrical domain Ω_ε with notation.

where $\varepsilon > 0$ is a small parameter and $\bar{x}_1 = (x_2, x_3)$. We denote by

$$\Gamma_\varepsilon := \partial\Omega_\varepsilon \cap \{x : 0 < x_1 < \ell\}$$

the lateral surface of Ω_ε , and by $Y_\varepsilon(y_1) := \{x : 0 \leq x_1 \leq \ell, \ \varepsilon^{-1}\bar{x}_1 \in \varpi\} \cap \{x : x_1 = y_1\}$ its cross-section at the point $y_1 \in [0, \ell]$, which is the domain $\varepsilon \varpi$ of diameter $\mathcal{O}(\varepsilon)$ (see Figure 2).

The thin cylinder Ω_ε is considered as a porous medium with dimensionless porosity $0 < \text{const} \leq \phi(x) < 1$ and the absolute permeability tensor (1), which is the symmetric, uniformly positive-definite tensor in $\bar{\Omega}_\varepsilon$ with smooth coefficients of class C^2 , where $\beta \in \mathbb{R}$. This means that $k_{23} = k_{32}$ and there are positive constants C_0, C_1, χ_1 , and χ_2 such that for all $x_1 \in [0, \ell]$ and $\bar{\xi}_1 = (\xi_2, \xi_3) \in \varpi$, and for all $\bar{\zeta}_1 = (\zeta_2, \zeta_3) \in \mathbb{R}^2$, we have

$$C_0 \leq k_1(x_1) \leq C_1, \quad \chi_1 |\bar{\zeta}_1|^2 \leq \sum_{i,j=2}^3 k_{ij}(x_1, \bar{\xi}_1) \zeta_i \zeta_j \leq \chi_2 |\bar{\zeta}_1|^2. \quad (3)$$

To reduce the technical calculations, we consider that the function ϕ depends only on the variable x_1 , and additionally assume that $\phi \in C^3([0, \ell])$.

Two immiscible and incompressible fluids (water and oil, denoted by w and o , respectively) flow through the cylinder. This type of movement is referred to as a two-phase flow. The key characteristics of such a flow are the saturation $S_b : \bar{\Omega}_\varepsilon \times [0, T] \mapsto [0, 1]$ and the velocity $\vec{V}_b : \bar{\Omega}_\varepsilon \times [0, T] \mapsto \mathbb{R}^3$ of the b -th fluid, where $b \in \{o, w\}$. The saturation S_b refers to the local proportion of the pore space occupied by the b -th phase, and

$$S_w + S_o = 1, \quad S_b \in [0, 1]. \quad (4)$$

Assuming that the density of each phase is constant, the equation for the conservation of mass for each phase is as follows:

$$\phi \partial_t S_b + \nabla \cdot \vec{V}_b = 0 \quad \text{in } \Omega_\varepsilon^T := \Omega_\varepsilon \times (0, T). \quad (5)$$

The dynamics of the two-phase porous flow process is characterized by Darcy's law (here, the acceleration of gravity is neglected)

$$\vec{V}_b = -\lambda_b \mathbb{K} \nabla P_b \quad \text{in } \Omega_\varepsilon^T, \quad (6)$$

where $\nabla = \text{grad}_x$, $P_b : \Omega_\varepsilon^T \mapsto \mathbb{R}$ is the corresponding phase pressure, $\lambda_b := \frac{k_{rb}}{\mu_i}$ is the phase mobility, k_{rb} is the relative permeability, and the positive constant μ_b is the dynamic viscosity.

The pressures in the two phases are generally not the same, owing to the surface tension involved. The relationship between them is given by a constitutive relation

$$P_o - P_w = p_c(S) \quad (S := S_w), \quad (7)$$

where $p_c : (0, 1) \mapsto \mathbb{R}$ is the capillary pressure. We assume that $p_c \in C^3([0, 1])$ and

$$p'_c(S) < 0 \quad \text{for } S \in [0, 1]. \quad (8)$$

The functions k_{ro} and k_{rw} are empirically defined non-negative functions of $S := S_w \in [0, 1]$, and we assume that they are smooth of class C^3 and satisfy

$$k'_{rw}(S) > 0, \quad k_{rw}(0) = 0, \quad (9)$$

$$k'_{ro}(S) < 0, \quad k_{ro}(1) = 0. \quad (10)$$

Then the total mobility $\lambda := \lambda_o + \lambda_w$ satisfies the inequalities

$$0 < c_1 := \min_{S \in [0,1]} \lambda(S) \leq \lambda(S) \leq c_2 := \max_{S \in [0,1]} \lambda(S) \quad \text{for } S \in [0, 1]; \quad (11)$$

obviously, $0 \leq \lambda_o \leq c_2$ and $0 \leq \lambda_w \leq c_2$.

Equations (4)–(7) represent the Muskat–Leverett two-phase flow model.

Remark 1. The assumptions of the non-singular capillary pressure curve (8) and the non-singular relative permeabilities (9) and (10) are key factors distinguishing this flow model in porous media from other nonlinear formulations. Such assumptions are commonly adopted in the mathematical modeling literature (see [1, 15, 21, 23, 24, 32, 33]) to facilitate mathematically rigorous analyses and numerical investigations, even though they are not satisfied by certain empirical models. In our setting, these assumptions ensure the regularity of the nonlinear model, enabling us to derive asymptotic results with full mathematical justification. Furthermore, as noted below, existence and uniqueness results have been established for two-phase degenerate models. Our findings therefore support the formal extension of these results to cases involving singular capillary pressure curves and more strongly degenerate relative permeabilities.

Boundary value problems for the system of Equations (4)–(7) are best posed after certain transformations that facilitate the simplification of the system while simultaneously revealing its basic mathematical structure [32], resulting in the so-called fractional flow formulation. We briefly recall these transformations below.

Based on (4) and (5), the total flow velocity $\vec{V} = \vec{V}_o + \vec{V}_w$ satisfies the equation $\nabla \cdot \vec{V} = 0$, and from (6) and (7), it follows that

$$-\vec{V} = \lambda_o \mathbb{K} \nabla P_o + \lambda_w \mathbb{K} \nabla P_w = \lambda_o \mathbb{K} (\nabla P_w + p'_c(S) \nabla S) + \lambda_w \mathbb{K} \nabla P_w = \lambda \mathbb{K} \nabla \left(P_w + \int_0^S \frac{\lambda_o(\eta)}{\lambda(\eta)} p'_c(\eta) d\eta \right). \quad (12)$$

Following [32] and introducing the “reduced” pressure

$$P = P_w + \int_0^S \frac{\lambda_o(\eta)}{\lambda(\eta)} p'_c(\eta) d\eta, \quad (13)$$

we get

$$-\vec{V} = \lambda \mathbb{K} \nabla P. \quad (14)$$

Now, we express $\vec{V}_w$ in terms of S and P :

$$-\vec{V}_w \stackrel{(6)}{=} \lambda_w \mathbb{K} \nabla P_w \stackrel{(13)}{=} \lambda_w \mathbb{K} \nabla \left(P - \int_0^S \frac{\lambda_o(\eta)}{\lambda(\eta)} p'_c(\eta) d\eta \right) = \lambda_w \mathbb{K} \nabla P - \frac{\lambda_w \lambda_o}{\lambda} p'_c \mathbb{K} \nabla S \stackrel{(14)}{=} \Lambda(S) \mathbb{K} \nabla S - b(S) \vec{V}, \quad (15)$$

where $b(S) := \frac{\lambda_w(S)}{\lambda(S)}$ is the fractional flow function (the fluidity) of the w -phase and

$$\Lambda(S) := -\frac{\lambda_w(S) \lambda_o(S)}{\lambda(S)} p'_c(S) \quad (16)$$

is the capillary diffusion coefficient. Due to (8)–(11),

$$\Lambda(S) > 0 \quad \text{for } S \in (0, 1) \quad \text{and} \quad \Lambda(0) = \Lambda(1) = 0. \quad (17)$$

Obviously, $\vec{V}_o = \Lambda(S) \mathbb{K} \nabla S + (1 - b(S)) \vec{V}$.

Function S satisfies the equation $\phi \partial_t S + \nabla \cdot \vec{V}_w = 0$ (see (5)). Therefore, based on (15), we get $\phi \partial_t S = \nabla \cdot (\Lambda(S) \mathbb{K} \nabla S - b(S) \vec{V})$.

Because the objective is to derive a one-dimensional model of two-phase flow as the cylinder Ω_ε shrinks to the interval $\mathcal{I} := \{x : x_1 \in (0, \ell), x_2 = x_3 = 0\}$ for $\varepsilon \rightarrow 0$, we mark the unknowns $S, P, \vec{V}$ and the permeability tensor $\mathbb{K}$ by the subscript ε . Matrix $\mathbb{K}_\varepsilon$ has a block-structure (see (1)), and to describe the heterogeneity of the material in the cross-section of Ω_ε at x_1 we use the variables $\frac{x_1}{\varepsilon}$. In addition, there is an intensity factor ε^β in the cross-sectional block of the permeability tensor. Thus, we are led to the system of two quasilinear differential equations

$$\nabla \cdot (\lambda(S_\varepsilon) \mathbb{K}_\varepsilon(x) \nabla P_\varepsilon) = 0 \quad \text{in } \Omega_\varepsilon^T, \quad (18)$$

$$\phi(x_1) \partial_t S_\varepsilon = \nabla \cdot \left(\Lambda(S_\varepsilon) \mathbb{K}_\varepsilon(x) \nabla S_\varepsilon - b(S_\varepsilon) \vec{V}_\varepsilon \right) \quad \text{in } \Omega_\varepsilon^T, \quad (19)$$

where $\vec{V}_\varepsilon = -\lambda(S_\varepsilon) \mathbb{K}_\varepsilon(x) \nabla P_\varepsilon$. Equation (18) is elliptic with respect to P_ε , whereas (19) is a degenerate parabolic equation with respect to S_ε .

We supplement systems (18) and (19) with typical boundary conditions commonly encountered in various applications:

$$S_\varepsilon|_{(x,t) \in Y_\varepsilon^T(0)} = S^0(0, t) \quad \text{and} \quad S_\varepsilon|_{(x,t) \in Y_\varepsilon^T(\ell)} = S^0(\ell, t), \quad (20)$$

$$P_\varepsilon|_{(x,t) \in Y_\varepsilon^T(0)} = q_0(t) \quad \text{and} \quad P_\varepsilon|_{(x,t) \in Y_\varepsilon^T(\ell)} = q_\ell(t), \quad (21)$$

$$-(\lambda(S_\varepsilon) \mathbb{K}_\varepsilon(x) \nabla P_\varepsilon) \cdot \nu_\varepsilon = \varepsilon^\alpha Q_\varepsilon(x, t), \quad (x, t) \in \Gamma_\varepsilon^T, \quad (22)$$

$$-\left(\Lambda(S_\varepsilon) \mathbb{K}_\varepsilon(x) \nabla S_\varepsilon - b(S_\varepsilon) \vec{V}_\varepsilon \right) \cdot \nu_\varepsilon = \varepsilon^\alpha b(S_\varepsilon) Q_\varepsilon(x, t), \quad (x, t) \in \Gamma_\varepsilon^T, \quad (23)$$

where $Y_\varepsilon^T(0) := Y_\varepsilon(0) \times (0, T)$, $Y_\varepsilon^T(\ell) := Y_\varepsilon(\ell) \times (0, T)$, $\Gamma_\varepsilon^T := \Gamma_\varepsilon \times (0, T)$, ν_ε is the outward unit normal to $\partial\Omega_\varepsilon$, the parameter $\alpha \geq 1$, and $Q_\varepsilon(x, t) := Q(x_1, \frac{x_1}{\varepsilon}, t)$.

A novel aspect of these boundary conditions is the introduction of intensity factor ε^α . Condition (22), which can be rewritten as $\vec{V}_\varepsilon \cdot \nu_\varepsilon = \varepsilon^\alpha Q_\varepsilon$ on Γ_ε^T , implies that the rate of flow of the mixture on Γ_ε^T is assumed to be known. Considering (15), condition (23) is reduced to $\vec{V}_{w,\varepsilon} \cdot \nu_\varepsilon = \varepsilon^\alpha b(S_\varepsilon) Q_\varepsilon$ and shows that the injection/extraction of the mixture on Γ_ε^T is carried out in proportion to the fluidity of the phases. Since $\vec{V}_\varepsilon = \vec{V}_{o,\varepsilon} + \vec{V}_{w,\varepsilon}$, we get $\vec{V}_{o,\varepsilon} \cdot \nu_\varepsilon = \varepsilon^\alpha (1 - b(S_\varepsilon)) Q_\varepsilon$ on Γ_ε^T . For the saturation $S_\varepsilon(x, t)$, it is also necessary to specify an initial condition

$$S_\varepsilon(x, 0) = S^0(x_1, 0), \quad x \in \Omega_\varepsilon. \quad (24)$$

The functions occurring on the right-hand sides of conditions (20)–(24) are assumed to be known and have the following properties:

- $S^0 \in C^{3,2}([0, \ell] \times [0, T])$ and $0 < S^0 < 1$ in $[0, \ell] \times [0, T]$;
- the functions $q_0, q_\ell \in C^2([0, T])$, $q_0(0) = q_\ell(0) = 0$, $q_0 > 0$, and $q_0(t) > q_\ell(t)$ for $t \in (0, T]$ (the last inequality indicates that the flow progresses from left to right);
- the function $Q(x_1, \frac{x_1}{\varepsilon}, t)$, $(x_1, \frac{x_1}{\varepsilon}, t) \in \{x_1 \in [0, \ell], \frac{x_1}{\varepsilon} \in \overline{\omega}, t \in [0, T]\}$, belongs to the class C^3 in the domain of definition, vanishes uniformly with respect to t and $\frac{x_1}{\varepsilon} = (\xi_2, \xi_3)$ in $[0, \delta]$ and $[\ell - \delta, \ell]$, where δ is a fixed small positive number, and $Q|_{t=0} = 0$.

To proceed, it is necessary to define a weak solution to problem (18)–(24), which will be further called problem $(\mathbb{P}_\varepsilon \mathbb{S}_\varepsilon)$. Before presenting the definition, we outline several assumptions, following [32, Chapter 5]. Since the capillary diffusion coefficient Λ may vanish at certain points (see (17)), one of our assumptions is that $\Lambda(S_\varepsilon) \nabla S_\varepsilon \in L^2(\Omega_\varepsilon^T)^3$, which, by virtue of (8), is equivalent to $\nabla u(S_\varepsilon) \in L^2(\Omega_\varepsilon^T)^3$, where $u : [0, 1] \rightarrow \mathbb{R}$ is an increasing function defined by

$$u(s) = \int_0^s \frac{\lambda_w(\xi) \lambda_o(\xi)}{\lambda(\xi)} d\xi.$$

If $S_\varepsilon \in L^\infty(\Omega_\varepsilon^T)$, then $u(S_\varepsilon) \in L^\infty(\Omega_\varepsilon^T)$ and admits traces on $Y_\varepsilon^T(0)$ and $Y_\varepsilon^T(\ell)$.

Definition 1. We say that a pair of functions P_ε and S_ε forms a weak solution to problem $(\mathbb{P}_\varepsilon \mathbb{S}_\varepsilon)$ if

1. $P_\varepsilon \in L^\infty(\Omega_\varepsilon^T)$, $\nabla P_\varepsilon \in L^\infty(0, T; L^2(\Omega_\varepsilon)^3)$, the boundary conditions (21) hold in the sense of traces;
2. $S_\varepsilon \in L^\infty(\Omega_\varepsilon^T)$, $0 \leq S_\varepsilon(x, t) \leq 1$ a.e. in Ω_ε^T , $\Lambda(S_\varepsilon) \nabla S_\varepsilon \in L^2(\Omega_\varepsilon^T)^3$, and

$$u(S_\varepsilon)|_{(x,t) \in Y_\varepsilon^T(0)} = u(S^0(0, t)) \quad \text{and} \quad u(S_\varepsilon)|_{(x,t) \in Y_\varepsilon^T(\ell)} = u(S^0(\ell, t));$$

3. for any function η from the Sobolev space $H^1(\Omega_\varepsilon^T)$ such that $\eta|_{Y_\varepsilon^T(0)} = \eta|_{Y_\varepsilon^T(\ell)} = 0$ and any function ζ from the Sobolev space $H^1(\Omega_\varepsilon)$ such that $\zeta|_{Y_\varepsilon(0)} = \zeta|_{Y_\varepsilon(\ell)} = 0$, we have for a.e. $t \in (0, T]$

$$\int_{\Omega_\varepsilon} \vec{V}_\varepsilon \cdot \nabla \zeta \, dx = \varepsilon^\alpha \int_{\Gamma_\varepsilon} Q_\varepsilon \zeta \, d\sigma_x \quad (25)$$

and

$$\int_{\Omega_\varepsilon^t} \phi S_\varepsilon \partial_t \eta \, dx d\tau + \int_{\Omega_\varepsilon^t} \vec{V}_{w,\varepsilon} \cdot \nabla \eta \, dx d\tau - \int_{\Omega_\varepsilon^t} \phi(x_1) S_\varepsilon(x, t) \eta(x, t) \, dx = \varepsilon^\alpha \int_{\Gamma_\varepsilon^t} b(S_\varepsilon) Q_\varepsilon \eta \, d\sigma_x d\tau - \int_{\Omega_\varepsilon^t} \phi(x_1) S^0(x_1, 0) \eta(x, 0) \, dx, \quad (26)$$

where $\vec{V}_\varepsilon$ and $\vec{V}_{w,\varepsilon}$ are defined by (14) and (15), respectively.

The above assumptions are sufficient to demonstrate the existence of a weak solution to problem $(\mathbb{P}_\varepsilon \mathbb{S}_\varepsilon)$ (see, e.g., [32, 33]). The issue of the uniqueness of weak solutions is more challenging; it has been established under specific conditions that guarantee some regularity of the weak solution: For degenerate elliptic–parabolic systems of two-phase flow in [33], and for nondegenerate ones in [32].

In our case, the absence of gravity and assumptions about the initial saturation S^0 and capillary pressure p_c ensure the maximum principle (see [32, Chapter 5, section 6])

$$0 < \delta_0 := \min_{(x,t) \in \overline{\Omega_\varepsilon^T}} S^0(x_1, t) \leq S_\varepsilon(x, t) \leq \max_{(x,t) \in \overline{\Omega_\varepsilon^T}} S^0(x_1, t) =: \delta_1 < 1 \quad (27)$$

for $(x, t) \in \Omega_\varepsilon^T$. The left inequality in (27) means that the parabolic equation (19) is regular, that is,

$$0 < \delta_2 \leq \Lambda(S_\varepsilon(x, t)) \quad \text{for a.e. } (x, t) \in \Omega_\varepsilon^T, \quad (28)$$

and there are no stagnant zones in the flow, where the limit values $\{0, 1\}$ of S_ε are reached. Thus, problem $(\mathbb{P}_\varepsilon \mathbb{S}_\varepsilon)$ is regular and admits a unique weak solution. Moreover, the boundary conditions (20) are satisfied in the sense of traces.

Our goal is to construct and justify the asymptotic approximation for the solution $(P_\varepsilon, S_\varepsilon)$ to problem $(\mathbb{P}_\varepsilon \mathbb{S}_\varepsilon)$ as $\varepsilon \rightarrow 0$, that is, when the thin cylinder Ω_ε shrinks to the interval

$$\mathcal{I} := \{x : x_1 \in (0, \ell), \bar{x}_1 = (0, 0)\}.$$

In addition, we study the influence of the parameters α and β on the asymptotic behavior of the solution.

3 | The Subcase $\alpha = 1$ and $\beta = 0$

Problem $(\mathbb{P}_\varepsilon \mathbb{S}_\varepsilon)$ involves three parameters: the small parameter ε , the intensity parameter α in the boundary conditions on Γ_ε , and the intensity parameter β in the absolute permeability tensor $\mathbb{K}_\varepsilon$. In such cases, as shown in [34, 35], constructing an approximation requires adjusting the asymptotic scale to these parameters and to the problem under consideration. We begin with this subcase of the first case in order to describe all stages of the analysis in detail. Moreover, the asymptotic estimates are more accurate in this subcase.

3.1 | Formal Asymptotic Analysis

From (27) it follows that the first term of the asymptotics of S_ε as $\varepsilon \rightarrow 0$ is a function $s_0 : \mathcal{I} \times [0, T] \mapsto \mathbb{R}$ that satisfies the inequalities

$$0 < \delta_0 \leq s_0(x_1, t) \leq \delta_1 < 1, \quad \text{for } (x_1, t) \in \mathcal{I}^T := \mathcal{I} \times [0, T]. \quad (29)$$

Using the inequality

$$\varepsilon \int_{\Gamma_\varepsilon} u^2 d\sigma_x \leq C \left(\varepsilon^2 \int_{\Omega_\varepsilon} |\nabla_{\bar{x}_1} u|^2 dx + \int_{\Omega_\varepsilon} u^2 dx \right) \quad \forall u \in H^1(\Omega_\varepsilon), \quad (30)$$

which was proved in [36, section 2], inequalities (3) and (11) and the assumptions for the functions q_0 , q_ℓ , and Q , we derive from (18), (21), and (22) the inequality

$$\max_{t \in [0, T]} \frac{1}{\sqrt{1} \Omega_\varepsilon \upharpoonright_3} \|P_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C_1. \quad (31)$$

Here $\upharpoonright_n$ is the n -dimensional Lebesgue measure of a set S .

Remark 2. Here and further, constants in all inequalities are positive and independent of the solution $(P_\varepsilon, S_\varepsilon)$ and the parameters ε , α , and β . In general, constants with the same indices in different inequalities are different.

Thus, the first term of the asymptotics of P_ε is of order $\mathcal{O}(1)$ in the norm of the Sobolev space $H^1(\Omega_\varepsilon)$ as $\varepsilon \rightarrow 0$. Taking this and (29) into account, we propose the following ansatzes:

$$\mathfrak{P}_\varepsilon(x, t) := p_0(x_1, t) + \varepsilon p_1(x_1, t) + \varepsilon^2 u_2\left(x_1, \frac{\bar{x}_1}{\varepsilon}, t\right) + \varepsilon^3 u_3\left(x_1, \frac{\bar{x}_1}{\varepsilon}, t\right) + \dots \quad (32)$$

and

$$\mathfrak{S}_\varepsilon(x, t) := s_0(x_1, t) + \varepsilon s_1(x_1, t) + \varepsilon^2 v_2\left(x_1, \frac{\bar{x}_1}{\varepsilon}, t\right) + \varepsilon^3 v_3\left(x_1, \frac{\bar{x}_1}{\varepsilon}, t\right) + \dots \quad (33)$$

to construct asymptotic approximations of P_ε and S_ε , respectively.

Substituting $\mathfrak{P}_\varepsilon$ and $\mathfrak{S}_\varepsilon$ in (18) instead of P_ε and S_ε , respectively, using Taylor's formula for λ , collecting terms of the same powers of ε and equating those sums to zero, we obtain differential equations in rescale variables $\bar{\xi}_1 = \frac{\bar{x}_1}{\varepsilon}$, $\bar{\xi}_1 = (\xi_2, \xi_3)$, for the coefficients u_2, u_3 , and v_2, v_3 in the rescale cross-section $Y(x_1) := \{\bar{\xi}_1 : \bar{\xi}_1 \in \varpi\}$.

Namely, at ε^0 we get

$$-\nabla_{\bar{\xi}_1} \cdot \left(\lambda(s_0(x_1, t)) \mathbf{K} \nabla_{\bar{\xi}_1} u_2(x_1, \bar{\xi}_1, t) \right) = \partial_{x_1} \left(\lambda(s_0(x_1, t)) k_1(x_1) \partial_{x_1} p_0(x_1, t) \right) \quad (34)$$

for $\bar{\xi}_1 \in Y(x_1)$, where

$$\mathbf{K}(x_1, \bar{\xi}_1) = \begin{pmatrix} k_{22}(x_1, \bar{\xi}_1) & k_{23}(x_1, \bar{\xi}_1) \\ k_{32}(x_1, \bar{\xi}_1) & k_{33}(x_1, \bar{\xi}_1) \end{pmatrix}, \quad (35)$$

and at ε^1

$$-\nabla_{\bar{\xi}_1} \cdot \left(\lambda(s_0) \mathbf{K} \nabla_{\bar{\xi}_1} u_3 + s_1 \lambda'(s_0) \mathbf{K} \nabla_{\bar{\xi}_1} u_2 \right) = \partial_{x_1} \left(\lambda(s_0) k_1 \partial_{x_1} p_1 + s_1 \lambda'(s_0) k_1 \partial_{x_1} p_0 \right). \quad (36)$$

Substituting $\mathfrak{P}_\varepsilon$ and $\mathfrak{S}_\varepsilon$ in (22) and equating the terms with the same order of ε , we derive

$$-\lambda(s_0) \left(\mathbf{K} \nabla_{\bar{\xi}_1} u_2 \right) \cdot \bar{\nu}(\bar{\xi}_1) = Q(x_1, \bar{\xi}_1, t), \quad \bar{\xi}_1 \in \partial Y_\varepsilon(x_1), \quad (37)$$

$$\left(\lambda(s_0) \mathbf{K} \nabla_{\bar{\xi}_1}^- u_3 + s_1 \lambda'(s_0) \mathbf{K} \nabla_{\bar{\xi}_1}^- u_2 \right) \cdot \bar{\nu}(\bar{\xi}_1) = 0, \quad \bar{\xi}_1 \in \partial Y_\varepsilon(x_1). \quad (38)$$

Here, $\bar{\nu}(\frac{\bar{x}_1}{\varepsilon}) = \left(\nu_2(\frac{\bar{x}_1}{\varepsilon}), \nu_3(\frac{\bar{x}_1}{\varepsilon}) \right)$ is the outward unit normal to the boundary of the disk $Y_\varepsilon(x_1)$; obviously, the outward unit normal ν_ε to the lateral surface of the thin cylinder Ω_ε is equal to $\left(0, \bar{\nu}(\frac{\bar{x}_1}{\varepsilon}) \right)$.

Equations (34) and (37), and Equations (36) and (38) are linear Neumann problems in $Y(x_1)$ with respect to the variables $\bar{\xi}_1$; x_1 and t are considered parameters. For uniqueness, we add the following condition to each of these problems:

$$\langle u_j(x_1, \cdot, t) \rangle_{Y(x_1)} = 0 \quad (j \in \{2, 3\}), \quad (39)$$

where

$$\langle u_j(x_1, \cdot, t) \rangle_{Y(x_1)} := \int_{Y(x_1)} u_j(x_1, \bar{\xi}_1, t) d\bar{\xi}_1.$$

Writing down the solvability condition for each of these problems, we obtain the following differential equations:

$$\partial_{x_1} \left(\lambda(s_0) k_1(x_1) \partial_{x_1} p_0 \right) = \hat{Q}(x_1, t), \quad (x_1, t) \in \mathcal{I}^T, \quad (40)$$

$$\partial_{x_1} \left(\lambda(s_0) k_1(x_1) \partial_{x_1} p_1 + s_1 \lambda'(s_0) k_1(x_1) \partial_{x_1} p_0 \right) = 0, \quad (x_1, t) \in \mathcal{I}^T. \quad (41)$$

Here

$$\hat{Q}(x_1, t) := \frac{1}{|\varpi|_2} \int_{\partial Y(x_1)} Q(x_1, \bar{\xi}_1, t) d\sigma_{\bar{\xi}_1}. \quad (42)$$

By performing the same calculations for Equation (19) and boundary condition (23), we obtain the following linear Neumann problems for determining the coefficients v_2 and v_3 in ansatz (33):

$$\begin{aligned} & -\nabla_{\bar{\xi}_1}^- \cdot \left(\Lambda(s_0) \mathbf{K}(x_1, \bar{\xi}_1) \nabla_{\bar{\xi}_1}^- v_2(x_1, \bar{\xi}_1, t) + \lambda_w(s_0) \mathbf{K}(x_1, \bar{\xi}_1) \nabla_{\bar{\xi}_1}^- u_2 \right) \\ & = \partial_{x_1} \left(\Lambda(s_0) k_1 \partial_{x_1} s_0 + \lambda_w(s_0) k_1 \partial_{x_1} p_0 \right) - \phi(x_1) \partial_t s_0, \quad \bar{\xi}_1 \in Y(x_1), \end{aligned} \quad (43)$$

$$-\left(\Lambda(s_0) \mathbf{K} \nabla_{\bar{\xi}_1}^- v_2 + \lambda_w(s_0) \mathbf{K} \nabla_{\bar{\xi}_1}^- u_2 \right) \cdot \bar{\nu}(\bar{\xi}_1) = b(s_0) Q(x_1, \bar{\xi}_1, t), \quad \bar{\xi}_1 \in \partial Y_\varepsilon(x_1), \quad (44)$$

$$\langle v_2(x_1, \cdot, t) \rangle_{Y(x_1)} = 0, \quad (45)$$

and

$$\begin{aligned} & -\nabla_{\bar{\xi}_1}^- \cdot \left(\Lambda(s_0) \mathbf{K} \nabla_{\bar{\xi}_1}^- v_3 + s_1 \Lambda'(s_0) \mathbf{K} \nabla_{\bar{\xi}_1}^- v_2 + \lambda_w(s_0) \mathbf{K} \nabla_{\bar{\xi}_1}^- u_3 + s_1 \lambda'_w(s_0) \mathbf{K} \nabla_{\bar{\xi}_1}^- u_2 \right) \\ & = \partial_{x_1} \left(\Lambda(s_0) k_1 \partial_{x_1} s_1 + s_1 \Lambda'(s_0) k_1 \partial_{x_1} s_0 + \lambda_w(s_0) k_1 \partial_{x_1} p_1 + s_1 \lambda'_w(s_0) k_1 \partial_{x_1} p_0 \right) - \phi(x_1) \partial_t s_1, \quad \bar{\xi}_1 \in Y(x_1), \end{aligned} \quad (46)$$

$$-\left(\Lambda(s_0) \mathbf{K} \nabla_{\bar{\xi}_1}^- v_3 + s_1 \Lambda'(s_0) \mathbf{K} \nabla_{\bar{\xi}_1}^- v_2 + \lambda_w(s_0) \mathbf{K} \nabla_{\bar{\xi}_1}^- u_3 + s_1 \lambda'_w(s_0) \mathbf{K} \nabla_{\bar{\xi}_1}^- u_2 \right) \cdot \bar{\nu}(\bar{\xi}_1) = b'(s_0) s_1 Q, \quad \bar{\xi}_1 \in \partial Y_\varepsilon(x_1), \quad (47)$$

$$\langle v_3(x_1, \cdot, t) \rangle_{Y(x_1)} = 0. \quad (48)$$

The solvability condition for the Neumann problem (43)–(45) is given by the differential equation

$$\phi \partial_t s_0 = \partial_{x_1} \left(\Lambda(s_0) k_1 \partial_{x_1} s_0 + \lambda_w(s_0) k_1 \partial_{x_1} p_0 \right) - b(s_0) \hat{Q}(x_1, t), \quad (x_1, t) \in \mathcal{I}^T, \quad (49)$$

while for the Neumann problem (46)–(48) it is given by the equation

$$\phi \partial_t s_1 = \partial_{x_1} (\Lambda(s_0) k_1 \partial_{x_1} s_1 + s_1 \Lambda'(s_0) k_1 \partial_{x_1} s_0 + \lambda_w(s_0) k_1 \partial_{x_1} p_1 + s_1 \lambda'_w(s_0) k_1 \partial_{x_1} p_0) - b'(s_0) s_1 \widehat{Q}(x_1, t), \quad (x_1, t) \in I^T. \quad (50)$$

3.2 | Limit Problem

Differential Equations (40) and (49) supplemented with appropriate boundary conditions and the initial condition from problem $(\mathbb{P}_\varepsilon \mathbb{S}_\varepsilon)$ form the following system:

$$\begin{cases} \partial_{x_1} (\lambda(s_0) k_1(x_1) \partial_{x_1} p_0) = \widehat{Q}(x_1, t) & \text{in } I^T, \\ \phi \partial_t s_0 = \partial_{x_1} (\Lambda(s_0) k_1 \partial_{x_1} s_0 + \lambda_w(s_0) k_1 \partial_{x_1} p_0) - b(s_0) \widehat{Q} & \text{in } I^T, \\ p_0(0, t) = q_0(t) \quad \text{and} \quad p_0(\ell, t) = q_\ell(t), \quad t \in [0, T], \\ s_0(0, t) = S^0(0, t) \quad \text{and} \quad s_0(\ell, t) = S^0(\ell, t), \quad t \in [0, T], \\ s_0(x_1, 0) = S^0(x_1, 0), \quad x_1 \in [0, \ell], \end{cases} \quad (51)$$

with the unknown functions $p_0(x_1, t)$ and $s_0(x_1, t)$, $(x_1, t) \in [0, \ell] \times [0, T]$. This problem is called *the limit problem* for problem $(\mathbb{P}_\varepsilon \mathbb{S}_\varepsilon)$ for subcase $\alpha = 1$ and $\beta = 0$.

From the first differential equation in problem (51) and the boundary conditions for p_0 , we derive the representation

$$p_0(x_1, t) = \int_0^{x_1} \frac{1}{\lambda(s_0) k_1(\eta)} \int_0^\eta \widehat{Q}(\zeta, t) d\zeta d\eta + \frac{q_\ell(t) - q_0(t) - \int_0^\ell \frac{1}{\lambda(s_0) k_1} \int_0^\eta \widehat{Q} d\zeta d\eta}{\int_0^\ell \frac{1}{\lambda(s_0) k_1} d\eta} \int_0^{x_1} \frac{1}{\lambda(s_0) k_1(\eta)} d\eta + q_0(t). \quad (52)$$

Using (52), the second differential equation in (51) is reduced to

$$\phi \partial_t s_0 = \partial_{x_1} (\Lambda(s_0) k_1(x_1) \partial_{x_1} s_0) + \tilde{a}(s_0, x_1, t) \partial_{x_1} s_0,$$

where

$$\tilde{a}(s_0, x_1, t) = b'(s_0) \left(\int_0^{x_1} \widehat{Q}(\zeta, t) d\zeta + \frac{q_\ell(t) - q_0(t) - \int_0^\ell \left(\frac{1}{\lambda(s_0) k_1} \int_0^\eta \widehat{Q} d\zeta \right) d\eta}{\int_0^\ell \frac{1}{\lambda(s_0) k_1} d\eta} \right),$$

or

$$\partial_t s_0 = \Lambda(s_0) \frac{k_1(x_1)}{\phi(x_1)} \partial_{x_1 x_1}^2 s_0 + a(\partial_{x_1} s_0, s_0, x_1, t), \quad (53)$$

where

$$a(\partial_{x_1} s_0, s_0, x_1, t) = \frac{\Lambda'(s_0) k_1(x_1)}{\phi(x_1)} (\partial_{x_1} s_0)^2 + \frac{\Lambda(s_0) k_1'(x_1)}{\phi(x_1)} \partial_{x_1} s_0 + \frac{1}{\phi(x_1)} \tilde{a}(s_0, x_1, t) \partial_{x_1} s_0.$$

Thus, for s_0 we obtain the following problem:

$$\begin{cases} \partial_t s_0 = \Lambda(s_0) \frac{k_1(x_1)}{\phi(x_1)} \partial_{x_1 x_1}^2 s_0 + a(\partial_{x_1} s_0, s_0, x_1, t) & \text{in } I^T, \\ s_0(0, t) = S^0(0, t) \quad \text{and} \quad s_0(\ell, t) = S^0(\ell, t), \quad t \in [0, T], \\ s_0(x_1, 0) = S^0(x_1, 0), \quad x_1 \in [0, \ell]. \end{cases} \quad (54)$$

Now we verify the conditions of Theorem 5.2, from [37, Chapter VI]. Owing to (3), (29), (17), and the assumptions for the function ϕ (see the beginning of Section 2), the coefficient $\Lambda \frac{k_1}{\phi}$ in (53) is bounded below and above by positive constants (this means that inequalities (5.10) from [37] with $m = 2$ are satisfied). Inequality (5.9) holds true for $\Phi = 1$.

Next, according to Remark 5.1 from [37], it is enough to show that inequality (5.6) holds. Due to the properties of the functions $p_c, \lambda_0, \lambda_w$, and λ (see (8)–(11)), the derivative Λ' is bounded, and recalling properties of the functions k_1, ϕ, q_0, q_ℓ , and Q , we derive

$$|\Lambda'(s) \frac{k_1(x_1)}{\phi(x_1)}| (1 + |p|)^2 + |\Lambda(s) \left(\frac{k_1(x_1)}{\phi(x_1)} \right)'| (1 + |p|) + \left| \frac{\Lambda'(s_0) k_1(x_1)}{\phi(x_1)} p^2 + \frac{\Lambda(s_0) k_1'(x_1)}{\phi(x_1)} p + \frac{1}{\phi(x_1)} \tilde{a}(s_0, x_1, t) p \right| \leq C_1 (1 + |p|)^2.$$

This inequality coincides with inequality (5.6) from [37].

In addition, the smoothness of the functions ϕ, k_1, q_0, q_ℓ , and Q ensures the necessary smoothness of the coefficients in Equation (53) to satisfy the smoothness conditions (c) in Theorem 5.2 from [37, Chapter VI]. The assumption for the smoothness of function S^0 ensures the fulfillment of the compatibility condition of zero order for problem (54). To satisfy the compatibility condition of the first order, we additionally assume that

$$\left(\phi(x_1) \partial_t S^0(x_1, 0) - \partial_{x_1} \left(\Lambda(S^0(x_1, 0)) k_1(x_1) \partial_{x_1} S^0(x_1, 0) \right) \right) \Big|_{x_1=0, \text{ and } x_1=\ell} = 0. \quad (55)$$

Thus, based on Theorem 5.2 [37], the following statement holds.

Proposition 1. *Problem (54) has a unique solution in the Hölder space $C^{2+\gamma, 1+\gamma}([0, \ell] \times [0, T])$ with $\gamma \in (0, 1)$.*

This statement and representation (52) indicate a unique classical solution to the limit problem (51) exists.

3.3 | Determination of the Remaining Coefficients in (32) and (33)

The differential equations in problem (51) are the solvability conditions for Neumann problems (34), (37), (39), and (43)–(45), respectively. Therefore, we first uniquely determine solution u_2 , and then solution v_2 .

Considering the first differential equation in (51), the Neumann problem (34), (37), (39) can now be rewritten as follows:

$$\begin{cases} -\nabla_{\bar{\xi}_1} \cdot \left(\mathbf{K} \nabla_{\bar{\xi}_1} u_2(x_1, \bar{\xi}_1, t) \right) = \frac{\hat{Q}(x_1, t)}{\lambda(s_0(x_1, t))}, & \bar{\xi}_1 \in Y(x_1), \\ -\left(\mathbf{K} \nabla_{\bar{\xi}_1} u_2 \right) \cdot \bar{\nu}(\bar{\xi}_1) = \frac{Q(x_1, \bar{\xi}_1, t)}{\lambda(s_0)}, & \bar{\xi}_1 \in \partial Y_\varepsilon(x_1), \\ \langle u_2(x_1, \cdot, t) \rangle_{Y(x_1)} = 0. \end{cases} \quad (56)$$

Since the function Q vanishes if $x_1 \in [0, \delta] \cup [\ell - \delta, \ell]$ (see the assumptions for Q), we conclude that $u_2(x_1, \bar{\xi}_1, t) = 0$ for those values of x_1 . In addition, because $Q|_{t=0} = q_0|_{t=0} = q_\ell|_{t=0} = 0$, coefficient u_2 also vanishes at $t = 0$.

Using the first two relations in (56) and the differential equation for s_0 in (51), the Neumann problem (43)–(45) is presented as follows:

$$\begin{cases} \nabla_{\bar{\xi}_1} \cdot \left(\Lambda(s_0) \mathbf{K}(x_1, \bar{\xi}_1) \nabla_{\bar{\xi}_1} v_2(x_1, \bar{\xi}_1, t) \right) = 0, & \bar{\xi}_1 \in Y(x_1), \\ \left(\Lambda(s_0) \mathbf{K}(x_1, \bar{\xi}_1) \nabla_{\bar{\xi}_1} v_2(x_1, \bar{\xi}_1, t) \right) \cdot \bar{\nu}(\bar{\xi}_1) = 0, & \bar{\xi}_1 \in \partial Y_\varepsilon(x_1), \\ \langle v_2(x_1, \cdot, t) \rangle_{Y(x_1)} = 0. \end{cases} \quad (57)$$

Obviously, $v_2 \equiv 0$.

For coefficients p_1 and s_1 we obtain the following linear system of differential equations:

$$\begin{cases} \partial_{x_1} \left(\lambda(s_0) k_1 \partial_{x_1} p_1 + s_1 \lambda'(s_0) k_1 \partial_{x_1} p_0 \right) = 0 & \text{in } I^T, \\ \phi \partial_t s_1 = \partial_{x_1} \left(\Lambda(s_0) k_1 \partial_{x_1} s_1 + a_1 s_1 + a_2 \partial_{x_1} p_1 \right) - b'(s_0) \hat{Q} s_1 & \text{in } I^T, \\ p_1(0, t) = 0 \quad \text{and} \quad p_1(\ell, t) = 0, & t \in [0, T], \\ s_1(0, t) = 0 \quad \text{and} \quad s_1(\ell, t) = 0, & t \in [0, T], \\ s_1(x_1, 0) = 0, & x_1 \in [0, \ell], \end{cases} \quad (58)$$

where

$$a_1(x_1, t) = \Lambda'(s_0) k_1 \partial_{x_1} s_0 + \lambda'_w(s_0) k_1 \partial_{x_1} p_0, \quad a_2(x_1, t) = \lambda_w(s_0) k_1(x_1). \quad (59)$$

It is evident that this problem has a trivial solution $p_1 \equiv s_1 \equiv 0$. It remains to be shown whether this solution is unique. From the first differential equation in (58) and the corresponding boundary conditions, it follows that

$$p_1(x_1, t) = d_1(t, s_1) \int_0^{x_1} \frac{d\eta}{\lambda(s_0(\eta, t)) k_1(\eta)} - \int_0^{x_1} \frac{s_1(\eta, t) \lambda'(s_0(\eta, t)) \partial_{x_1} p_0(\eta, t)}{\lambda(s_0(\eta, t))} d\eta, \quad (60)$$

where

$$d_1(t, s_1) = \int_0^\ell \frac{s_1 \lambda'(s_0) \partial_{x_1} p_0}{\lambda(s_0)} d\eta \left(\int_0^\ell \frac{d\zeta}{\lambda(s_0) k_1} \right)^{-1}.$$

Substituting (60) into the second differential equation of (58), we obtain

$$\phi \partial_t s_1 = \partial_{x_1} \left(\Lambda(s_0) k_1 \partial_{x_1} s_1 + A_1(x_1, t) s_1 \right) + \int_0^\ell A_2(x_1, \eta, t) s_1(\eta, t) d\eta + A_3(x_1, t) s_1, \quad (61)$$

where

$$A_1(x_1, t) = a_1(x_1, t) - b(s_0) \lambda'(s_0) k_1 \partial_{x_1} p_0, \quad A_3(x_1, t) = -b'(s_0) \hat{Q}$$

$$A_2(x_1, \eta, t) = \left(\int_0^\ell \frac{d\zeta}{\lambda(s_0) k_1} \right)^{-1} b'(s_0(x_1, t)) \partial_{x_1} s_0(x_1, t) \frac{\lambda'(s_0(\eta, t)) \partial_{x_1} p_0(\eta, t)}{\lambda(s_0(\eta, t))}.$$

Based on the assumptions for the coefficients of problem $(\mathbb{P}_\varepsilon \mathbb{S}_\varepsilon)$ and the smoothness of the solution (p_0, s_0) to the limit problem (51), functions A_1, A_2, A_3 are bounded in the uniform pointwise norm. Taking this into account, multiplying Equation (61) by s_1 , and integrating by parts over the domain $(0, \tau) \times (0, \ell)$, where τ is an arbitrary number from $(0, T)$, we obtain the inequality

$$c_0 \int_0^\ell s_1^2(x_1, \tau) dx_1 + c_1 \int_0^\tau \int_0^\ell |\partial_{x_1} s_1|^2 dx_1 dt \leq c_2 \int_0^\tau \int_0^\ell |\partial_{x_1} s_1| |s_1| dx_1 dt + c_3 \int_0^\tau \left(\int_0^\ell |s_1| dx_1 \right)^2 dt + c_4 \int_0^\tau \int_0^\ell |s_1|^2 dx_1 dt. \quad (62)$$

Applying Cauchy's inequality with $\delta = \frac{c_1}{2}$ ($ab \leq \delta a^2 + \frac{b^2}{4\delta}$) to the first integral on the right-hand side of (62) and the Cauchy–Bunyakovsky–Schwarz inequality to the second integral, we deduce that

$$c_0 \int_0^\ell s_1^2(x_1, \tau) dx_1 + \frac{c_1}{2} \int_0^\tau \int_0^\ell |\partial_{x_1} s_1|^2 dx_1 dt \leq c_5 \int_0^\tau \int_0^\ell |s_1|^2 dx_1 dt. \quad (63)$$

Then Gronwall's lemma, applied to the inequality

$$\int_0^\ell s_1^2(x_1, \tau) dx_1 \leq \frac{c_5}{c_0} \int_0^\tau \int_0^\ell |s_1|^2 dx_1 dt,$$

yields $\int_0^\ell s_1^2(x_1, \tau) dx_1 = 0$ for any $\tau \in (0, T)$. Thus, $s_1 \equiv 0$, and from (60), $p_1 \equiv 0$.

Remark 3. If the boundary conditions or right-hand sides in the differential equations of problem (58) are nonzero, then Gronwall's lemma provides an a priori estimate for the solution, and Galerkin's method proves its existence. For more general elliptic–parabolic problems, the existence and uniqueness of solutions were proven in [38].

Next, it follows from Neumann problem (36), (38), and (39) that solution u_3 is equal to zero, and from problem (46)–(48) that $v_3 \equiv 0$.

As a result, the approximation functions (32) and (33) for the solution to problem $(\mathbb{P}_\varepsilon \mathbb{S}_\varepsilon)$ now appear as follows:

$$\mathfrak{P}_\varepsilon(x, t) = p_0(x_1, t) + \varepsilon^2 u_2\left(x_1, \frac{\bar{x}_1}{\varepsilon}, t\right), \quad \mathfrak{S}_\varepsilon(x, t) = s_0(x_1, t). \quad (64)$$

Moreover, from the above calculations it follows that

$$\nabla \cdot (\lambda(s_0) \mathbb{K}_\varepsilon(x) \nabla \mathfrak{P}_\varepsilon) = \varepsilon^2 \mathcal{F}_\varepsilon^{(1)} \quad \text{in } \Omega_\varepsilon^T, \quad (65)$$

$$\phi(x_1) \partial_t s_0 - \nabla \cdot (\Lambda(s_0) \mathbb{K}_\varepsilon(x) \nabla s_0 - b(s_0) \vec{\mathfrak{B}}_\varepsilon) = \varepsilon^2 \mathcal{F}_\varepsilon^{(2)} \quad \text{in } \Omega_\varepsilon^T, \quad (66)$$

$$(\lambda(s_0) \mathbb{K}_\varepsilon(x) \nabla \mathfrak{P}_\varepsilon) \cdot \nu_\varepsilon + \varepsilon Q_\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon^T, \quad (67)$$

$$(\Lambda(s_0) \mathbb{K}_\varepsilon(x) \nabla s_0 - b(s_0) \vec{\mathfrak{B}}_\varepsilon) \cdot \nu_\varepsilon + \varepsilon b(s_0) Q_\varepsilon(x, t) = 0 \quad \text{on } \Gamma_\varepsilon^T, \quad (68)$$

$$\mathfrak{P}_\varepsilon|_{(x,t) \in Y_\varepsilon^T(0)} = q_0(t) \quad \text{and} \quad \mathfrak{P}_\varepsilon|_{(x,t) \in Y_\varepsilon^T(\ell)} = q_\ell(t), \quad (69)$$

$$s_0|_{(x,t) \in Y_\varepsilon^T(0)} = S^0(0, t), \quad s_0|_{(x,t) \in Y_\varepsilon^T(\ell)} = S^0(\ell, t), \quad s_0|_{t=0} = S^0(x_1, 0), \quad (70)$$

where the vector-function $\vec{\mathfrak{B}}_\varepsilon = -\lambda(s_0) \mathbb{K}_\varepsilon(x) \nabla \mathfrak{P}_\varepsilon$,

$$\mathcal{F}_\varepsilon^{(1)} = \partial_{x_1}(\lambda(s_0) k_1 \partial_{x_1} u_2) \quad \text{and} \quad \mathcal{F}_\varepsilon^{(2)} = -\partial_{x_1}(\lambda_w(s_0) k_1 \partial_{x_1} u_2).$$

Owing to the smoothness of the functions k_1 , λ , λ_w , s_0 (see Proposition 1), and u_2 , we have

$$\sup_{\Omega_\varepsilon^T} |\mathcal{F}_\varepsilon^{(1)}(x, t)| \leq C_1, \quad \sup_{\Omega_\varepsilon^T} |\mathcal{F}_\varepsilon^{(2)}(x, t)| \leq C_2. \quad (71)$$

3.4 | Justification

Here, we present and prove one of the main results.

Theorem 1 ($\alpha = 1$, $\beta = 0$). Assume that, in addition to the main assumptions made in Section 2, condition (55) is satisfied. Then, there are positive constants $\tilde{C}_0$, $\tilde{C}_1$, $\tilde{C}_2$, and ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$

$$\frac{1}{\sqrt{1} \Omega_\varepsilon \upharpoonright_3} \max_{t \in \times[0, T]} \|P_\varepsilon(\cdot, t) - \mathfrak{P}_\varepsilon(\cdot, t)\|_{H^1(\Omega_\varepsilon)} \leq \tilde{C}_0 \varepsilon^2, \quad (72)$$

$$\frac{1}{\sqrt{1} \Omega_\varepsilon \upharpoonright_3} \max_{t \in \times[0, T]} \|S_\varepsilon(\cdot, t) - s_0(\cdot, t)\|_{L^2(\Omega_\varepsilon)} + \frac{1}{\sqrt{1} \Omega_\varepsilon \upharpoonright_3} \|S_\varepsilon - s_0\|_{L^2(0, T; H^1(\Omega_\varepsilon))} \leq \tilde{C}_1 \varepsilon^2. \quad (73)$$

Remark 4. Because inequalities (72) and (73) contain integral norms and integrals taken over the thin cylinder Ω_ε , these norms must be rescaled, namely divided by the square of the volume of Ω_ε .

Proof. First, we specify relations for the differences

$$U_\varepsilon := \mathfrak{P}_\varepsilon - P_\varepsilon, \quad W_\varepsilon := s_0 - S_\varepsilon. \quad (74)$$

It is easy to see from (20), (21), (24), (69), and (70) that

$$U_\varepsilon|_{(x,t) \in Y_\varepsilon^T(0)} = 0 \quad \text{and} \quad U_\varepsilon|_{(x,t) \in Y_\varepsilon^T(\ell)} = 0, \quad (75)$$

$$W_\varepsilon|_{(x,t) \in Y_\varepsilon^T(0)} = 0, \quad W_\varepsilon|_{(x,t) \in Y_\varepsilon^T(\ell)} = 0, \quad W_\varepsilon|_{t=0} = 0. \quad (76)$$

Subtracting (18) from (65) and using the mean value theorem, we obtain

$$\nabla \cdot \left(\lambda(S_\varepsilon(x, t)) \mathbb{K}_\varepsilon(x) \nabla U_\varepsilon + W_\varepsilon \lambda'(\Theta_\varepsilon^{(1)}) \mathbb{K}_\varepsilon(x) \nabla \mathfrak{P}_\varepsilon \right) = \varepsilon^2 \mathcal{F}_\varepsilon^{(1)} \quad \text{in } \Omega_\varepsilon^T, \quad (77)$$

where $\Theta_\varepsilon^{(1)}(x, t) = \theta_1 s_0(x_1, t) + (1 - \theta_1) S_\varepsilon(x, t)$, $\theta_1 \in (0, 1)$.

Similarly, subtracting Equation (19) from (66), we deduce that

$$\phi(x_1) \partial_t W_\varepsilon - \nabla \cdot \left(\Lambda(S_\varepsilon) \mathbb{K}_\varepsilon(x) \nabla W_\varepsilon + W_\varepsilon \Lambda'(\Theta_\varepsilon^{(2)}) \mathbb{K}_\varepsilon(x) \nabla s_0 + \lambda_w(S_\varepsilon) \mathbb{K}_\varepsilon(x) \nabla U_\varepsilon + W_\varepsilon \lambda'_w(\Theta_\varepsilon^{(3)}) \mathbb{K}_\varepsilon(x) \nabla \mathfrak{P}_\varepsilon \right) = \varepsilon^2 \mathcal{F}_\varepsilon^{(2)} \quad \text{in } \Omega_\varepsilon^T. \quad (78)$$

The Neumann-type boundary conditions for U_ε and W_ε on Γ_ε^T are as follows:

$$\left(\lambda(S_\varepsilon) \mathbb{K}_\varepsilon(x) \nabla U_\varepsilon + W_\varepsilon \lambda'(\Theta_\varepsilon^{(1)}) \mathbb{K}_\varepsilon(x) \nabla \mathfrak{P}_\varepsilon \right) \cdot \nu_\varepsilon = 0 \quad (79)$$

and

$$\left(\Lambda(S_\varepsilon) \mathbb{K}_\varepsilon(x) \nabla W_\varepsilon + W_\varepsilon \Lambda'(\Theta_\varepsilon^{(2)}) \mathbb{K}_\varepsilon(x) \nabla s_0 + \lambda_w(S_\varepsilon) \mathbb{K}_\varepsilon(x) \nabla U_\varepsilon + W_\varepsilon \lambda'_w(\Theta_\varepsilon^{(3)}) \mathbb{K}_\varepsilon(x) \nabla \mathfrak{P}_\varepsilon \right) \cdot \nu_\varepsilon = 0. \quad (80)$$

Now, we multiply Equation (77) by U_ε and integrate it over Ω_ε . Given (75) and (79), we obtain

$$\int_{\Omega_\varepsilon} \lambda(S_\varepsilon) \mathbb{K}_\varepsilon \nabla U_\varepsilon \cdot \nabla U_\varepsilon \, dx = - \int_{\Omega_\varepsilon} W_\varepsilon \lambda'(\Theta_\varepsilon^{(1)}) \mathbb{K}_\varepsilon \nabla \mathfrak{P}_\varepsilon \cdot \nabla U_\varepsilon \, dx - \varepsilon^2 \int_{\Omega_\varepsilon} \mathcal{F}_\varepsilon^{(1)} U_\varepsilon \, dx.$$

Taking into account (3), (11), and (71), and the smoothness of p_0 and u_2 , we derive

$$\int_{\Omega_\varepsilon} |\nabla U_\varepsilon|^2 \, dx \leq C_1 \left(\int_{\Omega_\varepsilon} |W_\varepsilon| |\nabla U_\varepsilon| \, dx + \varepsilon^2 \int_{\Omega_\varepsilon} |U_\varepsilon| \, dx \right). \quad (81)$$

Due to (75) we have

$$\int_{\Omega_\varepsilon} |U_\varepsilon| \, dx \leq C_2 \varepsilon \sqrt{\int_{\Omega_\varepsilon} U_\varepsilon^2 \, dx} \leq C_3 \varepsilon \sqrt{\int_{\Omega_\varepsilon} |\nabla U_\varepsilon|^2 \, dx}.$$

As a result, from (81) we get

$$\int_{\Omega_\varepsilon} |\nabla U_\varepsilon(x, t)|^2 \, dx \leq C_4 \left(\varepsilon^6 + \int_{\Omega_\varepsilon} W_\varepsilon^2(x, t) \, dx \right) \quad \text{for any } t \in [0, T]. \quad (82)$$

Multiplying (78) by W_ε and integrating it over Ω_ε^τ , where $\tau \in (0, T)$, and taking into account (3) and (28) and the assumption for porosity, we obtain

$$\int_{\Omega_\varepsilon} W_\varepsilon^2(x, \tau) \, dx + \int_{\Omega_\varepsilon^\tau} |\nabla W_\varepsilon|^2 \, dx \, dt \leq C_0 \left(|I_1(\varepsilon)| + |I_2(\varepsilon)| + \varepsilon^2 \int_{\Omega_\varepsilon^\tau} \mathcal{F}_\varepsilon^{(2)} W_\varepsilon \, dx \, dt \right), \quad (83)$$

where

$$I_1(\varepsilon) = - \int_{\Omega_\varepsilon^\tau} W_\varepsilon \left(\Lambda'(\Theta_\varepsilon^{(2)}) \mathbb{K}_\varepsilon(x) \nabla s_0 + \lambda'_w(\Theta_\varepsilon^{(3)}) \mathbb{K}_\varepsilon(x) \nabla \mathfrak{P}_\varepsilon \right) \cdot \nabla W_\varepsilon \, dx \, dt, \quad (84)$$

$$I_2(\varepsilon) = - \int_{\Omega_\varepsilon^\tau} \lambda_w(S_\varepsilon) \mathbb{K}_\varepsilon(x) \nabla U_\varepsilon \cdot \nabla W_\varepsilon \, dx \, dt. \quad (85)$$

Using the smoothness of the coefficients of problem $(\mathbb{P}_\varepsilon \mathbb{S}_\mathcal{D})$, with the help of Cauchy's inequality with $\gamma > 0$ ($ab \leq \gamma a^2 + \frac{b^2}{4\gamma}$), we derive

$$|I_1(\varepsilon)| \leq \gamma_1 \int_0^\tau \int_{\Omega_\varepsilon} |\nabla W_\varepsilon|^2 dx dt + \frac{C_1}{\gamma_1} \int_0^\tau \int_{\Omega_\varepsilon} W_\varepsilon^2 dx dt, \quad (86)$$

$$|I_2(\varepsilon)| \leq \gamma_2 \int_0^\tau \int_{\Omega_\varepsilon} |\nabla W_\varepsilon|^2 dx dt + \frac{C_2}{\gamma_2} \int_0^\tau \int_{\Omega_\varepsilon} |\nabla U_\varepsilon|^2 dx dt \stackrel{(82)}{\leq} \gamma_2 \int_0^\tau \int_{\Omega_\varepsilon} |\nabla W_\varepsilon|^2 dx dt + \frac{C_3}{\gamma_2} \left(\varepsilon^6 + \int_0^\tau \int_{\Omega_\varepsilon} W_\varepsilon^2 dx dt \right). \quad (87)$$

Thanks to (71), we have

$$\left| \varepsilon^2 \int_0^\tau \int_{\Omega_\varepsilon} F_\varepsilon^{(2)} W_\varepsilon dx dt \right| \leq C_4 \left(\varepsilon^6 + \int_0^\tau \int_{\Omega_\varepsilon} W_\varepsilon^2 dx dt \right). \quad (88)$$

Then, taking suitable γ_1 and γ_2 in (86) and (87), we derive from (83) the inequality

$$\int_{\Omega_\varepsilon} W_\varepsilon^2(x, \tau) dx + \int_0^\tau \int_{\Omega_\varepsilon} |\nabla W_\varepsilon|^2 dx dt \leq C_5 \left(\varepsilon^6 + \int_0^\tau \int_{\Omega_\varepsilon} W_\varepsilon^2(x, t) dx dt \right) \quad (89)$$

for any $\tau \in (0, T)$. Applying Gronwall's lemma to the inequality

$$\int_{\Omega_\varepsilon} W_\varepsilon^2(x, \tau) dx \leq C_6 \left(\varepsilon^6 + \int_0^\tau \int_{\Omega_\varepsilon} W_\varepsilon^2(x, t) dx dt \right),$$

we obtain

$$\max_{t \in [0, T]} \int_{\Omega_\varepsilon} W_\varepsilon^2(x, t) dx \leq C_7 \varepsilon^6, \quad (90)$$

where constant C_7 depends on T .

To complete the proof, it should be noted that inequalities (90) and (89) imply estimates (73) and (148), and inequalities (90) and (82) imply estimate (72). $\square$

Denote by

$$\langle u \rangle_{\varepsilon\varpi}(x_1) := \frac{1}{\varepsilon^2 |\varpi|_2} \int_{\varepsilon\varpi} u(x_1, \bar{x}_1) d\bar{x}_1$$

the mean of a function $u \in L^2(\Omega_\varepsilon)$ over the cross-section $\varepsilon\varpi$ of the thin cylinder Ω_ε .

From (39), it is easy to verify that

$$\langle \mathfrak{P}_\varepsilon \rangle_{\varepsilon\varpi} = p_0(x_1, t), \quad \langle \partial_{x_1} \mathfrak{P}_\varepsilon \rangle_{\varepsilon\varpi} = \partial_{x_1} \langle \mathfrak{P}_\varepsilon \rangle_{\varepsilon\varpi} = \partial_{x_1} p_0(x_1, t). \quad (91)$$

The subsequent corollary is derived from Theorem 1.

Corollary 1 (case $\alpha = 1$, $\beta = 0$). *Let $(P_\varepsilon, S_\varepsilon)$ be a weak solution to problem $(\mathbb{P}_\varepsilon \mathbb{S}_\varepsilon)$, and (p_0, s_0) be a solution to the limit problem (51). Then the following estimates hold:*

$$\max_{t \in [0, T]} \|\langle P_\varepsilon \rangle_{\varepsilon\varpi} - p_0\|_{H^1(0, \ell)} + \|\langle P_\varepsilon \rangle_{\varepsilon\varpi} - p_0\|_{C([0, \ell] \times [0, T])} \leq C_1 \varepsilon^2, \quad (92)$$

$$\|\langle S_\varepsilon \rangle_{\varepsilon\varpi} - s_0\|_{L^2(0, T; H^1(0, \ell))} + \|\langle S_\varepsilon \rangle_{\varepsilon\varpi} - s_0\|_{L^2(0, T; C([0, \ell]))} \leq C_2 \varepsilon^2. \quad (93)$$

Proof. Using (91) and the Cauchy–Buniakovskii–Schwarz inequality, we deduce that

$$\begin{aligned} \|\langle P_\varepsilon \rangle_{\varepsilon\varpi}(\cdot, t) - p_0(\cdot, t)\|_{L^2(0, \ell)}^2 &= \frac{1}{\varepsilon^4 |\varpi|_2^2} \int_0^\ell \left(\int_{\varepsilon\varpi} (P_\varepsilon - \mathfrak{P}_\varepsilon) d\bar{x}_1 \right)^2 dx_1 \\ &\leq \frac{1}{\varepsilon^2 |\varpi|_2} \int_0^\ell \int_{\varepsilon\varpi} (P_\varepsilon - \mathfrak{P}_\varepsilon)^2 d\bar{x}_1 dx_1 \stackrel{(72)}{\leq} \frac{1}{\varepsilon^2 |\varpi|_2} |\Omega_\varepsilon|_3 \tilde{C}_0^2 \varepsilon^4 = C_1^2 \varepsilon^4 \end{aligned}$$

for any $t \in [0, T]$. Similarly, we demonstrate that

$$\max_{t \in [0, T]} \|\partial_{x_1} \langle P_\varepsilon \rangle_{\varepsilon \varpi} - \partial_{x_1} p_0\|_{L^2(0, \ell)} \leq C_1 \varepsilon^2.$$

Considering the continuous embedding of space $H^1(0, \ell)$ into $C([0, \ell])$, we obtain (92). Similarly, but now using (73), we derive (93). $\square$

4 | The Case $\alpha = 1$ and $\beta < 2$, $\beta \neq 0$

The principal challenge is to understand how the appearance of the value ε^β in the absolute permeability tensor (1) affects the overall result. To achieve this, we propose the use of the following approximation functions:

$$\mathfrak{P}_\varepsilon(x, t) = p_0(x_1, t) + \varepsilon^{2-\beta} u_{2-\beta} \left(x_1, \frac{\bar{x}_1}{\varepsilon}, t \right) \quad \text{and} \quad \mathfrak{S}_\varepsilon(x, t) = s_0(x_1, t), \quad (94)$$

where (p_0, s_0) is a solution to problem (51) and $u_{2-\beta}$ is a solution to problem (56).

Theorem 2 (the case $\alpha = 1$, $\beta < 2$ and $\beta \neq 0$). *Assume that, in addition to the main assumptions made in Section 2, condition (55) is satisfied. Then, there are positive constants $\tilde{C}_0$, $\tilde{C}_1$, $\tilde{C}_2$, and ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$ the differences $U_\varepsilon := \mathfrak{P}_\varepsilon - P_\varepsilon$ and $W_\varepsilon := \mathfrak{S}_\varepsilon - S_\varepsilon$ satisfy the estimates*

$$\frac{1}{\sqrt{1} \Omega_\varepsilon \upharpoonright_3} \max_{t \in \times[0, T]} \sqrt{\int_{\Omega_\varepsilon} |\partial_{x_1} U_\varepsilon(\cdot, t)|^2 dx + \varepsilon^\beta \int_{\Omega_\varepsilon} |\nabla_{\bar{x}_1} U_\varepsilon(\cdot, t)|^2 dx} \leq \tilde{C}_0 \varepsilon^{1-\frac{\beta}{2}}, \quad (95)$$

$$\frac{1}{\sqrt{1} \Omega_\varepsilon \upharpoonright_3} \max_{t \in \times[0, T]} \|W_\varepsilon(\cdot, t)\|_{L^2(\Omega_\varepsilon)} \leq \tilde{C}_1 \varepsilon^{1-\frac{\beta}{2}}, \quad (96)$$

and

$$\frac{1}{\sqrt{1} \Omega_\varepsilon \upharpoonright_3} \sqrt{\int_0^T \int_{\Omega_\varepsilon} |\partial_{x_1} W_\varepsilon|^2 dx dt + \varepsilon^\beta \int_0^T \int_{\Omega_\varepsilon} |\nabla_{\bar{x}_1} W_\varepsilon|^2 dx dt} \leq \tilde{C}_2 \varepsilon^{1-\frac{\beta}{2}}. \quad (97)$$

Proof. Compared with the proof of the previous theorem, the proof of this theorem requires certain changes, as shown below.

Upon substitution of functions (94) into problem $(\mathbb{P}_\varepsilon \mathbb{S}_\varepsilon)$, identical relations are found to those presented in (65)–(70). However, the orders of the residuals in differential Equations (65) and (66) will differ; they are of the order $\mathcal{O}(\varepsilon^{2-\beta})$. In addition, U_ε and W_ε satisfy relations (75)–(80) with the corresponding residuals in (77) and (78). Multiplying the corresponding Equation (77) by U_ε and integrating it over Ω_ε , we get

$$\int_{\Omega_\varepsilon} \lambda(S_\varepsilon) \mathbb{K}_\varepsilon \nabla U_\varepsilon \cdot \nabla U_\varepsilon dx = - \int_{\Omega_\varepsilon} W_\varepsilon \lambda'(\Theta_\varepsilon^{(1)}) \mathbb{K}_\varepsilon \nabla \mathfrak{P}_\varepsilon \cdot \nabla U_\varepsilon dx - \varepsilon^{2-\beta} \int_{\Omega_\varepsilon} \mathcal{F}_\varepsilon^{(1)} U_\varepsilon dx. \quad (98)$$

Taking into account the form of the absolute permeability tensor (1) for this case, (3), and (11), the left-hand side in (98) is estimated from below by the sum of two integrals $c \left(\int_{\Omega_\varepsilon} |\partial_{x_1} U_\varepsilon|^2 dx + \varepsilon^\beta \int_{\Omega_\varepsilon} |\nabla_{\bar{x}_1} U_\varepsilon|^2 dx \right)$. Thus, we deduce from (98) the inequality

$$c \left(\int_{\Omega_\varepsilon} |\partial_{x_1} U_\varepsilon|^2 dx + \varepsilon^\beta \int_{\Omega_\varepsilon} |\nabla_{\bar{x}_1} U_\varepsilon|^2 dx \right) \leq \left| \int_{\Omega_\varepsilon} W_\varepsilon \lambda'(\Theta_\varepsilon^{(1)}) \mathbb{K}_\varepsilon \nabla \mathfrak{P}_\varepsilon \cdot \nabla U_\varepsilon dx \right| + \varepsilon^{2-\beta} \int_{\Omega_\varepsilon} |\mathcal{F}_\varepsilon^{(1)}| |U_\varepsilon| dx. \quad (99)$$

The estimation of the right-hand side of (99) will now be undertaken in the following manner. The first integral is less than

$$\begin{aligned}
& C_2 \int_{\Omega_\varepsilon} |W_\varepsilon| \left| \left(k_1 \partial_{x_1} p_0 + \varepsilon^{2-\beta} \partial_{x_1} u_{2-\beta} \right) \partial_{x_1} U_\varepsilon + \varepsilon \left(\mathbf{K} \nabla_{\bar{x}_1} u_{2-\beta} \right) \cdot \nabla_{\bar{x}_1} U_\varepsilon \right| dx \\
& \leq C_3 \left(\gamma_1 \int_{\Omega_\varepsilon} |\partial_{x_1} U_\varepsilon|^2 dx + \frac{1}{\gamma_1} \int_{\Omega_\varepsilon} W_\varepsilon^2 dx + \int_{\Omega_\varepsilon} |W_\varepsilon|^2 dx + \varepsilon^2 \int_{\Omega_\varepsilon} |\nabla_{\bar{x}_1} U_\varepsilon|^2 dx \right)
\end{aligned}$$

for any $\gamma_1 > 0$, while the second one based on (71) is less than

$$C_1 \varepsilon^{2-\beta} \frac{1}{2} \int_{\Omega_\varepsilon} (1 + U_\varepsilon^2) dx \leq C_4 \left(\varepsilon^{4-\beta} + \varepsilon^{2-\beta} \int_{\Omega_\varepsilon} |\partial_{x_1} U_\varepsilon|^2 dx \right).$$

By choosing an appropriate value of γ_1 and ε_1 , we derive from (99) the inequality

$$\int_{\Omega_\varepsilon} |\partial_{x_1} U_\varepsilon|^2 dx + \varepsilon^\beta \int_{\Omega_\varepsilon} |\nabla_{\bar{x}_1} U_\varepsilon|^2 dx \leq C_1 \left(\varepsilon^{4-\beta} + \int_{\Omega_\varepsilon} W_\varepsilon^2 dx \right) \quad (100)$$

for any $\varepsilon \in (0, \varepsilon_1)$.

Now the analog of inequality (83) appears as follows:

$$\int_{\Omega_\varepsilon} W_\varepsilon^2(x, \tau) dx + \int_0^\tau \int_{\Omega_\varepsilon} (|\partial_{x_1} W_\varepsilon|^2 + \varepsilon^\beta |\nabla_{\bar{x}_1} W_\varepsilon|^2) dx dt \leq C_0 \left(|I_1(\varepsilon)| + |I_2(\varepsilon)| + \varepsilon^{2-\beta} \int_0^\tau \int_{\Omega_\varepsilon} \mathcal{F}_\varepsilon^{(2)} W_\varepsilon dx dt \right), \quad (101)$$

where I_1 and I_2 are determined using (84) and (85), respectively, $\tau \in (0, T)$. We proceed to evaluate the integrals on the right-hand side of (101) as follows:

$$\begin{aligned}
C_0 |I_1(\varepsilon)| & \leq C_1 \int_0^\tau \int_{\Omega_\varepsilon} |W_\varepsilon| (|\partial_{x_1} W_\varepsilon| + \varepsilon |\nabla_{\bar{x}_1} W_\varepsilon|) dx dt \leq \gamma_2 \int_0^\tau \int_{\Omega_\varepsilon} |\partial_{x_1} W_\varepsilon|^2 dx dt + \frac{C_2}{\gamma_2} \int_0^\tau \int_{\Omega_\varepsilon} W_\varepsilon^2 dx dt \\
& + C_3 \int_0^\tau \int_{\Omega_\varepsilon} W_\varepsilon^2 dx dt + C_3 \varepsilon^2 \int_0^\tau \int_{\Omega_\varepsilon} |\nabla_{\bar{x}_1} W_\varepsilon|^2 dx dt \quad \text{for any } \gamma_2 > 0, \\
C_0 |I_2(\varepsilon)| & \leq C_4 \int_0^\tau \int_{\Omega_\varepsilon} (|\partial_{x_1} U_\varepsilon| |\partial_{x_1} W_\varepsilon| + \varepsilon^\beta |\nabla_{\bar{x}_1} U_\varepsilon| |\nabla_{\bar{x}_1} W_\varepsilon|) dx dt \leq \gamma_3 \int_0^\tau \int_{\Omega_\varepsilon} |\partial_{x_1} W_\varepsilon|^2 dx dt + \frac{C_5}{\gamma_3} \int_0^\tau \int_{\Omega_\varepsilon} |\partial_{x_1} U_\varepsilon|^2 dx dt \\
& + \varepsilon^\beta \gamma_4 \int_0^\tau \int_{\Omega_\varepsilon} |\nabla_{\bar{x}_1} W_\varepsilon|^2 dx dt + \varepsilon^\beta \frac{C_6}{\gamma_4} \int_0^\tau \int_{\Omega_\varepsilon} |\nabla_{\bar{x}_1} U_\varepsilon|^2 dx dt
\end{aligned}$$

for any $\gamma_3 > 0$, $\gamma_4 > 0$,

$$C_0 \left| \varepsilon^{2-\beta} \int_0^\tau \int_{\Omega_\varepsilon} \mathcal{F}_\varepsilon^{(2)} W_\varepsilon dx dt \right| \stackrel{(71)}{\leq} C_7 \left(\varepsilon^{6-2\beta} + \int_0^\tau \int_{\Omega_\varepsilon} W_\varepsilon^2 dx dt \right).$$

Choosing the corresponding $\gamma_2, \gamma_3, \gamma_4$, and $\varepsilon_0 > 0$ ($\varepsilon_0 < \varepsilon_1$), we derive from (101) that for arbitrary $\varepsilon \in (0, \varepsilon_0)$ the following estimate holds:

$$\begin{aligned}
& \int_{\Omega_\varepsilon} W_\varepsilon^2(x, \tau) dx + \int_0^\tau \int_{\Omega_\varepsilon} (|\partial_{x_1} W_\varepsilon|^2 + \varepsilon^\beta |\nabla_{\bar{x}_1} W_\varepsilon|^2) dx dt \\
& \leq C_0 \left(\int_0^\tau \int_{\Omega_\varepsilon} W_\varepsilon^2 dx dt + \int_0^\tau \int_{\Omega_\varepsilon} |\partial_{x_1} U_\varepsilon|^2 dx dt + \varepsilon^\beta \int_0^\tau \int_{\Omega_\varepsilon} |\nabla_{\bar{x}_1} U_\varepsilon|^2 dx dt + \varepsilon^{6-2\beta} \right) \stackrel{(100)}{\leq} C_1 \left(\varepsilon^{4-\beta} + \int_0^\tau \int_{\Omega_\varepsilon} W_\varepsilon^2 dx dt \right). \quad (102)
\end{aligned}$$

The remaining step is to apply Gronwall's lemma, as presented at the end of Theorem 1, to derive estimates (95)–(97). $\square$

Similarly, the proof of Corollary 1 may also be applied to obtain the following corollary; however, now the asymptotic estimates from Theorem 2 should be applied.

Corollary 2 (case $\alpha = 1$, $\beta < 2$ and $\beta \neq 0$). Let $(P_\varepsilon, S_\varepsilon)$ be a weak solution to problem $(\mathbb{P}_\varepsilon \mathbb{S}_\varepsilon)$, and (p_0, s_0) be a solution to the limit problem (51). Then the following asymptotic estimates hold:

$$\max_{t \in [0, T]} \|\langle P_\varepsilon \rangle_{\varepsilon \varpi} - p_0\|_{H^1(0, \ell)} + \|\langle P_\varepsilon \rangle_{\varepsilon \varpi} - p_0\|_{C([0, \ell] \times [0, T])} \leq C_1 \varepsilon^{1-\frac{\beta}{2}}, \quad (103)$$

$$\|\langle S_\varepsilon \rangle_{\varepsilon \varpi} - s_0\|_{L^2(0, T; H^1(0, \ell))} + \|\langle S_\varepsilon \rangle_{\varepsilon \varpi} - s_0\|_{L^2(0, T; C([0, \ell]))} \leq C_2 \varepsilon^{1-\frac{\beta}{2}}. \quad (104)$$

Remark 5. The proofs of the estimates in Theorem 2 and in Corollary 2 also include the case where $\beta = 0$. However, the order of these estimates is worse than that obtained in Theorem 1 and Corollary 1. Therefore, the subcase $\alpha = 1$ and $\beta = 0$ was considered separately. To obtain better estimates in Theorem 2, especially for the value of $\beta \in (1, 2)$, additional terms of the asymptotics should be constructed. However, for nonlinear problems, as shown in [35], this requires additional cumbersome calculations, and for some problems it is almost impossible. In our case, to justify the one-dimensional models, as follows from estimates (103) and (104), this is sufficient.

5 | The Case $\alpha > \beta - 1$, $\alpha > 1$

In this case, we consider two subcases.

5.1 | Subcase $\beta \neq 0$

We now propose the following approximation functions:

$$\mathfrak{P}_\varepsilon(x, t) = p_0(x_1, t) + \varepsilon^{\alpha-1} p_{\alpha-1}(x_1, t) + \varepsilon^{\alpha-\beta+1} u_{\alpha-\beta+1}\left(x_1, \frac{\bar{x}_1}{\varepsilon}, t\right) \quad (105)$$

and

$$\mathfrak{S}_\varepsilon(x, t) = s_0(x_1, t) + \varepsilon^{\alpha-1} s_{\alpha-1}(x_1, t), \quad (106)$$

where (p_0, s_0) is a solution to the problem

$$\begin{cases} \partial_{x_1} \left(\lambda(s_0) k_1(x_1) \partial_{x_1} p_0 \right) = 0 & \text{in } \mathcal{I}^T, \\ \phi \partial_t s_0 = \partial_{x_1} \left(\Lambda(s_0) k_1 \partial_{x_1} s_0 + \lambda_w(s_0) k_1 \partial_{x_1} p_0 \right) & \text{in } \mathcal{I}^T, \\ p_0(0, t) = q_0(t) \quad \text{and} \quad p_0(\ell, t) = q_\ell(t), \quad t \in [0, T], \\ s_0(0, t) = S^0(0, t) \quad \text{and} \quad s_0(\ell, t) = S^0(\ell, t), \quad t \in [0, T], \\ s_0(x_1, 0) = S^0(x_1, 0), \quad x_1 \in [0, \ell], \end{cases} \quad (107)$$

$(p_{\alpha-1}, s_{\alpha-1})$ is a solution to the problem

$$\begin{cases} \partial_{x_1} \left(\lambda(s_0) k_1(x_1) \partial_{x_1} p_{\alpha-1} + \lambda'(s_0) k_1(x_1) \partial_{x_1} p_0 s_{\alpha-1} \right) = \widehat{Q}(x_1, t) & \text{in } \mathcal{I}^T, \\ \phi \partial_t s_{\alpha-1} = \partial_{x_1} \left(\Lambda(s_0) k_1 \partial_{x_1} s_{\alpha-1} + a_1 s_{\alpha-1} + a_2 \partial_{x_1} p_{\alpha-1} \right) - b(s_0) \widehat{Q} & \text{in } \mathcal{I}^T, \\ p_{\alpha-1}(0, t) = 0 \quad \text{and} \quad p_{\alpha-1}(\ell, t) = 0, \quad t \in [0, T], \\ s_{\alpha-1}(0, t) = 0 \quad \text{and} \quad s_{\alpha-1}(\ell, t) = 0, \quad t \in [0, T], \\ s_{\alpha-1}(x_1, 0) = 0, \quad x_1 \in [0, \ell], \end{cases} \quad (108)$$

and $u_{\alpha-\beta+1}$ is a solution to problem (56).

As in Section 3.2, we show that problem (107) has a unique solution from the Hölder space $C^{2+\gamma, 1+\gamma}([0, \ell] \times [0, T])$. Note that the compatibility condition of first order for problem (107) coincides with (55) because $Q|_{t=0} = 0$.

The coefficients a_1 and a_2 in the linear problem (108) are determined by formulas (59) through the solution (p_0, s_0) to problem (107). As in Section 3.3 (see Remark 3), we can show that problem (108) has a unique classical solution.

Substituting approximations (105) and (106) into problem $(\mathbb{P}_\varepsilon \mathbb{S}_\varepsilon)$, we find

$$\nabla \cdot (\lambda(\mathfrak{S}_\varepsilon) \mathbb{K}_\varepsilon(x) \nabla \mathfrak{P}_\varepsilon) = \varepsilon^{2\alpha-2} \mathcal{F}_{1,\varepsilon}^{(1)} + \varepsilon^{\alpha-\beta+1} \mathcal{F}_{2,\varepsilon}^{(1)} \quad \text{in } \Omega_\varepsilon^T, \quad (109)$$

$$\phi(x_1) \partial_t \mathfrak{S}_\varepsilon = \nabla \cdot \left(\Lambda(\mathfrak{S}_\varepsilon) \mathbb{K}_\varepsilon(x) \nabla \mathfrak{S}_\varepsilon - b(\mathfrak{S}_\varepsilon) \vec{\mathfrak{B}}_\varepsilon \right) + \varepsilon^{2\alpha-2} \mathcal{F}_{1,\varepsilon}^{(2)} + \varepsilon^{\alpha-\beta+1} \mathcal{F}_{2,\varepsilon}^{(2)} \quad \text{in } \Omega_\varepsilon^T, \quad (110)$$

$$(\lambda(\mathfrak{S}_\varepsilon) \mathbb{K}_\varepsilon(x) \nabla \mathfrak{P}_\varepsilon) \cdot \nu_\varepsilon + \varepsilon^\alpha Q_\varepsilon(x, t) = \varepsilon^{2\alpha-1} \Phi_\varepsilon^{(1)} \quad \text{on } \Gamma_\varepsilon^T, \quad (111)$$

$$\left(\Lambda(\mathfrak{S}_\varepsilon) \mathbb{K}_\varepsilon(x) \nabla \mathfrak{S}_\varepsilon - b(\mathfrak{S}_\varepsilon) \vec{\mathfrak{B}}_\varepsilon \right) \cdot \nu_\varepsilon + \varepsilon^\alpha b(\mathfrak{S}_\varepsilon) Q_\varepsilon = \varepsilon^{2\alpha-1} \Phi_\varepsilon^{(2)} \quad \text{on } \Gamma_\varepsilon^T, \quad (112)$$

$$\mathfrak{P}_\varepsilon|_{(x,t) \in Y_\varepsilon^T(0)} = q_0(t) \quad \text{and} \quad \mathfrak{P}_\varepsilon|_{(x,t) \in Y_\varepsilon^T(\ell)} = q_\ell(t), \quad (113)$$

$$\mathfrak{S}_\varepsilon|_{(x,t) \in Y_\varepsilon^T(0)} = S^0(0, t), \quad \mathfrak{S}_\varepsilon|_{(x,t) \in Y_\varepsilon^T(\ell)} = S^0(\ell, t), \quad \mathfrak{S}_\varepsilon|_{t=0} = S^0(x_1, 0). \quad (114)$$

Here $\vec{\mathfrak{B}}_\varepsilon = -\lambda(\mathfrak{S}_\varepsilon) \mathbb{K}_\varepsilon(x) \nabla \mathfrak{P}_\varepsilon$ and

$$\sup_{\Omega_\varepsilon^T} |\mathcal{F}_{i,\varepsilon}^{(1)}(x, t)| \leq C_1, \quad \sup_{\Omega_\varepsilon^T} |\mathcal{F}_{i,\varepsilon}^{(2)}(x, t)| \leq C_2, \quad i \in \{1, 2\}, \quad (115)$$

$$\sup_{\Gamma_\varepsilon^T} |\Phi_\varepsilon^{(1)}(x, t)| \leq C_3, \quad \sup_{\Gamma_\varepsilon^T} |\Phi_\varepsilon^{(2)}(x, t)| \leq C_4. \quad (116)$$

As we stated at the beginning of Section 3, all calculations are carried out in detail for the case $\alpha = 1$ and $\beta = 0$. To clarify, for example, the definition of $\Phi_\varepsilon^{(1)}$ we provide here is its explicit form. Using Taylor's formula and representation (106), we obtain:

$$\begin{aligned} & (\lambda(\mathfrak{S}_\varepsilon) \mathbb{K}_\varepsilon(x) \nabla \mathfrak{P}_\varepsilon) \cdot \nu_\varepsilon + \varepsilon^\alpha Q_\varepsilon(x, t) \\ &= \varepsilon^\alpha \left(\lambda(s_0) + \lambda'(s_0) \varepsilon^{\alpha-1} s_{\alpha-1} + \frac{1}{2} \lambda''(\theta) \varepsilon^{2\alpha-2} s_{\alpha-1}^2 \right) \left(\mathbf{K}(x_1, \bar{\xi}_1) \nabla_{\bar{\xi}_1} u_{\alpha-\beta+1} \right) \cdot \bar{\nu}(\bar{\xi}_1) \Big|_{\bar{\xi}_1 = \frac{\bar{x}_1}{\varepsilon}} + \varepsilon^\alpha Q_\varepsilon(x, t) \\ &\stackrel{(56)}{=} \varepsilon^{2\alpha-1} \left(\lambda'(s_0) s_{\alpha-1} + \frac{1}{2} \lambda''(\theta) \varepsilon^{\alpha-1} s_{\alpha-1}^2 \right) \left(\mathbf{K}(x_1, \bar{\xi}_1) \nabla_{\bar{\xi}_1} u_{\alpha-\beta+1} \right) \cdot \bar{\nu}(\bar{\xi}_1) \Big|_{\bar{\xi}_1 = \frac{\bar{x}_1}{\varepsilon}} =: \varepsilon^{2\alpha-1} \Phi_\varepsilon^{(1)}(x, t). \end{aligned}$$

Thanks to the smoothness assumptions on the coefficient λ and the entries of the matrix $\mathbf{K}$, the function $\Phi_\varepsilon^{(1)}$ satisfies the first estimate in (116). Analogously, the functions appearing on the right-hand sides of Equations (109), (110), and (112) are computed in the same manner.

Theorem 3 (case $\alpha > \beta - 1$, $\alpha > 1$, and $\beta \neq 0$). Assume that in addition to the main assumptions made in Section 2, condition (55) is satisfied. Then, there are positive constants $\tilde{C}_0$, $\tilde{C}_1$, $\tilde{C}_2$, and ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$ the differences $U_\varepsilon := \mathfrak{P}_\varepsilon - P_\varepsilon$ and $W_\varepsilon := \mathfrak{S}_\varepsilon - S_\varepsilon$ satisfy the estimates

$$\frac{1}{\sqrt{1} \Omega_\varepsilon \upharpoonright_3} \max_{t \in \times[0, T]} \sqrt{\int_{\Omega_\varepsilon} |\partial_{x_1} U_\varepsilon(\cdot, t)|^2 dx + \varepsilon^\beta \int_{\Omega_\varepsilon} |\nabla_{\bar{x}_1} U_\varepsilon(\cdot, t)|^2 dx} \leq \tilde{C}_0 \varepsilon^{\aleph_{\alpha, \beta}}, \quad (117)$$

$$\frac{1}{\sqrt{1} \Omega_\varepsilon \upharpoonright_3} \max_{t \in \times[0, T]} \|W_\varepsilon(\cdot, t)\|_{L^2(\Omega_\varepsilon)} \leq \tilde{C}_1 \varepsilon^{\aleph_{\alpha, \beta}} \quad (118)$$

and

$$\frac{1}{\sqrt{1} \Omega_\varepsilon \upharpoonright_3} \sqrt{\int_0^T \int_{\Omega_\varepsilon} |\partial_{x_1} W_\varepsilon|^2 dx dt + \varepsilon^\beta \int_0^T \int_{\Omega_\varepsilon} |\nabla_{\bar{x}_1} W_\varepsilon|^2 dx dt} \leq \tilde{C}_2 \varepsilon^{\aleph_{\alpha, \beta}}, \quad (119)$$

where $\aleph_{\alpha, \beta} = \min\{\alpha - 1, \frac{\alpha - \beta + 1}{2}\}$.

Proof. Compared with previous theorems, the proof of this theorem requires some changes. First, we note that because s_0 satisfies the bilateral inequality (29) and $s_{\alpha-1}$ is bounded and $\alpha > 1$, there exists an ε_1 such that for all $\varepsilon \in (0, \varepsilon_1)$

$$0 < \frac{\delta_0}{2} \leq \mathfrak{S}_\varepsilon(x, t) \leq \frac{1 + \delta_1}{2} < 1, \quad \text{for } (x, t) \in \Omega_\varepsilon^T, \quad (120)$$

where δ_0 and δ_1 are defined in (27).

The differences $U_\varepsilon = \mathfrak{P}_\varepsilon - P_\varepsilon$ and $W_\varepsilon = \mathfrak{S}_\varepsilon - S_\varepsilon$ satisfy the following relations:

$$\nabla \cdot \left(\lambda(S_\varepsilon(x, t)) \mathbb{K}_\varepsilon \nabla U_\varepsilon + W_\varepsilon \lambda'(\Theta_\varepsilon^{(1)}) \mathbb{K}_\varepsilon \nabla \mathfrak{P}_\varepsilon \right) = \varepsilon^{2\alpha-2} \mathcal{F}_{1,\varepsilon}^{(1)} + \varepsilon^{\alpha-\beta+1} \mathcal{F}_{2,\varepsilon}^{(1)} \quad \text{in } \Omega_\varepsilon^T, \quad (121)$$

$$\phi(x_1) \partial_t W_\varepsilon - \nabla \cdot \left(\Lambda(S_\varepsilon) \mathbb{K}_\varepsilon(x) \nabla W_\varepsilon + W_\varepsilon \Lambda'(\Theta_\varepsilon^{(2)}) \mathbb{K}_\varepsilon(x) \nabla \mathfrak{S}_\varepsilon + \lambda_w(S_\varepsilon) \mathbb{K}_\varepsilon(x) \nabla U_\varepsilon + W_\varepsilon \lambda'_w(\Theta_\varepsilon^{(3)}) \mathbb{K}_\varepsilon(x) \nabla \mathfrak{P}_\varepsilon \right) = \varepsilon^{2\alpha-2} \mathcal{F}_{1,\varepsilon}^{(2)} + \varepsilon^{\alpha-\beta+1} \mathcal{F}_{2,\varepsilon}^{(2)} \quad \text{in } \Omega_\varepsilon^T, \quad (122)$$

$$\left(\lambda(S_\varepsilon) \mathbb{K}_\varepsilon(x) \nabla U_\varepsilon + W_\varepsilon \lambda'(\Theta_\varepsilon^{(1)}) \mathbb{K}_\varepsilon(x) \nabla \mathfrak{P}_\varepsilon \right) \cdot \nu_\varepsilon = \varepsilon^{2\alpha-1} \Phi_\varepsilon^{(1)} \quad \text{on } \Gamma_\varepsilon^T, \quad (123)$$

$$\left(\Lambda(S_\varepsilon) \mathbb{K}_\varepsilon(x) \nabla W_\varepsilon + W_\varepsilon \Lambda'(\Theta_\varepsilon^{(2)}) \mathbb{K}_\varepsilon(x) \nabla \mathfrak{S}_\varepsilon + \lambda_w(S_\varepsilon) \mathbb{K}_\varepsilon(x) \nabla U_\varepsilon + W_\varepsilon \lambda'_w(\Theta_\varepsilon^{(3)}) \mathbb{K}_\varepsilon(x) \nabla \mathfrak{P}_\varepsilon \right) \cdot \nu_\varepsilon = \varepsilon^{2\alpha-1} \Phi_\varepsilon^{(2)} \quad \text{on } \Gamma_\varepsilon^T, \quad (124)$$

and (75) and (76). In (121), $\Theta_\varepsilon^{(1)}(x, t) = \theta_1 \mathfrak{S}_\varepsilon(x, t) + (1 - \theta_1) S_\varepsilon$, where $\theta_1 \in (0, 1)$.

The main difference from the previous theorems is the presence of discrepancies in boundary conditions (123) and (124). Multiplying the corresponding Equation (121) by U_ε and integrating it over Ω_ε , we get

$$\int_{\Omega_\varepsilon} \lambda(S_\varepsilon) \mathbb{K}_\varepsilon \nabla U_\varepsilon \cdot \nabla U_\varepsilon dx = - \int_{\Omega_\varepsilon} W_\varepsilon \lambda'(\Theta_\varepsilon^{(1)}) \mathbb{K}_\varepsilon \nabla \mathfrak{P}_\varepsilon \cdot \nabla U_\varepsilon dx + \varepsilon^{2\alpha-1} \int_{\Gamma_\varepsilon} \Phi_\varepsilon^{(1)} U_\varepsilon d\sigma_x - \varepsilon^{2\alpha-2} \int_{\Omega_\varepsilon} \mathcal{F}_{1,\varepsilon}^{(1)} U_\varepsilon dx - \varepsilon^{\alpha-\beta+1} \int_{\Omega_\varepsilon} \mathcal{F}_{2,\varepsilon}^{(1)} U_\varepsilon dx. \quad (125)$$

Taking into account the form of the absolute permeability tensor (1) and using Cauchy's inequality with any $\gamma > 0$, the first integral on the right-hand side of (125) is bounded from above by the sum

$$C_1 \left(\gamma \int_{\Omega_\varepsilon} |\partial_{x_1} U_\varepsilon|^2 dx + \frac{1}{\gamma} \int_{\Omega_\varepsilon} W_\varepsilon^2 dx + \varepsilon^{\alpha-1} \int_{\Omega_\varepsilon} W_\varepsilon^2 dx + \varepsilon^{\alpha+1} \int_{\Omega_\varepsilon} |\nabla_{\bar{x}_1} U_\varepsilon|^2 dx \right).$$

Considering (115), (116), and (75), the remaining integrals on the right-hand side of (125) can be estimated as follows:

$$\varepsilon^{2\alpha-2} \left| \int_{\Omega_\varepsilon} \mathcal{F}_{1,\varepsilon}^{(1)} U_\varepsilon dx \right| \leq C_2 \left(\varepsilon^{2\alpha} + \varepsilon^{2\alpha-2} \int_{\Omega_\varepsilon} |\partial_{x_1} U_\varepsilon|^2 dx \right), \quad (126)$$

$$\varepsilon^{\alpha-\beta+1} \left| \int_{\Omega_\varepsilon} \mathcal{F}_{2,\varepsilon}^{(1)} U_\varepsilon dx \right| \leq C_3 \left(\varepsilon^{\alpha-\beta+3} + \varepsilon^{\alpha-\beta+1} \int_{\Omega_\varepsilon} |\partial_{x_1} U_\varepsilon|^2 dx \right), \quad (127)$$

$$\varepsilon^{2\alpha-1} \left| \int_{\Gamma_\varepsilon} \Phi_\varepsilon^{(1)} U_\varepsilon d\sigma_x \right| \stackrel{(30)}{\leq} C_4 \left(\varepsilon^{2\alpha} + \varepsilon^{2\alpha} \int_{\Omega_\varepsilon} |\nabla_{\bar{x}_1} U_\varepsilon|^2 dx + \varepsilon^{2\alpha-2} \int_{\Omega_\varepsilon} |\partial_{x_1} U_\varepsilon|^2 dx \right). \quad (128)$$

Based on these estimates and taking suitable values for γ and ε_0 , we deduce from (125) the inequality

$$\int_{\Omega_\varepsilon} |\partial_{x_1} U_\varepsilon|^2 dx + \varepsilon^\beta \int_{\Omega_\varepsilon} |\nabla_{\bar{x}_1} U_\varepsilon|^2 dx \leq C_5 \left(\int_{\Omega_\varepsilon} W_\varepsilon^2 dx + \varepsilon^{2\alpha} + \varepsilon^{\alpha-\beta+3} \right) \quad (129)$$

for any $\varepsilon \in (0, \varepsilon_0)$.

Taking into account (1), (3), and (11), inequality (101) now looks like

$$\begin{aligned} & \int_{\Omega_\varepsilon} W_\varepsilon^2(x, \tau) dx + \int_0^\tau \int_{\Omega_\varepsilon} |\partial_{x_1} W_\varepsilon|^2 dx dt + \varepsilon^\beta \int_0^\tau \int_{\Omega_\varepsilon} |\nabla_{\bar{x}_1} W_\varepsilon|^2 dx dt \\ & \leq C_0 \left(|I_1(\varepsilon)| + |I_2(\varepsilon)| + \varepsilon^{2\alpha-1} \int_0^\tau \int_{\Gamma_\varepsilon} |\Phi_\varepsilon^{(2)}| |W_\varepsilon| d\sigma_x dt + \varepsilon^{2\alpha-2} \int_0^\tau \int_{\Omega_\varepsilon} |\mathcal{F}_{1,\varepsilon}^{(2)}| |W_\varepsilon| dx dt + \varepsilon^{\alpha-\beta+1} \int_0^\tau \int_{\Omega_\varepsilon} |\mathcal{F}_{2,\varepsilon}^{(2)}| |W_\varepsilon| dx dt \right), \quad (130) \end{aligned}$$

where $I_1(\varepsilon)$ and $I_2(\varepsilon)$ are determined using (84) and (85), respectively. The integrals on the right-hand side of (130) are estimated in the same way as integrals (126)–(128). Using Cauchy's inequality with any $\gamma > 0$, we bound $I_1(\varepsilon)$ and $I_2(\varepsilon)$ as follows:

$$|I_1(\varepsilon)| \leq \gamma_1 \int_0^\tau \int_{\Omega_\varepsilon} |\partial_{x_1} W_\varepsilon|^2 dx dt + \frac{C_1}{\gamma_1} \int_0^\tau \int_{\Omega_\varepsilon} |W_\varepsilon|^2 dx dt + \varepsilon^{\alpha-1} C_2 \int_0^\tau \int_{\Omega_\varepsilon} |W_\varepsilon|^2 dx dt + \varepsilon^{\alpha+1} C_3 \int_0^\tau \int_{\Omega_\varepsilon} |\nabla_{\bar{x}_1} W_\varepsilon|^2 dx dt,$$

$$|I_2(\varepsilon)| = \left| \int_0^\tau \int_{\Omega_\varepsilon} \lambda_w(S_\varepsilon) k_1(x_1) \partial_{x_1} U_\varepsilon \partial_{x_1} W_\varepsilon dx dt + \int_0^\tau \int_{\Omega_\varepsilon} \lambda_w(S_\varepsilon) \varepsilon^\beta (K \nabla_{\bar{x}_1} U_\varepsilon) \cdot \nabla_{\bar{x}_1} W_\varepsilon dx dt \right|$$

$$\leq \gamma_2 \int_0^\tau \int_{\Omega_\varepsilon} |\partial_{x_1} W_\varepsilon|^2 dx dt + \frac{C_4}{\gamma_2} \int_0^\tau \int_{\Omega_\varepsilon} |\partial_{x_1} U_\varepsilon|^2 dx dt + \varepsilon^\beta \gamma_3 \int_0^\tau \int_{\Omega_\varepsilon} |\nabla_{\bar{x}_1} W_\varepsilon|^2 dx dt + \varepsilon^\beta \frac{C_5}{\gamma_3} \int_0^\tau \int_{\Omega_\varepsilon} |\nabla_{\bar{x}_1} U_\varepsilon|^2 dx dt.$$

By choosing $\gamma_1, \gamma_2, \gamma_3$, and ε_1 to be suitably small, we derive from (130) the inequality

$$\int_{\Omega_\varepsilon} W_\varepsilon^2(x, \tau) dx + \int_0^\tau \int_{\Omega_\varepsilon} |\partial_{x_1} W_\varepsilon|^2 dx dt + \varepsilon^\beta \int_0^\tau \int_{\Omega_\varepsilon} |\nabla_{\bar{x}_1} W_\varepsilon|^2 dx dt$$

$$\leq C_3 \left(\varepsilon^{2\alpha} + \varepsilon^{\alpha-\beta+3} + \int_0^\tau \int_{\Omega_\varepsilon} |W_\varepsilon|^2 dx dt + \int_0^\tau \int_{\Omega_\varepsilon} |\partial_{x_1} U_\varepsilon|^2 dx dt + \varepsilon^\beta \int_0^\tau \int_{\Omega_\varepsilon} |\nabla_{\bar{x}_1} U_\varepsilon|^2 dx dt \right) \stackrel{(129)}{\leq} C_4 \left(\varepsilon^{2\alpha} + \varepsilon^{\alpha-\beta+3} + \int_0^\tau \int_{\Omega_\varepsilon} |W_\varepsilon|^2 dx dt \right) \quad (131)$$

for any $\varepsilon \in (0, \varepsilon_1)$.

Now applying Gronwall's lemma to the inequality

$$\int_{\Omega_\varepsilon} W_\varepsilon^2(x, \tau) dx \leq C_4 \left(\varepsilon^{2\alpha} + \varepsilon^{\alpha-\beta+3} + \int_0^\tau \int_{\Omega_\varepsilon} |W_\varepsilon|^2 dx dt \right),$$

we get

$$\max_{t \in [0, T]} \int_{\Omega_\varepsilon} W_\varepsilon^2(x, t) dx \leq C_5 (\varepsilon^{2\alpha} + \varepsilon^{\alpha-\beta+3}). \quad (132)$$

Estimates (117)–(119) follow directly from (132), (131), and (129). $\square$

The next corollary is a consequence of estimates (117)–(119).

Corollary 3 (the case $\alpha > \beta - 1$, $\alpha > 1$, and $\beta \neq 0$). *Let $(P_\varepsilon, S_\varepsilon)$ be a weak solution to problem $(\mathbb{P}_\varepsilon, \mathbb{S}_\varepsilon)$, and (p_0, s_0) be a solution to problem (107). Then the following asymptotic estimates hold:*

$$\max_{t \in [0, T]} \|\langle P_\varepsilon \rangle_{\varepsilon\omega} - p_0 - \varepsilon^{\alpha-1} p_{\alpha-1}\|_{H^1(0, \ell)} + \|\langle P_\varepsilon \rangle_{\varepsilon\omega} - p_0 - \varepsilon^{\alpha-1} p_{\alpha-1}\|_{C([0, \ell] \times [0, T])} \leq C_1 \varepsilon^{\aleph_{\alpha, \beta}}, \quad (133)$$

$$\|\langle S_\varepsilon \rangle_{\varepsilon\omega} - s_0 - \varepsilon^{\alpha-1} s_{\alpha-1}\|_{L^2(0, T; H^1(0, \ell))} + \|\langle S_\varepsilon \rangle_{\varepsilon\omega} - s_0 - \varepsilon^{\alpha-1} s_{\alpha-1}\|_{L^2(0, T; C([0, \ell]))} \leq C_2 \varepsilon^{\aleph_{\alpha, \beta}}, \quad (134)$$

where $\aleph_{\alpha, \beta} = \min\{\alpha - 1, \frac{\alpha - \beta + 1}{2}\}$.

5.2 | Subcase $\beta = 0$

This subcase is considered because it provides an opportunity to obtain more precise estimates than those presented in Theorem 3 and Corollary 3. It is obvious that the following approximation functions should be considered:

$$\mathfrak{P}_\varepsilon(x, t) = p_0(x_1, t) + \varepsilon^{\alpha-1} p_{\alpha-1}(x_1, t) + \varepsilon^{\alpha+1} u_{\alpha+1}\left(x_1, \frac{\bar{x}_1}{\varepsilon}, t\right) \quad (135)$$

and

$$\mathfrak{S}_\varepsilon(x, t) = s_0(x_1, t) + \varepsilon^{\alpha-1} s_{\alpha-1}(x_1, t), \quad (136)$$

where (p_0, s_0) and $(p_{\alpha-1}, s_{\alpha-1})$ are solutions to problems (107) and (108), respectively, $u_{\alpha+1}$ is a solution to problem (56).

Theorem 4 (case $\alpha > 1$, $\beta = 0$). Assume that, in addition to the main assumptions made in Section 2, condition (55) is satisfied. Then, there are positive constants $\tilde{C}_0$, $\tilde{C}_1$, and ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\frac{1}{\sqrt{1 - \Omega_\varepsilon \uparrow_3}} \max_{t \in [0, T]} \|P_\varepsilon(\cdot, t) - \mathfrak{P}_\varepsilon(\cdot, t)\|_{H^1(\Omega_\varepsilon)} \leq \tilde{C}_0 \varepsilon^{\aleph_\alpha} \quad (137)$$

and

$$\frac{1}{\sqrt{1 - \Omega_\varepsilon \uparrow_3}} \max_{t \in [0, T]} \|S_\varepsilon(\cdot, t) - \mathfrak{S}_\varepsilon(\cdot, t)\|_{L^2(\Omega_\varepsilon)} + \frac{1}{\sqrt{1 - \Omega_\varepsilon \uparrow_3}} \|S_\varepsilon - \mathfrak{S}_\varepsilon\|_{L^2(0, T; H^1(\Omega_\varepsilon))} \leq \tilde{C}_1 \varepsilon^{\aleph_\alpha}, \quad (138)$$

where $\aleph_\alpha := \min\{2(\alpha - 1), \alpha + 1\}$.

Proof. The proof of this theorem requires clarification compared to the proof of Theorem 1. Substituting functions (135) and (136) into problem $(\mathbb{P}_\varepsilon \mathbb{S}_\varepsilon)$, we find the same relations as in (109)–(114), but with $\beta = 0$. The same is true for the differences $U_\varepsilon = \mathfrak{P}_\varepsilon - P_\varepsilon$ and $W_\varepsilon = \mathfrak{S}_\varepsilon - S_\varepsilon$; they satisfy relations (121)–(124), in which $\beta = 0$.

The main difference with the corresponding stage in the proof of Theorem 1 is the presence of discrepancies in boundary conditions (79) and (80). Therefore, considering (116), the integral $\varepsilon^{2\alpha-1} \int_{\Gamma_\varepsilon} |U_\varepsilon| d\sigma_x$ appears on the right-hand side of inequality (81). Using (30) and (75), we obtain

$$\varepsilon^{2\alpha-1} \int_{\Gamma_\varepsilon} |U_\varepsilon| d\sigma_x \leq C \varepsilon^{2\alpha-1} \sqrt{\int_{\Omega_\varepsilon} |\nabla U_\varepsilon|^2 dx}.$$

As a result,

$$\int_{\Omega_\varepsilon} |\nabla U_\varepsilon(x, t)|^2 dx \leq C_1 \left(\varepsilon^{4\alpha-2} + \varepsilon^{\alpha+2} + \int_{\Omega_\varepsilon} W_\varepsilon^2(x, t) dx \right) \quad \text{for any } t \in [0, T]. \quad (139)$$

In addition, on the right side of inequality (130) there is an additional integral, which is estimated as follows:

$$\left| \varepsilon^{2\alpha-1} \int_0^\tau \int_{\Gamma_\varepsilon} \Phi_\varepsilon^{(2)} W_\varepsilon d\sigma_x dt \right| \stackrel{(116)}{\leq} C_2 \left(\varepsilon^{4\alpha-2} + \varepsilon \int_0^\tau \int_{\Gamma_\varepsilon} W_\varepsilon^2 d\sigma_x dt \right) \stackrel{(30)}{\leq} C_3 \left(\varepsilon^{4\alpha-2} + \varepsilon^2 \int_0^\tau \int_{\Omega_\varepsilon} |\nabla_{\bar{x}_1} W_\varepsilon|^2 dx dt + \int_0^\tau \int_{\Omega_\varepsilon} W_\varepsilon^2 dx dt \right). \quad (140)$$

We then choose ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$ the value $C_0 C_3 \varepsilon^2$ is less than a given positive number, so that we can move the integral $C_0 C_3 \varepsilon^2 \int_0^\tau \int_{\Omega_\varepsilon} |\nabla_{\bar{x}_1} W_\varepsilon|^2 dx dt$ to the left-hand side of the corresponding inequality (83) and obtain the inequality

$$\int_{\Omega_\varepsilon} W_\varepsilon^2(x, \tau) dx + \int_{\Omega_\varepsilon^\tau} |\nabla W_\varepsilon|^2 dx dt \leq C_4 \left(\varepsilon^{4\alpha-2} + \varepsilon^{\alpha+2} \int_{\Omega_\varepsilon^\tau} W_\varepsilon^2(x, t) dx dt \right) \quad (141)$$

for any $\tau \in (0, T)$. We then repeat the proof of Theorem 1 to derive estimates (137) and (138). $\square$

As we proved Corollary 1, from inequalities (137) and (138), we deduce the following corollary.

Corollary 4 (case $\alpha > 1$, $\beta = 0$). Let $(P_\varepsilon, S_\varepsilon)$ be a weak solution to problem $(\mathbb{P}_\varepsilon \mathbb{S}_\varepsilon)$, and let (p_0, s_0) and $(p_{\alpha-1}, s_{\alpha-1})$ be solutions to problems (107) and (108), respectively. Then the following estimates hold:

$$\max_{t \in [0, T]} \|\langle P_\varepsilon \rangle_{\varepsilon\omega} - p_0 - \varepsilon^{\alpha-1} p_{\alpha-1}\|_{H^1(0, \ell)} + \|\langle P_\varepsilon \rangle_{\varepsilon\omega} - p_0 - \varepsilon^{\alpha-1} p_{\alpha-1}\|_{C([0, \ell] \times [0, T])} \leq C_1 \varepsilon^{\aleph_\alpha}, \quad (142)$$

$$\|\langle S_\varepsilon \rangle_{\varepsilon\omega} - s_0 - \varepsilon^{\alpha-1} s_{\alpha-1}\|_{L^2(0, T; H^1(0, \ell))} + \|\langle S_\varepsilon \rangle_{\varepsilon\omega} - s_0 - \varepsilon^{\alpha-1} s_{\alpha-1}\|_{L^2(0, T; C([0, \ell]))} \leq C_2 \varepsilon^{\aleph_\alpha}, \quad (143)$$

where $\aleph_\alpha = \min\{2(\alpha - 1), \alpha + 1\}$.

6 | Approximations and Estimates for the Flow Velocities and Pressures

The obtained approximations for the solution to problem $(\mathbb{P}_\varepsilon \mathbb{S}_\varepsilon)$ for different values of the parameters α and β allow us to return to the Muskat–Leverett two-phase flow model (see Equations (4)–(7)), construct approximations for the flow velocities $\vec{V}_\varepsilon$, $\vec{V}_{w,\varepsilon}$, and $\vec{V}_{o,\varepsilon}$ and for the pressures $P_{w,\varepsilon}$ and $P_{o,\varepsilon}$, and prove the corresponding asymptotic estimates. As shown in the preceding sections, the best asymptotic estimates are derived when $\beta = 0$. Therefore, we will start with this case.

◆ If $\alpha = 1$ and $\beta = 0$, based on Equation (14) and using (64), we propose for the total velocity $\vec{V}_\varepsilon$ the approximation

$$\vec{V}_\varepsilon := - \left(\begin{array}{c} \lambda(s_0(x_1, t)) k_1(x_1) \partial_{x_1} p_0(x_1, t) \\ \varepsilon \lambda(s_0(x_1, t)) \mathbf{K}(x_1, \frac{\bar{x}_1}{\varepsilon}) \left(\nabla_{\bar{\xi}_1} u_2(x_1, \bar{\xi}_1) \right) \Big|_{\bar{\xi}_1 = \frac{\bar{x}_1}{\varepsilon}} \end{array} \right). \quad (144)$$

For the wetting phase velocity $\vec{V}_{w,\varepsilon}$, based on Equation (15), the following approximation is offered:

$$\vec{V}_{w,\varepsilon} := - \left(\begin{array}{c} \Lambda(s_0(x_1, t)) k_1(x_1) \partial_{x_1} s_0(x_1, t) \\ 0 \\ 0 \end{array} \right) + b(s_0(x_1, t)) \vec{V}_\varepsilon; \quad (145)$$

$$\vec{V}_{o,\varepsilon} := \vec{V}_\varepsilon - \vec{V}_{w,\varepsilon}. \quad (146)$$

Here, matrix $\mathbf{K}$ is given by (35), (p_0, s_0) is the solution to the limit problem (54), u_2 is the solution to problem (56).

Based on (13) and (7), we propose approximations for the pressures $P_{w,\varepsilon}$ and $P_{o,\varepsilon}$ in the following form:

$$\mathfrak{P}_{w,\varepsilon} := \mathfrak{P}_\varepsilon - \int_0^{s_0} \frac{\lambda_o(\eta)}{\lambda(\eta)} p'_c(\eta) d\eta, \quad \mathfrak{P}_{o,\varepsilon} := \mathfrak{P}_{w,\varepsilon} + p_c(s_0), \quad (147)$$

where $\mathfrak{P}_\varepsilon$ is taken from Equation (64).

Lemma 1 ($\alpha = 1, \beta = 0$). Suppose that the conditions of Theorem 1 are satisfied. Then, there are positive constants $\tilde{C}_1, \tilde{C}_2, \tilde{C}_3$, and ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\frac{1}{\sqrt{1} \Omega_\varepsilon \upharpoonright_3} \max_{t \in \times[0, T]} \left\| \vec{V}_\varepsilon(\cdot, t) - \vec{V}_\varepsilon(\cdot, t) \right\|_{(L^2(\Omega_\varepsilon))^3} \leq \tilde{C}_1 \varepsilon^2, \quad (148)$$

$$\frac{1}{\sqrt{1} \Omega_\varepsilon \upharpoonright_3} \left\| \vec{V}_{w,\varepsilon} - \vec{V}_{w,\varepsilon} \right\|_{(L^2(\Omega_\varepsilon^T))^3} \leq \tilde{C}_2 \varepsilon^2, \quad \frac{1}{\sqrt{1} \Omega_\varepsilon \upharpoonright_3} \left\| \vec{V}_{o,\varepsilon} - \vec{V}_{o,\varepsilon} \right\|_{(L^2(\Omega_\varepsilon^T))^3} \leq \tilde{C}_2 \varepsilon^2, \quad (149)$$

$$\frac{1}{\sqrt{1} \Omega_\varepsilon \upharpoonright_3} \|P_{w,\varepsilon} - \mathfrak{P}_{w,\varepsilon}\|_{L^2(0, T; H^1(\Omega_\varepsilon))} \leq \tilde{C}_3 \varepsilon^2, \quad (150)$$

$$\frac{1}{\sqrt{1} \Omega_\varepsilon \upharpoonright_3} \|P_{o,\varepsilon} - \mathfrak{P}_{o,\varepsilon}\|_{L^2(0, T; H^1(\Omega_\varepsilon))} \leq \tilde{C}_3 \varepsilon^2. \quad (151)$$

Proof. 1. According to (74) we have

$$\begin{aligned} \vec{V}_\varepsilon &= -\lambda(S_\varepsilon) \mathbb{K}_\varepsilon(x) \nabla P_\varepsilon = -\lambda(s_0 - W_\varepsilon) \mathbb{K}_\varepsilon(x) \nabla(\mathfrak{P}_\varepsilon - U_\varepsilon) \\ &= -\lambda(s_0) \mathbb{K}_\varepsilon \nabla \mathfrak{P}_\varepsilon + \lambda(s_0) \mathbb{K}_\varepsilon \nabla U_\varepsilon + \lambda'(\Theta_\varepsilon^{(1)}) W_\varepsilon \mathbb{K}_\varepsilon \nabla \mathfrak{P}_\varepsilon - \lambda'(\Theta_\varepsilon^{(1)}) W_\varepsilon \mathbb{K}_\varepsilon \nabla U_\varepsilon \\ &= \vec{V}_\varepsilon - \varepsilon^2 \left(\begin{array}{c} \lambda(s_0) k_1 \partial_{x_1} u_2 \\ 0 \\ 0 \end{array} \right) + \lambda(s_0) \mathbb{K}_\varepsilon \nabla U_\varepsilon + \lambda'(\Theta_\varepsilon^{(1)}) W_\varepsilon \mathbb{K}_\varepsilon \nabla \mathfrak{P}_\varepsilon - \lambda'(\Theta_\varepsilon^{(1)}) W_\varepsilon \mathbb{K}_\varepsilon \nabla U_\varepsilon. \end{aligned} \quad (152)$$

Now, using estimates (72) and (73) and considering (27) and (29), we derive estimate (148) from (152).

In the same way, we deduce

$$\begin{aligned}\vec{V}_{w,\varepsilon} &= -\Lambda(S_\varepsilon) \mathbb{K}_\varepsilon(x) \nabla S_\varepsilon + b(S_\varepsilon) \vec{V}_\varepsilon = -\Lambda(s_0 - W_\varepsilon) \mathbb{K}_\varepsilon(x) \nabla (s_0 - W_\varepsilon) + b(s_0 - W_\varepsilon) \vec{V}_\varepsilon \\ &= \vec{V}_{w,\varepsilon} + \Lambda(s_0) \mathbb{K}_\varepsilon \nabla W_\varepsilon + \Lambda'(\Theta_\varepsilon^{(2)}) W_\varepsilon \mathbb{K}_\varepsilon \nabla s_0 - \Lambda'(\Theta_\varepsilon^{(2)}) W_\varepsilon \mathbb{K}_\varepsilon \nabla W_\varepsilon + b(s_0) (\vec{V}_\varepsilon - \vec{V}_\varepsilon) - b'(\Theta_\varepsilon^{(3)}) W_\varepsilon \vec{V}_\varepsilon - b'(\Theta_\varepsilon^{(3)}) W_\varepsilon (\vec{V}_\varepsilon - \vec{V}_\varepsilon). \quad (153)\end{aligned}$$

Using the same inequalities as before and inequality (148), we obtain the first estimate in (149) from equality (153). The second estimate in (149) follows from the previous estimates, (146), and from the equality $\vec{V}_{o,\varepsilon} = \vec{V}_\varepsilon - \vec{V}_{w,\varepsilon}$.

2. From (13) we deduce that

$$P_{w,\varepsilon} - \mathfrak{P}_{w,\varepsilon} = P_\varepsilon - \mathfrak{P}_\varepsilon - \int_{s_0(x_1,t)}^{S_\varepsilon(x,t)} f(\eta) d\eta, \quad (154)$$

where $f(\eta) := \frac{\lambda_o(\eta)}{\lambda(\eta)} p'_c(\eta)$. Considering the assumptions for the relative permeabilities and capillary pressure (see (8)–(11)), f and its derivative are smooth and bounded in the domain of definition.

It is easy to see that

$$\int_{\Omega_\varepsilon^T} \left(\int_{s_0(x_1,t)}^{S_\varepsilon(x,t)} f(\eta) d\eta \right)^2 dx dt \leq C_1 \int_{\Omega_\varepsilon^T} (S_\varepsilon(x,t) - s_0(x_1,t))^2 dx dt, \quad (155)$$

$$\nabla_{\bar{x}_1} \left(\int_{s_0(x_1,t)}^{S_\varepsilon(x,t)} f(\eta) d\eta \right) = f(S_\varepsilon) \nabla_{\bar{x}_1} S_\varepsilon(x,t), \quad (156)$$

$$\partial_{x_1} \left(\int_{s_0(x_1,t)}^{S_\varepsilon(x,t)} f(\eta) d\eta \right) = f(S_\varepsilon) \partial_{x_1} S_\varepsilon(x,t) - f(s_0) \partial_{x_1} s_0(x_1,t) = f(S_\varepsilon) (\partial_{x_1} S_\varepsilon - \partial_{x_1} s_0) + \partial_{x_1} s_0 f'(\Theta_\varepsilon) (S_\varepsilon - s_0). \quad (157)$$

Considering (155)–(157) and using (72) and (73), we derive the first estimate in (151) from (154). The second follows from (7), the equality

$$P_{o,\varepsilon} - \mathfrak{P}_{o,\varepsilon} = P_{w,\varepsilon} - \mathfrak{P}_{w,\varepsilon} + p'_c(\Theta_\varepsilon^{(1)}) (S_\varepsilon - s_0),$$

and the boundedness of p'_c . □

◆ If $\alpha > 1$ and $\beta = 0$, the approximations of the velocities are as follows:

$$\vec{V}_\varepsilon := - \begin{pmatrix} \lambda(s_0) k_1 \partial_{x_1} p_0 + \varepsilon^{\alpha-1} (\lambda(s_0) k_1 \partial_{x_1} p_{\alpha-1} + \lambda'(s_0) k_1 \partial_{x_1} p_0 s_{\alpha-1}) \\ \varepsilon^\alpha \lambda(s_0(x_1,t)) \mathbf{K} \left(x_1, \frac{\bar{x}_1}{\varepsilon} \right) \left(\nabla_{\bar{\xi}_1} u_{\alpha+1}(x_1, \bar{\xi}_1) \right) \Big|_{\bar{\xi}_1 = \frac{\bar{x}_1}{\varepsilon}} \end{pmatrix}, \quad (158)$$

$$\vec{V}_{w,\varepsilon} := - \begin{pmatrix} \Lambda(s_0) k_1 \partial_{x_1} s_0 + \varepsilon^{\alpha-1} (\Lambda(s_0) k_1 \partial_{x_1} s_{\alpha-1} + \Lambda'(s_0) k_1 \partial_{x_1} s_0 s_{\alpha-1}) \\ 0 \\ 0 \end{pmatrix} + (b(s_0) + \varepsilon^{\alpha-1} b'(s_0) s_{\alpha-1}) \vec{V}_\varepsilon, \quad (159)$$

where (p_0, s_0) and $(p_{\alpha-1}, s_{\alpha-1})$ are solutions to problems (107) and (108), respectively, and $u_{\alpha+1}$ is a solution to problem (56); for the pressure $P_{w,\varepsilon}$ we take the approximation

$$\mathfrak{P}_{w,\varepsilon} := \mathfrak{P}_\varepsilon - \int_0^{\mathfrak{S}_\varepsilon} \frac{\lambda_o(\eta)}{\lambda(\eta)} p'_c(\eta) d\eta, \quad (160)$$

where $\mathfrak{P}_\varepsilon$ and $\mathfrak{S}_\varepsilon$ are determined using (135) and (136), respectively.

Using approximation formulas (137) and (138), similar to the previous one, we prove the lemma.

Lemma 2 ($\alpha > 1, \beta = 0$). Suppose that the conditions of Theorem 4 are satisfied. Let $\vec{V}_\varepsilon$ and $\vec{V}_{w,\varepsilon}$ be vector-valued functions determined by formulas (158) and (159), respectively, and $\mathfrak{P}_{w,\varepsilon}$ be a function defined by (160). Then, the following asymptotic estimates hold:

$$\frac{1}{\sqrt{1} \Omega_\varepsilon \upharpoonright_3} \max_{t \in [0, T]} \|\vec{V}_\varepsilon(\cdot, t) - \vec{\mathcal{V}}_\varepsilon(\cdot, t)\|_{(L^2(\Omega_\varepsilon))^3} + \frac{1}{\sqrt{1} \Omega_\varepsilon \upharpoonright_3} \|\vec{V}_{w, \varepsilon} - \vec{\mathcal{V}}_{w, \varepsilon}\|_{(L^2(\Omega_\varepsilon^T))^3} \leq \tilde{C}_0 \varepsilon^{\aleph_\alpha}, \quad (161)$$

$$\frac{1}{\sqrt{1} \Omega_\varepsilon \upharpoonright_3} \|P_{w, \varepsilon} - \mathfrak{P}_{w, \varepsilon}\|_{L^2(0, T; H^1(\Omega_\varepsilon))} \leq \tilde{C}_1 \varepsilon^{\aleph_\alpha}, \quad (162)$$

where $\aleph_\alpha := \min\{2(\alpha - 1), \alpha + 1\}$.

Remark 6. An approximation for $\vec{V}_{o, \varepsilon}$ is determined by formula (146), and for $P_{o, \varepsilon}$ by formula $\mathfrak{P}_{o, \varepsilon} := \mathfrak{P}_{w, \varepsilon} + p_c(\mathfrak{S}_\varepsilon)$. The corresponding asymptotic estimates follow from these formulas and estimates (161), (162), and (138). They are of the same order of magnitude. Therefore, we have not written about them.

◆ If $\alpha = 1$, $\beta < 2$, and $\beta \neq 0$, the approximation for the total velocity $\vec{V}_\varepsilon$ is given by the vector-function $\vec{\mathcal{V}}_\varepsilon = \left(\lambda(s_0) k_1 \partial_{x_1} p_0, 0, 0 \right)^\top$ ($\top$ is the sign of transposition), and the approximations for the velocity $\vec{V}_{w, \varepsilon}$ and pressure $P_{w, \varepsilon}$ are defined by Equations (145) and (147), respectively, where $\mathfrak{P}_\varepsilon$ is now determined by (94). For the same reasons as in Remark 6, we do not mention the functions $\vec{V}_{o, \varepsilon}$ and $P_{o, \varepsilon}$.

Lemma 3 ($\alpha = 1$, $\beta < 2$, and $\beta \neq 0$). Suppose that the conditions of Theorem 2 are satisfied, and denote by $U_{w, \varepsilon}$ the differences $P_{w, \varepsilon} - \mathfrak{P}_{w, \varepsilon}$. Then the following asymptotic estimates hold:

$$\frac{1}{\sqrt{1} \Omega_\varepsilon \upharpoonright_3} \sqrt{\int_0^T \int_{\Omega_\varepsilon} |\partial_{x_1} U_{w, \varepsilon}|^2 dx dt + \varepsilon^\beta \int_0^T \int_{\Omega_\varepsilon} |\nabla_{\bar{x}_1} U_{w, \varepsilon}|^2 dx dt} \leq \tilde{C}_1 \varepsilon^{1 - \frac{\beta}{2}}, \quad (163)$$

$$\frac{1}{\sqrt{1} \Omega_\varepsilon \upharpoonright_3} \max_{t \in [0, T]} \|\vec{V}_\varepsilon(\cdot, t) - \vec{\mathcal{V}}_\varepsilon(\cdot, t)\|_{(L^2(\Omega_\varepsilon))^3} + \frac{1}{\sqrt{1} \Omega_\varepsilon \upharpoonright_3} \|\vec{V}_{w, \varepsilon} - \vec{\mathcal{V}}_{w, \varepsilon}\|_{(L^2(\Omega_\varepsilon^T))^3} \leq \tilde{C}_2 \varepsilon^{\aleph_\beta}, \quad (164)$$

where $\aleph_\beta := \min\{1 - \frac{\beta}{2}, 1\}$.

Proof. First, note that from (154) it follows that $U_{w, \varepsilon}|_{x_1=0} = U_{w, \varepsilon}|_{x_1=\ell} = 0$. Then, similar to the second part of the proof of Lemma 1, but using estimates (95)–(97), we derive (163).

Next, as in the first part of the proof of Lemma 1, we prove that

$$\vec{V}_\varepsilon = \vec{\mathcal{V}}_\varepsilon - \varepsilon^{2-\beta} (\lambda(s_0) k_1 \partial_{x_1} u_{2-\beta}, 0, 0)^\top - \varepsilon (0, 0, \lambda(s_0) \mathbf{K} \nabla_{\bar{x}_1} u_{2-\beta})^\top + \lambda(s_0) \mathbb{K}_\varepsilon \nabla U_\varepsilon + \lambda'(\Theta_\varepsilon^{(1)}) W_\varepsilon \mathbb{K}_\varepsilon \nabla \mathfrak{P}_\varepsilon - \lambda'(\Theta_\varepsilon^{(1)}) W_\varepsilon \mathbb{K}_\varepsilon \nabla U_\varepsilon. \quad (165)$$

New in this proof is estimating the norm

$$\frac{1}{\sqrt{1} \Omega_\varepsilon \upharpoonright_3} \max_{t \in [0, T]} \|\mathbb{K}_\varepsilon \nabla U_\varepsilon\|_{(L^2(\Omega_\varepsilon))^3} \leq \frac{C_1}{\sqrt{1} \Omega_\varepsilon \upharpoonright_3} \max_{t \in [0, T]} \left(\|\partial_{x_1} U_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\varepsilon^\beta \nabla_{\bar{x}_1} U_\varepsilon\|_{(L^2(\Omega_\varepsilon))^2} \right) \stackrel{(95)}{\leq} C_2 \left(\varepsilon^{1 - \frac{\beta}{2}} + \varepsilon \right). \quad (166)$$

Then, using (95)–(97), we get inequality (164). $\square$

◆ If $\alpha > \beta - 1$, $\alpha > 1$, and $\beta \neq 0$, the approximation for the pressure $P_{w, \varepsilon}$ is determined by formula (160), but $\mathfrak{P}_\varepsilon$ and $\mathfrak{S}_\varepsilon$ are now determined in (105) and (106), respectively. The estimate for the difference $U_{w, \varepsilon} := P_{w, \varepsilon} - \mathfrak{P}_{w, \varepsilon}$ is derived in the same manner as in Lemmas 3 and 1, but it is necessary to use estimates (117)–(119).

Compared to the other cases, the order of the estimates was the smallest in this case. As follows from the previous case, the order of estimates for the flow velocities will become even smaller, indicating a complex influence of parameters α and β on the course of the process. We now propose the following approximations:

$$\vec{\mathcal{V}}_\varepsilon := -\left(\lambda(s_0) k_1 \partial_{x_1} p_0, 0, 0 \right)^\top, \quad \vec{\mathcal{V}}_{w, \varepsilon} := -\left(\Lambda(s_0) k_1 \partial_{x_1} s_0, 0, 0 \right)^\top + b(s_0) \vec{\mathcal{V}}_\varepsilon, \quad (167)$$

where (p_0, s_0) is a solution to problem (107).

Proceeding in the same way as before and using estimates (117)–(119), we derive

$$\vec{V}_\varepsilon = \vec{\mathcal{V}}_\varepsilon + \mathcal{O}(\varepsilon^{\alpha-1}) + \mathcal{O}(\varepsilon^{\aleph_{\alpha,\beta}}) + \mathcal{O}(\varepsilon^{\frac{\beta}{2} + \aleph_{\alpha,\beta}}) \quad \text{as } \varepsilon \rightarrow 0, \quad (168)$$

where $\aleph_{\alpha,\beta} = \min\{\alpha - 1, \frac{\alpha - \beta + 1}{2}\}$. So, if $\alpha \geq 3 - \beta$ (this means $\aleph_{\alpha,\beta} = \frac{\alpha - \beta + 1}{2}$), then

$$\vec{V}_\varepsilon = \vec{\mathcal{V}}_\varepsilon + \begin{cases} \mathcal{O}\left(\varepsilon^{\frac{\alpha - \beta + 1}{2}}\right), & \text{when } \beta > 0; \\ \mathcal{O}\left(\varepsilon^{\frac{\alpha + 1}{2}}\right), & \text{when } \beta < 0. \end{cases}$$

If $\alpha < 3 - \beta$, then $\vec{V}_\varepsilon = \vec{\mathcal{V}}_\varepsilon + \mathcal{O}(\varepsilon^{\alpha-1})$ for $\beta > 0$ (because of the other restrictions for α and β this means that $\beta \in (0, 2)$), and $\vec{V}_\varepsilon = \vec{\mathcal{V}}_\varepsilon + \mathcal{O}(\varepsilon^{\alpha-1+\frac{\beta}{2}})$ for $\beta < 0$ and $\alpha > -\frac{\beta}{2} + 1$. If $1 < \alpha < -\frac{\beta}{2} + 1$, additional asymptotic terms for P_ε and S_ε should be constructed (see Remark 5). In reality, parameter β is positive because the absolute permeability coefficients for different materials are very small (10^{-3} – 10^{-16} cm², see [1, section 5.5]). Therefore, in this case, we formulate the results for positive β .

Lemma 4 ($\alpha > \beta - 1$, $\alpha > 1$, and $\beta \neq 0$). Suppose that the conditions in Theorem 3 are satisfied. Then, the following asymptotic estimates hold:

$$\frac{1}{\sqrt{1} \Omega_\varepsilon \upharpoonright_3} \sqrt{\int_0^T \int_{\Omega_\varepsilon} |\partial_{x_1} U_{w,\varepsilon}|^2 dx dt + \varepsilon^\beta \int_0^T \int_{\Omega_\varepsilon} |\nabla_{\vec{x}_1} U_{w,\varepsilon}|^2 dx dt} \leq \tilde{C}_1 \varepsilon^{\aleph_{\alpha,\beta}},$$

where $U_{w,\varepsilon} = P_{w,\varepsilon} - \mathfrak{P}_{w,\varepsilon}$ and $\aleph_{\alpha,\beta} = \min\{\alpha - 1, \frac{\alpha - \beta + 1}{2}\}$;

$$\frac{1}{\sqrt{1} \Omega_\varepsilon \upharpoonright_3} \max_{t \in [0, T]} \|\vec{V}_\varepsilon(\cdot, t) - \vec{\mathcal{V}}_\varepsilon(\cdot, t)\|_{(L^2(\Omega_\varepsilon))^3} + \frac{1}{\sqrt{1} \Omega_\varepsilon \upharpoonright_3} \|\vec{V}_{w,\varepsilon} - \vec{\mathcal{V}}_{w,\varepsilon}\|_{(L^2(\Omega_\varepsilon^T))^3} \leq \tilde{C}_2 \varepsilon^{\frac{\alpha - \beta + 1}{2}}$$

for $\alpha \geq 3 - \beta$ and $\beta > 0$; and

$$\frac{1}{\sqrt{1} \Omega_\varepsilon \upharpoonright_3} \max_{t \in [0, T]} \|\vec{V}_\varepsilon(\cdot, t) - \vec{\mathcal{V}}_\varepsilon(\cdot, t)\|_{(L^2(\Omega_\varepsilon))^3} + \frac{1}{\sqrt{1} \Omega_\varepsilon \upharpoonright_3} \|\vec{V}_{w,\varepsilon} - \vec{\mathcal{V}}_{w,\varepsilon}\|_{(L^2(\Omega_\varepsilon^T))^3} \leq \tilde{C}_3 \varepsilon^{\alpha-1}$$

for $\alpha < 3 - \beta$ and $\beta \in (0, 2)$.

7 | Conclusions

1. In this work, we examined the nonlinear Muskat–Leverett two-phase flow model (4)–(7) in the thin cylinder Ω_ε , involving two additional parameters, α and β . The parameter α governs the flow rates of the mixture on the lateral surface Γ_ε , while β controls the scaling of the absolute permeability tensor in the transverse direction. Our results show that for a thin cylinder with cross-sectional diameter of order ε , the value $\alpha = 1$ is a critical threshold. In this case, the leading term of the asymptotics is described by problem (51) for $\beta < 2$, which corresponds to a one-dimensional Muskat–Leverett two-phase filtration model incorporating the specified lateral surface flows from the original problem. This limit problem can be reduced to the nonlinear parabolic Equation (53). For $\alpha > 1$ and $\beta < 2$, or $\alpha > \beta - 1$ with $\beta \geq 2$, the prescribed lateral flow rates on Γ_ε are of lower order, influencing only the second terms of the asymptotics, $p_{\alpha-1}$ and $s_{\alpha-1}$, which solve problem (108). In these cases, the limit problem is (107). For applications, it is advisable to consider both this limit problem and (108) to account for lateral exchange. In both parameter regimes analyzed, the derived asymptotics reveal a clear directional hierarchy in the flow within Ω_ε : The longitudinal component governs the leading-order behavior, while transversal contributions are asymptotically negligible at that order. This directional dominance arises naturally from the interplay between the thin-domain geometry, the scaling of the governing parameters, and the block-structured permeability tensor, which

together suppress cross-sectional transport to higher order corrections (see, e.g., the approximations for the total velocity in Section 6).

In practical applications such as the oil and gas industry, where fractured reservoirs are common, or environmental engineering, where fractured media often play a critical role in contaminant transport, lateral fluxes through fracture walls are typically not prescribed a priori but arise from interaction with the surrounding porous matrix. In a coupled fracture–matrix framework, these fluxes replace the function Q in our model as interface terms linking the fracture and matrix subproblems.

The one-dimensional Muskat–Leverett model developed here can be embedded into a coupled fracture–matrix framework, in which the fracture equations are solved together with a matrix flow model. In such a setting, the interface terms in the fracture equations, representing the lateral exchange fluxes, appear in the limit problem (51) in place of the function $\hat{Q}$. The velocity approximations for $\vec{V}_{w,\varepsilon}$ and $\vec{V}_{o,\varepsilon}$ obtained in Section 6 can then be used to evaluate these interface fluxes with high accuracy. This enables the fracture model to serve as a high-fidelity component in larger scale simulations of fractured reservoirs, where the surrounding medium is explicitly represented and the fracture–matrix exchange is resolved numerically (see, e.g., mixed-dimensional approaches in [10–14]). Such coupling enables the prediction of two-phase flow in realistic fractured systems, supporting informed and optimized decision-making in many applications, depending on the values of the parameters α and β . A rigorous mathematical justification of the reduced coupled fracture–matrix problem remains an interesting direction for future research.

Another potential application of our model relates to pore-network modeling, where the pore space is represented by one-dimensional pore throats connecting pore bodies (see, e.g., [39, Chapter 2]). Compared to full-dimensional pore-scale simulations, this geometric simplification significantly reduces computational cost and memory requirements. However, dimensionally reduced evolution equations on the pore throats have to be provided. Our model could exactly serve this purpose for two-phase flow.

If $\alpha < 1$ and $\beta < 2$, the term $\varepsilon^{\alpha-1}s_{\alpha-1}$ in (106) becomes unbounded as $\varepsilon \rightarrow 0$, violating the physical requirement that saturation remain in $[0,1]$. This corresponds to extremely high conductivity through the lateral surface, requiring a different modeling approach. The same applies to $\alpha < \beta - 1, \beta \geq 2$.

In the borderline case $\alpha = \beta - 1, \beta \geq 2$, the ansatz (105) indicates the onset of a “dual-porosity” regime, in which longitudinal and transverse two-phase flows occur at comparable rates. Consequently, the leading term in the expansion for the reduced pressure and saturation is no longer cross-sectionally uniform and will have the form

$$P_\varepsilon(x, t) \sim p_0\left(x_1, \frac{\bar{x}_1}{\varepsilon}, t\right) + \varepsilon^{\alpha-1}p_{\alpha-1}\left(x_1, \frac{\bar{x}_1}{\varepsilon}, t\right) + \cdots, \quad S_\varepsilon(x, t) \sim s_0\left(x_1, \frac{\bar{x}_1}{\varepsilon}, t\right) + \varepsilon^{\alpha-1}s_{\alpha-1}\left(x_1, \frac{\bar{x}_1}{\varepsilon}, t\right),$$

where $(p_0(x_1, \frac{\bar{x}_1}{\varepsilon}, t), s_0(p_0(x_1, \frac{\bar{x}_1}{\varepsilon}, t)))$ must solve a coupled system consisting of a nontrivial transverse profile determined by a cell two-phase flow problem posed on the cross-section with Neumann-type boundary conditions, and a modified one-dimensional Muskat–Leverett problem. The coefficients in both subproblems must encode the fractional-flow coupling in both the longitudinal and transverse directions. A direct application of the asymptotic expansion method used in this paper did not lead to unambiguous results. Therefore, two-scale convergence techniques appear to be more suitable for a rigorous derivation of the limit model. This represents an interesting and useful direction for future research. We also note that such two-scale techniques have been developed in the context of fluid flow in naturally fractured reservoirs, for example in petroleum engineering; see [40, 41].

2. This study relies on several assumptions. Smoothness of the given functions q_0, q_ℓ, S_0, Q , and of the coefficients of problem $(\mathbb{P}_\varepsilon \mathbb{S}_\varepsilon)$ is required to construct the asymptotic terms and estimate certain integrals. The assumption that Q vanishes near the bases of Ω_ε reflects many real situations, where penetration from solid rock into a fracture is local, and avoids the need to construct boundary layers near the bases.

To reduce technical complexity, we considered a rectilinear cylinder Ω_ε and assumed that the porosity ϕ depends only on the variable x_1 . The approach extends to curvilinear thin cylinders $\{x : 0 < x_1 < \ell_1, |\bar{x}_1| < \varepsilon h(x_1)\}$ with the smooth aperture function $h(x_1)$, $x_1 \in [0, \ell]$. Using the same approach as in [42], one can show that $h(x_1)$ will appear in the coefficients of the differential equations in the limit problem (51). The case of a nonconstant center-line of a curvilinear thin cylinder can be treated by the transformation method as in [20, section 2, Appendix A], although the analysis becomes substantially more intricate. If ϕ depends on all spatial variables, we can apply Taylor’s formula: $\phi(x) = \phi(x_1, 0, 0) + \varepsilon \nabla_{\bar{x}_1} \phi(x_1, 0, 0) \cdot \frac{\bar{x}_1}{\varepsilon} \Big|_{\frac{\bar{x}_1}{\varepsilon} = \frac{\bar{x}_1}{\varepsilon}} + \mathcal{O}(\varepsilon^2)$, and substitute this expansion together with the ansatzes (32)

and (33) into Equation (19). As a result, the coefficient $\phi(x_1, 0, 0)$ appears in the differential Equation (49) at $\partial_t s_0$.

In more general settings, the permeability tensor may be fully populated, and the coupling effects introduced by the off-diagonal terms can influence both the behavior of the solution and the form of the limit problem. In such cases, it may be necessary to modify the expansion strategy or employ alternative perturbation methods. This remains an

interesting direction for future research. However, in some cases, the asymptotic approach proposed here still applies. As shown in [6, section 4.2], even in non-thin regions, the off-diagonal components of the permeability tensor are often an order of magnitude smaller than the diagonal ones. For example, if we consider the permeability tensor

$$\mathbb{K}_\varepsilon = \begin{pmatrix} k_1(x_1) & \varepsilon k_{12}\left(x_1, \frac{\bar{x}_1}{\varepsilon}\right) & \varepsilon k_{13}\left(x_1, \frac{\bar{x}_1}{\varepsilon}\right) \\ \varepsilon k_{21}\left(x_1, \frac{\bar{x}_1}{\varepsilon}\right) & \varepsilon^\beta k_{22}\left(x_1, \frac{\bar{x}_1}{\varepsilon}\right) & \varepsilon^\beta k_{23}\left(x_1, \frac{\bar{x}_1}{\varepsilon}\right) \\ \varepsilon k_{31}\left(x_1, \frac{\bar{x}_1}{\varepsilon}\right) & \varepsilon^\beta k_{32}\left(x_1, \frac{\bar{x}_1}{\varepsilon}\right) & \varepsilon^\beta k_{33}\left(x_1, \frac{\bar{x}_1}{\varepsilon}\right) \end{pmatrix}$$

for $\beta \in [0, 1]$, then the approach presented here yields the same limit problem (51) for $\alpha = 1$ or (107) for $\alpha > 1$. We also note that convergence results for linear elliptic problems with a fully populated hydraulic conductivity matrix in a thin fracture were obtained in [20], and for nonlinear monotone problems in a junction of multiple thin cylinders in [43].

The main assumptions—regularity of capillary pressure and neglect of gravity in Darcy’s law—ensure the maximum principle (see (27)), which in turn guarantees the regularity of the problem in the sense of (28) and allows asymptotic estimates. Some studies assume regularity directly (e.g., [24]). Removing these assumptions would be an interesting challenge.

An important related problem is the analysis of a nonlinear Muskat–Leverett two-phase filtration model in thin graph-like networks that more accurately represent complex fracture geometries. In this setting, the principal mathematical difficulty lies in deriving the corresponding limit problem on a metric graph, equipped with appropriate Kirchhoff-type transmission conditions at the vertices to ensure mass conservation and correct coupling between the edges.

Conflicts of Interest

The authors declare no conflicts of interest.

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Data Availability Statement

The data that support the findings of this study are included within this article. A preprint of the manuscript is available on the arXiv platform at <https://doi.org/10.48550/arXiv.2411.02923>. No additional datasets were generated or analyzed during the current study.

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